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RAMANUJAN'S MASTER THEOREM FOR RIEMANNIAN SYMMETRIC SPACES

GESTUR ÓLAFSSON AND ANGELA PASQUALE

ABSTRACT. Ramanujan's Master theorem states that, under suitable conditions, the Mellin transform of a power series provides an interpolation formula for the coefficients of this series. Based on the duality of compact and non-compact reductive Riemannian symmetric spaces inside a common complexification, we prove an analogue of Ramanujan's Master Theorem for the spherical Fourier transform of a spherical Fourier series. This extends the results proven by Bertram for Riemannian symmetric spaces of rank-one.

INTRODUCTION

Ramanujan's Master Theorem asserts that if a function $f(x)$ can be expanded around $x = 0$ in a power series of the form

$$f(x) = \sum_{k=0}^{\infty} (-1)^k a(k) x^k$$

then

$$\int_0^{+\infty} x^{-\lambda-1} f(x) dx = -\frac{\pi}{\sin(\pi\lambda)} a(\lambda). \quad (1)$$

Of course, one needs additional assumptions for this formula to hold, as one can directly see from the example $a(\lambda) = \sin(\pi\lambda)$.

Formula (1) is contained in Ramanujan's First Quarterly Report to the Board of Studies of the University of Madras in 1913. These reports have never been published; see [2], p. 295. However, Hardy presents them in his book on Ramanujan's work [11] and provides a rigorous proof of formula (1) for a natural class of functions a and a natural set of parameters λ , by means of the Residue Theorem.

Let A, P, δ be real constants so that $A < \pi$ and $0 < \delta \leq 1$. Let $\mathcal{H}(\delta) = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\delta\}$. The Hardy class $\mathcal{H}(A, P, \delta)$ consists of all functions $a : \mathcal{H}(\delta) \rightarrow \mathbb{C}$ that are holomorphic on $\mathcal{H}(\delta)$ and satisfy the growth condition

$$|a(\lambda)| \leq C e^{-P(\operatorname{Re} \lambda) + A|\operatorname{Im} \lambda|}$$

for all $\lambda \in \mathcal{H}(\delta)$. Hardy's version of Ramanujan's Master theorem is the following, see [11], p. 189.

Theorem 0.1 (Ramanujan's Master Theorem). *Suppose $a \in \mathcal{H}(A, P, \delta)$. Then:*

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(1) *The power series*

$$f(x) = \sum_{k=0}^{\infty} (-1)^k a(k) x^k \quad (2)$$

converges for $0 < x < e^P$ and defines a real analytic function on this domain.

(2) *Let $0 < \sigma < \delta$. For $0 < x < e^P$ we have*

$$f(x) = \frac{1}{2\pi i} \int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{-\pi}{\sin(\pi\lambda)} a(\lambda) x^\lambda d\lambda. \quad (3)$$

The integral on the right hand side of (3) converges uniformly on compact subsets of $]0, +\infty[$ and is independent of the choice of σ .

(3) *Formula (1) holds for the extension of f to $]0, +\infty[$ and for all $\lambda \in \mathbb{C}$ with $0 < \operatorname{Re} \lambda < \delta$.*

The last part of Theorem 0.1 is obtained from its second part by applying Mellin's inversion formulas

$$\psi(\lambda) = \int_0^\infty \phi(x) x^{-\lambda} \frac{dx}{x}, \quad \phi(x) = \frac{1}{2\pi i} \int_{-\sigma-i\infty}^{-\sigma+i\infty} \psi(\lambda) x^\lambda d\lambda$$

to

$$\psi(\lambda) = \frac{-\pi}{\sin(\pi\lambda)} a(\lambda), \quad \phi(x) = f(x).$$

Formula (1) can be thought of as an interpolation formula allowing us to reconstruct $a(\lambda)$ from the discrete set of its values $\{a(k) : k \in \mathbb{Z}^+\}$. In particular, the function a must vanish identically whenever it vanishes on \mathbb{Z}^+ . As already noticed by Hardy ([11], p. 188), this fact and the bound π for the exponent A are related to the well-known theorem of Carlson.

An equivalent formulation of Ramanujan's Master Theorem is obtained by replacing $a(\lambda) \in \mathcal{H}(A, P, \delta)$ with $A(\lambda) = a(\lambda)\Gamma(\lambda+1)$ where Γ is the gamma function. Since $\Gamma(-\lambda)\Gamma(\lambda+1) = -\pi/\sin(\pi\lambda)$, this converts the power series (2) into

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{A(k)}{k!} x^k$$

and formula (1) becomes

$$\int_0^{+\infty} f(x) x^{-\lambda} \frac{dx}{x} = \Gamma(-\lambda) A(\lambda). \quad (4)$$

It holds for the extension of f to $]0, +\infty[$ and for all $\lambda \in \mathbb{C}$ with $0 < \operatorname{Re} \lambda < \delta$. In [11], §11.3, formula (1) is taken as the standard interpolation formula, though (4) often turns out to be more convenient in the applications.

In [4], Bertram provided the following group theoretic interpretation of (1). The functions x^λ ($\lambda \in \mathbb{C}$) are the spherical functions on $X_G = \mathbb{R}^+$ and the x^k ($k \in \mathbb{Z}$) are the spherical functions on the torus $X_U = U(1)$. Both X_G and X_U can be realized as real forms of their complexification $X_{\mathbb{C}} = \mathbb{C}^*$. Then (2) and (3) can be seen respectively as the restriction to X_U and X_G of a "good" holomorphic function f on $X_{\mathbb{C}}$. Let \tilde{f} and \hat{f} respectively denote the spherical Fourier transforms of f on X_G and X_U . Then Ramanujan's formula (1) becomes

$$\tilde{f}(\lambda) = \frac{-\pi}{\sin(\pi\lambda)} a(\lambda), \quad \hat{f}(k) = (-1)^k a(k).$$

By replacing the duality between $U(1)$ and \mathbb{R}^+ inside \mathbb{C}^* with the duality between symmetric spaces of the compact type $X_U = U/K$ and of noncompact type $X_G = G/K$ inside their complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, Bertram proved an analogue of Ramanujan's Master theorem for semisimple Riemannian symmetric spaces of rank one.

In this paper we extend Bertram's results first to semisimple Riemannian symmetric spaces of arbitrary rank and then to reductive symmetric spaces. Our main result is Theorem 2.1 which is Ramanujan's Master Theorem for the spherical Fourier transform on semisimple Riemannian symmetric spaces. The generalization to reductive symmetric spaces is then obtained by combining the semisimple case and a multivariable extension of the classical (abelian) Theorem 0.1; see Theorem 7.1.

Let X_U and X_G be dual Riemannian symmetric spaces, respectively of the compact and of the noncompact type, and let $X_{\mathbb{C}}$ be their complexification. Our theorem deals with spherical Fourier series on X_U of the form

$$f(x) = \sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \psi_{\mu}(x), \quad (5)$$

in which the coefficients $a(\mu + \rho)$ are obtained from a holomorphic function a belonging to a certain Hardy class $\mathcal{H}(A, P, \delta)$ associated with the pair (X_U, X_G) and depending on three real parameters A, P, δ . The function f defines a K -invariant holomorphic function on a neighborhood of X_U in $X_{\mathbb{C}}$. It is then shown to extend holomorphically to a neighborhood of X_G in $X_{\mathbb{C}}$ by means of the inverse spherical Fourier transform:

$$f(x) = \int_{\sigma + ia^*} a(\lambda) b(\lambda) \varphi_{\lambda}(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}. \quad (6)$$

Formulas (5) and (6) are the symmetric space analogue of Ramanujan's formulas (2) and (3). An interpolation formula extending (1) is then obtained from (6) using inversion theorems for the spherical Fourier transform. The function b occurring in (6) is a normalizing factor depending only on the pair (X_U, X_G) (and not on the Hardy class). It plays the rôle of the function $-(2i \sin(\pi x))^{-1}$ appearing in Ramanujan's formula (3). We refer the reader to Theorems 2.1 and 7.1 for more precise statements of our results and for the unexplained notation in (5) and (6).

As in the classical proof by Hardy, our principal tool to prove Ramanujan's Master Theorem in the semisimple case is the Residue Theorem. Our methods are a multivariable generalization of those of Bertram, and in fact we follow several hints to the general rank case that one can find in Section 0.6 of [4]. To deal with the task of taking residues in our multivariable setting, the main idea is to select the fundamental highest restricted weights as a basis of the space $\mathfrak{a}_{\mathbb{C}}^*$ of spectral parameters and then to work in the corresponding coordinates.

Besides [4], some other articles have dealt with the extension of Ramanujan's Master Theorem to special classes of semisimple or reductive symmetric spaces. Bertram [3] and Ding, Gross and Richards [8] have proven this theorem for the (reductive non semisimple) case of symmetric cones. The version considered in [8] corresponds to Ramanujan's formula (B) in Hardy's book [11], p. 186. Moreover, Ding [7] proved a slightly different version of Ramanujan's Master theorem for Hermitian symmetric spaces by a reduction to the case of symmetric cones from [8]. Because of this indirect proving method, the resulting theorem

has a more complicated structure than the special case of our Theorem 2.1 for the Hermitian symmetric situation.

This article is organized as follows. In Section 1 we recall the duality of Riemannian symmetric spaces of compact and noncompact type, we introduce the spherical transform in these contexts, and recall Lassalle's results on the holomorphic extension of spherical Fourier series. In section 2 we state Ramanujan's Master theorem for Riemannian symmetric spaces in duality. In the classical version of the Master Theorem, the interpolated coefficients of the power series are multiplied by a normalizing factor $b(\lambda)$ which is a constant multiple of the function $\sin(\pi\lambda)^{-1}$. Similarly, a normalizing function $b(\lambda)$ appears in the statement of Master's Theorem for symmetric spaces. In Section 3 we explain our choice of the function b . Its explicit expression is determined in Section 4, using the relation between the Plancherel densities for Riemannian symmetric spaces in duality. Examples for Riemannian symmetric spaces of rank-one or with even root multiplicities are presented. Section 5 collects several estimates which will be needed in the proof of Ramanujan's Master Theorem. This theorem is proven in Section 6. In Section 7 we extend our theorem to reductive Riemannian symmetric spaces. In the final Section 8, we discuss some further possible extensions of Ramanujan's Theorem for symmetric spaces to related settings and present some open problems.

Notation. We shall use the standard notation \mathbb{Z} , \mathbb{Z}^+ , \mathbb{R} , \mathbb{R}^+ and \mathbb{C} respectively for the integers, the nonnegative integers, the reals, the nonnegative reals, and the complex numbers.

1. PRELIMINARIES

1.1. Symmetric spaces and their structure. Let $X_U = U/K$ be a Riemannian symmetric space of the compact type. Hence U is a compact connected Lie group, and there is an involutive automorphism τ of U so that $U_0^\tau \subset K \subset U^\tau$. Here $U^\tau = \{u \in U : \tau(u) = u\}$ and U_0^τ is its connected component containing the unit element e of U . We will assume that K is connected and U is semisimple. (This assumption on U will be removed in Section 7.) As U is compact, it admits a finite dimensional faithful representation. So we can assume that $U \subset U(m) \subset \mathrm{GL}(m, \mathbb{C})$ for some m . Let $\mathfrak{u} \subset \mathrm{M}(m, \mathbb{C})$ denote the Lie algebra of U . The automorphism of \mathfrak{u} which is the differential of τ will be indicated by the same symbol. Then $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ where $\mathfrak{k} = \{X \in \mathfrak{u} : \tau(X) = X\}$ is the Lie algebra of K and $i\mathfrak{p} = \{X \in \mathfrak{u} : \tau(X) = -X\}$. Let $U_{\mathbb{C}}$ denote the analytic subgroup of $\mathrm{GL}(m, \mathbb{C})$ of Lie algebra $\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$. Set $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let G be analytic subgroup of $U_{\mathbb{C}}$ of Lie algebra \mathfrak{g} . Then G is a connected noncompact semisimple Lie group containing K , and $X_G = G/K$ is a symmetric space of the noncompact type. Since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}$, the group $U_{\mathbb{C}}$ is a complexification of G . We will write $G_{\mathbb{C}} = U_{\mathbb{C}}$. Let $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$ and let $K_{\mathbb{C}}$ be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{k}_{\mathbb{C}}$. Then $K_{\mathbb{C}}$ is a closed subgroup of $G_{\mathbb{C}}$. The symmetric spaces $X_U = U/K$ and $X_G = G/K$ embed in the complex homogeneous space $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ as totally real submanifolds.

A maximal abelian subspace \mathfrak{a} of \mathfrak{p} is called a Cartan subspace. The dimension of any Cartan subspace is a constant, called the *real rank* of G and the *rank* of X_U and X_G . Let \mathfrak{a}^* be the (real) dual space of \mathfrak{a} and let $\mathfrak{a}_{\mathbb{C}}^*$ be its complexification.

Let Σ be the set of (restricted) roots of the pair $(\mathfrak{g}, \mathfrak{a})$. It consists of all nonzero $\alpha \in \mathfrak{a}^*$ for which the vector space $\mathfrak{g}_{\alpha} := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for every } H \in \mathfrak{a}\}$ is nonzero. The dimension of \mathfrak{g}_{α} is called the *multiplicity* of the root α and is denoted by m_{α} . We fix a set Σ^+ of positive restricted roots. Then Σ is the disjoint union of Σ^+ and $-\Sigma^+$. Moreover,

$\mathfrak{a}^+ := \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+\}$ is an open polyhedral cone called the positive Weyl chamber.

A root $\alpha \in \Sigma$ is said to be *unmultipliable* if $2\alpha \notin \Sigma$. We respectively denote by Σ_* and $\Sigma_*^+ := \Sigma^+ \cap \Sigma_*$ the sets of unmultipliable roots and of positive unmultipliable roots in Σ . The half-sum of the positive roots counted with multiplicities is denoted by ρ : hence

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \frac{1}{2} \sum_{\beta \in \Sigma_*^+} \left(\frac{m_{\beta/2}}{2} + m_\beta \right) \beta. \quad (7)$$

Here we adopt the usual convention that the multiplicity $m_{\beta/2}$ is zero if $\beta/2$ is not a root. By classification, $m_{\beta/2}$ is always even.

The Cartan-Killing form B defines a Euclidean structure on the Cartan subspace \mathfrak{a} . We set $\langle X, Y \rangle := B(X, Y)$. We extend this inner product to \mathfrak{a}^* by duality, that is we set $\langle \lambda, \mu \rangle := \langle H_\lambda, H_\mu \rangle$ if H_γ is the unique element in \mathfrak{a} such that $\langle H_\gamma, H \rangle = \gamma(H)$ for all $H \in \mathfrak{a}$. The \mathbb{C} -bilinear extension of $\langle \cdot, \cdot \rangle$ to $\mathfrak{a}_\mathbb{C}^*$ will be denoted by the same symbol. We shall employ the notation

$$\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \quad (8)$$

for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $\alpha \in \mathfrak{a}^*$ with $\alpha \neq 0$. Notice that $2\lambda_\alpha = \lambda_{\alpha/2}$.

The *Weyl group* W of Σ is the finite group of orthogonal transformations of \mathfrak{a} generated by the reflections r_α with $\alpha \in \Sigma$, where

$$r_\alpha(H) := H - 2 \frac{\alpha(H)}{\langle \alpha, \alpha \rangle} H_\alpha, \quad H \in \mathfrak{a}.$$

The Weyl group action extends to \mathfrak{a}^* by duality, and to $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ by complex linearity.

We set $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Let $N = \exp \mathfrak{n}$ and $A = \exp \mathfrak{a}$ be the connected subgroups of G of Lie algebra \mathfrak{n} and \mathfrak{a} , respectively. The map $(k, a, n) \mapsto kan$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G . The resulting decomposition $G = KAN$ is the Iwasawa decomposition of G . Thus, for $g \in G$ we have $g = k(g) \exp H(g) n(g)$ for uniquely determined $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in G$. We will also need the polar decomposition $G = KAK$: every $g \in G$ can be written in the form $g = k_1 a k_2$ with $k_1, k_2 \in K$ and $a \in A$. The element a is unique up to W -invariance. It is therefore uniquely determined in $\overline{A^+}$ where $A^+ = \exp \mathfrak{a}^+$.

1.2. Normalization of measures. We adopt the normalization of measures from [15], Ch. II, §3.1. In particular, the Haar measures dk and du on the compact groups K and U are normalized to have total mass 1. The Haar measures da and $d\lambda$ on A and \mathfrak{a}^* , respectively, are normalized so that the Euclidean Fourier transform

$$(\mathcal{F}_A f)(\lambda) := \int_A f(a) e^{-i\lambda(\log a)} da, \quad \lambda \in \mathfrak{a}^*, \quad (9)$$

of a sufficiently regular function $f : A \rightarrow \mathbb{C}$ is inverted by

$$f(a) = \int_{\mathfrak{a}^*} (\mathcal{F}_A f)(\lambda) e^{i\lambda(\log a)} d\lambda, \quad a \in A. \quad (10)$$

The Haar measures dg and dn of G and N , respectively, are normalized so that $dg = e^{2\rho(\log a)} dk da dn$. Moreover, if L is a Lie group and P is a closed subgroup of L , with left Haar

measures dl and dp , respectively, then the L -invariant measure $d(lP)$ on the homogeneous space L/P (when it exists) is normalized so that

$$\int_L f(l) dl = \int_{L/P} \left(\int_P f(lp) dp \right) d(lP). \quad (11)$$

This condition normalizes the G -invariant measure $dx = d(gK)$ on $X_G = G/K$ and fixes the U -invariant measure $d(uK)$ on $X_U = U/K$ to have total mass 1.

