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# LYAPUNOV EXPONENTS FOR RANDOM CONTINUOUS-TIME SWITCHED SYSTEMS AND STABILIZABILITY

FRITZ COLONIUS\* AND GUILHERME MAZANTI†

**Abstract.** For linear systems in continuous time with random switching, the Lyapunov exponents are characterized using the Multiplicative Ergodic Theorem for an associated system in discrete time. An application to control systems shows that here a controllability condition implies that arbitrary exponential decay rates for almost sure stabilization can be obtained.

**Key words.** Random switching, Lyapunov exponents, Multiplicative Ergodic Theorem, almost sure stabilization, arbitrary rate of convergence

**AMS subject classifications.** 93C30, 37H15, 37A50, 93D15

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**1. Introduction.** Linear systems with switching coefficients are of considerable interest in theory and applications. The present paper considers systems in continuous time with random switching and develops methods to describe the exponential growth rates, i.e., the Lyapunov exponents. This is used to analyze stabilizability properties of control systems with random switching.

Systems with deterministic switching have been extensively studied, cf., e.g., the monograph Liberzon [20] and the surveys Lin and Antsaklis [21], and Shorten, Wirth, Mason, Wulff, and King [27]. An important motivation for our work comes from the theory of persistently excited control systems where switching means that the control is put on or off. These deterministic systems have been studied in a number of papers, with many results in special situations, cf. Chaillet, Chitour, Loria, and Sigalotti [5], Chitour and Sigalotti [10]. In particular, it is known that here, contrary to the situation for autonomous linear control systems, controllability does not imply stabilizability with arbitrary decay rates.

The analysis of random switched systems in the present paper is based on the classical Multiplicative Ergodic Theorem due to Oseledets (cf. Arnold [1]). It turns out that a direct application of this theorem to systems in continuous time with random switching is not feasible, since in general they do not define random dynamical systems in the sense of [1] (cf. Example 2.6). Instead, we apply the Multiplicative Ergodic Theorem to an associated system in discrete time and then deduce results for the Lyapunov exponents of the continuous-time system. We remark that Lyapunov exponents for continuous-time systems with random switching are also considered by Li, Chen, Lam, and Mao in [19]. They assume from the beginning that they have random dynamical systems and hence use the classical Multiplicative Ergodic Theorem.

The considered linear equations with random switching form Piecewise Deterministic Markov Processes (PDMP). These processes were introduced in Davis [12] and have since been extensively studied in the literature. For an analysis of their invariant measures, in particular, their supports, cf. Bakhtin and Hurth [2] and Benaïm, Le

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Borgne, Malrieu, and Zitt [3], also for further references. An important particular case which also attracts much research interest is that of Markovian jump linear systems (MJLS), in which one assumes that the random switching signal is generated by a continuous-time Markov chain. For more details, we refer to Bolzern, Colaneri, and De Nicolao [4], Fang and Loparo [13], and to the monograph Costa, Fragoso, and Todorov [11]. The case of nonlinear switched systems with random switching signals has also been considered in the literature, cf. e.g. Chatterjee and Liberzon [6], where multiple Lyapunov functions are used to derive a stability criterion under some slow switching condition that contains as a particular case switching signals coming from continuous-time Markov chains. We also remark that several different notions of stability for systems with random switching have been used in the literature; see, e.g., Feng, Loparo, Ji, and Chizeck [14] for a comparison between the usual notions in the context of MJLS. The one considered in this paper is that of almost sure stability.

The main results of this paper are (i) a Multiplicative Ergodic Theorem, Theorem 4.1, for linear continuous-time switching systems. This is based on a careful analysis of the relations between the Lyapunov exponents for an associated discrete-time system — which does define a random dynamical system — and those for the system in continuous time; and (ii) Theorem 6.1 showing that arbitrary decay rates may be achieved for linear control systems with random switching by choosing appropriate linear feedback laws. This is in contrast to the situation for deterministic switching by persistent excitations, as mentioned above.

The contents of this paper is as follows:

Section 2 constructs the random signals acting on the coefficients of the continuous-time system. Example 2.6 shows that, in general, one does not obtain a random dynamical system and Remark 2.7 discusses the relation to previous works in the literature. Section 3 introduces an associated system in discrete time, shows that it defines a random dynamical system, and discusses the relations between the Lyapunov exponents for continuous and discrete time. This leads to the formulation of a Multiplicative Ergodic Theorem for the continuous-time system in Section 4. Section 5 derives a formula for the maximal Lyapunov exponent. Finally, Section 6 presents the application to almost sure stabilization with arbitrary decay rate of linear control systems with random switching signals.

