HAL open science

Generalized analytic functions, Moutard-type transforms and holomorphic maps

Piotr Grinevich, Roman Novikov

To cite this version:

Piotr Grinevich, Roman Novikov. Generalized analytic functions, Moutard-type transforms and holomorphic maps. Functional Analysis and Its Applications, 2016, 50 (2), pp.150-152. hal-01234004

HAL Id: hal-01234004

https://hal.science/hal-01234004v1

Submitted on 26 Nov 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Generalized analytic functions, Moutard-type transforms and holomorphic maps *

P.G. Grinevich † R.G. Novikov‡

Abstract

We continue the studies of Moutard-type transform for generalized analytic functions started in [1]. In particular, we suggest an interpretation of generalized analytic functions as spinor fields and show that in the framework of this approach Moutard-type transforms for the aforementioned functions commute with holomorphic changes of variables.

Key words: generalized analytic functions, spinors, Moutard transforms.

We study the basic pair of conjugate equations of the generalized analytic function theory:

$$\partial_{\bar z}\psi = u\bar\psi \quad \text{in} \quad D, \tag{1}$$

$$\partial_{\bar z}\psi^+ = -\bar u\bar\psi^+ \quad \text{in} \quad D, \tag{2}$$

where D is an open domain in $\mathbb{C}$, $u = u(z)$ is a given function in D, $\partial_{\bar z} = \partial/\partial\bar z$; see [4]. Here and below, the notation $f = f(z)$ does not mean that f is holomorphic.

*The main part of the work was fulfilled during the visit of the first author to the IHES, France in November 2015. The first author was partially supported by the Russian Foundation for Basic Research, grant 13-01-12469 ofi-m2 and by the program "Fundamental problems of nonlinear dynamics", RAS.

†Landau Institute of Theoretical Physics, Chernogolovka, 142432, Russia; Lomonosov Moscow State University, Moscow, 119991, Russia; e-mail: pgg@landau.ac.ru

‡CNRS (UMR 7641), Centre de Mathématiques Appliquées, École Polytechnique, 91128, Palaiseau, France; IEPT RAS, 117997, Moscow, Russia; e-mail: novikov@cmap.polytechnique.fr

A new progress in the theory of generalized analytic functions was obtained very recently in [1] by showing that Moutard-type transforms can be applied to the pair of equations (1), (2). Note that ideas of Moutard-type transforms were developed and successfully used in the differential geometry, in the soliton theory in dimension 2+1, and in the spectral theory in dimension 2, see [1] for further references. In particular, our work [1] was essentially stimulated by recent articles by I.A. Taimanov [2], [3] on the Moutard-type transforms for the Dirac operators in the framework of the soliton theory in the dimension $2+1$. On the other hand, we were strongly motivated by some open problems of two-dimensional inverse scattering at fixed energy, where equation (1) arises as the $\bar\partial$-equation in spectral parameter.

A simple Moutard-type transform $\mathcal{M} = \mathcal{M}_{u,f,f^+}$ for the pair of conjugate equations (1), (2) is given by the formulas (see [1]):

$$\tilde u = \mathcal{M}u = u + \frac{f\overline{f^+}}{\omega_{f,f^+}}, \tag{3}$$

$$\tilde\psi = \mathcal{M}\psi = \psi - \frac{\omega_{\psi,f^+}}{\omega_{f,f^+}}\, f, \quad \tilde\psi^+ = \mathcal{M}\psi^+ = \psi^+ - \frac{\omega_{f,\psi^+}}{\omega_{f,f^+}}\, f^+, \tag{4}$$

where f and f^+ are some fixed solutions of equations (1) and (2), respectively, ψ and ψ^+ are arbitrary solutions of (1) and (2), respectively, and $\omega_{\psi,\psi^+} = \omega_{\psi,\psi^+}(z)$ denotes imaginary-valued function defined by:

$$\partial_z \omega_{\psi,\psi^+} = \psi\psi^+, \quad \partial_{\bar z}\omega_{\psi,\psi^+} = -\overline{\psi\psi^+} \ \text{ in } \ D, \tag{5}$$

where this definition is self-consistent, at least, for simply connected D, whereas a pure imaginary integration constant may depend on concrete situation. The point is that the functions $\tilde\psi$, $\tilde\psi^+$ defined in (4) satisfy the conjugate pair of Moutard-transformed equations:

$$\partial_{\bar z}\tilde\psi = \tilde u\, \overline{\tilde\psi} \qquad \text{in } \ D, \tag{6}$$

$$\partial_{\bar z}\tilde\psi^+ = -\overline{\tilde u}\, \overline{\tilde\psi^+} \qquad \text{in } \ D. \tag{7}$$

In addition, we have also the following new important result:

Proposition 1 *For a simple Moutard transform (3),(4) the following formula holds:*

$$\omega_{\tilde\psi,\tilde\psi^+} = \frac{\omega_{\psi,\psi^+}\omega_{f,f^+} - \omega_{\psi,f^+}\omega_{f,\psi^+}}{\omega_{f,f^+}} + c_{\tilde\psi,\tilde\psi^+}, \tag{8}$$

where $c_{\tilde\psi,\tilde\psi^+}$ is an imaginary constant.