1.3. Spherical functions on X_G and X_U . Let (π, V) be an irreducible unitary representation of G or of U , and let

$$V^K = \{v \in V : \pi(k)v = v \text{ for all } k \in K\}$$

be the subspace of the K -fixed vectors of V . The representation (π, V) is said to be *K-spherical* if $V^K \neq \{0\}$. In this case $\dim V^K = 1$.

The spherical functions on G are the matrix coefficients of the (non-unitary) spherical principal series representations. The spherical function on G of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is the K -biinvariant function $\varphi_\lambda : G \rightarrow \mathbb{C}$ given by

$$\varphi_\lambda(g) = \int_K e^{(\lambda - \rho)(H(gk))} dk \quad (12)$$

where $H : G \rightarrow \mathfrak{a}$ is the Iwasawa projection defined earlier. If e denotes the unit element of G , then $\varphi_\lambda(e) = 1$. Moreover, $\varphi_\lambda(g^{-1}) = \varphi_{-\lambda}(g)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $g \in G$. The function $\varphi_\lambda(g)$ is real analytic in $g \in G$ and W -invariant and entire in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

According to Helgason's theorem, the highest restricted weights of the finite-dimensional K -spherical representations of U are the dominant restricted weights, that is the elements of the set

$$\Lambda^+ = \{\mu \in \mathfrak{a}^* : \mu_\alpha \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+\}. \quad (13)$$

See [14], Ch. V, Theorem 4.1.

Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of \mathfrak{a}^* consisting of simple roots in Σ^+ . For $j = 1, \dots, l$ set

$$\beta_j = \begin{cases} \alpha_j & \text{if } 2\alpha_j \notin \Sigma \\ 2\alpha_j & \text{if } 2\alpha_j \in \Sigma \end{cases}. \quad (14)$$

Then $\Pi_* = \{\beta_1, \dots, \beta_l\}$ is a basis of \mathfrak{a}^* consisting of simple roots in Σ_*^+ . Define $\omega_1, \dots, \omega_l \in \mathfrak{a}^*$ by the conditions

$$(\omega_j)_{\beta_k} = \frac{\langle \omega_j, \beta_k \rangle}{\langle \omega_k, \beta_k \rangle} = \delta_{jk}. \quad (15)$$

Let $\mu \in \mathfrak{a}^*$. Then $\mu \in \Lambda^+$ if and only if

$$\mu = \sum_{j=1}^l \mu_j \omega_j \quad \text{with } \mu_j \in \mathbb{Z}^+, j = 1, \dots, l.$$

See [15], Ch. 2, Proposition 4.23.

The spherical functions on U are matrix coefficients of the finite-dimensional (non-unitary) spherical representations on U . As U is semisimple, the spherical functions on U are parametrized by a subset $\Lambda_K^+(U)$ of Λ^+ . If $X_U = U/K$ is simply connected, then $\Lambda_K^+(U) =$

Λ^+ . See [28], Ch. II, Theorem 8.2 and Corollary 1, Theorem 6.1(2), and p. 109 and 104. If $\mu \in \Lambda_K^+(U)$, the corresponding spherical function is given by

$$\psi_\mu(u) = \langle \pi_\mu(u)e_\mu, e_\mu \rangle, \quad (16)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the space V_μ of π_μ for which this representation is unitary and $e_\mu \in V_\mu^K$ is a unit vector.

The spherical functions on U are linked to the spherical functions on G by holomorphic continuation. More precisely, for every $\mu \in \Lambda_K^+(U)$, the spherical function ψ_μ on U with spectral parameter μ extends holomorphically to a $K_{\mathbb{C}}$ -biinvariant function on $G_{\mathbb{C}}$. Its restriction to G is K -biinvariant and coincides with the spherical function $\varphi_{\mu+\rho}$ of G . See [14], pp. 540–541, or [6], Lemma 2.5. The spherical function φ_λ of G extends holomorphically to a $K_{\mathbb{C}}$ -biinvariant function on $G_{\mathbb{C}}$ if and only if λ belongs to the W -orbit of $\Lambda_K^+(U) + \rho$. Notice that, as a matrix coefficient of a unitary representation, $\overline{\psi_\mu(u)} = \psi_\mu(u^{-1})$ for all $u \in U$.

If L is a group acting on a space X and $F(X)$ is a space of functions on X , then we shall denote by $F(X)^L$ the subspace of L -invariant elements of $F(X)$. In the following, the K -biinvariant functions on G (resp. on U) will be often identified with the K -invariant functions on $X_G = G/K$ (resp. on $X_U = U/K$). In particular, one can consider the φ_λ 's as K -invariant functions on X_G and the ψ_μ as K -invariant functions on $X_U = U/K$. In this case, one can think of the spherical functions on X_U as the restrictions of the holomorphic extension of the corresponding spherical functions on X_G :

$$\psi_\mu = \varphi_{\mu+\rho}|_{X_U}, \quad \mu \in \Lambda_K^+(U). \quad (17)$$

1.4. Spherical harmonic analysis on X_G and X_U . The *spherical Fourier transform* of a (sufficiently regular) K -invariant function $f : X_G \rightarrow \mathbb{C}$ is the function $\tilde{f} = \mathcal{F}_G f$ defined by

$$\tilde{f}(\lambda) = \mathcal{F}_G f(\lambda) = \int_{X_G} f(x)\varphi_{-\lambda}(x) dx \quad (18)$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which this integral exists. The Plancherel theorem states that the spherical Fourier transform \mathcal{F}_G extends as an isometry of $L^2(X_G)^K$ onto $L^2(i\mathfrak{a}^*, |W|^{-1}|c(\lambda)|^{-2}d\lambda)^W$. Here $|W|$ denotes the order of the Weyl group W . The function $c(\lambda)$ occurring in the Plancherel density is Harish-Chandra's c -function. It is the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ given explicitly by the Gindikin-Karpelevich product formula. In terms of Σ_*^+ , we have

$$c(\lambda) = c_0 \prod_{\beta \in \Sigma_*^+} c_\beta(\lambda) \quad (19)$$

where

$$c_\beta(\lambda) = \frac{2^{-2\lambda_\beta} \Gamma(\tilde{2\lambda}_\beta)}{\Gamma\left(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{1}{2}\right)\Gamma\left(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{m_\beta}{2}\right)} \quad (20)$$

and the constant c_0 is given by the condition $c(\rho) = 1$. Formula (20) looks slightly different from the usual formula for the c -function as found for instance in [14], Ch. IV, Theorem 6.4, where it is written in terms of positive indivisible roots ($\alpha \in \Sigma^+$ with $\alpha/2 \notin \Sigma^+$) rather than in terms of positive unmultipliable roots.

The spherical Fourier transform has the following inversion formula, which holds for instance a.e. if $f \in L^p(X_G)^K$, with $1 \leq p < 2$, and $\tilde{f} \in L^1(i\mathfrak{a}^*, |c(\lambda)|^{-2}d\lambda)^W$: for almost all

$x \in X_G$ we have

$$f(x) = \frac{1}{|W|} \int_{i\mathfrak{a}^*} \tilde{f}(\lambda) \varphi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}. \quad (21)$$

See [27], Theorem 3.3.

We shall also need some properties of the K -invariant L^p -Schwartz spaces $\mathcal{S}^p(X_G)^K$ on X_G . Let $1 < p \leq 2$ and let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. The K -invariant L^p -Schwartz space $\mathcal{S}^p(X_G)^K$ is the space of all C^∞ K -bi-invariant functions $f : G \rightarrow \mathbb{C}$ such that for any $D \in \mathfrak{U}(\mathfrak{g})$ and any integer $N \geq 0$ we have

$$\sup_{g \in G} (1 + \sigma(g))^N \varphi_0(g)^{-2/p} |(Df)(g)| < \infty.$$

Here $\sigma(g) = \|H\|$ if $g = k_1 \exp(H)k_2$ for $k_1, k_2 \in K$ and $H \in \mathfrak{a}^+$. Then $\mathcal{S}^p(X_G)^K \subset \mathcal{S}^2(X_G)^K$. Moreover, by identifying as usual K -biinvariant functions on G with K -invariant functions on X_G , we have $\mathcal{S}^p(X_G)^K \subset L^p(X_G)^K$.

Let $1 < p < 2$ and let $\varepsilon = \frac{2}{p} - 1$. Let $C(\varepsilon\rho)^0$ denote the interior of the convex hull of the W -orbit of $\varepsilon\rho$ in \mathfrak{a}^* and let $T_\varepsilon = C(\varepsilon\rho)^0 + i\mathfrak{a}^*$ be the tube domain in $\mathfrak{a}_\mathbb{C}^*$ of base $C(\varepsilon\rho)^0$. The W -invariant Schwartz space $\mathcal{S}(\mathfrak{a}_\mathbb{C}^*)^W$ consists of the W -invariant holomorphic functions $F : T_\varepsilon \rightarrow \mathbb{C}$ such that for every $u \in S(\mathfrak{a}_\mathbb{C})$ and every integer $N \geq 0$ we have

$$\sup_{\lambda \in T_\varepsilon} (1 + \|\lambda\|)^N |\partial(u)F(\lambda)| < \infty.$$

Then $\mathcal{S}(\mathfrak{a}_\mathbb{C}^*)^W \subset (L^1 \cap L^2)(i\mathfrak{a}^*, |W|^{-1}|c(\lambda)|^{-1} d\lambda)^W$. Moreover, the spherical Fourier transform \mathcal{F}_G is a bijection of $\mathcal{S}^p(X_G)^K$ onto the W -invariant Schwartz space $\mathcal{S}(\mathfrak{a}_\mathbb{C}^*)^W$. We refer the reader to [10], Ch. 7, and [1] for additional information.

The *compact spherical Fourier transform* $\hat{h} = \mathcal{F}_U h$ of a (sufficiently regular) K -invariant function $h : X_U \rightarrow \mathbb{C}$ is usually defined by integration against the spherical functions ψ_μ on X_U . Because of the relation (17), we consider $\hat{h} = \mathcal{F}_U h$ as the function defined for $\lambda \in \Lambda_K^+(U) + \rho$ by

$$\hat{h}(\lambda) = \mathcal{F}_U h(\lambda) = \int_{X_U} h(y) \overline{\varphi_\lambda(y)} dy = \int_{X_U} h(y) \varphi_{-\lambda}(y) dy. \quad (22)$$

The *spherical Fourier series* of h is the formal series on X_U given by

$$\sum_{\mu \in \Lambda_K^+(U)} d(\mu) \hat{h}(\mu + \rho) \psi_\mu = \sum_{\mu \in \Lambda_K^+(U)} d(\mu) \hat{h}(\mu + \rho) \varphi_{\mu + \rho}. \quad (23)$$

In (23), $d(\mu)$ denotes the dimension of the finite dimensional spherical representation π_μ of U of highest weight μ . According to Weyl dimension's formula, the function d is a polynomial function on $\mathfrak{a}_\mathbb{C}^*$. If $h \in L^2(X_U)^K$, then this series converges to h in L^2 -norm. The convergence is absolute and uniform, if h is smooth. The Plancherel theorem states that compact spherical Fourier transform \mathcal{F}_U extends to an isometry of $L^2(X_U)^K$ onto $L^2(\Lambda_K^+(U), d(\mu)d\mu)$ where $d\mu$ is the counting measure.

The spherical harmonic analysis on a general semisimple space U/K of the compact type can be reduced to the simply connected case. In fact, let G/K and U/K be Riemannian symmetric spaces in duality, as in Section 1.1. Suppose in that U/K is not simply connected. Let \tilde{U} denote the connected simply connected Lie group with Lie algebra \mathfrak{u} . Let $\tilde{\theta}$ be the involution on \tilde{U} with differential equal to the differential of the involution on U associated

with U/K . The subgroup \tilde{K} of fixed points of $\tilde{\theta}$ is connected. Hence \tilde{U}/\tilde{K} is a simply connected symmetric space of the compact type. Moreover, there is a subgroup S of the center of \tilde{U} so that $U = \tilde{U}/S$ and $K = K^*/S$ where K^* is a $\tilde{\theta}$ -invariant subgroup of \tilde{U} satisfying $\tilde{K}S \subset K^* \subset K_S = \{u \in \tilde{U} : u^{-1}\tilde{\theta}(u) \in S\}$. The space $U/K = \tilde{U}/K^*$ is then covered by \tilde{U}/\tilde{K} . See [13], Ch. VII, Theorem 9.1 and Corollary 9.3.

The group \tilde{U} is connected, simply connected, compact and semisimple. Hence its universal complexification is a connected, simply connected, semisimple, complex Lie group $\tilde{U}_{\mathbb{C}}$ of Lie algebra $\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$. Here, as in Section 1.1, we we have set $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ for the Lie algebras of G and U , respectively.

Let G^{\sharp} be the connected Lie subgroup of $\tilde{U}_{\mathbb{C}}$ with Lie algebra \mathfrak{g} . Then the inclusion of G^{\sharp} in $\tilde{U}_{\mathbb{C}}$ gives the universal complexification of G^{\sharp} . Moreover, G^{\sharp} has finite center. Since $\mathfrak{k} \subset \mathfrak{g}$ and \tilde{K} is connected, we also have $\tilde{K} \subset G^{\sharp}$. Hence \tilde{K} is a maximal compact subgroup of G^{\sharp} . Thus, G^{\sharp}/\tilde{K} is a symmetric space of the noncompact type, G^{\sharp} is a connected subgroup of its universal complexification $\tilde{U}_{\mathbb{C}}$, and the compact dual of G^{\sharp}/\tilde{K} is \tilde{U}/\tilde{K} , where \tilde{U} is the connected subgroup of $\tilde{U}_{\mathbb{C}}$ of Lie algebra \mathfrak{u} . This replaces the original dual pair $(G/K, U/K)$ with the dual pair $(G^{\sharp}/\tilde{K}, \tilde{U}/\tilde{K})$ with \tilde{U}/\tilde{K} simply connected.

Let $\pi : \tilde{U}/\tilde{K} \rightarrow \tilde{U}/K^* = U/K$ be the covering map. By composing with π , we can identify a K -invariant function f on U/K with a K^* -invariant function on \tilde{U}/\tilde{K} . The space of K -invariant functions on U/K can then be considered as the subspace of \tilde{K} -invariant functions on \tilde{U}/\tilde{K} that, moreover, are K^* -invariant. The compact spherical Fourier transform of f as a K -invariant function on U/K is then identified with the restriction to $\Lambda_K^+(U)$ of the compact spherical Fourier transform of f as K^* -invariant function on \tilde{U}/\tilde{K} . On the noncompact side, the canonical isomorphism of G/K and G^{\sharp}/\tilde{K} allows us to identify K -invariant functions on G/K with \tilde{K} -invariant functions on G^{\sharp}/\tilde{K} . Under these identifications, the noncompact spherical transforms on these two symmetric spaces agree.

1.5. Holomorphic extension of spherical Fourier series. Set $\mathfrak{b} := i\mathfrak{a}$, and let $\|\cdot\|$ be a W -invariant norm on \mathfrak{b} . Endow \mathfrak{a}^* with the dual norm, still denoted by the same symbol. Let $B = \{H \in \mathfrak{b} : \|H\| < 1\}$ be the open unit ball in \mathfrak{b} . Let $o_{\mathbb{C}} = eK_{\mathbb{C}}$ be the base point in $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. Under suitable exponential decay of their coefficients, Lassalle proved in [19] the normal convergence of the spherical Fourier series on U -invariant domains in $X_{\mathbb{C}}$ of the form $D_{\varepsilon} = U \exp(i\varepsilon B) \cdot o_{\mathbb{C}}$. Notice that D_{ε} is a neighborhood of X_U in $X_{\mathbb{C}}$ and that $D_{\varepsilon} \supset K \exp(i\varepsilon B) \cdot o_{\mathbb{C}}$. Moreover, $K \exp(i\varepsilon B) \cdot o_{\mathbb{C}}$ is an open neighborhood of $o_{\mathbb{C}}$ in $KA \cdot o_{\mathbb{C}}$, which is the image of X_G in its embedding in $X_{\mathbb{C}}$. For the reader's convenience, we collect the results which will be needed in the following.

