Self-triggered control via dynamic high-gain scaling (long version)

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Abstract

This paper focuses on the construction of self-triggered state feedback laws. The approach followed is a high-gain approach. The event which triggers an update of the control law is based on an dynamical system which state is the high-gain parameter. This approach allows to design control laws ensuring convergence to the origin for nonlinear systems with triangular structure an a specific upper bound on the nonlinearities.

Key words: Nonlinear systems, self-triggered control, continuous-discrete time controller, high-gain, dynamic scaling.

1 Introduction

The implementation of a control law on a process requires the use of an appropriate sampling scheme. In this regards, periodic control (with a constant sampling period) is the usual approach that is followed for practical implementation on digital platforms. Indeed, periodic control benefits from a huge literature, providing a mature theoretical background (see e.g. [8, 15, 2]) and numerous practical examples. The use of a constant sampling period makes easier the closed-loop analysis and the implementation, allowing solid theoretical results and a wide deployment in the industry. However, the rate of control execution being fixed by a worst case analysis (the chosen period must guarantee the stability for all possible operating conditions), this may lead to an unnecessary fast sampling rate and then to an overconsumption of available resources.

The recent growth of shared networked control systems for which communication and energy resources are often limited goes with an increasing interest in aperiodic control design. This can be observed in the comprehensive overview on event-triggered and self-triggered control presented in [11]. Event-triggered control strategies introduce a triggering condition assuming a continuous monitoring of the plant (that requires a dedicated hardware) while in self-triggered strategies, the control update time is based on predictions using previously received data. The main drawback of self-triggered control is the difficulty to guarantee an acceptable degree of robustness, especially in the case of uncertain systems.

Most of the existing results on event-triggered and self-triggered control for nonlinear systems are based on the input-to-state stability (ISS) assumption which implies the existence of a feedback control law ensuring an ISS property with respect to measurement errors ([19, 7, 1, 16]).

In this ISS framework, an emulation approach is followed: the knowledge of an existing robust feedback law in continuous time is assumed then some triggering conditions are proposed to preserve stability under sampling.

Another proposed approach consists in the redesign of a continuous time stabilizing control. For instance, the authors of [14] adapted the original *universal formula* introduced by Sontag for nonlinear systems affine in the control. The relevance of this method was experimentally shown in [20] where the regulation of an omnidirectional mobile robot was addressed.

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Although aperiodic control literature has proved an interesting potential, important fields still need to be further investigated to allow a wider practical deployment.

The high-gain approach is a very efficient tool to address the stabilizing control problem in the continuous time case. It has the advantage to allow uncertainties in the model and to remain simple.

Different approaches based on high-gain techniques have been followed in the literature to tackle the output feedback problem in the continuous-time case (see for instance [4], [13]) and more recently for the (periodic) discrete-in-time case (see [18]). In the context of observer design, [6] proposed the design of a continuous discrete time observer, revisiting high-gain techniques in order to give an adaptive sampling stepsize.

In this work, we extend the results obtained in [6] to self-triggered state feedback control. In high-gain designs, the asymptotic convergence is obtained by dominating the nonlinearities with high-gain techniques. In the proposed approach, the high-gain is dynamically adapted with respect to time varying nonlinearities in order to allow an efficient trade-off between the high-gain parameter and the sampling step size. Moreover, the proposed strategy is shown to ensure the existence of a minimum inter-execution time.

The paper is organized as follows. The control problem and the class of system considered is given in Section 2. In Section 3, some preliminary results concerning linear system are given. The main result is stated in Section 4 and its proof is given in Section 5. Finally Section 6 contains an illustrative example.

2 Problem Statement

2.1 Class of considered systems

In this work, we consider the problem of designing a self-triggered control for the class of uncertain nonlinear systems described by the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + f(x(t)), \tag{1}$$

where the state x is in $\mathbb{R}^n$; $u : \mathbb{R} \to \mathbb{R}$ is the control signal in $\mathbb{L}^\infty(\mathbb{R}_+, \mathbb{R})$, A is a matrix in $\mathbb{R}^{n\times n}$, B is a vector in $\mathbb{R}^{n\times 1}$ and $f : \mathbb{R}^n \times \mathbb{R}$ is a vector field having the following triangular structure

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2}$$

$$f(x) = \begin{pmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix}. \tag{3}$$

We consider the case in which the vector field f satisfies the following assumption.

Assumption 1 (Nonlinear bound) *There exist a non-negative continuous function c, positive real numbers c_0, c_1 and q such that for all $x \in \mathbb{R}^n$, we have*

$$|f_j(x(t))| \leq c(x_1)\left(|x_1| + |x_2| + \cdots + |x_j|\right), \tag{4}$$

with

$$c(x_1) = c_0 + c_1|x_1|^q. \tag{5}$$

Notice that Assumption 1 is more general than the incremental property introduced in [18] since the function c is not constant but depends on x_1. This bound can be related also to [17, 13] in which continuous output feedback law are designed. Note however that in these works no bounds are imposed on the function c. Note moreover that in our context we don't consider inverse dynamics.

2.2 Updated sampling time controller

The design of a self-triggered controller involves to compute the sequence of control values $u(t_k)$ where $(t_k)_{k\geq 0}$ is a sequence of times to be selected. We refer to the instants t_k as *execution times*. The existence of a *minimal inter-execution time*, which is some bound $\delta > 0$ such that $t_{k+1} - t_k \geq \delta$ for all $k \geq 0$, is needed to avoid zero inter-sampling time leading to Zeno phenomena.

In the sequel, we restrict ourselves to a classic sample-and-hold implementation. Hence, the input is constant between any two execution times:

$$u(t) = u(t_k), \quad \forall\, t \in [t_k, t_{k+1}).$$

Hence, in addition to a feedback controller that computes the control input, event-triggered and self-triggered control systems need a *triggering mechanism* that determines when the control input has to be updated again. This rule is said to be *static* if it only involves the current state of the system, and *dynamic* if it uses an additional internal dynamic variable [10].

For simplicity, we also assume that the process of measurement, computing the control $u(t_k)$ and updating the actuators can be neglected. This assumption reflects that in many implementations this time is much smaller than the time elapsed between the instants t_k and t_{k+1} ([12]).

2.3 Notation

In this paper, we denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product in $\mathbb{R}^n$ and by $\| \cdot \|$ the induced Euclidean norm; we use the same notation for the corresponding induced matrix norm. Also, we use the symbol $'$ to denote the transposition operation.

In the following, the notation $\xi(t^-)$ stands for $\lim_{\substack{\tau \to t \\ \tau < t}} \xi(\tau)$. Also, to simplify the presentation, we introduce the notations $\xi_k = \xi(t_k)$ and $\xi_k^- = \xi(t_k^-)$.

3 Preliminary results: the linear case

In high-gain design, the idea is to consider the nonlinear terms (the f_i's) as disturbances. A first step consists in synthesizing a robust control for the linear part of the system, neglecting the effects of the nonlinearities. Then, the convergence and robustness are amplified through a high gain parameter to deal with the nonlinearities.

Therefore, let us first focus on a general linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{6}$$

where the state x evolves in $\mathbb{R}^n$ and the control u is in $\mathbb{R}$. The matrix A is in $\mathbb{R}^{n \times n}$ and B is a column vector in $\mathbb{R}^n$.

In this preliminary case, we review a well known result concerning periodic sampling approaches. Indeed, an emulation approach is adopted for the stabilization of the linear part: a feedback law is designed in continuous time and a triggering condition is chosen to preserve stability under sampling.

