



HAL
open science

Approximation of stochastic processes by non-expansive flows and coming down from infinity

Vincent Bansaye

► **To cite this version:**

Vincent Bansaye. Approximation of stochastic processes by non-expansive flows and coming down from infinity. 2016. hal-01232373v2

HAL Id: hal-01232373

<https://hal.science/hal-01232373v2>

Preprint submitted on 24 May 2016 (v2), last revised 5 Jan 2017 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Approximation of stochastic processes by non-expansive flows and coming down from infinity

Vincent Bansaye*

May 24, 2016

Abstract

We approximate stochastic processes in finite dimension by dynamical systems. We provide trajectorial estimates which are uniform with respect to the initial condition for a well chosen distance. This relies on some non-expansivity property of the flow, which allows to deal with non-Lipschitz vector fields. We use the stochastic calculus and follow the martingale technics initiated in [5] to control the fluctuations. Our main applications deal with the short time behavior of stochastic processes starting from large initial values. We state general properties on the coming down from infinity of one-dimensional SDEs, with a focus on stochastically monotone processes. In particular, we recover and complement known results on Λ -coalescent and birth and death processes. Moreover, using Poincaré's compactification technics for dynamical systems close to infinity, we develop this approach in two dimensions for competitive stochastic models. We classify the coming down from infinity of Lotka-Volterra diffusions and provide uniform estimates for the scaling limits of competitive birth and death processes.

Key words: Approximation of stochastic processes, non-expansivity, dynamical system, coming down from infinity, martingales, scaling limits

Contents

1	Introduction	2
2	Random perturbation of dynamical systems	5
2.1	Trajectorial control for perturbed non-expansive dynamical systems	7
2.2	Non-expansive dynamical systems and perturbation by martingales	8
3	Uniform estimates for Stochastic Differential Equations	10
3.1	Main result	11
3.2	Adjunction of non-expansive domains	14

*CMAP, Ecole Polytechnique, CNRS, route de Saclay, 91128 Palaiseau Cedex-France; E-mail: vincent.bansaye@polytechnique.edu

4	Coming down from infinity for one-dimensional Stochastic Differential Equations	18
4.1	Weak convergence and coming down from infinity	20
4.2	Examples and applications	24
4.2.1	Λ -coalescent [26, 5]	24
4.2.2	Birth and death processes [29, 4]	26
4.2.3	Transmission Control Protocol	29
4.2.4	Logistic diffusions [9] and perspectives	31
5	Uniform estimates for two-dimensional competitive Lotka-Volterra processes	32
5.1	Uniform short time estimates for competitive Feller diffusions	33
5.2	Uniform scaling limits of competitive birth and death processes	35
5.3	Non-expansivity of the flow and Poincaré compactification	36
5.3.1	Non-expansivity in cones	37
5.3.2	Poincaré's compactification and coming down from infinity	39
5.3.3	Adjunction of open convex cones	43
5.4	Proofs of Theorem 5.1 and Corollary 5.2 and Theorem 5.3	45
6	Appendix	52

1 Introduction

The approximation of stochastic processes by dynamical systems has been largely developed, with a particular focus on random perturbation of dynamical systems (see e.g. [28, 17]) and the fluid and scaling limits of random models (see e.g. [16, 19, 12]). In this paper, we are interested in stochastic processes $(X_t : t \geq 0)$ taking values on a subset E of \mathbb{R}^d , which can be written as

$$X_t = X_0 + \int_0^t \psi(X_s) ds + R_t,$$

where R is a semimartingale. We aim at proving that X remains close to the dynamical system whose flow $\phi(x_0, t) = x_t$ is given by

$$x_t = x_0 + \int_0^t \psi(x_s) ds.$$

The point here is to estimate the probability of this event uniformly with respect to the initial condition $x_0 \in D$, when the drift term ψ may be non-Lipschitz on D . Our main motivation for such estimates is the description of the coming down from infinity, which amounts to let the initial condition x_0 go to infinity, and the uniform scaling limits of stochastic processes describing population models on unbounded domains.

Our approach relies on some contraction property of the flow, which provides stability on the dynamics. This notion is used in particular in control theory. More precisely, we say that the vector field ψ is non-expansive on a domain D when it prevents two trajectories from moving away for the euclidean norm on a subset D of \mathbb{R}^d . This amounts to

$$\forall x, y \in D, \quad (\psi(x) - \psi(y)) \cdot (x - y) \leq 0,$$

where \cdot is the usual scalar product on \mathbb{R}^d . Actually, the distance between two solutions may increase provided that this increase is not too fast. This is required for the applications considered here and we are working with (L, α) non-expansive vector fields, as defined below.

Definition 1.1. *The vector field $\psi : D \rightarrow \mathbb{R}^d$ is (L, α) non-expansive on $D \subset \mathbb{R}^d$ if for any $x, y \in D$,*

$$(\psi(x) - \psi(y)) \cdot (x - y) \leq L \|x - y\|_2^2 + \alpha \|x - y\|_2.$$

The non-expansivity property ensures that the drift term can not make the distance between the stochastic process X and the dynamical system x explode because of small fluctuations due to the perturbation R . To control the size of these fluctuations, we use classical martingale techniques in Section 2: let us refer to [16, 12] in the context of scaling limits and to [5] for a pioneering work on the speed of coming down from infinity of Λ -coalescents. In this latter, the short time behavior of the log of the number of blocks is captured and the non-expansivity argument for the flow is replaced by a technical result relying on the monotonicity of suitable functions in dimension 1 (Lemma 10 therein).

These results are developed and specified when X satisfies a Stochastic Differential Equation (SDE), in Section 3, which allows a diffusion component and random jumps given by a Poisson point measure. This covers the range of our applications. We estimate the probability that the stochastic process remains close to the dynamical system as soon as this latter is in a domain D when (L, α) -non-expansivity hold for a transformation of the process. These estimates hold for any $x_0 \in D$ and a well chosen distance d , which is bound to capture the fluctuations of X around the flow ϕ . Informally we obtain that

$$\mathbb{P}_{x_0} \left(\sup_{t \leq T \wedge T_D(x_0)} d(X_t, \phi(x_0, t)) \geq \varepsilon \right) \leq C_T \int_0^T \bar{V}_{d, \varepsilon}(x_0, t) dt, \quad (1)$$

where $T_D(x_0)$ corresponds to the exit time of the domain D for the flow ϕ started at x_0 . The distance d is of the form

$$d(x, y) = \|F(x) - F(y)\|_2,$$

where F is of class \mathcal{C}^2 , so that we can use the stochastic calculus. The perturbation needs to be controlled in a tube for this distance d around the trajectory of the dynamical system and

$$\bar{V}_{d, \varepsilon}(x_0, t) = \sup_{\substack{x \in E \\ d(x, \phi(x_0, t)) \leq \varepsilon}} \left\{ \varepsilon^{-2} \|V_F(x)\|_1 + \varepsilon^{-1} \|\tilde{b}_F(x)\|_1 \right\},$$

where V_F will be given by the quadratic variation of $F(X)$ and \tilde{b}_F will be an additional approximation term arising from Itô formula applied to $F(X)$.

The relevant choice of F will be illustrated in several examples. It is both linked to the geometry of the flow since a (L, α) non-expansivity property for the flow is required and to the size of the fluctuations induced by R which shall be controlled. The choice of F may thus be subtle, see in particular the role of the fluctuations for the examples of Section 4.2 and the adjunction procedure involved in the last section for non-expansivity.

The estimate (1) becomes uniform with respect to $x_0 \in D$ as soon as $\bar{V}_{d, \varepsilon}(\cdot, \cdot)$ can be bounded by an integrable function of the time. It allows then to characterize the coming

down from infinity for stochastic differential equations in \mathbb{R}^d . Roughly speaking, we consider an unbounded domain D and let T go to 0 to derive from (1) that for any $\varepsilon > 0$,

$$\lim_{T \rightarrow 0} \sup_{x_0 \in D} \mathbb{P}_{x_0} \left(\sup_{t \leq T} d(X_t, \phi(x_0, t)) \geq \varepsilon \right) = 0.$$

Letting then x_0 go to infinity enables to describe the coming down from infinity of processes in several ways. First, the control of the fluctuations of the process X for large initial values by a dynamical system gives a way to control its fluctuations and prove the tightness of \mathbb{P}_{x_0} for $x_0 \in D$. Moreover we can link the coming down from infinity of the process X to the coming down from infinity of the flow ϕ , in the vein of [5, 23, 4], which focus respectively on Λ coalescence, Ξ coalescent and birth and death processes.

In dimension 1, we use some monotonicity properties to identify the limiting values of \mathbb{P}_{x_0} as $x_0 \rightarrow \infty$ and to determine when the process comes down from infinity and how it comes down from infinity (Section 4). In particular, we recover the speed of coming down from infinity of Λ -coalescent [5] with $F = \log$ and in that case V_F is bounded. We also recover some results of [4] for birth and death processes and we can provide finer estimates for regularly varying death rates. Here F is polynomial and V_F is unbounded so this latter has to be controlled finely along the trajectory of the dynamical system. Finally, we consider the example of transmission control protocol which is non-stochastically monotone and $F(x) = \log(1 + \log(1 + x))$ is required to control its fluctuations for large values.

In higher dimension, the coming down from infinity of a dynamical system is a more delicate problem in general. Poincaré has initiated a theory to study dynamical systems close to infinity, which is particularly powerful for polynomial vector fields (see e.g. Chapter 5 in [15]). We develop this approach for competitive Lotka-Volterra models in dimension 2 in Section 5. We classify the ways the dynamical system can come down from infinity and describe the counterpart for the stochastic process, which differs when the dynamical system is getting close from the boundary of $(0, \infty)^2$.

The uniform estimates (1) can also be used to prove scaling limits of stochastic processes X^K to dynamical systems, which are uniform with respect to the initial condition, without Lipschitz property of the vector field ψ . It involves a suitable distance d as introduced above to capture the fluctuations of the process :

$$\lim_{K \rightarrow \infty} \sup_{x_0 \in D} \mathbb{P}_{x_0} \left(\sup_{t \leq T} d(X_t^K, \phi(x_0, t)) \geq \varepsilon \right) = 0,$$

for some fixed $T, \varepsilon > 0$. It is illustrated in this paper by the convergence of birth and death processes with competition to Lotka-Volterra competitive dynamical system in Section 5.

Let us end up with other motivations for this work, some of which being linked to works in progress.

First, our original motivation for studying the coming down from infinity is the description of the time for extinction for competitive models in varying environment. Roughly speaking, competitive periods make the size of the population quickly decrease, which can be captured by the coming down from infinity. Let us also note that the approach developed here could be extended to the varying environment framework by comparing the stochastic process to a non-autonome dynamical system.

Second, the coming down from infinity is linked to the uniqueness of the quasistationary distribution, see [29] for birth and death processes and [9] for some diffusions. Recently, the coming down from infinity has appeared as a key assumption for the geometric convergence of the conditioned process to the quasistationary distribution, uniformly with respect to the initial distribution. We refer to [10] for details, see in particular Assumption (A1) therein.

Notation. In the whole paper \cdot stands for the canonical scalar product on \mathbb{R}^d , $\|\cdot\|_2$ the associated euclidean norm and $\|\cdot\|_1$ the L^1 norm. We write $x = (x^{(i)} : i = 1, \dots, d) \in \mathbb{R}^d$ a row vector of real numbers. The product xy for $x, y \in \mathbb{R}^d$ is the vector $z \in \mathbb{R}^d$ such that $z_i = x_i y_i$. We denote by $\bar{B}(x, \varepsilon) = \{y \in \mathbb{R}^d : \|y - x\|_2 \leq \varepsilon\}$ the euclidean closed ball centered in x with radius ε . More generally, we note $\bar{B}_d(x, \varepsilon) = \{y \in O : d(x, y) \leq \varepsilon\}$ the closed ball centered in $x \in O$ with radius ε associated with the application $d : O \times O \rightarrow \mathbb{R}^+$. When χ is differentiable on an open set of \mathbb{R}^d and takes values in \mathbb{R}^d , we denote by J_χ its Jacobian and

$$(J_\chi(x))_{i,j} = \frac{\partial}{\partial x_j} \chi^{(i)}(x) \quad (i, j = 1, \dots, d).$$

We write F^{-1} the reciprocal function of a bijection F and A^{-1} the inverse of a non zero real number or invertible matrix A . If A is a matrix, its transpose is denoted by A^* .

By convention, we assume that $\sup \emptyset = 0$, $\inf \emptyset = \infty$ and if $x, y \in \mathbb{R} \cup \{\infty\}$, we write $x \wedge y$ for the smaller element of $\{x, y\}$ and $\infty \wedge x = x$.

We note $d(x) \sim_{x \rightarrow a} g(x)$ when $d(x)/g(x) \rightarrow 1$ as $x \rightarrow a$.

We also use the notation $\int_a^{\cdot} f(x) dx < \infty$ (resp. $= \infty$) for $a \in [0, \infty]$ when there exists $a_0 \in (0, \infty)$ such that $\int_a^{a_0} f(x) dx$ is well defined and finite (resp. infinite).

Finally, we denote by $\langle M \rangle$ the predictable quadratic variation of a continuous local martingale M and by $|A|$ the process giving the total variation of a process A and by $\Delta X_s = X_s - X_{s-}$ the jump at time s of a càdlàg process.

Outline of the paper. In the next Section, we provide general results for dynamical systems perturbed by semimartingales using the non-expansivity of the flow and martingale inequality. In Section 3, we derive approximations results for Markov process described by SDE. It relies on a transformation F of the process for which we apply the results of Section 2. An extension of the result by adjunction of non-expansive domains is provided and required for the applications of the last section. We then study the coming down from infinity for one dimensional SDEs in Section 4, with a focus on stochastically monotone processes. Finally we compare the coming down from infinity of two dimensional competitive Lotka-Volterra diffusions with the coming down from infinity of Lotka-Volterra dynamical systems and prove uniform approximations of these latter by birth and death processes.

2 Random perturbation of dynamical systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t \geq 0}$ a filtration of \mathcal{F} , which satisfies the usual conditions. We consider a \mathcal{F}_t -adapted càdlàg process X on $[0, \infty)$ which takes its values in a subset E of \mathbb{R}^d and satisfies for every $t \geq 0$,

$$X_t = X_0 + \int_0^t \psi(X_s) ds + R_t,$$

where $X_0 \in E$ a.s., ψ is a Borel measurable function from \mathbb{R}^d to \mathbb{R}^d locally bounded and $(R_t : t \geq 0)$ is a càdlàg \mathcal{F}_t -semimartingale. The process R is given by

$$R_t = A_t + M_t, \quad M_t = M_t^c + M_t^d,$$

with A_t a càdlàg \mathcal{F}_t -adapted process with a.s. bounded variations paths, M_t^c a continuous \mathcal{F}_t -local martingale, M_t^d a càdlàg \mathcal{F}_t -local martingale purely discontinuous and $R_0 = A_0 = M_0 = M_0^c = M_0^d = 0$.

Our aim is to compare the process X to the solution $x = \phi(x_0, \cdot)$ of the dynamical system associated with the vector field ψ and some initial condition x_0 :

$$x_t = x_0 + \int_0^t \psi(x_s) ds$$

For that purpose, we assume that ψ is locally Lipschitz on a (non-empty) open set E' of \mathbb{R}^d . Then the solution x of the equation above exists and is unique on a time interval $[0, T'(x_0))$, where $T'(x_0) \in (0, \infty]$. Moreover, for $t < T'(x_0)$, we define the maximal gap before t :

$$S_t := \sup_{s \leq t} \|X_s - x_s\|_2.$$

We also set

$$T_{D,\varepsilon}(x_0) = \sup\{t \in [0, T'(x_0)) : \forall s \leq t, x_s \in D \text{ and } \overline{B}(x_s, \varepsilon) \cap E \subset D\} \in [0, \infty] \quad (2)$$

the last time when x_t and its ε -neighborhood in E belong to a domain D . As mentioned in the introduction, the key property to control the distance between $(X_t : t \geq 0)$ and $(x_t : t \geq 0)$ before time $T_{D,\varepsilon}(x_0)$ is the (L, α) non-expansivity property of ψ on D , in the sense of Definition 1.1. When $\alpha = 0$, we simply say that ψ is L non-expansive on D . If additionally $L = 0$, we say that ψ is non-expansive on D . We first note that in dimension 1, the fact that ψ is non-expansive simply means that ψ is non-increasing. More generally, we observe that

$$\psi = A + \chi = A + f + g$$

is (L, α) non-expansive on D if A is a vector field whose euclidean norm is bounded by α on D and χ is L non-expansive on D . Moreover $\chi = f + g$ is L non-expansive on D if f is Lipschitz with constant L and g is non-expansive on D . Finally, when χ is differentiable on a convex open set O which contains D , χ is L non-expansive on D if for any $x \in O$,

$$\text{Sp}(J_\chi + J_\chi^*) \subset (-\infty, 2L],$$

where $\text{Sp}(J_\chi + J_\chi^*)$ is the spectrum of the symmetric linear operator (and hence diagonalisable) $J_\chi + J_\chi^*$, see table 1 in [2] for details and more general results and the last section for an application.

For convenience and use of Gronwall Lemma, we also introduce for $L, \alpha \geq 0$ and $\varepsilon > 0$,

$$T_\varepsilon^{L,\alpha} = \sup\{T \geq 0 : 4\alpha T \exp(2LT) \leq \varepsilon\} \in (0, \infty], \quad (3)$$

which is infinite if and only if $\alpha = 0$, i.e. as soon as the vector field ψ is L non-expansive.

2.1 Trajectorial control for perturbed non-expansive dynamical systems

The following lemma gives the trajectorial result which allows to control the gap between the stochastic process $(X_t : t \geq 0)$ and the dynamical system $(x_t : t \geq 0)$ by the size of the fluctuations of $(R_t : t \geq 0)$ and the gap between the initial positions. It relies on the (L, α) non-expansivity of the flow and the stochastic integral $\int_0^t (X_{s-} - x_s) \cdot dR_s$, which is the integral of the càglàd (thus predictable locally bounded) process $(X_{s-} - x_s : 0 \leq s < T'(x_0))$ with respect to the semimartingale $(R_t : t \geq 0)$, see [19] Chapter I, Theorem 4.31 for a classical reference.

We set for any $t < T'(x_0)$ and $\varepsilon > 0$,

$$\widetilde{R}_t^\varepsilon = \|X_0 - x_0\|_2^2 + \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \left[2 \int_0^t (X_{s-} - x_s) \cdot dR_s + \|[M]_t\|_1 \right],$$

where $[M] = [X]$ is the quadratic variation of the semimartingale X . More specifically,

$$\|[M]_t\|_1 = \|[X]_t\|_1 = \|M^c\|_1 + \sum_{s \leq t} \|\Delta X_s\|_2^2$$

and we refer to Chapter 1, Theorem 4.52 in [19] for a reference. Unless otherwise specified, the identities below hold *a.s.*

Lemma 2.1. *Assume that ψ is (L, α) non-expansive on some domain $D \subset E'$ and let $\varepsilon > 0$. Then for any $x_0 \in E'$ and $T < T_{D,\varepsilon}(x_0) \wedge T_\varepsilon^{L,\alpha}$, we have*

$$\{S_T \geq \varepsilon\} \subset \left\{ \sup_{t \leq T} \widetilde{R}_t^\varepsilon > \eta^2 \right\},$$

where $\eta = \varepsilon \exp(-LT)/\sqrt{2}$.

Proof. Let $x_0 \in E \cap D$. First, we consider the quadratic variation of $(X_t - x_t : 0 \leq t < T'(x_0))$. We have for $t < T'(x_0)$, (see Chapter 1, Definition 4.4.45 in [19] or use Itô formula),

$$[X - x]_t = [M]_t = (X_t - x_t)^2 - (X_0 - x_0)^2 - 2 \int_0^t (X_{s-} - x_s) d(X_s - x_s).$$

Summing the coordinates of $[M]$ and using the definitions of X and x , we get

$$\|X_t - x_t\|_2^2 = \|X_0 - x_0\|_2^2 + 2 \int_0^t (X_{s-} - x_s) \cdot (\psi(X_{s-}) - \psi(x_s)) ds + 2 \int_0^t (X_{s-} - x_s) \cdot dR_s + \|[M]_t\|_1.$$

Moreover for any $s < T_{D,\varepsilon}(x_0)$, $x_s \in D$ and $X_{s-} \in D$ on the event $\{S_{s-} \leq \varepsilon\}$. So using that ψ is (L, α) non-expansive on D ,

$$\mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot (\psi(X_{s-}) - \psi(x_s)) \leq \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \left(L \|X_{s-} - x_s\|_2^2 + \alpha \|X_{s-} - x_s\|_2 \right).$$

Then for any $t < T_{D,\varepsilon}(x_0)$,

$$\begin{aligned} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \|X_t - x_t\|_2^2 &\leq \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \left[2L \int_0^t \|X_s - x_s\|_2^2 ds + 2\alpha \int_0^t \|X_s - x_s\|_2 ds \right. \\ &\quad \left. + \|X_0 - x_0\|_2^2 + 2 \int_0^t (X_{s-} - x_s) \cdot dR_s + \|[M]_t\|_1 \right] \end{aligned}$$

and by definition of $\widetilde{R}^\varepsilon$,

$$\mathbf{1}_{\{S_{t-} \leq \varepsilon\}} S_t^2 \leq 2L \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} S_s^2 ds + 2\alpha t \varepsilon + \sup_{s \leq t} \widetilde{R}_s^\varepsilon.$$

By Gronwall lemma, we obtain for any $T < T_\varepsilon^{L,\alpha}$ and $t \leq T$,

$$\mathbf{1}_{\{S_{t-} \leq \varepsilon\}} S_t^2 \leq \left(2\alpha T \varepsilon + \sup_{s \leq T} \widetilde{R}_s^\varepsilon \right) e^{2LT}.$$

Moreover $2\alpha T e^{2LT} < \frac{\varepsilon}{2}$ and recalling that $\eta = \varepsilon / (\sqrt{2} \exp(LT))$, we have

$$(2\alpha T \varepsilon + \eta^2) e^{2LT} < \varepsilon^2.$$

Then

$$\left\{ \sup_{s \leq T} \widetilde{R}_s^\varepsilon \leq \eta^2 \right\} \subset \left\{ \sup_{t \leq T} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} S_t^2 < \varepsilon^2 \right\}. \quad (4)$$

Denoting

$$T_{exit} = \inf\{s < T_{D,\varepsilon}(x_0) \wedge T_\varepsilon^{L,\alpha} : S_s \geq \varepsilon\},$$

and recalling that S is càdlàg, we have $S_{T_{exit}-} \leq \varepsilon$ and $S_{T_{exit}} \geq \varepsilon$ on the event $\{T_{exit} \leq T\}$, so using (4) at time $t = T_{exit}$ ensures that

$$\{T_{exit} \leq T\} \subset \left\{ \sup_{s \leq T} \widetilde{R}_s^\varepsilon > \eta^2 \right\},$$

which ends up the proof. \square

2.2 Non-expansive dynamical systems and perturbation by martingales

We use now martingale maximal inequality to estimate the probability that the distance between the process $(X_t : t \geq 0)$ and the dynamical system $(x_t : t \geq 0)$ goes beyond some level $\varepsilon > 0$. Such arguments have been used in several contexts, see in particular [12] for scaling limits and [5] for the coming down from infinity of Λ -coalescent, which has inspired the results below.

Proposition 2.2. *Assume that ψ is (L, α) non-expansive on a domain $D \subset E'$ and let $\varepsilon > 0$. Then for any $x_0 \in E'$ and $T < T_{D,\varepsilon}(x_0) \wedge T_\varepsilon^{L,\alpha}$, we have for any $p \geq 1/2$ and $q \geq 0$,*

$$\begin{aligned} & \mathbb{P}(S_T \geq \varepsilon) \\ & \leq \mathbb{P}\left(\|X_0 - x_0\|_2 \geq \varepsilon \frac{e^{-LT}}{2\sqrt{2}}\right) + C_q \frac{e^{2qLT}}{\varepsilon^q} \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d\|A\|_s\right)^q\right] \\ & \quad + C_{p,d} \frac{e^{4pLT}}{\varepsilon^{2p}} \left[\mathbb{E}\left[\left(\int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d\|M^c\|_t\right)^p\right] + \mathbb{E}\left[\left(\sum_{t \leq T} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \|\Delta X_t\|_2^2\right)^p\right] \right], \end{aligned}$$

for some positive constants C_q (resp. $C_{p,d}$) which depend only on q (resp. p, d).