Theorem 1.1. (a) *Let $F : \Lambda_K^+(U) \rightarrow \mathbb{C}$. Suppose there are constants $C > 0$ and $\varepsilon > 0$ so that for all $\mu \in \Lambda_K^+(U)$*

$$|F(\mu)| \leq C e^{-\varepsilon \|\mu\|}.$$

Then the spherical Fourier series

$$\sum_{\mu \in \Lambda_K^+(U)} d(\mu) F(\mu) \psi_{\mu}(x)$$

converges normally on compact subsets of D_ε . Its sum is therefore a holomorphic K -invariant function on D_ε .

- (b) Conversely, suppose that h is a continuous K -invariant function on X_U admitting a holomorphic extension to a neighborhood of X_U in $X_{\mathbb{C}}$. Let $F(\mu) = \widehat{h}(\mu + \rho)$ be the Fourier coefficients of h . Then there are constants $C > 0$ and $\varepsilon > 0$ so that for all $\mu \in \Lambda_K^+(U)$

$$|F(\mu)| \leq C e^{-\varepsilon \|\mu\|}.$$

Proof. For the proof of part (a), we follow [19], p. 189. It is enough to prove the normal convergence of the series on compact sets of the form $\overline{D}_r = U \exp(ir\overline{B}) \cdot o_{\mathbb{C}}$ where $0 < r < \varepsilon$ and $\overline{B} = \{H \in \mathfrak{b} : \|H\| \leq 1\}$ is the closed unit ball in \mathfrak{b} . Set $\mathfrak{b}^+ = i\mathfrak{a}^+$. If $x \in X_{\mathbb{C}}$, then we can write $x = u \exp(iA(x)) \cdot o_{\mathbb{C}}$ for a unique $A(x) \in \overline{\mathfrak{b}^+}$ and some (non unique) $u \in U$; see [19], Théorème 1, p. 177. If $A(x) \in r\overline{B}$, then, by [19], Proposition 12, p. 184, one has

$$|\psi_\mu(x)| \leq e^{\mu(iA(x))} \leq e^{\sup_{H \in r\overline{B}} \mu(iH)} = e^{r\|\mu\|}.$$

Thus

$$|d(\mu)F(\mu)\psi_\mu(x)| \leq d(\mu)e^{-(\varepsilon-r)\|\mu\|},$$

which implies the convergence of the series as $d(\mu)$ is a polynomial in μ .

Part (b) is a special case of Proposition V.2.3 in [9]. It is proven using Cauchy's inequalities for the Fourier coefficients of h . \square

We observe that Theorem 1.1 holds in the general case where U is reductive. This will be needed in Section 7.

1.6. Coordinates in $\mathfrak{a}_{\mathbb{C}}^*$ and tubes domains around $i\mathfrak{a}^*$. We choose $\Pi_* = \{\omega_1, \dots, \omega_l\}$ as basis of \mathfrak{a}^* . For $\lambda \in \mathfrak{a}^*$ we have

$$\lambda = \sum_{j=1}^l \lambda_j \omega_j \quad \text{with} \quad \lambda_j := \lambda_{\beta_j} = \frac{\langle \lambda, \beta_j \rangle}{\langle \beta_j, \beta_j \rangle}. \quad (24)$$

Set $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^* : \lambda_\beta \geq 0 \text{ for all } \beta \in \Sigma_*^+\}$. By identifying $\lambda \equiv (\lambda_1, \dots, \lambda_l)$, we obtain the correspondences $\mathfrak{a}^* \equiv \mathbb{R}^l$, $\mathfrak{a}_+^* \equiv (\overline{\mathbb{R}^+})^l$ and $\Lambda^+ \equiv (\mathbb{Z}^+)^l$. For $\mu = \sum_{j=1}^l \mu_j \omega_j \in \Lambda^+$, we define

$$|\mu| = \mu_1 + \dots + \mu_l. \quad (25)$$

Set

$$\rho = \sum_{j=1}^l \rho_j \omega_j. \quad (26)$$

Since β_j is a multiple of a simple root, we have

$$\rho_j = \frac{1}{2} \left(m_{\beta_j} + \frac{m_{\beta_j/2}}{2} \right). \quad (27)$$

For an arbitrary $\beta \in \Sigma_*^+$, we set

$$\tilde{\rho}_\beta = \frac{1}{2} \left(m_\beta + \frac{m_{\beta/2}}{2} \right). \quad (28)$$

Notice that $\tilde{\rho}_\beta = \rho_j = \rho_{\beta_j}$ if $\beta = \beta_j$, but $\tilde{\rho}_\beta \neq \rho_\beta$ in general.

Let $\delta > 0$. We consider the following tube domains in $\mathfrak{a}_{\mathbb{C}}^*$ around the imaginary axis:

$$T_{\delta} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : |\operatorname{Re} \lambda_{\beta}| < \delta \tilde{\rho}_{\beta} \text{ for all } \beta \in \Sigma_*^+\}, \quad (29)$$

$$T'_{\delta} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : |\operatorname{Re} \lambda_j| < \delta \rho_j \text{ for all } j = 1, \dots, l\}, \quad (30)$$

$$T''_{\delta} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \operatorname{Re} \lambda_j < \delta \rho_j \text{ for all } j = 1, \dots, l\}. \quad (31)$$

In the following we shall denote by $B(T)$ the base in \mathfrak{a}^* of the tube domain T in $\mathfrak{a}_{\mathbb{C}}^*$. Hence $T = B(T) + i\mathfrak{a}^*$ and $B(T) = T \cap \mathfrak{a}^*$. The following lemma is rather standard; we provide a proof for the sake of completeness.

Lemma 1.2. *Let w_0 be the longest element of W . Then*

$$T'_{\delta} = T''_{\delta} \cap w_0(T''_{\delta}) \quad (32)$$

and

$$T_{\delta} = \bigcap_{w \in W} w(T'_{\delta}) = \bigcap_{w \in W} w(T''_{\delta}). \quad (33)$$

In particular, T_{δ} is the largest W -invariant tube domain contained in T'_{δ} . Moreover,

$$T_{\delta} = C(\delta\rho)^0 + i\mathfrak{a}^* \quad (34)$$

where $C(\nu)$ is the convex hull of the W -orbit $\{w\nu : w \in W\}$ of $\nu \in \mathfrak{a}^*$ and $C(\nu)^0$ is its interior.

Proof. It is enough to prove that the bases in \mathfrak{a}^* of the considered tube domains are equal.

The element w_0 maps Σ^+ into $-\Sigma^+$. Hence $w_0\rho = -\rho$. Moreover, w_0 maps the basis Π of simple roots in Σ^+ into the basis $-\Pi$ in $-\Sigma^+$. Hence it maps $\{\beta_1, \dots, \beta_l\}$ into $\{-\beta_1, \dots, -\beta_l\}$. Furthermore, it satisfies $w_0^{-1} = w_0$. Therefore

$$\begin{aligned} w_0(B(T''_{\delta})) &= \{w_0\lambda \in \mathfrak{a}^* : \langle \lambda - \delta\rho, \beta_j \rangle < 0 \text{ for all } j = 1, \dots, l\} \\ &= \{w_0\lambda \in \mathfrak{a}^* : \langle \lambda - \delta\rho, w_0\beta_j \rangle > 0 \text{ for all } j = 1, \dots, l\} \\ &= \{w_0\lambda \in \mathfrak{a}^* : \langle w_0\lambda - \delta w_0\rho, \beta_j \rangle > 0 \text{ for all } j = 1, \dots, l\} \\ &= \{\lambda \in \mathfrak{a}^* : \langle \lambda + \delta\rho, \beta_j \rangle > 0 \text{ for all } j = 1, \dots, l\} \\ &= \{\lambda \in \mathfrak{a}^* : \lambda_j > -\delta\rho_j \text{ for all } j = 1, \dots, l\}. \end{aligned}$$

This proves (32).

Set $C = B(\bigcap_{w \in W} w(T'_{\delta})) = \bigcap_{w \in W} w(B(T'_{\delta}))$. We have $B(T_{\delta}) \subset B(T'_{\delta})$ as $\tilde{\rho}_{\beta_j} = \rho_j$. Hence $B(T_{\delta}) \subset C$ as $B(T_{\delta})$ is W -invariant. Conversely, let $\lambda \in C$ and let $\beta \in \Sigma_*$. Since Σ_* is a reduced root system and the corresponding Weyl group is W , there is $j \in \{1, \dots, l\}$ and $w \in W$ so that $w\beta = \beta_j$. Since $w\lambda \in C \subset B(T'_{\delta})$, we obtain that $|\lambda_{\beta}| = |(w\lambda)_{w\beta}| = |(w\lambda)_{\beta_j}| < \delta\rho_j = \delta\tilde{\rho}_{\beta}$. Thus $\lambda \in B(T_{\delta})$. The second equality in (33) is a consequence of (32).

For the final result, we have by [14], Lemma 8.3 (i), that $C(\delta\rho) = \bigcap_{w \in W} w(\delta\rho - \mathfrak{a}_+^*)$. Notice that

$$\delta\rho - \mathfrak{a}_+^* = \{\delta\rho - \lambda \in \mathfrak{a}^* : \lambda_j \leq 0 \text{ for all } j = 1, \dots, l\} = \{\lambda \in \mathfrak{a}^* : \lambda_j \leq \delta\rho_j \text{ for all } j = 1, \dots, l\}.$$

Thus, by (33),

$$C(\delta\rho)^0 = \bigcap_{w \in W} w((\delta\rho - \mathfrak{a}_+^*)^0) = \bigcap_{w \in W} w(B(T''_{\delta})) = B(T_{\delta}).$$

□

By (34) and a theorem by Helgason and Johnson, $T_1 = C(\rho)^0 + i\mathfrak{a}^*$ is the interior of the set of parameters $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which the spherical function φ_λ is bounded. See [14], Theorem 8.1.

2. STATEMENT OF RAMANUJAN'S MASTER THEOREM FOR SYMMETRIC SPACES

Let $X_G = G/K$ and $X_U = U/K$ be semisimple symmetric spaces in duality inside their common complexification $X_{\mathbb{C}}$ as in Section 1.1. We suppose that X_U is simply connected.

Let A, P, δ be constants so that $A < \pi$, $P > 0$ and $0 < \delta \leq 1$. Let

$$\mathcal{H}(\delta) = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \operatorname{Re} \lambda_\beta > -\delta \tilde{\rho}_\beta \text{ for all } \beta \in \Sigma_*^+\}. \quad (35)$$

The *Hardy class* $\mathcal{H}(A, P, \delta)$ consists of the functions $a : \mathcal{H}(\delta) \rightarrow \mathbb{C}$ that are holomorphic on $\mathcal{H}(\delta)$ and so that

$$|a(\lambda)| \leq C \prod_{j=1}^l e^{-P(\operatorname{Re} \lambda_j) + A|\operatorname{Im} \lambda_j|} \quad (36)$$

for some constant $C \geq 0$ and for all $\lambda \in \mathcal{H}(\delta)$.

We denote by $\|X\|$ the norm of $X \in \mathfrak{a}$ with respect to the W -invariant norm which is induced by the Killing form. The same notation is also employed for the corresponding norm on \mathfrak{a}^* . Recall from Section 1.4 the notation d for the polynomial function on $\mathfrak{a}_{\mathbb{C}}^*$ given by Weyl dimension formula. An explicit expression of d in terms of the positive unmultipliable roots will be given in formula (58).

Theorem 2.1 (Ramanujan's Master Theorem for semisimple Riemannian symmetric spaces). *Let b be the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ defined by the equality*

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = \left(\frac{i}{2}\right)^l d(\lambda - \rho) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \quad (37)$$

and let

$$\begin{aligned} T_{\Sigma, m} = \{ & \lambda \in \mathfrak{a}_{\mathbb{C}}^* : |\operatorname{Re} \lambda_\beta| < 1 \text{ for all } \beta \in \Sigma_*^+ \text{ with } (m_{\beta/2})/2 \text{ even} \} \\ & \cap \{ \lambda \in \mathfrak{a}_{\mathbb{C}}^* : |\operatorname{Re} \lambda_\beta| < 1/2 \text{ for all } \beta \in \Sigma_*^+ \text{ with } (m_{\beta/2})/2 \text{ odd} \}. \end{aligned} \quad (38)$$

Suppose $a \in \mathcal{H}(A, P, \delta)$.

(1) Set

$$\Omega = \max_{j=1, \dots, l} \|\omega_j\|. \quad (39)$$

Then the spherical Fourier series

$$f(x) = \sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \psi_\mu(x) \quad (40)$$

converges normally on compact subsets of $D_{P/\Omega} = U \exp((P/\Omega)B) \cdot o_{\mathbb{C}}$ where $B = \{H \in \mathfrak{a} : \|H\| < 1\}$ is the open unit ball in \mathfrak{a} . Its sum is a K -invariant holomorphic function on the neighborhood $D_{P/\Omega}$ of X_U in $X_{\mathbb{C}}$.

- (2) Let T_δ be the tube domain in (29) and let $\sigma \in B(T_\delta)$. Then for $x = \exp H \in A$ with $\|H\| < P/\Omega$, we have

$$f(x) = \frac{1}{|W|} \int_{\sigma+ia^*} \left(\sum_{w \in W} a(w\lambda)b(w\lambda) \right) \varphi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}. \quad (41)$$

The integral on the right hand side of (41) is independent of the choice of σ . It converges uniformly on compact subsets of A and extends to a holomorphic K -invariant function on a neighborhood of X_G in $X_{\mathbb{C}}$.

- (3) The extension of f to X_G satisfies

$$\int_{X_G} |f(x)|^2 dx = \frac{1}{|W|} \int_{ia^*} \left| \sum_{w \in W} a(w\lambda)b(w\lambda) \right|^2 \frac{d\lambda}{|c(\lambda)|^2}.$$

Moreover,

$$\int_{X_G} f(x)\varphi_{-\lambda}(x) dx = \sum_{w \in W} a(w\lambda)b(w\lambda) \quad (42)$$

for all $\lambda \in T_\delta \cap T_{\Sigma,m}$. More precisely, the integral on the left-hand side of (42) converges in L^2 -sense and absolutely on ia^* . It defines a W -invariant holomorphic function on a W -invariant tube domain around ia^* , and (42) extends as an identity between holomorphic functions on $T_\delta \cap T_{\Sigma,m}$.

Remark 2.2. The function b occurring in Theorem 2.1 plays the role of the function $\frac{-1}{2\pi i} \frac{\pi}{\sin(\pi x)} = \frac{i}{2} \frac{1}{\sin(\pi x)}$ of the classical formula by Ramanujan. A more explicit formula for this function will be given in Corollary 4.4. The tube domain $T_{\Sigma,m}$ is linked to the singularities of b . Indeed, $T_{\Sigma,m} \cap T_\delta$ is the largest domain on which the function $\sum_{w \in W} a(w\lambda)b(w\lambda)$ is holomorphic for any $a \in \mathcal{H}(A, P, \delta)$. See Lemma 5.6. Furthermore, the condition $0 < \delta \leq 1$ guarantees the convergence of the integral in (41), as in the classical case.

Remark 2.3. Set $\|\lambda\|_1 = \sum_{j=1}^l |\lambda_j|$ for $\lambda = \sum_{j=1}^l \lambda_j \omega_j \in \mathfrak{a}^*$. Then $\|\cdot\|_1$ is a norm on \mathfrak{a}^* . The growth condition (36) can be written as $|a(\lambda)| \leq C e^{-P\|\operatorname{Re} \lambda\|_1 + A\|\operatorname{Im} \lambda\|_1}$. Since the spherical transform maps into W -invariant functions, it is sometimes more convenient to consider estimates with respect to the W -invariant norm $\|\cdot\|$ on \mathfrak{a}^* which is associated with the Killing form. This can of course be done by norm equivalence. We shall denote by c_1 and c_2 two positive constants so that $c_1\|\lambda\| \leq \|\lambda\|_1 \leq c_2\|\lambda\|$ for all $\lambda \in \mathfrak{a}^*$. By (24), one can choose for instance $c_1 = \Omega^{-1}$ with Ω as in (39) and $c_2 = \sum_{j=1}^l \|\beta_j\|^{-1}$.

Remark 2.4. Hardy's version of Ramanujan's Master Theorem holds for functions in $\mathcal{H}(A, P, \delta)$ when P is an arbitrary real number, whereas Theorem 2.1 is stated only for $P > 0$. This assumption cannot be removed. Notice first that if $a \in \mathcal{H}(A, P, \delta)$, then for all $\mu \in \Lambda^+$ we have $|a(\mu + \rho)| \leq C e^{-P\sum_{j=1}^l \mu_j} \leq C e^{-Pc\|\mu\|}$ where c is a positive constant (see Remark 2.3). According to Theorem 1.1 (b), $P > 0$ is a necessary condition for the elements $(-1)^{|\mu|} a(\mu + \rho)$ to be the Fourier coefficients of a continuous K -invariant function on X_U admitting a holomorphic extension to a neighborhood of X_U in $X_{\mathbb{C}}$. In this case, the Fourier series converges in an open domain of X_G containing the base point o . It is natural to ask whether a condition $P \geq 0$ could still allow the absolute convergence of the series $\sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \psi_\mu(x)$ for x in some open domain in X_G . The answer is negative. Indeed, by K -biinvariance, we

can restrict ourselves to domains in $\overline{A^+} \equiv \overline{A^+} \cdot o$. According to Proposition IV.5.2 in [9], one has the estimate

$$\psi_\mu(\exp H) \geq c(\mu + \rho)e^{\mu(H)}$$

for $\exp H \in \overline{A^+}$ and $\mu \in \Lambda^+$. Formulas (19) and (20) show that $c(\mu + \rho) > 0$ for $\mu \in \Lambda^+$. If $\sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \psi_\mu(\exp H)$ converges absolutely, then there is a constant $C_H > 0$ so that

$$|a(\mu + \rho)| c(\mu + \rho) e^{\mu(H)} \leq |a(\mu + \rho) \psi_\mu(\exp H)| \leq C_H$$

for all $\mu \in \Lambda^+$. Hence $|a(\mu + \rho)| \leq c(\mu + \rho)^{-1} e^{-\mu(H)}$ has exponential decay in μ for $H \neq 0$.

