



**HAL**  
open science

# Approximation of stochastic processes by non-expansive flows and coming down from infinity

Vincent Bansaye

► **To cite this version:**

Vincent Bansaye. Approximation of stochastic processes by non-expansive flows and coming down from infinity. 2015. hal-01232373v1

**HAL Id: hal-01232373**

**<https://hal.science/hal-01232373v1>**

Preprint submitted on 23 Nov 2015 (v1), last revised 5 Jan 2017 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Approximation of stochastic processes by non-expansive flows and coming down from infinity

Vincent Bansaye\*

November 23, 2015

## Abstract

We approximate stochastic processes in finite dimension by dynamical systems. We provide trajectorial estimates which are uniform with respect to the initial condition for a well chosen distance. Some non-expansivity property of the flow of the dynamical is required, which allows us to deal with non-Lipschitz vector fields. We use the stochastic calculus and we follow the martingale technic initiated in [5] to control the fluctuations. Our main applications deal with the short time behavior of stochastic processes starting from large initial values. We state some general properties on the coming down from infinity of one-dimensional SDE. In particular, we recover and complement known results on  $\Lambda$ -coalescent and birth and death processes. Moreover, using Poincaré's compactification for dynamical systems close to infinity, we develop this approach in two dimensions for competitive Lotka Volterra diffusions and classify the coming down from infinity. Finally, we provide uniform estimates for scaling limits of competitive birth and death processes.

*Key words:* Approximation of stochastic processes, non expansivity, dynamical system, coming down from infinity, scaling limits

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Random perturbation of dynamical systems</b>	<b>5</b>
2.1	Trajectorial control for perturbed non-expansive dynamical systems . . . . .	6
2.2	Non-expansive dynamical systems and perturbation by martingales . . . . .	8
<b>3</b>	<b>Uniform estimates for Stochastic Differential Equations</b>	<b>10</b>
3.1	Main result . . . . .	11
3.2	Adjunction of non-expansive domains . . . . .	14

---

\*CMAP, Ecole Polytechnique, CNRS, route de Saclay, 91128 Palaiseau Cedex-France; E-mail: [vincent.bansaye@polytechnique.edu](mailto:vincent.bansaye@polytechnique.edu)

<b>4</b>	<b>Coming down from infinity for one-dimensional Stochastic Differential Equation</b>	<b>18</b>
4.1	Weak convergence and coming down from infinity	20
4.2	Examples	24
4.2.1	$\Lambda$ -coalescent [24, 5].	25
4.2.2	Birth and death processes [27, 4].	27
4.2.3	Transmission Control Protocol	30
4.2.4	Logistic diffusions [7] and perspectives	31
<b>5</b>	<b>Uniform estimates for two-dimensional competitive Lotka Volterra diffusions</b>	<b>32</b>
5.1	Uniform short time estimates and coming back to compact sets	33
5.2	Uniform scaling limits of competitive birth and death processes	36
5.3	Non-expansivity of the flow and Poincaré compactification	37
5.4	Proofs of Theorem 5.1 and Corollary 5.2 and Theorem 5.3	45
<b>6</b>	<b>Appendix</b>	<b>52</b>

## 1 Introduction

The approximation of stochastic processes by dynamical systems has been largely developed, with a particular focus on random perturbation of dynamical systems (see e.g. [26, 15]) and the scaling limits of random models (see e.g. [14, 10]). In this paper, we are interested in stochastic processes  $(X_t : t \geq 0)$  taking values on a subset  $E$  of  $\mathbb{R}^d$ , which can be written as

$$X_t = X_0 + \int_0^t \psi(X_s) ds + R_t,$$

where  $R$  is a semi-martingale. Under some non-expansive property described below, we aim at proving that  $X$  remains close to the dynamical system whose flow  $\phi(x_0, t) = x_t$  is given by

$$x_t = x_0 + \int_0^t \psi(x_s) ds.$$

The point here is to estimate the probability of this event uniformly with respect to the initial condition  $x_0 \in D$ , when the drift term  $\psi$  may be non-Lipchitz on  $D$ . Our main motivation for such estimates is the the description of the coming down from infinity, which amounts to let the initial condition  $x_0$  goes to infinity, and the uniform scaling limits of stochastic processes describing populations models. We provide some illustrations and applications of the results for coalescent and competitive processes.

Our approach relies on some contraction property of the flow, which provides a stability property flow and is used in particular control theory. More precisely, we say in that paper that a vector field is non-expansive on a domain  $D$  when it makes two trajectories become closer and closer for the euclidian norm on a subset  $D$  of  $\mathbb{R}^d$ , which amounts to

$$\forall x, y \in D, \quad (\psi(x) - \psi(y)) \cdot (x - y) \leq 0,$$

where  $\cdot$  is the usual scalar product on  $\mathbb{R}^d$ . Actually, the distance between two solutions can increase provided that this increase is not too fast. This will required for the applications.

Thus, we are working with  $(L, \alpha)$  non-expansive vector field  $\psi$  :

$$\forall x, y \in D, \quad (\psi(x) - \psi(y)) \cdot (x - y) \leq L \|x - y\|_2^2 + \alpha \|x - y\|_2.$$

The non-expansivity property ensures that the drift term can not make the distance between the stochastic process  $X$  and the dynamical system  $x$  explode because of small fluctuations due to the perturbation  $R$ . To control the size of these fluctuations, we are using classical martingale technics : let us refer to [14, 10] in the context of scaling limits and to [5] for a pioneering work on the speed of coming down from infinity of  $\Lambda$ -coalescent.

Then we estimate the probability that the stochastic process remains close to the dynamical system as soon as this latter is in a domain  $D$  where  $\psi$  is  $(L, \alpha)$ - non-expansive. These estimates hold for a well chosen distance  $d$  and any  $x_0 \in D$  :

$$\mathbb{P}_{x_0} \left( \sup_{t \leq T \wedge T_D(x_0)} d(X_t, \phi(x_0, t)) \geq \varepsilon \right) \leq C_T \int_0^T \bar{V}_{d, \varepsilon}(x_0, t) dt, \quad (1)$$

where  $T_D(x_0)$  will correspond the exit time of the domain  $D$  for the flow  $\phi$  started at  $x_0$ . The distance  $d$  will be of the form

$$d(x, y) = \|F(x) - F(y)\|_2,$$

where  $F$  is of class  $\mathcal{C}^2$ , so that we can use the stochastic calculus. The perturbation needs to be controlled in a tube around the dynamical system and

$$\bar{V}_{d, \varepsilon}(x_0, s) = \sup_{\substack{x \in E \\ d(x, \phi(x_0, s)) \leq \varepsilon}} \left\{ \varepsilon^{-2} \|V_F(x)\|_1 + \varepsilon^{-1} \|\tilde{b}_F(x)\|_1 \right\},$$

where  $V_F$  will be given by the quadratic variation of  $F(X)$  and  $\tilde{b}_F$  will be an additional approximation term arising from Itô formula.

We are choosing  $F$  so that a  $(L, \alpha)$  non-expansivity property hold for the associated vector field and the size of the fluctuations is small enough. It will be illustrated in several examples. The estimate (1) becomes uniform with respect to  $x_0 \in D$  as soon as  $\bar{V}_{d, \varepsilon}(x_0, s)$  can be bounded by a function of time which is integrable. This choice of  $F$  is linked to the geometry of the flow and may be subtle, see in particular the last Section. In Section 3, the results are specified when  $X$  satisfies a Stochastic Differential Equation (SDE), which allows a diffusion component and random jumps given by a Poisson Point measure. This is covering the range of our application.

Then we use these estimates to characterize the coming down from infinity for some stochastic differentials equations in  $\mathbb{R}^d$ . Roughly speaking, we consider a domain  $D$  which is non-bounded and let  $T$  go to 0 to derive from (1) that for any  $\varepsilon > 0$ ,

$$\lim_{T \rightarrow 0} \sup_{x_0 \in D} \mathbb{P}_{x_0} \left( \sup_{t \leq T} d(X_t, \phi(x_0, t)) \geq \varepsilon \right) = 0.$$

Letting then  $x_0$  go to the boundary of  $D$  in the previous limit allows us to describe the coming down from infinity of processes in several ways.

First, the control of the fluctuations of the process  $X$  for large initial values by a dynamical system gives a way to prove the tightness of  $\mathbb{P}_{x_0}$  for  $x_0 \in D$ . We can then obtain the weak

convergence of  $\mathbb{P}_{x_0}$  as  $x_0 \rightarrow \infty$  for monotone SDE. Moreover we link the coming down from infinity of the process  $X$  to the coming down from infinity of the flow  $\phi$ .

In dimension 1, we use some monotonicity properties to determine when the process comes down from infinity and how it comes down from infinity (Section 4). In particular, we recover the speed of coming down from infinity of  $\Lambda$ -coalescent [5] with  $F = \log$  and in that case  $V_F$  is bounded. We also recover some results of [4] for birth and death processes and we can provide finer estimates for regularly varying deaths rate. Here  $F$  is polynomial and  $V_F$  is non-bounded so this latter has to be controlled finely along the trajectory of the dynamical system. Finally, we consider the transmission control protocol which is non stochastically monotone and requires to take  $F(x) = \log(1 + \log(1 + x))$  so that the fluctuations can be controlled.

In higher dimension, the coming down from infinity of a dynamical system is a more delicate problem in general. Poincaré has initiated a theory to study dynamical system close to infinity, which is particularly powerful for polynomial vector fields (see e.g. Chapter 5 in [13]). We develop this approach for competitive Lotka Volterra models in dimension 2 in Section 5. We classify then the ways the dynamical systems can come down from infinity and describe the counterpart for the stochastic process, which differs when the dynamical system is getting close from the boundary of  $(0, \infty)^2$ .

The uniform estimates given above can also be used to prove convergence of scaling limits of stochastic processes  $X^K$  to dynamical systems, which are uniform with respect to the initial condition on non-bounded domains, with a suitable distance  $d$  :

$$\lim_{K \rightarrow \infty} \sup_{x_0 \in D} \mathbb{P}_{x_0} \left( \sup_{t \leq T} d(X_t^K, \phi(x_0, t)) \geq \varepsilon \right) = 0,$$

for some fixed  $T, \varepsilon > 0$ . It is illustrated in this paper by the convergence of birth and death processes with competition to Lotka-Volterra competitive dynamical system in Section 5.

Let us end up with other motivations for this work, some of which being linked to works in progress.

First, our original motivation for studying the coming down from infinity is the description of the time for extinction for competitive models in varying environment. Roughly speaking, competitive periods make the size of the population quickly decrease, which can be captured by the coming down from infinity. Let us also note that the approach developed here could be extended to the varying environment framework by comparing the stochastic process to a non-autonome dynamical system.

Second, the coming down from infinity is linked to the unicity of quasistationary distribution, see [27] for birth and death processes and [7] for some diffusions. Recently, the coming down from infinity has appeared as a key assumption for the geometric convergence of the conditioned process to the quasistationary distribution. We refer to [8] for details.

Finally, several challenges open to deal with the coming down from infinity of other models in dimension 2 and more and with other scalings.

**Notations.** In the whole paper  $\cdot$  stands for the canonical scalar product on  $\mathbb{R}^d$ ,  $\|\cdot\|_2$  the associated euclidian norm and  $\|\cdot\|_1$  the  $L^1$  norm. We write  $x = (x^{(i)} : i = 1, \dots, d) \in \mathbb{R}^d$  a row vector of real numbers. The product  $xy$  for  $x, y \in \mathbb{R}^d$  is the vector  $z \in \mathbb{R}^d$  such that  $z_i = x_i y_i$ .

We denote by  $\bar{B}(x, \varepsilon) = \{y \in \mathbb{R}^d : \|y - x\|_2 \leq \varepsilon\}$  the euclidian closed ball centered in  $x$  with radius  $\varepsilon$ . More generally, we note  $\bar{B}_d(x, \varepsilon) = \{y \in \mathbb{R}^d : d(x, y) \leq \varepsilon\}$  the ball centered associated with the distance  $d$ .

When  $\chi$  is differentiable on a open set of  $\mathbb{R}^d$  and takes values in  $\mathbb{R}^d$ , we denote by  $J_\chi$  its Jacobian and

$$(J_\chi(x))_{i,j} = \frac{\partial}{\partial x_j} \chi^{(i)}(x) \quad (i, j = 1, \dots, d).$$

We write  $F^{-1}$  the reciprocal function of a bijection  $F$  and  $A^{-1}$  the inverse of a non zero real number or invertible matrix  $A$ . If  $A$  is a matrix, its transpose is denoted by  $A^*$ .

By convention  $\inf \emptyset = \infty$  and  $x \wedge y$  is the smaller element of  $\{x, y\}$  and  $\infty \wedge x = x$ .

We note  $d(x) \sim_{x \rightarrow a} g(x)$  when  $d(x)/g(x) \rightarrow 1$  as  $x \rightarrow a$ .

We also use the notation  $\int_a^{\cdot} f(x) dx < \infty$  (resp.  $= \infty$ ) for  $a \in [0, \infty]$  when there exists  $a_0 \in (0, \infty)$  such that  $\int_a^{a_0} f(x) dx$  is well defined and finite (resp. infinite).

Finally, we denote by  $\langle M \rangle$  the predictable quadratic variation of a continuous local martingale  $M$  and by  $|A|$  the total variation of a process  $A$  and by  $\Delta X_s = X_s - X_{s-}$  the jump at time  $s$  of a càdlàg process.

**Outline.** In the next Section, we provide general results for dynamical system perturbed by semimartingales using the non-expansivity of the flow and martingale inequality. In Section 3, we derive approximations results for Markov process described by SDE. It relies on a transformation  $F$  of the process for which we apply the result of Section 2. An extension of the result by adjunction of non-expansive domains is provided and required for applications. We then study the coming down from infinity for one dimensional SDE in Section 4, with a particular focus on stochastically monotone process. Finally we study the coming down from infinity of two dimensional competitive stochastic Lotka Volterra processes and their approximation by scaling limits of birth and death processes.

## 2 Random perturbation of dynamical systems

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $(\mathcal{F}_t)_{t \geq 0}$  a filtration of  $\mathcal{F}$ , which satisfies the usual conditions. We consider a  $\mathcal{F}_t$ -adapted càdlàg process  $X$  which takes its values in a subset  $E$  of  $\mathbb{R}^d$  and satisfies for every  $t \geq 0$ ,

$$X_t = X_0 + \int_0^t \psi(X_s) ds + R_t,$$

where  $X_0 \in E$  a.s.,  $\psi$  is locally Lipschitz on  $E$  and  $(R_t : t \geq 0)$  is a càdlàg  $\mathcal{F}_t$ -semimartingale. The process  $R$  is given by

$$R_t = A_t + M_t, \quad M_t = M_t^c + M_t^d,$$

with  $A_t$  a càdlàg  $\mathcal{F}_t$ -adapted process with a.s. bounded variations paths,  $M_t^c$  a continuous  $\mathcal{F}_t$ -local martingale,  $M_t^d$  a càdlàg  $\mathcal{F}_t$ -local martingale purely discontinuous and  $R_0 = A_0 = M_0 = M_0^c = M_0^d = 0$ .

Our aim is to compare the process  $X$  to the solution  $x = \phi(x_0, \cdot)$  of the dynamical system associated with the vector field  $\psi$  and some initial condition  $x_0$  :

$$x_t = x_0 + \int_0^t \psi(x_s) ds.$$

For that purpose, we assume that  $\psi$  can be extended to a domain  $E'$  such that  $x_0$  belongs to the interior of  $E'$  and  $\psi$  is still locally Lipschitz on  $E'$ . Then the solution  $x$  of the equation above exists and is unique on a time interval  $[0, T'(x_0))$ , where  $T'(x_0) \in (0, \infty]$ .

As briefly mentioned in the introduction, the key property to control the distance between  $(X_t : t \geq 0)$  and  $(x_t : t \geq 0)$  is some non-expansivity (or contraction) property of the flow. It means that the distance between two solutions is non-increasing. This notion is classical in dynamical system and control theory, it yields a stability property to the dynamical system. It amounts to require that the vector field  $\psi$  is non-increasing in dimension one. We consider here the euclidian distance to use the stochastic calculus and change this distance via a well chosen transformation in the next Sections. Thus, we need the following definition, which ensures the non-expansivity property of the flow for the euclidian distance, up to some term we can control.

**Definition 2.1.** *The vector field  $\psi : D \rightarrow \mathbb{R}^d$  is  $(L, \alpha)$  non-expansive on  $D \subset \mathbb{R}^d$  if for any  $x, y \in D$ ,*

$$(\psi(x) - \psi(y)) \cdot (x - y) \leq L \|x - y\|_2^2 + \alpha \|x - y\|_2.$$

When  $\alpha = 0$ , we simply say that  $\psi$  is  $L$  non-expansive on  $D$ . If additionally  $L = 0$ , we say that  $\psi$  is non-expansive on  $D$ . We observe that

$$\psi = A + \chi = A + f + g$$

is  $(L, \alpha)$  non-expansive on  $D$  if  $A$  is a vector field whose euclidian norm is bounded by  $\alpha$  on  $D$  and  $\chi$  is  $L$  non-expansive on  $D$ , or  $f$  is Lipschitz with constant  $L$  on  $D$  and  $g$  is non-expansive on  $D$ . Finally, when  $\chi$  is differentiable on a convex open set  $O$  which contains  $D$ ,  $\chi$  is  $L$  non-expansive on  $D$  if for any  $x \in O$ ,

$$\text{Sp}(J_\chi + J_\chi^*) \subset (-\infty, 2L],$$

where  $\text{Sp}(J_\chi + J_\chi^*)$  is the spectrum of the symmetric linear operator (and hence diagonalisable)  $J_\chi + J_\chi^*$ , see table 1 in [2] for details and more general results.

## 2.1 Trajectorial control for perturbed non-expansive dynamical systems

The following lemma gives the trajectorial result which allows us to control the gap between the stochastic process  $(X_t : t \geq 0)$  and the dynamical system  $(x_t : t \geq 0)$  by the size of the fluctuations of  $(R_t : t \geq 0)$  and the gap between the initial positions. It relies on the  $(L, \alpha)$  non-expansivity of the flow and the control of the stochastic integral  $\int_0^t (X_{s-} - x_s) \cdot dR_s$ , which is the integral of the càglàd predictable process (and thus locally bounded)  $(X_{s-} - x_s : 0 \leq s < T'(x_0))$  with respect to the semimartingale  $(R_t : t \geq 0)$ , see [17] Chapter I, Theorem 4.31 for a classical reference.

We set for any  $t < T'(x_0)$ ,

$$S_t := \sup_{s \leq t} \|X_s - x_s\|_2$$

and

$$\widetilde{R}_t = \|X_0 - x_0\|_2^2 + \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \left[ 2 \int_0^t (X_{s-} - x_s) \cdot dR_s + \|[M_t]\|_1 \right],$$

where  $[M] = [X]$  is the quadratic variation of the semimartingale  $X$  and

$$\|[M_t]\|_1 = \|[X_t]\|_1 = \|\langle M_t^c \rangle\|_1 + \sum_{s \leq t} \|\Delta X_s\|_2^2,$$

and we refer to Chapter 1, Theorem 4.52 in [17] for a classical reference. Finally, we define

$$T_{D,\varepsilon}(x_0) = \inf\{t \in [0, T'(x_0)) : \exists y \notin D, y \in \overline{B}(x_t, \varepsilon)\} \wedge T'(x_0)$$

the first time when the solution  $(x_t : t \in [0, T'(x_0)))$  is at distance  $\varepsilon$  of the boundary of  $D$ .

**Lemma 2.2.** *Assume that  $\psi$  is  $(L, \alpha)$  non-expansive on  $D$  and let  $\varepsilon > 0$ .*

*Then for any  $x_0 \in E \cap D$  and  $T < T_{D,\varepsilon}(x_0)$  such that  $4\alpha T \exp(2LT) \leq \varepsilon$ , we have*

$$\{S_T \geq \varepsilon\} \subset \left\{ \sup_{t \leq T} \widetilde{R}_t \geq \eta^2 \right\},$$

where  $\eta = \varepsilon \exp(-LT)/\sqrt{2}$ .

*Proof.* Let  $x_0 \in E \cap D$  and  $T < T'(x_0)$ . First, we consider the quadratic variation of  $(X_t - x_t : 0 \leq t < T'(x_0))$ . We have for  $t \leq T$  (see Chapter 1, Definition 4.4.45 in [17] or use Itô formula),

$$[X_t - x_t] = [M_t] = (X_t - x_t)^2 - (X_0 - x_0)^2 - 2 \int_0^t (X_{s-} - x_s) d(X_s - x_s).$$

Summing on the coordinates and using the definitions of  $\widetilde{R}$  and  $x$ , we get

$$\|X_t - x_t\|_2^2 = \|X_0 - x_0\|_2^2 + 2 \int_0^t (X_{s-} - x_s) \cdot (\psi(X_{s-}) - \psi(x_s)) ds + 2 \int_0^t (X_{s-} - x_s) \cdot dR_s + \|[M_t]\|_1.$$

Moreover for any  $s < T_{D,\varepsilon}(x_0)$ ,  $x_s \in D$  and  $X_{s-} \in D$  on the event  $\{S_{s-} \leq \varepsilon\}$ . So using that  $\psi$  is  $(L, \alpha)$  non-expansive on  $D$ ,

$$\mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot (\psi(X_{s-}) - \psi(x_s)) \leq \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \left( L \|X_{s-} - x_s\|_2^2 + \alpha \|X_{s-} - x_s\|_2 \right).$$

Then for any  $t < T_{D,\varepsilon}(x_0)$ ,

$$\begin{aligned} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \|X_t - x_t\|_2^2 &\leq \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \left[ 2L \int_0^t \|X_s - x_s\|_2^2 ds + 2\alpha \int_0^t \|X_s - x_s\|_2 ds \right. \\ &\quad \left. + \|X_0 - x_0\|_2^2 + 2 \int_0^t (X_{s-} - x_s) \cdot dR_s + \|[M_t]\|_1 \right] \end{aligned}$$

and by definition of  $\widetilde{R}$ ,

$$\mathbf{1}_{\{S_{t-} \leq \varepsilon\}} S_t^2 \leq 2L \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} S_s^2 ds + 2\alpha t \varepsilon + \sup_{s \leq t} \widetilde{R}_s.$$



By Gronwall lemma, we obtain for  $t \leq T$ ,

$$\mathbf{1}_{\{S_{t-} \leq \varepsilon\}} S_t^2 \leq \left( 2\alpha T \varepsilon + \sup_{s \leq T} \widetilde{R}_s \right) e^{2LT}.$$

We assume now that  $T$  satisfies

$$T e^{2LT} \leq \frac{\varepsilon}{4\alpha}$$

and choose  $\eta = \varepsilon / (\sqrt{2} \exp(LT))$ , so we have

$$(2\alpha T \varepsilon + \eta^2) e^{2LT} \leq \varepsilon^2.$$

Then for any  $T < T_{D,\varepsilon}(x_0)$ , we get

$$\mathbf{1}_{\{S_{T-} \leq \varepsilon\}} S_T^2 < \varepsilon^2 \quad \text{a.s. on the event } \left\{ \sup_{s \leq T} \widetilde{R}_s < \eta^2 \right\}.$$

Denoting

$$T_{exit} = \inf\{s < T_D(x_0) : S_s \geq \varepsilon\},$$

we have  $S_{T_{exit}-} \leq \varepsilon$  and  $S_{T_{exit}} \geq \varepsilon$  on the event  $\{T_{exit} \leq T\}$ , so that the last inequality at time  $T_{exit}$  ensures that

$$\{T_{exit} \leq T\} \subset \left\{ \sup_{s \leq T} \widetilde{R}_s \geq \eta^2 \right\},$$

which ends up the proof.  $\square$

## 2.2 Non-expansive dynamical systems and perturbation by martingales

We use now martingale maximal inequality to estimate the probability that the distance between the process  $(X_t : t \geq 0)$  and the dynamical system  $(x_t : t \geq 0)$  goes beyond some level  $\varepsilon > 0$ . Such arguments have been used in several contexts, see in particular [10] for scaling limits. For the coming down from infinity, we refer to [5], which is dedicated to  $\Lambda$ -coalescent and has inspired the results below.

**Proposition 2.3.** *Assume that  $\psi$  is  $(L, \alpha)$  non-expansive on  $D$  and let  $\varepsilon > 0$ .*

*Then for any  $x_0 \in E \cap D$  and  $T < T_{D,\varepsilon}(x_0)$  such that  $4\alpha T \exp(2LT) \leq \varepsilon$ , we have for  $p \geq 1/2, q \geq 0$ ,*

$$\begin{aligned} & \mathbb{P}(S_T \geq \varepsilon) \\ & \leq \mathbb{P}\left(\|X_0 - x_0\|_2 \geq \varepsilon \frac{e^{-LT}}{2\sqrt{2}}\right) + C_q \frac{e^{qLT}}{\varepsilon^q} \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d\|A|_s\|_1\right)^q\right] \\ & \quad + C_{p,d} \frac{e^{4pLT}}{\varepsilon^{2p}} \left[ \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d\|M_s^c\|_1\right)^p\right] + \mathbb{E}\left[\left(\sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \|\Delta X_s\|_2^2\right)^p\right] \right], \end{aligned}$$

for some positive constants  $C_q$  (resp.  $C_{p,d}$ ) which depend only on  $q$  (resp.  $p, d$ ).