In order to apply the Moutard-type transforms to studies of generalized-analytic functions with contour poles, we need to study, in particular, compositions of the former transforms and holomorphic maps.

Consider a holomorphic bijection W:

$$W : D \to D_*, \quad z \to \zeta(z), \tag{9}$$
$$W^{-1} : D_* \to D, \quad \zeta \to z(\zeta),$$

where D is the domain in (1), (2).

If we treat $\psi(z)$, $\psi^+(z)$ as scalar fields in equations (1), (2), then the conjugate property of these equations is not invariant with respect to holomorphic bijections . In the next theorem we give the proper transformation formulas for the conjugate pair of equations (1), (2) with respect to holomorphic bijections:

Theorem 1 *Let W be a holomorphic bijection as in (9). Let*

$$u_*(\zeta) = u(z(\zeta)) \sqrt{\frac{\partial z}{\partial \zeta} \frac{\partial \bar{z}}{\partial \bar{\zeta}}} = u(z(\zeta)) \left| \frac{\partial z}{\partial \zeta} \right|, \tag{10}$$

$$\psi_*(\zeta) = \psi(z(\zeta)) \sqrt{\frac{\partial z}{\partial \zeta}}, \quad \psi_*^+(\zeta) = \psi^+(z(\zeta)) \sqrt{\frac{\partial z}{\partial \zeta}}, \tag{11}$$

where $u(z)$, $\psi(z)$, $\psi^+(z)$ are the same that in equations (1), (2). Then:

$$\partial_{\bar{\zeta}} \psi_* = u_* \bar{\psi}_* \quad in \quad D_*, \tag{12}$$
$$\partial_{\bar{\zeta}} \psi_*^+ = -\bar{u}_* \bar{\psi}_*^+ \quad in \quad D_*. \tag{13}$$

In addition,

$$\omega_{\psi_*, \psi_*^+}(\zeta) = \omega_{\psi, \psi^+}(z(\zeta)), \tag{14}$$

where ω is defined according to (5).

Remark 1 *Formulas (10), (11) have the following natural interpretation: $\psi(z)$, $\psi^+(z)$ can be treated as spinors, i.e. differential forms of the type $\left(\frac{1}{2}, 0\right)$, and u can be treated as differential form of the type $\left(\frac{1}{2}, \frac{1}{2}\right)$. The corresponding forms can be written as:*

$$u = u(z)\sqrt{dz d\bar{z}}, \quad \psi = \psi(z)\sqrt{dz}, \quad \psi^+ = \psi^+(z)\sqrt{dz}. \tag{15}$$

It is very natural because the generalized analytic function equation (1) can be viewed as a special reduction of the two-dimensional Dirac system, see, for example [1].

Theorem 1 implies that W in (9) generates a map of the conjugate pair of equations (1), (2) into the conjugate pair of equations (12), (13). We also denote the latter map by W. Using this interpretation of W we obtain the following result:

Theorem 2 *The following formula holds:*

$$\mathcal{M}_{u_*,f_*,f_*^+} \circ W = W \circ \mathcal{M}_{u,f,f^+}, \tag{16}$$

where $\mathcal{M}_{u,f,f^+}$ and $\mathcal{M}_{u_*,f_*,f_*^+}$ are defined according to formulas (3), (4), and u_*, f_*, f_*^+, ω_{ψ_*,ψ_*^+} are defined according to (10), (11), (14).

Proposition 1 and Theorems 1 and 2 can be proved by direct calculations.

In the framework of the Moutard transform approach, using Theorem 2 we reduce local studies of generalized analytic functions with contour pole at a real-analytic curve to the case of contour pole at a straight line. These studies will be continued in a subsequent paper.

References

[1] P. G. Grinevich, R.G. Novikov, Moutard transform for generalized analytic functions, *Journal of Geometric Analysis*, DOI 10.1007/s12220-015-9657-8.

[2] I.A. Taimanov, Blowing up solutions of the modified Novikov-Veselov equation and minimal surfaces, *Theoretical and Mathematical Physics*, **182**:2 (2015), 173-181.

[3] I.A. Taimanov, The Moutard transformation of two-dimensional Dirac operators and Möbius geometry, *Mathematical Notes*, **97**:1 (2015), 124-135.

[4] I.N. Vekua, *Generalized Analytic Functions*, Pergamon Press Ltd. 1962.