Notice also that in the classical version of Ramanujan's Master Theorem, the series (2) converges normally on compact subsets of the domain $|z| < e^P$. This set is a neighborhood of $U(1) = \{z \in \mathbb{C} : |z| = 1\}$ if and only if $P > 0$. The restriction $P > 0$ is therefore intrinsically related to the geometric interpretation of Ramanujan's Master Theorem as a relation of mathematical objects on symmetric spaces in duality.

Remark 2.5. A slightly more general definition of Hardy class could be obtained by replacing the constants δ and P respectively by a W -invariant multiplicity function $\delta : \Sigma_* \rightarrow]0, 1]$ and a vector $P = (P_1, \dots, P_l)$ with $P_j > 0$ for all j . The version of Ramanujan's Master Theorem for this notion of Hardy class would not present any new difficulty with respect to the version proven here. We have nevertheless preferred to present the case where δ and P are constants, to keep the notation as simple as possible.

Remark 2.6. Suppose that $X_G = G/K$ and $X_U = U/K$ is a pair of symmetric spaces in duality, as in Section 1.1, with U/K not simply connected. As in Section 1.3, we can identify the K -invariant functions f on U/K with the K^* -invariant functions on its simply connected cover \tilde{U}/\tilde{K} . In this way, the compact spherical Fourier transform of f is the restriction to $\Lambda_K^+(U) \subset \Lambda^+$ of the transform of f as a function on \tilde{U}/\tilde{K} . Likewise, we identify the K -invariant functions on G/K with the \tilde{K} -invariant functions on G^\sharp/\tilde{K} . See Section 1.3. Ramanujan's Master theorem for X_G and X_U is then obtained from the one for \tilde{U}/\tilde{K} and G^\sharp/\tilde{K} by replacing the Hardy class $\mathcal{H}(A, P, \delta)$ with its subspace

$$\mathcal{H}_{U/K}(A, P, \delta) = \{a \in \mathcal{H}(A, P, \delta) : a(\mu) = 0 \text{ for all } \mu \in \Lambda^+ \setminus \Lambda_K^+(U)\}.$$

As in the classical case, there is an equivalent formulation of Ramanujan's Master Theorem using the gamma function. For simplicity, we only consider the case when X_U is simply connected. The general case can be dealt with as in Remark 2.6. Let $B(\lambda)$ be the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ defined by the equality

$$\frac{B(\lambda)}{c(\lambda)c(-\lambda)} = \left(\frac{1}{2\pi i}\right)^l d(\lambda - \rho) \prod_{j=1}^l \Gamma(-\lambda_j + \rho_j),$$

i.e.

$$B(\lambda) = b(\lambda) \prod_{j=1}^l \frac{1}{\Gamma(\lambda_j - \rho_j + 1)}.$$

Replace $a(\lambda) \in \mathcal{H}(A, P, \delta)$ with $A(\lambda) = a(\lambda) \prod_{j=1}^l \Gamma(\lambda_j - \rho_j + 1)$ in Theorem 2.1. Notice that A is in general no longer holomorphic on $\mathcal{H}(\delta)$. However, $A(\lambda)$ is holomorphic on

$\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \lambda_j > \rho_j - 1 \text{ for all } j = 1, \dots, l\}$ and $A(\lambda)B(\lambda) = a(\lambda)b(\lambda)$. The power series (40) now becomes

$$F(x) = \sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) \frac{A(\mu + \rho)}{\mu!} \psi_{\mu}(x) \quad (43)$$

where

$$\mu! = \mu_1! \cdots \mu_l! \quad \text{for} \quad \mu = \sum_{j=1}^l \mu_j \omega_j. \quad (44)$$

Moreover, the extension of F to X_G satisfies

$$\int_{X_G} F(x) \varphi_{-\lambda}(x) dx = \sum_{w \in W} A(w\lambda) B(w\lambda). \quad (45)$$

Other formulas can be deduced from (42) or (45) by formal manipulations. We mention for instance the following, which is the analogue Ramanujan's formula in [2], Corollary (i), p. 318.

Let $\tilde{B}(\lambda)$ be the meromorphic function defined on $\mathfrak{a}_{\mathbb{C}}^*$ by the equality

$$\frac{\tilde{B}(\lambda)}{c(\lambda)c(-\lambda)} = \left(\frac{1}{4\pi i} \right)^l \pi^{l/2} d(\lambda - \rho) \prod_{j=1}^l \frac{\Gamma(-\lambda_j + \rho_j)}{\Gamma((\lambda_j - \rho + 1)/2)},$$

i.e.

$$\tilde{B}(\lambda) = b(\lambda) \prod_{j=1}^l \frac{\cos(\pi(\lambda_j - \rho_j)/2)}{\Gamma(\lambda_j - \rho_j + 1)}.$$

Let

$$\tilde{A}(\lambda) = a(\lambda) \frac{\Gamma(\lambda_j - \rho_j + 1)}{\cos(\pi(\lambda_j - \rho_j)/2)}$$

with $a(\lambda) \in \mathcal{H}(A, P, \delta)$. The power series (40) becomes

$$\tilde{F}(x) = \sum_{\mu \in \Lambda^+} d(2\mu) \frac{\tilde{A}(2\mu + \rho)}{(2\mu)!} \psi_{2\mu}(x) \quad (46)$$

and the extension of \tilde{F} to X_G satisfies

$$\int_{X_G} \tilde{F}(x) \varphi_{-\lambda}(x) dx = \sum_{w \in W} \tilde{A}(w\lambda) \tilde{B}(w\lambda). \quad (47)$$

3. THE FUNCTION b

In this section we explain our choice of the function b occurring in the statement of Ramanujan's Master theorem, Theorem 2.1.

We are looking for a meromorphic function $b : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$ with the following property: For every function $a : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$ in the Hardy class, we have

$$\frac{1}{|W|} \int_{ia^*} \left(\sum_{w \in W} a(w\lambda) b(w\lambda) \right) \varphi_{\lambda}(x) \frac{d\lambda}{c(\lambda)c(-\lambda)} = \sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \varphi_{\mu+\rho}(x) \quad (48)$$

for all $x \in X_G$ sufficiently close to the base point $o = eK$.

The right-hand side of (48) will represent the holomorphic extension to a neighborhood of X_U in X_C of the K -invariant function $f_1 : X_U \rightarrow \mathbb{C}$ defined by the spherical Fourier series

$$f_1(y) = \sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \psi_\mu(y), \quad y \in X_U.$$

The right-hand side will give a K -invariant function $f_2 : X_G \rightarrow \mathbb{C}$ having the W -invariant function $\sum_{w \in W} a(w\lambda) b(w\lambda)$, $\lambda \in i\mathfrak{a}^*$, as noncompact spherical transform.

Observe that, since the spherical function φ_λ and the Plancherel density $[c(\lambda)c(-\lambda)]^{-1}$ are W -invariant in λ , the equality (48) can be rewritten as

$$\int_{i\mathfrak{a}^*} a(\lambda) b(\lambda) \varphi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)} = \sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \varphi_{\mu+\rho}(x) \quad (49)$$

for all $x \in X_G$ sufficiently close to the base point $o = eK$.

Under the identifications $\lambda \equiv (\lambda_1, \dots, \lambda_l)$ by means of the basis $\Pi_* = \{\omega_1, \dots, \omega_l\}$ of $\mathfrak{a}^* \equiv \mathbb{R}^l$, the right-hand side of (49) becomes

$$\sum_{\mu_1=0}^{+\infty} \cdots \sum_{\mu_l=0}^{+\infty} (-1)^{|\mu|} a(\mu + \rho) d(\mu) \varphi_{\mu+\rho}(x) \quad (50)$$

where

$$\mu = (\mu_1, \dots, \mu_l), \quad \mu + \rho = (\mu_1 + \rho_1, \dots, \mu_l + \rho_l), \quad |\mu| = \mu_1 + \cdots + \mu_l.$$

Moreover, the integral on the left-hand side becomes

$$\int_{i\mathbb{R}} \cdots \int_{i\mathbb{R}} a(\lambda) \varphi_\lambda(x) \left(\frac{b(\lambda)}{c(\lambda)c(-\lambda)} \right) d\lambda_1 \cdots d\lambda_l. \quad (51)$$

Under suitable decay and convergence conditions, subsequent applications of the 1-dimensional residue theorem to (51) yields (50) provided:

- (1) The function $\frac{b(\lambda)}{c(\lambda)c(-\lambda)}$ is meromorphic, with simple poles in the region $(\mathbb{R}^+)^l$ along the hyperplanes $\lambda_j = \mu_j + \rho_j$ with $\mu_j \in \mathbb{Z}^+$ and $j = 1, \dots, l$.
- (2) For $\mu = (\mu_1, \dots, \mu_l) \in (\mathbb{Z}^+)^l$, we have

$$(-2\pi i)^l \operatorname{Res}_{\lambda_1=\mu_1+\rho_1} \cdots \operatorname{Res}_{\lambda_l=\mu_l+\rho_l} \frac{b(\lambda)}{c(\lambda)c(-\lambda)} = (-1)^{|\mu|} d(\mu).$$

Based on the rank-one case in [4], we are therefore led to define $b(\lambda)$ by means of the equality

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = C_b d(\lambda - \rho) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \quad (52)$$

where C_b is a suitable constant. From the above arguments, we can compute that

$$C_b = \left(\frac{i}{2} \right)^l. \quad (53)$$

To make the definition (52) explicit, we need to further analyze the relation between the Plancherel density $[c(\lambda)c(-\lambda)]^{-1}$ for X_G and the Plancherel density $d(\mu)$ for X_U .

4. THE PLANCHEREL DENSITIES ON X_G AND X_U

Recall that if both $\beta/2$ and β are roots, then $m_{\beta/2}$ is even and m_β is odd. See e.g. [13], Chapter X, Ex. F. 4. For a fixed $\beta \in \Sigma_*^+$, the singularities of the function $[c_\beta(\lambda)c_\beta(-\lambda)]^{-1}$ are then described by distinguishing the following four cases:

- (a) m_β even, $m_{\beta/2} = 0$;
- (b) m_β odd, $m_{\beta/2} = 0$;
- (c) m_β odd, $m_{\beta/2}/2$ even;
- (d) m_β odd, $m_{\beta/2}/2$ odd.

Recall the constant $\tilde{\rho}_\beta$ attached to $\beta \in \Sigma_*^+$ from formula (28). Following the computations yielding to formula (25) in [16], we obtain the following lemma. See also [4], Proposition 1.4.1.

Lemma 4.1. *Let $\beta \in \Sigma_*^+$. Then*

$$\frac{1}{c_\beta(\lambda)c_\beta(-\lambda)} = C_\beta p_\beta(\lambda)q_\beta(\lambda) \quad (54)$$

where

- (1) C_β is a positive constant (depending on β and on the multiplicities), explicitly given by

$$C_\beta = 4\pi\varepsilon(\beta) \quad \text{where} \quad \varepsilon(\beta) = \begin{cases} (-1)^{m_{\beta/2}} & m_\beta \text{ is even} \\ (-1)^{(m_{\beta/2}+m_\beta-1)/2} & m_\beta \text{ is odd} \end{cases} \quad (55)$$

- (2) p_β is a polynomial. If $\tilde{\rho}_\beta > 1/2$, then

$$p_\beta(\lambda) = \lambda_\beta(\lambda_\beta + \tilde{\rho}_\beta - 1)(\lambda_\beta + \tilde{\rho}_\beta - 2) \cdots (\lambda_\beta - (\tilde{\rho}_\beta - 2))(\lambda_\beta - (\tilde{\rho}_\beta - 1)) \times \\ \times \left(\lambda_\beta + \left(\frac{m_{\beta/2}}{4} - \frac{1}{2}\right)\right) \left(\lambda_\beta + \left(\frac{m_{\beta/2}}{4} - \frac{3}{2}\right)\right) \cdots \left(\lambda_\beta - \left(\frac{m_{\beta/2}}{4} - \frac{3}{2}\right)\right) \left(\lambda_\beta - \left(\frac{m_{\beta/2}}{4} - \frac{1}{2}\right)\right),$$

and the product on the second line does not occur if $m_{\beta/2} = 0$. If $\tilde{\rho}_\beta = 1/2$, then $p_\beta(\lambda) = \lambda_\beta$.

- (3) $q_\beta(\lambda) = 1$ if m_β is even; if m_β is odd, then

$$q_\beta(\lambda) = -\tan\left(\pi\left(\lambda_\beta - \frac{m_{\beta/2}}{4}\right)\right) = \cot\left(\pi(\lambda_\beta - \tilde{\rho}_\beta)\right). \quad (56)$$

The relation between the Plancherel measures on X_G and X_U is given by the following lemma.

Lemma 4.2. *The dimension $d(\mu)$ of the finite-dimensional spherical representation of highest restricted weight $\mu \in \Lambda^+$ is given by*

$$d(\mu) = \left. \frac{c(\lambda - \mu)c(-\lambda + \mu)}{c(\lambda)c(-\lambda)} \right|_{\lambda=\mu+\rho}.$$

Proof. This is Theorem 9.10, p. 321, in [15]. □

The apparent singularities in the formula in Lemma 4.2 can be removed using Lemma 4.1 and the fact that the cotangent function is π -periodic. We obtain the following formula from [16], Proposition 3.5.

Proposition 4.3. For $\mu \in \Lambda^+$ we have

$$d(\mu) = \frac{P(\mu + \rho)}{P(\rho)}$$

where

$$P(\lambda) = \prod_{\beta \in \Sigma_*^+} p_\beta(\lambda) \quad (57)$$

and p_β is the polynomial from Lemma 4.1.

According to Proposition 4.3, the polynomial function on $\mathfrak{a}_{\mathbb{C}}^*$ extending $d(\mu)$ by means of Weyl integration formula can be written in terms of the positive unmultipliable roots as

$$d(\lambda) = \frac{P(\lambda + \rho)}{P(\rho)}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \quad (58)$$

where $P(\lambda)$ is as in (57).

Corollary 4.4. (a) Let $C_b = \left(\frac{i}{2}\right)^l$ be the constant introduced in (53). Then the function

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = \frac{C_b}{P(\rho)} P(\lambda) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \quad (59)$$

is meromorphic on $\mathfrak{a}_{\mathbb{C}}^*$ with simple poles located along the hyperplanes of equation

$$\pm \lambda_j - \rho_j = k_j$$

where $k_j \in \mathbb{Z}^+$ and $j \in \{1, \dots, l\}$.

(b) For all $w \in W$ the function $\frac{b(w\lambda)}{c(\lambda)c(-\lambda)}$ is holomorphic on the tube $T_1 = C(\rho)^0 + i\mathfrak{a}^*$ of Lemma 1.2.

(c) For all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have

$$b(\lambda) = K_b T(\lambda) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \quad (60)$$

where

$$T(\lambda) = \prod_{\beta \in \Sigma_*^+} t_\beta(\lambda) \quad (61)$$

and

$$t_\beta(\lambda) = \begin{cases} 1 & \text{if } m_\beta \text{ is even} \\ \tan(\pi(\lambda_\beta - \tilde{\rho}_\beta)) & \text{if } m_\beta \text{ is odd} \end{cases} \quad (62)$$

is the inverse of the function q_β from Lemma 4.1. Moreover, K_b is a constant depending on the multiplicities. It is given explicitly by

$$K_b = \left(\frac{i}{2}\right)^l \frac{c_0^2}{P(\rho)} \prod_{\beta \in \Sigma_*^+} C_\beta,$$

where C_β is as in (55) and c_0 is the constant appearing in the definition (19) of Harish-Chandra's c -function.

Proof. Immediate consequence of (52), Lemma 4.1 and Proposition 4.3. For the list of singular hyperplanes, notice that for fixed $j \in \{1, \dots, l\}$, the zeros of the function $\sin(\pi(\lambda_j - \rho_j))$ are located along the hyperplanes of equation $\lambda_j - \rho_j = k_j$ with $k_j \in \mathbb{Z}$. Recall that $\lambda_j = \lambda_{\beta_j}$ and $\tilde{\rho}_{\beta_j} = \rho_j$. The polynomial p_{β_j} is divided by

$$(\lambda_j - (\rho_j - 1))(\lambda_j - (\rho_j - 2)) \cdots (\lambda_j + (\rho_j - 2))(\lambda_j + (\rho_j - 1)).$$

Hence all singularities of $[\sin(\pi(\lambda_j - \rho_j))]^{-1}$ for $|\lambda_j| < \rho_j$ are canceled. No other singularities of this function are canceled by zeros of p_{β_j} .