Proof. By definition of $\widetilde{R}^\varepsilon$ and recalling that $R_s = A_s + M_s$,

$$\left\{ \sup_{t \leq T} \widetilde{R}_t^\varepsilon \geq \eta^2 \right\} \subset \left\{ \|X_0 - x_0\|_2^2 \geq \frac{\eta^2}{4} \right\} \cup B_\eta,$$

where

$$\begin{aligned} B_\eta = & \left\{ \sup_{t \leq T} \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dA_s \geq \frac{\eta^2}{16} \right\} \cup \left\{ \sup_{t \leq T} \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dM_s \geq \frac{\eta^2}{16} \right\} \\ & \cup \left\{ \int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d\| \langle M^c \rangle_t \|_1 \geq \frac{\eta^2}{4} \right\} \cup \left\{ \sum_{t \leq T} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \|\Delta X_t\|_2^2 \geq \frac{\eta^2}{4} \right\}. \end{aligned}$$

We know from Lemma 2.1 that

$$\{S_T \geq \varepsilon\} \subset \left\{ \sup_{s \leq T} \widetilde{R}_s \geq \eta^2 \right\}$$

and using Markov inequality yields

$$\begin{aligned} \mathbb{P}(S_T \geq \varepsilon) & \leq \mathbb{P}\left(\|X_0 - x_0\|_2^2 \geq \frac{\eta^2}{4}\right) + \mathbb{P}(B_\eta) \\ & \leq \mathbb{P}\left(\|X_0 - x_0\|_2^2 \geq \frac{\eta^2}{4}\right) + \left(\frac{16}{\eta^2}\right)^q \mathbb{E}\left(\left|\sup_{t \leq T} \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dA_s\right|^q\right) \\ & \quad + \left(\frac{16}{\eta^2}\right)^{2p} \mathbb{E}\left(\left|\sup_{t \leq T} \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dM_s\right|^{2p}\right) \\ & \quad + \left(\frac{4}{\eta^2}\right)^p \mathbb{E}\left(\left(\int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d\| \langle M^c \rangle_t \|_1\right)^p\right) + \left(\frac{4}{\eta^2}\right)^p \mathbb{E}\left(\left[\sum_{t \leq T} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \|\Delta X_t\|_2^2\right]^p\right). \end{aligned} \quad (5)$$

First using that $|f_s \cdot dg_s| \leq \|f_s\|_2 \|g_s\|_1$ since $|f_s^{(i)}| \leq \|f_s\|_2$, we have for $t \leq T$,

$$\int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dA_s \leq \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \|X_{s-} - x_s\|_2 dA_s^1 \leq \varepsilon \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} dA_s^1, \quad (6)$$

where $A_s^1 := \| |A_s| \|_1$ is the sum of the coordinates of the total variations of the process A . Second, Burkholder Davis Gundy inequality (see [14], 93, chap. VII, p. 287) for the local martingale

$$N_t = \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dM_s$$

ensures that there exists $C_p > 0$ such that

$$\mathbb{E}\left(\sup_{t \leq T} |N_t|^{2p}\right) \leq C_p \mathbb{E}([N]_T^p).$$

Writing the coordinates of X, M and x respectively $(X^{(i)} : i = 1, \dots, d)$, $(M^{(i)} : i = 1, \dots, d)$ and $(x^{(i)} : i = 1, \dots, d)$ and adding that

$$[N_T] = \int_0^T \sum_{i,j=1}^d \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-}^{(i)} - x_s^{(i)})(X_{s-}^{(j)} - x_s^{(j)}) d[M^{(i)}, M^{(j)}]_s \leq \varepsilon^2 \int_0^T \sum_{i,j=1}^d \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d[M^{(i)}, M^{(j)}]_s$$

and that $d[M^{(i)}, M^{(j)}]_s \leq d[M^{(i)}]_s + d[M^{(j)}]_s$, we obtain

$$\begin{aligned} & \mathbb{E} \left(\sup_{t \leq T} \left| \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dM_s \right|^{2p} \right) \\ & \leq C_{p,d} \varepsilon^{2p} \mathbb{E} \left(\left(\int_0^T \sum_{i=1}^d \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d[M_t^{(i)}] \right)^p \right) \\ & \leq C'_{p,d} \varepsilon^{2p} \left[\mathbb{E} \left(\left(\int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d \| \langle M_t^c \rangle \|_1 \right)^p \right) + \mathbb{E} \left(\left(\sum_{t \leq T} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \| \Delta X_t \|_2^2 \right)^p \right) \right], \quad (7) \end{aligned}$$

for some positive constants $C_{p,d}$ and $C'_{p,d}$, where we recall that $[M_t^{(i)}] = \langle M_t^{c,(i)} \rangle + \sum_{s \leq t} (\Delta X_s^{(i)})^2$. Plugging (6) and (7) in (5), we get

$$\begin{aligned} \mathbb{P}(S_T \geq \varepsilon) & \leq \mathbb{P} \left(\| X_0 - x_0 \|_2^2 \geq \frac{\eta^2}{4} \right) + \left(\frac{16\varepsilon}{\eta^2} \right)^q \mathbb{E} \left(\left(\int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} dA_s^1 \right)^q \right) \\ & \quad + \frac{C''_{p,d}}{\eta^{2p}} \left[\mathbb{E} \left(\left(\int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d \| \langle M^c \rangle \|_1 \right)^p \right) + \mathbb{E} \left(\left(\sum_{t \leq T} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \| \Delta X_t \|_2^2 \right)^p \right) \right] \end{aligned}$$

for some $C''_{p,d}$ positive. Recalling that $\eta = \varepsilon / (\sqrt{2} \exp(LT))$ ends up the proof. \square

3 Uniform estimates for Stochastic Differential Equations

In this section, we assume that $X = (X^{(i)} : i = 1, \dots, d)$ is a càdlàg Markov process which takes values in $E \subset \mathbb{R}^d$ and is the unique strong solution of the following SDE on $[0, \infty)$:

$$X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z) N(ds, dz) + \int_0^t \int_{\mathcal{X}} G(X_{s-}, z) \tilde{N}(ds, dz),$$

for any $x_0 \in E$ a.s., where $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ is a measurable space,

- $B = (B^{(i)} : i = 1, \dots, d)$ is a d -dimensional Brownian motion;
- N is a Poisson Point Measure (PPM) on $\mathbb{R}^+ \times \mathcal{X}$ with intensity $dsq(dz)$, where q is a σ -finite measure on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$; and \tilde{N} is the compensated measure of N .
- N and B are independent;
- $b = (b^{(i)} : i = 1, \dots, d)$, $\sigma = (\sigma_j^{(i)} : i, j = 1, \dots, d)$, H and G are Borel measurable functions locally bounded, which take values respectively \mathbb{R}^d , \mathbb{R}^{2d} , \mathbb{R}^d and \mathbb{R}^d .

Moreover, we follow the classical convention (see chapter II in [18]) and we assume that $HG = 0$, G is bounded and for any $t \geq 0$, $x + H(x, z) \in E$ for any $x \in E, z \in \mathcal{X}$ and

$$\int_0^t \int_{\mathcal{X}} |H(X_{s-}, z)| N(ds, dz) < \infty \quad \text{a.s.}, \quad \mathbb{E} \left(\int_0^t \int_{\mathcal{X}} \|G(X_{s- \wedge \sigma_n}, z)\|_2^2 ds q(dz) \right) < \infty,$$

for some sequence of stopping time $\sigma_n \uparrow \infty$. We do not discuss here the conditions which ensure the strong existence and uniqueness of this SDE for any initial condition. This will be standard results for the examples considered in this paper and we refer to [11] for some general statement relevant in our context.

3.1 Main result

We need a transformation F to construct a suitable distance and evaluate the gap between the process X and the associated dynamical system on a domain D .

Assumption 3.1. (i) The domain D is an open subset of \mathbb{R}^d and the function F is defined on an open set O which contains $\overline{D \cup E}$.

(ii) $F \in \mathcal{C}^2(O, \mathbb{R}^d)$ and F is a bijection from D into $F(D)$ and its Jacobian J_F is invertible on D .

(iii) For any $x \in E$,

$$\int_{\mathcal{X}} |F(x + H(x, z)) - F(x)| q(dz) < \infty.$$

and the function $x \in E \rightarrow h_F(x) = \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz)$ can be extended to the domain O . This function h_F is locally bounded on O and locally Lipschitz on D .

(iv) The function $x \rightarrow b(x)$ is locally Lipschitz on D .

Under this assumption, F is a C^2 diffeomorphism from D into $F(D)$ and $F(D)$ is an open subset of \mathbb{R}^d . We require in (iii) that the big jumps of $F(X)$ can be compensated. This assumption could be relaxed by letting the big jumps which could not be compensated in an additional term with finite variations, using the term A_t of the semimartingale R_t in the previous section. But that won't be useful for the applications given here. Under Assumption 3.1, we can set $b_F = b + J_F^{-1} h_F$, which is well defined and locally Lipschitz on D . We note that for any $x \in E \cap D$,

$$b_F(x) = b(x) + J_F(x)^{-1} \left(\int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz) \right).$$

We introduce the flow ϕ_F associated to b_F and defined for $x_0 \in D$ as the unique solution of

$$\phi_F(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \phi_F(x_0, t) = b_F(\phi_F(x_0, t)),$$

for $t \in [0, T(x_0))$, where $T(x_0)$ is the positive real number which gives the maximal time interval on which the solution exists and belongs to D . We observe that when $H = 0$, then $b_F = b$ and $\phi_F = \phi$ do not depend on the transformation F .

We introduce now the vector field ψ_F defined by

$$\psi_F = (J_F b_F) \circ F^{-1} = (J_F b + h_F) \circ F^{-1}$$

on the open set $F(D)$. We also set for any $x \in E$,

$$\widetilde{b}_F(x) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 F}{\partial x_i \partial x_j}(x) \sum_{k=1}^d \sigma_k^{(i)}(x) \sigma_k^{(j)}(x) + \int_{\mathcal{X}} [F(x + G(x, z)) - F(x) - J_F(x)G(x, z)] q(dz). \quad (8)$$

Let us note that the generator of X is given $\mathcal{L}F = \psi_F \circ F + \widetilde{b}_F$. The term \widetilde{b}_F is not contributing significantly to the coming down from infinity in the examples we consider here and thus considered as an approximation term. Moreover we need to introduce

$$V_F(x) = \sum_{i,j,k=1}^d \frac{\partial F}{\partial x_i}(x) \frac{\partial F}{\partial x_j}(x) \sigma_k^{(i)}(x) \sigma_k^{(j)}(x) + \int_{\mathcal{X}} [F(x + H(x, z) + G(x, z)) - F(x)]^2 q(dz). \quad (9)$$

for $x \in E$, to quantify the fluctuations of the process due to the martingale parts. Finally we are using the following application defined on O (and thus on $D \cup E$) to compare the process X and the flow ϕ_F :

$$d_F(x, y) = \|F(x) - F(y)\|_2.$$

We observe that d is indeed (at least) a distance on D and in the examples it will be a distance on the whole set $D \cup E$. We recall notation (3) and the counterpart of (2) is now defined by

$$T_{D,\varepsilon,F}(x_0) = \sup\{t \in [0, T(x_0)) : \forall s \leq t, \phi_F(x_0, s) \in D \text{ and } B_{d_F}(\phi_F(x_0, s), \varepsilon) \cap E \subset D\}. \quad (10)$$

Theorem 3.2. *Under Assumption 3.1, we assume that ψ_F is (L, α) non-expansive on $F(D)$. Then for any $\varepsilon > 0$ and $x_0 \in E \cap D$ and $T < T_{D,\varepsilon,F}(x_0) \wedge T_\varepsilon^{L,\alpha}$, we have*

$$\mathbb{P}_{x_0} \left(\sup_{t \leq T} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right) \leq C_d e^{4LT} \int_0^T \overline{V}_{F,\varepsilon}(x_0, s) ds,$$

where C_d is a positive constant depending only on the dimension d and

$$\overline{V}_{F,\varepsilon}(x_0, s) = \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, s)) \leq \varepsilon}} \left\{ \varepsilon^{-2} \|V_F(x)\|_1 + \varepsilon^{-1} \|\widetilde{b}_F(x)\|_1 \right\}.$$

We refer to the two next sections for examples and applications, which involve different choices for F and (L, α) non-expansivity with potentially α or L equal to 0. The key assumption here concerns the non-expansivity of ψ_F for a suitable choice of F . Before the proof of Theorem 3.2, let us give a useful criterion for L non-expansivity in the diffusion case ($q = 0$ and X continuous), which will be useful in Section 5.

Example. We recall from the first Section (or table 1 in [2]) that when $F(D)$ is convex and ψ_F is differentiable on $F(D)$, ψ_F is L non-expansive on $F(D)$ iff $\text{Sp}(J_{\psi_F}(y) + J_{\psi_F}^*(y)) \subset (-\infty, 2L]$ for any $y \in F(D)$. In the case $q = 0$, choosing

$$F(x) = (f_i(x_i) : i = 1, \dots, d)$$

and setting $A(x) = J_{\psi_F}(F(x))$, we have for any $i, j = 1, \dots, d$ such that $i \neq j$

$$A_{ij}(x) = \frac{f_i'(x_i)}{f_j'(x_j)} \frac{\partial}{\partial x_j} b^{(i)}(x), \quad A_{ii}(x) = \frac{\partial}{\partial x_i} b^{(i)}(x) + \frac{f_i''(x_i)}{f_i'(x_i)} b^{(i)}(x). \quad (11)$$

Then ψ_F is L non-expansive on $F(D)$ iff the largest eigenvalue of $A(x) + A^*(x)$ is less than $2L$ for any $x \in D$.

Proof of Theorem 3.2. Under Assumption 3.1, we can assume that $F \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$. Indeed, we can consider φF where $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ is equal to 0 on the complementary set of O and to 1 on $\overline{D \cup E}$, since these two sets are disjoint closed sets, using e.g. the smooth Urysohn lemma. This allows to extend F from $E \cup D$ to \mathbb{R}^d in such a way that $F \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$.

Applying now Itô's formula to F (see Chapter 2, Theorem 5.1 in [18]), we have :

$$\begin{aligned} F(X_t) &= F(x_0) + \int_0^t J_F(X_s) b(X_s) ds + \int_0^t \int_E [F(X_{s-} + H(X_{s-}, z)) - F(X_{s-})] N(ds, dz) \\ &\quad + \int_0^t \sum_{i,j=1}^d \frac{\partial F}{\partial x_i}(X_s) \sigma_j^{(i)}(X_s) dB_s^{(j)} + \int_0^t \int_E [F(X_{s-} + G(X_{s-}, z)) - F(X_{s-})] \tilde{N}(ds, dz) \\ &\quad + \int_0^t \tilde{b}_F(X_s) ds \end{aligned}$$

Then the \mathcal{F}_t -semimartingale $Y_t = F(X_t)$ takes values in $F(E)$ and can be written as

$$Y_t = F(x_0) + \int_0^t \psi(Y_s) ds + A_t + M_t^c + M_t^d, \quad (12)$$

where ψ, A, M^c and M^d are defined as follows. First, we set $\psi(y) = \mathbf{1}_{\{y \in F(D)\}} \psi_F(y)$ for $y \in \mathbb{R}^d$, so writing $\hat{b}_F(x) = J_F(x)b(x) + h_F(x)$ for $x \in E$, we have $\psi(Y_s) = \mathbf{1}_{\{Y_s \in F(D)\}} \hat{b}_F(X_s)$. Moreover,

$$A_t = \int_0^t (\tilde{b}_F(X_s) + \mathbf{1}_{\{Y_s \notin F(D)\}} \hat{b}_F(X_s)) ds$$

is a continuous \mathcal{F}_t -adapted process with a.s. bounded variations paths and

$$M_t^c = \int_0^t \sum_{i,j=1}^d \frac{\partial F}{\partial x_i}(X_s) \sigma_j^{(i)}(X_s) dB_s^{(j)}$$

is a continuous \mathcal{F}_t -local martingale and writing $K = G + H$ and using Assumption 3.1 (iii),

$$M_t^d = \int_0^t \int_{\mathcal{X}} [F(X_{s-} + K(X_{s-}, z)) - F(X_{s-})] \tilde{N}(ds, dz)$$

is a càdlàg \mathcal{F}_t -local martingale purely discontinuous.

We observe that the dynamical system $y_t = F(\phi_F(x_0, t))$ satisfies for $t < T(x_0)$,

$$y_0 = F(x_0), \quad y'_t = J_F(\phi_F(x_0, t)) b_F(\phi_F(x_0, t)) = \psi_F(y_t) = \psi(y_t)$$

and it is associated with the vector field ψ . Let us also note that $\psi_F = \psi$ is locally Lipschitz on $F(D)$. Moreover, recalling the definition (2) and setting $T'(y_0) = T(x_0)$, the first time $T_{F(D), \varepsilon}(y_0)$ when $(y_t)_{t \geq 0}$ starting from y_0 is at distance ε from the boundary of $F(D)$ for the euclidean distance is larger than $T_{D, \varepsilon, F}(x_0)$ defined by (10) :

$$T_{F(D), \varepsilon}(y_0) = \sup\{t \in [0, T'(y_0)) : \forall s \leq t, y_s \in F(D) \text{ and } \overline{B}(y_s, \varepsilon) \cap F(E) \subset F(D)\} \geq T_{D, \varepsilon, F}(x_0).$$

Adding that ψ is (L, α) non-expansive on $F(D)$, we apply now Proposition 2.2 to Y on $F(D)$ with $p = q = 1$ and $Y_0 = y_0 = F(x_0)$. Then, for any $T < T_{D, \varepsilon, F}(x_0) \wedge T_\varepsilon^{L, \alpha}$, we have

$$\begin{aligned} \mathbb{P}(S_T \geq \varepsilon) \leq & C_d e^{4LT} \left[\varepsilon^{-1} \mathbb{E} \left(\int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \| |A|_s \|_1 \right) \right. \\ & \left. + \varepsilon^{-2} \mathbb{E} \left(\int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \| \langle M^c \rangle_s \|_1 \right) + \varepsilon^{-2} \mathbb{E} \left(\sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \| \Delta Y_s \|_2^2 \right) \right] \end{aligned} \quad (13)$$

for some constant C_d positive, since $X_0 = x_0$ a.s. ensures that the first probability on the right hand side in Proposition 2.2 is null. Using now

$$\langle M^c \rangle_t = \int_0^t \sum_{i,j,k=1}^d \frac{\partial F}{\partial x_i}(X_s) \frac{\partial F}{\partial x_j}(X_s) \sigma_k^{(i)}(X_s) \sigma_k^{(j)}(X_s) ds,$$

we get

$$\int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d \| \langle M^c \rangle_t \|_1 \leq \int_0^T \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, t)) \leq \varepsilon}} \left\{ \sum_{i,j,k,l=1}^d \frac{\partial F^{(l)}}{\partial x_i}(x) \frac{\partial F^{(l)}}{\partial x_j}(x) \sigma_k^{(i)}(x) \sigma_k^{(j)}(x) \right\} dt,$$

since here $S_t = \sup_{s \leq t} \| Y_s - y_s \|_2 = \sup_{s \leq t} d_F(X_s, \phi_F(x_0, s))$. Similarly,

$$\begin{aligned} \mathbb{E} \left(\sum_{t \leq T} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \| \Delta Y_t \|_2^2 \right) &= \mathbb{E} \left(\int_0^T \int_{\mathcal{X}} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \| F(X_{t-} + K(X_{t-}, z)) - F(X_{t-}) \|_2^2 dt q(dz) \right) \\ &\leq \int_0^T \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, t)) \leq \varepsilon}} \int_{\mathcal{X}} \| F(x + K(x, z)) - F(x) \|_2^2 q(dz) dt \end{aligned}$$

and combining the two last inequalities we get

$$\mathbb{E} \left(\int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d \| \langle M^c \rangle_t \|_1 \right) + \mathbb{E} \left(\sum_{t \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \| \Delta Y_t \|_2^2 \right) \leq \int_0^T \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, t)) \leq \varepsilon}} \| V_F(x) \|_1 dt. \quad (14)$$

Finally, on the event $\{S_{t-} \leq \varepsilon\}$, $Y_{t-} = F(X_{t-}) \in F(D)$ for any $t \leq T$ since $T < T_{D, \varepsilon, F}(x_0)$, so

$$\mathbb{E} \left(\int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d \| |A|_t \|_1 \right) \leq \int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \| \tilde{b}_F(X_{t-}) \|_1 dt \leq \int_0^T \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, t)) \leq \varepsilon}} \| \tilde{b}_F(x) \|_1 dt \quad (15)$$

and the conclusion comes by plugging the two last inequalities in (13). \square

3.2 Adjunction of non-expansive domains

We provide now an extension of the conditions required for Theorem 3.2, which allows in particular to study cases where a family of transformations is used to get non-expansivity of the vector field along the whole trajectory of the dynamical system. It will be useful to study two-dimensional competitive processes in Section 5. First, we decompose the domain D into a family of subdomains $(D_i : i = 1, \dots, N)$ and require the following set of assumptions.

Assumption 3.3. (i) The domains D and $(D_i : i = 1, \dots, N)$ are open subsets of \mathbb{R}^d and the functions \mathbb{R}^d valued functions F_i are defined from an open set O_i which contains $\overline{D_i}$ and

$$D \subset \cup_{i=1}^N D_i, \quad F_i \in \mathcal{C}^2(O_i, \mathbb{R}^d).$$

Moreover F_i is a bijection from D_i into $F(D_i)$ whose Jacobian is invertible on D_i .

(ii) There exist a distance d on $\cup_{i=1}^N D_i \cup E$ and $c_1, c_2 > 0$ such that for any $i \in \{1, \dots, N\}$, $x, y \in D_i$,

$$c_1 d(x, y) \leq \|F_i(x) - F_i(y)\|_2 \leq c_2 d(x, y).$$

(iii) For each $i \in \{1, \dots, N\}$, for any $x \in E \cap D_i$,

$$\int_{\mathcal{X}} |F_i(x + H(x, z)) - F_i(x)| q(dz) < \infty.$$

and the function $x \in E \cap D_i \rightarrow h_{F_i}(x) = \int_{\mathcal{X}} [F_i(x + H(x, z)) - F_i(x)] q(dz)$ can be extended to $\overline{D_i}$.

Moreover this extension is locally bounded on $\overline{D_i}$ and locally Lipschitz on D_i .

(iv) The function $x \rightarrow b(x)$ is locally Lipschitz on $\cup_{i=1}^N D_i$.

Second, we consider the flow associated to the vector field b_{F_i} , where b_{F_i} is locally Lipschitz on the domain D_i and defined as previously by $b_{F_i}(x) = b(x) + J_{F_i}(x)^{-1} h_{F_i}(x)$. But now the flow ϕ may go from one domain to another. To glue the estimates obtained in the previous part by adjunction of domains, we need to bound the number of times κ the flow may change of domain D_i . More precisely, we consider a flow $\phi(\cdot, \cdot)$ such that $\phi(x_0, 0) = x_0$ for $x_0 \in D$ and let $\varepsilon_0 \in (0, 1)$, $\kappa \geq 1$ and $(t_k(\cdot) : k \leq \kappa)$ be a sequence of elements of $[0, \infty]$ such that $0 = t_0(x_0) \leq t_1(x_0) \leq \dots \leq t_\kappa(x_0)$ for $x_0 \in D$, which meet the following assumption.

Assumption 3.4. For any $x_0 \in D$, the flow $\phi(x_0, \cdot)$ is continuous on $[0, t_\kappa(x_0))$ and for any $k \leq \kappa - 1$, there exists $n_k(x_0) \in \{1, \dots, N\}$ such that for any $t \in (t_k(x_0), t_{k+1}(x_0))$,

$$\overline{B}_d(\phi(x_0, t), \varepsilon_0) \subset D_{n_k(x_0)} \quad \text{and} \quad \frac{\partial}{\partial t} \phi(x_0, t) = b_{F_{n_k(x_0)}}(\phi(x_0, t)).$$

This flow ϕ will be used in the continuous case in Section 5.1 and then we recall that $b_F = b$ does not depend on the transformation F . In that case, Assumption 3.4 simply means that $\overline{B}_d(\phi(x_0, t), \varepsilon)$ is included in one of the subdomains D_i for ε small enough, while the flow ϕ is directly given by $\phi(x_0, 0) = x_0$, $\frac{\partial}{\partial t} \phi(x_0, t) = b(\phi(x_0, t))$ as expected.

Recalling notation $\psi_F = (J_F b_F) \circ F^{-1}$ and the expressions of \widetilde{b}_F and V_F given respectively in (8) and (9), the result can be stated as follows.

Theorem 3.5. Under Assumptions 3.3 and 3.4, we assume that for each $i \in \{1, \dots, N\}$, ψ_{F_i} is (L_i, α_i) non-expansive on $F_i(D_i)$ and let $T_0 \in (0, \infty)$.

Then for any $\varepsilon > 0$ small enough, for any $T < \min\{T_\varepsilon^{L_i, \alpha_i} : i = 1, \dots, N\} \wedge T_0$ and $x_0 \in E \cap D$,

$$\mathbb{P}_{x_0} \left(\sup_{t \leq T \wedge t_\kappa(x_0)} d(X_t, \phi(x_0, t)) \geq \varepsilon \right) \leq C \sum_{k=0}^{\kappa-1} \int_{t_k(x_0) \wedge T}^{t_{k+1}(x_0) \wedge T} \overline{V}_{d, \varepsilon}(F_{n_k(x_0)}, x_0, t) dt,$$

where C is a positive constant which depends on d , $c_1, c_2, (L_i : i = 1, \dots, N)$, κ and T_0 and

$$\overline{V}_{d, \varepsilon}(F, x_0, s) = \sup_{\substack{x \in E \\ d(x, \phi(x_0, s)) \leq \varepsilon}} \left\{ \varepsilon^{-2} \|V_F(x)\|_1 + \varepsilon^{-1} \|\widetilde{b}_F(x)\|_1 \right\}.$$

The proof relies also on Proposition 2.2 but it is a technically more involved than the proof of Theorem 3.2. We observe that T_0 can be taken equal to ∞ in the previous statement in the case where $L_i = 0$ for any $i \in \{1, \dots, \kappa\}$. We need now the following constants.

$$b_k(x_0, T) = 2\sqrt{2} \exp(L_{n_k(x_0)} T), \quad a_k(x_0, T) = b_k(x_0, T) \frac{c_2}{c_1}, \quad \varepsilon_k(x_0, T) = \frac{c_1 \varepsilon_0}{c_2 b_k(x_0, T)} = \frac{\varepsilon_0}{a_k(x_0, T)},$$

for $k = 0, \dots, \kappa - 1$ and observe that $a_k(x_0, T) \geq 1$ and need the following lemma.

Lemma 3.6. *Under the assumptions of Theorem 3.5, for any $x_0 \in E \cap D$, $k \in \{0, \dots, \kappa - 1\}$, $T < T_\varepsilon^{L_{n_k(x_0)}, \alpha_{n_k(x_0)}}$ and $\varepsilon \leq \varepsilon_k(x_0, T)$, we have*

$$\mathbb{P}_{x_0} \left(\sup_{t_k(x_0) \leq t \leq t_{k+1}(x_0) \wedge T} d(X_t, \phi(x_0, t)) \geq \varepsilon a_k(x_0, T) \right) \leq \mathbb{P}(d(X_{t_k(x_0)}, \phi(x_0, t_k(x_0))) \geq \varepsilon) + C_{d, c_1, L_{n_k(x_0)}} \int_{t_k(x_0) \wedge T}^{t_{k+1}(x_0) \wedge T} \bar{V}_{d, \varepsilon a_k(x_0, T)}(F_{n_k(x_0)}, x_0, s) ds,$$

where $C_{d, c_1, L_{n_k(x_0)}}$ is a positive constant which depends only on d and c_1 and $L_{n_k(x_0)}$.