*Proof.* By definition of  $\widetilde{R}$  and recalling that  $R_s = A_s + M_s$ ,

$$\left\{ \sup_{t \leq T} \widetilde{R}_t \geq \eta^2 \right\} \subset \left\{ \|X_0 - x_0\|_2^2 \geq \frac{\eta^2}{4} \right\} \cup B_\eta,$$

where

$$\begin{aligned} B_\eta \subset & \left\{ \sup_{t \leq T} \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dA_s \geq \frac{\eta^2}{16} \right\} \cup \left\{ \sup_{t \leq T} \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} \int_0^t (X_{s-} - x_s) \cdot dM_s \geq \frac{\eta^2}{16} \right\} \\ & \cup \left\{ \int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d\|M_t^c\|_1 \geq \frac{\eta^2}{4} \right\} \cup \left\{ \sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \|\Delta X_s\|_2^2 \geq \frac{\eta^2}{4} \right\}. \end{aligned}$$

We know that

$$\{S_T \geq \varepsilon\} \subset \left\{ \sup_{s \leq T} \widetilde{R}_s \geq \eta^2 \right\}$$

from Lemma 2.2 and the Markov inequality yields

$$\begin{aligned} \mathbb{P}(S_T \geq \varepsilon) & \leq \mathbb{P}\left(\|X_0 - x_0\|_2^2 \geq \frac{\eta^2}{4}\right) + \mathbb{P}(B_\eta) \\ & \leq \mathbb{P}\left(\|X_0 - x_0\|_2^2 \geq \frac{\eta^2}{4}\right) + \left(\frac{16}{\eta^2}\right)^q \mathbb{E}\left(\left|\sup_{t \leq T} \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dA_s\right|^q\right) \\ & \quad + \left(\frac{16}{\eta^2}\right)^{2p} \mathbb{E}\left(\left|\sup_{t \leq T} \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dM_s\right|^{2p}\right) \\ & \quad + \left(\frac{4}{\eta^2}\right)^p \mathbb{E}\left(\left(\int_0^T \mathbf{1}_{\{S_{t-} \leq \varepsilon\}} d\|M_t^c\|_1\right)^p\right) + \left(\frac{4}{\eta^2}\right)^p \mathbb{E}\left(\left[\sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \|\Delta X_s\|_2^2\right]^p\right). \end{aligned}$$

First using that  $|f \cdot dg| \leq \|f\|_2 \|d\| \|g\|_1$ , we have for  $t \leq T$

$$\int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dA_s \leq \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \|X_{s-} - x_s\|_2 dA_s^1 \leq \varepsilon \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} dA_s^1,$$

where  $A_s^1 := \| |A_s| \|_1$ .

Second, Burkholder Davis Gundy inequality (see [12], 93, chap. VII, p. 287) for the local martingale

$$N_t = \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dM_s$$

ensures that there exists  $C_p > 0$  such that

$$\mathbb{E}\left(\sup_{t \leq T} |N_t|^{2p}\right) \leq C_p \mathbb{E}\left([N]_T^p\right)$$

Writing the coordinates of  $X$  and  $x$  respectively  $(X^{(i)} : i = 1, \dots, d)$  and  $(x^{(i)} : i = 1, \dots, d)$  and adding that

$$[N_T] = \int_0^T \sum_{i,j=1}^d \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-}^{(i)} - x_s^{(i)})(X_{s-}^{(j)} - x_s^{(j)}) d[M^{(i)}, M^{(j)}]_s \leq \varepsilon^2 \int_0^T \sum_{i,j=1}^d \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d[M^{(i)}, M^{(j)}]_s$$

and that  $d[M^{(i)}, M^{(j)}]_s \leq d[M^{(i)}]_s + d[M^{(j)}]_s$ , we obtain

$$\begin{aligned} & \mathbb{E} \left( \sup_{t \leq T} \left| \int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} (X_{s-} - x_s) \cdot dM_s \right|^{2p} \right) \\ & \leq C_{p,d} \varepsilon^{2p} \mathbb{E} \left( \left( \int_0^T \sum_{i=1}^d \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d[M^{(i)}]_s \right)^p \right) \\ & \leq C'_{p,d} \varepsilon^{2p} \left[ \mathbb{E} \left( \left( \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \ll M^c >_s \parallel_1 \right)^p \right) + \mathbb{E} \left( \left( \sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \|\Delta X_s\|_2^2 \right)^p \right) \right] \end{aligned}$$

for some positive constants  $C_{p,d}$  and  $C'_{p,d}$ , where we recall that  $[M^{(i)}]_t = \langle M^{c,(i)} \rangle_t + \sum_{s \leq t} \Delta X_s^{(i)}$ . Using now that

$$\{S_T \geq \varepsilon\} \subset \left\{ \sup_{s \leq T} \tilde{R}_t \geq \eta^2 \right\}$$

Putting pieces together, we get

$$\begin{aligned} \mathbb{P}(S_T \geq \varepsilon) & \leq \mathbb{P} \left( \|X_0 - x_0\|_2^2 \geq \frac{\eta^2}{4} \right) + \left( \frac{16\varepsilon}{\eta^2} \right)^q \mathbb{E} \left( \left( \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} dA_s^1 \right)^q \right) \\ & \quad + \frac{C''_{p,d}}{\eta^{2p}} \left[ \mathbb{E} \left( \left( \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \ll M_t^c > \parallel_1 \right)^p \right) + \mathbb{E} \left( \left( \sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \|\Delta X_s\|_2^2 \right)^p \right) \right] \end{aligned}$$

for some  $C''_{p,d}$  positive. Recalling that  $\eta = \varepsilon / (\sqrt{2} \exp(LT))$  ends up the proof.  $\square$

### 3 Uniform estimates for Stochastic Differential Equations

In this section, we assume that  $X = (X^{(i)} : i = 1, \dots, d)$  is a càdlàg Markov process which takes values in  $E \subset \mathbb{R}^d$  and is the unique strong solution of the following SDE on  $[0, \infty)$ :

$$X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z) N(ds, dz) + \int_0^t \int_{\mathcal{X}} G(X_{s-}, z) \tilde{N}(ds, dz),$$

for any  $x_0 \in E$ , where where  $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$  is a measurable space,

- $B = (B^{(i)} : i = 1, \dots, d)$  is a  $d$ -dimensional Brownian motion;
- $N$  is a Poisson Point Measure (PPM) on  $\mathbb{R}^+ \times \mathcal{X}$  with intensity  $dsq(dz)$ , where  $q$  is a sigma finite measure on  $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ ; and  $\tilde{N}$  is the compensated measure of  $N$ .
- $N, B$  and  $X_0$  are independent;
- $b = (b^{(i)} : i = 1, \dots, d)$  and  $\sigma = (\sigma_j^{(i)} : i, j = 1, \dots, d)$  and  $H$  and  $G$  are Borelian functions locally bounded in  $E$ .

Moreover, we follow the classical convention (see chapter II in [16]) and we assume that  $HG = 0$ ,  $G$  is bounded and for any  $t \geq 0$ ,

$$\int_0^t \int_{\mathcal{X}} |H(X_{s-}, z)| N(ds, dz) < \infty \quad \text{a.s.}, \quad \mathbb{E} \left( \int_0^t \int_{\mathcal{X}} \|G(X_{s- \wedge \sigma_n}, z)\|_2^2 dsq(dz) \right) < \infty,$$

for some sequence of stopping time  $\sigma_n \uparrow \infty$ . We do not discuss here the conditions which ensure the strong existence and uniqueness of the SDE for any initial condition. This will be classical results for the examples considered in this paper and we refer to [9] for some general statement relevant in our context.

### 3.1 Main result

We need a transformation  $F$  to construct a suitable distance and evaluate the gap between the process  $X$  and the associated dynamical system on a domain  $D$ .

**Assumption 3.1.** (i) The domain  $D$  is an open subset of  $\mathbb{R}^d$  and the function  $F$  is defined on an open set  $O$  which contains  $\overline{D \cup E}$ .

(ii)  $F \in \mathcal{C}^2(O, \mathbb{R}^d)$  and is a bijection from  $D$  into  $F(D)$  and its Jacobian  $J_F$  is invertible on  $D$ .

(iii) For any  $x \in E$ ,

$$\int_{\mathcal{X}} |F(x + H(x, z)) - F(x)| q(dz) < \infty.$$

(iv) The functions  $x \in E \rightarrow b(x)$  and  $x \in E \rightarrow \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz)$  are extendable to the domain  $D \cup E$ , so that this extension is locally Lipschitz on  $D$ .

Under this assumption,  $F$  is a  $C^2$  diffeomorphism from  $D$  into  $F(D)$ . We require in (iii) that the big jumps of  $F(X)$  can be compensated once we have applied the transformation  $F$ . This assumption could be relaxed by letting the big jumps in an additional term with finite variations, which is not contributing to the drift component  $b_F$ . This could be achieved by using the finite variation part  $A_t$  of the semi-martingale  $R_t$  in the previous Section. But that won't be used in the applications given here. Under Assumption 3.1, can set for any  $x \in D$ ,

$$b_F(x) = b(x) + J_F(x)^{-1} \left( \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz) \right)$$

and we observe that  $b_F$  is locally Lipschitz on  $D$ . Let us then define the associated flow  $\phi_F$  for  $x_0 \in D$  as the unique solution of

$$\phi_F(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \phi_F(x_0, t) = b_F(\phi_F(x_0, t)),$$

for  $t \in [0, T'(x_0))$ , where  $T'(x_0)$  is some positive real number which gives the maximal time interval on which the solution exists and belongs to  $D$ .

We introduce now the vector field  $\psi_F$  defined by

$$\psi_F = (J_F b_F) \circ F^{-1}$$

on the open set  $F(D)$ . We remark that  $\psi_F$  is also locally Lipschitz on  $F(D)$  and

$$\psi_F \circ F(x) = J_F(x) b(x) + \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz) = \mathcal{L}F(x),$$

where  $\mathcal{L}$  is the infinitesimal generator of  $X$ . We also set for any  $x \in D$ ,

$$\tilde{b}_F(x) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 F}{\partial x_i \partial x_j}(x) \sum_{k=1}^d \sigma_k^{(i)}(x) \sigma_k^{(j)}(x) + \int_{\mathcal{X}} [F(x + G(x, z)) - F(x) - J_F(x)G(x, z)] q(dz)$$

and

$$V_F(x) = \sum_{i,j,k=1}^d \frac{\partial F}{\partial x_i}(x) \frac{\partial F}{\partial x_j}(x) \sigma_k^{(i)}(x) \sigma_k^{(j)}(x) + \int_{\mathcal{X}} [F(x + H(x, z) + G(x, z)) - F(x)]^2 q(dz).$$

Finally we are using the following distance on  $D \cup E$  to compare the process and the dynamical system

$$d_F(x, y) = \|F(x) - F(y)\|_2$$

and we define

$$T_{D,\varepsilon,F}(x_0) = \inf\{t \in [0, T'(x_0)) : \exists x \notin D, d_F(x, \phi_F(x_0, t)) \leq \varepsilon\} \wedge T'(x_0) \quad (2)$$

which yields the time when the dynamical system is at distance  $\varepsilon$  from the boundary of  $D$  for the distance  $d_F$ .

**Theorem 3.2.** *Under Assumption 3.1, we assume that  $\psi_F$  is  $(L, \alpha)$  non-expansive on  $F(D)$ . Then for any  $\varepsilon > 0$  and  $x_0 \in E \cap D$  and  $T < T_{D,\varepsilon,F}(x_0)$  such that  $4\alpha T \exp(2LT) \leq \varepsilon$ , we have*

$$\mathbb{P}_{x_0} \left( \sup_{t \leq T} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right) \leq C_d e^{4LT} \int_0^T \bar{V}_{F,\varepsilon}(x_0, s) ds,$$

where  $C_d$  is a positive constant depending only on the dimension  $d$  and

$$\bar{V}_{F,\varepsilon}(x_0, s) = \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, s)) \leq \varepsilon}} \left\{ \varepsilon^{-2} \|V_F(x)\|_1 + \varepsilon^{-1} \|\tilde{b}_F(x)\|_1 \right\}.$$

We refer to the two next Sections for examples and applications, which involve different choices for  $F$  and  $(L, \alpha)$  non expansivity, with potentially  $\alpha$  or  $L$  equal to 0. The key assumption here concerns the non-expansivity of  $\psi_F$  for a suitable choice of  $F$ . Before the proof of the Theorem, let us give here a useful criterion for  $L$  non-expansivity in the diffusion case ( $q = 0$  and  $X$  continuous), which will be useful in Section 5.

*Example.* We recall from the first Section (or table 1 in [2]) that when  $F(D)$  is convex and  $\psi_F$  is differentiable on  $F(D)$ ,  $\psi_F$  is  $L$  non-expansive on  $F(D)$  iff  $\text{Sp}(J_{\psi_F}(y) + J_{\psi_F}^*(y)) \subset (-\infty, 2L]$  for any  $y \in F(D)$ . Choosing

$$F(x) = (f_i(x_i) : i = 1, \dots, d)$$

in the case  $q = 0$  and setting  $A(x) = J_{\psi_F}(F(x))$ , we have for any  $i, j = 1, \dots, d$  such that  $i \neq j$

$$A_{ij}(x) = \frac{f_i'(x_i)}{f_j'(x_j)} \frac{\partial}{\partial x_j} b^{(i)}(x), \quad A_{ii}(x) = \frac{\partial}{\partial x_i} b^{(i)}(x) + \frac{f_i''(x_i)}{f_i'(x_i)} b^{(i)}(x). \quad (3)$$

Then  $\psi_F$  is  $L$  non-expansive on  $F(D)$  iff the largest eigenvalue of  $A(x) + A^*(x)$  is less than  $2L$  for any  $x \in D$ .

*Proof of Theorem 3.2.* First, we construct a function  $\tilde{F} \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ , which coincide with  $F$  on  $E \cup D$ . It can be achieved by considering  $\varphi F$  where  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$  is equal to 0 on the complementary set of  $O$  and to 1 on  $\overline{D \cup E}$ , since these two sets are disjoint closed sets,

see the smooth Urysohn lemma. For simplicity we note  $F$  for  $\widetilde{F}$ . Using Itô's formula (see Chapitre 2, Theorem 5.1 in [16]), we have :

$$\begin{aligned}
F(X_t) &= F(x_0) + \int_0^t \sum_{i=1}^d \frac{\partial F}{\partial x_i}(X_s) b^{(i)}(X_s) ds + \sum_{i,j=1}^d \frac{\partial F}{\partial x_i}(X_s) \sigma_j^{(i)}(X_s) dB_s^{(j)} \\
&\quad + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 F}{\partial x_i \partial x_j} \sum_{k=1}^d \sigma_k^{(i)}(X_s) \sigma_k^{(j)}(X_s) ds \\
&\quad + \int_0^t \int_E [F(X_{s-} + H(X_{s-}, z)) - F(X_{s-})] N(ds, dz) \\
&\quad + \int_0^t \int_E [F(X_{s-} + G(X_{s-}, z)) - F(X_{s-})] \widetilde{N}(ds, dz) \\
&\quad + \int_0^t \int_E \left[ F(X_{s-} + G(X_s, z)) - F(X_s) - \sum_{i=1}^d \frac{\partial F}{\partial x_i}(X_s) G^{(i)}(X_s, z) \right] ds q(dz)
\end{aligned}$$

Then the  $\mathcal{F}_t$ -semimartingale  $Y_t = F(X_t)$  can be written as

$$Y_t = Y_0 + \int_0^t \psi_F(Y_s) ds + A_t + M_t^c + M_t^d, \quad (4)$$

where  $\psi_F(F(x)) = J_F(x) b_F(x) = J_F(x) b(x) + \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz)$ ;

$$A_t = \int_0^t \widetilde{b}_F(X_s) ds$$

is a continuous  $\mathcal{F}_t$ -adapted process with bounded variations paths;

$$M_t^c = \int_0^t \sum_{i,j=1}^d \frac{\partial F}{\partial x_i}(X_s) \sigma_j^{(i)}(X_s) dB_s^{(j)}$$

is a continuous  $\mathcal{F}_t$ -local martingale and writing  $K = G + H$ ,

$$M_t^d = \int_0^t \int_{\mathcal{X}} [F(X_{s-} + K(X_{s-}, z)) - F(X_{s-})] \widetilde{N}(ds, dz)$$

is a càdlàg  $\mathcal{F}_t$ -local martingale purely discontinuous thanks to Assumption 3.1 (iii).

We observe that the dynamical system  $y_t = F(\phi_F(x_0, t))$  satisfies

$$y_0 = F(x_0), \quad y'_t = J_F(\phi_F(x_0, t)) b_F(\phi_F(x_0, t)) = \psi_F(y_t)$$

and it is associated with the vector field  $\psi_F$ . Moreover, using the notations of Section 2.1 the first time  $T_{F(D), \varepsilon}^y(y_0)$  when  $y$  is at distance  $\varepsilon$  from the boundary of the domain  $F(D)$  coincide with  $T_{D, \varepsilon, F}(x_0)$  defined by (2):

$$T_{F(D), \varepsilon}^y(y_0) = \inf\{t \in [0, T'(x_0)) \exists x \notin F(D), F(\phi(x_0, t)) \in \overline{B}(x, \varepsilon)\} \wedge T'(x_0) = T_{D, \varepsilon, F}(x_0).$$

Adding that  $\psi_F$  is  $(L, \alpha)$  non-expansive on  $F(D)$  and  $4\alpha T \exp(2LT) \leq \varepsilon$ , we apply now Proposition 2.3 to  $Y$  on  $F(D)$  with  $p = q = 1$  and  $X_0 = x_0$ . Then, for any  $T < T_{D,\varepsilon,F}(x_0)$  such that

$$\begin{aligned} \mathbb{P}(S_T \geq \varepsilon) &\leq C_d e^{4LT} \left[ \varepsilon^{-1} \mathbb{E} \left( \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \| |A|_s \|_1 \right) \right. \\ &\quad \left. + \varepsilon^{-2} \mathbb{E} \left( \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \langle M^c \rangle_s \|_1 \right) + \varepsilon^{-2} \mathbb{E} \left( \sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \| \Delta Y_s \|_2^2 \right) \right] \end{aligned} \quad (5)$$

for some constant  $C_d$  positive, since  $X_0 = x_0$  ensures that the first probability on right hand side in Proposition 2.3 is null. Using now

$$\langle M^c \rangle_t = \int_0^t \sum_{i,j,k=1}^d \frac{\partial F}{\partial x_i}(X_s) \frac{\partial F}{\partial x_j}(X_s) \sigma_k^{(i)}(X_s) \sigma_k^{(j)}(X_s) ds,$$

we get

$$\int_0^t \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \langle M^c \rangle_s \|_1 \leq \int_0^t \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, s)) \leq \varepsilon}} \sum_{i,j,k,l=1}^d \frac{\partial F^{(l)}}{\partial x_i}(X_s) \frac{\partial F^{(l)}}{\partial x_j}(X_s) \sigma_k^{(i)}(X_s) \sigma_k^{(j)}(X_s) ds,$$

since here  $S_t = \sup_{s \leq t} \| Y_s - y_s \|_2 = \sup_{s \leq t} d_F(X_s, \phi_F(x_0, s))$ . Similarly,

$$\begin{aligned} \mathbb{E} \left( \sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \| \Delta Y_s \|_2^2 \right) &= \mathbb{E} \left( \int_0^T \int_{\mathcal{X}} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \| F(X_{s-} + K(X_{s-}, z)) - F(X_{s-}) \|_2^2 ds q(dz) \right) \\ &\leq \int_0^T \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, s)) \leq \varepsilon}} \int_{\mathcal{X}} \| F(x + K(x, z)) - F(x) \|_2^2 q(dz) ds \end{aligned}$$

and combining the two last inequalities we get

$$\mathbb{E} \left( \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \langle M^c \rangle_s \|_1 \right) + \mathbb{E} \left( \sum_{s \leq T} \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} \| \Delta Y_s \|_2^2 \right) \leq \int_0^T \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, s)) \leq \varepsilon}} \| V_F(x) \|_1 ds. \quad (6)$$

Finally,

$$\mathbb{E} \left( \int_0^T \mathbf{1}_{\{S_{s-} \leq \varepsilon\}} d \| |A|_s \|_1 \right) \leq \int_0^T \sup_{\substack{x \in E \\ d_F(x, \phi_F(x_0, s)) \leq \varepsilon}} \| \tilde{b}_F(x) \|_1 ds \quad (7)$$

and the conclusion comes by plugging the two last inequalities in (5).  $\square$

### 3.2 Adjunction of non-expansive domains

We provide now an extension of Theorem 3.2, which allows to study cases where a family of transformation is used to get non-expansivity of the vector field along the whole trajectory of the dynamical system. It will be useful to study two-dimensional competitive diffusions in Section 5. We use again the notation

$$b_F(x) = b(x) + J_F(x)^{-1} \left( \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz) \right)$$

when it is well defined and we decompose the domain  $D$  into a family of subdomains  $D_i$  and require the following set of assumptions.

**Assumption 3.3.** (i) The domains  $D$  and  $(D_i : i = 1, \dots, N)$  are open subsets of  $\mathbb{R}^d$  and the functions  $F_i$  are defined from an open set  $O_i$  which contains  $\overline{D_i}$  and

$$D \subset \cup_{i=1}^N D_i, \quad F_i \in \mathcal{C}^2(O_i, \mathbb{R}^d).$$

Moreover  $F_i$  is a bijection from  $D_i$  into  $F(D_i)$  whose Jacobian is invertible on  $D_i$ .

(ii) There exist a distance  $d$  on  $\cup_{i=1}^N D_i \cup E$  and  $c_1, c_2 > 0$  such that for any  $i \in \{1, \dots, N\}$ ,  $x, y \in D_i$ ,

$$c_1 d(x, y) \leq \|F_i(x) - F_i(y)\|_2 \leq c_2 d(x, y).$$

(iii) For each  $i \in \{1, \dots, N\}$ , for any  $x \in E \cap D_i$ ,

$$\int_{\mathcal{X}} |F_i(x + H(x, z)) - F_i(x)| q(dz) < \infty.$$

Moreover the functions  $x \in E \cap D_i \rightarrow b(x)$  and  $x \in E \cap D_i \rightarrow \int_{\mathcal{X}} [F_i(x + H(x, z)) - F_i(x)] q(dz)$  are extendable to the domain  $D_i$  and these extensions are still locally Lipschitz.

(iv) There exist a continuous flow  $\phi(\cdot, \cdot)$ ,  $\varepsilon_0 \in (0, 1)$ ,  $\kappa \geq 1$  and a positive function  $T(\cdot)$  such that for any  $x_0 \in D$  ( $t(k, x_0) : k = 0, \dots, \kappa$ ) and ( $n(k, x_0) : k = 1, \dots, \kappa - 1$ ) such that

$$0 = t(0, x_0) \leq t(1, x_0) \leq \dots \leq t(\kappa, x_0) = T(x_0), \quad n(k, x_0) \in \{1, \dots, N\}$$

and for any  $\varepsilon \in (0, \varepsilon_0]$ ,  $x_0 \in D$ ,  $k \leq \kappa - 1$  and  $t \in (t(k, x_0), t(k + 1, x_0))$

$$\phi(x_0, 0) = x_0, \quad \overline{B}_d(\phi(x_0, t), \varepsilon) \subset D_{n(k, x_0)} \quad \text{and} \quad \frac{\partial}{\partial t} \phi(x_0, t) = b_{F_{n(k, x_0)}}(\phi(x_0, t)).$$

This allows us to make an adjunction of domains  $D_i$  where we can find a transformation  $F_i$  to obtain a non-expansive vector field and then compare the process to a dynamical system which may change of domains a finite number of times. This adjunction will be used in the continuous case ( $H = 0$ ) in Section 5. Let us recall that in that case  $b_F = b$  does not depend on the transformation  $F$ , we require that  $\overline{B}_d(\phi(x_0, t), \varepsilon)$  is included in one of the subdomains, while  $\phi$  is directly given by

$$\phi(x_0, 0) = x_0 \quad \frac{\partial}{\partial t} \phi(x_0, t) = b(\phi(x_0, t)).$$

Recalling the notation  $\psi_F = (J_F b_F) \circ F^{-1}$ ,

$$\widetilde{b}_F(x) = \sum_{i,j=1}^d \frac{1}{2} \frac{\partial^2 F}{\partial x_i \partial x_j}(x) \sigma^{(i)}(x) \sigma^{(j)}(x) + \int_{\mathcal{X}} [F(x + G(x, z)) - F(x) - J_F(x)G(x, z)] q(dz)$$

and

$$V_F(x) = \sum_{i,j,k=1}^d \frac{\partial F}{\partial x_i}(x) \frac{\partial F}{\partial x_j}(x) \sigma_k^{(i)}(x) \sigma_k^{(j)}(x) + \int_{\mathcal{X}} [F(x + H(x, z) + G(x, z)) - F(x)]^2 q(dz),$$

the result can be stated as follows.



**Corollary 3.4.** Under Assumption 3.3, we assume that for each  $i \in \{1, \dots, N\}$ ,  $\psi_{F_i}$  is  $(L_i, \alpha_i)$  non-expansive on  $F_i(D_i)$ .

Then for any  $x_0 \in E \cap D$  and  $\varepsilon$  small enough and  $T \geq 0$  such that  $4\alpha_i T \exp(2L_i T) \leq \varepsilon$  for  $i \in \{1, \dots, N\}$ , we have

$$\mathbb{P}_{x_0} \left( \sup_{t \leq T \wedge T(x_0)} d(X_t, \phi(x_0, t)) \geq \varepsilon \right) \leq C \sum_{k=0}^{\kappa-1} \int_{t(k, x_0) \wedge T}^{t(k+1, x_0) \wedge T} \bar{V}_{d, \varepsilon}(F_{n(k, x_0)}, x_0, t) dt,$$

where  $C$  is a positive constant which depends on  $d, c_1, c_2, (L_i : i = 1, \dots, N)$  and  $\kappa$  and

$$\bar{V}_{d, \varepsilon}(F, x_0, s) = \sup_{\substack{x \in E \\ d(x, \phi(x_0, s)) \leq \varepsilon}} \left\{ \varepsilon^{-2} \|V_F(x)\|_1 + \varepsilon^{-1} \|\tilde{b}_F(x)\|_1 \right\}.$$

The proof follows the proof of Theorem 3.2 and rely on Proposition 2.3. More precisely, we need the following lemma which is an adaptation of the previous proof. We note

$$b_n(T) = 2\sqrt{2} \exp(L_n T), \quad a_n(T) = b_n(T) \frac{c_2}{c_1}, \quad \varepsilon_n(T) = \frac{c_1 \varepsilon_0}{c_2 b_n(T)} = \frac{\varepsilon_0}{a_n(T)},$$

for  $n = 1, \dots, N$  and observe that  $a_n(T) \geq 1$ .