It is well known that if there exists a feedback control law (continuous-in-time) $u(t) = Kx(t)$ that asymptotically stabilizes the system then there exists a strictly positive inter-execution time $\delta_k = t_{k+1} - t_k$ such that the discrete-in-time control law $u(t) = Kx(t_k)$ for t in $[t_k, t_{k+1})$ renders the system asymptotically stable. This result is rephrased in the following Lemma 1 whose proof is postponed in Appendix A.1 and for which we do not claim any originality.

Lemma 1 *Suppose the pair (A;B) is stabilizable, that is there exists a matrix K in $\mathbb{R}^n$ rendering $(A + BK)$ Hurwitz. Then there exists a strictly positive real number δ^* such that for all δ in $[0; \delta^*)$ the state feedback*

$$u(t) = Kx(t_k), \qquad \forall t \in [t_k, t_{k+1}), \quad \forall k \in \mathbb{N}, \tag{7}$$

where $(t_k)_{k \in \mathbb{N}}$ is the sequence defined as $t_{k+1} = t_k + \delta$ makes the origin of the dynamical system (6) *a globally and asymptotically stable equilibrium.*

This result which is based on robustness is valid for general matrices A and B. The proof is based on the fact that if $A+BK$ is Hurwitz, the origin of the discrete time linear system defined for all k in $\mathbb{N}$ as

$$x_{k+1} = \left[\exp(A\delta) + \int_0^\delta \exp(A(\delta - s))BK ds \right] x_k, \tag{8}$$

is asymptotically stable for δ sufficiently small.

However, when we consider the particular case in which A and B satisfy the triangular form as in (1) (integrator chain), it is shown in the following theorem that the inter-execution time can be selected arbitrarily large as long as the control is modified.

Theorem 1 (Chain of integrator) *Suppose the matrices A and B have the structure stated in* (2)*. Then for all gain matrix K in $\mathbb{R}^n$ such that $A + BK$ is Hurwitz there exists a strictly positive real number α^* such that for all α in $[0, \alpha^*)$ and for all $\delta > 0$ the state feedback control law*

$$u(t) = K\mathcal{L}x(t_k), \qquad \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N} \tag{9}$$

$$\mathcal{L} = \mathrm{diag}(L^n, L^{n-1}, ..., L), \tag{10}$$

$$L = \frac{\alpha}{\delta}, \tag{11}$$

where $(t_k)_{k \in \mathbb{N}}$ is the sequence defined as $t_{k+1} = t_k + \delta$ makes the origin of the dynamical system (6) *a globally asymptotically stable equilibrium.*

Before proving this theorem, we would like to emphasize that in the particular case of the chain of integrator the sampling period time δ can be selected arbitrarily large.

Proof of Theorem 1 : In order to analyze the behavior of the closed-loop system, let us mention the following algebraic properties of the matrix $\mathcal{L}$:

$$\mathcal{L}A = L A \mathcal{L}, \quad \mathcal{L}BK = LBK. \tag{12}$$

Let us introduce the following change of coordinates:

$$X = \frac{\mathcal{L}}{L^{n+1}}\, x = \left[\frac{x_1}{L} \;\; \frac{x_2}{L^2} \;\; \cdots \;\; \frac{x_n}{L^n} \right]'. \tag{13}$$

Employing (12), it yields that in the new coordinates the closed-loop dynamics are for all t in $[t_k, t_{k+1})$:

$$\dot{X}(t) = L(AX(t) + BKX_k). \tag{14}$$

By integrating the previous equality and employing (11)

it yields for all k in $\mathbb{N}$:

$$X_{k+1} = \left[\exp(AL\delta) + \int_0^\delta \exp(AL(\delta - s))LBKds\right] X_k,$$
$$= \left[\exp(A\alpha) + \int_0^\alpha \exp(A(\alpha - s))BKds\right] X_k.$$

In other word this is the same discrete dynamics then the one given in (8) for system (6) in closed loop with the state feedback KX_k. Consequently, from Lemma 1, there exists a strictly real number α^* such that $X = 0$ (and thus $x = 0$) is a GAS equilibrium for the system (14) provided $L\delta$ is in $[0, \alpha^*)$. □

4 Main result: the nonlinear case

We consider now the nonlinear system (1). Let K and α be chosen to stabilize the linear part of the system and consider the control

$$u(t) = K\mathcal{L}_k x(t_k), \qquad \forall t \in [t_k, t_k + \delta_k) \tag{15}$$
$$\mathcal{L}(t) = \text{diag}\left(L(t)^n, L(t)^{n-1}, ..., L(t)\right). \tag{16}$$

It remains to select the sequences L_k and δ_k to deal with the nonlinearities.

In the context of a linear growth condition (i.e. if the bound $c(x_1)$ defined in Assumption 1 is replaced by a constant c), the authors of [18] have shown that a (well chosen) constant parameter L_k can guarantee the global stability, provided that L_k is greater than a function of the bound. Here, we need to adapt the high-gain parameter to follow a function of the time varying bound.

Following the idea presented in [6] in the context of observer design, we consider the following update law for the high-gain parameter:

$$\dot{L}(t) = a_2 L(t)M(t)c(x_1(t)), \quad \forall t \in [t_k, t_k + \delta_k) \tag{17}$$
$$\dot{M}(t) = a_3 M(t)c(x_1(t)), \quad \forall t \in [t_k, t_k + \delta_k) \tag{18}$$
$$L_k = L_k^-(1 - a_1\alpha) + a_1\alpha \tag{19}$$
$$M_k = 1, \tag{20}$$

with initial condition $L(0) \geq 1$, $M(0) \geq 1$ and where a_1, a_2, a_3 are positive real numbers to be chosen. For a justification of this type of high-gain update law, the interested reader may refer to [6] where is it shown that this update law is a continuous discrete version of the high-gain parameter update law introduced in [17]. Finally, the execution times t_k are given by the following relations,

$$t_0 = 0, \quad t_{k+1} = t_k + \delta_k, \tag{21}$$

$$\delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}. \tag{22}$$

Equations (21)-(22) constitute the triggering mechanism of the self-triggered strategy. It not directly involves the state value x but the additional dynamic variable L and so can be referred as a dynamic triggering mechanism ([10]). The relationship between L_k and δ_k comes from equation (11). It highlights the trade-off between high-gain value and inter-execution time (see [9, 18]).

We are now ready to state our main result which proof is given in Section 5.

Theorem 2 (Global attractivity via self-triggered control) *Consider the class of uncertain nonlinear systems described by* (1) *such that the nonlinear functions f_i's satisfy Assumption 1. Then there exist positive numbers a_1, a_2, a_3, a gain matrix K and α^* such that for all α in $[0, \alpha^*]$, the self-triggered feedback* (15)-(22) *initiated from $L(0) \geq 1$ and $M(0) \geq 1$ renders $x = 0$ a globally attractive equilibrium. Moreover there exists a positive real number $\delta_{\min}$ such that $\delta_k > \delta_{\min}$ for all k and so ensures the existence of a minimal inter-execution time.*

5 Proof of Theorem 2

5.1 Selection of the gain matrix K

Let b be a positive real number such that $b < \frac{1}{q}$ with q given in Assumption 1. Let D be the diagonal matrix in $\mathbb{R}^{n\times n}$ defined by $D = \text{diag}(b, 1+b, \ldots, n+b-1)$. Let P be a symmetric positive definite matrix and K a vector in $\mathbb{R}^n$ such that (always possible, see [5])

$$P(A + BK) + (A + BK)^T P \leq -I, \tag{23}$$
$$p_1 I \leq P \leq p_2 I, \tag{24}$$
$$p_3 P \leq PD + DP \leq p_4 P, \tag{25}$$

with $p_1, \ldots, p_4$ positive real numbers.

