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Transient response of thermoelastic bodies linked by a thin layer of low stiffness and high thermal resistivity

Réponse instationnaire de corps thermoélastiques liées par une couche mince de faible rigidité et haute résistivité thermique

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A B S T R A C T

We extend to the thermoelastic case the study [1] devoted to the dynamic response of a structure made of two linearly elastic bodies linked by a thin soft adhesive linearly elastic layer. Once again, a formulation in terms of an evolution equation in a Hilbert space of possible states with finite energy makes it possible to identify the asymptotic behavior, when some geometrical and thermomechanical parameters tend to their natural limits, as the response of two bodies linked by a thermomechanical constraint. The genuine thermomechanical coupling remains in the constraint law only for a specific relative behavior of the parameters.

R É S U M É

On étend au cas thermoélastique l'étude [1] consacrée à la réponse dynamique d'un assemblage de deux corps linéairement élastiques liés par une couche adhésive linéairement élastique mince et molle. À nouveau, une formulation en terme d'équations d'évolution dans un espace de Hilbert d'états possibles d'énergie finie permet d'identifier le comportement asymptotique, lorsque des paramètres géométriques et thermomécaniques tendent vers leurs limites naturelles, comme la réponse de l'assemblage des deux corps par une liaison thermomécanique. Le couplage thermomécanique initial perdure dans la loi de la liaison uniquement pour un comportement relatif particulier des paramètres.

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1. Introduction

Adhesively bonded joints are an attractive way to put together the components of a structure. As in several situations thermal effects are not negligible, we extend a previous study [1] devoted to a linearly elastic material to the linearly thermoelastic case. Taking advantage of the coupling between mechanical and thermal effects, it is still possible to formulate the problem of determining the transient response of a structure made of two linearly thermoelastic bodies perfectly connected by a thin soft thermoelastic layer with high thermal resistivity in terms of an evolution equation in a Hilbert space of possible states (displacement, temperature, velocity) with finite energy. Hence it is possible to adopt the strategy of [1,2] in order to, first, obtain existence and uniqueness results and, then, to study the asymptotic behavior when some geometrical and thermomechanical data, now regarded as parameters, tend to their natural limits. The limit behavior which supports our proposal of a simplified but accurate enough model for the initial physical situation, corresponds to the dynamic response to the initial load of two linearly thermoelastic bodies connected by a thermomechanical constraint along the surface the adhesive layer shrinks to. The structure of the constitutive equations of the constraint is similar to the one of the layer with coefficients depending on the relative behaviors of the parameters but the thermomechanical coupling is maintained only for a particular relative behavior.

2. Setting the problem

We consider a structure consisting of two thermoelastic bodies (adherents) bonded by a thin thermoelastic layer (adhesive). The entire system occupies the domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. Let (Γ_0^M, Γ_1^M) and (Γ_0^T, Γ_1^T) be two partitions of $\partial\Omega$ with $\mathcal{H}_2(\Gamma_0^T) > 0$ and $\mathcal{H}_2(\Gamma_0^M) > 0$, where \mathcal{H}_2 is the two dimensional Hausdorff measure. We denote the orthonormal canonical basis of \mathbb{R}^3 , assimilated to the physical Euclidean space, by $\{e_1, e_2, e_3\}$ and for all (x_1, x_2, x_3) in \mathbb{R}^3 , \hat{x} stands for (x_1, x_2) . The intersection S of Ω with $\{x_3 = 0\}$ is supposed to have a positive Hausdorff measure and it is also assumed that there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0} = \{(\hat{x}, x_3) \in \Omega; |x_3| < \varepsilon_0\}$ is equal to $S \times (-\varepsilon_0, \varepsilon_0)$. Let $\varepsilon < \varepsilon_0$, then the adhesive occupies the layer B_ε , while each of the two adherents occupies $\Omega_\varepsilon^\pm := \{x \in \Omega; \pm x_3 > \varepsilon\}$ and let $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$. Adherents and adhesive are assumed to be perfectly stuck together along $S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-$, $S_\varepsilon^\pm := \{x \in \Omega; x_3 = \pm\varepsilon\}$. The structure is clamped on Γ_0^M , maintained at a uniform temperature T_0 on Γ_0^T , subjected to body forces of density f , to surface forces of density g_M on Γ_1^M and to thermal flux g^T on Γ_1^T .

The whole structure is modeled as linearly thermoelastic in the following way. Let $(\rho, \beta, \alpha, \kappa, a) \in L^\infty(\Omega; \mathbb{R} \times \mathbb{R} \times S^3 \times S^3 \times \text{Lin}(S^3))$ satisfying

$$\begin{cases} \exists(\rho_m, \beta_m, \kappa_m, a_m) \in (0, +\infty)^4 \\ \rho(x) \geq \rho_m, \quad \beta(x) \geq \beta_m, \quad \alpha(x) \geq 0, \quad \kappa(x)\xi \cdot \xi \geq \kappa_m|\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \\ a(x)e \cdot e \geq a_m|e|^2, \quad \forall e \in S^3, \quad \text{a.e. } x \in \Omega \end{cases} \quad (1)$$