Because of (a), the function $\frac{b(\lambda)}{c(\lambda)c(-\lambda)}$ is holomorphic on the tube T'_1 of Lemma 1.2. Hence $\frac{b(w\lambda)}{c(\lambda)c(-\lambda)}$ is holomorphic on the largest W -invariant tube domain contained in T'_1 , which is $T_1 = \cap_{w \in W} w(T'_1)$.

The formula for the constant K_b is obtained by comparing (59), (60), (19) and (54). \square

Remark 4.5. According to the four cases for m_β and $m_{\beta/2}$ listed in Section 4, we have $\tilde{\rho}_\beta \in \mathbb{Z}$ in cases (a) and (d), and $\tilde{\rho}_\beta \in \mathbb{Z} + 1/2$ in cases (b) and (c). We can therefore write (60) as

$$b(\lambda) = K'_b \left(\prod_{\substack{\beta \in \Sigma_*^+ \setminus \{\beta_1, \dots, \beta_l\} \\ \text{cases (b) or (c)}}} \cot(\pi\lambda_\beta) \right) \left(\prod_{\substack{\beta \in \Sigma_*^+ \setminus \{\beta_1, \dots, \beta_l\} \\ \text{case (d)}}} \tan(\pi\lambda_\beta) \right) \left(\prod_{\substack{j \in \{1, \dots, l\} \\ \text{cases (a), (b) or (c)}}} \frac{1}{\sin(\pi\lambda_j)} \right) \left(\prod_{\substack{j \in \{1, \dots, l\} \\ \text{case (d)}}} \frac{1}{\cos(\pi\lambda_j)} \right) \quad (63)$$

where $K'_b = \pm K_b$ and the sign depends on the parity of the multiplicities.

Remark 4.5 immediately implies the following corollary. Notice that the cases (a), (b) and (c) for the root multiplicities correspond to the situation in which $(m_{\beta/2})/2$ is even.

Corollary 4.6. *Let*

$$\Pi(\lambda) = \prod_{\beta \in \Sigma_*^+} \lambda_\beta \quad (64)$$

and let $T_{\Sigma, m}$ be as in (38). Then $\Pi(\lambda)b(\lambda)$ is holomorphic on $T_{\Sigma, m}$.

Example 4.7 (The rank-one case). In the (real) rank-one case, \mathfrak{a} is one dimensional. Then Σ^+ consists of at most two elements: β and, possibly, $\beta/2$. Hence $\Sigma_*^+ = \{\beta\}$, $l = 1$ and $\beta_1 = \beta$. According to Remark 4.5, we have

$$b(\lambda) = \begin{cases} \pm K_b [\sin(\pi\lambda_1)]^{-1} & \text{in cases (a), (b) and (c)} \\ \pm K_b [\cos(\pi\lambda_1)]^{-1} & \text{in case (d)}. \end{cases}$$

This case has been previously considered by Bertram in [4].

Example 4.8 (The even multiplicity case). Suppose that Σ is a reduced root system and that all roots multiplicities are even. Geometrically, even multiplicities correspond to Riemannian symmetric spaces of the noncompact type G/K with the property that all Cartan subalgebras in the Lie algebra \mathfrak{g} of G are conjugate under the adjoint group of \mathfrak{g} . The simplest examples occur when \mathfrak{g} admits a complex structure, in which case all root multiplicities are equal to 2. In the even multiplicity case, one has $b(\lambda) = K_b \prod_{j=1}^l [\sin(\pi\lambda_j)]^{-1}$. In the complex case, one can compute that $K_b = (i/2)^l$. In particular, for the complex rank-one case corresponding to the pair of symmetric spaces $G/K = \text{SL}(2, \mathbb{C})/\text{SU}(2)$ and $U/K \cong \text{SU}(2)$,

we have $b(\lambda) = \frac{i}{2} [\sin(\pi\lambda_1)]^{-1}$. Thus, in this case and with our choice of the coordinate λ_1 in $\mathfrak{a}_{\mathbb{C}}^*$, the function b agrees with the one of Ramanujan for the case of \mathbb{R}^+ .

Remark 4.9. In comparison to the classical version of the Master Theorem, the assumption $P > 0$ in the statement of Ramanujan's Theorem for symmetric spaces strongly restricts the class of spherical Fourier series to which our theorem can be applied. On the other hand, by replacing the pair $(U(1), \mathbb{R}^+)$ by a pair of Riemannian symmetric spaces in duality, we are considering a much richer class of different geometric situations where our theorem applies. According to the choice of the dual pair of symmetric spaces, the restrictions to the Cartan subspace A of the spherical functions $\varphi_{\mu+\rho}$ provide several classes of orthogonal polynomials of Jacobi type in several variables. Likewise, by the integral formulas corresponding to the decomposition $G = KAK$, the spherical Fourier transforms gives different specializations of Jacobi transforms in several variables. The simplest example of these specializations corresponds to the complex case considered above. In this case, $G = K_{\mathbb{C}}$ and $U = K \times K$ where K is a compact connected semisimple Lie group. The corresponding Riemannian symmetric spaces in duality are $X_G = K_{\mathbb{C}}/K$ and $X_U = (K \times K)/K$. The space X_U can be identified with K . In this way, the K -invariant functions on X_U correspond to the central functions on K . To simplify notation, we assume in the following that K is simply connected. The spherical representations of U are of the form $\pi_{\mu} = \delta_{\mu} \otimes \overline{\delta_{\mu}}$ where $\mu \in \Lambda^+$ is the highest weight of the irreducible representation δ_{μ} of K and $\overline{\delta_{\mu}}$ denotes the contragredient representation of δ_{μ} . The spherical functions on X_U are therefore the normalized characters $\frac{1}{\dim \delta} \chi_{\mu}$ where χ_{μ} is the character of δ_{μ} . On the noncompact side, one can obtain explicit formulas by using the integral formulas for the KAK decomposition of G and the explicit formulas on A for the spherical functions on Riemannian symmetric spaces of the noncompact type and G with complex structure; see e.g. [14], Ch. I, Theorem 5.8 and Ch. IV, Theorem 5.7. Since $\Sigma = \Sigma_*$ is reduced and all multiplicities are equal to 2, one has $\rho = \sum_{\beta \in \Sigma^+} \beta$. Set

$$f(x) = \sum_{\mu \in \Lambda^+} (-1)^{|\mu|} a(\mu + \rho) \chi_{\mu}(x) \quad x \in K_{\mathbb{C}}$$

with $a \in \mathcal{H}(A, P, \delta)$. Then Ramanujan's interpolation formula (42) becomes the Fourier integral formula

$$\int_{\mathfrak{a}^+} f(\exp H) \Delta(H) e^{\lambda(H)} dH = \sum_{w \in W} a(w\lambda) b(w\lambda)$$

where

$$\Delta(H) = \prod_{\beta \in \Sigma^+} (e^{\beta(H)} - e^{-\beta(H)}) = \sum_{w \in W} (\det w) e^{w\rho(H)},$$

$$b(\lambda) = c_b \prod_{j=1}^l \sin(\pi\lambda_j)^{-1}$$

and c_b is a suitable normalizing constant.

5. SOME ESTIMATES

In this section we collect some estimates which will be needed in the proof of Theorem 2.1.

For every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the spherical function φ_λ extends holomorphically as a $K_{\mathbb{C}}$ -invariant function on the domain $K_{\mathbb{C}} \exp(2\Omega_\pi) \cdot o$ in $X_{\mathbb{C}}$, where

$$\Omega_\pi = \{H \in \mathfrak{a}_{\mathbb{C}} : |\beta(\operatorname{Im} H)| < \pi/2 \text{ for all } \beta \in \Sigma\}. \quad (65)$$

The estimates of the holomorphically extended spherical functions that are given in Lemma 5.1 below will be sufficient to our purposes.

Recall that $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^* : \lambda_\beta \geq 0 \text{ for all } \beta \in \Sigma_+^*\}$. Notice that $\mathcal{H}(\delta) \supset \mathfrak{a}_+^* + i\mathfrak{a}^*$ for all $\delta > 0$. Recall also the constant Ω from (39).

Lemma 5.1. *There is a constant $C > 0$ so that*

$$|\varphi_\lambda(\exp H \cdot o)| \leq C e^{-\min_{w \in W} \operatorname{Im}(w\lambda(H_2)) + \max_{w \in W} \operatorname{Re}(w\lambda(H_1))} \quad (66)$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and all $H = H_1 + iH_2 \in \overline{\Omega_\pi}$ with $H_1, H_2 \in \mathfrak{a}$. In particular:

(a) for all $\lambda \in \mathfrak{a}_+^* + i\mathfrak{a}^*$ and $H \in \mathfrak{a}$ we have

$$|\varphi_\lambda(\exp H \cdot o)| \leq C e^{\Omega \|H\| (\sum_{j=1}^l \operatorname{Re} \lambda_j)}$$

(b) for all $H \in \overline{\Omega_\pi}$ and $\lambda \in \mathfrak{a}^*$ we have

$$|\varphi_\lambda(\exp H \cdot o)| \leq C e^{\|\operatorname{Im} H\| \|\operatorname{Im} \lambda\|}.$$

Proof. Estimates (66) are due to Opdam; see [24], Proposition 6.1(2) and Theorem 3.15. For (a), we can suppose by W -invariance that $H \in \mathfrak{a}^+$. In this case, for $\lambda \in \mathfrak{a}_+^* + i\mathfrak{a}^*$, we have

$$0 \leq \operatorname{Re} \lambda(H) = \sum_{j=1}^l \operatorname{Re} \lambda_j \omega_j(H) \leq \Omega \|H\| \left(\sum_{j=1}^l \operatorname{Re} \lambda_j \right).$$

Part (b) follows immediately from (66). \square

Lemma 5.2. *Let P be as in (57). Then there are positive constants C_0, C'_0 and C''_0 so that*

$$|P(\lambda)| \leq C_0 \prod_{\beta \in \Sigma_+^*} (1 + |\lambda_\beta|)^{m_{\beta/2} + m_\beta} \leq C'_0 (1 + \|\lambda\|)^M \leq C''_0 \prod_{j=1}^l (1 + |\lambda_j|)^M$$

where

$$M = \sum_{\beta \in \Sigma_+^*} (m_{\beta/2} + m_\beta). \quad (67)$$

Proof. The first inequality is an immediate consequence of the formula for p_β in Lemma 4.1, which gives p_β as polynomial of degree $m_{\beta/2} + m_\beta$ in λ_β . For the second, notice that $|\lambda_\beta| \leq \|\beta\|^{-1} \|\lambda\|$. The final inequality follows immediately from $\lambda = \sum_{j=1}^l \lambda_j \omega_j$. \square

We define

$$Q(\lambda) = \prod_{\beta \in \Sigma_+^*} q_\beta(\lambda) \quad (68)$$

where q_β is the function defined in Lemma 4.1.

Observe that there is a constant $K > 0$ so that

$$\left| \sin(\pi(\lambda_j - \rho_j)) \right|^{-1} \leq K e^{-\pi |\operatorname{Im} \lambda_j|} \quad (69)$$

for $|\operatorname{Im} \lambda_j| \geq 1$ or for $\operatorname{Re} \lambda_j = \rho_j + N + 1/2$ with $N \in \mathbb{Z}^+$.

The following lemma contains the estimates needed to apply the Residue Theorem.

Lemma 5.3. (a) Let N be a positive integer and let M be as in (67). Let $\lambda = \sum_{j=1}^l \lambda_j \omega_j \in \mathfrak{a}_{\mathbb{C}}^*$ with $|\operatorname{Im} \lambda_j| \geq 1$ or $\operatorname{Re} \lambda_j = \rho_j + N + 1/2$ or $\operatorname{Re} \lambda_j = 0$ for all $j = 1, \dots, l$. Then there is a positive constant C_1 , independent of N , so that

$$\left| \frac{b(\lambda)}{c(\lambda)c(-\lambda)} \right| \leq C_1 \prod_{j=1}^l [(1 + |\lambda_j|)^M e^{-\pi |\operatorname{Im} \lambda_j|}].$$

(b) Set

$$B = \left\{ \lambda = \sum_{j=1}^l \lambda_j \omega_j \in \mathfrak{a}_+^* + i\mathfrak{a}^* : |\operatorname{Im} \lambda_j| \geq 1 \text{ or } \operatorname{Re} \lambda_j \in (\rho_j + \mathbb{Z}^+ + 1/2) \cup \{0\} \text{ for all } j = 1, \dots, l \right\}. \quad (70)$$

Let $a \in \mathcal{H}(A, P, \delta)$. Then there is a constant $C_2 > 0$ so that for all $\lambda \in B$ and $H \in \overline{\mathfrak{a}^+}$ we have

$$\left| \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_\lambda(\exp H \cdot o) \right| \leq C_2 \prod_{j=1}^l [(1 + |\lambda_j|)^M e^{(A-\pi)|\operatorname{Im} \lambda_j| + (\|H\|\Omega - P)\operatorname{Re} \lambda_j}]. \quad (71)$$

Proof. For $|\operatorname{Im} \lambda_j| \geq 1$ or $\operatorname{Re} \lambda_j = \rho_j + N + 1/2$, the estimate in (a) is a consequence of (59), (69) and Lemma 5.2. The inequality holds also if $\operatorname{Re} \lambda_j = 0$ for some j , as the possible singularity of $\sin(\pi(\lambda_j - \rho_j))^{-1}$ at $\lambda_j = 0$ is cancelled by the factor λ_j in $p_{\beta_j}(\lambda)$; see formulas (59) and Lemma 4.1. Part (b) follows from (a) and Lemma 5.1.(a). \square

The next lemma will be useful to prove the independence on $\sigma \in B(T_\delta)$ for the integral occurring in Part (b) of Ramanujan's Theorem.

Lemma 5.4. Let $0 < \delta \leq 1$ and let T_δ be the tube domain from (33). Let M be the constant defined in (67).

(a) There is a constant $C_\delta > 0$ so that

$$\left| \frac{b(\lambda)}{c(\lambda)c(-\lambda)} \right| \leq C_\delta (1 + \|\lambda\|)^M e^{-\pi(\sum_{j=1}^l |\operatorname{Im} \lambda_j|)} \quad (72)$$

for all $\lambda \in T_\delta$.

(b) Let $a \in \mathcal{H}(A, P, \delta)$. For every $R > 0$ and every integer $N \geq 0$ there is a constant $C_{R,N,\delta} > 0$ so that for all $\lambda \in T_\delta$ and $H \in \mathfrak{a}$ with $\|H\| < R$, we have

$$\left| \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_\lambda(\exp H \cdot o) \right| \leq C_{R,N,\delta} (1 + \|\lambda\|)^{-N} \quad (73)$$

Consequently,

$$\left| \left(\sum_{w \in W} a(w\lambda)b(w\lambda) \right) \frac{\varphi_\lambda(\exp H \cdot o)}{c(\lambda)c(-\lambda)} \right| \leq C_{R,N,\delta} |W| (1 + \|\lambda\|)^{-N}. \quad (74)$$

Proof. The polynomial

$$p_j(\lambda) = (\lambda_j - (\rho_j - 1))(\lambda_j - (\rho_j - 2)) \cdots (\lambda_j - (-\rho_j + 2))(\lambda_j - (-\rho_j + 1))$$

is a divisor of $p_{\beta_j}(\lambda)$. Hence

$$\begin{aligned} \frac{b(\lambda)}{c(\lambda)c(-\lambda)} &= \frac{C_b}{P(\delta)} P(\lambda) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \\ &= \frac{C_b}{P(\delta)} \tilde{P}(\lambda) \prod_{j=1}^l \frac{p_j(\lambda)}{\sin(\pi(\lambda_j - \rho_j))} \end{aligned} \quad (75)$$

for a certain polynomial \tilde{P} . For fixed $\eta \in]0, 1[$, the function $\frac{z}{\sin(\pi z)}$ is bounded on $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq 1, |\operatorname{Re} z| \leq \eta\}$. By (69), we conclude that there is a constant $C'_\delta > 0$ so that for any fixed $j = 1, \dots, l$ and every $\lambda = \sum_h \lambda_h \omega_h$ with $|\operatorname{Re} \lambda_j| \leq \delta \rho_j$ and arbitrary $\lambda_h \in \mathbb{C}$ with $h \neq j$, we have

$$\left| \frac{p_j(\lambda)}{\sin(\pi(\lambda_j - \rho_j))} \right| \leq C'_\delta (1 + |\lambda_j|)^{\deg p_j} e^{-\pi |\operatorname{Im} \lambda_j|}.$$

We obtain Part (a) from these estimates and (75).