**Notation:** The sets  $\mathbb{N}^*$  and  $\mathbb{N}$  are used to denote the positive and nonnegative integers, respectively. For  $N \in \mathbb{N}^*$  we let  $\underline{N} := \{1, \dots, N\}$  and  $\mathbb{R}_+ := [0, \infty)$ ,  $\mathbb{R}_+^* := (0, \infty)$ .

**2. Continuous-time linear switched system and random switching signals.** Let  $N, d \in \mathbb{N}^*$  and  $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{R})$ . This paper considers the continuous-time linear switched system

$$\dot{x}(t) = A_{\alpha(t)}x(t), \quad (2.1)$$

where the switching signal  $\alpha$  belongs to the set  $\mathcal{P}$  defined by

$$\mathcal{P} := \{\alpha : \mathbb{R}_+ \rightarrow \underline{N} \text{ piecewise constant and right continuous}\}.$$

Recall that a piecewise constant function has only finitely many discontinuity points on any bounded interval. Given an initial condition  $x_0 \in \mathbb{R}^d$  and  $\alpha \in \mathcal{P}$ , system (2.1) admits a unique solution defined on  $\mathbb{R}_+$ , which we denote by  $\varphi_c(\cdot; x_0, \alpha)$ . In order to simplify the notation, for  $i \in \underline{N}$ , we denote by  $\Phi^i$  the linear flow defined by the matrix  $A_i$ , i.e.,  $\Phi_t^i = e^{A_i t}$  for every  $t \in \mathbb{R}_+$ .

We consider in this paper that the signal  $\alpha$  is randomly generated according to a Markov process which we describe now. Let  $M \in \mathcal{M}_N(\mathbb{R})$  be a stochastic matrix, i.e.,  $M$  has nonnegative entries and  $\sum_{j=1}^N M_{ij} = 1$  for every  $i \in \underline{N}$ . Let  $p$  be a probability vector in  $\mathbb{R}^N$ , i.e.,  $p \in [0, 1]^N$  and  $\sum_{i=1}^N p_i = 1$ . When necessary, we will regard  $p$  as a row vector  $p = (p_1, \dots, p_N)$ . We assume in this paper that  $p$  is invariant under  $M$ , i.e., that  $pM = p$ . Finally, let  $\mu_1, \dots, \mu_N$  be probability measures on  $\mathbb{R}_+^*$  with the Borel  $\sigma$ -algebra  $\mathfrak{B}$  and with finite expectation, i.e.,  $\int_{\mathbb{R}_+^*} td\mu_i(t) < \infty$  for every  $i \in \underline{N}$ . Whenever necessary, we will use that  $\mu_1, \dots, \mu_N$  define probability measures on  $\mathbb{R}_+$  with its Borel  $\sigma$ -algebra, that we also denote by  $\mathfrak{B}$  for simplicity.

The random model for the signal  $\alpha$  can be described as follows. We choose an initial state  $i \in \underline{N}$  according to the probability law defined by  $p$ . Then, at every time the system switches to a state  $i$ , we choose a random positive time  $T$  according to the probability law  $\mu_i$  and stay in  $i$  during the time  $T$ , before switching to the next state, which is chosen randomly according to the probability law corresponding to the  $i$ -th row  $(M_{ij})_{j=1}^N$  of the matrix  $M$ . Let us perform this construction more precisely. Recall the construction of product  $\sigma$ -algebras (see, e.g., Halmos [17, §38, §49]).

**DEFINITION 2.1.** *Let  $\Omega = (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^*}$  and provide  $\Omega$  with the product  $\sigma$ -algebra  $\mathfrak{F} = (\mathfrak{P}(\underline{N}) \times \mathfrak{B})^{\mathbb{N}^*}$ . Endow  $(\Omega, \mathfrak{F})$  with the probability measure  $\mathbb{P}$  defined, for  $n \in \mathbb{N}^*$ ,  $i_1, \dots, i_n \in \underline{N}$ , and  $U_1, \dots, U_n \in \mathfrak{B}$  by*

$$\begin{aligned} & \mathbb{P} \left( \{i_1\} \times U_1 \times \{i_2\} \times U_2 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}} \right) \\ &= p_{i_1} \mu_{i_1}(U_1) M_{i_1 i_2} \mu_{i_2}(U_2) \cdots M_{i_{n-1} i_n} \mu_{i_n}(U_n). \end{aligned}$$

For a given measurable space  $X$ , we denote by  $\text{Pr}(X)$  the set of all probability measures on  $X$ . The next result shows that the construction from Definition 2.1 is actually a Markov chain in the state space  $\underline{N} \times \mathbb{R}_+$ . For the definitions of Markov process and its transition probability, initial law, and transition operator, we refer to Hairer [16].