where S^3 is the space of 3×3 symmetric matrices with the usual inner product and norm denoted by \cdot and $|\cdot|$, as in \mathbb{R}^3 , and $\text{Lin}(S^3)$ is the space of linear mapping from S^3 to S^3 . The mass density, the specific heat coefficient, the thermal dilatation, the heat conductivity and the elasticity coefficients in the adherents are $\rho, \beta, \alpha, \kappa$ and a , respectively, while the positive numbers $\tilde{\rho}, \tilde{\beta}, \tilde{\alpha}, \tilde{\kappa}, \lambda$ and μ denote the mass density, the specific heat coefficient, the thermal dilatation, the heat conductivity and the Lamé coefficients in the adhesive assumed to be isotropic and homogeneous. Thus problem (\mathcal{P}_h) of determining the evolution of the assembly involves the quintuplet $h = (\varepsilon, \lambda, \mu, \kappa, \gamma)$ of data where $\gamma = (3\lambda + 2\mu)\tilde{\alpha}$ and thereafter all the fields will be indexed by h . In the following, the upper dot denotes the differentiation with respect to time t , $e(u)$ is the linearized strain tensor associated with the displacement field u . Hence problem (\mathcal{P}_h) reads as:

$$(\mathcal{P}_h) \begin{cases} \rho \ddot{u}_h = \text{div } \sigma_h + f, & T_0 \beta \dot{\theta}_h = \text{div } q_h - T_0 \alpha \alpha \cdot e(\dot{u}_h) \quad \text{in } \Omega_\varepsilon \\ \sigma_h = a(e(u_h) - \theta_h \alpha), & q_h = \kappa \nabla \theta_h \quad \text{in } \Omega_\varepsilon \\ \tilde{\rho} \ddot{u}_h = \text{div } \sigma_h + f, & T_0 \tilde{\beta} \dot{\theta}_h = \text{div } q_h - T_0 \gamma \text{tr } e(\dot{u}_h) \quad \text{in } B_\varepsilon \\ \sigma_h = \lambda \text{tr } e(u_h) Id + 2\mu e(u_h) - \gamma \theta_h Id, & q_h = \tilde{\kappa} \nabla \theta_h \quad \text{in } B_\varepsilon \\ \sigma_h \nu = g^M \quad \text{on } \Gamma_1^M, & q_h \cdot \nu = g^T \quad \text{on } \Gamma_1^T, \quad u_h = 0 \quad \text{on } \Gamma_0^M, \quad \theta_h = 0 \quad \text{on } \Gamma_0^T \\ u_h(x, 0) = u_h^0(x), & \dot{u}_h(x, 0) = v_h^0(x), \quad \theta_h(x, 0) = \theta_h^0(x), \quad \text{a.e. } x \in \Omega \end{cases} \quad (2)$$

where u_h, θ_h, σ_h and q_h are the fields of displacement, temperature increment with respect to T_0 , the stress tensor and the heat flux vector, respectively, while $u_h^0, v_h^0(x), \theta_h^0$ are the initial conditions. The symbols Id and ν refer to the 3×3 identity matrix and the outward unitary normal to $\partial\Omega$.

3. Existence and uniqueness result for (\mathcal{P}_h)

Assuming that

$$(H_1) : (f, g^M, g^T) \in C^{0,1}([0, T]; L^2(\Omega; \mathbb{R}^3)) \times C^{2,1}([0, T]; L^2(\Gamma_1^M; \mathbb{R}^3)) \times C^{2,1}([0, T]; L^2(\Gamma_1^T))$$

we seek $z_h := (u_h, \theta_h)$ in the form

$$z_h = z_h^e + z_h^r \quad (3)$$

where z_h^e is the unique solution to

$$z_h^e(t) \in Z_h; \quad \Phi_h(z_h^e(t), z) = L(t)(z), \quad \forall z \in Z_h, \quad \forall t \in [0, T] \quad (4)$$

with

$$\begin{cases} Z_h = H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3) \times H_{\Gamma_0^T}^1(\Omega) \\ H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3) := \{v \in H^1(\Omega; \mathbb{R}^3); v = 0 \text{ in the sense of traces on } \Gamma_0^M\} \\ H_{\Gamma_0^T}^1(\Omega) := \{v \in H^1(\Omega); v = 0 \text{ in the sense of traces on } \Gamma_0^T\} \end{cases} \quad (5)$$

$$\begin{cases} \Phi_h(z, z') = (u, u')_{h,1} + (\theta, \theta')_{h,3} - (u', \theta)_{h,5} + (u, \theta')_{h,5}, \quad \forall z = (u, \theta), \quad \forall z' = (u', \theta') \in Z_h \\ (u, u')_{h,1} := \int_{\Omega_\varepsilon} a e(u) \cdot e(u') \, dx + \int_{B_\varepsilon} (\lambda \operatorname{tr} e(u) \operatorname{tr} e(u') + 2\mu e(u) \cdot e(u')) \, dx, \quad \forall u, u' \in H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3) \\ (\theta, \theta')_{h,3} := \int_{\Omega_\varepsilon} \frac{\kappa}{T_0} \nabla \theta \cdot \nabla \theta' \, dx + \int_{B_\varepsilon} \frac{\tilde{\kappa}}{T_0} \nabla \theta \cdot \nabla \theta' \, dx, \quad \forall \theta, \theta' \in H_{\Gamma_0^T}^1(\Omega) \\ (u, \theta)_{h,5} := \int_{\Omega_\varepsilon} \alpha \alpha \cdot e(u) \theta \, dx + \int_{B_\varepsilon} \gamma \operatorname{tr} e(u) \theta \, dx, \quad \forall (u, \theta) \in H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3) \times H_{\Gamma_0^T}^1(\Omega) \end{cases} \quad (6)$$

and

$$L(t)(z) := \int_{\Gamma_1^M} g^M(x, t) \cdot u(x) \, d\mathcal{H}_2 + \int_{\Gamma_1^T} g^T(x, t) \theta(x) \, d\mathcal{H}_2, \quad \forall z = (u, \theta) \in Z_h \quad (7)$$