Proof. Let us fix $k \in \{0, \dots, \kappa - 1\}$ and $x_0 \in E \cap D$ and $T < T_\varepsilon^{L_{n_k(x_0)}, \alpha_{n_k(x_0)}}$. We write $F = F_{n_k(x_0)}$ and $D = D_{n_k(x_0)}$ for simplicity. As at the beginning of the previous proof, we can assume that $F \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ and recall that F is bijection from D into $F(D)$. We note that $z_0 = \phi(t_k(x_0), x_0) \in D$ by Assumption 3.4 and the solution z of $z'_t = b_F(z_t)$ is well defined on a non-empty (maximal) time interval since b_F is locally Lipschitz on D using Assumption 3.3. By uniqueness in Cauchy Lipschitz theorem, $z_t = \phi(t_k(x_0) + t, x_0)$ for $t \in [t_k(x_0), t_{k+1}(x_0))$.

We write now $\tilde{X}_t = X_{t_k(x_0)+t}$ and the counterpart of (12) for $Y_t = F(X_{t_k(x_0)+t}) = F(\tilde{X}_t)$. We get :

$$Y_t = Y_0 + \int_0^t \psi(Y_s) ds + A_t + M_t^c + M_t^d, \quad (16)$$

for $t \geq 0$, where $\psi(y) = \mathbf{1}_{\{y \in F(D)\}} \psi_F(y)$,

$$M_t^c = \int_0^t \sum_{i,j=1}^d \frac{\partial F}{\partial x_i}(\tilde{X}_s) \sigma_j^{(i)}(\tilde{X}_s) dB_s^{(j)},$$

and we make here the following decomposition for A and M^d . Using Assumption 3.3 (iii) for the compensation of jumps when $\tilde{X}_{s-} \in D$ and writing again $\hat{b}_F(x) = J_F(x)b(x) + h_F(x)$, we set

$$A_t = \int_0^t \left(\bar{b}_F(\tilde{X}_s) - \mathbf{1}_{\{\tilde{X}_s \notin D, F(\tilde{X}_s) \in F(D)\}} \hat{b}_F(\tilde{X}_s) \right) ds + \int_0^t \int_{\mathcal{X}} \mathbf{1}_{\{\tilde{X}_{s-} \notin D\}} \left[F(\tilde{X}_{s-} + H(\tilde{X}_{s-}, z)) - F(\tilde{X}_{s-}) \right] N(ds, dz),$$

which is a process with a.s. finite variations paths; and

$$M_t^d = \int_0^t \int_{\mathcal{X}} \left[F(\tilde{X}_{s-} + G(\tilde{X}_{s-}, z)) - F(\tilde{X}_{s-}) \right] \tilde{N}(ds, dz) + \int_0^t \int_{\mathcal{X}} \mathbf{1}_{\{\tilde{X}_{s-} \in D\}} \left[F(\tilde{X}_{s-} + H(\tilde{X}_{s-}, z)) - F(\tilde{X}_{s-}) \right] \tilde{N}(ds, dz)$$

is a càdlàg \mathcal{F}_t -local martingale purely discontinuous.

Moreover by Assumptions 3.4 and 3.3 (ii), for any $t < t_{k+1}(x_0) - t_k(x_0)$, $x_t \in D$, $y_t = F(x_t) \in F(D)$ satisfies $y'_t = \psi(y_t)$ and for any $\varepsilon \in (0, c_1 \varepsilon_0]$,

$$\bar{B}(y_t, \varepsilon) \cap F(E) = F(\bar{B}_{d_F}(\phi(x_0, t_k(x_0) + t), \varepsilon)) \subset F(\bar{B}_d(\phi(x_0, t_k(x_0) + t), \varepsilon/c_1)) \subset F(D).$$

Adding that $\psi = \psi_F$ is $(\alpha_{n_k(x_0)}, L_{n_k(x_0)})$ non-expansive on $F(D)$, we can then apply Proposition 2.2 to the process Y for $p = q = 1$ and $E' = F(D)$ and get for any $\varepsilon \in (0, c_1 \varepsilon_0]$,

$$\begin{aligned} & \mathbb{P}_{x_0} \left(\sup_{t \leq T_1} \|Y_t - y_t\|_2 \geq \varepsilon \right) \\ & \leq \mathbb{P}(\|Y_0 - y_0\|_2 \geq \varepsilon/b_k(x_0, T_0)) + C\varepsilon^{-1} \mathbb{E} \left(\int_0^{T_1} \mathbf{1}_{\{S_s \leq \varepsilon\}} d \| |A|_s \|_1 \right) \\ & \quad + C\varepsilon^{-2} \left[\mathbb{E} \left(\int_0^{T_1} \mathbf{1}_{\{S_t \leq \varepsilon\}} d \| < M^c >_t \|_1 \right) + \mathbb{E} \left(\sum_{t \leq T_1} \mathbf{1}_{\{S_t \leq \varepsilon\}} \| \Delta Y_t \|_2^2 \right) \right] \end{aligned}$$

for any $T_1 < T_\varepsilon^{L_{n_k(x_0)}, \alpha_{n_k(x_0)}} \wedge (t_{k+1}(x_0) - t_k(x_0))$, where C is positive constant depending on $L_{n_k(x_0)}$ and d . Following (14) and (15), we obtain

$$\begin{aligned} & \mathbb{P}_{x_0} \left(\sup_{[t_k(x_0), t_{k+1}(x_0) \wedge T]} d_F(X_t, \phi(x_0, t)) \geq \varepsilon \right) \\ & \leq \mathbb{P}(d_F(X_{t_k(x_0)}, x_{t_k(x_0)}) \geq \varepsilon/b_k(x_0, T)) + C' \int_{t_k(x_0)}^{t_{k+1}(x_0) \wedge T} \bar{V}_{d_F, \varepsilon}(x_0, s) ds. \quad (17) \end{aligned}$$

for some constant C' depending also only of $L_{n_k(x_0)}$ and d . Using again Assumption 3.3 (ii) to replace d_F by d above, we have

$$\left\{ d(X_{t_k(x_0)}, x_{t_k(x_0)}) < \varepsilon/(c_2 b_k(x_0, T)) \right\} \subset \left\{ d_F(X_{t_k(x_0)}, x_{t_k(x_0)}) < \varepsilon/b_k(x_0, T) \right\}$$

and

$$\bar{V}_{d_F, \varepsilon}(x_0, s) \leq (c_1^{-1} + c_1^{-2}) \bar{V}_{d, \varepsilon/c_1}(F_{n_k(x_0)}, x_0, s).$$

and we obtain

$$\begin{aligned} & \mathbb{P}_{x_0} \left(\sup_{[t_k(x_0), t_{k+1}(x_0) \wedge T]} d(X_t, \phi(x_0, t)) \geq \varepsilon/c_1 \right) \\ & \leq \mathbb{P} \left(d(X_{t_k(x_0)}, x_{t_k(x_0)}) \geq \frac{\varepsilon}{c_2 b_k(x_0, T)} \right) + C'(c_1^{-1} + c_1^{-2}) \int_{t_k(x_0)}^{T \wedge t_{k+1}(x_0)} \bar{V}_{d, \varepsilon/c_1}(F, x_0, s) ds. \end{aligned}$$

Using finally the quasi-left continuity of X , this inequality can be extended to the closed interval $[t_k(x_0), T \wedge t_{k+1}(x_0)]$, which ends up the proof by replacing ε by $\varepsilon c_2 b_k(x_0, T)$. \square

Proof of Theorem 3.5. We write $T_m = T_0 \wedge \min \{ T_\varepsilon^{L_i, \alpha_i} : i = 1, \dots, N \} \in (0, \infty]$ and set

$$\underline{\varepsilon} = \inf \{ \varepsilon_k(x_0, T) : k = 1, \dots, N; T < T_0 \} \in (0, \infty).$$

Lemma 3.6 and Markov property at time $t_k(x_0) \wedge T$ ensure that for any $\varepsilon \in (0, \underline{\varepsilon}]$, $x_0 \in E \cap D$, $T \in (0, T_m)$,

$$\begin{aligned} \mathbb{P}_{x_0} \left(\sup_{[t_k(x_0), t_{k+1}(x_0) \wedge T]} d(X_t, \phi(x_0, t)) \geq \varepsilon a_k(x_0, T), \quad \sup_{[0, t_k(x_0) \wedge T]} d(X_t, \phi(x_0, t)) < \varepsilon \right) \\ \leq C \int_{t_k(x_0) \wedge T}^{t_{k+1}(x_0) \wedge T} \bar{V}_{d, \varepsilon/a_k(x_0, T)}(F_{n_k(x_0)}, x_0, s) ds. \end{aligned}$$

for each $k = 0, \dots, \kappa - 1$, by setting $C = \max\{C_{d, c_1, L_i} : i = 1, \dots, N\}$.

Denoting $A_k(x_0, T) = \prod_{i \leq k} a_i(x_0, T)$ and recalling that $a_i(x_0, T) \geq 1$, by iteration we obtain for $\varepsilon \leq \underline{\varepsilon}/A_\kappa(x_0, T)$,

$$\begin{aligned} \mathbb{P}_{x_0} \left(\bigcup_{k=0}^{\kappa-1} \left\{ \sup_{[t_k(x_0), t_{k+1}(x_0) \wedge T]} d(X_t, \phi(x_0, t)) \geq \varepsilon A_k(x_0, T) \right\} \right) \\ \leq C \sum_{k=0}^{\kappa-1} \int_{t_k(x_0) \wedge T}^{t_{k+1}(x_0) \wedge T} \bar{V}_{d, \varepsilon A_k(x_0, T)}(F_{n_k(x_0)}, x_0, s) ds, \end{aligned}$$

since $X_0 = x_0 = \phi(0, x_0)$. This ensures that

$$\begin{aligned} \mathbb{P}_{x_0} \left(\sup_{0 \leq t \leq T \wedge t_\kappa(x_0)} d(X_t, \phi(x_0, t)) \geq \varepsilon A_\kappa(x_0, T) \right) \\ \leq C A_\kappa(x_0, T)^2 \sum_{k=0}^{\kappa-1} \int_{t_k(x_0) \wedge T}^{t_{k+1}(x_0) \wedge T} \bar{V}_{d, \varepsilon A_k(x_0, T)}(F_{n_k(x_0)}, x_0, s) ds, \end{aligned}$$

Recalling that $(n_k(x_0) : k = 0, \dots, \kappa)$ takes value in a finite set, $A_\kappa(x_0, T)$ is bounded for $x_0 \in E \cap D$ and $T \in [0, T_0)$ by a constant depending only on κ , $(L_i : i = 1, \dots, N)$, c_1 and c_2 . This yields the result. \square

4 Coming down from infinity for one-dimensional Stochastic Differential Equations

In this section, we assume that $E \subset \mathbb{R}$ and $+\infty$ is a limiting value of E and $D = (a, \infty)$ for some $a \in (0, \infty)$. Following the beginning of the previous section, we consider a càdlàg Markov process X which takes values in E and assume that it is the unique strong solution of the following SDE on $[0, \infty)$:

$$X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z) N(ds, dz) + \int_0^t \int_{\mathcal{X}} G(X_{s-}, z) \tilde{N}(ds, dz),$$

for any $x_0 \in E$, where in particular we recall that $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ is a measurable space; B is a Brownian motion; N is a Poisson point measure on $\mathbb{R}^+ \times \mathcal{X}$ with intensity $dsq(dz)$; N and B are independent. We make the following assumption, which is a convenient counterpart of Assumption 3.1 for the study of the coming down infinity in dimension 1.

Assumption 4.1. Let $F \in \mathcal{C}^2((a', \infty), \mathbb{R})$, for some $a' \in [-\infty, a)$ such that $\bar{E} \subset (a', \infty)$.

(i) For any $x > a$, $F'(x) > 0$ and $F(x) \rightarrow \infty$ as $x \rightarrow \infty$.

(ii) For any $x \in E$, $\int_{\mathcal{X}} |F(x + H(x, z)) - F(x)| q(dz) < \infty$.

The function $x \in E \rightarrow h_F(x) = \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz)$ can be extended to (a', ∞) .

This extension is locally bounded on (a', ∞) and locally Lipschitz on (a, ∞) .

(iii) b is locally Lipschitz on (a, ∞) .

(iv) The function $b_F = b + \hat{b}_F/F'$ is negative on (a, ∞) .

Following the previous sections, we consider now the flow ϕ_F given for $x_0 \in (a, \infty)$ by

$$\phi_F(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \phi_F(x_0, t) = b_F(\phi_F(x_0, t)),$$

which is well and uniquely defined and belongs to (a, ∞) on a maximal time interval denoted by $[0, T(x_0))$, where $T(x_0) \in (0, \infty]$. We first observe that $x_0 \rightarrow \phi_F(x_0, t)$ is increasing where it is well defined. This can be seen by recalling that the local Lipschitz property ensures the uniqueness of solutions and thus prevents the trajectories from intersecting. Then $T(x_0)$ is increasing and its limit when $x_0 \uparrow \infty$ is denoted by $T(\infty)$ and belong to $(0, \infty]$. Moreover,

$$\phi_F(\infty, t) = \lim_{x_0 \rightarrow \infty} \phi_F(x_0, t)$$

is well defined for any $t \in [0, T(\infty))$. Under Assumption 4.1 above, $b_F(x) < 0$ for $x \in (a, \infty)$ and for any $x_0 \in (a, \infty)$, and $t < T(x_0)$, $\int_{x_0}^{\phi_F(x_0, t)} \frac{1}{b_F(x)} dx = t$, which yields the following classification.

Either

$$\int_{\infty}^{\cdot} \frac{1}{b_F(x)} < +\infty,$$

and then

$$\phi_F(\infty, t) = \inf \left\{ s \geq 0 : \int_{\infty}^s \frac{1}{b_F(x)} dx < t \right\} < \infty$$

for any $t \in (0, T(\infty))$ and $t \in [0, T(\infty)) \rightarrow \phi(\infty, t) \in \bar{\mathbb{R}}$ is continuous, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is endowed with

$$\bar{d}(x, y) = |e^{-x} - e^{-y}|. \quad (18)$$

We then say that the dynamical system *instantaneously comes down from infinity*.

Otherwise, $T(\infty) = \infty$ and $\phi(\infty, t) = \infty$ for any $t \in [0, \infty)$.

Our aim now is to derive an analogous criterion for stochastic differential equations using the results of the previous section. Letting the process start from infinity requires some additional work. We give first a condition useful for the identification of the limiting values of $(\mathbb{P}_x : x \in E)$.

Definition 4.2. The process X is stochastically monotone if for all $x_0, x_1 \in E$ such that $x_0 \leq x_1$, for all $t > 0$ and $x \in \mathbb{R}$, we have

$$\mathbb{P}_{x_0}(X_t \geq x) \leq \mathbb{P}_{x_1}(X_t \geq x).$$

The Λ -coalescent, the birth and death process, continuous diffusions with strong pathwise uniqueness and several of their extensions satisfy this property, while e.g. the Transmission Control Protocol does not and we refer to the examples of forthcoming Section 4.2 for details.

4.1 Weak convergence and coming down from infinity

We recall that $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ endowed with \bar{d} defined by (18) is polish and the notation of the previous section become $\psi_F = (F'b_F) \circ F^{-1}$, $\tilde{b}_F(x) = F''(x)\sigma(x)^2 + \int_{\mathcal{X}} [F(x + G(x, z)) - F(x) - F'(x)G(x, z)]q(dz)$ and $V_F(x) = (F'(x)\sigma(x))^2 + \int_{\mathcal{X}} [F(x + H(x, z) + G(x, z)) - F(x)]^2 q(dz)$.

In this section, we introduce

$$\hat{V}_{F,\varepsilon}(a, t) = \sup_{x \in E \cap \mathcal{D}_{F,\varepsilon}(a, t)} \left\{ \varepsilon^{-2} V_F(x) + \varepsilon^{-1} \tilde{b}_F(x) \right\},$$

for convenience, where using the extension $F(\infty) = \infty$, $F^{-1}(\infty) = \infty$, we set

$$\mathcal{D}_{F,\varepsilon}(a, t) = \left(a, F^{-1}(F(\phi(\infty, t)) + \varepsilon) \right) = \{x \in (a, \infty) : F(x) \leq F(\phi(\infty, t)) + \varepsilon\}.$$

Finally, we introduce the following key assumption to use the results of the previous section.

Assumption 4.3. *The vector field ψ_F is (L, α) non-expansive on $(F(a), \infty)$ and for any $\varepsilon > 0$,*

$$\int_0^\infty \hat{V}_{F,\varepsilon}(a, t) dt < \infty. \quad (19)$$

Let us remark that ψ_F is (L, α) non-expansive on $(F(a), \infty)$ iff for all $x > y > F(a)$, $\psi_F(x) \leq \psi_F(y) + L(x - y) + \alpha$, i.e. for all $x > y > a$, $F'(x)b(x) + h_F(x) \leq F'(y)b(y) + h_F(y) + L(F(x) - F(y)) + \alpha$.

First, we give sufficient conditions for the convergence of $(\mathbb{P}_x)_{x \in E}$ as $x \rightarrow \infty$. For that purpose, we introduce the modulus

$$w'(f, \delta, [A, B]) = \inf_{\mathbf{b}} \max_{\ell=0, \dots, L-1} \sup_{b_\ell \leq s, t < b_{\ell+1}} \bar{d}(f_s, f_t) \quad (20)$$

where the infimum extends over all subdivisions $\mathbf{b} = (b_\ell, \ell = 0, \dots, L)$ of $[A, B]$ which are δ -sparse and we refer to Chapter 3 in [7] for details on the Skorokhod topology.

Proposition 4.4. *We assume that X is stochastically monotone.*

(i) *If $E = \{0, 1, 2, \dots\}$, then $(\mathbb{P}_x)_{x \in E}$ converges weakly as $x \rightarrow \infty$ in the space of probability measures on $\mathcal{ID}([0, T], \bar{\mathbb{R}})$.*

(ii) *If Assumptions 4.1 and 4.3 hold and $\int_\infty \frac{1}{b_F(x)} < +\infty$ and for any $K > 0$,*

$$\lim_{\delta \rightarrow 0} \sup_{x \in E, x \leq K} \mathbb{P}_x(w'(X, \delta, [0, T]) \geq \varepsilon) = 0, \quad (21)$$

then $(\mathbb{P}_x)_{x \in E}$ converges weakly as $x \rightarrow \infty$ in the space of probability measures on $\mathcal{ID}([0, T], \bar{\mathbb{R}})$.

The convergence result (i) concerns the discrete case $\sigma = 0$. It has been obtained in [13] when the limiting probability \mathbb{P}_∞ is known a priori and the process comes down from infinity. The proof of the tightness of (i) follows [13] and relies on the monotonicity and the fact that the

states are non instantaneous, which is here due to our càdlàg assumption for any initial state space. The identification of the limit is derived directly from the monotonicity and the proof of (i) is actually a direct extension of Lemma 2.1 in [4]. This proof is omitted.

The tightness argument for (ii) is different and can be applied to processes with a continuous part and extended to larger dimensions. The control of the fluctuations of the process for large values relies on the approximation by a continuous dynamical system using Assumption 4.3 and the previous section. Then the tightness on compacts sets is guaranteed by (21). We refer to the proof below.

In the next result, we assume that $(\mathbb{P}_x)_{x \in E}$ converges weakly and \mathbb{P}_∞ is then well defined as the limiting probability. We determine now under our assumptions when (and how) the process comes down from infinity and more precisely we link the coming down from infinity of the process X to that of the flow ϕ .

Theorem 4.5. *We assume that Assumptions 4.1 and 4.3 hold and that $(\mathbb{P}_x : x \in E)$ converges weakly as $x \rightarrow \infty$ in the space of probability measures on $\mathbb{D}([0, T], \bar{\mathbb{R}})$ to \mathbb{P}_∞ .*

(i) If

$$\int_\infty \frac{1}{b_F(x)} < +\infty,$$

then

$$\mathbb{P}_\infty(\forall t > 0 : X_t < +\infty) = 1 \quad \text{and} \quad \mathbb{P}_\infty\left(\lim_{t \downarrow 0+} F(X_t) - F(\phi_F(\infty, t)) = 0\right) = 1.$$

(ii) Otherwise $\mathbb{P}_\infty(\forall t \geq 0 : X_t = +\infty) = 1$.

After the proof coming just below, we are considering several examples. For the Λ -coalescent, we recover the speed of coming down from infinity of [5] using $F = \log$ and in that case V_F is bounded. For birth and death processes with polynomial death rates, fluctuations are smaller and we can use $F(x) = x^\beta$ ($\beta < 1$) and get a finer approximation of the process coming down from infinity by a dynamical system. But V_F is no longer bounded and has to be controlled finely along the dynamical system coming down from infinity. When proving that some birth and death processes or Transmission Control Protocol do not come down from infinity, we are looking for F increasing slowly enough so that V_F is bounded to check (19) and use the result above on unbounded domains, see the next section for details.

Let us turn to the proofs of the two last results and start with the following lemma, where we recall notation $D = (a, \infty)$, $d_F(x, y) = |F(x) - F(y)|$ from the previous section and the definitions of $T_{D, \varepsilon, F}(x_0)$ and $T_\varepsilon^{L, \alpha}$ given respectively in (10) and (3).

Lemma 4.6. *Under Assumptions 4.1 and 4.3, for any $x_0 \in E \cap D$, $\varepsilon > 0$ and $T < T_{D, \varepsilon, F}(x_0) \wedge T_\varepsilon^{L, \alpha}$, we have*

$$\mathbb{P}_{x_0}\left(\sup_{t \leq T} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon\right) \leq C(\varepsilon, T),$$

where

$$C(\varepsilon, T) = C \exp(4LT) \int_0^T \hat{V}_{F, \varepsilon}(a, t) dt$$

goes to 0 when $T \rightarrow 0$ and C is a positive constant.

Proof. Assumption 3.1 and the (L, α) non-expansivity of ψ_F are guaranteed respectively by Assumptions 4.1 and 4.3, with here $O = (a', \infty)$ and $D = (a, \infty)$. Thus, we can apply Theorem 3.2 on the domain (a, ∞) and for any $x_0 \in D \cap E$ and $\varepsilon > 0$, we have for any $T < T_{D, \varepsilon, F}(x_0) \wedge T_\varepsilon^{L, \alpha}$,

$$\mathbb{P}_{x_0} \left(\sup_{t < T} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right) \leq C \exp(4LT) \int_0^T \bar{V}_{F, \varepsilon}(x_0, s) ds.$$

Now let $t < T_{D, \varepsilon, F}(x_0)$ and $x \in E$ such that $d_F(x, \phi_F(x_0, t)) \leq \varepsilon$. Then $F(a) < F(x) \leq F(\phi_F(x_0, t)) + \varepsilon$ and combining the monotonicities of the flow ϕ_F and the function F ,

$$F(a) < F(x) \leq F(\phi_F(\infty, t)) + \varepsilon.$$

Thus $x \in \mathcal{D}_{F, \varepsilon}(a, t)$ and

$$\bar{V}_{F, \varepsilon}(x_0, t) \leq \hat{V}_{F, \varepsilon}(a, t),$$

which ends up the proof, since the behavior of $C(\varepsilon, T)$ when $T \rightarrow 0$ comes from (19). \square

Proof of the Proposition 4.4 (ii). The fact that X is a stochastically monotone Markov process ensures that for all $x_0, x_1 \in E$, $x_0 \leq x_1$, $k \geq 0$, $0 \leq t_1 \leq \dots \leq t_k$, $a_1, \dots, a_k \in \mathbb{R}$,

$$\mathbb{P}_{x_0}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k) \leq \mathbb{P}_{x_1}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k).$$

It can be shown by induction for $k \geq 1$ by using the Markov property at time t_1 and writing $X_{t_1}^{x_1} = X_{t_1}^{x_0} + B$, where X^x is the process X starting at x and B is a non-negative random variable \mathcal{F}_{t_1} measurable. Then

$$\mathbb{P}_{x_0}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k)$$

converges as $x_0 \rightarrow \infty$ ($x_0 \in E$) by monotonicity, which identifies the finite dimensional limiting distributions of $(\mathbb{P}_x : x \in E)$ when $x \rightarrow \infty$.