**PROPOSITION 2.2.** *For  $n \in \mathbb{N}^*$ , let  $x_n : \Omega = (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^*} \rightarrow \underline{N} \times \mathbb{R}_+$  denote the canonical projection onto the  $n$ -th coordinate. Then  $(x_n)_{n=1}^\infty$  is the unique Markov process in  $\underline{N} \times \mathbb{R}_+$  with transition probability  $P : \underline{N} \times \mathbb{R}_+ \rightarrow \text{Pr}(\underline{N} \times \mathbb{R}_+)$  defined by*

$$P(i, t)(\{j\} \times U) = M_{ij} \mu_j(U), \quad \forall i, j \in \underline{N}, \forall t \in \mathbb{R}_+, \forall U \in \mathfrak{B}, \quad (2.2)$$

and with initial law  $\nu_1$  given by

$$\nu_1(\{j\} \times U) = p_j \mu_j(U), \quad \forall j \in \underline{N}, \forall U \in \mathfrak{B}. \quad (2.3)$$

The transition operator  $T : \text{Pr}(\underline{N} \times \mathbb{R}_+) \rightarrow \text{Pr}(\underline{N} \times \mathbb{R}_+)$  of this chain is given by

$$T\nu(\{j\} \times U) = \sum_{i=1}^N \nu(\{i\} \times \mathbb{R}_+) M_{ij} \mu_j(U), \quad \forall j \in \underline{N}, \forall U \in \mathfrak{B}. \quad (2.4)$$

*Proof.* Observe that  $\underline{N} \times \mathbb{R}_+$  is a complete separable metric space. Then by [16, Proposition 2.38], it suffices to show that, for every  $n \in \mathbb{N}^*$ ,  $i_1, \dots, i_n \in \underline{N}$ , and  $U_1, \dots, U_n \in \mathfrak{B}$ ,

$$\begin{aligned} & \mathbb{P} \left( \{i_1\} \times U_1 \times \{i_2\} \times U_2 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}} \right) \\ &= \int_{\{i_1\} \times U_1} \int_{\{i_2\} \times U_2} \cdots \int_{\{i_{n-1}\} \times U_{n-1}} P(i_{n-1}, t_{n-1})(\{i_n\} \times U_n) \\ & \quad dP(i_{n-2}, t_{n-2})(i_{n-1}, t_{n-1}) \cdots dP(i_1, t_1)(i_2, t_2) d\nu_1(i_1, t_1). \quad (2.5) \end{aligned}$$

The definitions (2.2) and (2.3) of  $P$  and  $\nu_1$  immediately give

$$\begin{aligned}
& \int_{\{i_1\} \times U_1} \int_{\{i_2\} \times U_2} \cdots \int_{\{i_{n-1}\} \times U_{n-1}} P(i_{n-1}, t_{n-1})(\{i_n\} \times U_n) \\
& \quad dP(i_{n-2}, t_{n-2})(i_{n-1}, t_{n-1}) \cdots dP(i_1, t_1)(i_2, t_2) d\nu_1(i_1, t_1) \\
&= \int_{U_1} \int_{U_2} \cdots \int_{U_{n-1}} M_{i_{n-1}i_n} \mu_{i_n}(U_n) \\
& \quad M_{i_{n-2}i_{n-1}} d\mu_{i_{n-1}}(t_{n-1}) \cdots M_{i_1i_2} d\mu_{i_2}(t_2) p_{i_1} d\mu_{i_1}(t_1) \\
&= M_{i_{n-1}i_n} \mu_{i_n}(U_n) M_{i_{n-2}i_{n-1}} \mu_{i_{n-1}}(U_{n-1}) \cdots M_{i_1i_2} \mu_{i_2}(U_2) p_{i_1} \mu_{i_1}(U_1),
\end{aligned}$$

and thus (2.5) holds. The expression of the transition operator follows immediately from its definition (see, e.g., [16, Definition 2.31]).  $\square$