As $(g^M, g^T) \rightarrow z_h^e$ is linear continuous from $L^2(\Gamma_1^M; \mathbb{R}^3) \times L^2(\Gamma_1^T; \mathbb{R})$ into Z_h , we have

$$z_h^e \in C^{2,1}([0, T]; Z_h) \quad (8)$$

The remaining part z_h^r will be involved in an evolution equation governed by a m -dissipative operator A_h in a Hilbert space of possible states “with finite energy” defined by

$$H_h := H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3) \quad (9)$$

and endowed with the inner product and its norm defined respectively by:

$$(U^1, U^2)_h := (u^1, u^2)_{h,1} + K_h((v_1, \theta_1), (v_2, \theta_2)), \quad |U^i|_h^2 := (U^i, U^i)_h, \quad \forall U^i = (u^i, \theta^i, v^i) \in H_h, \quad i = 1, 2 \quad (10)$$

with

$$\begin{cases} K_h((v, \theta), (v', \theta')) = (v, v')_{h,2} + (\theta, \theta')_{h,4} \\ (v, v')_{h,2} := \int_{\Omega_\varepsilon} \rho v \cdot v' \, dx + \int_{B_\varepsilon} \tilde{\rho} v \cdot v' \, dx, \quad \forall v, v' \in L^2(\Omega; \mathbb{R}^3) \\ (\theta, \theta')_{h,4} := \int_{\Omega_\varepsilon} \beta \theta \theta' \, dx + \int_{B_\varepsilon} \tilde{\beta} \theta \theta' \, dx, \quad \forall \theta, \theta' \in L^2(\Omega) \end{cases} \quad (11)$$

Operator A_h is defined by:

$$\begin{cases} D(A_h) = \left\{ U = (u, \theta, v) \in H_h; \begin{cases} \text{(i)} & (v, \theta) \in Z_h \text{ and} \\ \text{(ii)} & \exists!(w, \tau) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega) \text{ such that} \end{cases} \right. \\ (w, v')_{h,2} + (u, v')_{h,1} - (v', \theta)_{h,5} = 0, \quad \forall v' \in H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3) \\ \left. \begin{cases} (\tau, \theta')_{h,4} + (\theta, \theta')_{h,3} + (v, \theta')_{h,5} = 0, \quad \forall \theta' \in H_{\Gamma_0^T}^1(\Omega) \\ A_h U = (v, \tau, w) \end{cases} \right\} \quad (12) \end{cases}$$

It is straightforward to check the following.

Proposition 3.1. Operator A_h is m -dissipative and for all $\phi_h = (\phi_h^1, \phi_h^2, \phi_h^3)$ in H_h

$$\begin{cases} \bar{U}_h = (\bar{u}_h, \bar{\theta}_h, \bar{v}_h) \text{ s.t.} \\ \bar{U}_h - A_h \bar{U}_h = \phi_h \end{cases} \Leftrightarrow \begin{cases} \bar{u}_h = \bar{v}_h + \phi_h^1 \\ \bar{z}_h = (\bar{v}_h, \bar{\theta}_h) \in Z_h; & \Psi_h(\bar{z}_h, z) = \mathcal{L}_h(z), \quad \forall z = (v, \theta) \in Z_h \\ \text{with} \\ \Psi_h = \Phi_h + K_h \\ \mathcal{L}_h(z) = (\phi_h^1, u)_{h,1} + (\phi_h^2, u)_{h,2} + (\phi_h^3, \theta)_{h,3}, \quad \forall z = (v, \theta) \in Z_h \end{cases} \quad (13)$$

Then taking (H_1) , (3), (4), (8) and (12) into account, it is clear that (\mathcal{P}_h) is formally equivalent to

$$\begin{cases} \frac{dU_h^r}{dt} - A_h U_h^r = F_h := \left(u^e - \frac{du^e}{dt}, -\frac{d\theta^e}{dt}, -\frac{dv^e}{dt} + f_h \right) \\ U^r(0) = U_h^0 - (z_h^e(0), 0), \quad U_h^0 := (u_h^0, \theta_h^0, v_h^0) \end{cases} \quad (14)$$

with

$$f_h = (1/\rho \mathbb{1}_{\Omega_\varepsilon} + 1/\tilde{\rho}(1 - \mathbb{1}_{\Omega_\varepsilon}))f \quad (15)$$

where $\mathbb{1}_{\Omega_\varepsilon}$ denotes the characteristic function of Ω_ε . Consequently [3] the following holds:

Theorem 3.1. If (f, g^M, g^T) satisfies (H_1) and if U_h^0 belongs to $(z_h^e(0), 0) + D(A_h)$, then (14) has a unique solution in $C^1([0, T]; H_h)$. Hence there exists a unique (u_h, θ_h) in

$$(C^1([0, T]; H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3)) \cap C^2([0, T]; L^2(\Omega; \mathbb{R}^3))) \times (C^1([0, T]; L^2(\Omega)) \cap C^0([0, T]; H_{\Gamma_0^T}^1(\Omega)))$$

which satisfies:

$$\begin{cases} \left[\left(\frac{d^2 u_h}{dt^2}, u \right)_{h,2} + (u_h, u)_{h,1} - (u, \theta_h)_{h,5} = \int_{\Omega} f \cdot u \, dx + \int_{\Gamma_1^M} g^M \cdot u \, d\mathcal{H}_2, \quad \forall u \in H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3) \right. \\ \left. \left[\left(\frac{d\theta_h}{dt}, \theta \right)_{h,4} + (\theta_h, \theta)_{h,3} + \left(\frac{du_h}{dt}, \theta \right)_{h,5} = \int_{\Gamma_1^T} g^T \theta \, d\mathcal{H}_2, \quad \forall \theta \in H_{\Gamma_0^T}^1(\Omega) \right] \end{cases} \quad (16)$$

We set:

$$U_h^e = (z_h^e, u_h^e) \quad (17)$$

4. Asymptotic behavior

Regarding the quintuplet h of geometrical and thermomechanical data as a quintuplet of parameters taking values in a countable subset of $(0, \varepsilon_0) \times (0, +\infty)^4$ with a single cluster point $h^* = (0, \lambda^*, \mu^*, \kappa^*, \gamma^*)$, we now study the asymptotic behavior of z_h in order to suggest a simplified but accurate enough model for the initial transient thermoelastic problem. We make the following assumptions on the relative magnitudes of the parameters:

$$(H_2) \begin{cases} \text{i) } h^* \in \{0\} \times [0, +\infty)^4 \\ \text{ii) } \exists (\bar{\lambda}, \bar{\mu}, \bar{\kappa}) \in [0, +\infty)^3 \text{ s.t. } (\lambda/2\varepsilon, \mu/2\varepsilon, \tilde{\kappa}/2\varepsilon) \rightarrow (\bar{\lambda}, \bar{\mu}, \bar{\kappa}) \\ \text{iii) } \bar{\mu} \in (0, +\infty] \text{ if } \min\{\mathcal{H}_2(\Gamma_0^{M+}), \mathcal{H}_2(\Gamma_0^{M-})\} = 0 \\ \quad \bar{\kappa} \in (0, +\infty] \text{ if } \min\{\mathcal{H}_2(\Gamma_0^{T+}), \mathcal{H}_2(\Gamma_0^{T-})\} > 0 \\ \text{iv) } \limsup_{s \rightarrow s^*} \varepsilon^2/\mu = \limsup_{s \rightarrow s^*} \varepsilon^2/\tilde{\kappa} = 0 \end{cases}$$

4.1. A candidate for describing the limit problem

Depending on the finiteness of $(\bar{\lambda}, \bar{\mu}, \bar{\kappa})$, we will have six distinct cases indexed by $I = (I_1, I_2) \in \{1, 2, 3\} \times \{1, 2\}$. The case $I_1 = 1$ corresponds to $(\bar{\lambda}, \bar{\mu}) \in [0, +\infty)^2$, $I_1 = 2$ to $(\bar{\lambda}, \bar{\mu}) \in \{+\infty\} \times [0, +\infty)$, $I_1 = 3$ to $\bar{\mu} = +\infty$, $I_2 = 1$ to $\bar{\kappa} \in [0, +\infty)$ and $I_2 = 2$ to case $\bar{\kappa} = +\infty$. We introduce the following spaces:

$$\begin{aligned}
{}^1Z &:= {}^1V \times {}^1W \\
{}^1V &:= H_{\Gamma_0^M}^1(\Omega \setminus S; \mathbb{R}^3), \quad {}^2V := \{v \in H_{\Gamma_0^M}^1(\Omega \setminus S; \mathbb{R}^3); [v]_3 = 0 \text{ on } S\}, \quad {}^3V := H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3) \\
{}^1W &:= H_{\Gamma_0^T}^1(\Omega \setminus S), \quad {}^2W := H_{\Gamma_0^T}^1(\Omega)
\end{aligned} \tag{18}$$

As any element $z = (u, \theta)$ of $H_{\Gamma_0^M}^1(\Omega \setminus S; \mathbb{R}^3) \times H_{\Gamma_0^T}^1(\Omega \setminus S)$ (spaces defined as $H_{\Gamma_0^M}^1(\Omega; \mathbb{R}^3)$ and $H_{\Gamma_0^T}^1(\Omega)$) has restrictions $z^\pm = (u^\pm, \theta^\pm)$ to Ω^\pm in $H^1(\Omega^\pm; \mathbb{R}^3) \times H^1(\Omega^\pm)$, the difference between the traces on S of z^+ and z^- , denoted by $[z] = ([u], [\theta])$, belongs to $L^2(S; \mathbb{R}^3 \times \mathbb{R})$ and represents the relative displacement and the jump of temperature across S .