With the matrix K selected it remains to select the parameters a_1, a_2, a_3 and α^*. This is done in Proposition 1 and 2. Proposition 1 focuses on the existence of the sequence (x_k, L_k) for all k in $\mathbb{N}$, whereas, based on a Lyapunov analysis, Proposition 2 shows that a sequence of quadratic function of scaled coordinates is decreasing. Based on these two propositions, the proof of Theorem 2 is given in Section 5.4 where it is shown that the time function L is bounded.

5.2 Existence of the sequence $(t_k, x_k, L_k)_{k\in\mathbb{N}}$

The first step of the proof is to show that the sequence $(x_k, L_k)_{k\in\mathbb{N}} = (x(t_k), L(t_k))_{k\in\mathbb{N}}$ is well defined. Note that it doesn't imply that $x(t)$ is defined for all t since for the time being it has not been shown that the sequence t_k is unbounded. This is obtained in Section 5.4 when proving Theorem 2.

Proposition 1 (Existence of the sequence) *For all $a_1 > 0$, $a_2 \geq \frac{2n}{p_3}$, $a_3 > 0$ and $\alpha > 0$ the sequence $(t_k, x_k, L_k)_{k\in\mathbb{N}}$ is well defined.*

Proof of Proposition 1: To show that the sequence is well defined let us assume that this is not the case and that there exists k^* such that (t_k, x_k, L_k) is well defined but not $(t_{k+1}, x_{k+1}, L_{k+1})$. This implies that there exists a time $t^* > t_k$ such that $x(\cdot)$ and $L(\cdot)$ are well defined for all t in $[t_k, t^*)$ and such that

$$\lim_{t\to t^*} |x(t)| + |L(t)| = +\infty. \tag{26}$$

Since for all t in $[t_k, t^*)$ we have with (22) $L(t) \leq \frac{\alpha}{(t-t_k)}$, this implies:

$$L^* = \lim_{t\to t^*} L(t) \leq \frac{\alpha}{(t^* - t_k)} < +\infty. \tag{27}$$

On another hand, let us introduce the following scaled coordinates for system (1) (see also equation (13))

$$X(t) = \mathcal{S}(t)x(t), \tag{28}$$

$$\mathcal{S}(t) = \text{diag}\left(\frac{1}{L(t)^b}, \cdots, \frac{1}{L(t)^{n+b-1}}\right) = \frac{\mathcal{L}(t)}{L(t)^{n+b}}. \tag{29}$$

Denoting $V(X) = X(t)'PX(t)$, we have along the solution of (1) and for all t in $[t_k, t^*)$

$$\dot{\overline{V(X)}} = \dot{X}(t)'PX(t) + X(t)'P\dot{X}(t), \tag{30}$$

where

$$\begin{aligned}\dot{X}(t) &= \dot{\mathcal{S}}(t)x(t) + \mathcal{S}(t)\dot{x}(t)\\ &= -\frac{\dot{L}(t)}{L(t)}DS(t)x(t)\\ &\qquad + S(t)\left[Ax(t) + BK\mathcal{L}_k x_k + f(x(t))\right]\\ &= -\frac{\dot{L}(t)}{L(t)}DX(t)\\ &\qquad + L(t)AX(t) + L(t)BKX_k + S(t)f(x(t)).\end{aligned}$$

With the preceding equality, (30) becomes for all t in $[t_k, t^*)$

$$\begin{aligned}\dot{\overline{V(X)}} &= -\frac{\dot{L}(t)}{L(t)}X(t)'(PD + DP)X(t)\\ &\quad + L(t)[X(t)'(A'P + PA)X(t) + 2X(t)'PBKX_k]\\ &\qquad + 2X(t)'PS(t)f(x(t)).\end{aligned} \tag{31}$$

Since $M > 1$, we have with (25) for all t in $[t_k, t^*)$

$$\begin{aligned}-\frac{\dot{L}(t)}{L(t)}X(t)'(PD + DP)X(t) &\leq -p_3\frac{\dot{L}(t)}{L(t)}X(t)'PX(t)\\ &= -p_3a_2Mc(x_1(t))V(X)\\ &\leq -p_3a_2c(x_1(t))V(X).\end{aligned}$$

Moreover, using Young's inequality, we have

$$2X(t)'PBKX_k \leq X(t)'PX(t) + X_k(K'B'P + PBK)X_k.$$

Hence we have, for all t in $[t_k, t^*)$

$$\begin{aligned}\dot{\overline{V(X)}} &\leq -p_3a_2c(x_1(t))V(X) + L[X(t)'(A'P + PA)X(t)\\ &\qquad + X_k'(K'B'P + PBK)X_k] + 2nc(x_1(t))V(X)\\ &\leq \left(-p_3a_2c(x_1(t)) + L(t)\lambda_1 + 2nc(x_1(t))\right)V(X)\\ &\qquad + L(t)\lambda_2V_k,\end{aligned}$$

where [1] $\lambda_1 = \max\{0, \frac{\lambda_{\max}(A'P+PA)}{\lambda_{\min}(P)}\}$ and $\lambda_2 = \max\{0, \frac{\lambda_{\max}(K'B'P+PBK)}{\lambda_{\min}(P)}\}$.
Bearing in mind that $L(t) \leq L^*$ for all t in $[t_k, t^*)$ and since $a_2 \geq \frac{2n}{p_3}$, the previous inequality becomes

$$\dot{\overline{V(X)}} \leq L^*\lambda_1V(X) + L^*\lambda_2V_k.$$

This gives for all t in $[t_k, t^*)$

$$\begin{aligned}V(t) &\leq \exp(\lambda_1L^*(t - t_k))V_k\\ &\qquad + \int_0^{t-t_k}\exp(\lambda_1L^*(t - t_k - s))\lambda_2V_k ds\\ &\leq \exp(\lambda_1\alpha)V_k + (\exp(\lambda_1\alpha) - 1)\frac{\lambda_2}{\lambda_1}V_k.\end{aligned}$$

Hence, $\lim_{t\to t^*}|X(t)| < +\infty$. This, with (27), contradict (26). Hence, there exist $(t_{k+1}, x_{k+1}, L_{k+1})$ and proves that the sequence is well defined for all k in $\mathbb{N}$. □

5.3 Lyapunov analysis

This section is devoted to the Lyapunov analysis. It is shown that a good selection of the parameters a_1, a_2 and a_3 in the high-gain update law (17)-(20) yields the decrease of the sequences $V(X_k)_{k\geq 0}$.

Remark 1 *Due to the jumps of the high-gain L at instants t_k in equation (19), the Lyapunov function $t \mapsto V(X(t))$ does not decrease continuously as illustrated in Fig. 1. However, the sequences $V(X_k)_{k\geq 0}$ is decreasing.*

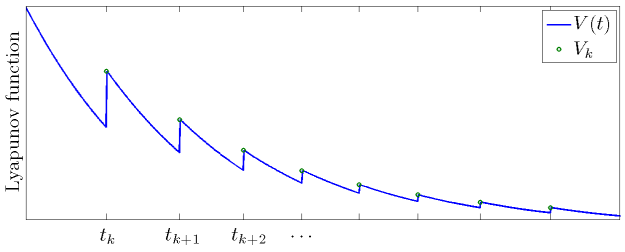

Fig. 1. Time evolution of Lyapunov function V.