To prove (b), observe first that by Lemma 1.2, the function $\frac{b(w\lambda)}{c(\lambda)c(-\lambda)}$ is holomorphic on $T_1 \supset T_\delta$. Moreover, T_δ is a W -invariant subset of $\mathcal{H}(\delta)$. Hence $\frac{a(w\lambda)b(w\lambda)}{c(\lambda)c(-\lambda)} \varphi_\lambda(\exp H \cdot o)$ is holomorphic on T_δ . Notice that $|\varphi_\lambda(\exp H \cdot o)| \leq \varphi_{\operatorname{Re} \lambda}(\exp H \cdot o)$. Let $R > 0$ be fixed. Since the basis $B(T_\delta)$ of the tube domain T_δ has compact closure, it follows by continuity, that there is a constant $C_{R,\delta} > 0$ so that

$$|\varphi_\lambda(\exp H \cdot o)| \leq C_{R,\delta} \quad (76)$$

for all $\lambda \in T_\delta$ and $H \in \mathfrak{a}$ with $\|H\| \leq R$. (This can also be obtained from Lemma 5.1.)

Suppose that $a \in \mathcal{H}(A, P, \delta)$. By (76) and Part (a), there is a constant $C'_{R,\delta} > 0$ so that

$$\left| \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_\lambda(\exp H \cdot o) \right| \leq C'_{R,\delta} (1 + \|\lambda\|)^M e^{(A-\pi) \sum_{j=1}^l |\operatorname{Im} \lambda_j|}.$$

This implies (73) as $A < \pi$. □

We shall also need the following result, which is a local version of a classical argument by Malgrange (see [15], p. 278, and [26], Lemma 4.2).

Lemma 5.5. *Let V' be an open domain in $\mathfrak{a}_{\mathbb{C}}^*$ and let $H : V' \rightarrow \mathbb{C}$ a holomorphic function satisfying the following property: there exist constants $R \in \mathbb{R}$, $s > 0$ and $C > 0$ so that*

$$|H(\lambda)| \leq C(1 + \|\lambda\|)^s e^{R \|\operatorname{Im} \lambda\|}$$

for all $\lambda \in V'$. Let $\tau > 0$ and let V be an open domain in $\mathfrak{a}_{\mathbb{C}}^*$ such that

$$V_\tau := \{\nu \in \mathfrak{a}_{\mathbb{C}}^* : \exists \lambda \in V \text{ with } \|\lambda - \nu\| \leq \tau\} \subset V'.$$

Let p be a polynomial such that $F(\lambda) = \frac{H(\lambda)}{p(\lambda)}$ is holomorphic on V' . Then there is a constant $C_\tau > 0$ (depending also on C, R, s) such that

$$|F(\lambda)| \leq C_\tau (1 + \|\lambda\|)^s e^{R \|\operatorname{Im} \lambda\|}$$

for all $\lambda \in V$.

Proof. Let $m = \deg p$. By Cauchy's integral formula, for any multiindex α there is a constant $C_{m,\alpha} > 0$ so that for every $\lambda \in V$ we have

$$|F(\lambda)(\partial^\alpha p)(\lambda)| \leq C_{m,\alpha} \int_{\|\xi\| \leq \tau} |F(\lambda + \xi)| |p(\lambda + \xi)| d\xi.$$

Choose α so that $\partial^\alpha p$ is a constant $d \neq 0$. We obtain:

$$\begin{aligned} |F(\lambda)| &\leq d^{-1} C_{m,\alpha} \int_{\|\xi\| \leq \tau} |H(\lambda + \xi)| d\xi \\ &\leq d^{-1} C C_{m,\alpha} \int_{\|\xi\| \leq \tau} (1 + \|\lambda + \xi\|)^s e^{R(\|\operatorname{Im} \lambda + \operatorname{Im} \xi\|)} d\xi \\ &\leq C_\tau (1 + \|\lambda\|)^s e^{R\|\operatorname{Im} \lambda\|}, \end{aligned}$$

where $C_\tau = d^{-1} C C_{m,\alpha} (1 + \tau)^s e^{R|\tau|} \int_{\|\xi\| \leq \tau} d\xi$. \square

The following lemma shows that if $a \in \mathcal{H}(A, P, \delta)$, then there is a constant $\varepsilon \in]0, 1[$ (depending on A, P, δ) so that the function

$$\tilde{a}(\lambda) = \sum_{w \in W} a(w\lambda) b(w\lambda) \quad (77)$$

belongs to the W -invariant Schwartz space $\mathcal{S}(\mathfrak{a}_\varepsilon^*)^W$ on the tube domain T_ε around $i\mathfrak{a}^*$; see (29) for the definition of T_ε and section 1.3 for the definition of the W -invariant Schwartz space.

For $0 \leq \eta < 1/2$ set

$$\begin{aligned} T_{\Sigma, m, \eta} &= \{ \lambda \in \mathfrak{a}_\mathbb{C} : |\operatorname{Re} \lambda_\beta| < 1 - \eta, \beta \in \Sigma_*^+ \text{ with } (m_{\beta/2})/2 \text{ even} \} \\ &\quad \cap \{ \lambda \in \mathfrak{a}_\mathbb{C} : |\operatorname{Re} \lambda_\beta| < 1 - \eta, \beta \in \Sigma_*^+ \text{ with } (m_{\beta/2})/2 \text{ odd} \}. \end{aligned}$$

So $T_{\Sigma, m, 0} = T_{\Sigma, m}$ is the tube domain on which $\Pi(\lambda)b(\lambda)$ is holomorphic; see Corollary 4.6.

Lemma 5.6. *Set $s = |\Sigma_*^+|$.*

(a) *Let $0 < \eta < 1/2$. Then there is a constant $C_\eta > 0$ so that*

$$|\Pi(\lambda)b(\lambda)| \leq C_\eta (1 + \|\lambda\|)^s e^{-\pi(\sum_{j=1}^l |\operatorname{Im} \lambda_j|)}$$

for all $\lambda \in T_{\Sigma, m, \eta}$.

(b) *Let $a \in \mathcal{H}(A, P, \delta)$ and set $\tilde{a}(\lambda) = \sum_{w \in W} a(w\lambda)b(w\lambda)$. Then \tilde{a} is holomorphic in $T_{\Sigma, m} \cap T_\delta$. Moreover, let $0 < \eta < \min\{1/2, \delta\}$. Then there is a constant $C_{\eta, a} > 0$ so that*

$$|\tilde{a}(\lambda)| \leq C_{\eta, a} (1 + \|\lambda\|)^s e^{(A-\pi)c_2 \|\operatorname{Im} \lambda\|}$$

for all $\lambda \in T_{\Sigma, m, \eta} \cap T_{\delta-\eta}$. Here c_2 is the positive constant introduced in Remark 2.3.

(c) *Let*

$$\gamma = \min \left\{ \delta, \min_{\beta \in \Sigma_*^+, (m_{\beta/2})/2 \text{ even}} (\tilde{\rho}_\beta)^{-1}, \min_{\beta \in \Sigma_*^+, (m_{\beta/2})/2 \text{ odd}} (2\tilde{\rho}_\beta)^{-1} \right\} \in]0, 1[.$$

Then $T_\gamma \subset T_{\Sigma, m, \eta} \cap T_\delta$. Moreover, let $0 < \varepsilon < \gamma$. Then $\tilde{a} \in \mathcal{S}(\mathfrak{a}_\varepsilon^)^W$, the W -invariant Schwartz space on the tube domain T_ε .*

Proof. Since $\Pi(\lambda)b(\lambda)$ is bounded on $T_{\Sigma,m,\eta}$, the proof of the estimate in (a) follows the same argument as in part (a) of Lemma 5.4.

To prove part (b), notice first that, by Corollary 4.6, on $T_{\Sigma,m}$ the function $b(\lambda)$ has at most simple poles on hyperplanes of the form $\lambda_\beta = 0$ with $\beta \in \Sigma_*^+$. The same property holds on $T_{\Sigma,m} \cap T_\delta$ for $b(w\lambda)a(w\lambda)$, with $w \in W$, and hence for $\tilde{a}(\lambda)$. But $\tilde{a}(\lambda)$, as W -invariant function, cannot admit first order singularities on root hyperplanes through the origin. Thus \tilde{a} is holomorphic on $T_{\Sigma,m} \cap T_\delta$.

Let $0 < \eta < \eta' < \min\{1/2, \delta\}$. By (a), there is a constant $C_{\eta'} > 0$ so that

$$|\Pi(\lambda)b(\lambda)a(\lambda)| \leq C_{\eta'}(1 + \|\lambda\|)^s e^{(A-\pi)\|\operatorname{Im} \lambda\|_1} \leq C_{\eta'}(1 + \|\lambda\|)^s e^{(A-\pi)c_2\|\operatorname{Im} \lambda\|}$$

for all $\lambda \in T_{\Sigma,m,\eta'}$. Since $\Pi(\lambda)$ is W -skew-invariant, we obtain, on $T_{\Sigma,m,\eta'}$:

$$|\pi(\lambda)\tilde{a}(\lambda)| \leq C_{\eta'}|W|(1 + \|\lambda\|)^s e^{(A-\pi)c_2\|\operatorname{Im} \lambda\|}.$$

The estimate for \tilde{a} on $T_{\Sigma,m,\eta}$ follows then from Lemma 5.5.

To show that $T_\gamma \subset T_{\Sigma,m} \cap T_\delta$, notice that $\gamma \leq \delta$ and that if $\lambda \in T_\gamma$, then $|\operatorname{Re} \lambda_\beta| < \gamma \tilde{\rho}_\beta \leq 1$ if $(m_{\beta/2})/2$ is even and $\leq 1/2$ if $(m_{\beta/2})/2$ odd. Hence $T_\gamma \subset T_{\Sigma,m}$. The property that $\tilde{a} \in \mathcal{S}(\mathfrak{a}_\mathbb{C}^*)^W$ for $0 < \varepsilon < \gamma$ is a consequence of Cauchy's estimates. Indeed, let $\tilde{\varepsilon} = (\gamma - \varepsilon)/2$. For $\lambda \in T_\varepsilon$, let $D = \{\nu \in \mathfrak{a}_\mathbb{C}^* : |\nu_j - \lambda_j| \leq \tilde{\varepsilon} \text{ for all } j = 1, \dots, l\}$ be the closed polydisc with center $\lambda = (\lambda_1, \dots, \lambda_l)$ and multiradius $(\varepsilon, \dots, \varepsilon)$. Then $D \subset T_\gamma$. According to Cauchy's estimates (see e.g. [18], Lemma 2.3.9), for every multiindex α ,

$$|\partial^\alpha \tilde{a}(\lambda)| \leq \frac{\alpha!}{\tilde{\varepsilon}^{|\alpha|}} \sup_{\nu \in D} |\tilde{a}(\nu)|.$$

Since $|\tilde{a}(\nu)| \leq C_\gamma(1 + \|\nu\|)^s e^{(A-\pi)c_2\|\operatorname{Im} \nu\|}$, by estimating $\|\nu\|$ and $\|\operatorname{Im} \nu\|$ in terms of $\|\lambda\|$ and $\|\operatorname{Im} \lambda\|$, respectively, we obtain for a constant $C_{\tilde{\varepsilon},\gamma}$ depending on $\tilde{\varepsilon}$ and γ but not on λ :

$$|\tilde{a}(\lambda)| \leq C_{\tilde{\varepsilon},\gamma}(1 + \|\lambda\|)^s e^{(A-\pi)c_2\|\operatorname{Im} \lambda\|}.$$

Since $A < \pi$, we conclude the required rapid decay. \square

6. PROOF OF RAMANUJAN'S MASTER THEOREM FOR SYMMETRIC SPACES

In this section we prove Theorem 2.1. Part 1 is an immediate consequence of Lassalle's Theorem 1.1 with $F(\mu) = (-1)^{|\mu|}a(\mu + \rho)$ and $\varepsilon = P/\Omega$. Indeed, for $\mu \in \Lambda^+$ we have $\Omega(\sum_{j=1}^l \mu_j) = \Omega\|\mu\|_1 \geq \|\mu\|$. Hence, for $a \in \mathcal{H}(\delta)$,

$$|a(\mu + \rho)| \leq C \prod_{j=1}^l e^{-P(\mu_j + \rho_j)} = C' e^{-P \sum_{j=1}^l \mu_j} \leq C' e^{-\frac{P}{\Omega}\|\mu\|},$$

where $C' = C \prod_{j=1}^l e^{-P\rho_j}$.

To prove Part 2, let N be a positive integer. For $j = 1, \dots, l$, let $C_{j,N}$ be the closed rectangular contour in the λ_j -plane passing clockwise through its vertices $-iN$, iN , $iN + (\rho_j + N + 1/2)$ and $-iN + (\rho_j + N + 1/2)$. Set

$$f(\lambda) = C_b a(\lambda) \varphi_\lambda(x) d(\lambda - \rho).$$

Recall that the spherical function φ_λ is an entire function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Hence f is holomorphic on $\mathcal{H}(\delta)$. Suppose $\lambda_2, \dots, \lambda_l$ are fixed values so that $\lambda_j - \rho_j \notin \mathbb{Z}^+$. Then, by the residue

theorem, we have

$$\begin{aligned} \oint_{C_{1,N}} f(\lambda) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} d\lambda_1 &= (-2\pi i) \sum_{\mu_1=0}^N \operatorname{Res}_{\lambda_1=\rho_1+\mu_1} \left(f(\lambda) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \right) \\ &= (-2\pi i) \sum_{\mu_1=0}^N f(\rho_1 + \mu_1, \lambda_2, \dots, \lambda_l) \frac{(-1)^{\mu_1}}{\pi} \prod_{j=2}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))}. \end{aligned}$$

Iterating, we obtain

$$\begin{aligned} \oint_{C_{1,N}} \cdots \oint_{C_{l,N}} f(\lambda) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} d\lambda_1 \cdots d\lambda_l &= \\ &= (-2\pi i)^l \sum_{\mu_1=0}^N \cdots \sum_{\mu_l=0}^N f(\rho_1 + \mu_1, \dots, \rho_l + \mu_l) \frac{(-1)^{\mu_1 + \cdots + \mu_l}}{\pi^l} \\ &= (-2i)^l C_b \sum_{\mu_1=0}^N \cdots \sum_{\mu_l=0}^N (-1)^{|\mu|} d(\mu) a(\rho + \mu) \varphi_{\rho+\mu}(x). \end{aligned}$$

Thus, by (53),

$$\oint_{C_{1,N}} \cdots \oint_{C_{l,N}} a(\lambda) \varphi_\lambda(x) \frac{b(\lambda)}{c(\lambda)c(-\lambda)} d\lambda_1 \cdots d\lambda_l = \sum_{\mu_1=0}^N \cdots \sum_{\mu_l=0}^N (-1)^{|\mu|} d(\mu) a(\rho + \mu) \varphi_{\rho+\mu}(x). \quad (78)$$

By Part 1, the right-hand side of (78) converges as $N \rightarrow \infty$ to $\sum_{\mu \in \Lambda^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \varphi_{\mu+\rho}(x)$ for $x = \exp H \cdot o$ with $\|H\| < P/\Omega$, the convergence being normal on compacta of $\exp((P/\Omega)B) \cdot o$.

For the limit of the left-hand side of (78), we shall use the estimate (71). Observe first that the domain of integration for the left-hand side of (78) is

$$D(C_{1,N}, \dots, C_{l,N}) = \left\{ \lambda = \sum_{j=1}^l \lambda_j \omega_j \in \mathfrak{a}_{\mathbb{C}}^* : \lambda_j \in C_{j,N} \text{ for all } j = 1, \dots, l \right\}$$

and

$$\bigcup_{N=1}^{\infty} D(C_{1,N}, \dots, C_{l,N}) \subset B$$

where B is the set on which the estimate (71) holds.