Let us turn to the proof of the tightness in the Skorokhod space $\mathbb{D}([0, T], \bar{\mathbb{R}})$ and fix $\eta > 0$. The flow ϕ_F comes down instantaneously from infinity since $\int_\infty 1/b_F(x) < \infty$. Thus, we can choose $T_0 \in (0, T(\infty))$ such that $\phi_F(\infty, T_0) \in (a, \infty)$. Using also that F tends to ∞ , let us now fix $K_1 \in [\phi_F(\infty, T_0), \infty)$ and $\varepsilon \in (0, \eta)$ such that $\bar{d}(K_1, \infty) \leq \eta$ and for any $x \geq K_1$ and $y \in \mathbb{R}$ such that $d_F(x, y) < \varepsilon$, we have $\bar{B}_{d_F}(x, \varepsilon) \subset (a, \infty)$ and $\bar{d}(x, y) < \eta$. By continuity and monotonicity of $t \rightarrow \phi(\infty, t)$, there exists $T_1 \in (0, T_0]$ such that $\phi_F(\infty, T_1) = K_1 + 1$. Adding that $T(x_0) \uparrow T(\infty)$ and $\phi_F(x_0, T_1) \uparrow \phi(\infty, T_1)$ as $x_0 \uparrow \infty$, we have $\phi_F(x_0, T_1) \geq K_1$ for any x_0 large enough and then $T_{D, \varepsilon, F}(x_0) \geq T_1$. Thus, Lemma 4.6 ensures that for any x_0 large enough and $T < T_1 \wedge T_\varepsilon^{L, \alpha}$,

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} \left(\sup_{t \leq T} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right) \leq C(\varepsilon, T), \quad (22)$$

where $C(\varepsilon, T) \rightarrow 0$ as $T \rightarrow 0$. Let now $T_2 \in (0, T_1 \wedge T_\varepsilon^{L, \alpha})$ such that $C(\varepsilon, T_2) \leq \eta$. Using that for any $t \in [0, T_2]$, $\phi_F(x_0, t) \geq K_1$ and $\bar{d}(\phi_F(x_0, t), \infty) \leq \eta$ and $d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon$ as soon as $\bar{d}(X_t, \phi_F(x_0, t)) \geq \eta$, we obtain

$$\left\{ \sup_{t \leq T_2} \bar{d}(X_t, \infty) \geq 2\eta \right\} \subset \left\{ \sup_{t \leq T_2} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right\}.$$

Adding that $\phi(x_0, T_2) \uparrow \phi(\infty, T_2) \in [K_1, \infty)$ and F' positive on $[K_1, \infty)$, we have also

$$\{F(X_{T_2}) \geq F(\phi(\infty, T_2)) + \eta\} \subset \{d_F(X_{T_2}, \phi_F(x_0, T_2)) \geq \eta\}.$$

Writing $K = F^{-1}(F(\phi(\infty, T_2)) + \eta) \in [K_1, \infty)$, (22) and the two last inclusions ensure that

$$\mathbb{P}_{x_0} \left(\left\{ \sup_{t \leq T_2} \bar{d}(X_t, \infty) \geq 2\eta \right\} \cup \{X_{T_2} \geq K\} \right) \leq \eta.$$

for x_0 large enough. Moreover, by (21), for any $T \geq T_2$, for δ small enough,

$$\sup_{x \in E; x \leq K} \mathbb{P}_x(w'(X, \delta, [0, T - T_2]) \geq 2\eta) \leq \eta.$$

Combining these two last bounds by Markov property at time T_2 , we get that for x_0 large enough and δ small enough, $\mathbb{P}_{x_0}(w'(X, \delta, [0, T]) \geq 2\eta) \leq 2\eta$. The tightness is proved. \square

Proof of Theorem 4.5. We fix $\varepsilon > 0$ and let $T_0 \in (0, T(\infty))$ such that $\bar{B}_{d_F}(\phi(T_0, \infty), 2\varepsilon) \subset D$. We observe that $T_{D, \varepsilon, F}(x_0) \geq T_0$ for x_0 large enough since $\phi(x_0, T_0) \uparrow \phi(\infty, T_0)$ as $x_0 \uparrow \infty$ and $t \in [0, T(x_0)) \rightarrow \phi(x_0, t)$ decreases. We apply Lemma 4.6 and get for any $T \leq T_0$,

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} \left(\sup_{t \leq T} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right) \leq C(\varepsilon, T), \quad (23)$$

where $C(\varepsilon, T) \rightarrow 0$ as $T \rightarrow 0$.

We first consider the case (i) and fix now also $t_0 \in (0, T_0)$. The flow ϕ comes down from infinity instantaneously and is continuous, so $\phi(\infty, t) < \infty$ on $[t_0, T]$ and $\phi(x_0, \cdot)$ converges to $\phi(\infty, \cdot)$ uniformly on $[t_0, T]$, using the monotonicity of the convergence and the continuity of the limit. We obtain from (23) that for any $T \leq T_0$,

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} \left(\sup_{t_0 \leq t \leq T} d_F(X_t, \phi_F(\infty, t)) > \varepsilon \right) \leq C(\varepsilon, T),$$

and the weak convergence of $(\mathbb{P}_x : x \in E)$ to \mathbb{P}_∞ yields

$$\mathbb{P}_\infty \left(\sup_{t_0 \leq t \leq T} d_F(X_t, \phi_F(\infty, t)) > \varepsilon \right) \leq C(\varepsilon, T).$$

Letting $t_0 \downarrow 0$ and then $T \downarrow 0$ ensures that

$$\lim_{T \rightarrow 0} \mathbb{P}_\infty \left(\sup_{0 < t \leq T} d_F(X_t, \phi_F(\infty, t)) > \varepsilon \right) = 0.$$

Then $\mathbb{P}_\infty(\lim_{t \downarrow 0+} F(X_t) - F(\phi_F(\infty, t)) = 0) = 1$ and $\mathbb{P}_\infty(\forall t > 0 : X_t < \infty) = 1$, which proves (i).

For the case (ii), i.e. $\int_\infty 1/b_F(x) = \infty$, we recall that $T(\infty) = \infty$, so (23) yields

$$\mathbb{P}_\infty \left(F(X_T) < \limsup_{x_0 \rightarrow \infty} F(\phi(x_0, T)) - A \right) \leq C(A, T)$$

for any $T > 0$. Adding that $F(\phi(x_0, T)) \uparrow F(\phi(\infty, T)) = F(\infty) = \infty$ as $x_0 \uparrow \infty$,

$$\mathbb{P}_\infty(X_T < \infty) \leq C(A, T).$$

Since $\phi(\infty, t) = \infty$ for any $t \geq 0$, $\mathcal{D}_{F, A}(a, t) = (a, \infty)$ for any $A > 0$. Then $C(A, T) \leq \frac{1}{A}C(1, T)$ for $A \geq 1$ and $C(A, T) \rightarrow 0$ as $A \rightarrow \infty$, since $C(1, T) < \infty$ by (19). We get $\mathbb{P}_\infty(X_t = \infty) = 1$, which ends up the proof recalling that X is a càdlàg Markov process under \mathbb{P}_∞ . \square

4.2 Examples and applications

We consider here examples of processes in one dimension and recover some known results. We also get new estimates and we illustrate the assumptions required and the choice of F . Thus, we recover classical results on the coming down from infinity for Λ -coalescent and refine some of them for birth and death processes. Here $b, \sigma = 0$ and the condition allowing the compensation of jumps (Assumption 4.1 (ii)) will be obvious. We also provide a criterion for the coming down from infinity of the Transmission Control Protocol, which is piecewise deterministic markov process ($b \neq 0, \sigma = 0$). Several extensions of these results could be achieved, such as mixing branching coalescing processes or additional catastrophes. They are left for future works, while the next section will consider a class of diffusions in higher dimension.

4.2.1 Λ -coalescent [26, 5]

Pitman [26] has given a Poissonian representation of Λ -coalescent. We recall that Λ is a finite measure on $[0, 1]$ and we set $\nu(dy) = y^{-2}\Lambda(dy)$. Without loss of generality, we assume that $\Lambda[0, 1] = 1$ and for simplicity, we focus on coalescent without Kingman part and assume $\Lambda(\{0\}) = 0$. We consider a Poisson Point Process on $(\mathbb{R}^+)^2$ with intensity $dt\nu(dy)$: each atom (t, y) yields a coalescence event where each block is picked independently with probability y and all the blocks picked merge into a single block. Then the numbers of blocks jumps from n to $B_{n,y} + 1_{B_{n,y} < n}$, where $B_{n,y}$ follows a binomial distribution with parameter $(n, 1 - y)$. Thus, the number of blocks X_t at time t is the solution of the SDE

$$X_t = X_0 - \int_0^t \int_0^1 \int_{[0,1]^{\mathbb{N}}} \left(-1 + \sum_{1 \leq i \leq X_{s-}} 1_{u_i \leq y} \right)^+ N(ds, dy, du),$$

where N is a PPM with intensity on $\mathbb{R}^+ \times \mathcal{X}$ with intensity $dtq(dz)$ and here $E = \mathbb{N} = \{1, 2, \dots\}$, $\mathcal{X} = [0, 1] \times [0, 1]^{\mathbb{N}}$ endowed with the cylinder σ -algebra of borelian sets of $[0, 1]$, $q(dz) = q(dydu) = \nu(dy)du$ where du is the uniform measure on $[0, 1]^{\mathbb{N}}$, $\sigma = 0$, and

$$H(x, z) = H(x, (y, u)) = - \left(-1 + \sum_{1 \leq i \leq x} 1_{u_i \leq y} \right)^+.$$

We follow [5] and we denote for $x \in (1, \infty)$,

$$F(x) = \log(x), \quad \psi(x) = \int_{[0,1]} (e^{-xy} - 1 + xy)\nu(dy).$$

In particular F meets the Assumption 4.1 (i) with $a > 0$ and $a' = 0$. Moreover for every $x \in \mathbb{N}$,

$$\begin{aligned} h_F(x) &= \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)]q(dz) \\ &= \int_{\mathcal{X}} \log\left(\frac{x + H(x, z)}{x}\right)q(dz) \\ &= \int_{[0,1]} \nu(dy)\mathbb{E}\left(\log\left(\frac{B_{x,y} + 1_{B_{x,y} < x}}{x}\right)\right) = -\frac{\psi(x)}{x} + h(x), \end{aligned}$$

where h is bounded thanks to Proposition 7 in [5]. Thus h can be extended to a bounded C^1 function on $(0, \infty)$ and Assumption 4.1 (ii) is satisfied. Moreover,

$$\psi_F(x) = h_F(F^{-1}(x)) = -\frac{\psi(\exp(x))}{\exp(x)} + h(\exp(x))$$

and Lemma 9 in [5] ensures that $x \in (1, \infty) \rightarrow \psi(x)/x$ is increasing. Then ψ_F is $(0, 2 \|h\|_\infty)$ non-expansive on $(0, \infty)$. Moreover here

$$b_F(x) = F'(x)^{-1}h_F(x) = -\psi(x) + xh(x).$$

Adding that $\psi(x)/x \rightarrow \infty$ as $x \rightarrow \infty$, we get $b_F(x) < 0$ for x large enough and Assumption 4.1 (iv) is checked. Finally, $\tilde{b}_F = 0$ since $\sigma = 0$ and $G = 0$ and the second part of Proposition 7 in [5] ensures that

$$V_F(x) = \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)]^2 q(dz) = \int_{[0,1]} \nu(dy) \mathbb{E} \left(\left(\log \left(\frac{B_{x,y} + 1_{B_{x,y} < x}}{x} \right) \right)^2 \right)$$

is bounded for $x \in \mathbb{N}$. Then Assumptions 4.1 and 4.3 are satisfied with $F = \log$, $a' = 0$ and a large enough. Moreover $(\mathbb{P}_x : x \in \mathbb{N})$ converges weakly to \mathbb{P}_∞ , which can be derived from Proposition 4.4 (i) since X is stochastically monotone. Thus Theorem 4.5 can be applied and writing $w_t = \phi_F(\infty, t)$, we have

(i) If $\int_\infty \frac{1}{b_F(x)} < +\infty$, then $w_t \in C^1((0, \infty), (0, \infty))$, $w'_t = -\psi(w_t) + w_t h(w_t)$ for $t > 0$ and

$$\mathbb{P}_\infty(\forall t > 0 : X_t < \infty) = 1 \quad \text{and} \quad \mathbb{P}_\infty \left(\lim_{t \downarrow 0+} \frac{X_t}{w_t} = 0 \right) = 1.$$

(ii) Otherwise $\mathbb{P}_\infty(\forall t \geq 0 : X_t = +\infty) = 1$.

To compare with known results, let us note that $b_F(x) \sim \psi(x)$ as $x \rightarrow \infty$ and

$$\int^\infty \frac{1}{\psi(x)} dx < \infty \Leftrightarrow \int_\infty \frac{1}{b_F(x)} < \infty,$$

so that we recover here the criterion of coming down from infinity obtained in [6]. This latter is equivalent to the criterion initially proved in [27] :

$$\sum_{n=2}^{\infty} \gamma_n^{-1} < \infty,$$

where

$$\gamma_n = - \int_{[0,1]} H(n, z) q(dz) = \sum_{k \geq 2} (k-1) \binom{n}{k} \int_{[0,1]} y^k (1-y)^{n-k} \nu(dy).$$

Remark 1 : let us note that this condition can be rewritten as $\int_\infty 1/b(x) dx < \infty$, where $b(x)$ is a locally Lipschitz function, which is non-increasing and equal to $-\gamma_n$ for any $n \in \mathbb{N}$. But the proof cannot be achieved using $F = Id$, even if $b(x)$ is non-expansive since $V_{Id}(x)$ cannot be controlled.

Finally, following [5] let us consider the flow associated to the vector field $-\psi(\exp(x))/\exp(x)$ and write v_t the flow starting from ∞ . In the case (i) when the process comes down from infinity, we can use Lemma 6.1 in Appendix to check that $\log(w_t) - \log(v_t)$ goes to 0 as $t \rightarrow 0$ since $\psi_F(x) + \psi(\exp(x))/\exp(x)$ is bounded. Thus

$$w_t \sim_{t \downarrow 0^+} v_t, \quad \text{where} \quad v_t = \inf \left\{ s > 0 : \int_s^\infty \frac{1}{\psi(x)} dx < t \right\}$$

satisfies $v'_t = \psi(v_t)$ for $t > 0$. We recover here the speed of coming down from infinity of [5].

Remark 2 : We have here proved that the speed of coming down from infinity is w using Theorem 4.5 and [5] and then observe that this speed function is equivalent to v . It is possible to recover directly that v is the speed of coming down from infinity by using Proposition 2.2 and a slightly different decomposition of the process X following [5] :

$$\log(X_t) = \log(X_0) - \int_0^t \psi(X_s) ds + \int_0^t \int_{\mathcal{X}} \log \left(\frac{X_{s-} + (-1 + \sum_{i \leq X_{s-}} 1_{u_i \leq y})^+}{X_{s-}} \right) \tilde{N}(ds, dy, du) + A_t,$$

where $A_t = \int_0^t h(X_s) ds$ is a process with finite variations. This could also be done directly by extending the result of Section 3 and adding a term with finite variations in the decomposition using Section 2.

Remark 3 : let us also mention that the speed of coming down from infinity for some Ξ coalescent has been obtained in [23] with a similar method than [5]. The reader could find there other and detailed information about the coming down from infinity of coalescent processes. Finally, we mention [24, 25] for stimulating recent results on the fluctuations of the Λ - coalescent around the dynamical system v_t for small times.

4.2.2 Birth and death processes [29, 4]

We consider a birth and death process X and we denote by λ_k (resp. μ_k) the birth rate (resp. the death rate) when the population size is equal to $k \in E = \{0, 1, 2, \dots\}$. We assume that $\mu_0 = \lambda_0 = 0$ and $\mu_k > 0$ for $k \geq 1$ and we denote

$$\pi_1 = \frac{1}{\mu_1}, \quad \pi_k = \frac{\lambda_1 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} \quad (k \geq 2).$$

We also assume that

$$\sum_{k \geq 1} \frac{1}{\lambda_k \pi_k} = \infty. \quad (24)$$

Then the process X is well defined on E and eventually becomes extinct a.s. [20, 21], i.e. $T_0 = \inf\{t > 0 : X_t = 0\} < \infty$ p.s. It is the unique strong solution on E of the following SDE

$$X_t = X_0 + \int_0^t \int_0^\infty (1_{z \leq \lambda_{X_{s-}}} - 1_{\lambda_{X_{s-}} < z \leq \lambda_{X_{s-}} + \mu_{X_{s-}}}) N(ds, dz)$$

where N is a Poisson Point Measure with intensity $ds dz$ on $[0, \infty)^2$. Lemma 2.1 in [4] ensures that $(\mathbb{P}_x : x \in E)$ converges weakly to \mathbb{P}_∞ . It can also be derived from Proposition 4.4 (i) since

X is stochastically monotone. Under the extinction assumption (24), the following criterion for the coming down from infinity is well known [1]:

$$S = \lim_{n \rightarrow \infty} \mathbb{E}_n(T_0) = \sum_{i \geq 1} \pi_i + \sum_{n \geq 1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i < +\infty. \quad (25)$$

The speed of coming down from infinity of birth and death processes has been obtained in [4] for regularly varying rate (with index $\rho > 1$) and a birth rate negligible compared to the death rate. Let us here get a finer result for a relevant subclass which allows rather simple computations, which is a competitive model in population dynamics and contains in particular the logistic birth and death process.

Proposition 4.7. *We assume that there exist $b \geq 0$, $c > 0$ and $\rho > 1$ such that*

$$\lambda_k = bk, \quad \mu_k = ck^\rho \quad (k \geq 0).$$

Then for any $\alpha \in (0, 1/2)$,

$$\mathbb{P}_\infty \left(\lim_{t \downarrow 0+} t^{\alpha/(1-\rho)} (X_t/w_t - 1) = 0 \right) = 1,$$

where

$$w_t \sim_{t \downarrow 0+} [ct/(\rho - 1)]^{1/(1-\rho)}.$$

This complements the results obtained in [4], where it was shown that $X_t/w_t \rightarrow 1$ as $t \downarrow 0$. The proof used the decomposition of the trajectory in terms of the first hitting time of integers, which works well when multiple death can not occur simultaneously. The fact that X satisfies a central limit theorem when $t \rightarrow 0$ under \mathbb{P}_∞ (see Theorem 5.1 in [4]) ensures that the previous result is sharp in the sense that it does not hold for $\alpha \geq 1/2$.

Remark. Using (26) and Lemma 6.3 in Appendix, a more explicit form of the previous result can be given for $\alpha < (\rho - 1) \wedge 1/2$:

$$\mathbb{P}_\infty \left(\lim_{t \downarrow 0+} t^{\alpha/(1-\rho)} \left(\frac{X_t}{[ct/(\rho - 1)]^{1/(1-\rho)}} - 1 \right) = 0 \right) = 1.$$

Before the proof, we also consider the critical case where the competition rate is slightly larger than the birth rate. We recover here the criterion for the comes down from infinity using Theorem 4.5. We complement this result by providing estimates both when the process comes and does not come from infinity. The function f_β defined by

$$f_\beta(x) = \int_2^{2+x} 1/\sqrt{y \log(y)^\beta} dy.$$

provides the best distance (i.e. the fastest increasing function going to infinity) allowing to compare the process and the flow by bounding the quadratic variation. It allows in particular to capture the fluctuations when they do not come down from infinity, see (ii) below.

Proposition 4.8. *We assume that there exist $b \geq 0$, $c > 0$ and $\beta > 0$ such that*

$$\lambda_k = bk, \quad \mu_k = ck \log(k+1)^\beta \quad (k \geq 0).$$

(i) *If $\beta > 1$, then $\mathbb{P}_\infty(\forall t > 0 : X_t < +\infty) = 1$ and*

$$\mathbb{P}_\infty \left(\lim_{t \downarrow 0^+} f_\beta(X_t) - f_\beta(w_t) = 0 \right) = 1,$$

where $w_t = \phi_{f_\beta}(\infty, t) \in \mathcal{C}^1((0, \infty), (0, \infty))$.

(ii) *If $\beta \leq 1$, $\mathbb{P}_\infty(\forall t \geq 0 : X_t = +\infty) = 1$ and for any $\varepsilon > 0$,*

$$\lim_{T \rightarrow 0} \limsup_{x_0 \rightarrow \infty, x_0 \in \mathbb{N}} \mathbb{P}_{x_0} \left(\sup_{t \leq T} |f_\beta(X_t) - f_\beta(\phi_{f_\beta}(x_0, t))| \geq \varepsilon \right) = 0.$$

We do not provide more explicit expressions of the flow ϕ_{f_β} or the behavior of w_t in short time for that case and we turn to the proof of the two previous propositions. Let us specify notation for the birth and death process. Here $\chi = [0, \infty)$, $q(dz) = dz$ and

$$H(x, z) = 1_{z \leq \lambda_x} - 1_{\lambda_x < z \leq \lambda_x + \mu_x}.$$

Letting $F \in \mathcal{C}^1((-1, \infty), \mathbb{R})$, we have $\int_\chi |F(x + H(x, z)) - F(x)| q(dz) < \infty$ and here

$$h_F(x) = (F(x+1) - F(x))\lambda_x + (F(x-1) - F(x))\mu_x$$

for $x \in \{0, 1, \dots\}$. For the classes of birth and death rates λ, μ considered in the two previous propositions, h_F is well defined on $(-1, \infty)$ by the identity above and $h_F \in \mathcal{C}^1((-1, \infty), \mathbb{R})$. Assumption 4.1 can thus checked with $a' = -1$. Finally

$$V_F(x) = (F(x+1) - F(x))^2 \lambda_x + (F(x) - F(x-1))^2 \mu_x.$$

Proof of Proposition 4.7. We consider now $\alpha \in (0, 1/2)$ and

$$F(x) = (1+x)^\alpha \quad (\alpha \in (0, 1/2)).$$

Then $h_F(x) = ((x+2)^\alpha - (x+1)^\alpha)bx + (x^\alpha - (x+1)^\alpha)cx^\rho$ and there exists $a > 0$ such that $h'_F(x) < 0$ for $x \geq a$. This ensures that Assumption 4.1 is checked with $a' = -1$ and a . Moreover $\psi_F = h_F \circ F^{-1}$ is non-increasing on (a, ∞) and ψ_F is non-expansive on $(F(a), \infty)$.

Adding that here

$$h_F(x) \sim_{x \rightarrow \infty} -c\alpha x^{\rho+\alpha-1}$$

we get

$$b_F(x) = F'(x)^{-1} h_F(x) = -c(1+x)^\rho + O(x^{\max(\rho-1, 1)}) \quad (x \rightarrow \infty) \quad (26)$$

and one can use Lemma 6.2 in Appendix with $\psi_1(x) = b_F(x)$ and $\psi_2(x) = -cx^\rho$ to check that

$$\phi_F(\infty, t) \sim_{t \downarrow 0^+} [ct/(\rho-1)]^{1/(1-\rho)}. \quad (27)$$

Finally

$$V_F(x) = ((x+2)^\alpha - (x+1)^\alpha)^2 bx + ((x+1)^\alpha - x^\alpha)^2 cx^\rho \sim \alpha^2 cx^{\rho+2\alpha-2} \quad (x \rightarrow \infty).$$

Adding that for any $T > 0$, there exists $c_0 > 0$ such that $\phi(\infty, t) \leq c_0 t^{1/(1-\rho)}$ for $t \in [0, T]$ and that $F^{-1}(y) = y^{1/\alpha} - 1$, then for any $\varepsilon > 0$, there exists $c'_0 > 0$ such that for any $t \leq T$,

$$\hat{V}_{F,\varepsilon}(a, t) \leq \varepsilon^{-2} \sup \left\{ V_F(x) : 0 \leq x \leq ((\phi_F(\infty, t) + 1)^\alpha + \varepsilon)^{1/\alpha} - 1 \right\} \leq c'_0 (t^{1/(1-\rho)})^{\rho+2\alpha-2}.$$

Using that $(\rho + 2\alpha - 2)/(1 - \rho) = -1 + (2\alpha - 1)/(1 - \rho) > -1$ since $\alpha < 1/2$, we obtain

$$\int_0^\cdot \hat{V}_{F,\varepsilon}(a, t) dt < \infty.$$

Thus Assumptions 4.1 and 4.3 are satisfied and Theorem 4.5 (i) can be applied and defining $w_t = \phi_F(\infty, t)$, we have for any $\alpha \in (0, 1/2)$, $\mathbb{P}_\infty \left(\lim_{t \downarrow 0^+} X_t^\alpha - w_t^\alpha = 0 \right) = 1$. This ends up the proof recalling (27). \square

Proof of Proposition 4.8. The criterion $\beta > 1$ for the coming down from infinity can be derived easily from the criterion $S < \infty$ recalled in (25) and the fact that \mathbb{P}_n converges to \mathbb{P}_∞ . It is also a consequence of Theorem 4.5 using the function $F(x) = (1+x)^\alpha$ as in the previous proof. Let us turn to the proof of the estimates (i – ii) and take $F = f_\beta$. Then $F(x) \rightarrow \infty$ as $x \rightarrow \infty$,

$$h_F(x) = bx \int_{2+x}^{3+x} \frac{1}{\sqrt{y} \log(y)^\beta} dy - cx \log(x)^\beta \int_{1+x}^{2+x} \frac{1}{\sqrt{y} \log(y)^\beta} dy$$

and its derivative is negative for x large enough. Then Assumptions 4.1 is satisfied with again $a' = 1$ and a large enough. So $\psi_F(x) = h_F(F^{-1}(x))$ is decreasing and thus non-expansive for x large enough. Moreover there exists $C > 0$ such that

$$V_F(x) \leq Cx \log(x)^\beta \left(\int_{1+x}^{2+x} \frac{1}{\sqrt{y} \log(y)^\beta} dy \right)^2.$$

So V_F is bounded and Assumption 4.3 is satisfied. Then (i) comes from Theorem 4.5 (i) and (ii) comes from Lemma 4.6 observing that $T_{D,\varepsilon,F}(x_0) \rightarrow \infty$ as $x_0 \rightarrow \infty$. \square

4.2.3 Transmission Control Protocol

The Transmission Control Protocol [3] is a model for transmission of data, mixing a continuous (positive) drift which describes the growth of the data transmitted and jumps due to congestions, where the size of the data are divided by two. Then the size X_t of data at time t is given by the unique strong solution on $[0, \infty)$ of

$$X_t = x_0 + bt - \int_0^t \mathbf{1}_{\{u \leq r(X_{s-})\}} \frac{X_{s-}}{2} N(ds, du),$$

where $x_0 \geq 0$, $b > 0$, $r(x)$ is a continuous positive non-decreasing function and N is PPM on $[0, \infty)^2$ with intensity $dsdu$. This is a classical example of Piecewise Deterministic Markov process. Usually, $r(x) = cx^\beta$, with $\beta \geq 0$, $c > 0$. First, let us note that the process is not stochastically monotone and the convergence of $(\mathbb{P}_x : x \geq 0)$ is left open. The choice of F is a bit more delicate here owing to the size and intensity of the fluctuations. Consider F such that $F'(x) > 0$ for $x > 0$. Here $E = [0, \infty)$, $h_F(x) = r(x)(F(x/2) - F(x))$,

$$b_F = b + F'^{-1} h_F, \quad \psi_F = (bF' + h_F) \circ F^{-1}.$$

Finally

$$V_F(x) = r(x)(F(x/2) - F(x))^2$$

and we cannot use $F(x) = (1+x)^\gamma$ or $F(x) = \log(1+x)^\gamma$ since then the second part of Assumption 4.3 does not hold. We need to reduce the size of the jumps even more and take $F(x) = \log(1 + \log(1+x))$. The model is not stochastically monotone but Lemma 4.6 can be used to get the following result, which yields a criterion for the coming down from infinity.