[1] If Z is a symmetric matrix, $\lambda_{\max}(Z)$ and $\lambda_{\min}(Z)$ denotes respectively its largest and its smallest eigenvalue.

Remark 2 *Using the results obtained in [17] on lower triangular systems, the dynamic scaling* (29) *includes a number b. Although the decreases of $V(X_k)$ can be obtained with $b = 1$, it will be required that $bq < 1$ in order to ensure the boundedness of $L(\cdot)$ (see equation (46) in Section 5.4).*

The aim of this subsection is to show the following intermediate result.

Proposition 2 (Decrease of scaled coordinates) *There exist positive numbers a_1 (sufficiently small), a_2 (sufficiently large), and α^* such that with $a_3 = 2n$ and for all α in $[0, \alpha^*]$ the following property is satisfied:*

$$V(X_{k+1}) - V(X_k) \leq -\frac{\alpha p_1}{4p_2^2} \left\|\mathcal{S}_{k+1}^- \mathcal{S}_k^{-1}\right\|^2 V(X_k). \quad (32)$$

Proof of Proposition 2 : First of all, we assume that $a_2 \geq \frac{2n}{p_3}$. Hence, with Proposition 1, we know that the sequence (t_k, x_k, L_k) is well defined for all k in $\mathbb{N}$. Let k be in $\mathbb{N}$. The nonlinear system (1) with the control (15) gives the closed-loop dynamics

$$\dot{x}(t) = Ax(t) + BK\mathcal{L}_k x_k + f(x(t)), \qquad \forall t \in [t_k, t_k + \delta_k).$$

Integrating the preceding equality between t_k and t_{k+1} yields

$$x_{k+1} = \exp(A\delta_k)x_k + \int_0^{\delta_k} \exp(A(\delta_k - s))BK\mathcal{L}_k x_k ds + \int_0^{\delta_k} \exp(A(\delta_k - s))f(x(t_k + s))ds. \quad (33)$$

Let us first mention the following algebraic properties satisfied by the matrix function $\mathcal{L}(\cdot)$:

$$\mathcal{L}(s)A = L(s)A\mathcal{L}(s), \quad (34)$$
$$\mathcal{L}(s)\exp(As) = \exp(L(s)As)\mathcal{L}(s), \quad (35)$$
$$\mathcal{L}(s)BK = L(s)BK. \quad (36)$$

Employing the algebraic properties (34)-(36) we get, when left multiplying (33) by $\mathcal{S}_{k+1}^-$,

$$\mathcal{S}_{k+1}^- x_{k+1} = R + Q(\alpha)\mathcal{S}_{k+1}^- x_k, \quad (37)$$

where

$$R = \int_0^{\delta_k} \exp(L_{k+1}^- A(\delta_k - s))\mathcal{S}_{k+1}^- f(x(t_k + s))ds, \quad (38)$$

$$Q(\alpha) = \exp(A\alpha) + L_{k+1}^- \int_0^{\delta_k} \exp(L_{k+1}^- A(\delta_k - s))BKds = \exp(A\alpha) + \int_0^{\alpha} \exp(A(\alpha - \tau))BKd\tau, \quad (39)$$

by the time scaling $\tau = L_{k+1}^- s$.

Note that, since we have $X_{k+1} = \Psi\mathcal{S}_{k+1}^- x_{k+1}$ with $\Psi = \mathcal{S}_{k+1}(\mathcal{S}_{k+1}^-)^{-1}$, (37) yields

$$\begin{aligned} V(X_{k+1}) =& X_{k+1}' P X_{k+1} \\ =& (\Psi\mathcal{S}_{k+1}^- x_{k+1})' P \Psi \mathcal{S}_{k+1}^- x_{k+1} \\ =& V(X_k) + T_1 + T_2, \end{aligned}$$

with

$$\begin{aligned} T_1 =& X_k' \mathcal{S}_k^{-1} \mathcal{S}_{k+1}^- Q(\alpha)' \Psi P \Psi Q(\alpha) \mathcal{S}_{k+1}^- \mathcal{S}_k^{-1} X_k - V(X_k), \\ T_2 =& 2X_k' \mathcal{S}_k^{-1} \mathcal{S}_{k+1}^- Q(\alpha)' \Psi P \Psi R + R' \Psi P \Psi R. \end{aligned}$$

The following two steps are devoted to upper bound the two terms T_1 and T_2. The fact that the Lyapunov function is decreasing is due to the term T_1 which will be shown to be negative. The second term is handled by robustness.

Lemma 2 *Let $a_1 = \frac{1}{2p_2p_4}$. There exists $\alpha^* > 0$ sufficiently small such that for all α in $[0, \alpha^*)$*

$$T_1 \leq -\left[\frac{p_3 p_1 a_2}{a_3}\left(\exp\left(a_3 \int_{t_k}^{t_k+\delta_k} c(r)dr\right) - 1\right) + \frac{\alpha p_1}{4p_2}\right] \times \|\mathcal{S}_{k+1}^- x_k\|^2, \quad (40)$$

where $c(r) = c(x_1(r))$.

The proof of Lemma 2 is inspired from [6] in which similar computations are derived. This one is postponed in Appendix A.2.

Lemma 3 *There exist two real positive continuous functions N_1 and N_2 such that the following inequality holds*

$$T_2 \leq \|\mathcal{S}_{k+1}^- x_k\|^2 N_1(\alpha)\left[\exp\left(n\int_0^{\delta_k} c(t_k + r)dr\right) - 1\right]^2 + \left\|\mathcal{S}_{k+1}^- x_k\right\|^2 N_2(\alpha)\left[\exp\left(n\int_0^{\delta_k} c(t_k + r)dr\right) - 1\right].$$

The proof of Lemma 3 is postponed in Appendix A.3.

With the two bounds obtained for T_1 and T_2, we get

$$V(X_{k+1}) - V(X_k) \leq \left\|\mathcal{S}_{k+1}^- x_k\right\|^2 \left[N_1(\alpha)\left[e^\beta - 1\right] + N_2(\alpha)\left[e^\beta - 1\right]^2 - \frac{p_3 p_1 a_2}{a_3}\left[e^{\frac{a_3}{n}\beta} - 1\right] - \frac{\alpha p_1}{4p_2}\right],$$

where we have put

$$\beta = n \int_0^{\delta_k} c(t_k + r)dr.$$

If we take $a_3 = 2n$, the previous inequality becomes

$$V(X_{k+1}) - V(X_k) \leq \left\|\mathcal{S}_{k+1}^- x_k\right\|^2 \left[-\frac{\alpha p_1}{4p_2} + \left[e^\beta - 1\right] \left[N_1(\alpha) + N_2(\alpha)\left[e^\beta - 1\right] - \frac{p_3 p_1 a_2}{2n}\left[e^\beta + 1\right]\right]\right]$$

$$\leq \left\|\mathcal{S}_{k+1}^- x_k\right\|^2 \left[-\frac{\alpha p_1}{4p_2} + \left[e^\beta - 1\right]\left[e^\beta + 1\right]\left[N_1(\alpha) + N_2(\alpha) - \frac{p_3 p_1 a_2}{2n}\right]\right].$$

For $a_2 \geq = 2n\dfrac{N_1(\alpha) + N_2(\alpha)}{p_3 p_1}$ it yields

$$V(X_{k+1}) - V(X_k) \leq -\frac{\alpha p_1}{4p_2}\left\|\mathcal{S}_{k+1}^- x_k\right\|^2, \leq -\frac{\alpha p_1}{4p_2^2}\left\|\mathcal{S}_{k+1}^- \mathcal{S}_k^{-1}\right\|^2 V(X_k).$$

□

5.4 Boundedness of L and proof of Theorem 2

Although the construction of the updated law for the high-gain parameter (17)-(20) follows the idea developed in [6], the study of the behavior of the high-gain parameter is more involved. Indeed, in the context of observer design of [6], the nonlinear function c was assumed to be essentially bounded while in the present work, c is depending on x_1. This implies that the interconnection structure between state and high-gain dynamics must be further investigated.