REMARK 2.3. *The canonical projection of  $\underline{N} \times \mathbb{R}_+$  onto  $\underline{N}$  transforms the Markov chain from Proposition 2.2 into a discrete Markov chain in the finite state space  $\underline{N}$  with transition matrix  $M$  and initial distribution  $p$ .*

To construct a random switching signal  $\alpha$  from a certain  $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega$ , we regard  $(i_n)_{n=1}^\infty$  as the sequence of states taken by  $\alpha$  and  $t_n$  as the time spent in the state  $i_n$ . For this construction to be well-defined, one needs to check that the switching times of such  $\alpha$  tend to  $\infty$ . The next proposition shows that this is the case in a subset of  $\Omega$  of full measure.

PROPOSITION 2.4. *The subset  $\Omega_0$  of  $\Omega$  defined by*

$$\Omega_0 := \left\{ (i_n, t_n)_{n=1}^\infty \in \Omega \mid \sum_{n=1}^\infty t_n = \infty \text{ and } t_n > 0 \text{ for every } n \in \mathbb{N}^* \right\}$$

satisfies  $\mathbb{P}(\Omega_0) = 1$ .

*Proof.* We write  $\Omega_0 = \Omega' \cap \Omega''$ , with

$$\begin{aligned}
\Omega' &= \left\{ (i_n, t_n)_{n=1}^\infty \in \Omega \mid \sum_{n=1}^\infty t_n = \infty \right\}, \\
\Omega'' &= \{(i_n, t_n)_{n=1}^\infty \in \Omega \mid t_n > 0 \text{ for every } n \in \mathbb{N}^*\}.
\end{aligned}$$

Then it follows that

$$\mathbb{P}(\Omega'') = \mathbb{P} \left( \bigcap_{n=1}^\infty \{(i_j, t_j)_{j=1}^\infty \in \Omega \mid t_n > 0\} \right) = 1,$$

since for every  $n$

$$\mathbb{P}\{(i_j, t_j)_{j=1}^\infty \in \Omega \mid t_n > 0\} = \sum_{(i_1, \dots, i_n) \in \underline{N}^n} p_{i_1} M_{i_1i_2} \cdots M_{i_{n-1}i_n} \mu_{i_n}((0, \infty)) = 1.$$

Denoting by  $\Omega'^c$  the complement of  $\Omega'$  in  $\Omega$ , we have

$$\begin{aligned}
\Omega'^c &= \left\{ (i_n, t_n)_{n=1}^\infty \in \Omega \mid \sum_{n=1}^\infty t_n < \infty \right\} \subset \left\{ (i_n, t_n)_{n=1}^\infty \in \Omega \mid \lim_{n \rightarrow \infty} t_n = 0 \right\} \\
&= \bigcap_{k=1}^\infty \bigcup_{r=1}^\infty \bigcap_{m=r}^\infty \left\{ (i_n, t_n)_{n=1}^\infty \in \Omega \mid t_m < \frac{1}{k} \right\}.
\end{aligned} \tag{2.6}$$

For  $k, r, K \in \mathbb{N}^*$  with  $K \geq r$ , let

$$\begin{aligned} E_K^{r,k} &= \bigcap_{m=r}^K \left\{ (i_n, t_n)_{n=1}^\infty \in \Omega \mid t_m < \frac{1}{k} \right\} \\ &= (\underline{N} \times \mathbb{R}_+)^{r-1} \times (\underline{N} \times [0, 1/k])^{K-r+1} \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus K} \\ &= \bigcup_{(i_1, \dots, i_K) \in \underline{N}^K} \prod_{j=1}^{r-1} (\{i_j\} \times \mathbb{R}_+) \times \prod_{j=r}^K (\{i_j\} \times [0, 1/k]) \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus K}. \end{aligned}$$