Let the continuous bilinear forms:

$$\left\{ \begin{aligned}
{}^1(u, u')_1 &:= \int_{\Omega \setminus S} ae(u) \cdot e(u') \, dx + \int_S (\bar{\lambda}[u]_3[u']_3 + 2\bar{\mu}([u] \otimes_s e_3) \cdot ([u'] \otimes_s e_3)) \, d\hat{x}, \quad \forall u, u' \in {}^1V \\
{}^2(u, u')_1 &:= \int_{\Omega \setminus S} ae(u) \cdot e(u') \, dx + \int_S 2\bar{\mu}([u] \otimes_s e_3) \cdot ([u'] \otimes_s e_3) \, d\hat{x}, \quad \forall u, u' \in {}^1V \\
{}^3(u, u')_1 &:= \int_{\Omega} ae(u) \cdot e(u') \, dx, \quad \forall u, u' \in {}^3V \\
{}^1(\theta, \theta')_3 &:= \int_{\Omega \setminus S} \frac{\kappa}{T_0} \nabla \theta \cdot \nabla \theta' \, dx + \int_S \frac{\bar{\kappa}}{T_0} [\theta][\theta'] \, d\hat{x}, \quad \forall \theta, \theta' \in {}^1W \\
{}^2(\theta, \theta')_3 &:= \int_{\Omega} \frac{\kappa}{T_0} \nabla \theta \cdot \nabla \theta' \, dx, \quad \forall \theta, \theta' \in {}^2W \\
(u, u')_2 &:= \int_{\Omega} \rho u \cdot u' \, dx, \quad \forall u, u' \in L^2(\Omega; \mathbb{R}^3) \\
(\theta, \theta')_4 &:= \int_{\Omega} \beta \theta \theta' \, dx, \quad \forall \theta, \theta' \in L^2(\Omega) \\
{}^1(v, \theta)_5 &:= \int_{\Omega \setminus S} \alpha \alpha \cdot e(v) \theta \, dx + \int_S \gamma^* [v]_3 \frac{(\theta^+ + \theta^-)}{2} \, dx, \quad \forall (v, \theta) \in {}^1Z
\end{aligned} \right. \tag{19}$$

Hence

$${}^1H := \{U = (u, \theta, v) \in {}^1V \times L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3)\} \tag{20}$$

is a Hilbert space if equipped with

$${}^1|U|^2 := {}^1(u, u)_1 + (v, v)_2 + (\theta, \theta)_4 \tag{21}$$

As for A_h , it is straightforward to check that operator 1A defined by

$$\left\{ \begin{aligned}
D({}^1A) &= \left\{ U = (u, \theta, v) \in {}^1H; \begin{cases} \text{(i)} & (v, \theta) \in {}^1Z \text{ and} \\ \text{(ii)} & \exists! (w, \tau) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega) \text{ such that} \end{cases} \right. \\
(w, v')_2 + {}^1(u, v')_1 - {}^1(v', \theta)_5 &= 0, \quad \forall v' \in {}^1V \\
(\tau, \theta')_4 + {}^1(\theta, \theta')_3 + {}^1(v, \theta')_5 &= 0, \quad \forall \theta' \in {}^1W \\
{}^1AU &= (v, \tau, w)
\end{aligned} \right. \tag{22}$$

is m -dissipative and to note that for all $\phi = (\phi^1, \phi^2, \phi^3)$ in 1H

$$\left\{ \begin{aligned}
{}^1\bar{U} &= ({}^1\bar{u}, {}^1\bar{\theta}, {}^1\bar{v}) \text{ s.t.} \\
{}^1\bar{U} - {}^1A'{}^1\bar{U} &= \phi \quad \Leftrightarrow \quad \begin{cases} {}^1\bar{u} = {}^1\bar{v} + \phi^1 \\ {}^1\bar{z} = ({}^1\bar{v}, {}^1\bar{\theta}) \in {}^1Z; \quad {}^1\Psi({}^1\bar{z}, z) = {}^1\mathcal{L}(z), \quad \forall z \in {}^1Z \end{cases}
\end{aligned} \right. \tag{23}$$

with, for any $z = (v, \theta)$ and $z' = (v', \theta')$ in 1Z

$$\begin{cases} {}^I\Psi(z, z') := {}^I\Phi(z, z') + (\theta, \theta')_2 + (v, v')_4 \\ {}^I\Phi(z, z') := {}^I_1(v, v')_1 + {}^I_2(\theta, \theta')_3 - {}^I(v', \theta)_5 + {}^I(v, \theta')_5 \\ {}^I\mathcal{L}(z') := {}^I(\phi^1, v')_1 + (\phi^2, v')_2 + (\phi^3, \theta')_4 \end{cases} \quad (24)$$

Thus a similar statement as that of [Theorem 3.1](#) is valid for the following equation, which will be shown to describe the asymptotic behavior of z_h :

$$\begin{cases} \frac{d^I U^r}{dt} - {}^I A^I U^r = {}^I F := \left({}^I u^e - \frac{d^I u^e}{dt}, -\frac{d^I \theta^e}{dt}, -\frac{d^I u^e}{dt} + \frac{f}{\rho} \right) \\ {}^I U^r(0) = {}^I U^{r,0} \end{cases} \quad (25)$$

with

$${}^I z^e = ({}^I u^e, {}^I \theta^e) \in C^{2,1}([0, T], {}^I Z); \quad {}^I \Psi({}^I z^e(t), z) = L(t)(z), \quad \forall z \in {}^I Z, \forall t \in [0, T] \quad (26)$$

We set

$${}^I U^e = ({}^I z^e, {}^I u^e), \quad {}^I U = {}^I U^e + {}^I U^r \quad (27)$$

4.2. Convergence

To prove the convergence of z_h toward ${}^I z = {}^I z^e + {}^I z^r$, with ${}^I z^r = ({}^I u^r, {}^I \theta^r)$, we use Trotter's theory of convergence of semi-groups of linear operators acting on variable spaces [\[4\]](#) because z_h^r and ${}^I z^r$ do not inhabit the same space.