Proof of Theorem 2 : Assume a_1, a_2, a_3 and α^* have been selected as in Proposition 1 and 2. With Proposition 1, it yields that for all k in $\mathbb{N}$, the triple (t_k, X_k, L_k) is well defined. Moreover, in the proof of Proposition 1, it has been shown that the following inequality holds for all s in $[0, \delta_k)$

$$V(t_k + s) \leq \left[\exp(\lambda_1\alpha) + (\exp(\lambda_1\alpha) - 1)\frac{\lambda_2}{\lambda_1}\right] V_k. \quad (41)$$

With Proposition 2, $(V_k)_{k\in\mathbb{N}}$ is a non increasing sequence, hence $V(t)$ and $\frac{|x_1(t)|}{L(t)^b} = X_1(t)$ are also bounded on the time of existence of the solution.

With this result in hand, let us analyze the high-gain dynamics. We have for all k and all t in $[t_k, t_{k+1})$, $\dot{L}(t) = \frac{a_2}{a_3}L(t)\dot{M}(t)$. This implies that for all s in $[t_k, t_{k+1})$

$$L(s) = \exp\left(\frac{a_2}{a_3}\int_{t_k}^s \dot{M}(\nu)d\nu\right) L_k, = \exp\left(\frac{a_2}{a_3}M(s) - \frac{a_2}{a_3}\right) L_k. \quad (42)$$

Consequently, from (19) and (22)

$$L_{k+1} = \exp\left(\frac{a_2}{a_3}(M_{k+1}^- - 1)\right) L_k(1 - a_1\alpha) + a_1\alpha, \quad (43)$$

and δ_k satisfies

$$\exp\left(\frac{a_2}{a_3}(M_{k+1}^- - 1)\right)\delta_k L_k = \alpha.$$

Since $M_{k+1}^- \geq 1, a_2 \geq 0$ and $a_3 \geq 0$ the previous equality implies

$$\delta_k L_k \leq \alpha\ . \quad (44)$$

Moreover, we have

$$\dot{M}(s) = a_3 M(s)c(x_1(s)), = a_3 M(s)(c_0 + c_1|x_1|^q), = a_3 M(s)\left(c_0 + c_1\left(\frac{|x_1(s)|}{L(s)^b}\right)^q L(s)^{bq}\right), \leq a_3 M(s)(c_0 + c_1 c_\infty L(s)^{bq}),$$

where c_∞ is a bound which comes from the Lyapunov analysis and the fact that as long as the solution exists $\frac{|x_1(s)|}{L(s)^b}$ is bounded. Since $L(s) \geq 1$, this implies that $\dot{M}(s) \leq a_3(c_0 + c_1c_\infty)M(s)L(s)^{bq}$, giving with (42)

$$\dot{M}(s) \leq CM(s)\exp\left(\frac{a_2}{a_3}bq(M(s) - 1)\right) L_k^{bq},$$

where $C = a_3(c_0 + c_1c_\infty)$. Let $\psi(s)$ be a solution to the scalar dynamical system

$$\psi(0) = 1\ ,\ \dot{\psi} = C\psi\exp\left(\frac{a_2}{a_3}bq(\psi - 1)\right).$$

This dynamical system is defined in $[0, T)$. Note that we have employing Gronwall Lemma that for all s such that $0 \leq (s - t_k)L_k^{bq} < T$

$$M(s) \leq \psi((s - t_k)L_k^{bq}).$$

Consequently for all s such that $0 \leq (s - t_k)L_k^{bq} < T$

$$M_{k+1}^- = M(t_k + \delta_k^-) \leq \psi\left(\delta_k L_k^{bq}\right).$$

From this we get employing (44) that for all k such that $L_k^{bq-1} < \frac{T}{\alpha}$

$$1 \leq M_{k+1}^- \leq \psi\left(\alpha L_k^{bq-1}\right), \tag{45}$$

and we get employing (43) that for all k such that $L_k^{bq-1} < \frac{T}{\alpha}$

$$L_{k+1} \leq F(L_k), \tag{46}$$

where

$$F(L_k) = \exp\left(\psi\left(\alpha L_k^{bq-1}\right) - 1\right) L_k(1 - a_1\alpha) + a_1\alpha.$$

Note that since $bq < 1$ and $\psi(0) = 1$, it yields that for L large enough $L^{bq-1} < \frac{T}{\alpha}$ and

$$\lim_{L \to +\infty} \frac{F(L)}{L} = 1 - a_1\alpha < 1.$$

Consequently, there exists $\bar{L}$ such that for all $L > \bar{L}$, $F(\bar{L}) < \bar{L}$. To show that the sequence L_k is bounded, it remains to look the maximal value that can be reached starting from $L_k < \bar{L}$ in one iteration. From (43) and (45), we get that L_{k+1} is a continuous function of L_k and then admit a maximum.

Hence there exists a positive real number $L_{\max}$ such that for all t, $1 \leq L(t) \leq L_{\max}$. This implies that for all k, $\frac{L_k}{L_{k+1}^-} \geq \frac{1}{L_{\max}}$. Consequently inequality (32) becomes

$$V(X_{k+1}) - V(X_k) \leq -\frac{\alpha p_1}{4p_2^2} \frac{1}{L_\infty^{n+b-1}} V(X_k).$$

Hence, with (41), this implies

$$\lim_{t \to +\infty} V(X(t)) = 0.$$

Employing the fact that $L(\cdot)$ is bounded once again, this yields

$$\lim_{t \to +\infty} \|x(t)\| = 0.$$

Moreover, from inequality (44) we can conclude that there exists a positive real number $\delta_{\min} = \frac{\alpha}{L_{\max}}$ such that $\delta_k > \delta_{\min}$ for all k.

6 Illustrative example

We apply our approach to the following uncertain third-order system proposed in [13]

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = \theta x_1^2 x_3 + u \end{cases} \tag{47}$$

where θ is a constant parameter which only a magnitude bound θ_{max} is known. The stabilization of this problem is not trivial even in the case of a continuous-in-time controller. The difficulties come from the nonlinear term $x_1^2 x_3$ that makes x_3 dynamics not globally Lipschitz, and from the uncertainty on θ value, preventing the use of a feedback to cancel the nonlinearity.