Lemma 6.1. *Let $j \in \{1, \dots, l\}$ be fixed. For a positive integer N , let $\gamma_{j,N}$ be the portion of the contour $C_{j,N}$ from iN to $-iN$, and let $\eta_{j,N}$ be the vertical portion of $C_{j,N}$ from $-iN$ to iN (see Figure 1). Let $\tau < 0$ and $\sigma < 0$ be fixed constants. Then*

$$\begin{aligned} \lim_{N \rightarrow +\infty} \int_{\gamma_{j,N}} (1 + |z|)^M e^{\sigma |\operatorname{Im} z| + \tau \operatorname{Re} z} dz &= 0, \\ \lim_{N \rightarrow +\infty} \int_{\eta_{j,N}} (1 + |z|)^M e^{\sigma |\operatorname{Im} z| + \tau \operatorname{Re} z} dz &= \int_{-\infty}^{+\infty} (1 + |y|)^M e^{\sigma |y|} dy < \infty \end{aligned}$$

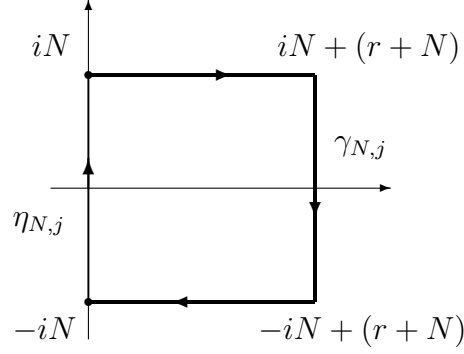


FIGURE 1. $C_{j,N} = \gamma_{j,N} \cup \eta_{j,N}$

Write $C_{j,N} = \gamma_{j,N} \cup \eta_{j,N}$ as above. Then

$$\begin{aligned} \oint_{C_{1,N}} \cdots \oint_{C_{l,N}} &= \left(\int_{\eta_{1,N}} - \int_{-\gamma_{1,N}} \right) \cdots \left(\int_{\eta_{l,N}} - \int_{-\gamma_{l,N}} \right) \\ &= \int_{\eta_{1,N}} \cdots \int_{\eta_{l,N}} \pm \sum_{\nu} \int_{\nu_{1,N}} \cdots \int_{\nu_{l,N}} \end{aligned}$$

where the sum is over all possible combinations $\nu = (\nu_{1,N}, \dots, \nu_{l,N})$, with $\nu_{j,N} \in \{\eta_{j,N}, -\gamma_{j,N}\}$, which are different from $(\eta_{1,N}, \dots, \eta_{l,N})$. In the following we write $d\lambda$ for $d\lambda_1 \cdots d\lambda_l$ and assume that the integration is performed in that order. Then

$$\begin{aligned} \oint_{C_{1,N}} \cdots \oint_{C_{l,N}} \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_{\lambda}(\exp H \cdot o) d\lambda &= \int_{\eta_{1,N}} \cdots \int_{\eta_{l,N}} \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_{\lambda}(\exp H \cdot o) d\lambda \\ &\pm \sum_{\nu} \int_{\nu_{1,N}} \cdots \int_{\nu_{l,N}} \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_{\lambda}(\exp H \cdot o) d\lambda. \quad (79) \end{aligned}$$

Here $H \in \overline{\mathfrak{a}^+}$ is fixed and we suppose that $\|H\| < P/\Omega$. Lemma 6.1 and the estimate (71) prove that the first integral on the right-hand side of (79) converges to

$$\int_{i\mathfrak{a}^*} \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_{\lambda}(\exp H \cdot o) d\lambda$$

as $N \rightarrow +\infty$. The estimate (71) also shows that for every $\nu = (\nu_{1,N}, \dots, \nu_{l,N})$ we have

$$\int_{\eta_{1,N}} \cdots \int_{\eta_{l,N}} \left| \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_{\lambda}(\exp H \cdot o) \right| d\lambda \leq C_{\nu} \prod_{j=1}^l \int_{\nu_{j,N}} (1 + |\lambda_j|)^M e^{\sigma|\operatorname{Im} \lambda_j| + \tau \operatorname{Re} \lambda_j} d\lambda_j,$$

with

$$\sigma = A - \pi < 0 \quad \text{and} \quad \tau = \|H\|\Omega - P < 0.$$

According to Lemma 6.1, for all $j = 1, \dots, l$, we have

$$\lim_{N \rightarrow +\infty} \int_{\nu_{j,N}} (1 + |\lambda_j|)^M e^{\sigma|\operatorname{Im} \lambda_j| + \tau \operatorname{Re} \lambda_j} d\lambda_j = \begin{cases} 0 & \text{if } \nu_{j,N} = \gamma_{j,N} \\ \int_{-\infty}^{\infty} (1 + |y|)^M e^{\sigma|y|} dy < \infty & \text{if } \nu_{j,N} = \eta_{j,N} \end{cases}$$

Each $\nu = (\nu_{1,N}, \dots, \nu_{l,N})$ contains at least one index j for which $\nu_{j,N} = \gamma_{j,N}$. So

$$\lim_{N \rightarrow +\infty} \int_{\nu_{1,N}} \cdots \int_{\nu_{l,N}} \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_\lambda(\exp H \cdot o) d\lambda = 0.$$

Thus

$$\lim_{N \rightarrow +\infty} \oint_{C_{1,N}} \cdots \oint_{C_{l,N}} \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_\lambda(\exp H \cdot o) d\lambda = \int_{i\mathfrak{a}^*} \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} \varphi_\lambda(\exp H \cdot o) d\lambda.$$

By the W -invariance of $[c(\lambda)c(-\lambda)]^{-1} \varphi_\lambda$, this concludes the proof of Part 2 of Theorem 2.1 in the case $\sigma = 0$.

To replace the integration along the imaginary axis $i\mathfrak{a}^*$ by the integration along any translate $\sigma + i\mathfrak{a}^*$ with $\sigma \in B(T_\delta)$, it suffices to use (74) and the following lemma, which is a consequence of Cauchy's theorem.

Lemma 6.2. *Let $D \subset \mathfrak{a}^*$ be nonempty, compact and connected, and let $F : T_D = D + i\mathfrak{a}^* \rightarrow \mathbb{C}$ be holomorphic. Suppose that, for every compact subset $\omega \subset D$ and every integer $N \geq 0$, we have*

$$\sup_{\operatorname{Re} \lambda \in D} (1 + \|\lambda\|)^N |F(\lambda)| < \infty.$$

Then, for any $\sigma \in D$, the integral $\int_{i\mathfrak{a}^} F(\sigma + \lambda) d\lambda$ exists and is independent of σ .*

Proof. See e.g. [10], Lemma 6.6.2. □

We now prove that the function $f(x) = \int_{i\mathfrak{a}^*} \tilde{a}(\lambda) \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda$ extends as a $K_{\mathbb{C}}$ -invariant holomorphic function of x on a neighborhood of X_G in $X_{\mathbb{C}}$. Here we have put

$$\tilde{a}(\lambda) = \sum_{w \in W} a(w\lambda) b(w\lambda).$$

Recall that $\varphi_\lambda(x)$ extends holomorphically as a $K_{\mathbb{C}}$ -invariant function on the domain $K_{\mathbb{C}} \exp(2\Omega_\pi) \cdot o$ with Ω_π given in (65). By $K_{\mathbb{C}}$ -invariance, it therefore suffices to consider the holomorphic extension of f inside $\exp(\Omega_\pi) \cdot o$. On this domain, Opdam's estimates (66) are available.

By estimates (72) and Remark 2.3, there is a constant $C_\delta > 0$ so that

$$\begin{aligned} |\tilde{a}(\lambda)| |c(\lambda)|^{-2} &\leq C_\delta (1 + \|\lambda\|)^M e^{(A-\pi) \sum_{j=1}^l |\operatorname{Im} \lambda_j|} \\ &\leq C_\delta (1 + \|\lambda\|)^M e^{(A-\pi)c_2 \|\operatorname{Im} \lambda\|} \end{aligned}$$

for all $\lambda \in i\mathfrak{a}^*$. For $\varepsilon > 0$ we set

$$T_{\varepsilon, \pi} = \{H \in \mathfrak{a}_{\mathbb{C}} : \|\operatorname{Im} H\| < \varepsilon\}.$$

Suppose we have chosen ε so that $(\pi - A)c_2 > \varepsilon$ and $T_{\varepsilon, \pi} \subset \Omega_\pi$. By (b) of Lemma 5.1, there is a constant $C_{\delta, \varepsilon} > 0$ such that

$$|\tilde{a}(\lambda) \varphi_\lambda(\exp H \cdot o)| |c(\lambda)|^{-2} \leq C_{\delta, \varepsilon} (1 + \|\lambda\|)^M e^{((A-\pi)c_2 + \varepsilon) \|\operatorname{Im} \lambda\|}.$$

The right-hand side of this inequality is an exponentially decaying function of $\operatorname{Im} \lambda$, hence integrable on $Q \times i\mathfrak{a}^*$ where Q is any compact subset of $T_{\varepsilon, \Omega}$. This allows us to apply to the $f(\exp H \cdot o)$ the theorems of Morera and Fubini, and the claim follows.

Finally, to prove the third part of Ramanujan's Master theorem, we use Lemma 5.6, which states that if $a \in \mathcal{H}(A, P, \delta)$ then $\tilde{a}(\lambda) \in \mathcal{S}(\mathfrak{a}_\varepsilon^*)^W$ for a certain $\varepsilon \in]0, 1[$. This in fact implies that $\mathcal{F}_G^{-1} \tilde{a} \in \mathcal{S}^p(X_G)^K \subset L^p(X_G)^K \cap L^2(X_G)^K$ with $p = 2/(\varepsilon + 1) \in]1, 2[$. Let f be the

spherical Fourier series associated with a as in (40). Comparison of the inversion formula (21) with (41) shows that $\mathcal{F}_G^{-1}\tilde{a}$ is a smooth K -invariant extension of f to all of G/K . Formula (42) states then that \tilde{a} is the spherical Fourier transform of $\mathcal{F}_G^{-1}\tilde{a}$. The identity holds pointwise for $\lambda \in T_\varepsilon$ and in L^2 sense for $\lambda \in i\mathfrak{a}^*$. The right-hand side of (42) provides a holomorphic extension of the spherical Fourier transform of $\mathcal{F}_G^{-1}\tilde{a}$ to all of $T_{\Sigma,m} \cap T_\delta$. Finally the equality of L^2 -norms of f and \tilde{a} is an immediate consequence of the Plancherel theorem for \mathcal{F}_G . This concludes the proof of Theorem 2.1.

7. THE REDUCTIVE CASE

In this section we extend Ramanujan's Master Theorem to reductive Riemannian symmetric spaces. As in Section 1, we consider Riemannian symmetric spaces in duality $X_U = U/K$ and $X_G = G/K$ inside their complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. We still assume that K is connected, but we now remove the assumption that U is semisimple. References for the following structures are Chapter II in [28], Part II §1 in [29], and Sections 1 and 2 in [6].

Let \mathfrak{z} be the center of the Lie algebra \mathfrak{u} of U . Then $\mathfrak{u} = \mathfrak{z} \oplus \mathfrak{u}'$ where $\mathfrak{u}' = [\mathfrak{u}, \mathfrak{u}]$ is semisimple. As in Section 1, let τ be the involution associated with X_U , and let $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ be the corresponding decomposition of \mathfrak{u} . Notice that τ preserves \mathfrak{z} and \mathfrak{u}' . We shall assume that $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, i.e. that the symmetric pair $(\mathfrak{u}, \mathfrak{k})$ is effective. Hence $\mathfrak{u}' = \mathfrak{k} \oplus i\mathfrak{p}'$ with $\mathfrak{p}' = \mathfrak{p} \cap (i\mathfrak{u}')$ and $i\mathfrak{p} = \mathfrak{z} \oplus (i\mathfrak{p}')$. The Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of G is reductive, and $\mathfrak{g} = i\mathfrak{z} \oplus \mathfrak{g}'$ where $i\mathfrak{z}$ is the center of \mathfrak{g} and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{k} \oplus \mathfrak{p}'$ is semisimple.

Set $\Gamma_0 = \{X \in \mathfrak{z} : \exp X = e\}$ where e is the identity of U . Then Γ_0 is a full rank lattice in \mathfrak{z} and $T = \mathfrak{z}/\Gamma_0 = \exp \mathfrak{z}$ is isomorphic to the identity component of the center of U . Let U' be the analytic subgroup of U with Lie algebra \mathfrak{u}' . Then U' is a compact connected semisimple Lie group with finite center containing K . Moreover $U = TU' \cong T \times_F U'$ where $F = T \cap U'$ is a finite central subgroup of U . We shall assume for simplicity that F is trivial. Hence $U \cong T \times U'$.

The involutive automorphism τ leaves K invariant and $X_{U'} = U'/K$ is a semisimple Riemannian symmetric space of the compact type. Moreover

$$U/K \cong T \times U'/K \cong \exp(\mathfrak{z}) \times U'/K. \quad (80)$$

Let G' be the analytic subgroup of G with Lie algebra \mathfrak{g}' . Then G' is a noncompact connected semisimple Lie group with finite center, and K is maximal compact in G' . The subgroup $V = \exp(i\mathfrak{z})$ is the split component of G . We have $G = G'V$ with $G' \cap V = \{e\}$. Hence

$$G/K \cong V \times G'/K = \exp(i\mathfrak{z}) \times G'/K. \quad (81)$$

Furthermore, $G_{\mathbb{C}} = (G'V)_{\mathbb{C}} \cong V_{\mathbb{C}} \times G'_{\mathbb{C}}$, where $V_{\mathbb{C}} = \exp(\mathfrak{z} \oplus i\mathfrak{z})$. So

$$G_{\mathbb{C}}/K_{\mathbb{C}} \cong V_{\mathbb{C}} \times G'_{\mathbb{C}}/K_{\mathbb{C}}. \quad (82)$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Then $\mathfrak{a} = i\mathfrak{z} \oplus \mathfrak{a}'$ where $\mathfrak{a}' \subset \mathfrak{p}'$ is maximal abelian. We fix an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{a} by setting it on \mathfrak{a}' equal to the one associated with the Killing form, equal on $i\mathfrak{z}$ to a fixed inner product, and by declaring that \mathfrak{a}' and $i\mathfrak{z}$ are orthogonal in \mathfrak{a} . Extend then $\langle \cdot, \cdot \rangle$ by duality on \mathfrak{a}^* and by \mathbb{C} -bilinearity on $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$. We denote by $\|\cdot\|$ the norms on \mathfrak{a} and \mathfrak{a}^* associated with the inner product $\langle \cdot, \cdot \rangle$. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $a = \exp H \in \exp(\mathfrak{a}_{\mathbb{C}}) \subset G_{\mathbb{C}}$, then we write $a^\lambda = e^{\lambda(H)}$, provided this is well defined.

Let Σ be the set of restricted roots of $(\mathfrak{g}', \mathfrak{a}')$. Denote by Σ^+ a choice of positive roots and by $(\mathfrak{a}')^+$ the corresponding positive Weyl chamber. Finally set $\mathfrak{a}^+ = i\mathfrak{z} \oplus (\mathfrak{a}')^+$. The

Weyl group W of $(\mathfrak{g}, \mathfrak{a})$ is the finite group generated by the reflections relative to Σ . It acts trivially on $i\mathfrak{z}$. A fundamental domain for the action of W on \mathfrak{a} is $\overline{\mathfrak{a}^+} = i\mathfrak{z} \oplus \overline{(\mathfrak{a}')^+}$.

The definitions of spherical functions on semisimple Riemannian symmetric spaces via (12) and (16) extend to the reductive case. Notice also that, by Remark 2.6, we can suppose that U'/K is simply connected.

As proven in [6], Section 2, the set parametrizing the K -spherical representations of U is in this case $\Lambda_K^+(U) = i\Gamma_0^* \oplus \Lambda_K^+(U')$ where

$$i\Gamma_0^* = \{\mu^0 \in i\mathfrak{z}^* : \mu^0(H) \in 2\pi i\mathbb{Z} \text{ for all } H \in \Gamma_0\} \quad (83)$$

and $\Lambda_K^+(U') = \{\mu \in (\mathfrak{a}')^* : \mu_\alpha \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+\}$ is as in (13). Here the direct sum symbol means that every element $\mu \in \Lambda_K^+(U)$ admits a unique decomposition as a sum of an element of $\mu^0 \in i\Gamma_0^*$ and an element of $\mu' \in \Lambda_K^+(U')$. If $\mu = \mu^0 + \mu'$, then $d(\mu) = d(\mu')$. Moreover, the spherical function of spectral parameter $\mu = \mu^0 + \mu'$ is

$$\psi_\mu(tu') = t^{\mu^0} \psi_{\mu'}(u'), \quad t \in T = \exp(\mathfrak{z}), u' \in U', \quad (84)$$

where $\psi_{\mu'}(u')$ is the spherical function of spectral parameter μ' on the semisimple Riemannian symmetric space of the compact type U'/K . A similar property holds for the spherical functions on G/K : if $\lambda = \lambda^0 + \lambda' \in \mathfrak{a}_\mathbb{C}^* = \mathfrak{z}_\mathbb{C}^* \oplus (\mathfrak{a}')_\mathbb{C}^*$, then the spherical function φ_λ on G/K is given by

$$\varphi_\lambda(xg') = x^{\lambda^0} \varphi_{\lambda'}(g'), \quad x \in V = \exp(i\mathfrak{z}), g' \in G', \quad (85)$$

where $\varphi_{\lambda'}(g')$ is the spherical function of spectral parameter λ' on the semisimple Riemannian symmetric space of the noncompact type G'/K . In particular, as in the semisimple case, the spherical functions on U/K extend holomorphically to $G_\mathbb{C}$ and

$$\psi_\mu|_G = \varphi_{\mu+\rho} \quad (86)$$

where $\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in (\mathfrak{a}')^*$. Since $\langle \lambda, \alpha \rangle = 0$ for $\lambda \in \mathfrak{z}_\mathbb{C}^*$ and $\alpha \in \Sigma$, we can extend the definition of the c -function to $\mathfrak{a}_\mathbb{C}^*$ by the same formula (19) as in the semisimple case. We then have $c(\lambda) = c(\lambda')$ if $\lambda = \lambda^0 + \lambda' \in \mathfrak{z}_\mathbb{C}^* \oplus (\mathfrak{a}')_\mathbb{C}^*$.