Proposition 4.9. (i) *If there exists $c > 0$ and $\beta > 1$ such that $r(x) \geq c \log(1+x)^\beta$ for any $x \geq 1$, then for any $T > 0$, $\eta > 0$, there exists K such that*

$$\inf_{x_0 \geq 0} \mathbb{P}_{x_0}(\exists t \leq T : X_t \leq K) \geq 1 - \eta.$$

(ii) *If there exists $c > 0$ and $\beta \leq 1$ such that $r(x) \leq c \log(1+x)^\beta$ for any $x \geq 0$, then for any $T, K > 0$,*

$$\lim_{x_0 \rightarrow \infty} \mathbb{P}_{x_0}(\exists t \leq T : X_t \leq K) = 0.$$

Thus, in the first regime, the process comes down instantaneously and a.s. from infinity, while in the second regime it stays at infinity. In particular, if $r(x) = cx^\beta$ and $\beta, c > 0$, the process comes down instantaneously from infinity. If $\beta = 0$, it does not, which can actually be seen easily since in the case $r(\cdot) = c$, $X_t \geq (x_0 + bt)/2^{N_t}$, where N_t is a Poisson Process with rate c and the right hand side goes to ∞ as $x_0 \rightarrow \infty$ for any $t \geq 0$.

Proof. Here $E = [0, \infty)$ and we consider

$$F(x) = \log(1 + \log(1+x))$$

on (a', ∞) where $a' \in (-1, 0)$ is chosen such that $\log(1+a') > -1$. Then

$$F'(x) = \frac{1}{(1+x)(1+\log(1+x))} > 0.$$

Moreover

$$F(x/2) - F(x) = \log(1 - \varepsilon(x)),$$

where

$$\varepsilon(x) = 1 - \frac{1 + \log(1+x/2)}{1 + \log(1+x)} = \frac{\log(2) + O(1/(1+x))}{1 + \log(1+x)}.$$

We consider now

$$r(x) = c \log(1+x)^\beta$$

with $c > 0$ and $\beta \in [0, 2]$. We get

$$b_F(x) = b + c \log(1+x)^\beta (1+x)(1+\log(1+x)) \log(1 - \varepsilon(x)) \sim -cx \log(x)^\beta$$

as $x \rightarrow \infty$. Thus, Assumptions 4.1 is satisfied for a' and a large enough. Moreover

$$\int_{\infty}^{\cdot} \frac{1}{b_F(x)} dx < +\infty \text{ if and only if } \beta > 1.$$

Adding that $h'_F(x) = c\beta \log(1+x)^{\beta-1} (F(x/2) - F(x)) + c \log(1+x)^\beta (F'(x/2)/2 - F'(x))$, we get $(bF' + h_F)'(x) < 0$ for $x > a$ where a is large enough. Then, $bF' + h_F$ is non-increasing on (a, ∞)

and F^{-1} is non-increasing on $(F(a), \infty)$, then $\psi_F = h_F \circ F^{-1}$ is non-expansive on $(F(a), \infty)$. Finally

$$V_F(x) = c \log(1+x)^\beta \log(1-\varepsilon(x))^2 \sim c \log(x)^{\beta-2}$$

as $x \rightarrow \infty$ and V_F is bounded since $\beta \leq 2$. So Assumptions 4.1 and 4.3 are satisfied for a' and a large enough and we can apply Lemma 4.6. We get for any $x_0 \geq 0$ and $A > 0$,

$$\mathbb{P}_{x_0} \left(\sup_{t \leq T} |F(X_t) - F(\phi_F(x_0, t))| \geq A \right) \leq C(A, T), \quad (28)$$

where $C(A, T) \rightarrow 0$ as $T \rightarrow 0$ and $C(A, T) \leq C.T. \sup_{x \geq 0} V_F(x)/A^2$.

We can now prove (i). By a simple coupling argument, $X_t \leq \tilde{X}_t$ a.s. for $t \geq 0$, where \tilde{X}_t is a TCP associated with the rate of jumps

$$\tilde{r}(x) = c \log(1+x)^{\beta \wedge 2}$$

where $\beta > 1$. Then $\phi_F(x_0, t) \leq \phi_F(\infty, t) < \infty$ since $\beta > 1$ ensures that the dynamical system comes down from infinity. Letting $T \rightarrow 0$ in (28) yields (i).

To prove (ii), we use similarly the coupling $X_t \geq \tilde{X}_t$ with $\tilde{r}(x) = c \log(1+x)^\beta$ and $\beta < 1$ and let now $A \rightarrow \infty$ in (28). This ends up the proof since V_F bounded ensures that $C(A, T) \rightarrow 0$. \square

4.2.4 Logistic diffusions [9] and perspectives

The coming down from infinity of diffusions of the form

$$dZ_t = \sqrt{\gamma Z_t} dB_t + h(Z_t) dt$$

has been studied in [9] and is linked to the uniqueness of the quasistationary distribution (Theorem 7.3). Writing $X_t = 2\sqrt{Z_t/\gamma}$, it becomes

$$dX_t = dB_t - q(X_t) dt,$$

where $q(x) = x^{-1}(1/2 - 2h(\gamma x^2/4)/\gamma)$. Under some assumptions (see Remark 7.4 in [9]), the coming down from infinity is indeed equivalent to

$$\int^\infty \frac{1}{q(x)} dx < \infty,$$

which can be compared to our criterion in Theorem 4.5. Several extensions and new results could be obtained using the results of this section. In particular one may be interested to mix a diffusion part for competition, negative jumps due to coalescence and branching events. In that vein, let us mention [22]. This is one motivation to take into account the compensated Poisson measure in the definition of the process X , so that Lévy processes and CSBP may be considered in general. It is left for future stimulating works. Let us here simply mention that a class of particular interest is given by the logistic Feller diffusion :

$$dZ_t = \sqrt{\gamma Z_t} dB_t + (\tau Z_t - a Z_t^2) dt.$$

The next part is proving new results on the speed of coming down of this diffusion. This part actually deals more generally with the two dimensional version of this diffusion, where non-expansivity and the behavior of the dynamical system are more delicate.

5 Uniform estimates for two-dimensional competitive Lotka-Volterra processes

We consider the flow $\phi : [0, \infty)^2 \times [0, \infty) \rightarrow [0, \infty)^2$ denoted by

$$x_t = \phi(x_0, t) = (x_t^{(1)}, x_t^{(2)})$$

and starting from $x_0 = (x_0^{(1)}, x_0^{(2)}) \in [0, \infty)^2$, which is the unique solution of the competitive Lotka-Volterra ODE on $[0, \infty)$:

$$\begin{aligned} (x_t^{(1)})' &= x_t^{(1)}(\tau_1 - ax_t^{(1)} - cx_t^{(2)}) \\ (x_t^{(2)})' &= x_t^{(2)}(\tau_2 - bx_t^{(2)} - dx_t^{(1)}), \end{aligned} \quad (29)$$

where $a, b, c, d \geq 0$. The coefficients a and b are the intraspecific competition rates and c, d are the interspecific competition rates. We assume that

$$a, b, c, d > 0$$

or $a, b > 0$ and $c = d = 0$, so that our results cover the (simpler) case of one-single competitive (logistic) model. In this section, we prove that this flow gives the deterministic approximation of stochastic processes in two situations. These results rely on the application of the results of Section 3 to a well chosen family of transformations

$$F_{\beta, \gamma}(x) = \begin{pmatrix} x_1^\beta \\ \gamma x_2^\beta \end{pmatrix}, \quad x \in (0, \infty)^2, \quad \beta \in (0, 1], \quad \gamma > 0, \quad (30)$$

using the adjunction procedure and Poincaré compactification technics for flows to control and describe the dynamical system coming down from infinity.

In Section 5.1, we study the small time behavior starting from large values of the stochastic process $X = (X^{(1)}, X^{(2)})$ defined as the unique strong solution of the following SDE,

$$\begin{aligned} X_t^{(1)} &= x_0^{(1)} + \int_0^t X_s^{(1)}(\tau_1 - aX_s^{(1)} - cX_s^{(2)})ds + \int_0^t \sigma_1 \sqrt{X_s^{(1)}} dB_s^{(1)} \\ X_t^{(2)} &= x_0^{(2)} + \int_0^t X_s^{(2)}(\tau_2 - bX_s^{(2)} - dX_s^{(1)})ds + \int_0^t \sigma_2 \sqrt{X_s^{(2)}} dB_s^{(2)}, \end{aligned} \quad (31)$$

for $t \geq 0$, where B is two dimensional Brownian motion. This is the classical continuous stochastic (Lotka-Volterra) model for two competitive species, see e.g. [8] for related issues on quasi-stationary distributions. We compare for a suitable distance the diffusion X to the flow $\phi(x_0, t)$. We then derive the behavior of the process X coming from infinity.

Second, in Section 5.2, we can prove that usual scaling limits of competitive birth and death processes (see (36) for a definition) hold uniformly with respect to the initial values, for suitable distances and parameters.

These results give answers to two issues which have motivated this work : first, how classical competitive stochastic models regulate large populations (see in particular forthcoming Corollary 5.2); second, can we extend individual based-models approximations of Lotka-Volterra dynamical system to arbitrarily large initial values and if yes, when and for which distance. We believe that the techniques developed here also allow to study the coming down from infinity of these competitive birth and death processes and other multi-dimensional stochastic processes.

5.1 Uniform short time estimates for competitive Feller diffusions

We consider the domain

$$\mathcal{D}_\alpha = (\alpha, \infty)^2$$

and the distance d_β on $[0, \infty)^2$ defined for $\beta > 0$ by

$$d_\beta(x, y) = \sqrt{|x_1^\beta - y_1^\beta|^2 + |x_2^\beta - y_2^\beta|^2} = \|F_{\beta,1}(x) - F_{\beta,1}(y)\|_2. \quad (32)$$

We recall that $a, b, c, d > 0$ or $(a = b > 0$ and $c = d = 0)$ and we define

$$T_D(x_0) = \inf\{t \geq 0 : \phi(x_0, t) \notin D\} \quad (33)$$

the first time when the flow ϕ starting from x_0 exits D .

Theorem 5.1. *For any $\beta \in (0, 1)$, $\alpha > 0$ and $\varepsilon > 0$,*

$$\lim_{T \rightarrow 0} \sup_{x_0 \in \mathcal{D}_\alpha} \mathbb{P}_{x_0} \left(\sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d_\beta(X_t, \phi(x_0, t)) \geq \varepsilon \right) = 0.$$

This yields a control of the stochastic process X defined by (31) by the dynamical system for large initial values and times small enough. We are not expecting that this control hold outside \mathcal{D}_α . Indeed, the next result shows that the process and the dynamical system have a different behavior close to the boundary $(0, \infty) \times \{0\} \cup \{0\} \times (0, \infty)$. The proof below can not be achieved for $\beta = 1$ since then the associated quadratic variations are not integrable at time 0. Heuristically, $\sqrt{Z_t} dB_t$ is of order $\sqrt{1/t} dB_t$, which is not becoming small when $t \rightarrow 0$ and the fluctuations are not vanishing for d_1 in short time.

We denote $\widehat{(x, y)} \in (-\pi, \pi]$ the oriented angle in the trigonometric sense between two non-zero vectors of \mathbb{R}^2 and if $ab \neq cd$, we write

$$x_\infty = \frac{1}{ab - cd}(b - c, a - d). \quad (34)$$

The following classification yields the way the diffusion comes down from infinity.

Corollary 5.2. *We assume that $\sigma_1 > 0, \sigma_2 > 0$ and let $x_0 \in (0, \infty)^2$.*

(i) *If $a > d$ and $b > c$, then for any $\eta \in (0, 1)$ and $\varepsilon > 0$,*

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left(\sup_{\eta T \leq t \leq T} \|tX_t - x_\infty\|_2 \geq \varepsilon \right) = 0,$$

If furthermore x_0 is collinear to x_∞ , the previous limit holds also for $\eta = 0$.

(ii) *If $d > a$ and $c > b$ and $\widehat{(x_\infty, x_0)} \neq 0$, then for any $T > 0$,*

$$\lim_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left(\inf\{t \geq 0 : X_t^{(i)} = 0\} \leq T \right) = 1,$$

where $i = 1$ when $\widehat{(x_0, x_\infty)} \in (0, \pi/2]$ and $i = 2$ when $\widehat{(x_0, x_\infty)} \in [-\pi/2, 0)$.

(iii) *If $a \leq d$ and $b > c$, then for any $T > 0$,*

$$\lim_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left(\inf\{t \geq 0 : X_t^{(2)} = 0\} \leq T \right) = 1.$$

(iv) If $a = d$ and $b = c$, then

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left(\sup_{t \leq T} \|tX_t - (ax_0^{(1)} + bx_0^{(2)})^{-1}x_0\|_2 \geq \varepsilon \right) = 0.$$

In the first case (i), the diffusion X and the dynamical system x come down from infinity in a single direction x_∞ , with speed proportional to $1/t$. They only need a short time at the beginning of the trajectory to find this direction. This short time quantified by η here could be made arbitrarily small when x_0 becomes large. Let us also observe that the one-dimensional logistic Feller diffusion X_t is given by $X_t^{(1)}$ for $c = d = 0$. Thus, taking x_0 collinear to x_∞ , (i) yields the speed of coming down from infinity of one-dimensional logistic Feller diffusions:

$$\lim_{T \rightarrow 0} \lim_{r \rightarrow \infty} \mathbb{P}_r \left(\sup_{t \leq T} |atX_t - 1| \geq \varepsilon \right) = 0. \quad (35)$$

In the second case (ii), the direction taken by the dynamical system and the process depends on the initial direction. The dynamical system then goes to the boundary of $(0, \infty)^2$ without reaching it. But the fluctuations of the process make it reach the boundary and one species die. When the process starts in the direction of x_∞ , additional work would be required to describe its behavior, linked to the behavior of the dynamical system around the associated unstable variety coming from infinity.

In the third case (iii), the dynamical system ϕ goes to the boundary $(0, \infty) \times \{0\}$ when coming down from infinity, wherever it comes from. Then, as above, the diffusion $X^{(2)}$ hits 0. Let us note that the dynamical system may then go to a coexistence fixed point or even to a fixed point where only the species 2 survive. This latter case occurs when

$$\tau_2/b > \tau_1/c, \quad \tau_2/d > \tau_1/a$$

and is illustrated in the third simulation below. Obviously, the symmetric situation happens when $b \leq c$ and $d < a$.

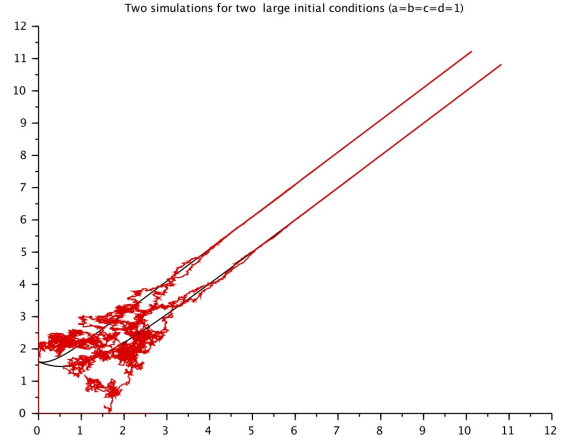
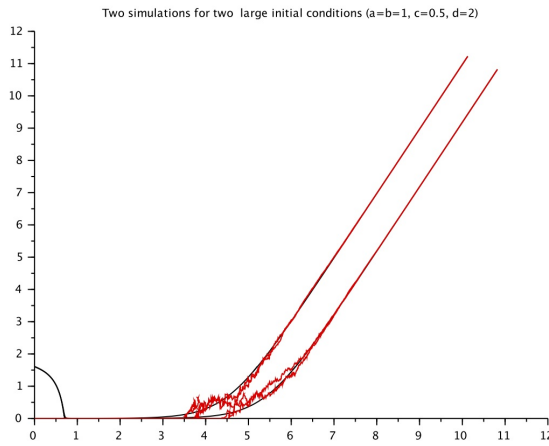
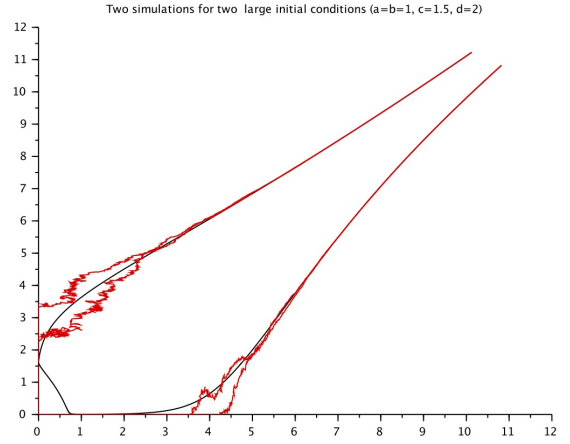
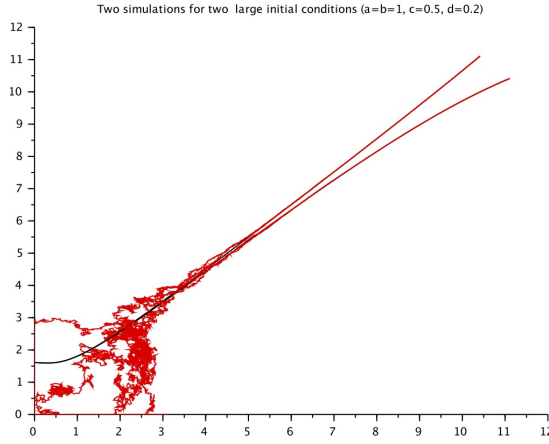
Moreover, in both cases (ii – iii), when the diffusion hits the boundary, it becomes a one-dimensional Feller logistic diffusion whose coming down infinity has been given above (35). In the case (iv), the process comes down from infinity in the direction of its initial value, at speed $1/t$.

Finally, let us note that this raises several questions on the characterization of a process starting from infinity in dimension 2. In particular, informally, the process coming down from infinity in a direction x_0 which is not x_∞ has a discontinuity at time 0 in the cases (i – ii – iii).

Simulations. We consider two large initial values x_0 such that $\|x_0\|_1 = 10^5$. We plot the dynamical system (in black line) and two diffusions (in red line) starting from these two initial values. In each simulation, $\tau_1 = 1$, $\tau_2 = 4$ and the solutions of the dynamical system converge to the fixed point where only the second species survive. The coefficient diffusion terms are $\sigma_1 = \sigma_2 = 10$. We plot here $G(x_t)$ and $G(X_t)$, where

$$G(x, y) = (X, Y) = (\log(1 + x), \log(1 + y))$$

to zoom on the behavior of the processes when coming close to one of the axes. The four regimes (i – ii – iii – iv) of the corollary above, which describe the ways the process can come down from infinity, are successively illustrated. One can also compare with the pictures of Section 5.3 describing the dynamical system.



5.2 Uniform scaling limits of competitive birth and death processes

Let us deal finally with competitive birth and death processes and consider their scaling limits to the Lotka-Volterra dynamical system ϕ given by (29). We provide here estimates which are uniform with respect to the initial values in a cone in the interior of $(0, \infty)^2$. These scaling limits are usual approximations in large populations of dynamical system by individual based model, see in particular Section 6 in [16]. The birth and death rates are given for population sizes $n_1, n_2 \geq 0$ and $K \geq 1$ by

$$\lambda_1^K(n_1, n_2) = \lambda_1 n_1, \quad \mu_1^K(n_1, n_2) = \mu_1 n_1 + a n_1 \cdot \frac{n_1}{K} + c n_1 \cdot \frac{n_2}{K}$$

for the first species and by

$$\lambda_2^K(n_1, n_2) = \lambda_2 n_2, \quad \mu_2^K(n_1, n_2) = \mu_2 n_2 + b n_2 \cdot \frac{n_2}{K} + d n_2 \cdot \frac{n_1}{K}$$

for the second species and we assume that

$$\lambda_1 - \mu_1 = \tau_1, \quad \lambda_2 - \mu_2 = \tau_2.$$

Dividing the number of individuals by K , the normalized population size X^K satisfies

$$X_t^K = x_0 + \int_0^t \int_{[0, \infty)} H^K(X_{s-}, z) N(ds, dz), \quad (36)$$

where writing $\tau_1^K = \lambda_1^K + \mu_1^K$ for convenience,

$$H^K(x, z) = \frac{1}{K} \left(\begin{array}{c} \mathbf{1}_{\{z \leq \lambda_1^K(Kx)\}} - \mathbf{1}_{\{\lambda_1^K(Kx) \leq z \leq \tau_1^K(Kx)\}} \\ \mathbf{1}_{\{0 \leq z - \tau_1^K(Kx) \leq \lambda_2^K(Kx)\}} - \mathbf{1}_{\{\lambda_2^K(Kx) \leq z - \tau_1^K(Kx) \leq \lambda_2^K(Kx) + \mu_2^K(Kx)\}} \end{array} \right). \quad (37)$$

and N is a PPM on $[0, \infty) \times [0, \infty)$ with intensity $dsdz$. We set

$$\mathcal{D}_\alpha = \{(x_1, x_2) \in (\alpha, \infty)^2 : x_1 \geq \alpha x_2, x_2 \geq \alpha x_1\},$$

which will be required both for the control of the flow and the the control of the fluctuations. We only consider here the case

$$(b > c > 0 \text{ and } a > d > 0) \quad \text{or} \quad (a, b > 0 \text{ and } c = d = 0) \quad \text{or} \quad (a = d > 0 \text{ and } b = c > 0) \quad (38)$$

since we know from the previous Corollary that it gives the cases when the flow does not go instantaneously to the boundary of $(0, \infty)^2$ in short time when coming from infinity and thus it does not exit from \mathcal{D}_α , which would prevent the uniformity below. One can also the cases $x_l = x_\infty$ and $x_l = \hat{x}_0$ in the forthcoming Lemma 5.7 (ii) and Figure 1.

Theorem 5.3. *For any $T > 0$, $\beta \in (0, 1/2)$ and $\alpha, \varepsilon > 0$, there exists $C > 0$ such that for any $K \geq 0$,*

$$\sup_{x_0 \in \mathcal{D}_\alpha} \mathbb{P}_{x_0} \left(\sup_{t \leq T} d_\beta(X_t^K, \phi(x_0, t)) \geq \varepsilon \right) \leq \frac{C}{K}.$$

The proof, which is given below, rely on (L, α_K) non-expansivity of the flow associated with X^K , with $\alpha_K \rightarrow 0$. Additional work should allow to make T go to infinity when K goes to infinity. The critical power $\beta = 1/2$ is reminiscent from results obtained for one dimensional logistic birth and death process in Proposition 4.7 in Section 4.2.2.

5.3 Non-expansivity of the flow and Poincaré compactification

The proofs of the three previous statements of this section rely on the following lemmas. The first one provides the domains where the transformation $F_{\beta, \gamma}$ yields a non-expansive vector field. It is achieved by determining the spectrum of the symmetrized operator of the Jacobian matrix of $\psi_{F_{\beta, \gamma}}$. This is the key ingredient to use the results of Section 3 for the study of the coming down from infinity of Lotka-Volterra diffusions (Theorem 5.1) and the proof of the scaling limits of birth and death processes (Theorem 5.3).

We also need to control the flow ϕ when it comes down from infinity. The next lemmas classify the different behaviors in terms of values of the coefficients of the dynamical system and provide some additional qualitative results useful for the proofs given in the next section. These proofs of these lemmas rely on the extension of the flow on the boundary at infinity given by Poincaré's technics, which is detailed for the flow considered here.

As one can see on spectral computations below, the domains where non-expansivity properties hold for $\psi_{F_{\beta, \gamma}}$ are cones. We recall that a cone is a subset \mathcal{C} of \mathbb{R}^2 such that for all

$x \in \mathcal{C}$ and $\lambda > 0$, $\lambda x \in \mathcal{C}$. We are using in particular the convex components of open cones, which are open convex cones. For S a subset of \mathbb{R}^2 , we denote by \bar{S} the closure of S . Recalling notations of Section 3, we have here $E = [0, \infty)^2$, $d = 2$ and

$$\psi_F = (J_F b) \circ F^{-1},$$

where

$$b(x) = b(x_1, x_2) = \begin{pmatrix} \tau_1 x_1 - a x_1^2 - c x_1 x_2 \\ \tau_2 x_2 - b x_2^2 - d x_1 x_2 \end{pmatrix}. \quad (39)$$

5.3.1 Non-expansivity in cones

Let us write $\bar{\tau} = \max(\tau_1, \tau_2)$ and

$$q_\beta = 4ab(1 + \beta)^2 + 4(\beta^2 - 1)cd$$

for convenience and consider the open cones of $(0, \infty)^2$,

$$D_{\beta, \gamma} = \left\{ x \in (0, \infty)^2 : 4\beta(1 + \beta)(a d x_1^2 + b c x_2^2) + q_\beta x_1 x_2 - \left(c \gamma^{-1} x_1^\beta x_2^{1-\beta} - d \gamma x_1^{1-\beta} x_2^\beta \right)^2 > 0 \right\}. \quad (40)$$

Lemma 5.4. *Let $\beta \in (0, 1]$ and $\gamma > 0$.*

The vector field $\psi_{F_{\beta, \gamma}}$ is $\bar{\tau}$ non-expansive on each convex component of the open cone $F_{\beta, \gamma}(D_{\beta, \gamma})$.

In the particular case $a, b > 0$ and $c = d = 0$, for any $\beta \in (0, 1]$ and $\gamma > 0$, $D_{\beta, \gamma} = (0, \infty)^2$. But this fact does hold in general. We need the transformations $F_{\beta, \gamma}$ for well chosen values of γ to get the non-expansivity property of the flow on unbounded domains, while $\beta < 1$ will be required to control the fluctuations. Let us finally note that $(0, \infty)^2$ may not be coverable by a single domain of the form $D_{\beta, \gamma}$ and the adjunction procedure of Section 3.2 will be needed.