**Lemma 3.5.** Under the assumptions of Corollary 3.4, for any  $x_0 \in E \cap D$ ,  $k = 0, \dots, \kappa - 1$ ,  $\varepsilon \in (0, \varepsilon_{n(k, x_0)})$  and  $T \geq 0$  such that  $4\alpha_{n(k, x_0)} T \exp(2L_{n(k, x_0)} T) \leq \varepsilon$ , we have

$$\begin{aligned} & \mathbb{P}_{x_0} \left( \sup_{t(k, x_0) \leq t \leq t(k+1, x_0) \wedge T} d(X_t, \phi(x_0, t)) \geq \varepsilon a_{n(k, x_0)}(T) \right) \\ & \leq \mathbb{P}(d(X_{t(k, x_0)}, \phi(x_0, t(k, x_0))) \geq \varepsilon) + C_{d, c_1} \int_{t(k, x_0) \wedge T}^{t(k+1, x_0) \wedge T} \bar{V}_{d, \varepsilon a_{n(k, x_0)}(T)}(F_{n(k, x_0)}, x_0, s) ds \end{aligned}$$

where  $C_{d, c_1}$  is a positive constant which depends on  $d$  and  $c_1$ .

*Proof.* We apply Proposition 2.3 to the process  $F_{n(k, x_0)}(X)$  on the time interval  $[t(k, x_0), t(k+1, x_0) \wedge T)$ . For that purpose we first note that by Assumption 3.3 (iv), we have

$$\frac{\partial}{\partial t} \phi(x_0, t) = b_{F_{n(k, x_0)}}(\phi(x_0, t))$$

on the time interval  $(t(k, x_0), t(k+1, x_0) \wedge T)$ . Thus  $t \rightarrow \phi(x_0, t)$  can be derived on the right at time  $t(k, x_0)$  and the equality hold on the interval  $[t(k, x_0), t(k+1, x_0) \wedge T)$ . As in the previous proof, we use a function  $\tilde{F}_i \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$  which coincides with  $F_i$  on  $D_i$ . Then we write the semimartingale decomposition of  $\tilde{F}_i(Y_t)$  on the time interval  $[t(k, x_0), t(k+1, x_0) \wedge T)$  and we get the counterpart of (4) :

$$Y_t = F_{n(k, x_0)}(X_t) = Y_{t(k, x_0)} + \int_{t(k, x_0)}^t \psi_{F_{n(k, x_0)}}(Y_s) ds + A_t + M_t^c + M_t^d. \quad (8)$$

Let us note

$$\begin{aligned} T(k, x_0, \varepsilon) &= T_{D_{n(k, x_0)}, \varepsilon, \tilde{F}_{n(k, x_0)}} \wedge t(k+1, x_0) \\ &= \inf \left\{ t \in [t(k, x_0), T(x_0)] : \exists x \notin D_{n(k, x_0)}, d_{\tilde{F}_{n(k, x_0)}}(x, \phi(x_0, t)) \leq \varepsilon \right\} \wedge t(k+1, x_0) \end{aligned}$$

and

$$I(k, x_0, T) = [t(k, x_0), T \wedge T(k, x_0, \varepsilon)].$$

Moreover  $\widetilde{F}_n = F_n$  and  $\psi_{\widetilde{F}_n} = \psi_{F_n}$  on  $\widetilde{F}_n(D_n) = F_n(D_n)$  and this vector field is  $(\alpha_n, L_n)$  non-expansive on  $F_n(D_n)$ . By Proposition 2.3 for  $p = q = 1$ , we get for  $\varepsilon > 0$  and  $T$  such that  $4\alpha_n(k, x_0)T \exp(2L_n(k, x_0)T) \leq \varepsilon$ ,

$$\begin{aligned} & \mathbb{P}_{x_0} \left( \sup_{t \in I(k, x_0, T)} d_{\widetilde{F}_n(k, x_0)}(X_t, \phi(x_0, t)) \geq \varepsilon \right) \\ & \leq \mathbb{P} \left( d_{\widetilde{F}_n(k, x_0)}(X_{t(k, x_0)}, x_{t(k, x_0)}) \geq \varepsilon / b_n(k, x_0)(T) \right) + C_1 \varepsilon^{-1} \mathbb{E} \left( \int_{I(k, x_0, T)} \mathbf{1}_{\{S_s \leq \varepsilon\}} d \|A|_s\|_1 \right) \\ & \quad + C_{1,d} \varepsilon^{-2} \left[ \mathbb{E} \left( \int_{I(k, x_0, T)} \mathbf{1}_{\{S_s \leq \varepsilon\}} d \| \langle M^c \rangle_s \|_1 \right) + \mathbb{E} \left( \sum_{s \in I(k, x_0, T)} \mathbf{1}_{\{S_s \leq \varepsilon\}} \|\Delta Y_s\|_2^2 \right) \right] \end{aligned}$$

Recalling (6) and (7), we obtain

$$\begin{aligned} & \mathbb{P}_{x_0} \left( \sup_{t \in I(k, x_0, T)} d_{\widetilde{F}_n(k, x_0)}(X_t, \phi(x_0, t)) \geq \varepsilon \right) \\ & \leq \mathbb{P}(d_{\widetilde{F}_n(k, x_0)}(X_{t(k, x_0)}, x_{t(k, x_0)}) \geq \varepsilon / b_n(k, x_0)(T)) + C'_d \int_{I(k, x_0, T)} \overline{V}_{\widetilde{F}_n(k, x_0), \varepsilon}(x_0, s) ds. \quad (9) \end{aligned}$$

for some constant  $C'_d$  positive. Now we use Assumption 3.3 (ii), which we write

$$c_1 d(x, y) \leq d_{\widetilde{F}_n(k, x_0)}(x, y) \leq c_2 d(x, y)$$

for  $x, y \in D_n(k, x_0)$  since then  $\widetilde{F}_n(k, x_0) = F_n(k, x_0)$ . By Assumption 3.3 (iv), for any  $t \in [t(k, x_0), t(k+1, x_0)]$  and  $\varepsilon \in (0, c_1 \varepsilon_0)$ ,

$$\overline{B}_{d_{\widetilde{F}_n(k, x_0)}}(\phi(x_0, t), \varepsilon) \subset D_n(k, x_0).$$

It ensures that

$$T_{D_n(k, x_0), \varepsilon, \widetilde{F}_n(k, x_0)} \geq t(k+1, x_0)$$

and  $I(k, x_0, T) = [t(k, x_0), T \wedge t(k+1, x_0)]$ . Moreover, for any  $t \in I(k, x_0, T)$ ,

$$\left\{ \sup_{t \in I(k, x_0, T)} d_{\widetilde{F}_n(k, x_0)}(X_t, \phi(x_0, t)) < \varepsilon \right\} \subset \left\{ \sup_{t \in I(k, x_0, T)} d(X_t, \phi(x_0, t)) < \varepsilon / c_1 \right\}$$

and

$$\left\{ d(X_{t(k, x_0)}, x_{t(k, x_0)}) < \varepsilon / (c_2 b_n(k, x_0)(T)) \right\} \subset \left\{ d_{\widetilde{F}_n(k, x_0)}(X_{t(k, x_0)}, x_{t(k, x_0)}) < \varepsilon / b_n(k, x_0)(T) \right\}$$

and

$$\overline{V}_{\widetilde{F}_n(k, x_0), \varepsilon}(x_0, s) \leq (c_1^{-1} + c_1^{-2}) \overline{V}_{d, \varepsilon / c_1}(F_n(k, x_0), x_0, s).$$

We obtain from (9)

$$\begin{aligned} & \mathbb{P}_{x_0} \left( \sup_{t \in I(k, x_0, T)} d(X_t, \phi(x_0, t)) \geq \varepsilon / c_1 \right) \\ & \leq \mathbb{P} \left( d(X_{t(k, x_0)}, x_{t(k, x_0)}) \geq \frac{\varepsilon}{c_2 b_n(k, x_0)(T)} \right) + C'_d (c_1^{-1} + c_1^{-2}) \int_{t(k, x_0)}^{T \wedge t(k+1, x_0)} \overline{V}_{d, \varepsilon / c_1}(F_n(k, x_0), x_0, s) ds. \end{aligned}$$

Using finally the quasi left continuity of  $X$ , this inequality can be extended to the closed interval  $I(k, x_0, T) = [t(k, x_0), T \wedge t(k+1, x_0)]$ , which ends up the proof by replacing  $\varepsilon$  by  $\varepsilon c_2 b_{n(k, x_0)}(T)$ .  $\square$

*Proof of Corollary 3.4.* Let  $x_0 \in E \cap D$  and  $\varepsilon < \min(\varepsilon_i : i = 1, \dots, N)$  and  $T > 0$  such that  $4\alpha_i T \exp(2LT) \leq \varepsilon$  for each  $i = 1, \dots, N$ . Lemma 3.5 and Markov property at time  $t(k, x_0) \wedge T$  ensure that

$$\begin{aligned} \mathbb{P}_{x_0} \left( \sup_{t(k, x_0) \leq t \leq t(k+1, x_0) \wedge T} d(X_t, \phi(x_0, t)) \geq \varepsilon a_{n(k, x_0)}(T), \quad \sup_{t \leq t(k, x_0) \wedge T} d(X_t, \phi(x_0, t)) < \varepsilon \right) \\ \leq C_{d, c_1} \int_{t(k, x_0) \wedge T}^{t(k+1, x_0) \wedge T} \bar{V}_{d, \varepsilon/a_{n(k, x_0)}(T)}(F_{n(k, x_0)}, x_0, s) ds. \end{aligned}$$

for each  $k = 0, \dots, \kappa - 1$ . Noting  $A_k(x_0, T) = \prod_{i \leq k} a_{n(i, x_0)}(T)$  and recalling that  $a_i(T) \geq 1$ , by iteration we obtain for  $\varepsilon \leq \min(\varepsilon_i : i = 1, \dots, N)/A_\kappa(x_0, T)$ ,

$$\begin{aligned} \mathbb{P}_{x_0} \left( \bigcup_{k=0}^{\kappa-1} \left\{ \sup_{t(k, x_0) \leq t \leq t(k+1, x_0) \wedge T} d(X_t, \phi(x_0, t)) \geq \varepsilon A_k(x_0, T) \right\} \right) \\ \leq \sum_{k=0}^{\kappa-1} C_{d, c_1} \int_{t(k, x_0) \wedge T}^{t(k+1, x_0) \wedge T} \bar{V}_{d, \varepsilon A_k(x_0, T)}(F_{n(k, x_0)}, x_0, s) ds. \end{aligned}$$

This ensures that

$$\begin{aligned} \mathbb{P}_{x_0} \left( \sup_{0 \leq t \leq T \wedge T(x_0)} d(X_t, \phi(x_0, t)) \geq \varepsilon A_\kappa(x_0, T) \right) \\ \leq C_{d, c_1} A_\kappa(x_0, T)^2 \sum_{k=0}^{\kappa-1} \int_{t(k, x_0) \wedge T}^{t(k+1, x_0) \wedge T} \bar{V}_{d, \varepsilon A_\kappa(x_0, T)}(F_{n(k, x_0)}, x_0, s) ds, \end{aligned}$$

Adding that  $(n(k, x_0) : k = 0, \dots, \kappa)$  takes value in a finite set and that  $A_\kappa(x_0, T)$  is bounded with respect to  $x_0$  by a constant depending only on  $\kappa, (L_i : i = 1, \dots, N), c_1$  and  $c_2$  yields the result.  $\square$

## 4 Coming down from infinity for one-dimensional Stochastic Differential Equation

In this section  $E \subset \mathbb{R}$ ,  $D = (a, \infty)$  for some  $a > 0$  and following the beginning of the previous Section, we assume that  $X$  is a càdlàg Markov process which takes values in  $E$  and is the unique strong solution of the following SDE on  $[0, \infty)$  :

$$X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z) N(ds, dz) + \int_0^t \int_{\mathcal{X}} G(X_{s-}, z) \tilde{N}(ds, dz),$$

for any  $x_0 \in E$ , where  $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$  is a measurable space;  $B$  is a Brownian motion;  $N$  is a Punctual Point Measure (PPM) on  $\mathbb{R}^+ \times \mathcal{X}$  with intensity  $dsq(dz)$ ;  $N, B$  and  $X_0$  are independent. We make the following Assumption, which is the counterpart of Assumption 3.1 and will be convenient.

**Assumption 4.1.** Let  $F \in \mathcal{C}^2((a', \infty), \mathbb{R})$ , for some  $a' \in [-\infty, a)$  such that  $\bar{E} \subset (a', \infty)$ .

(i) For any  $x > a$ ,  $F'(x) > 0$  and  $F(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

(ii) For any  $x \in E$ ,

$$\int_{\mathcal{X}} |F(x + H(x, z)) - F(x)| q(dz) < \infty$$

and

$$b_F(x) = b(x) + F'(x)^{-1} \left( \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz) \right)$$

can be extended to  $E \cup (a, \infty)$  and that this extension is locally Lipschitz on  $(a, \infty)$ .

(iii) For any  $x > a$ ,  $b_F(x) < 0$ .

Let us first note that  $F$  is then a bijection from  $(a, \infty)$  into  $(F(a), \infty)$ .

As in the previous Sections, we consider now the flow  $\phi_F$  given for  $x_0 \in (a, \infty)$  by

$$\phi_F(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \phi_F(x_0, t) = b_F(\phi_F(x_0, t)),$$

which is well and uniquely defined and belongs to  $(a, \infty)$  on a maximal time interval denoted by  $[0, T'(x_0))$ , where  $T'(x_0) \in (0, \infty]$ . We first observe that  $\phi_F(x_0, t)$  is increasing with respect to  $x_0$ . It can be seen by recalling that the local Lipschitz property ensures the uniqueness of solutions and thus it prevents the trajectories from intersecting. Then  $T'(x_0)$  is increasing and its limit when  $x_0 \uparrow \infty$  is denoted  $T'(\infty)$ . Moreover

$$\phi_F(\infty, x_0) = \lim_{x_0 \rightarrow \infty} \phi_F(x_0, t)$$

is well defined for any  $t \in [0, T'(\infty))$ .

Using that  $b_F(x) < 0$  for  $x \in (a, \infty)$ , we get for any  $x_0 \in (a, \infty)$ , and  $t < T'(x_0)$ ,

$$\int_{x_0}^{\phi(x_0, t)} \frac{1}{b_F(x)} dx = t.$$

Either

$$\int_{\infty}^{\cdot} \frac{1}{b_F(x)} < +\infty$$

and then  $\phi(\infty, t) < \infty$  for any  $t \in (0, T'(\infty))$  and  $t \in [0, T'(\infty)) \rightarrow \phi(\infty, t)$  is continuous. We then say that the dynamical system *instantaneously comes down from infinity* and we have

$$\phi(\infty, t) = \inf \left\{ s \geq 0 : \int_{\infty}^s \frac{1}{b_F(x)} dx < t \right\}.$$

Otherwise,  $T'(\infty) = +\infty$  and  $\phi(\infty, t) = +\infty$  for any  $t \geq 0$ .

Our aim now is to derive an analogous criterion for stochastic differential equations using the results of the previous Section. Letting the process start from infinity requires some additional work. Let us give a condition useful for the identification of the adherence values of  $(\mathbb{P}_x : x \in E)$ .

**Definition 4.2.** The process  $X$  is stochastically monotone if for all  $x_0, x_1 \in E$  such that  $x_0 \leq x_1$ , for all  $t > 0$  and  $x \in \mathbb{R}$ , we have

$$\mathbb{P}_{x_0}(X_t \geq x) \leq \mathbb{P}_{x_1}(X_t \geq x).$$

The  $\Lambda$ -coalescent, the birth and death process, diffusions and several of their extensions satisfy this property. But the Transmission Control Protocol does not and we refer to the examples of forthcoming Section 4.2 for details.

#### 4.1 Weak convergence and coming down from infinity

We note  $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  which we endow with the distance

$$\bar{d}(x, y) = |e^{-x} - e^{-y}|$$

and recall that  $\bar{\mathbb{R}}$  is then polish. We also note  $w(f, \delta, [0, T]) = \sup_{s, t \leq T, |t-s| \leq \delta} \bar{d}(f_s, f_t)$ . The notations of the previous section become

$$\psi_F = (F' b_F) \circ F^{-1},$$

$$\tilde{b}_F(x) = F''(x)\sigma(x)^2 + \int_{\mathcal{X}} [F(x + G(x, z)) - F(x) - F'(x)G(x, z)]q(dz)$$

and

$$V_F(x) = (F'(x)\sigma(x))^2 + \int_{\mathcal{X}} [F(x + H(x, z) + G(x, z)) - F(x)]^2 q(dz).$$

In this Section, for convenience we introduce

$$\hat{V}_{F, \varepsilon}(a, t) = \sup_{x \in E \cap \mathcal{D}_{F, \varepsilon}(a, t)} \left\{ \varepsilon^{-2} V_F(x) + \varepsilon^{-1} \tilde{b}_F(x) \right\},$$

where using the convention  $F(\infty) = \infty$ ,  $F^{-1}(\infty) = \infty$ , we set

$$\mathcal{D}_{F, \varepsilon}(a, t) = \left( a, F^{-1}(F(\phi(\infty, t)) + \varepsilon) \right).$$

Finally, we introduce the following key assumption to use the results of the previous Section.

**Assumption 4.3.** *The vector field  $\psi_F$  is  $(L, \alpha)$  non-expansive on  $(F(a), \infty)$  and for any  $\varepsilon > 0$*

$$\int_0^\cdot \hat{V}_{F, \varepsilon}(a, t) dt < \infty. \quad (10)$$

We note that in dimension 1,  $\psi_F$  is  $(L, \alpha)$  non-expansive on  $(F(a), \infty)$  iff for all  $x > y > F(a)$ ,

$$\psi_F(x) - \psi_F(y) \leq L(x - y) + \alpha.$$

First we give sufficient conditions for the convergence of  $(\mathbb{P}_x)_{x \in E}$ .

**Proposition 4.4.** *We assume that  $X$  is stochastically monotone.*

(i) *If  $E = \{0, 1, 2, \dots\}$ , then  $(\mathbb{P}_x)_{x \in E}$  converges weakly as  $x \rightarrow \infty$  in the space of probability measures on  $\mathbb{D}([0, T], \bar{\mathbb{R}})$ .*

(ii) *If Assumptions 4.1 and 4.3 hold and  $\int_0^\cdot \frac{1}{b_F(x)} < +\infty$  and for any  $K > 0$ ,*

$$\lim_{\delta \rightarrow 0} \sup_{x_0 \in E, x_0 \leq K} \mathbb{P}_{x_0}(w(X, \delta, [0, T]) \geq \varepsilon) = 0, \quad (11)$$

*then  $(\mathbb{P}_x)_{x \in E}$  converges weakly as  $x \rightarrow \infty$  in the space of probability measures on  $\mathbb{D}([0, T], \bar{\mathbb{R}})$ .*

The convergence result (i) concern the discrete case  $\sigma = 0$ . It has been obtained in [11] when the limiting probability  $\mathbb{P}_\infty$  is known a priori and the process comes down from infinity. The proof of the tightness of (i) follows [11] and rely on the monotonicity and the fact that the states are non instantaneous, which is here due to our càdlàg assumption for any initial state space. The identification of the limit is derived directly from the monotonicity and the proof of (i) is actually a direct extension of Lemma in [4] for the case of birth and death processes. This proof is omitted.

The tightness argument for (ii) is different and can be applied to processes with a continuous part. Here the control of the fluctuations of the process for large values relies on the approximation by a continuous dynamical system using Assumption 4.3 and the previous Section. Then the tightness on compacts sets is guaranteed by (11) and we refer to the proof below.

In the next result, we assume that  $(\mathbb{P}_x)_{x \in E}$  converges and  $\mathbb{P}_\infty$  is then well defined as the limiting probability. We then determine under Assumption 4.3 when (and how) the process comes down from infinity.

**Theorem 4.5.** *We assume that Assumptions 4.1 and 4.3 hold and that  $(\mathbb{P}_x : x \in E)$  converges weakly as  $x \rightarrow \infty$  in the space of probability measures on  $\mathcal{D}([0, T], \mathbb{R})$  to  $\mathbb{P}_\infty$ .*

(i) If

$$\int_\infty \frac{1}{b_F(x)} < +\infty,$$

then

$$\mathbb{P}_\infty(\forall t > 0 : X_t < +\infty) = 1 \quad \text{and} \quad \mathbb{P}_\infty\left(\lim_{t \downarrow 0+} F(X_t) - F(\phi_F(\infty, t)) = 0\right) = 1.$$

(ii) Otherwise  $\mathbb{P}_\infty(\forall t \geq 0 : X_t = +\infty) = 1$ .

After the proof coming just below, we are considering several examples.

For the  $\Lambda$ -coalescent, we recover the speed of coming down from infinity of [5] using  $F = \log$  and in that case  $V$  is bounded.

For birth and death processes with polynomial death rates, we can use  $F(x) = x^\beta$  ( $\beta < 1$ ) and get a finer approximation of the process by a dynamical system. Then  $V$  is no longer bounded and has to be controled along the dynamical system coming down from infinity.

When proving that birth and death processes do not come down from infinity, we are looking for  $F$  such that  $V_F$  is bounded to use the result above, see the examples.

Let us turn to the proofs of the two last results and start with the following lemma, where we recall the notation  $D = (a, \infty)$ ,  $d_F(x, y) = |F(x) - F(y)|$  and

$$T_{D, \varepsilon, F}(x_0) = \inf\{t \in [0, T'(x_0)) : \exists x \notin D, d_F(x, \phi_F(x_0, t)) \leq \varepsilon\} \wedge T'(x_0).$$

from the previous Section.

**Lemma 4.6.** *Under Assumptions 4.1 and 4.3, for any  $x_0 \in E \cap D$ ,  $\varepsilon > 0$  and  $T < T_{D, \varepsilon, F}(x_0)$  such that  $4\alpha T \exp(2LT) \leq \varepsilon$ , we have*

$$\mathbb{P}_{x_0}\left(\sup_{t \leq T} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon\right) \leq C \exp(4LT) \int_0^T \hat{V}_{F, \varepsilon}(a, t) dt,$$

where  $C$  is a positive constant.

*Proof.* Assumption 3.1 and the  $(L, \alpha)$  non-expansivity of  $\psi_F$  are guaranteed respectively by Assumptions 4.1 and 4.3. Thus, we can apply Theorem 3.2 on the domain  $D = (a, \infty)$  and for any  $x_0 \in D \cap E$  and  $\varepsilon > 0$ , we have

$$\mathbb{P}_{x_0} \left( \sup_{t < T \wedge T_{D, \varepsilon, F}(x_0)} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right) \leq C \exp(4LT) \int_0^T \bar{V}_{F, \varepsilon}(x_0, s) ds.$$

Now let  $t < T_{D, \varepsilon, F}(x_0)$  and  $x \in E$  such that  $d_F(x, \phi_F(x_0, t)) \leq \varepsilon$ . Then

$$F(a) < F(x) \leq F(\phi_F(x_0, t)) + \varepsilon$$

and combining the monotonicities of the flow  $\phi_F$  and the function  $F$ ,

$$F(a) < F(x) \leq F(\phi(\infty, t)) + \varepsilon$$

Thus  $x \in \mathcal{D}_{F, \varepsilon}(a, t)$  and

$$\bar{V}_{F, \varepsilon}(x_0, t) \leq \hat{V}_{F, \varepsilon}(a, t),$$

which ends up the proof.  $\square$

*Proof of the Proposition 4.4 (ii).* The fact that  $X$  is a stochastically monotone Markov process ensures that for all  $x_0, x_1 \in E$ ,  $x_0 \leq x_1$ ,  $k \geq 0$ ,  $0 \leq t_1 \leq \dots \leq t_k$ ,  $a_1, \dots, a_k \in \mathbb{R}$ ,

$$\mathbb{P}_{x_0}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k) \leq \mathbb{P}_{x_1}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k).$$

It can be shown by induction by using the Markov property at time  $t_1$  and writing  $X_{t_1}^{x_1} = X_{t_1}^{x_0} + B$ , where  $X^x$  is the process  $X$  starting at  $x$  and  $B$  is a non-negative random variable  $\mathcal{F}_{t_1}$  measurable. Then

$$\mathbb{P}_{x_0}(X_{t_1} \geq a_1, \dots, X_{t_k} \geq a_k)$$

converges by monotonicity as  $x_0 \rightarrow \infty$ , which identifies the finite dimensional limiting distributions of  $(\mathbb{P}_x : x \in E)$  when  $x \rightarrow \infty$ .