Formulas (84) and (85) reduce the spherical harmonic analysis on the pair of reductive symmetric spaces $U/K, G/K$ to the harmonic analysis on the abelian spaces T, V together with the spherical harmonic analysis on the semisimple symmetric spaces $U'/K, G'/K$.

Let $v = \dim V$ be the dimension of the split component of G . Since Γ_0 is a full rank lattice in \mathfrak{z} we can choose linearly independent vectors $e_1, \dots, e_v \in \mathfrak{z}$ so that $\Gamma_0 = \sum_{k=1}^v \mathbb{Z} e_k$. Define $\varepsilon_1, \dots, \varepsilon_v \in i\Gamma_0^*$ by $\varepsilon_k(e_h) = 2\pi i \delta_{k,h}$. Then $\{\varepsilon_1, \dots, \varepsilon_v\}$ is a basis of $(i\mathfrak{z})^*$. We fix $\{\varepsilon_1, \dots, \varepsilon_v, \omega_1, \dots, \omega_l\}$ as a basis of $\mathfrak{a}^* = (i\mathfrak{z})^* \oplus (\mathfrak{a}')^*$. The corresponding decomposition of $\lambda = \lambda^0 + \lambda' \in \mathfrak{a}_\mathbb{C}^* = \mathfrak{z}_\mathbb{C}^* \oplus (\mathfrak{a}')_\mathbb{C}^*$ will be written either as

$$\lambda = \sum_{j=1}^v \lambda_j \varepsilon_j + \sum_{j=v+1}^{v+l} \lambda_j \omega_j \quad \text{or as} \quad \lambda = \sum_{k=1}^v \lambda_k^0 \varepsilon_k + \sum_{j=1}^l \lambda_j' \omega_j. \quad (87)$$

Define

$$\Gamma_0^+ = \sum_{k=1}^v \mathbb{Z}^+ e_k, \quad (88)$$

$$(i\Gamma_0^*)^+ = \{\mu^0 \in i\mathfrak{z}^* : \mu^0(H) \in 2\pi i\mathbb{Z}^+ \text{ for all } H \in \Gamma_0^+\} \quad (89)$$

and

$$\Lambda^{++} = (i\Gamma_0^*)^+ \oplus \Lambda_K^+(U'). \quad (90)$$

If $\mu = \mu^0 + \mu' \in \Lambda^{++}$, then $\mu_k^0 \in \mathbb{Z}^+$ for all $k = 1, \dots, v$ and $\mu'_j \in \mathbb{Z}^+$ for all $j = 1, \dots, l$. We set

$$|\mu| = \mu_1^0 + \dots + \mu_v^0 + \mu'_1 + \dots + \mu'_l.$$

Moreover, if $x = \exp X$ with $X = i \sum_k X_k e_k \in i\mathfrak{z}$ and $\lambda^0 = \sum_k \lambda_k^0 \varepsilon_k \in \mathfrak{z}_{\mathbb{C}}^*$, then

$$x^{\lambda^0} = \prod_{k=1}^v e^{\lambda_k^0 X_k} = \prod_{k=1}^v x_k^{\lambda_k^0} \quad \text{with} \quad x_k = e^{X_k} \in]0, +\infty[.$$

Let A, P, δ be constants so that $A < \pi$, $P > 0$ and $0 < \delta \leq 1$. Set

$$\mathcal{H}(\delta) = \left\{ \lambda = \lambda^0 + \lambda' \in \mathfrak{a}_{\mathbb{C}}^* : \operatorname{Re} \lambda'_\beta > -\delta \tilde{\rho}_\beta \text{ for } \beta \in \Sigma_*^+ \text{ and } \operatorname{Re} \lambda_k^0 > -\delta \text{ for } k = 1, \dots, v \right\}. \quad (91)$$

The Hardy class $\mathcal{H}(A, P, \delta)$ is the space of all holomorphic functions $a : \mathcal{H}(\delta) \rightarrow \mathbb{C}$ satisfying the growth condition: there exists a constant $C > 0$ so that

$$|a(\lambda)| \leq C \prod_{j=1}^{v+l} e^{-P(\operatorname{Re} \lambda_j) + A|\operatorname{Im} \lambda_j|} \quad (92)$$

for all $\lambda \in \mathcal{H}(\delta)$. Furthermore, set

$$b(\lambda) = b^0(\lambda^0) b'(\lambda'), \quad \lambda = \lambda^0 + \lambda' \in \mathfrak{a}_{\mathbb{C}}^*, \quad (93)$$

where $b'(\lambda')$ is the function b of the semisimple Riemannian symmetric spaces U'/K , G'/K , as in (37), and

$$b^0(\lambda^0) = \left(\frac{i}{2} \right)^v \prod_{k=1}^v \frac{1}{\sin(\pi \lambda_k^0)}. \quad (94)$$

Observe that, since W acts trivially on $\mathfrak{z}_{\mathbb{C}}^*$, we have

$$\sum_{w \in W} a(w\lambda) b(w\lambda) = b^0(\lambda^0) \sum_{w \in W} a(w\lambda) b'(w\lambda'), \quad \lambda = \lambda^0 + \lambda'. \quad (95)$$

Let T'_δ be the tube domain in $(\mathfrak{a}')_{\mathbb{C}}^*$ defined by (29), and let

$$T_\delta^0 = \{ \lambda^0 \in \mathfrak{z}_{\mathbb{C}}^* : 0 < \operatorname{Re} \lambda_k^0 < \delta \text{ for all } k = 1, \dots, v \}.$$

Consider the tube domain in $\mathfrak{a}_{\mathbb{C}}^*$ given by

$$T_\delta = T_\delta^0 \oplus T'_\delta. \quad (96)$$

Its base in \mathfrak{a}^* is $B(T_\delta) = \{ \lambda^0 \in (i\mathfrak{z})^* : 0 < \lambda_k^0 < \delta \text{ for all } k = 1, \dots, v \} \oplus B(T'_\delta)$. Finally, let $T'_{\Sigma, m}$ be as in (38) and let $T_{\Sigma, m} = \mathfrak{z}_{\mathbb{C}}^* \oplus T'_{\Sigma, m}$.

The following theorem combines the semisimple and the multivariable abelian versions of Ramanujan's Master Theorem. Because of formulas (87) to (96), its proof reduces to a straightforward combination of the arguments used in these two cases, and is omitted.

Theorem 7.1 (Ramanujan's Master Theorem for reductive Riemannian symmetric spaces). *Keep the above assumptions and notation, and let $a \in \mathcal{H}(A, P, \delta)$. Then the following properties hold:*

(1) *The spherical Fourier series*

$$f(x) = \sum_{\mu \in \Lambda^{++}} (-1)^{|\mu|} d(\mu) a(\mu + \rho) \psi_{\mu}(x) \quad (97)$$

converges normally on compact subsets of $D_{P/\Omega} = U \exp((P/\Omega)B) \cdot o_{\mathbb{C}}$ where $B = \{H \in \mathfrak{a} : \|H\| < 1\}$ is the open unit ball in \mathfrak{a} . Its sum is a K -invariant holomorphic function on the neighborhood $D_{P/\Omega}$ of X_U in $X_{\mathbb{C}}$.

(2) Let T_{δ} be the tube domain in (96) and let $\sigma \in B(T_{\delta})$. Then for $x = \exp H \in A$ with $\|H\| < P/\Omega$, we have

$$f(x) = \frac{1}{|W|} \int_{\sigma + i\mathfrak{a}^*} \left(\sum_{w \in W} a(w\lambda) b(w\lambda) \right) \varphi_{\lambda}(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}. \quad (98)$$

(3) *The formula*

$$\int_{X_G} f(x) \varphi_{-\lambda}(x) dx = \sum_{w \in W} a(w\lambda) b(w\lambda) \quad (99)$$

holds for the extension of f to X_G and for all $\lambda \in T_{\delta} \cap T_{\Sigma, m} = T_{\delta}^0 \oplus T'_{\Sigma, m}$.

8. FURTHER REMARKS AND OPEN PROBLEMS

In [4], Bertram presents an interesting alternative approach to Ramanujan's Master Theorem for the rank-one case, by means of the kernel function $k(u, w) = (z(u) + z(w))^{-2\rho}$. In this formula, z is the holomorphic continuation to $X_{\mathbb{C}}$ of the function defined on X_U by $z(x) = \cos(d(x, K))$, where $d(x, K)$ is the distance of the point x from the base point in X_U . Moreover, $2\rho = m_{\beta/2}/2 + m_{\beta}$ for the unique root $\beta \in \Sigma_{*}^{+}$. The function z is natural, as every K -invariant function on X_U factors through z . By considering $k(x, y) = k_x(y) = k_y(x)$, Bertram proves analogues to Mehler's and Neumann's formulas, stating that there exists meromorphic functions $d(\lambda) = d(-\lambda)$ and $e(\lambda)$ (explicitly given as ratios of products of gamma functions) so that

$$(\mathcal{F}_G k_y)(\lambda) = d(\lambda) \varphi_{\lambda}(y) \quad \text{for } |\operatorname{Re} \lambda_{\beta}| < \rho$$

and

$$(\mathcal{F}_U k_x)(\mu + \rho) = (-1)^{\mu} e(\mu + \rho) \Phi_{-\mu - \rho}(x) \quad \text{for } \mu \in \mathbb{Z}^{+}.$$

Here Φ_{λ} denotes Harish-Chandra's series. By means of Harish-Chandra's relation $\varphi_{\lambda} = c(\lambda)\Phi_{\lambda} + c(-\lambda)\Phi_{-\lambda}$, he finds $b(\lambda) = \frac{d(\lambda)}{e(\lambda)} c(-\lambda)$. The explicit formulas for d, e and c allow him to recover the formula for b . It would be interesting to find a similar approach in higher rank as well. Unfortunately, we do not know higher dimensional analogues neither of the kernel function nor of the formulas by Mehler or Neumann, and the knowledge of b provides information only on the ratio $\frac{d(\lambda)}{e(\lambda)}$.

Another open question is on the nature of the functions on X_G, X_U and $X_{\mathbb{C}}$ satisfying the assumptions stated in Ramanujan's theorem. For instance, what are the functions on X_U having Fourier coefficients coming from elements of $\mathcal{H}(A, P, \delta)$? This kind of questions are related to theorems of Paley-Wiener type. Notice that one can use Laplace transforms to obtain elements of $\mathcal{H}(A, P, \delta)$. Indeed, identify $\mathfrak{a}_{\mathbb{C}}^{*}$ with \mathbb{C}^l by means of the basis Π_{*} , as in

section 1.6. Let $h : \mathfrak{a}_{\mathbb{C}}^* \equiv \mathbb{C}^l \rightarrow \mathbb{C}$ be integrable and having support inside a domain of the form $[P, R]^l$ with $P < R < +\infty$, and set

$$\mathcal{L}h(\lambda) = \int_0^{+\infty} \dots \int_0^{+\infty} h(x) e^{-\sum_{j=1}^l \lambda_j x_j} dx.$$

Then $\mathcal{L}h(\lambda)$ is an entire function on $\mathfrak{a}_{\mathbb{C}}^*$. Moreover, for every $d > 0$ there is a constant $C_d > 0$ so that $|(\mathcal{L}h)(\lambda)| \leq C_d e^{-P \sum_{j=1}^l \operatorname{Re} \lambda_j}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re} \lambda_j > -d$ for all j . Thus $\mathcal{L}h \in \mathcal{H}(A, P, \delta)$ for all $0 \leq A < \pi$.

Finally, the spherical Fourier transform on Riemannian symmetric spaces of the noncompact and of the compact type has been extended by the works of Heckman, Opdam and Cherednik to the setting of hypergeometric functions associated with root systems. See [24], [25] and references therein. A natural question is therefore a generalization of Ramanujan's Master Theorem in this setting. The necessary L^p -harmonic analysis on root systems needed for instance to generalize the proof of the final part of Theorem 2.1 has been recently developed in [20]. There are nevertheless technical difficulties, for instance the fact that at some points we use the classification of the root multiplicities, listed at the beginning of section 4. These kinds of arguments have to be replaced. A root system situation where all problems should be easily overpassed corresponds to the even multiplicity case. Indeed, the shift operators techniques developed by Opdam allow in this case a reduction to a Euclidean W -invariant situation where a multivariable analogue of the classical Ramanujan's theorem can be easily proven. See [12], Part 1, and [21, 22]. We will come back to these issues in future work.

REFERENCES

- [1] J.-P. Anker: *The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan.* *J. Funct. Anal.* **96** (1991), no. 2, 331–349.
- [2] B. Berndt: *Ramanujan's Notebooks, Part I*, Springer-Verlag, New York, 1985.
- [3] W. Bertram: Généralisation d'une formule de Ramanujan dans le cadre de la transformation de Fourier sphérique associée à la complexification d'un espace symétrique compact. *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), no. 11, 1161–1166
- [4] ———: Ramanujan's master theorem and duality of symmetric spaces. *J. Funct. Anal.* **148** (1997), no. 1, 117–151.
- [5] N. Bourbaki: *Elements of mathematics: Lie groups and Lie algebras*, Chapters 1–3, Springer Verlag, 1989.
- [6] T. Branson, G. Ólafsson and A. Pasquale: The Paley-Wiener theorem and the local Huygens' principle for compact symmetric spaces: the even multiplicity case, *Indag. Math. (N.S.)* **16** (2005), no. 3-4, 393–428.
- [7] H. Ding: Ramanujan's master theorem for Hermitian symmetric spaces. *Ramanujan J.* **1** (1997), no. 1, 35–52.
- [8] H. Ding, K.I. Gross and D.St.P. Richards: Ramanujan's master theorem for symmetric cones. *Pacific J. Math.* **175** (1996), no. 2, 447–490.
- [9] J. Faraut: Espaces hilbertiens invariants de fonctions holomorphes. In: J. Faraut, F. Rouvière, M. Vergne, *Analyse sur les groupes de Lie et théorie des représentations*, Séminaires et Congrès **7** (2003), 101–167.
- [10] R. Gangolli and V.S. Varadarajan: *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Springer Verlag, 1988.
- [11] G.H. Hardy: *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work*. Chelsea Publishing Company, 1991.

- [12] G. Heckman and H. Schlichtkrull. *Harmonic Analysis and Special Functions on Symmetric Spaces*. Academic Press, 1994.
- [13] S. Helgason: *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, New York, 1978.
- [14] ———: *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*. Pure and Applied Mathematics, 113. Academic Press, Inc., Orlando, FL, 1984.
- [15] ———: *Geometric Analysis on Symmetric Spaces*. American Mathematical Society, Providence, 1994.
- [16] J. Hilgert and A. Pasquale: Resonances and residue operators for symmetric spaces of rank one. *J. Math. Pures Appl.* **91** (2009), 495–507.
- [17] G. Hochschild: *The Structure of Lie Groups*. Holden-Day, Inc., 1965.
- [18] T. Krantz: *Several Complex Variable*, AMS Chelsea, 2001.
- [19] M. Lassalle: Séries de Laurent des fonctions holomorphes dans la complexification d’un espace symétrique compact, *Ann. Sci. Ec. Norm. Sup.* **11** (1978), 167–210.
- [20] E. K. Narayanan, A. Pasquale, and S. Pusti: Asymptotics of Harish-Chandra expansions, bounded hypergeometric functions associated with root systems, and applications, 30 pages, arXiv:1201.3891
- [21] G. Ólafsson and A. Pasquale: A Paley-Wiener Theorem for the Θ -hypergeometric transform: the even multiplicity case. *J. Math. Pures Appl.* (9) **83** (2004), no. 7, 869–927.
- [22] ———: Paley-Wiener theorems for the Θ -spherical transform: an overview. *Acta Appl. Math.* **81** (2004), 275–309.
- [23] G. Ólafsson and H. Schlichtkrull: A local Paley-Wiener theorem for compact symmetric spaces, *Adv. Math.* **218** (2008), 202–215.
- [24] E.M. Opdam: Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.* **175** (1995), no. 1, 75–121.
- [25] ———: *Lecture Notes on Dunkl Operators for Real and Complex Reflection Groups*. MSJ Memoirs, 8. Mathematical Society of Japan, Tokyo, 2000.
- [26] A. Pasquale: Asymptotic analysis of Θ -hypergeometric functions, *Invent. Math.* **157** (2004), 71–122.
- [27] R. J. Stanton and P. A. Tomas: Pointwise inversion of the spherical transform on $L^p(G/K)$, $1 \leq p < 2$. *Proc. Amer. Math. Soc.* **73** (1979), 398–404.
- [28] M. Takeuchi: *Modern Spherical Functions*. Translations of Mathematical Monographs, Amer. Math. Soc, 1994.
- [29] V. S. Varadarajan: *Harmonic Analysis on Real Reductive Groups*, Lecture Notes in Mathematics **576**, Springer Verlag, 1977.
- [30] G. Warner: *Harmonic Analysis on Semi-Simple Lie Groups. I*. Die Grundlehren der mathematischen Wissenschaften, Band **188**. Springer-Verlag, 1972.

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