First, to define a linear operator ${}^I P_h$ from ${}^I H$ to H_h suitable for comparing the elements of ${}^I H$ and H_h we classically [\[1,2,5\]](#) use the smoothing linear continuous operator R_ε from $H^1(\Omega \setminus S)$ to $H^1(\Omega)$ defined by

$$R_\varepsilon \varphi(x) = \begin{cases} \varphi^S(x) + \text{Min}\{|x_3|/\varepsilon, 1\} \varphi^A(x), & \forall x \text{ in } B_\varepsilon \\ \varphi(x), & \forall x \text{ in } \Omega_\varepsilon \end{cases} \quad (28)$$

where $2\varphi^S(x) = \varphi(\hat{x}, x_3) + \varphi(\hat{x}, -x_3)$ and $2\varphi^A(x) = \varphi(\hat{x}, x_3) - \varphi(\hat{x}, -x_3)$. Hence the operator ${}^I P_h$ defined by

$$\begin{cases} {}^I H \ni U = (u, \theta, v) \mapsto {}^I P_h U = ({}^I_1 P_h u, {}^I_2 P_h \theta, v) \in H_h \\ {}^I_1 = 1: \quad ({}^I_1 P_h u)_i = R_\varepsilon u_i, \quad i = 1, 2, 3 \\ {}^I_1 = 2: \quad ({}^I_2 P_h u)_\alpha = R_\varepsilon u_\alpha, \quad \alpha = 1, 2, ({}^I_2 P_h u)_3 = u_3 \\ {}^I_1 = 3: \quad {}^I_3 P_h u = u \\ {}^I_2 = 1: \quad {}^I_1 P_h \theta = R_\varepsilon \theta \\ {}^I_2 = 2: \quad {}^I_2 P_h \theta = \theta \end{cases} \quad (29)$$

satisfies.

Proposition 4.1.

i) There exists a strictly positive constant c independent of h such that

$$|{}^I P_h U|_h \leq c |U|, \quad \forall U \in {}^I H, \quad (30)$$

ii) when h goes to h^* , $|{}^I P_h U|_h$ converges toward $|U|$ for all U in ${}^I H$,

iii)

$$\begin{aligned} \left| e({}^I_1 P_h u) - \frac{[u]}{2\varepsilon} \otimes_s e_3 \right|_{L^2(B_\varepsilon; S^2)}^2 &\leq c\varepsilon^2 |e(u)|_{L^2(\Omega \setminus S; S^3)}^2, \quad \forall u \in H_{\Gamma_0^M}^1(\Omega \setminus S; \mathbb{R}^3) \\ \left| \nabla({}^I_1 P_h \theta) - \frac{[\theta]}{2\varepsilon} \otimes_s e_3 \right|_{L^2(B_\varepsilon)}^2 &\leq c\varepsilon^2 |\nabla \theta|_{L^2(\Omega \setminus S; \mathbb{R}^3)}^2, \quad \forall \theta \in H_{\Gamma_0^T}^1(\Omega \setminus S) \\ \left| R_\varepsilon \theta - \frac{\gamma_0(\theta^+) + \gamma_0(\theta^-)}{2} \right|_{L^2(B_\varepsilon)}^2 &\leq c\varepsilon^2 |\nabla \theta|_{L^2(B_\varepsilon; \mathbb{R}^3)}^2, \quad \forall \theta \in H_{\Gamma_0^T}^1(\Omega \setminus S) \end{aligned} \quad (31)$$

where $\gamma_0(\theta^\pm)$ denotes the trace on S of θ^\pm .

Next we state that U_h in H_h converges in the sense of Trotter toward U in ${}^I H$ if

$$\lim_{h \rightarrow h^*} |{}^I P_h U - U_h| = 0 \quad (32)$$

As in [5], such a suitable notion of convergence implies the following proposition.

Proposition 4.2. For all $U = (u, \theta, v)$ in ${}^I H$, if $U_h = (u_h, \theta_h, v_h)$ in H_h converges toward U in the sense of Trotter, then:

- i) $\mathbb{1}_{\Omega_\varepsilon}(e(u_h), \nabla \theta_h)$ converges strongly in $L^2(\Omega \setminus S; S^3 \times \mathbb{R}^3)$ toward $(e(u), \nabla \theta)$ and, for all positive $\eta < \varepsilon_0$, the sequence (u_h, θ_h) converges strongly in $H^1(\Omega_\eta; \mathbb{R}^3 \times \mathbb{R})$ toward (u, θ) ,
- ii) the traces on S_ε^\pm of (u_h, θ_h) identified with elements of $L^2(S; \mathbb{R}^3 \times \mathbb{R})$, converge strongly in $L^2(S; \mathbb{R}^3 \times \mathbb{R})$ toward the traces on S of (u^\pm, θ^\pm) ,
- iii) $\int_{-\varepsilon}^\varepsilon (e(u_h), \nabla \theta_h) dx_3$ converges strongly in $L^2(S; S^3 \times \mathbb{R}^3)$ toward $([u] \otimes_s e_3, [\theta] e_3)$ if $(\bar{\mu}, \bar{\kappa}) \in (0, +\infty]^2$,
- iv) (u_h, θ_h) converges strongly in $L^2(\Omega; \mathbb{R}^3 \times \mathbb{R})$ toward (u, θ) ,
- v) v_h converges strongly in $L^2(\Omega; \mathbb{R}^3)$ toward v .