Proof. We write for $y = (y_1, y_2) \in [0, \infty)^2$,

$$\psi_{F_{\beta, \gamma}}(y) = \psi_1(y) + \psi_{2, \beta, \gamma}(y), \quad (41)$$

where

$$\psi_1(y) = \begin{pmatrix} \beta \tau_1 y_1 \\ \beta \tau_2 y_2 \end{pmatrix}, \quad \psi_{2, \beta, \gamma}(y) = - \begin{pmatrix} \beta y_1 \left(a y_1^{1/\beta} + c \gamma^{-1/\beta} y_2^{1/\beta} \right) \\ \beta y_2 \left(b \gamma^{-1/\beta} y_2^{1/\beta} + d y_1^{1/\beta} \right) \end{pmatrix}.$$

First, ψ_1 is Lipschitz on $[0, \infty)^2$ with constant $\bar{\tau}$ since $\beta \in (0, 1]$. Moreover, writing $A_{\beta, \gamma}(x) = J_{\psi_{2, \beta, \gamma}}(F_{\beta, \gamma}(x))$, we have for any $x \in [0, \infty)^2$,

$$A_{\beta, \gamma}(x) + A_{\beta, \gamma}^*(x) = - \begin{pmatrix} 2a(1 + \beta)x_1 + 2c\beta x_2 & c\gamma^{-1} x_1^\beta x_2^{1-\beta} + d\gamma x_2^\beta x_1^{1-\beta} \\ c\gamma^{-1} x_1^\beta x_2^{1-\beta} + d\gamma x_2^\beta x_1^{1-\beta} & 2b(1 + \beta)x_2 + 2d\beta x_1 \end{pmatrix}.$$

This can be seen using (11) or by a direct computation. We consider now the trace and the determinant of this matrix :

$$T(x) = \text{Tr}(A_{\beta, \gamma}(x) + A_{\beta, \gamma}^*(x)), \quad D(x) = \det(A_{\beta, \gamma}(x) + A_{\beta, \gamma}^*(x)). \quad (42)$$

As $\beta > 0$ and $x \in (0, \infty)^2$, $T(x) < 0$, while

$$D(x) = (2a(1 + \beta)x_1 + 2c\beta x_2)(2b(1 + \beta)x_2 + 2d\beta x_1) - \left(c\gamma^{-1}x_1^\beta x_2^{1-\beta} + d\gamma x_2^\beta x_1^{1-\beta}\right)^2. \quad (43)$$

It is positive when $x = (x_1, x_2) \in D_{\beta, \gamma}$ and then the spectrum of $A_{\beta, \gamma}(x) + A_{\beta, \gamma}^*(x)$ is included in $(-\infty, 0]$. Recalling table 1 in [2] or the beginning of Section 2, this ensures that $\psi_{2, \beta, \gamma}$ is non-expansive on the open convex components of $F_{\beta, \gamma}(D_{\beta, \gamma})$. Then $\psi_{F_{\beta, \gamma}}$ is $\bar{\tau}$ non-expansive on the open convex components of $F_{\beta, \gamma}(D_{\beta, \gamma})$. Let us finally observe that $D_{\beta, \gamma}$ and thus $F_{\beta, \gamma}(D_{\beta, \gamma})$ are open cones, which ends up the proof of (i). \square

We define now

$$\mathcal{C}_{\eta, \beta, \gamma} = \left\{x \in (0, \infty)^2 : x_1/x_2 \in (0, \eta) \cup (x_{\beta, \gamma} - \eta, x_{\beta, \gamma} + \eta) \cup (1/\eta, \infty)\right\},$$

writing $x_{\beta, \gamma} = (d\gamma^2/c)^{1/(2\beta-1)}$ when it is well defined. The next result ensures that these domains provide a covering by non-expansive cones.

Lemma 5.5. *Let $\gamma > 0$, $\beta \in (0, 1) - \{1/2\}$, $c \neq 0$ and $d \neq 0$.*

(i) *There exists $\eta > 0$ such that $\mathcal{C}_{\eta, \beta, \gamma} \subset D_{\beta, \gamma}$.*

(ii) *There exist $\eta > 0$, $A > 0$ and $\mu > 0$ such that for any x, y which belong both to the complementary set of $B(0, A)$ and to a same convex component of the cone $\mathcal{C}_{\eta, \beta, \gamma}$, then*

$$(\psi_{F_{\beta, \gamma}}(x) - \psi_{F_{\beta, \gamma}}(y)) \cdot (x - y) \leq -\mu(\|x\|_2 \wedge \|y\|_2) \|x - y\|_2^2.$$

Proof. (i) The inclusion $\{x \in (0, \infty)^2 : x_1 = x_\gamma x_2\} \subset D_{\beta, \gamma}$ comes from the fact that

$$x_1 = (d\gamma^2/c)^{1/(2\beta-1)}x_2 \quad \text{implies that} \quad \left(c\gamma^{-1}x_1^\beta x_2^{1-\beta} - d\gamma x_1^{1-\beta} x_2^\beta\right)^2 = 0$$

and the fact that $q_\beta > 0$. The inclusion $\{x \in (0, \infty)^2 : x_1/x_2 \in (0, \eta) \cup (1/\eta, \infty)\} \subset D_{\beta, \gamma}$ is obtained by bounding

$$\left(c\gamma^{-1}x_1^\beta x_2^{1-\beta} - d\gamma x_1^{1-\beta} x_2^\beta\right)^2 \leq \left(c\gamma^{-1}\eta^{1-\beta} + d\gamma\eta^\beta\right)^2 x_1^2$$

when $x_2 \leq \eta x_1$. Indeed, in the case $c, d \neq 0$, we have $c, d > 0$ and $a > 0$ since $q_\beta > 0$ and letting η be small enough such that $4\beta(1 + \beta)ad > (c\gamma^{-1}\eta^{1-\beta} + d\gamma\eta^\beta)^2$ yields the result since $\beta \in (0, 1)$.

(ii) Recalling notation (42), for any $x \in [0, \infty)^2 - \{(0, 0)\}$, $T(x) < 0$ and the value of $D(x)$ is given by (43). Let $x_0 \neq 0$ such that $D(x_0) > 0$, then there exist $v_1, v_2 > 0$ and some open ball $\mathcal{V}(x_0)$ centered in x_0 , such that for any $x \in \mathcal{V}(x_0)$, we have $-v_1 \leq T(x) < 0$ and $D(x) \geq v_2$. So for any $\lambda > 0$ and $x \in \mathcal{V}(x_0)$,

$$T(\lambda x) = \lambda T(x) \in [-\lambda v_1, 0), \quad D(\lambda x) = \lambda^2 D(x) \in [\lambda^2 v_2, \infty).$$

Writing $E(\cdot)$ the minimal eigenvalue of $A_{\beta, \gamma}(\cdot) + A_{\beta, \gamma}^*(\cdot)$, we have for any $x \in \mathcal{V}$,

$$E(\lambda x) \leq \frac{D(\lambda x)}{T(\lambda x)} \leq -\lambda \frac{v_2}{v_1} < 0,$$

since D (resp. T) gives the product (resp. the sum) of the two eigenvalues. We obtain that there exists $\mu > 0$ such that for any x in the convex cone $\mathcal{C}(x_0)$ generated by \mathcal{V} , the spectrum of $A_{\beta,\gamma}(x) + A_{\beta,\gamma}^*(x)$ is included in $(-\infty, -2\mu \|x\|]$. We use now the decomposition (41) and

$$(\psi_{2,\beta,\gamma}(x) - \psi_{2,\beta,\gamma}(y)) \cdot (x - y) \leq -\mu \|x\|_2 \|x - y\|_2^2,$$

for any $x, y \in \mathcal{C}(x_0)$, see again Table 1 in [2] for details. Moreover ψ_1 is Lipschitz with constant $\bar{\tau}$ and we get

$$(\psi_{F_{\beta,\gamma}}(x) - \psi_{F_{\beta,\gamma}}(y)) \cdot (x - y) \leq (\bar{\tau} - \mu \|x\|_2) \|x - y\|_2^2.$$

We conclude by choosing $\eta > 0$ such that $\mathcal{C}_{\eta,\beta,\gamma} \subset \cup_{x_0 \in \{x_\gamma, (0,1), (1,0)\}} \mathcal{C}(x_0)$. \square

5.3.2 Poincaré's compactification and coming down from infinity

To describe the coming down from infinity of the flow ϕ , we use the following compactification K of $[0, \infty)^2$:

$$K(x) = K(x_1, x_2) = \left(\frac{x_1}{1 + x_1 + x_2}, \frac{x_2}{1 + x_1 + x_2}, \frac{1}{1 + x_1 + x_2} \right) = (y_1, y_2, y_3)$$

The application K is a bijection from $[0, \infty)^2$ into the simplex \mathcal{S} defined by

$$\mathcal{S} = \{y \in [0, 1]^2 \times (0, 1] : y_1 + y_2 + y_3 = 1\} \subset \bar{\mathcal{S}} = \{y \in [0, 1]^3 : y_1 + y_2 + y_3 = 1\}.$$

We note $\partial\mathcal{S}$ the outer boundary of \mathcal{S} :

$$\partial\mathcal{S} = \bar{\mathcal{S}} - \mathcal{S} = \{(y_1, 1 - y_1, 0) : y_1 \in [0, 1]\} = \left\{ \lim_{r \rightarrow \infty} K(rx) : x \in [0, \infty)^2 - \{(0, 0)\} \right\}.$$

The key point to describe the direction of the dynamical system ϕ coming from infinity is the following change of time. It allows to extend the flow on the boundary and is an example of Poincaré's compactification technics [15]. More precisely, we consider the flow Φ of the dynamical system on $\bar{\mathcal{S}}$ given for $z_0 \in \bar{\mathcal{S}}$ and $t \geq 0$ by

$$\Phi(z_0, 0) = z_0, \quad \frac{\partial}{\partial t} \Phi(z_0, t) = H(\Phi(z_0, t)), \quad (44)$$

where H is the Lipschitz function on $\bar{\mathcal{S}}$ defined by

$$\begin{aligned} H^1(y_1, y_2, y_3) &= y_1 y_2 [(b - c)y_2 + (d - a)y_1] + y_1 y_3 [(\tau_1 - \tau_2 - c)y_2 - a y_1 + y_3 \tau_1] \\ H^2(y_1, y_2, y_3) &= y_1 y_2 [(a - d)y_1 + (c - b)y_2] + y_2 y_3 [(\tau_2 - \tau_1 - d)y_1 - b y_2 + y_3 \tau_2] \\ H^3(y_1, y_2, y_3) &= y_3 (a y_1^2 + b y_2^2 + (c + d)y_1 y_2 - \tau_1 y_1 y_3 - \tau_2 y_2 y_3). \end{aligned} \quad (45)$$

The study of Φ close to $\partial\mathcal{S}$ is giving us the behavior of ϕ close to infinity using the change of time $\varphi \in \mathcal{C}^1([0, \infty)^2 \times [0, \infty), [0, \infty))$ defined by

$$\varphi(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \varphi(x_0, t) = 1 + \|\phi(x_0, t)\|_1.$$

Lemma 5.6. For any $x_0 \in [0, \infty)^2$ and $t \geq 0$,

$$K(\phi(x_0, t)) = \Phi(K(x_0), \varphi(x_0, t)).$$

Proof. We denote by $(y_t : t \geq 0)$ the image of the dynamical system $(x_t : t \geq 0)$ through K :

$$y_t = K(x_t) = K(\phi(x_0, t)).$$

Then

$$y'_t = G(x_t) = G \circ K^{-1}(y_t)$$

where

$$G^{(1)}(x_1, x_2) = \frac{(d-a)x_1^2 x_2 + (b-c)x_1 x_2^2 + (\tau_1 - \tau_2 - c)x_1 x_2 - ax_1^2 + \tau_1 x_1}{(1+x_1+x_2)^2}$$

and

$$G^{(2)}(x_1, x_2) = \frac{(c-b)x_2^2 x_1 + (a-d)x_2 x_1^2 + (\tau_2 - \tau_1 - d)x_2 x_1 - bx_2^2 + \tau_2 x_2}{(1+x_1+x_2)^2}$$

and

$$G^{(3)}(x_1, x_2) = \frac{ax_1^2 + bx_2^2 + (c+d)x_1 x_2 - \tau_1 x_1 - \tau_2 x_2}{(1+x_1+x_2)^2}.$$

Using that $x_1 = y_1/y_3$ and $x_2 = y_2/y_3$ and recalling the definition (45) of H , we have

$$G \circ K^{-1}(y) = \frac{1}{y_3} H(y)$$

for $y = (y_1, y_2, y_3) \in \mathcal{S}$. The key point now of the theory of Poincaré is that H can be extended continuously to $\bar{\mathcal{S}}$ and that the trajectories of the dynamical system $(z_t : t \geq 0)$ associated to the vector field H :

$$z'_t = H(z_t)$$

are the same than the trajectories of $(y_t : t \geq 0)$ whose vector field is $G \circ K^{-1}$. Indeed the positive scalar $1/y_3$ only change the norm of the vector field and thus the speed at which the same trajectory is covered. The associated change of time $v_t = \varphi(x_0, t)$ such that

$$z_{v_t} = y_t = K(x_t)$$

can now be simply computed :

$$v'_t = \frac{1}{K^{(3)}(\phi(x_0, t))} = 1 + \|\phi(x_0, t)\|_1,$$

which completes the proof. □

To describe the direction from which the flow ϕ comes down from infinity, we introduce the hitting times of cones :

$$t_-(x_0, x, \varepsilon) = \inf_{s \geq 0} \{(\widehat{x_s, x}) \in [-\varepsilon, +\varepsilon]\}, \quad t_+(x_0, x, \varepsilon) = \inf_{s \geq t_-(x_0, x, \varepsilon)} \{(\widehat{x_s, x}) \notin [-2\varepsilon, +2\varepsilon]\}, \quad (46)$$

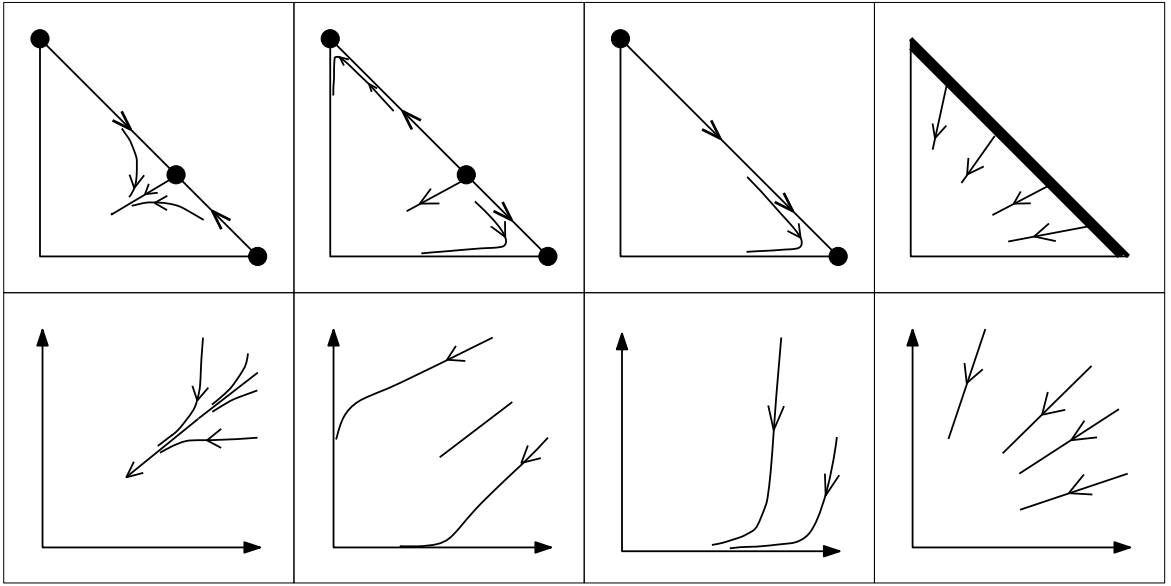
where we recall that $x_s = \phi(x_0, s)$ and $\inf \emptyset = \infty$. The directions x_l involved in the coming down from infinity are defined by

- $x_l = x_\infty$ if $b > c$ and $a > d$, where x_∞ has been defined in (34).
- $x_l = (1/a, 0)$ if $b > c$ and $a \leq d$; or if $c > b$ and $d > a$ and $(\widehat{x_0, x_\infty}) > 0$.

- $x_l = (0, 1/b)$ if $a > d$ and $b \leq c$; or if $c > b$ and $d > a$ and $(\widehat{x_0, x_\infty}) < 0$.
- $x_l = \widehat{x_0}$ if $a = d$ and $b = c$, where for any $x_0 \in (0, \infty)^2$, $\widehat{x_0} = x_0 / (ax_0^{(1)} + bx_0^{(2)})$.

We prove below that x_l gives the direction from which the dynamical system comes down from infinity using the previous compactification result. We can then specify the speed of coming down from infinity of the flow ϕ since the problem is reduced to the one dimension where computations can be easily lead.

Figure 1 : flow close to infinity. We draw the four regimes of the compactified flow Φ starting close or on the boundary $\partial\mathcal{S}$ and below the associated behavior of the original flow ϕ on $[0, \infty)^2$. The fixed points of the boundary are fat.



Lemma 5.7. (i) For any $T > 0$, there exists $c_T > 0$ such that $\|\phi(x_0, t)\|_1 \leq c_T/t$ for all $x_0 \in [0, \infty)^2$ and $t \in (0, T]$.

(ii) For all $x_0 \in (0, \infty)^2$ and $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} t_-(rx_0, x_l, \varepsilon) = 0, \quad \lim_{r \rightarrow \infty} t_+(rx_0, x_l, \varepsilon) > 0.$$

(iii) Moreover,

$$\lim_{t \rightarrow 0} \limsup_{r \rightarrow \infty} \left| \|t\phi(rx_0, t)\|_1 - \|x_l\|_1 \right| = 0.$$

Proof. (i) Using $a > 0$, we first observe that

$$(x_t^{(1)})' \leq -\frac{a}{2}(x_t^{(1)})^2$$

in the time intervals when $x_t^{(1)} \geq 2\tau_1/a$. Solving $(x_t^{(1)})' = -(x_t^{(1)})^2 a/2$ proves (i).

(ii) We use the notation (44) and (45) above and the dynamics of $z_t = \Phi(z_0, t)$ on the invariant set $\partial\mathcal{S}$ is simply given by the vector field $H(y_1, y_2, 0)$ for $y_1 \in [0, 1]$, $y_1 + y_2 = 1$:

$$H^{(1)}(y_1, y_2, 0) = -H^{(2)}(y_1, y_2, 0) = y_1 y_2 [(b-c)y_2 + (d-a)y_1].$$

The two points $(1, 0, 0)$ and $(0, 1, 0)$ on $\partial\mathcal{S}$ are invariant for the dynamical system $(z_t : t \geq 0)$. Let us first consider the case when $a \neq d$ or $b \neq c$. There is an additional invariant point in $\partial\mathcal{S}$ if and only if

$$(b - c)(a - d) > 0.$$

Thus, if $(b - c)(a - d) \leq 0$, $H^{-1}((0, 0, 0)) \cap \partial\mathcal{S} = \{(1, 0, 0), (0, 1, 0)\}$ and z_t starting from the boundary $\partial\mathcal{S}$ goes either to $(1, 0, 0)$ whatever its initial value z_0 ; or to $(0, 1, 0)$ whatever its initial value z_0 . Then by Lemma 5.6 the dynamical system $z_{\varphi(x_0, t)} = K(x_t)$ starting close to the boundary $\partial\mathcal{S}$ goes

- either to $(1, 0, 0)$; and then $(\widehat{x_t, x_l})$ becomes small, where $x_l = (1/a, 0)$.
- or to $(0, 1, 0)$; and then $(\widehat{x_t, x_l})$ becomes small, where $x_l = (0, 1/b)$.

The fact that $t_-(rx_0, x_l, \varepsilon) \rightarrow 0$ as r goes to infinity is then due to the fact that z goes to a neighborhood of $(1, 0, 0)$ or $(0, 1, 0)$ in a finite time and the fact that $\partial\varphi(x_0, t)/\partial t = 1 + \|rx_0\|_1 \rightarrow \infty$ as $r \rightarrow \infty$ makes this time arbitrarily small for $(x_t : t \geq 0)$. Moreover $t_+(rx_0, x_l, \varepsilon)$ is not becoming close to 0 as r goes to infinity since the speed of the dynamical system $(x_t : t \geq 0)$ is bounded on the compact sets of $[0, \infty)^2$.

Otherwise $(b - c)(a - d) > 0$ and

$$H^{-1}((0, 0, 0)) \cap \partial\mathcal{S} = \{(1, 0, 0), (0, 1, 0), z_\infty\},$$

where z_∞ is the unique invariant point in the interior of the boundary :

$$z_\infty = \frac{1}{b - c + a - d} (b - c, a - d, 0).$$

Then we need to see if z_∞ is repulsive or attractive on the invariant set $\partial\mathcal{S}$. In the case $c > b$ and $d > a$, this point is attractive and z_∞ is a saddle and

$$z_\infty = \lim_{r \rightarrow \infty} K(rx_\infty).$$

So Lemma 5.6 now ensures that the dynamical system x_t takes the direction $x_l = x_\infty$ when starting from a large initial value. Similarly, $t_-(rx_0, x_l, \varepsilon) \rightarrow 0$ and $t_+(rx_0, x_l, \varepsilon)$ does not.

In the case $b < c$ and $a < d$, y_∞ is a source and the dynamical system z_t either goes to $(1, 0, 0)$ (and then $x_l = (1/a, 0)$) or to $(0, 1, 0)$ (and then $x_l = (0, 1/b)$). This depends on the position of the initial value with respect to the second unstable variety and thus on the sign of $(\widehat{x_0, x_\infty})$.

Finally, the case $a = d, b = c$ is handled similarly noting that the whole set $\partial\mathcal{S}$ is invariant.

(iii) We know from (ii) that the direction of the dynamical system coming from infinity is x_l and we reduce now its dynamics close to infinity to a one-dimensional and solvable problem. Indeed, let us write

$$x_t(r) = \phi(rx_0, t)$$

and focus on the case $x_l^{(1)} \neq 0$. In that case, using again the compactification of Lemma 5.6 and the behavior in short time of $x_t(r)$ obtained in (ii), we obtain that $(x_t(r) : t \geq 0)$ does not come close from $\{0\} \times [0, \infty)$ on finite time intervals. More precisely, there exists $M_T > 0$ such

that $x_t^{(2)}(r) \leq M_T x_t^{(1)}(r)$ for $t \in [0, T]$ and $r \geq 1$. Using this bound in (29) provides a lower bound for $x_t^{(1)}(r)$ and we obtain for any $\varepsilon > 0$,

$$t_1(r, \varepsilon) := \inf \left\{ t \geq 0 : x_t^{(1)}(r) < |\tau_1|/\varepsilon \right\} \in (0, \infty], \quad t_1(\varepsilon) = \liminf_{r \rightarrow \infty} t_1(r, \varepsilon) \in (0, \infty].$$

Moreover, by definition (46), for any $\varepsilon > 0$ and $r > 0$, and $t \in [t_-(rx_0, x_l, \varepsilon), t_+(rx_0, x_l, \varepsilon)]$, then $(x_t(r), x_l) \leq 2\varepsilon$ and for ε small enough,

$$\left| \frac{x_t^{(2)}(r)}{x_t^{(1)}(r)} - \frac{x_l^{(2)}}{x_l^{(1)}} \right| \leq u(\varepsilon), \quad (47)$$

where $u(\varepsilon) \in [0, \infty)$ and $u(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Writing

$$\theta_l = \frac{x_l^{(2)}}{x_l^{(1)}} \quad t_-(r, \varepsilon) = t_-(rx_0, x_l, \varepsilon), \quad t_+(r, \varepsilon) = t_+(rx_0, x_l, \varepsilon) \wedge t_1(u(\varepsilon))$$

for convenience, plugging (47) in the first equation of (29) yields for any ε positive small enough and for any $t \in [t_-(r, \varepsilon), t_+(r, \varepsilon)]$

$$-(a + c\theta_l + (1 + c)u(\varepsilon)) \leq \frac{(x_t^{(1)}(r))'}{(x_t^{(1)}(r))^2} \leq -(a + c\theta_l - (1 + c)u(\varepsilon)).$$

We get by integration

$$\frac{1}{(a + c\theta_l + (1 + c)u(\varepsilon))(t - t_-(r)) + 1/x_{t_-(r)}^{(1)}(r)} \leq x_t^{(1)}(r) \leq \frac{1}{(a + c\theta_l - (1 + c)u(\varepsilon))(t - t_-(r)) + 1/x_{t_-(r)}^{(1)}(r)}.$$

Using (ii), $t_-(r, \varepsilon) \rightarrow 0$ and $t_+(\varepsilon) = \liminf_{r \rightarrow \infty} t_+(r, \varepsilon) > 0$ as $r \rightarrow \infty$. Moreover $x_l^{(1)} \neq 0$ ensures that $x_{t_-(r, \varepsilon)}^{(1)}(r) \rightarrow \infty$ as $r \rightarrow \infty$. Then for any ε positive small enough and $t \leq t_+(\varepsilon)$,

$$\frac{1}{a + c\theta_l + (1 + c)u(\varepsilon)} \leq \liminf_{r \rightarrow \infty} tx_t^{(1)}(r) \leq \limsup_{r \rightarrow \infty} tx_t^{(1)}(r) \leq \frac{1}{a + c\theta_l - (1 + c)u(\varepsilon)}.$$

Letting finally $\varepsilon \rightarrow 0$, we obtain

$$\lim_{t \rightarrow 0} \limsup_{r \rightarrow \infty} |tx_t^{(1)}(r) - 1/(a + c\theta_l)| = 0.$$

Using again (47) provides the counterpart for $tx_t^{(2)}$ and ends the proof in the case $x_l^{(1)} \neq 0$. The case $x_l^{(2)} \neq 0$ is treated similarly. \square

5.3.3 Adjunction of open convex cones

Finally, we need the following additional results on the dynamical system coming from infinity. It allows to decompose the trajectory in $\mathcal{D}_\alpha = (\alpha, \infty)^2$ into time intervals when it belongs to a subdomain for which we know a transformation giving a non-expansive flow. Recall from (33) that $T_D(x_0)$ is the exit time of D for the flow started from x_0 and the distance $d_\beta(x, y) = \|F_{\beta,1}(x) - F_{\beta,1}(y)\|_2$ from (32) and the definition of x_l in the middle of Section 5.3.2.