However, system (47) belongs to the class of systems (1) and the Assumption 1 is satisfied with $c(x_1) = \theta_{max} x_1^2$. Hence, by Theorem 2, self-triggered feedback controller (15)-(22) can be constructed. Simulation were conducted with gain matrix K and coefficient α selected as

$$K = \begin{bmatrix} -1 & -3 & -3 \end{bmatrix}', \quad \alpha = 0.4$$

to stabilize the linear part of the system (47). Parameters a_1, a_2 and a_3 have then been selected through a trial and error procedure as follows:

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 1.$$

Simulation results are given in Fig. 2 and Fig. 3. The evolution of the control and state trajectories are displayed in Fig. 2 for a particular initial condition. The corresponding evolution of the Lyapunov function V and the high-gain L are shown in Fig. 2. We can see how the inter-execution times δ_k adapts to the nonlinearity. Interestingly, it allows a significant increase of δ_k when the state is close to the origin: $L(t)$ then goes to 1 and consequently δ_k increases toward α value (that was selected as $\alpha = 0.4$ in this simulation).

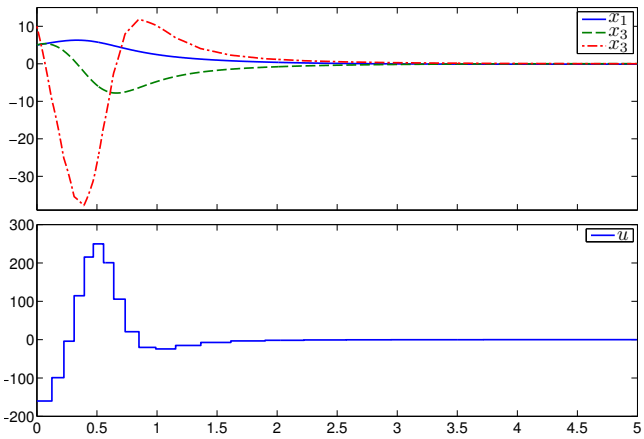

Fig. 2. Control signal and state trajectories of (47) with $(x_1, x_2, x_3) = (5, 5, 10)$ as initial conditions.

7 conclusion

In this paper a novel self-triggered state feedback law has been given. This one is based on a high-gain methodology. The event which triggers an update of the control law is based on an dynamical system which state

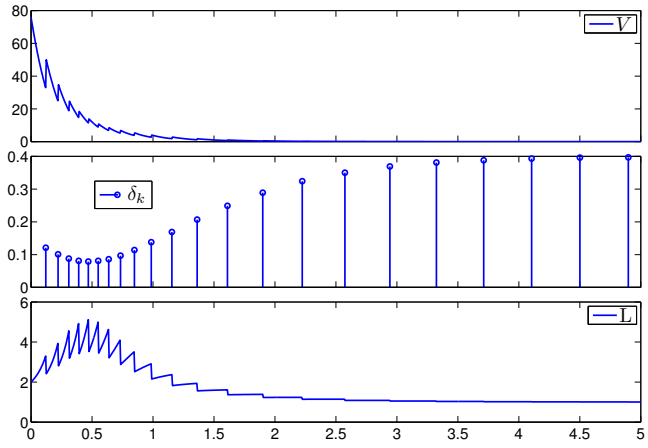

Fig. 3. Simulation results

is the high-gain parameter. This approach allows to design control laws ensuring convergence to the origin for nonlinear systems with triangular structure an a specific upper bound on the nonlinearities.

A Proofs of Lemmas

A.1 Proof of Lemma 1

Let P be a symmetric positive definite matrix. K rendering $(A+BK)$ Hurwitz, we have

$$P(A+BK)+(A+BK)^T P \leq -\lambda P, \tag{A.1}$$
$$pI \leq P \leq I, \tag{A.2}$$

with λ, p and some positive real numbers.

For all τ in $[0;\delta)$, the solution of the closed loop is

$$x(t_k+\tau) = \left[\exp(A\tau) + \int_0^\tau \exp(A(\tau-s))BKds\right] x(t_k). \tag{A.3}$$

In order to proof that the origin is the GAS equilibrium, we shall show that the Lyapunov function $V(x(t)) = x(t)'Px(t)$ is decreasing toward zero along the solution to the system. From (A.3), it comes for all τ in $[0;\delta)$

$$\begin{aligned} V(t_k+\tau) &= x(t_k+\tau)'Px(t_k+\tau) \\ &= x(t_k)'\Psi(\tau)'P\Psi(\tau)x(t_k) \end{aligned}$$

where

$$\Psi(\tau) = \exp(A\tau) + \int_0^\tau (\exp(A(\tau-s))BK)ds.$$

Given v in $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$, consider the function

$$\nu(\tau, v) = v'\Psi(\tau)'P\Psi(\tau)v.$$

We have

$$\begin{aligned} \nu(0,v) &= v'Pv, \\ \frac{\partial \nu}{\partial \tau}(0,v) &= v'[P(A+BK)+(A+BK)'P]v \\ &\leq -\lambda v'Pv. \end{aligned} \tag{A.4}$$

Now, we can write

$$\nu(\tau, v) = v'Pv + \tau\frac{\partial \nu}{\partial \tau}(0,v) + \rho(\tau, v),$$

with $\lim_{\tau\to 0}\frac{\rho(\tau,v)}{\tau} = 0$. This equality together with (A.4) imply that

$$\nu(\tau, v) \leq (1-\tau\lambda)v'Pv + \rho(\tau, v).$$

The vector v being in a compact set and the function ρ being continuous, there exists δ^* such that for all δ in $[0;\delta^*)$ we have $\rho(\delta, v) \leq \frac{\delta\lambda}{2}v'Pv$ for all v. This gives

$$\nu(\delta, v) \leq \left(1-\frac{\delta\lambda}{2}\right)v'Pv, \qquad \forall\ \delta \in [0,\delta^*), \forall\ v \in S^{n-1}.$$

This property being true for every v in S^{n-1}, we have

$$\Psi(\delta)'P\Psi(\delta) \leq \left(1-\frac{\delta\lambda}{2}\right)P,$$

and there exists δ^* such that for all δ in $[0;\delta^*)$ we have

$$\begin{aligned} V(t_k+\delta) &\leq \left(1-\frac{\delta\lambda}{2}\right)x(t_k)'Px(t_k) \\ &= \left(1-\frac{\delta\lambda}{2}\right)V(t_k). \end{aligned}$$

A.2 Proof of Lemma 2

The proof of Lemma 2 uses the following lemma whose proof is given in Appendix A.4.

Lemma 4 *Taking a_1 sufficiently small, there exists $\alpha_1 > 0$ such that for all $\alpha < \alpha_1$ we have*

$$Q(\alpha)'\Psi P\Psi Q(\alpha) \leq P - \frac{\alpha}{4p_2}P.$$

Proof of Lemma 2: We have, for all v in $\mathbb{R}^n$

$$\begin{aligned} &v'\mathcal{S}_{k+1}^{-}P\mathcal{S}_{k+1}^{-}v - v\mathcal{S}_k'P\mathcal{S}_k v = \\ &\qquad v'\left(\int_{t_k}^{t_{k+1}} \frac{d\mathcal{S}(s)}{ds}P\mathcal{S}(s) + \mathcal{S}P\frac{d\mathcal{S}(s)}{ds}ds\right)v. \end{aligned}$$

However, we have for all s in $[t_k, t_{k+1})$

$$\frac{d\mathcal{S}(s)}{ds} = -\frac{\dot{L}(s)}{L(s)} D\mathcal{S}(s).$$

Consequently, it yields

$$\begin{aligned} &v'\mathcal{S}_{k+1}^{-} P \mathcal{S}_{k+1}^{-} v - v\mathcal{S}_k' P \mathcal{S}_k v \\ &\quad = v' \left(\int_{t_k}^{t_{k+1}} -\frac{\dot{L}(s)}{L(s)} \mathcal{S}(s)[DP + PD]\mathcal{S}(s) ds \right) v. \end{aligned}$$