Lastly we conclude by using the classical Trotter's theory of convergence of linear semi-groups [4], where it suffices to make hypothesis (H_4) on the initial state and loading and to establish the following convergence result involving stationary problems:

Proposition 4.3. Under the additional assumption

$$(H_3) : \quad \text{supp}(g^M) \cap \bar{B}_{\varepsilon_0} = \text{supp}(g^T) \cap \bar{B}_{\varepsilon_0} = \emptyset, \quad \forall t \in [0, T] \quad (33)$$

and if

$$\begin{aligned} \min\{\mathcal{H}^2(\Gamma_0^{M+}), \mathcal{H}^2(\Gamma_0^{M-})\} = 0, \quad \text{say } \mathcal{H}^2(\Gamma_0^{M-}) = 0, \quad \text{then } \text{supp}(g^M) \cap (\partial\Omega_{\varepsilon_0}^-) = \emptyset \\ \min\{\mathcal{H}^2(\Gamma_0^{T+}), \mathcal{H}^2(\Gamma_0^{T-})\} = 0, \quad \text{say } \mathcal{H}^2(\Gamma_0^{T-}) = 0, \quad \text{then } \text{supp}(g^T) \cap (\partial\Omega_{\varepsilon_0}^-) = \emptyset \end{aligned}$$

we have

$$\left\{ \begin{array}{l} \text{i) } \lim_{h \rightarrow h^*} |{}^I P_h (I - {}^I A)^{-1} \phi - (I - A_h)^{-1} P_h \phi|_h = 0, \quad \forall \phi \in {}^I H \\ \text{ii) } \lim_{h \rightarrow h^*} |{}^I P_h {}^I U^e(t) - U_h^e(t)|_h = 0 \quad \text{uniformly on } [0, T] \\ \text{iii) } \lim_{h \rightarrow h^*} \int_0^T |{}^I P_h {}^I F(t) - F_h(t)|_h dt = 0 \end{array} \right. \quad (34)$$

Proof. Points ii) and iii) are obtained by an obvious variant, taking into account the increments of temperature $\theta_h^e, {}^I \theta$, of the proof given in [2]. By taking advantage of Proposition 3.1 and (23) it remains to establish the convergence of $\bar{z}_h = (\bar{v}_h, \bar{\theta}_h)$ solution to (13) with $\phi_h = {}^I P_h \phi$ toward ${}^I \bar{z} = ({}^I \bar{v}, {}^I \bar{\theta})$ solution to (23). By choosing $(v, \theta) = (\bar{v}_h, \bar{\theta}_h)$ in (13) we deduce that $\sum_{i=1}^2 (\bar{v}_h, \bar{v}_h)_{h,i} + (\bar{\theta}_h, \bar{\theta}_h)_{h,i+2}$ remains bounded and consequently that there exists $z^* = (v^*, \theta^*)$ in ${}^I Z$ and a non relabeled subsequence such that

$$\left\{ \begin{array}{l} \text{i) } \bar{z}_h \text{ converges strongly in } L^2(\Omega; \mathbb{R}^3 \times \mathbb{R}) \text{ toward } z^* \\ \text{ii) } \mathbb{1}_{\Omega_\varepsilon}(e(\bar{v}_h), \nabla \bar{\theta}_h) \text{ weakly converges in } L^2(\Omega \setminus S; S^3 \times \mathbb{R}^3) \text{ toward } (e(v^*), \nabla \theta^*) \\ \text{iii) } \int_{-\varepsilon}^\varepsilon (e(\bar{v}_h), \nabla \bar{\theta}_h) dx_3 \text{ converges strongly in } L^2(S; S^3 \times \mathbb{R}^3) \text{ toward } ([v^*] \otimes_s e_3, [\theta^*] e_3) \\ \text{iv) } \text{the traces on } S_\varepsilon^\pm \text{ of } \bar{z}_h, \text{ identified with elements of } L^2(S; \mathbb{R}^3 \times \mathbb{R}), \text{ converge strongly in } \\ L^2(S; S^3 \times \mathbb{R}^3) \text{ to the traces of } z^{*\pm} \end{array} \right. \quad (35)$$

Thus it is easy, taking into account Proposition 4.1-iii) to go to the limit on the various terms of $\Psi_h(\bar{z}_h, ({}^I P_h v, {}^I P_h \theta))$ for all (v, θ) in ${}^I Z$. The novelty, induced by the thermoelastic coupling, lies in terms like $\gamma \int_{B_\varepsilon} \text{tre}(\bar{v}_h) {}^I P_h \theta dx$ and $\gamma \int_{B_\varepsilon} \bar{\theta}_h \text{tre}({}^I P_h v) dx$. The convergence of the first term toward $\gamma^* \int_S [v^*]_3 \frac{(\theta^+ + \theta^-)}{2} d\hat{x}$ stems from Proposition 4.1-iii) and

(35)-iii) while the convergence of the second one toward $\gamma^* \int_S \frac{(\theta^{*+} + \theta^{*-})}{2} [v]_3 d\hat{x}$ stems from an integration by part, [Proposition 4.1-iii\)](#) and (35)-iii), iv). Hence one deduces that $z^* = {}^I\bar{z}$, so that the whole sequence \bar{z}_h converges toward ${}^I\bar{z}$. Then it is straightforward to complete the proof by using the definition of ${}^I P_h$, [Proposition 4.1](#) and (35). \square