Lemma 5.8. (i) Let $\alpha > 0$, $\beta \in (0, 1)$, $N \in \mathbb{N}$ and $(C_i)_{i=1, \dots, N}$ be a family of open convex cones of $(0, \infty)^2$ such that

$$(0, \infty)^2 = \cup_{i=1}^N C_i.$$

Then, there exists $\kappa \in \mathbb{N}$ and $(t_k(x_0) : k = 0, \dots, \kappa)$ and $(n_k(x_0) : k = 1, \dots, \kappa - 1)$ such that for any $x_0 \in \mathcal{D}_\alpha$,

$$0 = t_0(x_0) \leq t_1(x_0) \leq \dots \leq t_\kappa(x_0) = T_{\mathcal{D}_\alpha}(x_0), \quad n_k(x_0) \in \{1, \dots, N\}$$

and for ε small enough, $k \leq \kappa - 1$ and $t \in [t_k(x_0), t_{k+1}(x_0))$, we have

$$\overline{B}_{d_\beta}(\phi(x_0, t), \varepsilon) \subset C_{n_k(x_0)}.$$

(ii) In the case, $x_1 = x_\infty \in (0, \infty)^2$, for any $x_0 \in (0, \infty)^2$ and $\varepsilon > 0$,

$$\liminf_{r \rightarrow \infty} T_{\mathcal{D}_\varepsilon}(rx_0) > 0.$$

(iii) In the case $x_1 = (1/a, 0)$, for any $x_0 \in (0, \infty)^2$ and $\varepsilon > 0$ and $T > 0$, for r large enough,

$$T_{\mathcal{D}_\varepsilon}(rx_0) = \inf\{t \geq 0 : \phi(rx_0, t) \in [0, \infty) \times [0, \varepsilon]\} \leq T.$$

Proof. (i) We define

$$C_i^\varepsilon = \{x \in \mathcal{D}_\alpha \cap C_i : \overline{B}_{d_\beta}(x, \varepsilon) \subset C_i\}$$

and we first observe that for ε small enough,

$$\cup_{i=1}^N C_i^{2\varepsilon} = \mathcal{D}_\alpha,$$

since the open convex cones C_i are domains between two half-lines of $(0, \infty)^2$ and their collection for $i = 1, \dots, N$ covers $(0, \infty)^2$. We define

$$u_0^i(x_0) = \inf\{t \geq 0 : \phi(x_0, t) \in C_i^{2\varepsilon}\}, \quad v_0^i(x_0) = \inf\{t \geq u_0^i(x_0) : \phi(x_0, t) \notin C_i^\varepsilon\}$$

and by recurrence for $k \geq 1$,

$$u_k^i(x_0) = \inf\{t \geq v_{k-1}^i(x_0) : \phi(x_0, t) \in C_i^{2\varepsilon}\}, \quad v_k^i(x_0) = \inf\{t \geq u_k^i(x_0) : \phi(x_0, t) \notin C_i^\varepsilon\}.$$

Let us then note that

$$\overline{\mathcal{S}} = \cup_{i=1}^N \overline{K(C_i)}, \quad \partial\mathcal{S} = \cup_{i=1}^N \partial K(\overline{C_i}),$$

where

$$\partial K(\overline{C_i}) = \overline{K(C_i)} - K(\overline{C_i}) = \{(t, 1-t, 0) : t \in [a_i, b_i]\}$$

for some $0 \leq a_i \leq b_i \leq 1$. Recall that $z_t = \Phi(z_0, t)$ has been introduced in (44) and is defined on $\overline{\mathcal{S}}$. On the boundary $\partial\mathcal{S}$, it is given by $(z_t^{(1)}, 1 - z_t^{(1)}, 0)$ where $z_t^{(1)}$ is monotone. Outside this boundary, $(z_t : t \geq 0)$ goes to a fixed point since the competitive Lotka-Volterra dynamical system $(x_t : t \geq 0)$ does. This ensures that

$$M^i(x_0) = \max\{k : v_k^i(x_0) < \infty\}$$

is bounded for $x_0 \in \mathcal{D}_\alpha$ and $i \in \{1, \dots, N\}$. This yields the result using the time intervals $[u_k^i(x_0), v_k^i(x_0)]$ which provides a covering $[0, T_{\mathcal{D}_\alpha}(x_0)]$ for $i = 1, \dots, N$ whose cardinal is bounded

with respect to $x_0 \in \mathcal{D}_\alpha$.

(ii) comes simply from Lemma 5.6 which ensures that in the case $x_l = x_\infty$, the dynamical system comes down from infinity in the interior of $(0, \infty)^2$, see also the first picture in Figure 1 above.

(iii) We use again the dynamical system $(z_t : t \geq 0)$ given by Φ and defined in (44). More precisely, the property here comes from the continuity of the associated flow with respect to the initial condition. Indeed, in the case $x_l = (1/a, 0)$, the trajectories of $(z_t : t \geq 0)$ starting from r large go close to $(1, 0, 0)$ and then remain close to $\{(t, 0, 1-t) : t \in [0, 1]\}$ until getting close to the fixed point on this boundary. This ensures that $(x_t : t \geq 0)$ goes close to the boundary $(0, \infty) \times \{0\}$ and exits from \mathcal{D}_ε through $(0, \infty) \times \{\varepsilon\}$. The fact that this exit time $T_{\mathcal{D}_\varepsilon}(rx_0)$ goes to zero as $r \rightarrow \infty$ is due to the fact that the dynamics of $(x_t : t \geq 0)$ is an acceleration of that of $(z_t : t \geq 0)$ when starting close to infinity, with time change $1 + \|\phi(x_0, t)\|_1$. \square

5.4 Proofs of Theorem 5.1 and Corollary 5.2 and Theorem 5.3

We can now prove the Theorem 5.1 for the diffusion X defined by (31) using the results of Section 3. Here $E = [0, \infty)^2$, $d = 2$, $q = 0$ (or $H = G = 0$), $\sigma_j^{(i)} = 0$ if $j \neq i$ and

$$\sigma_1^{(1)}(x) = \sigma_1 \sqrt{x_1}, \quad \sigma_2^{(2)}(x) = \sigma_2 \sqrt{x_2}.$$

Moreover $b_{F_{\beta, \gamma}} = b$ is given by (39), $\psi_{F_{\beta, \gamma}} = (J_{F_{\beta, \gamma}} b_{F_{\beta, \gamma}}) \circ F_{\beta, \gamma}^{-1}$ and

$$\tilde{b}_{F_{\beta, \gamma}}(x) = \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2 F_{\beta, \gamma}}{\partial^2 x_i}(x) \sigma_i^{(i)}(x)^2 = \frac{1}{2} \beta(\beta-1) \begin{pmatrix} \sigma_1^2 x_1^{\beta-1} \\ \gamma \sigma_2^2 x_2^{\beta-1} \end{pmatrix} \quad (48)$$

and

$$V_{F_{\beta, \gamma}}(x) = \sum_{i=1}^2 \left(\frac{\partial F_{\beta, \gamma}}{\partial x_i}(x) \sigma_i^{(i)}(x) \right)^2 = \beta^2 \begin{pmatrix} \sigma_1^2 x_1^{2\beta-1} \\ (\gamma \sigma_2)^2 x_2^{2\beta-1} \end{pmatrix}. \quad (49)$$

Proof of Theorem 5.1. To use Theorem 3.5, we first find a suitable covering of $(0, \infty)^2$ by a finite number of convex open sets $(C_i : i = 1, \dots, N)$ for which a non-expansivity transformation exists. For that purpose, let us deal with the case $c \neq 0$ (and then $d \neq 0$), while we recall from Lemma 5.4 that the case $c = d = 0$ is obvious.

Let $\beta \in (1/2, 1)$ close enough to 1 so that $q_\beta = 4ab(1+\beta)^2 + 4cd(\beta^2-1) > 0$. By Lemma 5.5 (i), for any $\gamma > 0$, there exists $\eta = \eta(\beta, \gamma) > 0$ such that $\mathcal{C}_{\eta, \beta, \gamma} \subset D_{\beta, \gamma}$. The collection of the convex components of this family of open convex cones $(\mathcal{C}_{\eta, \beta, \gamma} : \gamma > 0)$ provides a covering of $(0, \infty)^2$, since it contains the half line $\{x_1 = x_2 x_\gamma\}$ and $\{x_\gamma : \gamma > 0\} = (0, \infty)$. We underline that this collection contains also non empty cones $\{(x_1, x_2) \in (0, \infty)^2 : x_1 < \eta x_2\}$ and $\{(x_1, x_2) \in (0, \infty)^2 : x_2 < \eta x_1\}$ for some $\eta > 0$. Then, by a compactness argument, we can extract a finite family of open convex cones from this collection which covers $(0, \infty)^2$, which we denote by $(C_i^\beta : i = 1, \dots, N)$. Thus, for each i , there exists $\gamma_i > 0$ such that

$$C_i^\beta \subset C_{\eta_i, \beta, \gamma_i} \subset D_{\beta, \gamma_i} \quad (0, \infty)^2 = \cup_{i=1}^N C_i^\beta,$$

where $\eta_i = \eta(\beta, \gamma_i)$. Writing $F_i = F_{\beta, \gamma_i}$ for convenience, Lemma 5.4 ensures that the vector field ψ_{F_i} is $\bar{\tau}$ non-expansive on $F_i(C_i^\beta)$.

We let now $\alpha > 0$ and we use Lemma 5.8 (i) for the covering $(C_i^\beta : i = 1, \dots, N)$ of $(0, \infty)^2$. Then there exist $\kappa \in \mathbb{N}$ and $(t_k(x_0) : k = 0, \dots, \kappa)$ and $(n_k(x_0) : k = 1, \dots, \kappa - 1)$ such that for any $x_0 \in \mathcal{D}_\alpha$,

$$0 = t_0(x_0) \leq t_1(x_0) \leq \dots \leq t_\kappa(x_0) = T_{\mathcal{D}_\alpha}(x_0), \quad n_k(x_0) \in \{1, \dots, N\}$$

and for ε small enough, $k \leq \kappa - 1$ and $t \in [t_k(x_0), t_{k+1}(x_0))$, we have

$$\bar{B}_{d_\beta}(\phi(x_0, t), \varepsilon) \subset C_{n_k(x_0)}^\beta.$$

We consider now the open convex sets

$$D_i^{\beta, \alpha} = C_i^\beta \cap \mathcal{D}_{\alpha/2}.$$

Then for ε small enough, for any $x_0 \in \mathcal{D}_\alpha$ and $t \leq T_{\mathcal{D}_\alpha}(x_0)$, we have $\phi(x_0, t) \in \mathcal{D}_\alpha$ and

$$\bar{B}_{d_\beta}(\phi(x_0, t), \varepsilon) \subset D_{n_k(x_0)}^{\beta, \alpha}, \quad \mathcal{D}_\alpha \subset \bigcup_{i=1}^N D_i^{\beta, \alpha}.$$

Assumptions 3.3 and 3.4 are thus checked with $D = \mathcal{D}_\alpha$, $D_i = D_i^{\beta, \alpha}$, $O_i = D_i^{\beta, \alpha/2}$ ($i = 1, \dots, N$), $d = d_\beta$ and ϕ defined by (29). Moreover ψ_{F_i} is $\bar{\tau}$ non-expansive on $F_i(D_i)$. We can now apply Theorem 3.5 and get

$$\mathbb{P}_{x_0} \left(\sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d(X_t, \phi(x_0, t)) \geq \varepsilon \right) \leq C \sum_{k=0}^{\kappa-1} \int_{t_k(x_0) \wedge T}^{t_{k+1}(x_0) \wedge T} \bar{V}_{d_\beta, \varepsilon}(F_{n_k(x_0)}, x_0, t) dt$$

for some positive constant C . We need now to control \bar{V} . First we use (48) to see that \tilde{b}_{F_i} is bounded on $\mathcal{D}_{\alpha/2}$. Then for ε small enough,

$$c'_i(\varepsilon) := \sup_{\substack{x_0 \in \mathcal{D}_\alpha, t \leq T_{\mathcal{D}_\alpha}(x_0) \\ d_\beta(x, \phi(x_0, t)) \leq \varepsilon}} \|\tilde{b}_{F_i}(x)\|_1 < \infty.$$

Moreover combining Lemma 5.7 (i) and (49), there exists $c''_i(\varepsilon) > 0$ such that for any $x_0 \in \mathcal{D}_\alpha$ and $t \leq T_{\mathcal{D}_\alpha}(x_0)$,

$$\bar{V}_{d, \varepsilon}(F_i, x_0, t) = \sup_{\substack{x \in [0, \infty)^2 \\ d_\beta(x, \phi(x_0, t)) \leq \varepsilon}} \left\{ \varepsilon^{-2} \|V_{F_i}(x)\|_1 + \varepsilon^{-1} \|\tilde{b}_{F_i}(x)\|_1 \right\} \leq \varepsilon^{-2} \frac{c''_i(\varepsilon)}{t^{2\beta-1}} + \varepsilon^{-1} c'_i(\varepsilon).$$

Adding that $\int_0^\infty \left(\varepsilon^{-2} \frac{c''_i(\varepsilon)}{t^{2\beta-1}} + \varepsilon^{-1} c'_i(\varepsilon) \right) dt < \infty$ for $\beta < 1$, we get

$$\limsup_{T \downarrow 0} \sup_{x_0 \in \mathcal{D}_\alpha} \mathbb{P}_{x_0} \left(\sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d_\beta(X_t, \phi(x_0, t)) \geq \varepsilon \right) = 0$$

for ε small enough. This ends up the proof for $\beta < 1$ close enough to 1, which is enough to deal with \mathcal{D}_α . \square

We can now describe the coming down from infinity of the two-dimensional competitive Lotka-Volterra diffusion X .

Proof of Corollary 5.2. Let us deal with (i), so $x_l = x_\infty \in (0, \infty)^2$ and we fix $x_0 \in (0, \infty)^2$. Recalling (33) and Lemma 5.8 (ii),

$$\liminf_{r \rightarrow \infty} T_{\mathcal{D}_\varepsilon}(rx_0) > 0$$

and we use Theorem 5.1 to get for any $\beta \in (0, 1)$,

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left(\sup_{t \leq T} d_\beta(X_t, x_t(r)) \geq \varepsilon \right) = 0,$$

where we write again $x_t(r) = \phi(rx_0, t)$ for convenience. Then, using that $d_\beta(tx, ty) = t^\beta d_\beta(x, y)$,

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{xr_0} \left(\sup_{t \leq T} d_\beta(tX_t, tx_t(r)) \geq \varepsilon \right) = 0. \quad (50)$$

Fix $\eta \in (0, 1)$. Combining Lemma 5.7 (ii) and (iii) and

$$\|tx_t(r) - x_\infty\|_2 \leq \left| \|tx_t(r)\|_1 - \|x_\infty\|_1 \right| + \min(\|tx_t(r)\|_2, \|x_\infty\|_2) |\sin(\widehat{x_t, x_\infty})|$$

ensures that

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\eta T \leq t \leq T} \|tx_t(r) - x_\infty\|_2 = 0.$$

Using the two last limits displayed and $\|tX_t - x_\infty\|_2 \leq \|tX_t - tx_t\|_2 + \|tx_t - x_\infty\|_2$, we get

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left(\sup_{\eta T \leq t \leq T} \|tX_t - x_\infty\|_2 \geq \alpha \right) = 0,$$

for any $\alpha > 0$, since the euclidean distance is uniformly continuous from the bounded sets of $[0, \infty)^2$ endowed with d_β to \mathbb{R}^+ endowed with the absolute value. This proves the first part of (i). The second part of (i) (resp. the proof of (iv)) is obtained similarly just by noting that $t_-(rx_0, x_\infty, \varepsilon) = 0$ (resp. $t_-(rx_0, \widehat{x_0}, \varepsilon) = 0$) if x_0 is collinear to x_∞ .

For the cases (ii – iii), we know from Lemma 5.7 that the dynamical system is going to the boundary of $(0, \infty)^2$ in short time. Let us deal with the case

$$x_l = (1/a, 0)$$

and the case $x_l = (0, 1/b)$ would be handled similarly. We fix $x_0 \in (0, \infty)^2$, $T_0 > 0$, $\varepsilon \in (0, 1]$, $\eta > 0$ and $\beta \in (0, 1)$. By Theorem 5.1, there exists $T \leq T_0$ such that for r large enough

$$\mathbb{P}_{rx_0} \left(\sup_{t \leq T \wedge T_{\mathcal{D}_\varepsilon}(rx_0)} d_\beta(X_t, \phi(rx_0, t)) \geq \varepsilon \right) \leq \eta.$$

By Lemma 5.8 (iii), for r large enough, we have $T_{\mathcal{D}_\varepsilon}(rx_0) = \inf\{t \geq 0 : \phi^{(2)}(rx_0, t) \leq \varepsilon\} \leq T$. Thus,

$$\mathbb{P}_{rx_0} \left(d_\beta(X_{T_{\mathcal{D}_\varepsilon}(rx_0)}, \phi(rx_0, T_{\mathcal{D}_\varepsilon}(rx_0))) \geq \varepsilon \right) \leq \eta \quad \text{and} \quad \phi^{(2)}(rx_0, T_{\mathcal{D}_\varepsilon}(rx_0)) = \varepsilon.$$

Fix now $c \geq 1$ such that $c^\beta \geq 2$. We get

$$\begin{aligned} \mathbb{P}_{rx_0} \left(X_{T_{D_\varepsilon}(rx_0)}^{(2)} \geq c\varepsilon \right) &= \mathbb{P}_{rx_0} \left(\left(X_{T_{D_\varepsilon}(rx_0)}^{(2)} \right)^\beta - \varepsilon^\beta \geq (c^\beta - 1)\varepsilon^\beta \right) \\ &\leq \mathbb{P}_{rx_0} \left(d_\beta(X_{T_{D_\varepsilon}(rx_0)}, \phi(rx_0, T_{D_\varepsilon}(rx_0))) \geq \varepsilon \right) \leq \eta, \end{aligned}$$

since $\varepsilon^\beta \geq \varepsilon$. By Markov property and the fact that the boundaries of $[0, \infty)^2$ are absorbing, we obtain for r large enough

$$\begin{aligned} \mathbb{P}_{rx_0} \left(X_{2T_0}^{(2)} = 0 \right) &\geq \mathbb{P} \left(X_{T_{D_\varepsilon}(rx_0)}^{(2)} \leq c\varepsilon, \exists t \in [T_{D_\varepsilon}(rx_0), T_{D_\varepsilon}(rx_0) + T_0] : X_t^{(2)} = 0 \right) \\ &\geq (1 - \eta)p(c\varepsilon), \end{aligned}$$

where

$$p(x) = \mathbb{P}_x \left(X_{T_0}^{(2)} = 0 \right).$$

Moreover $X^{(2)}$ is stochastically smaller than a one-dimensional Feller diffusion and $\sigma_2 \neq 0$, so $\lim_{x \downarrow 0+} p(x) = 1$. Letting $\varepsilon \rightarrow 0$ in the previous inequality yields

$$\liminf_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left(X_{2T_0}^{(2)} = 0 \right) \geq 1 - \eta.$$

Letting $\eta \rightarrow 0$ ends up the proof of (ii – iii). \square

We finally prove the scaling limits stated in Theorem 5.3. Here we use the notations of Sections 3 for $X^K = (X^{K,i} : i = 1, \dots, d)$ and we have $\chi = [0, \infty)$, $q(dz) = dz$ and

$$h_F^K(x) = \int_0^\infty [F(x + H^K(x, z)) - F(x)] dz.$$

Recalling that $F_{\beta, \gamma}$ is defined in (30) and H^K defined in (37), we have

$$h_{F_{\beta, \gamma}}^K(x) = \left(\begin{array}{l} \lambda_1 K x_1 \left((x_1 + 1/K)^\beta - x_1^\beta \right) + K x_1 (\mu_1 + a x_1 + c x_2) \left((x_1 - 1/K)^\beta - x_1^\beta \right) \\ \gamma \lambda_2 K x_2 \left((x_2 + 1/K)^\beta - x_2^\beta \right) + \gamma K x_2 (\mu_2 + b x_2 + d x_1) \left((x_2 - 1/K)^\beta - x_2^\beta \right) \end{array} \right). \quad (51)$$

We consider

$$\psi_{F_{\beta, \gamma}}^K = h_{F_{\beta, \gamma}}^K \circ F_{\beta, \gamma}^{-1}, \quad b_{F_{\beta, \gamma}}^K = J_{F_{\beta, \gamma}}^{-1} h_{F_{\beta, \gamma}}^K$$

and to compare these quantities, we introduce

$$\Delta_{\beta, \gamma}^K(x) = \frac{\beta(\beta - 1)}{2K} \left(\begin{array}{l} (a x_1 + c x_2) x_1^{\beta-1} \\ \gamma (b x_2 + d x_1) x_2^{\beta-1} \end{array} \right).$$

We finally recall that $\mathcal{D}_\alpha = \{(x_1, x_2) \in (\alpha, \infty)^2 : x_1 \geq \alpha x_2, x_2 \geq \alpha x_1\}$ and

$$b(x) = \left(\begin{array}{l} \tau_1 x_1 - a x_1^2 - c x_1 x_2 \\ \tau_2 x_2 - b x_2^2 - d x_1 x_2 \end{array} \right), \quad J_{F_{\beta, \gamma}}(x) = \left(\begin{array}{cc} \beta x_1^{\beta-1} & 0 \\ 0 & \gamma \beta x_2^{\beta-1} \end{array} \right), \quad \psi_{F_{\beta, \gamma}} = (J_{F_{\beta, \gamma}} b) \circ F_{\beta, \gamma}^{-1},$$

to prove the following result.

Lemma 5.9. For any $\alpha > 0$ and $\beta \in (0, 1)$ and $\gamma > 0$, there exists $C > 0$ such that for any $x \in \mathcal{D}_\alpha$ and $K \geq 2/\alpha$,

(i)

$$\|h_{F_{\beta,\gamma}}^K(x) - J_{F_{\beta,\gamma}}(x)b(x) - \Delta_{\beta,\gamma}^K(x)\|_2 \leq \frac{C}{K} \|x\|_2^{\beta-1}.$$

(ii)

$$\|b_{F_{\beta,\gamma}}^K(x) - b(x)\|_2 \leq \frac{C}{K} \|x\|_2.$$

(iii)

$$\psi_{F_{\beta,\gamma}}^K(x) = \psi_{F_{\beta,\gamma}}(x) + \Delta_{\beta,\gamma}^K(F_{\beta,\gamma}^{-1}(x)) + R_{\beta,\gamma}^K(F_{\beta,\gamma}^{-1}(x)),$$

where $\|R_{\beta,\gamma}^K(x)\|_2 \leq C/K$.

(iv) Moreover $\psi_{F_{\beta,\gamma}}^K$ is $(C, C/K)$ non-expansive on $F_{\beta,\gamma}(D_{\beta,\gamma} \cap \mathcal{D}_\alpha)$, where we recall that $D_{\beta,\gamma}$ is defined in (40).

(v) Finally,

$$\|\psi_{F_{\beta,\gamma}}^K(x) - \psi_{F_{\beta,\gamma}}(x)\|_2 \leq C \frac{1 + \|x\|}{K}$$

Proof. First, by Taylor-Lagrange formula applied to $(1+h)^\beta$, there exists $c_0 > 0$ such that

$$\left| \left(z + \frac{\delta}{K} \right)^\beta - z^\beta - \frac{\delta}{K} \beta z^{\beta-1} - \frac{\delta^2}{2K^2} \beta(\beta-1) z^{\beta-2} \right| \leq \frac{c_0}{K^2} z^{\beta-3}$$

for any $z > \alpha$ and $K \geq 2/\alpha$ and $\delta \in \{-1, 1\}$, since $h = \delta/(Kz) \in (-1/2, 1/2)$. Using then (51) and

$$J_{F_{\beta,\gamma}}(x)b(x) = \begin{pmatrix} \beta x_1^{\beta-1} x_1(\tau_1 - ax_1 - cx_2) \\ \gamma \beta x_2^{\beta-1} x_2(\tau_2 - bx_2 - dx_1) \end{pmatrix}$$

yields (i), since $\|x\|_2$, x_1 and x_2 are equivalent up to a positive constant when $x \in \mathcal{D}_\alpha$. We immediately get (iii) since $\|x\|_2^{1-\beta}$ is bounded on $[\alpha, \infty)^2$ when $\beta \leq 1$.

Then (i) and the fact that there exists $c_0 > 0$ such that for any $x \in \mathcal{D}_\alpha$ and $u \in [0, \infty)^2$,

$$\|(J_{F_{\beta,\gamma}}(x))^{-1} \Delta_{\beta,\gamma}^K(x)\|_2 \leq c_0 \frac{\|x\|_2}{K}, \quad \|(J_{F_{\beta,\gamma}}(x))^{-1} u\|_2 \leq c_0 \|x\|_2^{1-\beta} \|u\|_2$$

proves (ii).