Writing  $T^a = T'(\infty)/2 > 0$ , we note that  $\phi_F(\infty, T^a) \in (a, \infty)$  and  $\phi_F(x_0, T^a) \uparrow \phi_F(\infty, T^a)$ . Using the monotonicity of  $t \rightarrow \phi_F(x_0, t)$  herited from  $b_F < 0$ , there exists  $\varepsilon_0 \in (0, 1]$  and  $\bar{a} > a$  such that for any  $\varepsilon \leq \varepsilon_0$  and  $x_0 \in E \cap [\bar{a}, \infty)$ ,

$$T_{D, \varepsilon, F}(x_0) > T^a. \quad (12)$$

Then, for any  $\varepsilon \in (0, \varepsilon_0]$ , Lemma 4.6 yields

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} \left( \sup_{t \leq T^a \wedge T_0} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right) \leq C(\varepsilon, T_0), \quad (13)$$

and Assumption 4.1 guarantees that for each  $\varepsilon > 0$ ,  $C(\varepsilon, T_0) \rightarrow 0$  as  $T_0 \rightarrow 0$ .

Let us turn to the proof of the tightness. Without loss of generality we can assume  $a > 0$ . Noting that  $d(x, y) \leq |x - y|$  on  $[0, \infty)$  and writing  $\bar{d}_F(x, y) = \bar{d}(F(x), F(y))$ , we get for any  $T_0 \leq T^a$ ,

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} \left( \left\{ \sup_{t \leq T_0} \bar{d}_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right\} \cup \left\{ d_F(X_{T_0}, \phi_F(x_0, T_0)) \geq \varepsilon \right\} \right) \leq C(\varepsilon, T_0). \quad (14)$$

By monotone convergence of  $\phi_F(x_0, t)$  to  $\phi_F(\infty, t)$  and continuity of  $\phi_F(\infty, t)$  on  $[0, T'(\infty))$ , we obtain the uniform convergence of  $F(\phi_F(x_0, t))$  to  $F(\phi_F(\infty, t))$  in  $(\overline{\mathbb{R}}, d)$  for  $t \in [0, T^a]$ . Then

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} \left( \left\{ \sup_{t \leq T_0} \overline{d}_F(X_t, \phi_F(\infty, t)) > \varepsilon \right\} \cup \{d_F(X_{T_0}, \phi_F(\infty, T_0)) > \varepsilon\} \right) \leq C(\varepsilon, T_0).$$

Using the continuity of  $t \in [0, T^a] \rightarrow F(\phi_F(\infty, t))$  in  $(\overline{\mathbb{R}}, d)$ , we can find  $\delta > 0$  such that  $w(F(\phi(\infty, \cdot)), \delta, [0, T_0]) \leq \varepsilon$ . Then,

$$\{w(F(X), \delta, [0, T_0]) \geq 2\varepsilon\} \subset \left\{ \sup_{t \leq T_0} \overline{d}_F(X_t, \phi_F(\infty, t)) \geq \varepsilon \right\}.$$

Adding that  $\inf_{x \in (a, b)} F'(x) > 0$  for any  $b > a$  and  $d(x, y) \rightarrow 0$  as  $x, y \rightarrow \infty$ , for any  $\varepsilon, \eta > 0$ , there exist  $T, \delta > 0$  such that

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} \left( \{w(X, \delta, [0, T_0]) \geq \varepsilon\} \cup \{d_F(X_{T_0}, \phi_F(\infty, T_0)) > \varepsilon\} \right) \leq \eta.$$

The assumption  $\int_{\infty} \frac{1}{b_F(x)} < +\infty$  ensures that  $\phi_F(\infty, T) < \infty$ . Then  $K := F^{-1}(F(\phi_F(\infty, T_0)) + \varepsilon)$  is finite and recalling that  $\phi(x_0, T_0) \uparrow \phi(\infty, T_0)$  as  $x_0 \uparrow \infty$ ,

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} \left( \{w(X, \delta, [0, T_0]) \geq \varepsilon\} \cup \{X_{T_0} \geq K\} \right) \leq \eta.$$

Moreover (11) ensures that,

$$\lim_{\delta \rightarrow 0} \sup_{x_0 \in [0, K]} \mathbb{P}_{x_0} (w(X, \delta, [0, T - T_0]) \geq 2\varepsilon) = 0$$

Combining these limits we get

$$\lim_{\delta \rightarrow 0} \limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} (w(X, \delta, [0, T]) \geq 2\varepsilon) \leq 2\eta.$$

and the tightness is proved.  $\square$

*Proof of Theorem 4.5.* Using Assumptions 4.1 and 4.3, we can apply Lemma 4.6. We recall the notation  $T^a = T'(\infty)/2$  and  $T_{D, \varepsilon, F} > T^a$  for  $\varepsilon$  small enough and  $x_0$  large enough from (12) in the previous proof. We get for any  $t_0 \in (0, T^a)$  and  $T \leq T^a$  and  $\varepsilon$  small enough,

$$\limsup_{x_0 \rightarrow \infty} \mathbb{P}_{x_0} \left( \sup_{t_0 \leq t \leq T} d_F(X_t, \phi_F(x_0, t)) \geq \varepsilon \right) \leq C(\varepsilon, T),$$

where  $C(\varepsilon, T) \rightarrow 0$  as  $T \rightarrow 0$ .

We first consider the case (i) and the dynamical comes down from infinity. Then  $\phi(\infty, t) < \infty$  on  $[t_0, T^a]$  and  $\phi(x_0, \cdot)$  converges to  $\phi(\infty, \cdot)$  uniformly on  $[t_0, T^a]$  using again the monotonicity of the convergence and the continuity of the limit. We obtain for  $T \leq T^a$ ,

$$\limsup_{x_0 \rightarrow \infty} \mathbb{P}_{x_0} \left( \sup_{t_0 \leq t \leq T} d_F(X_t, \phi_F(\infty, t)) > \varepsilon \right) \leq C(\varepsilon, T),$$



and the weak convergence of  $(\mathbb{P}_x : x \in E)$  to  $\mathbb{P}_\infty$  yields

$$\mathbb{P}_\infty \left( \sup_{t_0 \leq t \leq T^a \wedge T} d_F(X_t, \phi_F(\infty, t)) > \varepsilon \right) \leq C(\varepsilon, T),$$

As the right hand side does not depend on  $t_0$ , letting  $t_0 \downarrow 0$  and then  $T \downarrow 0$  yields

$$\lim_{T \rightarrow 0} \mathbb{P}_\infty \left( \sup_{0 < t \leq T} d_F(X_t, \phi_F(\infty, t)) > \varepsilon \right) = 0,$$

where the instantaneous coming down from infinity of  $\phi_F$  ensures that  $\phi_F(\infty, t) < \infty$  for any  $t \in (0, T]$ . Then

$$\mathbb{P}_\infty \left( \lim_{t \downarrow 0+} F(X_t) - F(\phi_F(\infty, t)) = 0 \right) = 1$$

and  $\mathbb{P}_\infty(\forall t > 0 : X_t < \infty) = 1$ , which proves (i).

For the case

$$\int_\infty \frac{1}{b_F(x)} = +\infty,$$

we use first that  $T'(\infty) = \infty$ . Moreover  $\phi_F(x_0, t) \uparrow \infty$  as  $x_0 \uparrow \infty$  and  $A > 0$  and for any  $t > 0$ , for  $x_0$  large enough,  $T_{D,A,F}(x_0) > t$ . Then using again Lemma 4.6, for any  $t > 0$ , we have

$$\limsup_{x_0 \rightarrow \infty, x_0 \in E} \mathbb{P}_{x_0} (d_F(X_t, \phi_F(x_0, t)) \geq A) \leq C(A, t)$$

and

$$\mathbb{P}_\infty(X_t < \infty) \leq C(A, t),$$

where  $C(A, t) \rightarrow 0$  as  $A \rightarrow \infty$ . Then

$$\mathbb{P}_\infty(X_t = \infty) = 1,$$

which ends up the proof since  $X$  is càdlàg under  $\mathbb{P}_\infty$ . □

## 4.2 Examples

We consider now some examples, namely  $\Lambda$ -coalescent and birth and death processes, which are discrete Markov processes, and the TCP which is a PDMP (Piecewise Deterministic Markov Process). This allows us to apply the previous results in different ways and illustrate the assumptions required and the choice of  $F$ . In particular, we recover classical results on the coming down from infinity and refine some of them. Here  $\sigma = 0$  and the condition allowing the compensation of jumps (Assumption 4.1 (ii)) will be obvious.

Several extensions of these results could be achieved, such as mixing branching coalescing processes or additional catastrophes. They are left for future works, while the next Section will consider a class of diffusions in higher dimension.

#### 4.2.1 $\Lambda$ -coalescent [24, 5].

Pitman [24] has given a Poissonian representation of  $\Lambda$ -coalescent. We recall that  $\Lambda$  is a finite measure on  $[0, 1]$  and we set  $\nu(dy) = y^{-2}\Lambda(dy)$ . Without loss of generality, we assume that  $\Lambda[0, 1] = 1$  and for simplicity, we focus on coalescent without Kingman part and assume  $\Lambda(\{0\}) = 0$ . We consider a Poisson Point Process on  $(\mathbb{R}^+)^2$  with intensity  $dt\nu(dy)$ : each atom  $(t, y)$  yields a coalescence event where each block is picked independently with probability  $y$  and all the blocks picked merge into a single block. Then the numbers of blocks jumps from  $n$  to  $B_{n,y} + 1_{B_{n,y} < n}$ , where  $B_{n,y}$  follows a binomial distribution with parameter  $(n, 1 - y)$ . Thus, the number of blocks  $X_t$  at time  $t$  is the solution of the SDE

$$X_t = X_0 - \int_0^t \int_0^1 \int_{[0,1]^{\mathbb{N}}} \left( -1 + \sum_{1 \leq i \leq X_{s-}} 1_{u_i \leq y} \right)^+ N(ds, dy, du)$$

where  $N(dt, dy, du)$  is a PPM with intensity  $dt\nu(dy)du$  and here  $E = \mathbb{N} = \{1, 2, \dots\}$ ,  $\mathcal{X} = [0, 1]^{\mathbb{N}}$ ,  $q(dz) = q(dydu) = \nu(dy)du$ ,  $\sigma = 0$ , and

$$H(x, z) = H(x, (y, u)) = - \left( -1 + \sum_{1 \leq i \leq x} 1_{u_i \leq y} \right)^+.$$

We follow [5] and we denote for  $x \in (1, \infty)$ ,

$$D = (1, \infty], \quad F(x) = \log(x), \quad \psi(x) = \int_{[0,1]} (e^{-xy} - 1 + xy)\nu(dy).$$

In particular  $F$  meets the Assumption 4.1 (i) with  $a = 1$  and  $a' = 0$ . Moreover for every  $x \in \mathbb{N}$ ,

$$\begin{aligned} h_F(x) &= \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)]q(dz) \\ &= \int_{\mathcal{X}} \log\left(\frac{x + H(x, z)}{x}\right)q(dz) \\ &= \int_{[0,1]} \nu(dy)\mathbb{E}\left(\log\left(\frac{B_{x,y} + 1_{B_{x,y} < x}}{x}\right)\right) = -\frac{\psi(x)}{x} + h(x), \end{aligned}$$

where  $h$  is bounded thanks to Proposition 7 in [5]. Thus,

$$\psi_F(x) = h_F(F^{-1}(x)) = -\frac{\psi(\exp(x))}{\exp(x)} + h(\exp(x)).$$

Moreover  $h$  can be extended to a bounded function on  $(0, \infty)$ , which is  $C^1$  and such that the previous identity hold for any  $x > 0$ . Then Assumption 4.1 (ii) is satisfied.

Lemma 9 in [5] ensures that  $x \in (1, \infty) \rightarrow \psi(x)/x$  is increasing. Then  $\psi_F$  is  $(0, 2 \|h\|_{\infty})$  non-expansive on  $(0, \infty)$ . Moreover

$$b_F(x) = F'(x)^{-1}h_F(x) = -\psi(x) + xh(x)$$

is also locally Lipschitz on  $(1, \infty)$  and the associated flow  $\phi_F$  is well defined on  $(1, \infty)$ . Adding that  $\psi(x)/x \rightarrow \infty$  as  $x \rightarrow \infty$ , we get  $b_F(x) < 0$  for  $x$  large enough and Assumption 4.1 (iii) is

checked.

Finally,  $\widetilde{b}_F = 0$  since  $\sigma = 0$  and  $G = 0$  and the second part of Proposition 7 in [5] ensures that

$$V_F(x) = \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)]^2 q(dz) = \int_{[0,1]} \nu(dy) \mathbb{E} \left( \left( \log \left( \frac{B_{x,y} + 1_{B_{x,y} < x}}{x} \right) \right)^2 \right)$$

is bounded for  $x \in \mathbb{N}$ . Then Assumptions 4.1 and 4.3 are satisfied with  $a = 1, a' = 0$ . Moreover  $(\mathbb{P}_x : x \in \mathbb{N})$  converges weakly to  $\mathbb{P}_\infty$ , which can be derived from Proposition 4.4 (i) since  $X$  is stochastically monotone. Thus Theorem 4.5 can be applied and defining  $w_t = \phi_F(\infty, t)$ , we have

(i) If

$$\int_{\infty}^{\cdot} \frac{1}{b_F(x)} < +\infty,$$

then  $w_t \in \mathcal{C}^1((0, \infty), (0, \infty))$  and  $w'_t = \psi(w_t) + w_t h(w_t)$  for  $t > 0$  and

$$\mathbb{P}_\infty(\forall t > 0 : X_t < \infty) = 1 \quad \text{and} \quad \mathbb{P}_\infty \left( \lim_{t \downarrow 0+} \frac{X_t}{w_t} = 0 \right) = 1.$$

(ii) Otherwise  $\mathbb{P}_\infty(\forall t \geq 0 : X_t = +\infty) = 1$ .

To compare with known results, let us note that  $b_F(x) \sim \psi(x)$  as  $x \rightarrow \infty$  and

$$\int_{\infty}^{\cdot} \frac{1}{\psi(x)} dx < \infty \Leftrightarrow \int_{\infty}^{\cdot} \frac{1}{b_F(x)} < +\infty$$

so that we recover here the criterion of coming down from infinity obtained in [6]. This latter is equivalent to the criterion initially proved in [25] :

$$\sum_{n=2}^{\infty} \gamma_n^{-1} < \infty,$$

where

$$\gamma_n = - \int_{[0,1]} H(n, z) q(dz) = \sum_{k \geq 2} (k-1) \binom{n}{k} \int_{[0,1]} y^k (1-y)^{n-k} \nu(dy).$$

*Remark 1 : let us note that this condition can be rewritten as  $\int_{\infty}^{\cdot} 1/b(x) dx < \infty$ , where  $b(x)$  is a locally Lipschitz function, which is non-increasing and equal to  $-\gamma_n$  for any  $n \in \mathbb{N}$ . But the proof cannot be achieved using  $F = Id$ , even if  $b(x)$  is non-expansive since  $V_{Id}(x)$  cannot be controlled.*

Finally, when the process comes down from infinity, we can use Lemma 6.1 to check that  $b_F(x) \sim \psi(x)$  as  $x \rightarrow \infty$  ensures that

$$w_t \sim_{t \downarrow 0+} v_t, \quad \text{where} \quad v_t = \inf \left\{ s > 0 : \int_s^{\infty} \frac{1}{\psi(x)} dx < t \right\}$$

satisfies  $v'_t = \psi(v_t)$  for  $t > 0$ . Thus, we recover here the speed of coming down from infinity of [5].

*Remark 2 :* We have here proved that the speed of coming down from infinity is  $w$  using the general result given above and [5] and then observe that this speed function is equivalent to  $v$ . It is possible to recover directly that  $v$  is the speed of coming down from infinity by using Proposition 2.3 and a slightly different decomposition of the process  $X$  following [5] :

$$\log(X_t) = \log(X_0) - \int_0^t \psi(X_s) ds + \int_0^t \int_0^1 \int_{[0,1]^N} \log\left(\frac{X_{s-} + (-1 + \sum_{i \leq X_{s-}} 1_{u_i \leq y})^+}{X_{s-}}\right) \tilde{N}(ds, dy, du) + A_t,$$

where  $A_t = \int_0^t h(X_s) ds$  is a process with finite variations. This could also be done directly by extending the result of Section 3 and adding a term with finite variations in the decomposition using Section 2.

*Remark 3 :* let us also mention that the speed of coming down from infinity for some  $\Xi$  coalescents has been obtained in [21] with a similar method than [5]. The reader could find there other and detailed information about the coming down from infinity of coalescent processes. Finally, we mention [22, 23] for stimulating recent results on the fluctuations the  $\Lambda$ - coalescent around the dynamical system  $v_t$  for small times.

#### 4.2.2 Birth and death processes [27, 4] .

We consider a birth and death process  $X$  and we denote by  $\lambda_k$  (resp.  $\mu_k$ ) the birth rate when the population size is equal to  $k \in E = \{0, 1, 2, \dots\}$ . We assume that  $\mu_0 = \lambda_0 = 0$  and  $\mu_k > 0$  for  $k \geq 1$  and we note

$$\pi_1 = \frac{1}{\mu_1}, \quad \pi_k = \frac{\lambda_1 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} \quad (k \geq 2).$$

We also assume that

$$\sum_{k \geq 1} \frac{1}{\lambda_k \pi_k} = \infty. \quad (15)$$

Then the process  $X$  is well defined on  $E$  and eventually become extinct [18, 19]. It is the unique strong solution on  $E$  of the following SDE

$$X_t = X_0 + \int_0^t \int_0^\infty (1_{z \leq \lambda_{X_{s-}}} - 1_{\lambda_{X_{s-}} < z \leq \lambda_{X_{s-}} + \mu_{X_{s-}}}) N(ds, dz)$$

where  $N$  is a Poisson Point Measure with intensity  $ds dz$  on  $[0, \infty)^2$ . Lemma 2.1 in [4] ensures that  $(\mathbb{P}_x : x \in E)$  converges weakly to  $\mathbb{P}_\infty$ . It can also be derived from Proposition 4.4 (i) since  $X$  is stochastically monotone. Under the extinction assumption (15), the following criterion for the coming down from infinity is well known [1] :

$$S = \lim_{n \rightarrow \infty} \mathbb{E}_n(T_0) = \sum_{i \geq 1} \pi_i + \sum_{n \geq 1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i < +\infty.$$

The speed of coming down from infinity of birth and death process has been obtained in [4] for regularly varying rate (with index  $\rho > 1$ ) and a birth rate negligible compared to the death rate. Let us here get a finer result for a relevant subclass which allows rather simple computations. In population dynamics, it is a classical model for independent birth and

polynomial competitive death rate, which contains in particular the logistic birth and death process. It can be extended easily to larger classes provided that the birth and the death rates have the same asymptotic behavior and in particular it could include Kingman coalescent or other coalescent without multiple collisions.

**Corollary 4.7.** *We assume that there exist  $c > 0$ ,  $b \geq 0$  and  $\rho > 1$  such that*

$$\lambda_k = bk, \quad \mu_k = ck^\rho \quad (k \geq 1).$$

*Then for any  $\alpha \in (0, 1/2)$ ,*

$$\mathbb{P}_\infty \left( \lim_{t \downarrow 0+} t^{\alpha/(1-\rho)} (X_t/w_t - 1) = 0 \right) = 1,$$

*where*

$$w_t \sim_{t \downarrow 0+} [ct/(\rho - 1)]^{1/(1-\rho)}.$$

This complements the results obtained in [4], where it was shown that  $X_t/w_t \rightarrow 1$  as  $t \downarrow 0$ . The proof used the decomposition of the trajectory in terms of the first hitting time of integers. The fact that  $X$  satisfies a central limit theorem when  $t \rightarrow 0$  under  $\mathbb{P}_\infty$  (see Theorem 5.1 in [4]) ensures that the previous result is sharp in the sense that it does not hold for  $\alpha \geq 1/2$ .

*Proof.* We consider  $\alpha \in (-1, \infty)$  and the  $C^2$  function

$$F(x) = (1+x)^\alpha \quad (\alpha \in (0, 1/2)).$$

Assumption 4.1 can be checked with  $a' = -1$ . We have here  $\chi = [0, \infty)$ ,  $q(dz) = dz$ ,

$$H(x, z) = 1_{z \leq \lambda_x} - 1_{\lambda_x < z \leq \lambda_x + \mu_x}$$

and

$$h_F(x) = \int_{\mathcal{X}} [F(x + H(x, z)) - F(x)] q(dz) = ((x+2)^\alpha - (x+1)^\alpha) \lambda_x + (x^\alpha - (x+1)^\alpha) \mu_x$$

for  $x \in \{0, 1, \dots\}$ . This function can be extended in a locally Lipschitz function on  $(-1, \infty)$  such that for  $x \geq 0$ ,

$$h_F(x) = ((x+2)^\alpha - (x+1)^\alpha) bx + (x^\alpha - (x+1)^\alpha) cx^\rho$$

Moreover

$$h_F(x) \sim_{x \rightarrow \infty} -c\alpha x^{\rho+\alpha-1}$$

and  $h'_F(x)$  is negative for  $x$  large enough since  $\rho > 1 - \alpha$ . Thus, there exists  $a > 0$  such that  $\psi_F = h_F \circ F^{-1}$  is non-increasing on  $(a, \infty)$  and  $\psi_F$  is non-expansive on  $(F(a), \infty)$ . Here

$$b_F(x) = F'(x)^{-1} h_F(x) = \alpha^{-1} (1+x)^{1-\alpha} h_F(x) = -c(1+x)^\rho + O(x^{\max(\rho-1, 1)}) \quad (16)$$

as  $x \rightarrow \infty$  and using Lemma 6.1 in Appendix with  $\psi_1(x) = b_F(x)$  and  $\psi_2(x) = -cx^\rho$  (or making direct computations)

$$\phi_F(\infty, t) \sim_{t \downarrow 0+} [ct/(\rho - 1)]^{1/(1-\rho)}.$$

Finally

$$V_F(x) = ((x+2)^\alpha - (x+1)^\alpha)^2 \lambda_x + ((x+1)^\alpha - x^\alpha)^2 \mu_x \sim \alpha^2 c x^{\rho+2\alpha-2}$$

as  $x \rightarrow \infty$ . Adding that for any  $T > 0$ , there exists  $c_0 > 0$  such that  $\phi(\infty, t) \leq c_0 t^{1/(1-\rho)}$  for  $t \in [0, T]$  and that

$$F^{-1}(y) = y^{1/\alpha} - 1,$$

then for any  $\varepsilon > 0$ , there exists  $c'_0 > 0$  such that for any  $t \leq T$ ,

$$\hat{V}_{F,\varepsilon}(a, t) \leq \varepsilon^{-2} \sup \left\{ V_F(x) : 0 \leq x \leq ((\phi_F(\infty, t) + 1)^\alpha + \varepsilon)^{1/\alpha} - 1 \right\} \leq c'_0 (t^{1/(1-\rho)})^{\rho+2\alpha-2}.$$

Then, for  $\alpha < 1/2$ ,

$$\int_0^T \hat{V}_{F,\varepsilon}(a, t) dt < \infty,$$

since  $(\rho + 2\alpha - 2)/(1 - \rho) = -1 + (2\alpha - 1)/(1 - \rho) > -1$ . Assumptions 4.1 and 4.3 are satisfied and Theorem 4.5 (i) can be applied and defining  $w_t = \phi_F(\infty, t)$ , we have for any  $\alpha \in (0, 1/2)$ ,

$$\mathbb{P}_\infty(\lim_{t \downarrow 0+} X_t^\alpha - w_t^\alpha = 0).$$

This ends up the proof. □

*Remark.* Finally, we note that using (16) and forthcoming Lemma 6.2, the previous result can be stated for  $\alpha < (\rho - 1) \wedge 1/2$  as

$$\mathbb{P}_\infty \left( \lim_{t \downarrow 0+} t^{\alpha/(1-\rho)} \left( \frac{X_t}{[ct/(\rho-1)]^{1/(1-\rho)}} - 1 \right) = 0 \right) = 1.$$

Let us finally illustrate the criterion to know when the process comes down from infinity for the critical case where the competition rate is slightly larger than the birth rate.