Thus we deduce the convergence, uniformly on $[0, T]$, in the sense of Trotter of the solution to (14) toward that to (25) with ${}^I U^{r,0} = {}^I U^0 - {}^I U^e(0)$ and the additional condition of convergence and compatibility between the initial state and loading:

$$(H_4): \quad {}^I U^0 \in {}^I U^e(0) + D({}^I A); \quad U_h^0 \in U_h^e + D(A_h) \quad \text{and} \quad \lim_{h \rightarrow h^*} |{}^I P_h {}^I U^0 - U_h^0|_h = 0$$

This can be rephrased in a more explicit way with respect to (\mathcal{P}_h) as

Theorem 4.1. *The solution to*

$$\frac{dU_h}{dt} + A_h(U_h - U_h^e) = (u_h^e, 0, f_h), \quad U_h(0) = U_h^0 \quad (36)$$

converges toward

$$\frac{d{}^I U}{dt} + {}^I A({}^I U - {}^I U^e) = \left({}^I u^e, 0, \frac{f}{\rho} \right), \quad {}^I U(0) = {}^I U^0 \quad (37)$$

in the sense

$$\lim_{h \rightarrow h^*} |{}^I P_h U(t) - {}^I U(t)|_h = 0, \quad \lim_{h \rightarrow h^*} |U_h(t)|_h = |{}^I U(t)|, \quad \text{uniformly on } [0, T] \quad (38)$$

5. Concluding remarks

[Theorem 4.1](#) implies that ${}^I z = ({}^I u, {}^I \theta)$ satisfies

$$\begin{aligned} \left(\frac{d^2 {}^I u}{dt^2}, u \right)_2 + {}^I ({}^I u, u)_1 - {}^I (u, {}^I \theta)_5 &= \int_{\Omega} f \cdot u \, dx + \int_{\Gamma_1^M} g^M \cdot u \, d\mathcal{H}_2, \quad \forall u \in {}^I V \\ \left(\frac{d{}^I \theta}{dt}, \theta \right)_4 + {}^I ({}^I \theta, \theta)_3 + \left(\frac{d{}^I u}{dt}, \theta \right)_5 &= \int_{\Gamma_1^T} g^T \cdot \theta \, d\mathcal{H}_2, \quad \forall \theta \in {}^I W \end{aligned} \quad (39)$$

Thus the limit behavior deals with the transient response to the initial loading (f, g^M, g^T) of the assembly of two linearly thermoelastic bodies occupying Ω^\pm as reference configuration and linked along S by a thermomechanical constraint that strongly depends on the relative magnitude of the parameters $(\varepsilon, \lambda, \mu, \kappa, \gamma)$. If we denote by ${}^I \sigma$ and ${}^I q$ the stress vector on S and the heat flux across S and by w_T and w_N the tangential and normal components of any vector w ($w_N = w \cdot e_3$, $w_T = w - w_N e_3$), the constitutive equations of the constraint read as:

$$\begin{aligned} I = (1, 1): \quad {}^I \sigma_T &= \bar{\mu} [{}^I u]_T, \quad {}^I \sigma_N = (\bar{\lambda} + 2\bar{\mu}) [{}^I u]_N - \gamma^* \frac{({}^I \theta^+ + {}^I \theta^-)}{2}, \quad {}^I q_N = \bar{\kappa} [{}^I \theta] + \frac{\gamma^*}{2} \left[\frac{d{}^I u}{dt} \right]_N \\ I = (2, 1): \quad [{}^I u]_N &= 0, \quad {}^I \sigma_T = \bar{\mu} [{}^I u]_T, \quad {}^I q_N = \bar{\kappa} [{}^I \theta] \\ I = (3, 1): \quad [{}^I u] &= 0, \quad {}^I q_N = \bar{\kappa} [{}^I \theta] \\ I = (1, 2): \quad {}^I \sigma_T &= \bar{\mu} [{}^I u]_T, \quad {}^I \sigma_N = (\bar{\lambda} + 2\bar{\mu}) [{}^I u]_N - \gamma^* {}^I \theta, \quad [{}^I \theta] = 0, \quad {}^I q_N = \frac{\gamma^*}{2} \left[\frac{d{}^I u}{dt} \right]_N \\ I = (2, 2): \quad [{}^I u]_N &= 0, \quad {}^I \sigma_T = \bar{\mu} [{}^I u]_T, \quad [{}^I \theta] = 0, \quad {}^I q_N = 0 \\ I = (3, 2): \quad [{}^I u] &= 0, \quad [{}^I \theta] = 0, \quad {}^I q_N = 0 \end{aligned}$$

When $I = (1, 1)$, a term of thermal nature involving the average temperature has to be added to the classical expression of the normal stress in pure elasticity, while it appears a thermal surface source term proportional to the jumps of temperature and also of normal velocity. When $I = (1, 2)$, there are no jump of temperature, the additional term in the normal stress is then proportional to the surface temperature and it appears a thermal surface source proportional to the jump of normal velocity. Note that in these sole cases where thermoelastic coupling is still present in the constraint, Lamé coefficients λ and μ should be of order ε and thus $\tilde{\alpha}$ of order $1/\varepsilon$ in a way that $\gamma^* = \lim(3\lambda + 2\mu)\tilde{\alpha}$ is finite and positive.

Lastly, as usual, our proposal of a simplified but accurate enough model for the behavior of the real structure is the one obtained in the case $I = (1, 1)$ by replacing $\bar{\lambda}$, $\bar{\mu}$, $\bar{\kappa}$ and γ^* by the real values $\lambda/2\varepsilon$, $\mu/2\varepsilon$, $\tilde{\kappa}/2\varepsilon$ and $(3\lambda + 2\mu)\tilde{\alpha}$.

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