Note that since $L(0) > 1$, it yields that $L(t) > 1$ on the time of existence of the solution. Moreover, we have also $\dot{L} \geq 0$ and taking into account the bounds on P in (24) and on $DP + PD$ in (25), we get

$$\begin{aligned} &v'\mathcal{S}_{k+1}^{-} P \mathcal{S}_{k+1}^{-} v - v\mathcal{S}_k' P \mathcal{S}_k v \\ &\quad \leq v' \left(p_3 \int_{t_k}^{t_{k+1}} -\frac{\dot{L}(s)}{L(s)} \mathcal{S}(s) P \mathcal{S}(s) ds \right) v \\ &\quad = v' \left(p_3 \int_{t_k}^{t_{k+1}} -a_2 M(s) c(s) \mathcal{S}(s) P \mathcal{S}(s) ds \right) v \\ &\quad = v' \left(p_3 \int_{t_k}^{t_{k+1}} -a_2 \exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr \right) \right. \\ &\qquad\qquad \left. \times c(s) \mathcal{S}(s) P \mathcal{S}(s) ds \right) v \\ &\quad \leq -p_3 p_1 a_2 v' \left(\int_{t_k}^{t_{k+1}} \exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr \right) \right. \\ &\qquad\qquad \left. \times c(s) \|\mathcal{S}(s)\|^2 ds \right) v. \end{aligned}$$

Note that since $L_k \leq L_{k+1}^{-}$, we finally get

$$\begin{aligned} &v'\mathcal{S}_{k+1}^{-} P \mathcal{S}_{k+1}^{-} v - v\mathcal{S}_k' P \mathcal{S}_k v \\ &\quad \leq -p_3 p_1 a_2 v' \left(\int_{t_k}^{t_{k+1}} \exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr \right) c(s) \right. \\ &\qquad\qquad \left. \times \|\mathcal{S}_{k+1}^{-}\|^2 ds \right) v \\ &\quad = -\frac{p_3 p_1 a_2}{a_3} v' \left(\exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr \right) - 1 \right) \|\mathcal{S}_{k+1}^{-}\|^2 v \\ &\quad \leq -\frac{p_3 p_1 a_2}{a_3} \left(\exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr \right) - 1 \right) \|\mathcal{S}_{k+1}^{-} v\|^2. \end{aligned}$$

The previous inequality with $v = x_k$, together with Lemma 4, ensures the bound (40). This ends the proof of Lemma 2. □

A.3 Proof of Lemma 3

First, we seek for an upper bound of the norm of $\mathcal{S}_{k+1}^{-} f(x(t_k + s))$. We have

$$\begin{aligned} &\|\mathcal{S}_{k+1}^{-} f(x(t_k + s))\|^2 \\ &\quad = \sum_{j=1}^{n} \left((L_{k+1}^{-})^{-b-j+1} f_j(x(t_k + s)) \right)^2 \\ &\quad \leq \sum_{j=1}^{n} (L_{k+1}^{-})^{2(-b-j+1)} \left(\sum_{i=1}^{j} c(t_k + s) |x_i(t_k + s)| \right)^2 \\ &\quad = c(t_k + s)^2 \sum_{j=1}^{n} \left(\sum_{i=1}^{j} (L_{k+1}^{-})^{-b-j+1} |x_i(t_k + s)| \right)^2. \end{aligned}$$

Since $L_{k+1}^{-} \geq 1$, we have $(L_{k+1}^{-})^{-b-j+1} \leq (L_{k+1}^{-})^{-b-i+1}$ whenever $1 \leq i \leq j$. It yields

$$\begin{aligned} &\|\mathcal{S}_{k+1}^{-} f(x(t_k + s))\|^2 \\ &\quad \leq c(t_k + s)^2 \sum_{j=1}^{n} \left(\sum_{i=1}^{n} (L_{k+1}^{-})^{-b-i+1} |x_i(t_k + s)| \right)^2 \\ &\quad \leq c(t_k + s)^2 \sum_{j=1}^{n} n \|\mathcal{S}_{k+1}^{-} x(t_k + s)\|^2 \\ &\quad = n^2 c(t_k + s)^2 \|\mathcal{S}_{k+1}^{-} x(t_k + s)\|^2. \end{aligned} \tag{A.5}$$

Hence, from (38) and (A.5), we get

$$\begin{aligned} \|R\| &\leq \int_0^{\delta_k} \exp\left(L_{k+1}^{-} \|A\| (\delta_k - s) \right) nc(t_k + s) \\ &\qquad \times \|\mathcal{S}_{k+1}^{-} x(t_k + s)\| ds \\ &= \exp(\|A\| \alpha) \int_0^{\delta_k} \exp\left(-L_{k+1}^{-} \|A\| s \right) nc(t_k + s) \\ &\qquad \times \|\mathcal{S}_{k+1}^{-} x(t_k + s)\| ds. \end{aligned} \tag{A.6}$$

Moreover, we have for all s in $[0; \delta_k)$

$$\begin{aligned} &\mathcal{S}_{k+1}^{-} \dot{x}(t_k + s) \\ &= \mathcal{S}_{k+1}^{-} A x(t_k + s) + \mathcal{S}_{k+1}^{-} BK \mathcal{L}_k x_k + \mathcal{S}_{k+1}^{-} f(x(t_k + s)). \end{aligned}$$

Denoting by $w(s)$ the expression $\mathcal{S}_{k+1}^{-} x(t_k + s)$, this gives

$$\begin{aligned} \frac{d}{ds}\|w(s)\| &= \frac{\langle \dot{w}(s), w(s) \rangle}{\|w(s)\|} \\ &\leq \|\dot{w}(s)\| \\ &\leq \left\| L_{k+1}^{-} A w(s) \right\| + \left\| \mathcal{S}_{k+1}^{-} BK \mathcal{L}_k x_k \right\| \\ &\qquad + \left\| \mathcal{S}_{k+1}^{-} f(x(t_k + s)) \right\| \\ &\leq \left(L_{k+1}^{-} \|A\| + nc(t_k + s) \right) \|w(s)\| \\ &\qquad + \left\| BK (L_{k+1}^{-})^{-b-n+1} \mathcal{L}_k x_k \right\|, \text{ by (A.5)} \\ &\leq \left(L_{k+1}^{-} \|A\| + nc(t_k + s) \right) \|w(s)\| \\ &\qquad + \|BK\| \|w(0)\|. \end{aligned}$$

Hence, integrating the previous inequality, we obtain

$$\|w(s)\| \leq \int_0^s (L_{k+1}^- \|A\| + nc(t_k+s))\|w(s)\|ds + \|BK\|\|w(0)\| s + \|w(0)\|.$$

Since $(L_{k+1}^- \|A\| + nc(t_k+s))$ is a continuous non-negative function and $(\|BK\| s+1)\|w(0)\|$ is non-decreasing, applying a variant of the Gronwall-Bellman inequality (see [3, Theorem 1.3.1]), it comes

$$\|w(s)\| \leq (\|BK\| s+1)\|w(0)\| \times \exp\left(\int_0^s (L_{k+1}^- \|A\| + nc(t_k+r)dr\right),$$

and we have

$$\begin{aligned}\|\mathcal{S}_{k+1}^- x(t_k+s)\| \leq & (\|BK\| s+1)\exp\left(\int_0^s L_{k+1}^- \|A\| + nc(t_k+r)dr\right)\left\|\mathcal{S}_{k+1}^- x_k\right\| \\ = & (\|BK\| s+1)\exp\left(L_{k+1}^- \|A\| s\right) \times \exp\left(\int_0^s nc(t_k+r)dr\right)\left\|\mathcal{S}_{k+1}^- x_k\right\|. \end{aligned} \tag{A.7}$$