**Corollary 4.8.** *We assume that there exist  $b \geq 0$ ,  $\beta \geq 0$  and  $c > 0$  such that*

$$\lambda_k = bk, \quad \mu_k = ck \log(k)^\beta \quad (k \geq 1).$$

*If  $\beta > 1$ , then  $\mathbb{P}_\infty(\forall t > 0 : X_t < +\infty) = 1$ .*

*Otherwise,  $\mathbb{P}_\infty(\forall t \geq 0 : X_t = +\infty) = 1$ .*

*Proof.* The proof can be achieved using the functions  $F(x) = \log(1+x)$  or  $F(x) = (1+x)^\alpha$  on  $(-1, \infty)$ , for some  $\alpha \in (0, 1/2)$ . Indeed the function

$$h_F(x) = \int_{\mathcal{X}} [F(x+H(x,z)) - F(x)] q(dz) = bx(F(x+1) - F(x)) + c(F(x-1) - F(x))x \log(x)^\beta$$

is well defined on  $(0, \infty)$  and its derivative is negative for  $x$  large enough. So

$$\psi_F(x) = h_F(F^{-1}(x))$$

is decreasing and thus non-expansive for  $x$  large enough. Moreover

$$V_F(x) \leq (\lambda_x + \mu_x)(F(x) - F(x-1))^2 \leq Cx \log(x)^\beta (F(x) - F(x-1))^2$$

is bounded and Assumption 4.3 is satisfied. Then the result comes from Theorem 4.5 by noting that

$$b_F(x) = (F'(x))^{-1} h_F(x) = -O(x \log(x)^\beta)$$

as  $x \rightarrow \infty$ . Then  $\int_\infty^x \frac{1}{b_F(x)} dx = +\infty$  iff  $\beta > 1$ , which ends up the proof. □

### 4.2.3 Transmission Control Protocol

The Transmission Control Protocol (TCP) [3] is a model for transmission of data, mixing a continuous (positive) drift which describes the growth of the data transmitted and jumps due to congestions, where the size of the data are divided by two. Then the size  $X_t$  of data at time  $t$  is given by the unique strong solution of

$$X_t = x_0 + bt - \int_0^t \mathbf{1}_{\{u \leq r(X_{s-})\}} \frac{X_{s-}}{2} N(ds, du)$$

where  $x_0 \geq 0$ ,  $b > 0$ ,  $r(x)$  is a continuous positive non-decreasing function and  $N$  is PPM on  $[0, \infty)^2$  with intensity  $dsdu$ . This is a classical example of Piecewise Deterministic Markov process. Usually,  $r(x) = cx^\beta$ , with  $\beta \geq 0$ ,  $c > 0$ . First, let us note that the process is not stochastically monotone and the convergence of  $(\mathbb{P}_x : x \geq 0)$  is left open. The choice of  $F$  is a bit more delicate here owing to the size and intensity of the fluctuations. Consider  $F$  such that  $F'(x) > 0$  for  $x > 0$ . Here

$$b_F(x) = b + r(x)F'(x)^{-1}(F(x/2) - F(x)),$$

$h_F(x) = bF'(x) + r(x)(F(x/2) - F(x))$  and  $\psi_F(x) = h_F \circ F^{-1}$ . As soon as  $r$  is differentiable, we have

$$h'_F(x) = bF''(x) + r'(x)(F(x/2) - F(x)) + r(x)(F'(x/2)/2 - F'(x)).$$

Finally

$$V_F(x) = r(x)(F(x/2) - F(x))^2$$

and we cannot use  $F(x) = (1+x)^\gamma$  or  $F(x) = \log(1+x)^\gamma$  since then the second part of Assumption 4.3 does not hold. We need to reduce the size of the jump even more and take  $F(x) = \log(1 + \log(1+x))$ . The model is not stochastically monotone but Lemma 4.6 can be used to get the following result, which yields a criterion for the coming down from infinity depending on the growth of  $r(\cdot)$ .

**Corollary 4.9.** (i) *If there exists  $c > 0$  and  $\beta > 1$  such that  $r(x) \geq c \log(1+x)^\beta$  for any  $x \geq 1$ , then for any  $T > 0$ ,  $\eta > 0$ , there exists  $K$  such that*

$$\inf_{x_0 \geq 0} \mathbb{P}_{x_0}(\exists t \leq T : X_t \leq K) \geq 1 - \eta.$$

(ii) *If there exists  $c > 0$  and  $\beta \leq 1$  such that  $r(x) \leq c \log(1+x)^\beta$  for any  $x \geq 0$ , then for any  $T, K > 0$ ,*

$$\lim_{x_0 \rightarrow \infty} \mathbb{P}_{x_0}(\exists t \leq T : X_t \leq K) = 0.$$

Thus, in the first regime, the process comes down instantaneously and a.s. from infinity, while in the second regime it stays at infinity. In particular, if  $r(x) = cx^\beta$  and  $\beta, c > 0$ , the process comes down instantaneously from infinity. If  $\beta = 0$ , it does not, which can actually be seen easily since in the case  $r(\cdot) = c$ ,  $X_t \geq (x_0 + bt)/2^{N_t}$ , where  $N_t$  is a Poisson Process with rate  $c$  and the right hand side goes to  $\infty$  as  $x_0$  goes to infinity for any  $t \geq 0$ .

*Proof.* Here  $E = [0, \infty)$  and we consider

$$F(x) = \log(1 + \log(1+x))$$

on  $(a', \infty)$  where  $a' \in (-1, 0)$  is chosen such that  $\log(1 + a') > -1$ . Then

$$F'(x) = \frac{1}{(1+x)(1+\log(1+x))} > 0.$$

Moreover

$$F(x/2) - F(x) = \log(1 - \varepsilon(x)),$$

where

$$\varepsilon(x) = 1 - \frac{1 + \log(1 + x/2)}{1 + \log(1 + x)} = \frac{\log(2) + O(1/(1+x))}{1 + \log(1 + x)}.$$

We consider now

$$r(x) = c \log(1 + x)^\beta$$

with  $c > 0$  and  $\beta \in [0, 2]$ . We get

$$b_F(x) = b + c \log(1 + x)^\beta (1 + x)(1 + \log(1 + x)) \log(1 - \varepsilon(x)) \sim -cx \log(x)^\beta$$

as  $x \rightarrow \infty$ . Thus,

$$\int_{\infty}^{\cdot} \frac{1}{b_F(x)} dx < +\infty$$

if and only if  $\beta > 1$ . Moreover  $h'_F(x) < 0$  for  $x \geq a$  where  $a$  is chosen large enough. Thus,  $h_F$  is non-increasing on  $(a, \infty)$   $\psi_F = h_F \circ F^{-1}$  is non-expansive on  $(F(a), \infty)$ . Finally

$$V_F(x) = c \log(1 + x)^\beta \log(1 - \varepsilon(x))^2 \sim c \log(x)^{\beta-2}$$

as  $x \rightarrow \infty$  and  $V_F$  is bounded since  $\beta \leq 2$ . So Assumptions 4.1 and 4.3 are satisfied and we can apply Lemma 4.6. Then we use Lemma 4.6 to get for any  $x_0 \geq 0$  and  $A > 0$ ,

$$\mathbb{P}_{x_0} \left( \sup_{t \leq T} |F(X_t) - F(\phi_F(x_0, t))| \geq A \right) \leq \frac{C(T)}{A^2}, \quad (17)$$

where  $C(T) \rightarrow 0$  as  $T \rightarrow 0$ .

We can now prove (i). By a simple coupling argument,  $X_t \leq \tilde{X}_t$  a.s. for  $t \geq 0$ , where  $\tilde{X}_t$  is a TCP associated with the rate of jumps

$$\tilde{r}(x) = c \log(1 + x)^{\beta \wedge 2}$$

where  $\beta > 1$ . Then  $\phi_F(x_0, t) \leq \phi(\infty, t)$  and this latter is finite for any  $t > 0$  since  $\beta > 1$  ensures that the dynamical system comes down from infinity. Letting  $T \rightarrow 0$  in (17) then yields (i). To prove (ii), we use similarly the coupling  $X_t \geq \tilde{X}_t$  with  $\tilde{r}(x) = c \log(1 + x)^\beta$  and  $\beta < 1$  and let  $A \rightarrow \infty$  in (17).  $\square$

#### 4.2.4 Logistic diffusions [7] and perspectives

The coming down from infinity of diffusions of the form

$$dZ_t = \sqrt{\gamma Z_t} dB_t + h(Z_t) dt$$



has been studied in [7] and are linked to the uniqueness of the quasistationary distribution (Theorem 7.3). Writing  $X_t = 2\sqrt{Z_t/\gamma}$ , it becomes

$$dX_t = dB_t - q(X_t)dt,$$

where  $q(x) = x^{-1}(1/2 - 2h(\gamma x^2/4)/\gamma)$ . Under some assumptions (see Remark 7.4 in [7]), the coming down from infinity is indeed equivalent to

$$\int^{\infty} \frac{1}{q(x)} dx < \infty,$$

which can be compared to our criterion in Theorem 4.5. Several extensions and new results could be obtained using the results of this Section. In particular one may be interested to mix a diffusion part for competition, negative jumps due to coalescence and branching events. In that vein, let us mention [20]. This is one motivation to take into account the compensated Poisson measure in the definition of the process  $X$ , so that Lévy processes and CSBP may be considered in general. It is left for future stimulating works.

Let us here simply mention that a class of particular interest is given by the logistic Feller diffusion :

$$dZ_t = \sqrt{\gamma Z_t} dB_t + (\tau Z_t - aZ_t^2) dt.$$

The next part is proving new results both on the speed of coming down of this diffusion and the scaling limits of birth and death process to this diffusion. This part actually deals more generally with the two dimensional version of this diffusion, where non-expansivity and the behavior of the dynamical system are more delicate.

## 5 Uniform estimates for two-dimensional competitive Lotka Volterra diffusions

We consider the 2 dimensional continuous stochastic process  $X = (X^{(1)}, X^{(2)})$  defined as the unique strong solution of the following SDE

$$\begin{aligned} dX_t^{(1)} &= X_t^{(1)}(\tau_1 - aX_t^{(1)} - cX_t^{(2)})dt + \sigma_1 \sqrt{X_t^{(1)}} dB_t^{(1)} \\ dX_t^{(2)} &= X_t^{(2)}(\tau_2 - bX_t^{(2)} - dX_t^{(1)})dt + \sigma_2 \sqrt{X_t^{(2)}} dB_t^{(2)}, \end{aligned}$$

for  $t \geq 0$ , where  $a, b, c, d \geq 0$ . This is the classical stochastic (Lotka Volterra) model for two competitive species. The coefficients  $a$  and  $b$  are the intraspecific competition rates and  $c, d$  are the interspecific competition rates. We assume that

$$a, b, c, d > 0$$

or  $a, b > 0$  and  $c = d = 0$  so that our results cover the (simpler) case of one-dimensional logistic diffusions. Here again for simplicity we write

$$x_t = \phi(x_0, t) = (x_t^{(1)}, x_t^{(2)})$$

the dynamical system starting from  $x_0$  solution of

$$\begin{aligned} (x_t^{(1)})' &= x_t^{(1)}(\tau_1 - ax_t^{(1)} - cx_t^{(2)}) \\ (x_t^{(2)})' &= x_t^{(2)}(\tau_2 - bx_t^{(2)} - dx_t^{(1)}). \end{aligned} \tag{18}$$

We compare for a suitable distance the diffusion  $X$  to the flow  $\phi(x_0, t)$ . It relies on the application of the results of Section 3 to a well chosen family of transformations

$$F_{\beta, \gamma}(x) = \begin{pmatrix} x_1^\beta \\ \gamma x_2^\beta \end{pmatrix}, \quad x \in (0, \infty)^2, \quad \beta \in (0, 1], \quad \gamma > 0,$$

and we use Poincaré compactification technics for flows to describe the dynamical system coming from infinity and derive the behavior of the process  $X$  coming from infinity.

## 5.1 Uniform short time estimates and coming back to compact sets

We consider the domain

$$\mathcal{D}_\alpha = \{x \in (0, \infty)^2 : x_1 > \alpha, x_2 > \alpha\}$$

and the distance  $d_\beta$  on  $[0, \infty)^2$  defined for  $\beta > 0$  by

$$d_\beta(x, y) = \sqrt{|x_1^\beta - y_1^\beta|^2 + |x_2^\beta - y_2^\beta|^2} = \|F_{\beta, 1}(x) - F_{\beta, 1}(y)\|_2$$

and we note

$$T_D(x_0) = \inf\{t \geq 0 : \phi(x_0, t) \notin D\}$$

the first time when the flow  $\phi$  starting from  $x_0$  exits  $D$ .

**Theorem 5.1.** *For any  $\beta \in (0, 1)$ ,  $\alpha > 0$  and  $\varepsilon > 0$ ,*

$$\lim_{T \rightarrow 0} \sup_{x_0 \in \mathcal{D}_\alpha} \mathbb{P}_{x_0} \left( \sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d_\beta(X_t, \phi(x_0, t)) \geq \varepsilon \right) = 0.$$

This yields a control of the stochastic process  $X$  by the dynamical system for large initial values and times small enough. We are not expecting that this control hold outside  $\mathcal{D}_\alpha$ . Indeed, the next result shows that the process and the dynamical system have a different behavior close to the boundary  $(0, \infty) \times \{0\} \cup \{0\} \times (0, \infty)$ . The proof below can not be achieved for  $\beta = 1$  since then the associated quadratic variations are not integrable at time 0. Heuristically,  $\sqrt{Z_t} dB_t$  is of order  $\sqrt{1/t} dB_t$ , which is not becoming small when  $t \rightarrow 0$  and the fluctuations are not vanishing for  $d_1$  in short time.

We note  $\widehat{(x, y)} \in (-\pi, \pi)$  the oriented angle in the trigonometric sense between two non-zero vectors of  $\mathbb{R}^2$  and we write

$$x_\infty = \frac{1}{ab - cd}(b - c, a - d).$$

when  $ab \neq cd$ . We now describe the way the process comes down from infinity, which yields the following classification.

**Corollary 5.2.** *We assume that  $\sigma_1 > 0, \sigma_2 > 0$  and let  $x_0 \in (0, \infty)^2$ .*

(i) *If  $a > d$  and  $b > c$ , then for any  $\eta \in (0, 1)$  and  $\varepsilon > 0$ ,*

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left( \sup_{\eta T \leq t \leq T} \|tX_t - x_\infty\|_2 \geq \varepsilon \right) = 0,$$

If furthermore  $x_0$  is colinear to  $x_\infty$ , the previous limit hold also for  $\eta = 0$ .

(ii) If  $d > a$  and  $c > b$  and  $(\widehat{x_\infty, x_0}) \neq 0$ , then for any  $T > 0$ ,

$$\lim_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left( \inf\{t \geq 0 : X_t^{(i)} = 0\} \leq T \right) = 1,$$

where  $i = 1$  when  $(\widehat{x_0, x_\infty}) \in (0, \pi/2]$  and  $i = 2$  when  $(\widehat{x_0, x_\infty}) \in [-\pi/2, 0)$ .

(iii) If  $a \leq d$  and  $b > c$ , then for any  $T > 0$ ,

$$\lim_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left( \inf\{t \geq 0 : X_t^{(2)} = 0\} \leq T \right) = 1.$$

(iv) If  $a = d$  and  $b = c$ , then

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{rx_0} \left( \sup_{t \leq T} \|tX_t - (ax_0^{(1)} + bx_0^{(2)})^{-1}x_0\|_2 \geq \varepsilon \right) = 0.$$

In the first case (i), the diffusion  $X$  (and the dynamical system) comes down from infinity in a single direction  $x_\infty$ , with speed proportional to  $1/t$ . They only need a short time at the beginning of the trajectory to find this direction. This short time quantified by  $\eta$  here could be made arbitrarily small when  $x_0$  becomes large. Let us also observe that the one-dimensional logistic Feller diffusion  $X_t$  is given by  $X_t^{(1)}$  for  $c = d = 0$ . Thus, taking  $x_0$  collinear to  $x_\infty$ , (i) yields the speed of coming down from infinity of one-dimensional logistic Feller diffusions:

$$\lim_{T \rightarrow 0} \lim_{r \rightarrow \infty} \mathbb{P}_r \left( \sup_{t \leq T} |atX_t - 1| \geq \varepsilon \right) = 0. \quad (19)$$

In the second case (ii), the direction taken by the dynamical system and the process depends on the initial direction. The dynamical system then goes to the boundary of  $(0, \infty)^2$  without reaching it. But the fluctuations of the process make it reach 0. When the process starts in the direction of  $x_\infty$ , additional work is required to describe its behavior, linked to the behavior of the dynamical system on the associated unstable variety coming from infinity.

In the third case (iii), the dynamical system  $\phi$  goes to the boundary  $(0, \infty) \times \{0\}$  when coming down from infinity, wherever it comes from. Then, as above, the diffusion  $X^{(2)}$  hits 0. Let us note that the dynamical system may then go to a coexistence fixed point or even to a fixed point where only the species 2 survives. This latter case occurs when

$$\tau_2/b > \tau_1/c, \quad \tau_2/d > \tau_1/a$$

and is illustrated in the third simulation below. Obviously, the symmetric situation happens when  $b \leq c$  and  $d < a$ .

Moreover, in both cases (ii – iii), when the diffusion hits the boundary, it becomes a one-dimensional Feller logistic diffusion whose coming down infinity has been given above (19). In the case (iv), the process comes down from infinity in the direction of its initial value, at speed  $1/t$ .

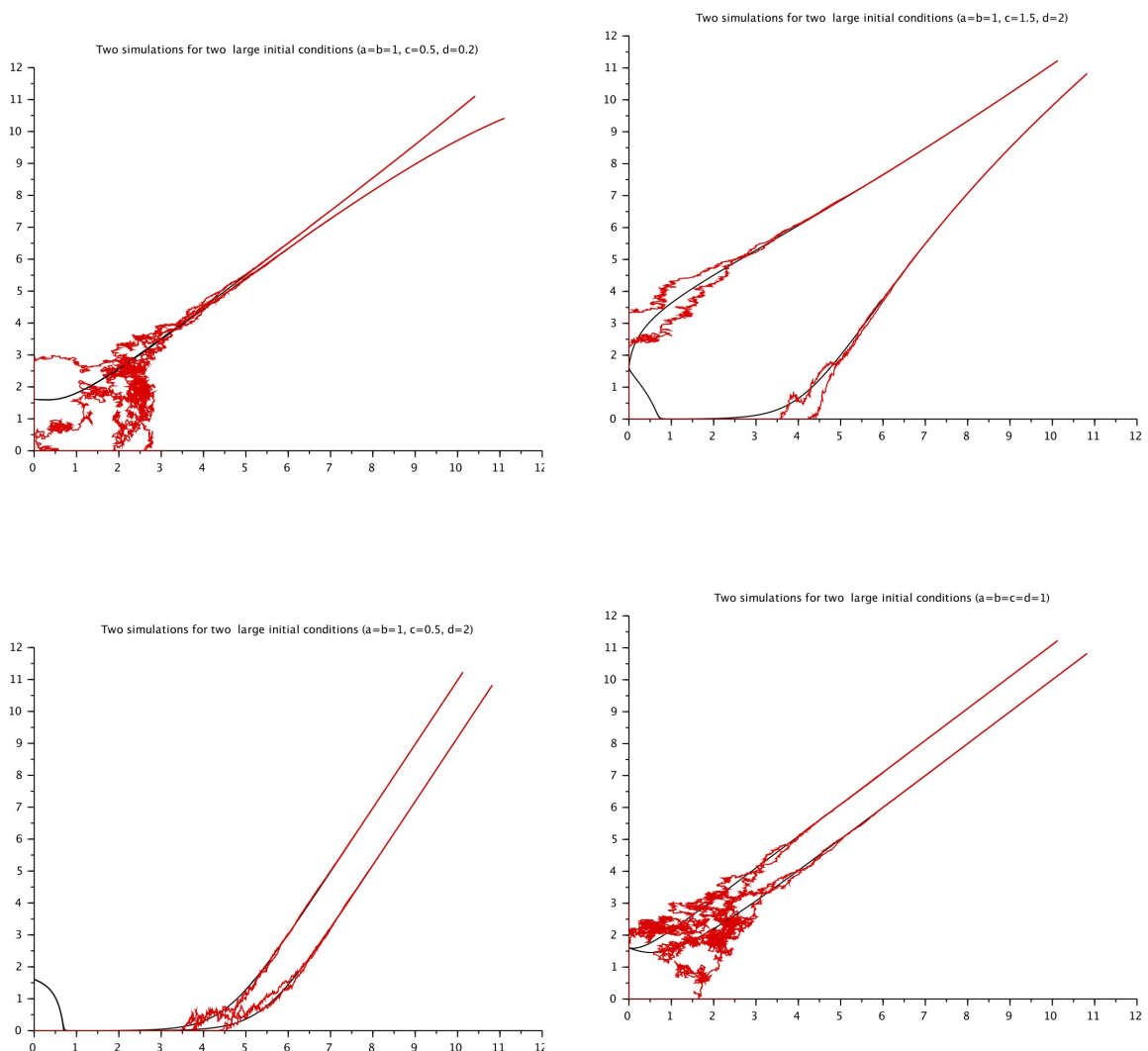
Finally, let us note that this raises several questions on the characterization of a process starting from infinity in dimension 2. In particular, informally, the process coming down from infinity in a direction  $x_0$  which is not  $x_\infty$  has a discontinuity at time 0 in the cases (i – ii – iii).

*Simulations.* We consider two large initial values  $x_0$  such that  $\|x_0\|_1 = 10^5$ . We plot the dynamical system (in black line) and two diffusions (in red line) starting from these two initial values. In each simulation,  $\tau_1 = 1, \tau_2 = 4$  and the solutions of the dynamical system converge to the fixed point where only the second species survives. The coefficient diffusion terms are  $\sigma_1 = \sigma_2 = 10$ . We plot here  $F(x_t)$  and  $F(X_t)$ , where

$$F(x, y) = (X, Y) = (\log(1 + x), \log(1 + y))$$

to zoom on the behavior of the processes when coming close to one of the axis. The four regimes (i – ii – iii – iv) of the Corollary, which give the ways the process can come down from infinity, are successively illustrated.

One can also compare with the pictures of Section 5.3 describing the dynamical system.



## 5.2 Uniform scaling limits of competitive birth and death processes

Let us deal now with competitive birth and death processes and consider the scaling limits to the dynamical system  $\phi$  given by (18). We provide here estimates for these convergences which are uniform with respect to the initial values in a cone in the interior of  $(0, \infty)^2$ . The birth and death rates are given for population sizes  $n_1, n_2 \geq 0$  and  $K \geq 1$  by

$$\lambda_1^K(n_1, n_2) = \lambda_1 n_1, \quad \mu_1^K(n_1, n_2) = \mu_1 n_1 + a n_1 \cdot \frac{n_1}{K} + c n_1 \cdot \frac{n_2}{K}$$

for the first species and by

$$\lambda_2^K(n_1, n_2) = \lambda_2 n_2, \quad \mu_2^K(n_1, n_2) = \mu_2 n_2 + b n_2 \cdot \frac{n_2}{K} + d n_2 \cdot \frac{n_1}{K}$$

for the second species and we assume that

$$\lambda_1 - \mu_1 = \tau_1, \quad \lambda_2 - \mu_2 = \tau_2.$$

Dividing the number of individuals by  $K$ , the normalized population size  $X^K$  satisfies

$$X_t^K = x_0 + \int_0^t \int_{[0, \infty)} H^K(X_{s-}, z) N(ds, dz),$$

where writing  $\tau_1^K = \lambda_1^K + \mu_1^K$  for convenience,

$$H^K(x, z) = \frac{1}{K} \left( \mathbf{1}_{\{z \leq \lambda_1^K(Kx)\}} - \mathbf{1}_{\{\lambda_1^K(Kx) \leq z \leq \tau_1^K(Kx)\}} \right. \\ \left. - \mathbf{1}_{\{z - \tau_1^K(Kx) \leq \lambda_2^K(Kx)\}} - \mathbf{1}_{\{\lambda_2^K(Kx) \leq z - \tau_1^K(Kx) \leq \lambda_2^K(Kx) + \mu_2^K(Kx)\}} \right).$$

and  $N$  is a PPM with intensity  $dsdz$  on  $[0, \infty) \times [0, \infty)$ . We set

$$\mathbb{D}_\alpha = \{(x_1, x_2) \in (\alpha, \infty)^2 : x_1 \geq \alpha x_2, x_2 \geq \alpha x_1\},$$

which will be required both for the control of the flow and the the control of the fluctuations. We only consider here the case

$$b > c \geq 0, \quad a > d \geq 0,$$

or  $a = d > 0$  and  $b = c > 0$ , since we know from the previous Corollary (see also forthcoming Lemma 5.6) that it gives the cases when the flow does not go instantaneously to the boundary of  $(0, \infty)^2$  in short time when coming from infinity and thus it does not exit  $\mathbb{D}_\alpha$  in short time.

**Theorem 5.3.** *For any  $T > 0$ ,  $\beta \in (0, 1/2)$  and  $\alpha, \varepsilon > 0$ , there exists  $C > 0$  such that for any  $K \geq 0$ ,*

$$\sup_{x_0 \in \mathbb{D}_\alpha} \mathbb{P}_{x_0} \left( \sup_{t \leq T} d_\beta(X_t^K, \phi(x_0, t)) \geq \varepsilon \right) \leq \frac{C}{K}.$$

We use here  $(L, \alpha_K)$  non-expansivity of the flow, see the proof below. The statement could be refined and in particular additional work should allow to make  $T$  go to infinity when  $K$  goes to infinity. The optimal power  $\beta = 1/2$  is reminiscent from results obtain for one dimensional logistic birth and death process in Section 4.2.2.

### 5.3 Non-expansivity of the flow and Poincaré compactification

The proofs of the three previous statements of this section rely on the following lemmas. The first one provides the domains where the transformation  $F_{\beta,\gamma}$  yields a non-expansive vector field and is the key ingredient to use the results of Section 3 for the proof the Theorem 5.1 and Theorem 5.3. The next lemmas describe the behavior of the dynamical system starting from infinity. This relies on the extension of the flow on the boundary at infinity using Poincaré's technics. These lemmas will be used in the proofs of the three statements.

We recall that a cone is a subset  $\mathcal{C}$  of  $\mathbb{R}^2$  such that for all  $x \in \mathcal{C}$  and  $\lambda > 0$ ,  $\lambda x \in \mathcal{C}$ . We are using in particular the convex components of open cones, which are open convex cones. For  $S$  a subset of  $\mathbb{R}^2$ , we denote by  $\overline{S}$  the closure of  $S$ . Recalling notations of Section 3, we have here

$$\psi_F = (J_F b_F) \circ F^{-1} = (J_F \circ b) \circ F^{-1}$$

where

$$b(x) = b(x_1, x_2) = \begin{pmatrix} \tau_1 x_1 - a x_1^2 - c x_1 x_2 \\ \tau_2 x_2 - b x_2^2 - d x_1 x_2 \end{pmatrix}$$

and we define  $\overline{\tau} = \max(\tau_1, \tau_2)$ . Let us write

$$q_\beta = 4ab(1 + \beta)^2 + 4(\beta^2 - 1)cd$$

for convenience and consider the open cones

$$D_{\beta,\gamma} = \{x \in (0, \infty)^2 : 4\beta(1 + \beta)(ad x_1^2 + bc x_2^2) + q_\beta x_1 x_2 - (c\gamma^{-1} x_1^\beta x_2^{1-\beta} - d\gamma x_1^{1-\beta} x_2^\beta)^2 > 0\}$$

and

$$\mathcal{C}_{\eta,\gamma} = \{x \in (0, \infty)^2 : x_1/x_2 \in (0, \eta) \cup (x_\gamma - \eta, x_\gamma + \eta) \cup (1/\eta, \infty)\},$$

writing  $x_\gamma = (d\gamma^2/c)^{1/(2\beta-1)}$  when it is well defined.