We observe that $\Delta_{\beta,\gamma}^K \circ F_{\beta,\gamma}^{-1}$ is uniformly Lipschitz on $F_{\beta,\gamma}(\mathcal{D}_\alpha)$ with constant L since its partial derivative are bounded on this domain. Recalling then from Lemma 5.4 (i) that $\psi_{F_{\beta,\gamma}}$ is $\bar{\tau}$ non expansive on $F_{\beta,\gamma}(D_{\beta,\gamma})$, the decomposition (iii) ensures that $\psi_{F_{\beta,\gamma}}^K$ is $(\bar{\tau} + L, C/K)$ non-expansive on $F_{\beta,\gamma}(D_{\beta,\gamma} \cap \mathcal{D}_\alpha)$. So (iv) holds.

Finally, using (iii) and adding that

$$\sup_{x \in \mathcal{D}_\alpha} \frac{\|\Delta_{\beta,\gamma}^K(F_{\beta,\gamma}^{-1}(x))\|_2}{K \|x\|_2} < \infty$$

proves (v) and ends up the proof. \square

Proof of Theorem 5.3. We first observe that assumption (5.4) ensures that $q_\beta = 4ab(1 + \beta)^2 + 4cd(\beta^2 - 1) > 0$. Using Lemma 5.5 (ii) and recalling that $\cup_{\gamma > 0} \mathcal{C}_{\eta(\beta, \gamma), \beta, \gamma} = (0, \infty)^2$, we can find finite collections of open convex cones $(D_i : i = 1, \dots, N)$ positive real numbers $(\gamma_i : i = 1, \dots, N)$ and $(\mu_i : i = 1, \dots, N)$ and $A > 0$ such that

$$\cup_{i=1}^N D_i = (0, \infty)^2$$

and, writing $F_i = F_{\beta, \gamma_i}$,

$$(\psi_{F_i}(x) - \psi_{F_i}(y)) \cdot (x - y) \leq -\mu_i (\|x\|_2 \wedge \|y\|_2) \|x - y\|_2^2, \quad (52)$$

for any $x, y \in D_i$ such that $\|x\|_2 \geq A$ and $\|y\|_2 \geq A$.

Combining this inequality with Lemma 5.9 (v) and observing that $\|x\|_2 \wedge \|y\|_2 \geq \|x\|_2 (1 - \varepsilon/A)$ when $y \in B(x, \varepsilon)$ and $\|x\|_2 \geq A$, the assumptions of Lemma 6.4 in Appendix are met. It ensures that the flow $\tilde{\phi}_i^K$ defined by

$$\tilde{\phi}_i^K(y_0, t) = y_0, \quad \frac{\partial}{\partial t} \tilde{\phi}_i^K(y_0, t) = \psi_{F_i}^K(\tilde{\phi}_i^K(y_0, t))$$

is close to the flow $\tilde{\phi}_i$ associated to ψ_{F_i}

$$\tilde{\phi}_i(y_0, t) = y_0, \quad \frac{\partial}{\partial t} \tilde{\phi}_i(y_0, t) = \psi_{F_i}(\tilde{\phi}_i(y_0, t)).$$

More precisely, letting $T > 0$, $\|\tilde{\phi}_i^K(y_0, t) - \tilde{\phi}_i(y_0, t)\|_2 \rightarrow 0$ as $K \rightarrow \infty$, uniformly for $y_0 \in F_i(D_i)$ and $t \in [0, T_i(y_0))$, where $T_i(y_0)$ is the maximal time before T when the flow $\tilde{\phi}_i(y_0, t)$ remains in $F_i(D_i)$. Then the flow $\phi_i^K(x_0, t) = F_i^{-1}(\tilde{\phi}_i^K(x_0, t))$ is uniformly close to the flow $\phi_i(x_0, t) = F_i^{-1}(\tilde{\phi}_i(x_0, t))$ for $x_0 \in D_i$ and $t \in [0, T_i(F_i(x_0))$ for the distance d_{F_i} and thus for the distance d_β . Moreover

$$\frac{\partial}{\partial t} \phi_i^K(x_0, t) = b_{F_i}^K(\phi_i^K(x_0, t)), \quad \frac{\partial}{\partial t} \phi_i(x_0, t) = b(\phi_i(x_0, t)),$$

since $b_{F_i} = b$ and $\phi_i = \phi$.

We note that $\mathcal{D}_\alpha \subset D_\alpha$ and using Lemma 5.8 (i), we introduce $\kappa \in \mathbb{N}$, $\varepsilon_0 > 0$ and the sequences $(t_k(x_0) : k = 0, \dots, \kappa)$ and $(n_k(x_0) : k = 1, \dots, \kappa - 1)$ such that

$$0 = t_0(x_0) \leq t_1(x_0) \leq \dots \leq t_\kappa(x_0) = T_{\mathcal{D}_\alpha}(x_0), \quad n_k(x_0) \in \{1, \dots, N\}$$

and for any $\varepsilon < \varepsilon_0$, $x_0 \in \mathcal{D}_\alpha$ and $t \in [t_k(x_0), t_{k+1}(x_0))$

$$\bar{B}_{d_\beta}(\phi(x_0, t), \varepsilon) \subset D_{n_k(x_0)}.$$

We use then that ϕ_i^K is close to ϕ for the distance d_β when $K \rightarrow \infty$, uniformly with respect to the initial condition on finite time interval. Then for K large enough we can define the continuous flow ϕ^K by adjunction on the time intervals as follows

$$\phi^K(x_0, 0) = x_0, \quad \bar{B}_{d_\beta}(\phi^K(x_0, t), \varepsilon) \subset D_{n_k(x_0)} \quad \text{and} \quad \frac{\partial}{\partial t} \phi^K(x_0, t) = b_{F_{n_k(x_0)}}^K(\phi^K(x_0, t))$$

for $t \in (t_k(x_0), t_{k+1}(x_0))$ and $k = 0, \dots, \kappa - 1$. Assumptions 3.3 and 3.4 are satisfied for the process X^K , with $D = \mathcal{D}_\alpha$ and D_i, F_i, t_k, n_k given above. From Lemma 5.9 (iv), we know that $\psi_{F_i}^K$ is $(C_i, C_i/K)$ non-expansive on $F_i(D_i)$ and for K large enough, we have

$$4C_i T \exp(2L_i T) \leq K\varepsilon, \quad \text{i.e. } T \leq T_\varepsilon^{C_i, C_i/K} \quad \text{for } i = 1, \dots, N.$$

Thus, we apply Theorem 3.5 and get for $x_0 \in \mathcal{D}_\alpha$ and K large enough,

$$\mathbb{P}_{x_0} \left(\sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d_\beta(X_t^K, \phi^K(x_0, t)) \geq \varepsilon \right) \leq C \sum_{k=0}^{\kappa-1} \int_{t_k(x_0) \wedge T}^{t_{k+1}(x_0) \wedge T} \bar{V}_{d, \varepsilon}(F_{n_k(x_0)}, x_0, t) dt, \quad (53)$$

where C is positive constant which does not depend on K, x_0, T, α .

Let us now consider $\alpha_0 > 0$ and observe that under Assumption ()

$$T_\alpha = \inf_{x_0 \in \mathcal{D}_{\alpha_0}} T_{\mathcal{D}_\alpha}(x_0) \xrightarrow{\alpha \rightarrow 0} +\infty,$$

since we know from Lemma 5.7 (ii) that the process comes down from infinity along the vector x_∞ or \widehat{x}_0 . So we can choose $\alpha > 0$ small enough so that $T_\alpha \geq T$. Moreover,

$$V_{F_{\beta, \gamma}}^K(x) = (V_{F_{\beta, \gamma}}^{K, (1)}(x), V_{F_{\beta, \gamma}}^{K, (2)}(x)) = \int_0^\infty (F_{\beta, \gamma}(x + H^K(x, z)) - F_{\beta, \gamma}(x + H^K(x, z)))^2 dz$$

and recalling (37) and writing $\gamma_1 = 1, \gamma_2 = \gamma$, we have for $i \in \{1, 2\}$

$$\begin{aligned} V_{F_{\beta, \gamma}}^{K, (i)}(x) &= \gamma_i \left[\lambda_i^K(Kx) \left((x_i + 1/K)^\beta - x_i^\beta \right)^2 + \mu_i^K(Kx) \left((x_i - 1/K)^\beta - x_i^\beta \right)^2 \right] \\ &\leq \frac{c'}{K} x_i^{2\beta-2} x_i (1 + x_1 + x_2) \end{aligned}$$

for some $c' > 0$. Then for $x \in \mathcal{D}_\alpha$,

$$\|V_{F_{\beta, \gamma}}^K(x)\|_1 \leq \frac{c''}{K} (x_1^{2\beta} (1 + x_2/x_1) + x_2^{2\beta} (1 + x_1/x_2)) \leq \frac{c''_\alpha}{K} (x_1^{2\beta} + x_2^{2\beta}).$$

Moreover we know from Lemma 5.7 (i) that $x_t^{(i)} \leq c_T/t$ for $t \in [0, T]$. Then we have for $x_0 \in \mathcal{D}_{\alpha_0}$ and ε small enough,

$$\int_0^T \bar{V}_{d, \varepsilon}^K(F_{\beta, \gamma}, x_0, t) dt \leq \varepsilon^{-2} \frac{c'''_\alpha}{K} \int_0^T t^{-2\beta} dt.$$

Using the fact that $\int_0^T t^{-2\beta} dt < \infty$ for $\beta < 1/2$, we get

$$\sum_{k=0}^{\kappa-1} \int_{t_k(x_0) \wedge T}^{t_{k+1}(x_0) \wedge T} \bar{V}_{d, \varepsilon}(F_{n_k(x_0)}, x_0, t) dt \leq \varepsilon^{-2} \frac{c''''_\alpha}{K}.$$

Recalling that $T_{\mathcal{D}_\alpha}(x_0) \geq T$ when $x_0 \in \mathcal{D}_{\alpha_0}$ and that the flow ϕ^K is uniformly close to the flow ϕ for the distance d_β when K goes to infinity, for any $T > 0$ and $\varepsilon > 0$ and $\alpha_0 > 0$, (53) becomes

$$\sup_{x_0 \in \mathcal{D}_{\alpha_0}} \mathbb{P}_{x_0} \left(\sup_{t \leq T} d_\beta(X_t^K, \phi(x_0, t)) \geq \varepsilon \right) \leq \varepsilon^{-2} \frac{C_{\alpha_0, T}}{K},$$

for K large enough, where $C_{\alpha_0, T}$ is some positive constant depending on α_0 and T . \square

Remark. Let us mention an alternative approach. Using Proposition 2.2 (or extending the Corollary of Section 3), one could try to compare directly the process X and to the flow ϕ (instead of ϕ^K) and put the remaining term R^K in a finite variation part A_t .

6 Appendix

We give here first three technical results to study the coming down from infinity of dynamical systems in one dimension. Let ψ_1 and ψ_2 be two locally Lipschitz function defined on $(0, \infty)$ which are negative for x large enough. Let ϕ_1 and ϕ_2 the flows associated respectively to ψ_1 and ψ_2 . We state here simple conditions to guarantee that two such flows are close or equivalent near $+\infty$, when ϕ_1 comes down from infinity.

Lemma 6.1. *We assume that ψ_1 is (L, α) non-expansive and $\int_{\infty}^{\cdot} \frac{1}{\psi_1(x)} dx < \infty$ and*

$$\psi_1(x) = \psi_2(x) + h(x)$$

where L is Lipschitz and h is bounded. Then ϕ_2 comes down from infinity and

$$\lim_{t \downarrow 0^+} \phi_2(\infty, t) - \phi_1(\infty, t) = 0.$$

Proof. This result can be proved using Lemma 2.1 or mimicking its proof in this particular case without stochastic components. Indeed, we set

$$X_t = \phi_2(x_0, t) = x_0 + \int_0^t \psi_1(\phi_2(x_0, s)) ds + R_t,$$

where $R_t = \int_0^t h(X_s) ds = \int_0^t h(\phi_2(x_0, s)) ds$. Then

$$|\widetilde{R}_t| = \mathbf{1}_{\{S_t \leq \varepsilon\}} \left| \int_0^t (X_s - x_s) dR_s \right| \leq 2\varepsilon t \|h\|_{\infty}$$

and Lemma 2.1 ensures that for any $\varepsilon > 0$, for T small enough,

$$\sup_{s \leq T, x_0 \geq 1} |\phi_2(x_0, t) - \phi(x_0, t)| \leq \varepsilon.$$

Letting $x_0 \rightarrow \infty$ yields the result, recalling that $\int_{\infty}^{\cdot} \frac{1}{\psi_1(x)} dx < \infty$ ensures that $\phi_1(\infty, t) < \infty$ for any $t > 0$. \square

Lemma 6.2. *If $\int_{\infty}^{\cdot} \frac{1}{\psi_1(x)} dx < \infty$ and $\psi_1(x) \sim_{x \rightarrow \infty} \psi_2(x)$, then $\int_{\infty}^{\cdot} \frac{1}{\psi_2(x)} dx < \infty$ and ϕ_2 comes down from infinity.*

If additionally $\phi_1(\infty, t) \sim ct^{-\alpha}$ as $t \downarrow 0^+$ for some $\alpha > 0$ and $c > 0$, then

$$\phi_2(\infty, t) \sim_{t \rightarrow 0} ct^{-\alpha}.$$

Proof. Let $\varepsilon \in (0, 1)$ and choose $x_1 > 0$ such that

$$(1 + \varepsilon)\psi_2(x) \leq \psi_1(x) < 0,$$

for $x \geq x_1$. Then for any $x_0 > x_1$,

$$\phi_1(x_0, t) \geq (1 + \varepsilon) \int_0^t \psi_2(\phi_1(x_0, s)) ds$$

for t small enough. Then, $\phi_1(x_0, t) \geq \phi_2(\infty, (1 + \varepsilon)t)$ and

$$\phi_1(\infty, t/(1 + \varepsilon)) \geq \phi_2(\infty, t)$$

for t small enough. Proving the symmetric inequality ends up the proof. \square

In the case of polynomial drift, we specify here the error term when coming from infinity.

Lemma 6.3. *Let $\rho > 1, c > 0, \alpha > 0, \varepsilon > 0$ and*

$$\psi(x) = -cx^\rho(1 + r(x)x^{-\alpha}),$$

where r is locally Lipschitz and bounded on (x_0, ∞) for some $x_0 > 0$.

Denoting by ϕ the flow associated to ψ , we have

$$\phi(\infty, t) = (ct/(\rho - 1))^{1/(1-\rho)}(1 + \tilde{r}(t)t^{\alpha/(\rho-1)}),$$

where \tilde{r} is a bounded function.

Proof. As \tilde{r} is bounded, there exists $c_1 > c_2$ such that

$$-cx^\rho(1 + c_1x^{-\alpha}) \leq \psi(x) \leq -cx^\rho(1 + c_2x^{-\alpha}),$$

so for some $c'_1 > c'_2$ and x large enough

$$-cx^{-\rho}(1 - c'_2x^{-\alpha}) \leq \frac{1}{\psi(x)} \leq -cx^{-\rho}(1 - c'_1x^{-\alpha}).$$

Then

$$-c \int_{\phi(x_0,0)}^{\phi(x_0,t)} x^{-\rho}(1 - c'_2x^{-\alpha})dx \leq \int_{\phi(x_0,0)}^{\phi(x_0,t)} \frac{dx}{\psi(x)} \leq -c \int_{\phi(x_0,0)}^{\phi(x_0,t)} x^{-\rho}(1 - c'_1x^{-\alpha})dx,$$

where the middle term is equal to t . Letting $x_0 \rightarrow \infty$

$$c''_2 \phi(\infty, t)^{-\rho-\alpha+1} \leq t - \frac{c}{\rho-1} \phi(\infty, t)^{-\rho+1} \leq c''_1 \phi(\infty, t)^{-\rho-\alpha+1}.$$

We know from the previous lemma that $\phi(\infty, t) \sim (c\rho^{-1}t)^{1/(1-\rho)}$ as $t \rightarrow 0$ and we get here

$$\phi(\infty, t) = (ct/(\rho - 1))^{1/(1-\rho)}(1 + O(t^{-1+(-\rho+1-\alpha)/(1-\rho)})) = (ct/(\rho - 1))^{1/(1-\rho)}(1 + O(t^{\alpha/(\rho-1)})),$$

which ends up the proof. \square

We need also the following estimates. We assume here that ψ and ψ^K are locally Lipschitz vectors fields on the closure \bar{D} of an open domain $D \subset \mathbb{R}^d$ and their respective flows on D are denoted by ϕ and ϕ^K . We write here $T_{D,\varepsilon}(x_0) = \sup\{t \geq 0 : \forall s < T(x_0) : \bar{B}(\phi(x_0, s), \varepsilon) \subset D\}$.

Lemma 6.4. *We assume that there exist $A \geq 1, c, \mu > 0$ and $\varepsilon \in (0, 1]$ such that*

$$(\psi(x) - \psi(y)) \cdot (x - y) \leq -\mu \|x\|_2 \|x - y\|_2^2, \quad (54)$$

for any $x \in D \cap B(0, A)^c$ and $y \in \bar{B}(x, \varepsilon)$. and

$$\|\psi(x) - \psi^K(x)\|_2 \leq c \frac{1 + \|x\|_2}{K} \quad (55)$$

for any $x \in D$ and $K \geq 1$.

Then, there exists $L \geq 0$ such that for all $T \geq 0, K \geq 2 \max\left(L, 3\frac{c}{\mu}\right) e^{2LT/\varepsilon}$, $x_0 \in D$ and $t < T_{D,\varepsilon}(x_0) \wedge T$,

$$\|\phi(x_0, t) - \phi^K(x_0, t)\|_2 \leq \frac{e^{2LT}}{K} \max\left(L, 3\frac{c}{\mu}\right).$$

Proof. Let $T > 0$ and $K \geq 2 \max(L, 3\frac{c}{\mu})e^{2LT}/\varepsilon$.

STEP 0. Using that on the closure of $D \cap B(0, A+2)$, ψ is Lipschitz and that $K \|\psi^K(\cdot) - \psi(\cdot)\|_2$ is bounded on $D \cap B(0, A+2)$ by (55), there exists $L > 0$ such that

$$\|\psi(x) - \psi^K(y)\|_2 \leq L(\|x - y\|_2 + 1/K), \quad (56)$$

for any $x, y \in D \cap B(0, A+2)$ and $K \geq 1$.

STEP 1. When $x_t = \phi(x_0, t) \in D \cap B(0, A)^c$ and $x_t^K = \phi^K(x_0, t) \in \bar{B}(x_t, \varepsilon)$, then (54) and (55) and Cauchy-Schwarz inequality give

$$\begin{aligned} \frac{d}{dt} \|x_t - x_t^K\|_2^2 &= 2(\psi(x_t) - \psi^K(x_t^K)) \cdot (x_t - x_t^K) \\ &= 2(\psi(x_t) - \psi(x_t^K)) \cdot (x_t - x_t^K) + 2(\psi(x_t^K) - \psi^K(x_t^K)) \cdot (x_t - x_t^K) \\ &\leq 2\left(-\mu \|x_t\|_2 \|x_t - x_t^K\|_2 + c \frac{1 + \|x_t^K\|_2}{K}\right) \|x_t - x_t^K\|_2. \end{aligned}$$

Moreover $\|x_t\|_2 \geq A$ and $x_t^K \in \bar{B}(x_t, \varepsilon)$ give

$$\frac{1 + \|x_t^K\|_2}{\|x_t\|_2} \leq \frac{1}{A} + 1 + \frac{\varepsilon}{A} \leq 3,$$

so $\|x_t - x_t^K\|_2 \in [3c/(K\mu), \varepsilon]$ implies

$$\frac{d}{dt} \|x_t - x_t^K\|_2^2 \leq 0.$$

This means that when $x_t \in D \cap B(0, A)^c$, the gap $\|x_t - x_t^K\|_2$ tends to decrease when it is larger than $3c/(K\mu)$ but smaller than ε .

STEP 2. Let us consider $t_0 \leq t_1 \leq T_{D,\varepsilon}(x_0) \wedge T$ and write $M_K := \max(L, 3c/\mu)/K$. We assume here that for any $t \in [t_0, t_1)$, $x_t \in B(0, A+1)$ and

$$\|x_{t_0} - x_{t_0}^K\|_2 = M_K, \quad \|x_t - x_t^K\|_2 \in [M_K, \varepsilon].$$

Observing that $x_t^K \in B(0, A+2)$ for $t \in [t_0, t_1)$ and using (56) and Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{d}{dt} \|x_t - x_t^K\|_2^2 &\leq 2 \|x_t - x_t^K\|_2 \|\psi(x_t) - \psi^K(x_t^K)\|_2 \\ &\leq 2L \|x_t - x_t^K\|_2^2 + \frac{2}{K} \|x_t - x_t^K\|_2 \\ &\leq 4L \|x_t - x_t^K\|_2^2. \end{aligned}$$

since $\|x_t - x_t^K\|_2 \geq 1/M_K$. We can apply Gronwall lemma to get that

$$\|x_t - x_t^K\|_2 \leq e^{2L(t-t_0)} M_K \leq \varepsilon/2$$

for $t \in [t_0, t_1)$, since $K \geq 2Le^{2LT}/\varepsilon$.

STEP 3. Using the two previous steps, we first check that $\|x_t - x_t^K\|_2 \leq \varepsilon$ for any $t \leq T_{D,\varepsilon}(x_0) \wedge T$. Then we use STEP 1 to see that $\|x_t - x_t^K\|_2$ can go beyond M_K only when $x_t \in B(0, A+1)$ and then STEP 2 ensures that it cannot beyond $M_K \exp(2LT)$, which yields the result. \square

Acknowledgment. The author wishes to thank warmly Pierre Collet, Sylvie Méléard and Stéphane Gaubert for stimulating and fruitful discussions in Ecole polytechnique along this work. The author is also grateful to thank Clément Foucart and Sylvie Méléard for their reading of a previous version of this paper and their various comments. This work was partially funded by the Chaire Modélisation Mathématique et Biodiversité VEOLIA-École Polytechnique-MNHN-F.X.

References

- [1] W. J. Anderson. *Continuous-time Markov chains*. Springer Series in Statistics: Probability and its Applications. Springer-Verlag, New York, 1991. An applications-oriented approach.
- [2] Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: a brief introduction and some open problems. *53rd IEEE Conference on Decision and Control* December 15-17, 2014. Los Angeles, California, USA.
- [3] R. Azais, J-B Bardet, A. Génadot, N. Krell and P-A Zitt. ESAIM: Proceedings, January 2014, Vol. 44, p. 276-290 SMAI Groupe MAS - Journées MAS 2012 - Session thématique.
- [4] V. Bansaye, S. Méléard and M. Richard. The speed of coming down from infinity for birth and death processes. To appear in *Adv. Appl. Probab.*, 2016.
- [5] J. Berestycki, N. Berestycki, and V. Limic. The Lambda-coalescent speed of coming down from infinity. *Ann. Probab.*, 38(1):207–233, 2010.
- [6] J. Bertoin and J-F LeGall. Stochastic flows associated to coalescent processes. III. Limit theorems. *Illinois J. Math.* (2006), Vol. 50, p. 147-181.
- [7] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999.
- [8] P. Cattiaux, S. Méléard. Competitive or weak cooperative stochastic Lotka-Volterra systems conditioned to non extinction. *J. Math. Biology.* 60(6):797-829, 2010.
- [9] P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, 37(5):1926–1969, 2009.
- [10] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. To appear in *Probab. Th. Rel. Field*, 2015.
- [11] D. A. Dawson and Z. Li. Stochastic equations, flows and measure-valued processes. *Ann. Probab.* 40 (2012), no. 2, 813-857.
- [12] R. Darling and J. Norris. Differential equation approximations for Markov chains. *Probab. Surv.* 5 (2008), 37-79.

- [13] P. Donnelly. Weak convergence to a Markov chain with an entrance boundary: ancestral processes in population genetics. *Ann. Probab.*, 19(3):1102–1117, 1991.
- [14] C. Dellacherie and P. A. Meyer. *Probabilities and potential. B. Theory of martingales*. North-Holland Mathematics Studies, 72. North-Holland Publishing Co., Amsterdam, 1982.
- [15] F. Dumortier, J. Llibre and J. C. Artés. *Qualitative Theory of Planar Differential Systems*. Springer-Verlag, Berlin, 2006.
- [16] S. N. Ethier and T. G. Kurtz. *Markov processes. Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, New York, 1986. .
- [17] M. I. Frendlin and A. D. Wentzell. *Random perturbations of dynamical systems* . Third edition. Grundlehren der Mathematischen Wissenschaften, 260. Springer, Heidelberg, 2012
- [18] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes, 2nd ed.* North-Holland, 1989.
- [19] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, 1987.
- [20] S. Karlin and J. L. McGregor. The differential equations of birth-and-death processes, and the Stieltjes moment problem. *Trans. Amer. Math. Soc.*, 85:489–546, 1957.
- [21] S. Karlin and H. M. Taylor. *A first course in stochastic processes*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, second edition, 1975.
- [22] V. Le and E. Pardoux Height and the total mass of the forest of genealogical trees of a large population with general competition. To appear in *ESAIM P & S*.
- [23] V. Limic. On the speed of coming down from infinity for Ξ -coalescent processes. *Electron. Journal Probab.* Vol. 15, 217–240, 2010.
- [24] V. Limic and A. Talarczyk. Diffusion limits for mixed with Kingman coalescents at small times. *Electron. J. Probab.*, vol. 20, Paper 45, 2015.
- [25] V. Limic and A. Talarczyk. Second-order asymptotics for the block counting process in a class of regularly varying Lambda-coalescents. *Ann. Probab.*, Vol. 43, 2015.
- [26] J. Pitman. Coalescents with multiple collisions. *Ann. Probab.* 27 (1999), no. 4, 1870-1902.
- [27] J. Schweinsberg. A necessary and sufficient condition for the Λ -coalescent to come down from infinity. *Electron. Comm. Probab.* (2000) Vol. 5, P. 1-11.
- [28] A. V. Skorokhod, F. C. Hoppensteadt and H. Salehi. *Random perturbation methods with applications in science and engineering* . Applied Mathematical Sciences, 150. Springer-Verlag, New York, 2002.
- [29] E. A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, 23 (4) : 683–700, 1991.