Consequently, according to (A.6) and (A.7), we get

$$\begin{aligned}\|R\| \leq & \exp(\|A\|\alpha)\int_0^{\delta_k} nc(t_k+s)(\|BK\| s+1) \times \exp\left(\int_0^s (nc(t_k+r)dr\right)\left\|\mathcal{S}_{k+1}^- x_k\right\| ds \\ \leq & \exp(\|A\|\alpha)\int_0^{\delta_k} nc(t_k+s)(\|BK\|\delta_k+1) \times \exp\left(\int_0^s (nc(t_k+r)dr\right)\left\|\mathcal{S}_{k+1}^- x_k\right\| ds \\ \leq & \exp(\|A\|\alpha)(\alpha\|BK\|+1)\int_0^{\delta_k} nc(t_k+s) \times \exp\left(\int_0^s (nc(t_k+r)dr\right) ds \left\|\mathcal{S}_{k+1}^- x_k\right\| \\ = & \exp(\|A\|\alpha)(\alpha\|BK\|+1) \times \left[\exp\left(\int_0^{\delta_k} (nc(t_k+r)dr\right) - 1\right]\left\|\mathcal{S}_{k+1}^- x_k\right\|. \end{aligned}$$

Hence, employing Lemma 5 this gives the existence of two continuous function N_1 and N_2 such that

$$\begin{aligned}T_2 = & R'\Psi P\Psi R + 2X_k'\mathcal{S}_k^{-1}\mathcal{S}_{k+1}^- Q(\alpha)'\Psi P\Psi R, \\ \leq & \|\mathcal{S}_{k+1}^- x_k\|^2 N_1(\alpha)\left[\exp\left(n\int_0^{\delta_k} c(t_k+r)dr\right) - 1\right]^2 \\ & + \left\|\mathcal{S}_{k+1}^- x_k\right\|^2 N_2(\alpha)\left[\exp\left(n\int_0^{\delta_k} c(t_k+r)dr\right) - 1\right], \end{aligned}$$

where

$$N_1(\alpha) = \exp(2\|A\|\alpha)(\alpha\|BK\|+1)^2\frac{\|P\|}{(1-a_1\alpha)^{2(n-b+1)}},$$
$$N_2(\alpha) = 2\exp(\|A\|\alpha)(\alpha\|BK\|+1)\frac{\|Q(\alpha)\|\|P\|}{(1-a_1\alpha)^{2(n-b+1)}}.$$

A.4 Proof of Lemma 4

In order to prove Lemma 4, we need the following lemma which will be proved in the next section. and which is an adaptation du lemme 2 de [6].

Lemma 5 *The matrix P satisfies the following property for all a_1 and α such that $a_1\alpha < 1$*

$$\Psi P\Psi \leq \psi_0(\alpha)P\psi_0(\alpha), \tag{A.8}$$

where

$$\psi_0(\alpha) = \operatorname{diag}\left(\frac{1}{(1-a_1\alpha)^b}, \ldots, \frac{1}{(1-a_1\alpha)^{n+b-1}}\right).$$

Given v in $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$, consider the function

$$\nu(\alpha, v) = v'Q(\alpha)'\psi_0(\alpha)P\psi_0(\alpha)Q(\alpha)v.$$

We have

$$\begin{aligned} Q(\alpha) &= \exp(A\alpha) + \int_0^\alpha \exp(A(\alpha-s))BKds, \\ Q(0) &= I, \\ \frac{\partial Q}{\partial\alpha}(0) &= A+BK, \\ \psi_0(0) &= I, \\ \frac{\partial\psi_0}{\partial\alpha}(0) &= a_1 D, \end{aligned}$$

then

$$\begin{aligned}\nu(0,v) &= v'Pv, \\ \frac{\partial\nu}{\partial\alpha}(0,v) &= v'\left[P[A+BK+a_1D] + [A+BK+a_1D]'P\right]v. \end{aligned}$$

So using the inequalities in (23)-(25) and setting $a_1 = \frac{1}{2p_2p_4}$, we get

$$\begin{aligned}\frac{\partial \nu}{\partial \alpha}(0,v) &\leq v'\left(a_1p_4P - \frac{1}{p_2}P\right)v \\ &= -\frac{1}{2p_2}v'Pv. \end{aligned} \tag{A.9}$$

Now, we can write

$$\nu(\alpha,v) = v'Pv + \alpha\frac{\partial \nu}{\partial \alpha}(0,v) + \rho(\alpha,v)$$

with $\lim_{\alpha\to 0}\frac{\rho(\alpha,v)}{\alpha} = 0$. This equality together with (A.9) imply that

$$\nu(\alpha,v) \leq v'Pv\left[1 - \alpha\frac{1}{2p_2}\right] + \rho(\alpha,v).$$

The vector v being in a compact set and the function r being continuous, there exists α^* such that for all α in $[0,\alpha^*)$ we have $r(\alpha,v) \leq \alpha\frac{1}{4p_2}v'Pv$ for all v. This gives

$$\nu(\alpha,v) \leq v'Pv\left[1 - \alpha\frac{1}{4p_2}\right], \forall\ \alpha \in [0,\alpha^*), \forall\ v \in S^{n-1}.$$

This property being true for every v, this ends the proof of Lemma 4.

A.5 Proof of Lemma 5

Consider the matrix function defined as

$$\mathcal{P}(s) = \operatorname{diag}(s^b,\ldots,s^{n+b-1})P\operatorname{diag}(s^b,\ldots,s^{n+b-1}).$$

Note that for all v in $\mathbb{R}^n$

$$\begin{aligned}\frac{d}{ds}v'\mathcal{P}(s)v =& \frac{1}{s}v'\operatorname{diag}(s^b,\ldots,s^{n+b-1})(D'P+PD) \\ &\times \operatorname{diag}(s^b,\ldots,s^{n+b-1})v \\ >& 0.\end{aligned}$$

Hence, $\mathcal{P}$ is an increasing function. Furthermore, we have

$$\begin{aligned}\Psi P\Psi =& \mathcal{S}_{k+1}(\mathcal{S}^-_{k+1})^{-1}P\mathcal{S}_{k+1}(\mathcal{S}^-_{k+1})^{-1} \\ =& \operatorname{diag}\left(\left(\frac{L^-_{k+1}}{L_{k+1}}\right)^b,\ldots,\left(\frac{L^-_{k+1}}{L_{k+1}}\right)^{n+b-1}\right)P \\ &\times \operatorname{diag}\left(\left(\frac{L^-_{k+1}}{L_{k+1}}\right)^b,\ldots,\left(\frac{L^-_{k+1}}{L_{k+1}}\right)^{n+b-1}\right) \\ =& \mathcal{P}\left(\frac{L^-_{k+1}}{L^-_{k+1}(1-a_1\alpha)+a_1\alpha}\right),\end{aligned}$$

Hence, as

$$\frac{L^-_{k+1}}{L^-_{k+1}(1-a_1\alpha)+a_1\alpha} \leq \frac{1}{1-a_1\alpha},$$

we get the inequality of Lemma 5, i.e., $\Psi P\Psi \leq \mathcal{P}\left(\frac{1}{1-a_1\alpha}\right)$.

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