**Lemma 5.4.** *Let  $\beta \in (0, 1]$  and  $\gamma > 0$ .*

- (i) *The vector field  $\psi_{F_{\beta,\gamma}}$  is  $\overline{\tau}$  non-expansive on each convex component of the open cone  $F(D_{\beta,\gamma})$ .*
- (ii) *If  $q_\beta > 0$  and  $\beta \neq 1/2$  and  $c \neq 0$ , then*

$$\{(x_1, x_2) \in (0, \infty)^2 : x_1 = x_\gamma x_2\} \subset D_{\beta,\gamma}.$$

*If  $q_\beta > 0$  and  $\beta < 1$ , then there exists  $\eta > 0$  such that*

$$\{(x_1, x_2) \in (0, \infty)^2 : x_2 < \eta x_1\} \cup \{(x_1, x_2) \in (0, \infty)^2 : x_1 < \eta x_2\} \subset D_{\beta,\gamma}.$$

- (iii) *If  $q_\beta > 0$  and  $\beta \neq 1/2$  and  $\beta \neq 1$  and  $c \neq 0$  and  $d \neq 0$ , then there exist  $\eta > 0$ ,  $A > 0$  and  $\mu > 0$  such that for any  $x, y$  which belong both to the complementary set of  $B(0, A)$  and to the same convex component of the cone  $\mathcal{C}_{\eta,\gamma}$ , then*

$$(\psi_{F_{\beta,\gamma}}(x) - \psi_{F_{\beta,\gamma}}(y)) \cdot (x - y) \leq -\mu \|x\|_2 \|x - y\|_2^2.$$

In the particular case  $a, b > 0$  and  $c = d = 0$ , for any  $\beta \in (0, 1]$  and  $\gamma > 0$ ,  $D_{\beta,\gamma} = (0, \infty)^2$ . But this fact does hold in general. We need the transformations  $F_{\beta,\gamma}$  for well chosen values of  $\gamma$  to get the non-expansivity property of the flow on non-bounded domains, while  $\beta < 1$  will be required to control the fluctuations. Let us finally note that  $(0, \infty)^2$  may not be coverable by a single domain of the form  $D_{\beta,\gamma}$  and the adjunction procedure of Section 3.2 will be needed.

*Proof of Lemma 5.4.* (i) We write for  $y \in [0, \infty)^2$ ,

$$\psi_{F_{\beta,\gamma}}(y) = \psi_1(y) + \psi_{2,\beta,\gamma}(y),$$

where

$$\psi_1(y) = \begin{pmatrix} \beta \tau_1 y_1 \\ \beta \tau_2 y_2 \end{pmatrix}, \quad \psi_{2,\beta,\gamma}(y) = - \begin{pmatrix} \beta y_1 \left( a y_1^{1/\beta} + c \gamma^{-1/\beta} y_2^{1/\beta} \right) \\ \beta y_2 \left( b \gamma^{-1/\beta} y_2^{1/\beta} + d y_1^{1/\beta} \right) \end{pmatrix}.$$

First,  $\psi_1$  is Lipschitz on  $[0, \infty)^2$  with constant  $\bar{\tau}$  since  $\beta \in (0, 1]$ . Moreover, writing  $A_{\beta,\gamma}(x) = J_{\psi_{2,\beta,\gamma}}(F_{\beta,\gamma}(x))$  we have

$$A_{\beta,\gamma}(x) + A_{\beta,\gamma}^*(x) = - \begin{pmatrix} 2a(1+\beta)x_1 + 2c\beta x_2 & c\gamma^{-1}x_1^\beta x_2^{1-\beta} + d\gamma x_2^\beta x_1^{1-\beta} \\ c\gamma^{-1}x_1^\beta x_2^{1-\beta} + d\gamma x_2^\beta x_1^{1-\beta} & 2b(1+\beta)x_2 + 2d\beta x_1 \end{pmatrix}.$$

This can be seen using (3) or by a direct computation. As  $\beta > 0$  and  $x \in (0, \infty)^2$ , the trace of  $A_{\beta,\gamma}(x) + A_{\beta,\gamma}^*(x)$  is negative, while its determinant is equal to

$$(2a(1+\beta)x_1 + 2c\beta x_2)(2b(1+\beta)x_2 + 2d\beta x_1) - \left( c\gamma^{-1}x_1^\beta x_2^{1-\beta} + d\gamma x_2^\beta x_1^{1-\beta} \right)^2. \quad (20)$$

It is positive when  $x = (x_1, x_2) \in D_{\beta,\gamma}$  and then the spectrum of  $A_{\beta,\gamma}(x) + A_{\beta,\gamma}^*(x)$  is included in  $(-\infty, 0]$ . Recalling table 1 in [2] or the beginning of Section 2, we obtain that  $\psi_{2,\beta,\gamma}$  is non-expansive on the open convex components of  $F_{\beta,\gamma}(D_{\beta,\gamma})$ . Then  $\psi_{F_{\beta,\gamma}}$  is  $\bar{\tau}$  non-expansive on the open convex components of  $F_{\beta,\gamma}(D_{\beta,\gamma})$ . Let us finally observe that  $D_{\beta,\gamma}$  and thus  $F_{\beta,\gamma}(D_{\beta,\gamma})$  are open cones, which ends up the proof of (i).

(ii) The first inclusion comes from the fact that  $x_1 = (d\gamma^2/c)^{1/(2\beta-1)}x_2$  implies that

$$\left( c\gamma^{-1}x_1^\beta x_2^{1-\beta} - d\gamma x_1^{1-\beta} x_2^\beta \right)^2 = 0,$$

while the second inclusion is obtained by bounding

$$\left( c\gamma^{-1}x_1^\beta x_2^{1-\beta} - d\gamma x_1^{1-\beta} x_2^\beta \right)^2 \leq \left( c\gamma^{-1}\eta^{1-\beta} + d\gamma\eta^\beta \right)^2 x_1^2$$

when  $x_2 \leq \eta x_1$ . Letting  $\eta$  be small enough such that  $4\beta(1+\beta)ad > (c\gamma^{-1}\eta^{1-\beta} + d\gamma\eta^\beta)^2$  ends up the proof in the case  $c, d > 0$ , while the case  $c = d = 0$  is obvious.

(iii) The proof uses the notations and computations of (i, ii). We write

$$D(x) = \det(A_{\beta,\gamma}(x) + A_{\beta,\gamma}^*(x)), \quad T(x) = \text{trace}(A_{\beta,\gamma}(x) + A_{\beta,\gamma}^*(x))$$

and recall that the value of  $D(x)$  is given by (20) and for any  $x \in [0, \infty)^2 - \{(0, 0)\}$ ,  $T(x) < 0$ . Let  $x_0 \neq 0$  such that  $D(x_0) > 0$ , then there exist  $v_1, v_2 > 0$  such that for any  $x$  in a ball  $\mathcal{V}$  centered in  $x_0$ , we have  $-v_1 \leq T(x) < 0$  and  $D(x) \geq v_2$ . So for any  $\lambda > 0$  and  $x \in \mathcal{V}$ ,

$$T(\lambda x) = \lambda T(x) \in [-\lambda v_1, 0), \quad D(\lambda x) = \lambda^2 D(x) \in [\lambda^2 v_2, \infty).$$

Writing  $E(\cdot)$  the minimal eigenvalue of  $A_{\beta,\gamma}(\cdot) + A_{\beta,\gamma}^*(\cdot)$ , we have for any  $x \in \mathcal{V}$ ,

$$E(\lambda x) \leq \frac{D(\lambda x)}{T(\lambda x)} \leq -\lambda \frac{v_2}{v_1} < 0,$$

since  $D$  (resp.  $T$ ) gives the product (resp. the sum) of the two eigenvalues. We obtain that there exists  $\mu > 0$  such that for any  $x$  in the convex cone  $\mathcal{C}$  generated by  $\mathcal{V}$ , the spectrum of  $A_{\beta,\gamma}(x) + A_{\beta,\gamma}^*(x)$  is included in  $(-\infty, -2\mu \|x\|]$ . Then for any  $x, y \in \mathcal{C}$ ,

$$(\psi_{2,\beta,\gamma}(x) - \psi_{2,\beta,\gamma}(y)) \cdot (x - y) \leq -\mu \|x\|_2 \|x - y\|_2^2,$$

using again [2]. Moreover  $\psi_1$  is Lipschitz with constant  $\bar{\tau}$  and we get

$$(\psi_{\beta,\gamma}(x) - \psi_{\beta,\gamma}(y)) \cdot (x - y) \leq (\bar{\tau} - \mu \|x\|_2) \|x - y\|_2^2.$$

We conclude using (ii) and respectively  $x_0 = x_\gamma$  and  $x_0 = (0, 1)$  and  $x_0 = (1, 0)$ .  $\square$

To describe the coming down from infinity of the dynamical system, we use the following compactification of  $[0, \infty)^2$

$$F(x) = F(x_1, x_2) = \left( \frac{x_1}{1 + x_1 + x_2}, \frac{x_2}{1 + x_1 + x_2}, \frac{1}{1 + x_1 + x_2} \right) = (y_1, y_2, y_3)$$

The application  $F$  is a bijection from  $[0, \infty)^2$  into the simplex  $\mathcal{S}$  defined by

$$\mathcal{S} = \{y \in [0, 1]^2 \times (0, 1] : y_1 + y_2 + y_3 = 1\} \subset \bar{\mathcal{S}} = \{y \in [0, 1]^3 : y_1 + y_2 + y_3 = 1\}.$$

We note  $\partial\mathcal{S}$  the outer boundary of  $\mathcal{S}$  :

$$\partial\mathcal{S} = \bar{\mathcal{S}} - \mathcal{S} = \{(y_1, 1 - y_1, 0) : y_1 \in [0, 1]\} = \left\{ \lim_{r \rightarrow \infty} F(rx) : x \in [0, \infty)^2 - \{(0, 0)\} \right\}.$$

The key point to describe the direction of the dynamical system  $\phi$  coming from infinity is the following change of time. It allows to extend the flow on the boundary and is an example of Poincaré's compactification technics [13]. More precisely, we consider the flow  $\Phi$  of the dynamical system on  $\bar{\mathcal{S}}$  given for  $z_0 \in \bar{\mathcal{S}}$  and  $t \geq 0$  by

$$\Phi(z_0, 0) = z_0, \quad \frac{\partial}{\partial t} \Phi(z_0, t) = H(\Phi(z_0, t)), \quad (21)$$

where  $H$  is the Lipschitz function on  $\bar{\mathcal{S}}$  defined by

$$\begin{aligned} H^1(y_1, y_2, y_3) &= y_1 y_2 [(b - c)y_2 + (d - a)y_1] + y_1 y_3 [(\tau_1 - \tau_2 - c)y_2 - a y_1 + y_3 \tau_1] \\ H^2(y_1, y_2, y_3) &= y_1 y_2 [(a - d)y_1 + (c - b)y_2] + y_2 y_3 [(\tau_2 - \tau_1 - d)y_1 - b y_2 + y_3 \tau_2] \\ H^3(y_1, y_2, y_3) &= y_3 (a y_1^2 + b y_2^2 + (c + d)y_1 y_2 - \tau_1 y_1 y_3 - \tau_2 y_2 y_3). \end{aligned}$$

The study of  $\Phi$  close to  $\partial\mathcal{S}$  is giving us the behavior of  $\phi$  close to infinity using the change of time  $\varphi \in \mathcal{C}^1([0, \infty)^2 \times [0, \infty), [0, \infty))$  defined by

$$\varphi(x_0, 0) = x_0, \quad \frac{\partial}{\partial t} \varphi(x_0, t) = 1 + \|\phi(x_0, t)\|_1.$$



**Lemma 5.5.** For any  $x_0 \in [0, \infty)^2$  and  $t \geq 0$ ,

$$F(\phi(x_0, t)) = \Phi(F(x_0), \varphi(x_0, t)).$$

*Proof.* We denote by  $(y_t : t \geq 0)$  the image of the dynamical system  $(x_t : t \geq 0)$  through  $F$  :

$$y_t = F(x_t) = F(\phi(x_0, t)).$$

Then

$$y'_t = G(x_t) = G \circ F^{-1}(y_t)$$

where

$$G^{(1)}(x_1, x_2) = \frac{(d-a)x_1^2 x_2 + (b-c)x_1 x_2^2 + (\tau_1 - \tau_2 - c)x_1 x_2 - ax_1^2 + \tau_1 x_1}{(1+x_1+x_2)^2}$$

and

$$G^{(2)}(x_1, x_2) = \frac{(c-b)x_2^2 x_1 + (a-d)x_2 x_1^2 + (\tau_2 - \tau_1 - d)x_2 x_1 - bx_2^2 + \tau_2 x_2}{(1+x_1+x_2)^2}$$

and

$$G^{(3)}(x_1, x_2) = \frac{ax_1^2 + bx_2^2 + (c+d)x_1 x_2 - \tau_1 x_1 - \tau_2 x_2}{(1+x_1+x_2)^2}.$$

Using that  $x_1 = y_1/y_3$  and  $x_2 = y_2/y_3$ , we have

$$G \circ F^{-1}(y) = \frac{1}{y_3} H(y)$$

for  $y = (y_1, y_2, y_3) \in \mathcal{S}$ . The key point now of the theory of Poincaré is that  $H$  can be extended continuously to  $\bar{\mathcal{S}}$  and that the trajectories of the dynamical system  $(z_t : t \geq 0)$  associated to the vector field  $H$  :

$$z'_t = H(z_t)$$

are the same than the trajectories of  $(y_t : t \geq 0)$  whose vector field is  $G \circ F^{-1}$ . Indeed the positive scalar  $1/y_3$  only change the norm of the vector field and thus the speed at which the same trajectory is covered. The associated change of time  $v_t = \varphi(x_0, t)$  such that

$$z_{v_t} = y_t = F(x_t)$$

can now be simply computed :

$$v'_t = \frac{1}{F^{(3)}(\phi(x_0, t))} = 1 + \|\phi(x_0, t)\|_1,$$

which completes the proof. □

We introduce now the hitting times of cones to control the direction of the flow  $\phi$ ,

$$t_-(x_0, x, \varepsilon) = \inf_{s \geq 0} \{(\widehat{x_s}, x) \in [-\varepsilon, +\varepsilon]\}, \quad t_+(x_0, x, \varepsilon) = \inf_{s \geq t_-(x_0, x, \varepsilon)} \{(\widehat{x_s}, x) \notin [-2\varepsilon, +2\varepsilon]\},$$

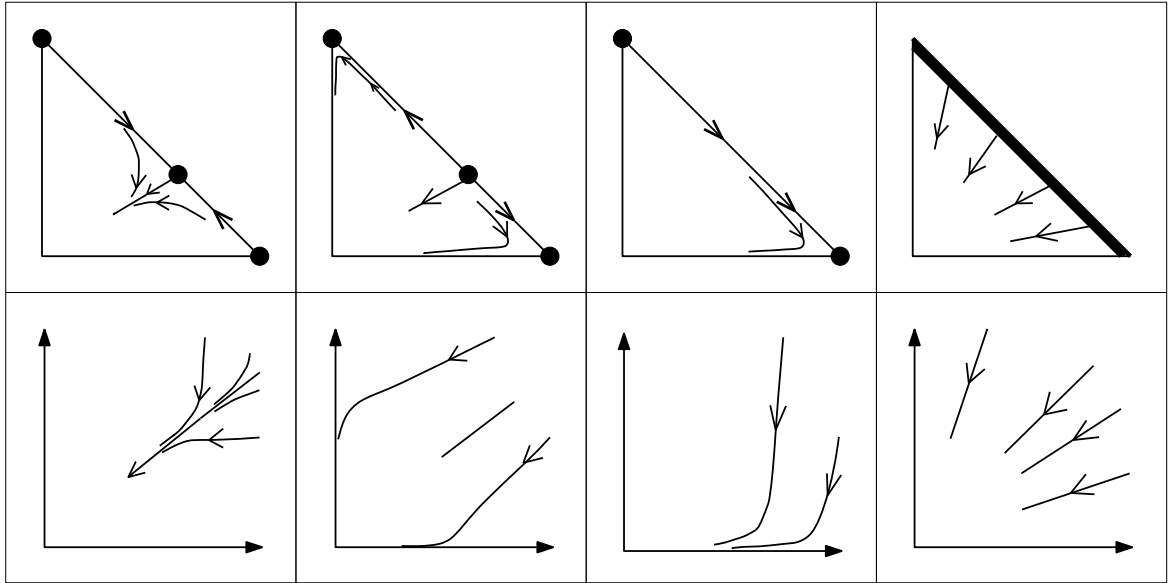
where we recall that  $x_s = \phi(x_0, s)$  and with the convention  $\inf \emptyset = \infty$ . We also introduce the direction  $x_l$  defined by

- $x_l = x_\infty$  if  $b > c$  and  $a > d$ .
- $x_l = (1/a, 0)$  if  $b > c$  and  $a \leq d$ ; or if  $c > b$  and  $d > a$  and  $(\widehat{x_0}, \widehat{x_\infty}) > 0$ .
- $x_l = (0, 1/b)$  if  $a > d$  and  $b \leq c$ ; or if  $c > b$  and  $d > a$  and  $(\widehat{x_0}, \widehat{x_\infty}) < 0$ .
- $x_l = \widehat{x_0}$  if  $a = d$  and  $b = c$ , where for  $x_0 \in (0, \infty)^2$ ,

$$\widehat{x_0} = \frac{x_0}{ax_0^{(1)} + bx_0^{(2)}}.$$

We prove below that  $x_l$  gives the direction from which the dynamical system comes down from infinity using the previous compactification result. We can then specify the speed of coming down from infinity of the dynamical system  $\phi$ .

Figure 1 : dynamical system close to infinity. We draw below the four regimes of the compactified flow  $\Phi$  starting close or on the boundary  $\partial S$  and the associated behavior of the original flow  $\phi$ . The fixed points of the boundary are fat.



**Lemma 5.6.** (i) For any  $T > 0$ , there exists  $c_T > 0$  such that  $\|\phi(x_0, t)\|_1 \leq c_T/t$  for all  $x_0 \in [0, \infty)^2$  and  $t \in (0, T]$ .

(ii) For all  $x_0 \in (0, \infty)^2$  and  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow \infty} t_-(rx_0, x_l, \varepsilon) = 0, \quad \lim_{r \rightarrow \infty} t_+(rx_0, x_l, \varepsilon) > 0.$$

(iii) Moreover,

$$\lim_{t \rightarrow 0} \limsup_{r \rightarrow \infty} \left| \|t\phi(rx_0, t)\|_1 - \|x_l\|_1 \right| = 0.$$

*Proof.* (i) Using  $a > 0$ , we first note that

$$(x_t^{(1)})' \leq -\frac{a}{2}(x_t^{(1)})^2$$

in the time intervals when  $x_t^{(1)} \geq 2\tau_1/a$ . Solving  $(x_t^{(1)})' = -(x_t^{(1)})^2 a/2$  proves (i).

(ii) We use the notations of Lemma 5.5 and the dynamics of  $z_t = \Phi(z_0, t)$  on the invariant set  $\partial\mathcal{S}$  is simply given by the vector field  $H(y_1, y_2, 0)$  for  $y_1 \in [0, 1]$ ,  $y_1 + y_2 = 1$ :

$$H^{(1)}(y_1, y_2, 0) = -H^{(2)}(y_1, y_2, 0) = y_1 y_2 [(b-c)y_2 + (d-a)y_1].$$

The two points  $(1, 0, 0)$  and  $(0, 1, 0)$  on  $\partial\mathcal{S}$  are invariant for the dynamical system. Let us first consider the case when  $a \neq d$  or  $b \neq c$ . There is an additional invariant point in the interior of  $\partial\mathcal{S}$  if and only if

$$(b-c)(a-d) > 0.$$

This invariant point is then unique and given by

$$z_\infty = \frac{1}{b-c+a-d} (b-c, a-d, 0).$$

Thus, if  $(b-c)(a-d) \leq 0$ ,  $H^{-1}((0, 0, 0)) \cap \partial\mathcal{S} = \{(1, 0, 0), (0, 1, 0)\}$  and  $z_t$  starting from the boundary  $\partial\mathcal{S}$  goes either to  $(1, 0, 0)$  or to  $(0, 1, 0)$ . Then by Lemma 5.5 the dynamical system  $z_{v_t} = F(x_t)$  starting from  $z_0$  close to the boundary  $\partial\mathcal{S}$  goes either to  $(1, 0, 0)$  (and then  $x_l = (1/a, 0)$ ); or to  $(0, 1, 0)$  (and then  $x_l = (0, 1/b)$ ). The fact that  $t_-(rx_0, x_l, \varepsilon)$  goes to 0 as  $r$  goes to infinity is then due to  $v'_0 = 1 + \|rx_0\|_1 \rightarrow \infty$  as  $r \rightarrow \infty$ . Moreover  $t_+(rx_0, x_l, \varepsilon)$  is not becoming close to 0 as  $r$  goes to infinity since the speed of the dynamical system  $(x_t : t \geq 0)$  is bounded on the compact sets of  $[0, \infty)^2$ .

Otherwise  $(b-c)(a-d) > 0$  and

$$H^{-1}((0, 0, 0)) \cap \partial\mathcal{S} = \{(1, 0, 0), (0, 1, 0), z_\infty\}.$$

Then we need to see if  $z_\infty$  is repulsive or attractive on the invariant set  $\partial\mathcal{S}$ . In the case  $c > b$  and  $d > a$ , this point is attractive and  $z_\infty$  is a saddle and

$$z_\infty = \lim_{r \rightarrow \infty} F(rx_\infty).$$

So Lemma 5.5 now ensures that the dynamical system  $x_t$  takes the direction  $x_l = x_\infty$  when starting from a large initial value. Similarly,  $t_-(rx_0, x_l, \varepsilon)$  goes to 0 and  $t_+(rx_0, x_l, \varepsilon)$  does not. In the case  $b < c$  and  $a < d$  case,  $y_\infty$  is a source and the dynamical system  $z_t$  either goes to  $(1, 0, 0)$  (and then  $x_l = (1/a, 0)$ ) or to  $(0, 1, 0)$  (and then  $x_l = (0, 1/b)$ ). This depends on the position of the initial value with respect to the second unstable variety and thus on the sign of  $(\widehat{x_0, x_\infty})$ .

Finally, the case  $a = d$  and  $b = c$  is handled similarly noting the whole set  $\partial\mathcal{S}$  is invariant.

(iii) We know from (ii) the direction of the dynamical system coming from infinity so we can reduce its dynamics close to infinity to a one-dimensional and solvable problem. Indeed, in the case  $x_l^{(1)} \neq 0$ , we set  $\theta_l = x_l^{(2)}/x_l^{(1)} \in [0, \infty)$ . We write

$$x_t = \phi(rx_0, t)$$

and for any  $\varepsilon > 0$  small enough and  $t \in [t_-(rx_0, x_l, \varepsilon), t_+(rx_0, x_l, \varepsilon)]$ , then  $(\widehat{x_t, x_l}) \leq 2\varepsilon$  and

$$\left| x_t^{(2)}/x_t^{(1)} - \theta_l \right| \leq u(\varepsilon), \quad (22)$$

where  $u(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . It ensures that

$$\tau_1 x_t^{(1)} - (a + c\theta_l + u(\varepsilon))(x_t^{(1)})^2 \leq (x_t^{(1)})' \leq \tau_1 x_t^{(1)} - (a + c\theta_l - u(\varepsilon))(x_t^{(1)})^2$$

and if additionally  $x_t^{(1)} \geq |\tau_1|/u(\varepsilon)$ , we get

$$-(a + c\theta_l + 2u(\varepsilon)) \leq \frac{(x_t^{(1)})'}{(x_t^{(1)})^2} \leq -(a + c\theta_l - 2u(\varepsilon)).$$

Since  $x_l^{(1)} > 0$ , Lemma 5.5 ensures that ratio  $x_t^{(2)}/x_t^{(1)}$  is bounded uniformly for  $r \geq 1$  in compact time intervals. Then there exists  $M_T > 0$  such that  $x_t^{(2)} \leq M_T x_t^{(1)}$  for  $t \in [0, T]$ . Using this bound to control  $x_t^{(1)}$  from below in (18), we obtain for any  $\varepsilon > 0$ ,

$$t_1(r) := \inf \left\{ t \geq 0 : x_t^{(1)} < |\tau_1|/u(\varepsilon) \right\} \in (0, \infty]$$

and

$$t_1 = \liminf_{r \rightarrow \infty} t_1(r) > 0.$$

Writing  $t_-(r) = t_-(rx_0, x_l, \varepsilon)$ ,  $t_+(r) = t_+(rx_0, x_l, \varepsilon)$  for simplicity, we have

$$\frac{1}{(a + c\theta_l + 2u(\varepsilon))(t - t_-(r)) + 1/x_{t_-(r)}^{(1)}} \leq x_t^{(1)} \leq \frac{1}{(a + c\theta_l - 2u(\varepsilon))(t - t_-(r)) + 1/x_{t_-(r)}^{(1)}},$$

for any  $t \in [t_-(r), t_1(r) \wedge t_+(r)]$ . Using (ii),  $t_-(r) \rightarrow 0$  and  $t_0 = \liminf_{r \rightarrow \infty} t_+(r) > 0$  as  $r \rightarrow \infty$ . Using (i) and  $x_l^{(1)} \neq 0$ , we get  $x_{t_-(r)}^{(1)} \rightarrow \infty$  as  $r \rightarrow \infty$ . Then for any  $t \leq t_0 \wedge t_1$ ,

$$\frac{1}{a + c\theta_l + 2u(\varepsilon)} \leq \liminf_{r \rightarrow \infty} t x_t^{(1)} \leq \limsup_{r \rightarrow \infty} t x_t^{(1)} \leq \frac{1}{a + c\theta_l - 2u(\varepsilon)}.$$

Letting  $\varepsilon$  go to 0, we obtain

$$\lim_{t \rightarrow 0} \limsup_{r \rightarrow \infty} |t x_t^{(1)} - 1/(a + c\theta_l)| = 0.$$

Recalling (22) ends the proof in the case  $x_l^{(1)} \neq 0$ . The case  $x_l^{(2)} \neq 0$  is treated similarly.  $\square$

Finally, we need the following additional results on the dynamical system coming from infinity. It allows to decompose the trajectory of the dynamical system in  $\mathcal{D}_\alpha = (\alpha, \infty)^2$  in time intervals when it belongs to a subdomain for which we know a transformation giving a non-expansive flow. Recall that  $T_D(x_0) = \inf\{t \geq 0 : \phi(x_0, t) \notin D\}$  and

$$d_\beta(x, y) = \sqrt{|x_1^\beta - y_1^\beta|^2 + |x_2^\beta - y_2^\beta|^2} = \|F_{\beta,1}(x) - F_{\beta,1}(y)\|_2.$$

**Lemma 5.7.** (i) Let  $\alpha > 0$ ,  $\beta \in (0, 1)$ ,  $N \in \mathbb{N}$  and  $(D_i)_{i=1, \dots, N}$  be a family of open convex cones of  $(0, \infty)^2$  such that

$$(0, \infty)^2 = \cup_{i=1}^N D_i.$$

Then, there exists  $\kappa \in \mathbb{N}$  and  $(t(k, x_0) : k = 0, \dots, \kappa)$  and  $(n(k, x_0) : k = 1, \dots, \kappa - 1)$  such that for any  $x_0 \in \mathcal{D}_\alpha$ ,

$$0 = t(0, x_0) \leq t(1, x_0) \leq \dots \leq t(\kappa, x_0) = T_{\mathcal{D}_\alpha}(x_0), \quad n(\cdot, x_0) \in \{1, \dots, N\}$$

and for  $\varepsilon$  small enough,  $k \leq \kappa - 1$  and  $t \in [t(k, x_0), t(k + 1, x_0))$ , we have

$$\overline{B}_{d_\beta}(\phi(x_0, t), \varepsilon) \subset D_{n(k, x_0)}.$$

(ii) In the case,  $x_1 = x_\infty \in (0, \infty)^2$ , for any  $x_0 \in (0, \infty)^2$  and  $\varepsilon > 0$ ,

$$\liminf_{r \rightarrow \infty} T_{\mathcal{D}_\varepsilon}(rx_0) > 0.$$

(iii) In the case  $x_1 = (1/a, 0)$ , for any  $x_0 \in (0, \infty)^2$  and  $\varepsilon > 0$  and  $T > 0$ , for  $r$  large enough,

$$T_{\mathcal{D}_\varepsilon}(rx_0) = \inf\{t \geq 0 : \phi(rx_0, t)^{(2)} \leq \varepsilon\} \leq T.$$

*Proof.* (i) We define

$$D_i^\varepsilon = \{x \in \mathcal{D}_\alpha \cap D_i : \overline{B}_{d_\beta}(x, \varepsilon) \subset D_i\}$$

and we first observe that for  $\varepsilon$  small enough,

$$\cup_{i=1}^N D_i^{2\varepsilon} = \mathcal{D}_\alpha,$$

since the open convex cone  $D_i$  is the domain between two half lines of  $(0, \infty)^2$  and their collection for  $i = 1, \dots, N$  covers  $(0, \infty)^2$ . We define

$$u_0^i(x_0) = \inf\{t \geq 0 : \phi(x_0, t) \in D_i^{2\varepsilon}\}, \quad v_0^i(x_0) = \inf\{t \geq u_0^i(x_0) : \phi(x_0, t) \notin D_i^\varepsilon\}$$

and by recurrence for  $k \geq 1$ ,

$$u_k^i(x_0) = \inf\{t \geq v_{k-1}^i(x_0) : \phi(x_0, t) \in D_i^{2\varepsilon}\}, \quad v_k^i(x_0) = \inf\{t \geq u_k^i(x_0) : \phi(x_0, t) \notin D_i^\varepsilon\}.$$

Let us then note that

$$\overline{\mathcal{S}} = \cup_{i=1}^N \overline{F(D_i)}, \quad \partial\mathcal{S} = \cup_{i=1}^N \partial F(\overline{D_i}),$$

where

$$\partial F(\overline{D_i}) = \overline{F(D_i)} - F(\overline{D_i}) = \{(t, 1 - t, 0) : t \in [a_i, b_i]\}$$

for some  $0 \leq a_i \leq b_i \leq 1$ . The dynamical system  $(z_t : t \geq 0)$  given by  $\Phi$  has been introduced in (21) and is defined on  $\overline{\mathcal{S}}$ . On the boundary  $\partial\mathcal{S}$ , it is given by  $(z_t^{(1)}, 1 - z_t^{(1)}, 0)$  where  $z_t^{(1)}$  is monotone. Outside this boundary,  $(z_t : t \geq 0)$  goes to a fixed point since the competitive Lotka Volterra dynamical system  $(x_t : t \geq 0)$  does.

Then the numbers

$$M^i(x_0) = \max\{k : v_k^i(x_0) < \infty\}$$

are bounded for  $x_0 \in \mathcal{D}_\alpha$  and  $i \in \{1, \dots, N\}$ . This yields the result using the time intervals  $[u_k^i(x_0), v_k^i(x_0)]$  which provides a covering  $[0, T_{\mathcal{D}_\alpha}(x_0)]$  for  $i = 1, \dots, N$  whose cardinal is bounded with respect to  $x_0 \in \mathcal{D}_\alpha$ .

(ii) comes simply from Lemma 5.5 which ensures that in the case  $x_1 = x_\infty$ , the dynamical system comes down from infinity in the interior of  $(0, \infty)^2$ , see also the first picture in Figure

1 above.

(iii) We use again the dynamical system  $(z_t : t \geq 0)$  given by  $\Phi$  and more precisely here the continuity of the associated flow with respect to the initial condition. Indeed, in the case  $x_t = (1/a, 0)$ , the trajectories of  $(z_t : t \geq 0)$  starting from  $r$  large go close to  $(0, 1, 0)$  and then remain close to the boundary  $(0, \infty) \times \{0\}$  until getting close to the fixed point on this boundary. This ensures that  $(x_t : t \geq 0)$  exists from  $\mathcal{D}_\varepsilon$  through  $(0, \infty) \times \{\varepsilon\}$ . The fact that this exit time  $T_{\mathcal{D}_\varepsilon}(x_0)$  goes to zero is due to the fact that the dynamics of  $(x_t : t \geq 0)$  is an acceleration of that of  $(z_t : t \geq 0)$  when starting close to infinity, the change of time being given by  $1 + \|\phi(x_0, t)\|_1$ .  $\square$

#### 5.4 Proofs of Theorem 5.1 and Corollary 5.2 and Theorem 5.3

We can now prove the Theorem 5.1 using the results of Section 3. Here  $d = 2$ ,  $H = G = 0$ ,  $\sigma_j^{(i)} = 0$  if  $j \neq i$  and

$$\sigma_1^{(1)}(x) = \sigma_1 \sqrt{x_1}, \quad \sigma_2^{(2)}(x) = \sigma_2 \sqrt{x_2}.$$

Moreover

$$\tilde{b}_{F_{\beta, \gamma}}(x) = \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2 F_{\beta, \gamma}}{\partial^2 x_i}(x) \sigma_i^{(i)}(x)^2 = \frac{1}{2} \beta(\beta - 1) \begin{pmatrix} \sigma_1^2 x_1^{\beta-1} \\ \gamma \sigma_2^2 x_2^{\beta-1} \end{pmatrix} \quad (23)$$

and

$$V_{F_{\beta, \gamma}}(x) = \sum_{i=1}^2 \left( \frac{\partial F_{\beta, \gamma}}{\partial x_i}(x) \sigma_i^{(i)}(x) \right)^2 = \beta^2 \begin{pmatrix} \sigma_1^2 x_1^{2\beta-1} \\ (\gamma \sigma_2)^2 x_2^{2\beta-1} \end{pmatrix}. \quad (24)$$

*Proof of Theorem 5.1.* To use Corollary 3.4, we first find a suitable covering of  $(0, \infty)^2$  by a finite number of convex open sets  $(D_i : i = 1, \dots, N)$  for which the non-expansivity hold. For that purpose, let us deal with the case  $c \neq 0$  (and then  $d \neq 0$ ), while the case  $c = d = 0$  is obvious. We fix  $\beta \in (1/2, 1)$ , which is close enough to 1 so that  $q_\beta = 4ab(1+\beta)^2 + 4cd(\beta^2 - 1) > 0$ . By Lemma 5.4 (ii), we consider the convex components of the open cones  $(D_{\beta, \gamma} : \gamma > 0)$  which contain

$$\{x_1 < \eta x_2\} \quad \text{or} \quad \{x_1 = x_\gamma x_2\} \quad \text{or} \quad \{x_2 < \eta x_1\}$$

for some  $\eta > 0$  and we recall that  $x_\gamma = (d\gamma^2/c)^{1/(2\beta-1)}$ . These convex components are open convex cones and the collection of these cones for  $\gamma > 0$  yields a covering of  $(0, \infty)^2$ . By a compactness argument, we can extract a finite family of open convex cones from this collection, which we denote by  $(D_i^\beta : i = 1, \dots, N)$  and which satisfies

$$(0, \infty)^2 = \cup_{i=1}^N D_i^\beta, \quad D_i^\beta \subset D_{\beta, \gamma_i},$$

for some  $\gamma_i > 0$ . Writing  $F_i = F_{\beta, \gamma_i}$  for convenience, Lemma 5.4 (i) ensures that the vector field  $\psi_{F_i}$  is  $\bar{\tau}$  non-expansive on  $F_i(D_i^\beta)$ .

We let now  $\alpha > 0$  and we use Lemma 5.7 (i) for the covering  $(D_i^\beta : i = 1, \dots, N)$ , so there exist  $\kappa \in \mathbb{N}$  and  $(t(k, x_0) : k = 0, \dots, \kappa)$  and  $(n(k, x_0) : k = 1, \dots, \kappa - 1)$  such that for any  $x_0 \in \mathcal{D}_\alpha$ ,

$$0 = t(0, x_0) \leq t(1, x_0) \leq \dots \leq t(\kappa, x_0) = T_{\mathcal{D}_\alpha}(x_0) = T(x_0), \quad n(k, x_0) \in \{1, \dots, N\}$$

and for  $\varepsilon$  small enough,  $k \leq \kappa - 1$  and  $t \in [t(k, x_0), t(k+1, x_0))$ , we have

$$\bar{B}_{d_\beta}(\phi(x_0, t), \varepsilon) \subset D_{n(k, x_0)}^\beta.$$

We consider now the open convex sets

$$D_i^{\beta, \alpha} = D_i^\beta \cap \mathcal{D}_{\alpha/2}.$$

Then for  $\varepsilon$  small enough, for any  $x_0 \in \mathcal{D}_\alpha$  and  $t \leq T_{\mathcal{D}_\alpha}(x_0)$ , we have  $\phi(x_0, t) \in \mathcal{D}_\alpha$  and

$$\bar{B}_{d_\beta}(\phi(x_0, t), \varepsilon) \subset D_{n(k, x_0)}^{\beta, \alpha}, \quad \mathcal{D}_\alpha \subset \cup_{i=1}^N D_i^{\beta, \alpha}.$$

Asumption 3.3 is thus checked with  $D = \mathcal{D}_\alpha$ ,  $D_i = D_i^{\beta, \alpha}$ ,  $O_i = D_i^{\beta, \alpha/2}$  ( $i = 1, \dots, N$ ),  $d = d_\beta$  and  $\phi$  defined by (18). We can now apply Corollary 3.4 and get

$$\mathbb{P}_{x_0} \left( \sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d(X_t, \phi(x_0, t)) \geq \varepsilon \right) \leq C \sum_{k=0}^{\kappa-1} \int_{t(k, x_0) \wedge T}^{t(k+1, x_0) \wedge T} \bar{V}_{d_\beta, \varepsilon}(F_{n(k, x_0)}, x_0, t) dt$$

for some positive constant  $C$ . We need now to control  $\bar{V}$ . First we use (23) to see that  $\tilde{b}_{F_i}$  is bounded on  $\mathcal{D}_{\alpha/2}$ . Then for  $\varepsilon$  small enough,

$$C'_i(\varepsilon) := \sup_{\substack{x_0 \in \mathcal{D}_\alpha, t \leq T_{\mathcal{D}_\alpha}(x_0) \\ d_\beta(x, \phi(x_0, t)) \leq \varepsilon}} \|\tilde{b}_{F_i}(x)\|_1 < \infty.$$

Moreover by Lemma 5.6 (i), we have

$$x_t^{(1)} \leq c_T/t, \quad x_t^{(2)} \leq c_T/t,$$

for  $t \in (0, T]$  and  $x_0 \in (0, \infty)^2$ . So using (24), there exists  $C''_i(\varepsilon) > 0$  such that for any  $x_0 \in \mathcal{D}_\alpha$  and  $t \leq T_{\mathcal{D}_\alpha}(x_0)$ ,

$$\bar{V}_{d, \varepsilon}(F_i, x_0, t) = \sup_{\substack{x \in [0, \infty)^2 \\ d_\beta(x, \phi(x_0, t)) \leq \varepsilon}} \left\{ \varepsilon^{-2} \|V_{F_i}(x)\|_1 + \varepsilon^{-1} \|\tilde{b}_{F_i}(x)\|_1 \right\} \leq \varepsilon^{-2} \frac{C''_i(\varepsilon)}{t^{2\beta-1}} + \varepsilon^{-1} C'_i(\varepsilon).$$

Adding that

$$\int_0^{\cdot} \left( \varepsilon^{-2} \frac{C''_i(\varepsilon)}{t^{2\beta-1}} + \varepsilon^{-1} C'_i(\varepsilon) \right) dt < \infty$$

for  $\beta < 1$ , we get

$$\limsup_{T \downarrow 0} \sup_{x_0 \in \mathcal{D}_\alpha} \mathbb{P}_{x_0} \left( \sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d_\beta(X_t, \phi(x_0, t)) \geq \varepsilon \right) = 0$$

for  $\varepsilon$  small enough. This ends up the proof.  $\square$

We can now describe the coming down from infinity of the two dimensional competitive Lotka Volterra diffusion.

*Proof of Corollary 5.2.* Let us deal with (i), so  $x_l = x_\infty \in (0, \infty)^2$  and we fix  $x_0 \in (0, \infty)^2$ . By Lemma 5.7 (ii),

$$\liminf_{r \rightarrow \infty} T_{\mathcal{D}_\varepsilon}(rx_0) > 0$$

and we use Theorem 5.1 to get for  $\beta \in (0, 1]$ ,

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{x_0} \left( \sup_{t \leq T} d_\beta(X_t, x_t) \geq \varepsilon \right) = 0,$$

where we write here for convenience

$$x_t = \phi(rx_0, t).$$

Then, using that  $d_\beta(tx, ty) = t^\beta d_\beta(x, y)$ ,

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{x_0} \left( \sup_{t \leq T} d_\beta(tX_t, tx_t) \geq \varepsilon \right) = 0. \quad (25)$$

Fix  $\eta \in (0, 1)$ . By Lemma 5.6 (ii),

$$\lim_{r \rightarrow \infty} t_-(rx_0, x_\infty, \varepsilon) = 0, \quad t_\varepsilon = \liminf_{r \rightarrow \infty} t_+(rx_0, x_\infty, \varepsilon) > 0.$$

Then for  $T < t_\varepsilon$  which is small enough, for  $r$  large enough, we have both

$$t_-(rx_0, x_\infty, \varepsilon) \leq \eta T \leq T \leq t_+(rx_0, x_\infty, \varepsilon)$$

and

$$\left| \|tx_t\|_1 - \|x_\infty\|_1 \right| \leq \varepsilon.$$

using Lemma 5.6 (iii). We get for  $t \in [\eta T, T]$ ,

$$\begin{aligned} \|tx_t - x_\infty\|_2 &\leq \left| \|tx_t\|_1 - \|x_\infty\|_1 \right| + \min(\|tx_t\|_2, \|x_\infty\|_2) |\sin(\widehat{x_t, x_\infty})| \\ &\leq 2(1 + \|x_\infty\|_2)\varepsilon, \end{aligned} \quad (26)$$

since for all  $r_1 > r_2 > 0$  and  $\theta_1 - \theta_2 \in (-\pi/2, \pi/2)$ ,

$$\left| r_1 e^{i\theta_1} - r_2 e^{i\theta_2} \right| \leq |r_1 \cos(\theta_1 - \theta_2) - r_2| + r_1 |\sin(\theta_1 - \theta_2)| \leq 2(r_1 - r_2) + r_2 |\sin(\theta_1 - \theta_2)|.$$

Combining (25) and (26) and

$$\|tX_t - x_\infty\|_2 \leq \|tX_t - tx_t\|_2 + \|tx_t - x_\infty\|_2,$$

we get

$$\lim_{T \rightarrow 0} \limsup_{r \rightarrow \infty} \mathbb{P}_{x_0} \left( \sup_{\eta T \leq t \leq T} \|tX_t - x_\infty\|_2 \geq \alpha \right) = 0,$$

for any  $\alpha > 0$ , since the euclidian distance is uniformly continuous from the bounded sets of  $(0, \infty)^2$  endowed with  $d_\beta$  to  $\mathbb{R}^+$ . This proves the first part of (i). The second part of (i) (resp. the proof of (iv)) is obtained similarly just by noting that  $t_-(rx_0, x_\infty, \varepsilon) = 0$  (resp.  $t_-(rx_0, \widehat{x_0}, \varepsilon) = 0$ ) if  $x_0$  is collinear to  $x_\infty$ .



For the cases (ii – iii), we know from Lemma 5.6 that the dynamical system is going to the boundary of  $(0, \infty)^2$  in short time. Let us deal with the case

$$x_l = (1/a, 0)$$

and the case  $x_l = (0, 1/b)$  would be handled similarly. We fix  $x_0 \in (0, \infty)^2$ ,  $T_0 > 0$ ,  $\epsilon, \eta > 0$  and  $\beta \in (0, 1)$ . By Theorem 5.1, there exists  $T \leq T_0$  such that for  $r$  large enough

$$\mathbb{P}_{rx_0} \left( \sup_{t \leq T \wedge T_{D_\epsilon}(rx_0)} d_\beta(X_t, \phi(rx_0, t)) \geq \epsilon \right) \leq \eta.$$

By Lemma 5.7 (iii), for  $r$  large enough, we have  $T_{D_\epsilon}(rx_0) = \inf\{t \geq 0 : \phi^{(2)}(rx_0, t) \leq \epsilon\} \leq T$ . Thus,

$$\mathbb{P}_{rx_0} (d_\beta(X_{T_{D_\epsilon}(rx_0)}, \phi(rx_0, T_{D_\epsilon}(rx_0))) \geq \epsilon) \leq \eta \quad \text{and} \quad \phi^{(2)}(rx_0, T_{D_\epsilon}(rx_0)) = \epsilon.$$

Fix now  $c \geq 1$  such that  $c^\beta \geq 2$ . We get

$$\begin{aligned} \mathbb{P}_{rx_0} (X_{T_{D_\epsilon}(rx_0)}^{(2)} \geq c\epsilon) &= \mathbb{P}_{rx_0} \left( \left( X_{T_{D_\epsilon}(rx_0)}^{(2)} \right)^\beta - \epsilon^\beta \geq (c^\beta - 1)\epsilon^\beta \right) \\ &\leq \mathbb{P}_{rx_0} (d_\beta(X_{T_{D_\epsilon}(rx_0)}, \phi(rx_0, T_{D_\epsilon}(rx_0))) \geq \epsilon) \leq \eta, \end{aligned}$$

since  $c^\beta - 1 \geq 1$  and  $\epsilon^\beta \geq \epsilon$ . By Markov property, we obtain for  $r$  large enough

$$\begin{aligned} \mathbb{P}_{rx_0} (X_{2T_0}^{(2)} = 0) &\geq \mathbb{P} (X_{T_{D_\epsilon}(rx_0)}^{(2)} \leq c\epsilon, \exists t \in [T_{D_\epsilon}(rx_0), T_{D_\epsilon}(rx_0) + T_0] : X_t^{(2)} = 0) \\ &\geq (1 - \eta)p(c\epsilon), \end{aligned}$$

where

$$p(x) = \mathbb{P}_x (X_{T_0}^{(2)} = 0).$$

Moreover  $X^{(2)}$  is stochastically smaller than a one-dimensional Feller diffusion and  $\sigma_2 \neq 0$ , so

$$\lim_{x \downarrow 0} p(x) = 1.$$

Letting  $\epsilon \rightarrow 0$  in the previous inequality yields

$$\liminf_{r \rightarrow \infty} \mathbb{P}_{rx_0} (X_{2T_0}^{(2)} = 0) \geq 1 - \eta.$$

Letting  $\eta \rightarrow 0$  ends up the proof of (ii – iii). □

We finally prove the scaling limits. Here we use the notations of Sections 3 for  $X^K = (X^{K,i} : i = 1, \dots, d)$  and we have  $\chi = [0, \infty)$ ,  $q(dz) = dz$  and

$$h_F^K(x) = \int_0^\infty [F(x + H^K(x, z)) - F(x)] dz.$$

Here we consider

$$F(x) = \begin{pmatrix} f^{(1)}(x_1) \\ f^{(2)}(x_2) \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and  $h_F^K$  is given for  $x \in [0, \infty)^2$  and  $i \in \{1, 2\}$  by

$$h_F^{K,(i)}(x) = \lambda_i^K(Kx)(F^{(i)}(x + e_i/K) - F^{(i)}(x)) + \mu_i^K(Kx)(F^{(i)}(x - e_i/K) - F^{(i)}(x)).$$

We recall that  $\mathcal{D}_\alpha = \{(x_1, x_2) \in (\alpha, \infty)^2 : x_1 \geq \alpha x_2, x_2 \geq \alpha x_1\}$  and

$$b(x) = b_{Id}^K(x) = \begin{pmatrix} \tau_1 x_1 - ax_1^2 - cx_1 x_2 \\ \tau_2 x_2 - bx_2^2 - dx_1 x_2 \end{pmatrix}.$$

We also recall that

$$\psi_{F_{\beta,\gamma}}^K = h_{F_{\beta,\gamma}}^K \circ F_{\beta,\gamma}^{-1}, \quad \psi_{F_{\beta,\gamma}} = (J_{F_{\beta,\gamma}} b) \circ F_{\beta,\gamma}^{-1}, \quad b_{F_{\beta,\gamma}}^K(x) = (J_{F_{\beta,\gamma}}(x))^{-1} h_{F_{\beta,\gamma}}^K(x)$$

and we introduce

$$D_{\beta,\gamma}^K(x) = \frac{\beta(\beta-1)}{2K} \begin{pmatrix} (ax_1 + cx_2)x_1^{\beta-1} \\ \gamma(bx_2 + dx_1)x_2^{\beta-1} \end{pmatrix}.$$

**Lemma 5.8.** *For any  $\alpha > 0$  and  $\beta \in (0, 1)$  and  $\gamma > 0$ , there exists  $C > 0$  such that for any  $x \in \mathcal{D}_\alpha$  and  $K \geq 2/\alpha$ ,*

(i)

$$\|h_{F_{\beta,\gamma}}^K(x) - J_{F_{\beta,\gamma}}(x)b(x) - D_{\beta,\gamma}^K(x)\|_2 \leq \frac{C}{K} \|x\|_2^{\beta-1}.$$

(ii)

$$\|b_{F_{\beta,\gamma}}^K(x) - b(x)\|_2 \leq \frac{C}{K} \|x\|_2.$$

(iii)

$$\psi_{F_{\beta,\gamma}}^K(x) = \psi_{F_{\beta,\gamma}}(x) + D_{\beta,\gamma}^K(F_{\beta,\gamma}^{-1}(x)) + R_{\beta,\gamma}^K(F_{\beta,\gamma}^{-1}(x)),$$

where  $\|R_{\beta,\gamma}^K(x)\|_2 \leq C/K$ .

(iv) Moreover  $\psi_{F_{\beta,\gamma}}^K$  is  $(C, C/K)$  non-expansive on  $F_{\beta,\gamma}(D_{\beta,\gamma} \cap \mathcal{D}_\alpha)$ .

*Proof.* First, by Taylor expansion, there exists  $C > 0$  such that

$$\left| \left(z + \frac{\delta}{K}\right)^\beta - z^\beta - \frac{\delta}{K} \beta z^{\beta-1} - \frac{\delta^2}{2K^2} \beta(\beta-1) z^{\beta-2} \right| \leq \frac{C}{K^2} z^{\beta-3}$$

for any  $z > \alpha$  and  $K \geq 2/\alpha$  and  $\delta \in \{-1, 1\}$ , since  $1 + \delta/(Kz) > 1/2$ . Using then that

$$h_{F_{\beta,\gamma}}^K(x) = \begin{pmatrix} \lambda_1 K x_1 \left( (x_1 + 1/K)^\beta - x_1^\beta \right) + K x_1 (\mu_1 + ax_1 + cx_2) \left( (x_1 - 1/K)^\beta - x_1^\beta \right) \\ \gamma \lambda_2 K x_2 \left( (x_2 + 1/K)^\beta - x_2^\beta \right) + \gamma K x_2 (\mu_2 + bx_2 + dx_1) \left( (x_2 - 1/K)^\beta - x_2^\beta \right) \end{pmatrix}$$

and

$$J_{F_{\beta,\gamma}}(x)b(x) = \begin{pmatrix} \beta x_1^{\beta-1} x_1 (\tau_1 - ax_1 - cx_2) \\ \gamma \beta x_2^{\beta-1} x_2 (\tau_2 - bx_2 - dx_1) \end{pmatrix}$$

yields (i), since  $\|x\|_2$ ,  $x_1$  and  $x_2$  are equivalent up to constant when  $x \in \mathcal{D}_\alpha$ .

Then the previous lemma and the fact that there exists  $c_0 > 0$  such that

$$\|J_{F_{\beta,\gamma}}(x)^{-1} D_{\beta,\gamma}^K(x)\|_2 \leq c_0 \frac{\|x\|_2}{K}$$

for  $x \in \mathcal{D}_\alpha$  ensures that there exists  $c' > 0$  such that

$$\|b_{F_{\beta,\gamma}}^K(x) - b(x)\|_2 \leq c' \frac{\|x\|_2}{K}.$$

Moreover

$$\psi_{F_{\beta,\gamma}}^K = \psi_{F_{\beta,\gamma}} + D_{\beta,\gamma}^K(F_{\beta,\gamma}^{-1}) + R_{\beta,\gamma}^K(F_{\beta,\gamma}^{-1}),$$

where  $\|R_{\beta,\gamma}^K(x)\|_2 \leq c''/K$  for some  $c'' > 0$ , since  $\beta < 1$ . This proves (ii – iii).

Adding that  $D_{\beta,\gamma}^K \circ F_{\beta,\gamma}^{-1}$  is uniformly Lipschitz on  $F_{\beta,\gamma}(\mathcal{D}_\alpha)$  with constant  $L$  and using Lemma 5.4 (i), we get that  $\psi_{F_{\beta,\gamma}}^K$  is  $(\bar{\tau} + L, c''/K)$  non-expansive on  $F_{\beta,\gamma}(D_{\beta,\gamma} \cap \mathcal{D}_\alpha)$ . So (iv) hold.  $\square$

*Proof of Theorem 5.3.* We first observe that  $(a > d$  and  $b > c)$  or  $(a = d, b = c)$  ensures that

$$q_\beta = 4ab(1 + \beta)^2 + 4cd(\beta^2 - 1) > 0.$$

Using now Lemma 5.4 (iii) and following the beginning of the proof of Theorem 5.1, we can find a covering of  $\mathcal{D}_\alpha$  by a finite collection of open convex cones  $D_i = D_{\beta,\gamma_i}$  ( $i = 1, \dots, N$ ) and noting  $F_i = F_{\beta,\gamma_i}$ , there exist  $\mu_i > 0$  such that

$$(\psi_{F_i}(x) - \psi_{F_i}(y)) \cdot (x - y) \leq -\mu_i \|x\|_2 \|x - y\|_2^2, \quad (27)$$

for any  $x, y \in D_i$  which are large enough.

Let us fix  $\varepsilon > 0$ . We note that  $\mathcal{D}_\alpha \subset D_\alpha$  and using Lemma 5.7 (i), we introduce the sequences  $(t(k, x_0) : k = 0, \dots, \kappa)$  and  $(n(k, x_0) : k = 1, \dots, \kappa - 1)$  such that

$$0 = t(0, x_0) \leq t(1, x_0) \leq \dots \leq t(\kappa, x_0) = T_{\mathcal{D}_\alpha}(x_0), \quad n(k, x_0) \in \{1, \dots, N\}$$

and for  $\varepsilon$  small enough, for any  $x_0 \in \mathcal{D}_\alpha$ ,

$$\phi(x_0, 0) = x_0, \quad \bar{B}_{d_\beta}(\phi(x_0, t), \varepsilon) \subset D_{n(k, x_0)} \quad \text{and} \quad \frac{\partial}{\partial t} \phi(x_0, t) = b(\phi(x_0, t)).$$

Using Lemma 5.8 (iii) and adding that  $\|D_{\beta,\gamma}^K(F_{\beta,\gamma}^{-1}(x))\|_2 / \|x\|_2$  is bounded for  $x \in \mathcal{D}_\alpha$ , there exists  $C > 0$  such that for any  $i = 1, \dots, N$ , for any  $y \in F_i(D_i)$ ,

$$\|\psi_{F_i}^K(y) - \psi_{F_i}(y)\|_2 \leq C \frac{1 + \|y\|}{K}$$

Using again Lemma 5.8 (iii) and now the fact that on the compact domains of  $(0, \infty)^2$ ,  $K \|\psi_{F_i}^K(\cdot) - \psi_{F_i}(\cdot)\|_2$  is bounded and  $\psi_{F_i}$  is Lipschitz, there exists  $L$  such that

$$\|\psi_{F_i}(x) - \psi_{F_i}^K(y)\|_2 \leq L(\|x - y\|_2 + 1/K)$$

for any  $x, y$  in a compact domain.

The assumptions of Lemma 6.3 in Appendix are met and this Lemma ensures that the flow  $\widetilde{\phi}_i^K$  defined by

$$\widetilde{\phi}_i^K(y_0, t) = y_0, \quad \frac{\partial}{\partial t} \widetilde{\phi}_i^K(y_0, t) = \psi_{F_i}^K(\widetilde{\phi}_i^K(y_0, t))$$

is close to the flow  $\tilde{\phi}_i$  associated to  $\psi_{F_i}$

$$\tilde{\phi}_i(y_0, t) = y_0, \quad \frac{\partial}{\partial t} \tilde{\phi}_i(y_0, t) = \psi_{F_i}(\tilde{\phi}_i(y_0, t)).$$

More precisely,  $\|\tilde{\phi}_i^K(y_0, t) - \tilde{\phi}_i(y_0, t)\|_2 \rightarrow 0$  as  $K \rightarrow \infty$ , uniformly for  $t \in [0, T]$  and  $y_0 \in F_i(D_i)$ . Then the flow  $\phi_i^K(x_0, t) = F_i^{-1}(\tilde{\phi}_i^K(x_0, t))$  is uniformly close to the flow  $\phi_i(x_0, t) = F_i^{-1}(\tilde{\phi}_i(x_0, t))$  for  $t \in [0, T]$  and  $x_0 \in D_i$  for the distance  $d_{F_i}$  and thus for the distance  $d_\beta$ . Moreover

$$\frac{\partial}{\partial t} \phi_i^K(x_0, t) = b_{F_i}^K(\phi_i^K(x_0, t)), \quad \frac{\partial}{\partial t} \phi_i(x_0, t) = b(\phi_i(x_0, t)),$$

since  $b_{F_i} = b$ . Then  $\phi_i^K$  is close to  $\phi_i = \phi$  for the distance  $d_\beta$  and for  $K$  large enough, we can define the continuous flow  $\phi^K$  by adjunction on the time intervals as follows

$$\phi^K(x_0, 0) = x_0, \quad \bar{B}_{d_\beta}(\phi^K(x_0, t), \varepsilon) \subset D_{n(k, x_0)} \quad \text{and} \quad \frac{\partial}{\partial t} \phi^K(x_0, t) = b_{F_{n(k, x_0)}}^K(\phi^K(x_0, t))$$

for  $t \in (t(k, x_0), t(k+1, x_0))$  and  $k = 0, \dots, \kappa - 1$ . Assumption 3.3 is satisfied for  $D = \mathcal{D}_\alpha$ . From Lemma 5.8 (iv), we know that  $\psi_{F_i}^K$  is  $(C_i, C_i/K)$  non-expansive on  $F_i(D_i)$ . Thus, following the proof of Theorem 5.1, we apply Corollary 3.4 and get for  $x_0 \in \mathcal{D}_\alpha$  and  $T \geq 0$  and  $K \geq 2/\alpha$  such that  $4C_i T \exp(2L_i T) \leq K\varepsilon$  for  $i = 1, \dots, N$ ,

$$\mathbb{P}_{x_0} \left( \sup_{t \leq T \wedge T_{\mathcal{D}_\alpha}(x_0)} d_\beta(X_t, \phi^K(x_0, t)) \geq \varepsilon \right) \leq C \sum_{k=0}^{\kappa-1} \int_{t(k, x_0) \wedge T}^{t(k+1, x_0) \wedge T} \bar{V}_{d, \varepsilon}(F_{n(k, x_0)}^K x_0, t) dt,$$

where  $C$  is positive constant which does not depend on  $K, x_0$ . Moreover

$$\begin{aligned} V_{F_{\beta, \gamma}}^{K, (i)}(x) &= \lambda_i^K(x) \left( (x_i + 1/K)^\beta - x_i^\beta \right)^2 + \mu_i^K(x) \left( (x_i - 1/K)^\beta - x_i^\beta \right)^2 \\ &\leq \frac{C'}{K} x_i^{2\beta-2} x_i (1 + x_1 + x_2) \end{aligned}$$

for some  $C' > 0$ . Then for  $x \in \mathcal{D}_\alpha$ ,

$$\|V_{F_{\beta, \gamma}}^K(x)\|_1 \leq \frac{C''}{K} \left( x_1^{2\beta} (1 + x_2/x_1) + x_2^{2\beta} (1 + x_1/x_2) \right) \leq \frac{C''_\alpha}{K} \left( x_1^{2\beta} + x_2^{2\beta} \right).$$

Moreover we know from Lemma 5.6 (i) that  $x_t^{(1)} \leq C_T/t$  for  $t \in [0, T]$ . Then we have for  $x_0 \in \mathcal{D}_\alpha$  and  $\varepsilon$  small enough and  $T \leq T_{\mathcal{D}_\alpha}$ ,

$$\int_0^T \bar{V}_{d, \varepsilon}^K(F_{\beta, \gamma}, x_0, t) dt \leq \frac{C'''_\alpha \varepsilon^{-2}}{K} \int_0^T t^{-2\beta} dt.$$

Using the fact that  $\int_0^T t^{-2\beta} dt < \infty$  for  $\beta < 1/2$ , we get

$$\sum_{k=0}^{\kappa-1} \int_{t(k, x_0) \wedge T}^{t(k+1, x_0) \wedge T} \bar{V}_{d, \varepsilon}(F_{n(k, x_0)}, x_0, t) dt \leq \varepsilon^{-2} \frac{C''''_\alpha}{K}$$

Recalling that the flow  $\phi^K$  is uniformly close to the flow  $\phi$  for the distance  $d_\beta$  when  $K$  goes to infinity, we obtain for any  $\varepsilon > 0$  (small enough),

$$\sup_{x_0 \in \mathbb{D}_\alpha} \mathbb{P}_{x_0} \left( \sup_{t \leq T \wedge T_{\mathbb{D}_\alpha}(x_0)} d_\beta(X_t^K, \phi(x_0, t)) \geq \varepsilon \right) \leq \varepsilon^{-2} \frac{C_\alpha}{K}.$$

for some  $C_\alpha$  positive. To conclude, let us fix  $\alpha_0, T > 0$  and observe that

$$T_\alpha = \inf_{x_0 \in \mathbb{D}_{\alpha_0}} T_{\mathbb{D}_\alpha}(x_0) \xrightarrow{\alpha \rightarrow 0} \infty$$

since in the case  $b > c \geq 0$ ,  $a > d \geq 0$ , we know from Lemma 5.6 (ii) that the process comes down from infinity along the vector  $x_\infty$ . So we can choose  $\alpha > 0$  small enough so that  $T_\alpha \geq T$ . The last inequality becomes

$$\sup_{x_0 \in \mathbb{D}_{\alpha_0}} \mathbb{P}_{x_0} \left( \sup_{t \leq T} d_\beta(X_t^K, \phi(x_0, t)) \geq \varepsilon \right) \leq \varepsilon^{-2} \frac{C_\alpha}{K},$$

and hold for  $K \geq \max(2/\alpha, 4C_i T \exp(2L_i T)/\varepsilon : 1 \leq i \leq N)$ , which ends up the proof.  $\square$

*Remark.* Let us mention an alternative approach. Using Proposition 2.3 (or extending the Corollary of Section 3), one could try to compare directly the process  $X$  and to the flow  $\phi$  (instead of  $\phi^K$ ) and put the remaining term  $R^K$  in a finite variation part  $A_t$ .

## 6 Appendix

We give here first two technical results to study the coming down from infinity of dynamical systems in one dimension with polynomial drift.

Let  $\psi_1$  and  $\psi_2$  be two locally Lipschitz function defined on  $(0, \infty)$  which are negative for  $x$  large enough. Let  $\phi_1$  and  $\phi_2$  the flows associated respectively to  $\psi_1$  and  $\psi_2$ . Let us state a simple condition to guarantee that two such flows are equivalent near  $+\infty$ .

**Lemma 6.1.** *If*

$$\psi_1(x) \sim_{x \rightarrow \infty} \psi_2(x) \quad \text{and} \quad \int_\infty \frac{1}{\psi_1(x)} dx < \infty$$

*then*

$$\int_\infty \frac{1}{\psi_2(x)} dx < \infty$$

*and if additionally  $\phi_1(\infty, t) \sim ct^{-\alpha}$  as  $t \rightarrow 0$  for some  $\alpha < 0$  and  $c > 0$ , then*

$$\phi_2(\infty, t) \sim_{t \rightarrow 0} ct^{-\alpha}.$$

*Proof.* Let  $\varepsilon \in (0, 1)$  and choose  $x_1 > 0$  such that

$$0 > \psi_1(x) \geq (1 + \varepsilon)\psi_2(x)$$

for  $x \geq x_1$ . Then for any  $x_0 > x_1$ ,

$$\phi_1(x_0, t) \leq (1 + \varepsilon) \int_0^t \psi_2(\phi_1(x_0, s)) ds$$

for  $t$  small enough. Then,  $\phi_1(x_0, t) \leq \phi_2(\infty, (1 + \varepsilon)t)$  and

$$\phi_1(\infty, t/(1 + \varepsilon)) \leq \phi_2(\infty, t)$$

for  $t$  small enough. Proving the symmetric inequality ends up the proof.  $\square$

In the case of polynomial drift, we specify here the error term when coming from infinity.

**Lemma 6.2.** *Let  $\rho > 1, c > 0, \alpha > 0, \varepsilon > 0$  and*

$$\psi(x) = -cx^\rho(1 + r(x)x^{-\alpha}),$$

where  $r$  is locally Lipschitz and bounded on  $(x_0, \infty)$ .

Denoting by  $\phi$  the flow associated to  $\psi$ , we have

$$\phi(\infty, t) = (ct/(\rho - 1))^{1/(1-\rho)}(1 + \tilde{r}(t)t^{\alpha/(\rho-1)}),$$

where  $\tilde{r}$  is a bounded function.

*Proof.* As  $\tilde{r}$  is bounded, there exists  $c_1 > c_2$  such that

$$-cx^\rho(1 + c_1x^{-\alpha}) \leq \psi(x) \leq -cx^\rho(1 + c_2x^{-\alpha}),$$

so for some  $c'_1 > c'_2$  and  $x$  large enough

$$-cx^{-\rho}(1 - c'_2x^{-\alpha}) \leq \frac{1}{\psi(x)} \leq -cx^{-\rho}(1 - c'_1x^{-\alpha}).$$

Then

$$-c \int_{\phi(x_0,0)}^{\phi(x_0,t)} x^{-\rho}(1 - c'_2x^{-\alpha})dx \leq \int_{\phi(x_0,0)}^{\phi(x_0,t)} \frac{dx}{\psi(x)} \leq -c \int_{\phi(x_0,0)}^{\phi(x_0,t)} x^{-\rho}(1 - c'_1x^{-\alpha})dx,$$

where the middle term is equal to  $t$ . Letting  $x_0 \rightarrow \infty$

$$c''_2\phi(\infty, t)^{-\rho-\alpha+1} \leq t - \frac{c}{\rho-1}\phi(\infty, t)^{-\rho+1} \leq c''_1\phi(\infty, t)^{-\rho-\alpha+1}.$$

We know from the previous lemma that  $\phi(\infty, t) \sim (c\rho^{-1}t)^{1/(1-\rho)}$  as  $t \rightarrow 0$  and we get here

$$\phi(\infty, t) = (ct/(\rho - 1))^{1/(1-\rho)}(1 + O(t^{-1+(-\rho+1-\alpha)/(1-\rho)})) = (ct/(\rho - 1))^{1/(1-\rho)}(1 + O(t^{\alpha/(\rho-1)})),$$

which ends up the proof.  $\square$

We need also the following estimates. Let  $\psi$  and  $\psi^K$  be locally Lipschitz vectors fields on an open domain  $D \subset \mathbb{R}^d$  with respective flows  $\phi$  and  $\phi^K$ . We write again  $T_D(x_0)$  the maximal time when the flow  $\phi$  started at  $x_0$  is well defined and belongs to  $D$ .

**Lemma 6.3.** Let  $T > 0$ . We assume that there exist  $A \geq 1, c, L, \mu > 0$  and  $\varepsilon \in (0, 1]$  such that

$$\|\psi(x) - \psi^K(x)\|_2 \leq c \frac{1 + \|x\|_2}{K}$$

for any  $x \in D$  and  $K \geq 1$ , and

$$\|\psi(x) - \psi^K(y)\|_2 \leq L(\|x - y\|_2 + 1/K)$$

for any  $x, y \in D \cap B(0, A + 2)$  and  $K \geq 1$ , while

$$(\psi(x) - \psi(y)) \cdot (x - y) \leq -\mu \|x\|_2 \|x - y\|_2^2,$$

for any  $x \in D \cap B(0, A)^c$  and  $y \in \bar{B}(x, \varepsilon)$ .

Then, for every  $K$  such that  $K \geq \max\left(2\frac{L}{\varepsilon}e^{2LT}, 3\frac{c}{\varepsilon\mu}\right)$ , for all  $x_0 \in D$  and  $t < T_D(x_0)$ ,

$$\|\phi(x_0, t) - \phi^K(x_0, t)\|_2 \leq \frac{1}{K} \max\left(2Le^{2LT}, 3\frac{c}{\mu}\right).$$

*Proof.* Let  $T > 0$  and  $K \geq \max\left(2Le^{2LT}, 3\frac{c}{\mu}\right)$ .

When  $x_t = \phi(x_0, t) \in D \cap B(0, A)^c$  and  $x_t^K = \phi^K(x_0, t) \in \bar{B}(x_t, \varepsilon)$ , then the assumptions above and Cauchy Schwarz inequality give

$$\begin{aligned} \frac{d}{dt} \|x_t - x_t^K\|_2^2 &= 2(\psi(x_t) - \psi^K(x_t^K)) \cdot (x_t - x_t^K) \\ &= 2(\psi(x_t) - \psi(x_t^K)) \cdot (x_t - x_t^K) + 2(\psi(x_t^K) - \psi^K(x_t^K)) \cdot (x_t - x_t^K) \\ &\leq 2\left(-\mu \|x_t\|_2 \|x_t - x_t^K\|_2 + c \frac{1 + \|x_t^K\|_2}{K}\right) \|x_t - x_t^K\|_2. \end{aligned}$$

Moreover  $\|x_t\|_2 \geq A$  and  $x_t^K \in \bar{B}(x_t, \varepsilon)$  give

$$\frac{1 + \|x_t^K\|_2}{\|x_t\|_2} \leq \frac{1}{A} + 1 + \frac{\varepsilon}{A} \leq 3,$$

so  $\|x_t - x_t^K\|_2 \in [3c/(K\mu), \varepsilon]$  implies

$$\frac{d}{dt} \|x_t - x_t^K\|_2^2 \leq 0.$$

This means that when  $x_t \in D \cap B(0, A)^c$ , the gap  $\|x_t - x_t^K\|_2$  tends to decrease when it is larger than  $3c/(K\mu)$  but smaller than  $\varepsilon$ .

Moreover by assumption and Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{d}{dt} \|x_t - x_t^K\|_2^2 &\leq 2 \|x_t - x_t^K\|_2 \|\psi(x_t) - \psi^K(x_t^K)\|_2 \\ &\leq 2L \|x_t - x_t^K\|_2^2 + \frac{2}{K} \|x_t - x_t^K\|_2 \end{aligned}$$

as soon as  $x_t$  and  $x_t^K$  both belong to  $B(0, A + 2)$ . Let now  $x_0 \in D \cap B(0, A)$ . On the time interval when  $x_t \in B(0, A + 1)$  and

$$\|x_t - x_t^K\|_2 \geq \frac{L}{K}$$

we have

$$\frac{d}{dt} \|x_t - x_t^K\|_2^2 \leq 4L \|x_t - x_t^K\|_2^2.$$

We can apply Gronwall lemma to get

$$\|x_t - x_t^K\|_2 \leq 2 \frac{L}{K} e^{2Lt} \leq \varepsilon$$

and  $x_t \in B(0, A + 1)$  (and thus  $x_t^K \in B(0, A + 2)$ ).

Gluing these estimates yields

$$\|x_t - x_t^K\|_2 \leq \max\left(2 \frac{L}{K} e^{2LT}, 3 \frac{c}{K\mu}\right)$$

for  $t \leq T$ , which completes the proof. □

**Acknowledgment.** I wish to thank warmly Pierre Collet, Sylvie Méléard and Stéphane Gaubert for stimulating and fruitful discussions in Ecole polytechnique along this work. I also want to thank Clément Foucart for its reading of a previous version of this paper and its comments.

This work was partially funded by the Chaire Modélisation Mathématique et Biodiversité VEOLIA-École Polytechnique-MNHN-FX.

## References

- [1] W. J. Anderson. *Continuous-time Markov chains*. Springer Series in Statistics: Probability and its Applications. Springer-Verlag, New York, 1991. An applications-oriented approach.
- [2] Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: a brief introduction and some open problems. *53rd IEEE Conference on Decision and Control* December 15-17, 2014. Los Angeles, California, USA.
- [3] R. Azais, J-B Bardet, A. Génadot, N. Krell and P-A Zitt. ESAIM: Proceedings, January 2014, Vol. 44, p. 276-290 SMAI Groupe MAS - Journées MAS 2012 - Session thématique.
- [4] V. Bansaye, S. Méléard and M. Richard. The speed of coming down from infinity for birth and death processes. *Preprint*, 2014.
- [5] J. Berestycki, N. Berestycki, and V. Limic. The Lambda-coalescent speed of coming down from infinity. *Ann. Probab.*, 38(1):207–233, 2010.



- [6] J. Bertoin and J-F LeGall. Stochastic flows associated to coalescent processes. III. Limit theorems. *Illinois J. Math.* (2006), Vol. 50, p. 147-181.
- [7] P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, 37(5):1926–1969, 2009.
- [8] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. To appear in *Probab. Th. Rel. Field*, 2015.
- [9] D. A. Dawson and Z. Li. Stochastic equations, flows and measure-valued processes. *Ann. Probab.* 40 (2012), no. 2, 813-857.
- [10] R. Darling and J. Norris. Differential equation approximations for Markov chains. *Probab. Surv.* 5 (2008), 37-79.
- [11] P. Donnelly. Weak convergence to a Markov chain with an entrance boundary: ancestral processes in population genetics. *Ann. Probab.*, 19(3):1102–1117, 1991.
- [12] C. Dellacherie, P. A. Meyer. *Probabilities and potential. B. Theory of martingales*. North-Holland Mathematics Studies, 72. North-Holland Publishing Co., Amsterdam, 1982.
- [13] F. Dumortier, J. Llibre and J. C. Artés. *Qualitative Theory of Planar Differential Systems*. Springer-Verlag, Berlin, 2006.
- [14] S. N. Ethier and T. G. Kurtz. *Markov processes. Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, New York, 1986. .
- [15] M. I. Frendlin and A. D. Wentzell. *Random perturbations of dynamical systems* . Third edition. Grundlehren der Mathematischen Wissenschaften, 260. Springer, Heidelberg, 2012
- [16] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes, 2nd ed.* North-Holland, 1989.
- [17] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, 1987.
- [18] S. Karlin and J. L. McGregor. The differential equations of birth-and-death processes, and the Stieltjes moment problem. *Trans. Amer. Math. Soc.*, 85:489–546, 1957.
- [19] S. Karlin and H. M. Taylor. *A first course in stochastic processes*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, second edition, 1975.
- [20] V. Le and E. Pardoux Height and the total mass of the forest of genealogical trees of a large population with general competition. To appear in *ESAIM P & S*.
- [21] V. Limic. On the speed of coming down from infinity for  $\Xi$ -coalescent processes. *Electron. Journal Probab.* Vol. 15, 217–240, 2010.
- [22] V. Limic and A. Talarczyk. Diffusion limits for mixed with Kingman coalescents at small times <http://arxiv.org/abs/1409.6200>, 2014.

- [23] V. Limic and A. Talarczyk (2014). Second-order asymptotics for the block counting process in a class of regularly varying Lambda-coalescents. To appear in *Ann. Probab.*
- [24] J. Pitman. Coalescents with multiple collisions. *Ann. Probab.* 27 (1999), no. 4, 1870-1902.
- [25] J. Schweinsberg. A necessary and sufficient condition for the  $\Lambda$ -coalescent to come down from infinity. *Electron. Comm. Probab.* (2000) Vol. 5, P. 1-11.
- [26] A. V. Skorokhod, F. C. Hoppensteadt and H. Salehi. *Random perturbation methods with applications in science and engineering* . Applied Mathematical Sciences, 150. Springer-Verlag, New York, 2002.
- [27] E. A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, 23 (4) : 683–700, 1991.