This union is disjoint, and thus

$$\mathbb{P} \left( E_K^{r,k} \right) = \sum_{(i_1, \dots, i_K) \in \underline{N}^K} p_{i_1} \prod_{j=2}^K M_{i_{j-1} i_j} \prod_{j=r}^K \mu_{i_j}([0, 1/k]) \leq \mu_{\max}(k)^{K-r+1},$$

where  $\mu_{\max}(k) = \max_{i \in \underline{N}} \mu_i([0, 1/k])$ . Then  $\mu_{\max}(k) \rightarrow 0$  as  $k \rightarrow \infty$ , and hence there exists  $k_* \in \mathbb{N}^*$  such that  $\mu_{\max}(k_*) < 1$ . Since, for every  $r, k \in \mathbb{N}^*$ , the sequence of sets  $(E_K^{r,k})_{K=r}^\infty$  is decreasing, we obtain that

$$\mathbb{P} \left( \bigcap_{m=r}^\infty \left\{ (i_n, t_n)_{n=1}^\infty \in \Omega \mid t_m < \frac{1}{k_*} \right\} \right) = \lim_{K \rightarrow \infty} \mathbb{P} \left( E_K^{r,k_*} \right) = 0.$$

This shows that  $\mathbb{P}(\Omega') = 1$  thanks to (2.6).  $\square$

We now associate to each  $\omega \in \Omega_0$  a signal  $\alpha \in \mathcal{P}$ .

**DEFINITION 2.5.** *We define the map  $\alpha : \Omega_0 \rightarrow \mathcal{P}$  as follows: for  $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega_0$ , we set  $s_0 = 0$ ,  $s_n = \sum_{k=1}^n t_k$  for  $n \in \mathbb{N}^*$ , and  $\alpha(\omega)(t) = i_n$  for every  $n \in \mathbb{N}^*$  and  $t \in [s_{n-1}, s_n)$ .*

Notice that  $\alpha$  is well-defined since  $\sum_{n=1}^\infty t_n = \infty$  for every  $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega_0$ . When necessary, we regard  $\alpha$  as a function  $\alpha : \Omega \rightarrow \mathcal{P}$  defined almost everywhere.

In order to consider solutions of (2.1) for signals  $\alpha$  chosen randomly according to the previous construction, we use the solution map  $\varphi_c$  of (2.1) to define the map

$$\varphi_{rc} : \begin{cases} \mathbb{R}_+ \times \mathbb{R}^d \times \Omega_0 & \rightarrow \mathbb{R}^d \\ (t; x_0, \omega) & \mapsto \varphi_c(t; x_0, \alpha(\omega)). \end{cases} \quad (2.7)$$

A natural idea to study the exponential behavior of the switched system with random switching signals described by  $\varphi_{rc}$  would be to apply the continuous-time Oseledets' Multiplicative Ergodic Theorem (see, e.g., Arnold [1, Theorem 3.4.1]) to obtain information on the Lyapunov exponents for  $\varphi_{rc}$ . To do so,  $\varphi_{rc}$  should define a random dynamical system on  $\mathbb{R}^d \times \Omega$ , i.e., one would have to provide a metric dynamical system  $\theta$  on  $\Omega$  — a measurable dynamical system  $\theta : \mathbb{R}_+ \times \Omega \rightarrow \Omega$  on  $(\Omega, \mathfrak{F}, \mathbb{P})$  such that  $\theta_t$  preserves  $\mathbb{P}$  for every  $t \geq 0$  — in such a way that  $\varphi_{rc}$  becomes a cocycle over  $\theta$  (for the precise definitions of random dynamical system, metric dynamical system, and cocycle, see, e.g., [1]). However, in general the natural choice for  $\theta$  to obtain the cocycle property for  $\varphi_{rc}$ , namely the time shift, does not define such a measure preserving map, as shown in the following example.

**EXAMPLE 2.6.** *For  $t \geq 0$ , let  $\theta_t : \Omega \rightarrow \Omega$  be defined for almost every  $\omega \in \Omega$  as follows. For  $\omega = (i_j, t_j)_{j=1}^\infty \in \Omega_0$ , set  $s_0 = 0$ ,  $s_k = \sum_{j=1}^k t_j$  for  $k \in \mathbb{N}^*$ . Let  $n \in \mathbb{N}^*$  be the unique integer such that  $t \in [s_{n-1}, s_n)$ . We define  $\theta_t(\omega) = (i_j^*, t_j^*)_{j=1}^\infty$  by*

$i_j^* = i_{n+j-1}$  for  $j \in \mathbb{N}^*$ ,  $t_1^* = s_n - t$ ,  $t_j^* = t_{n+j-1}$  for  $j \geq 2$ . One immediately verifies that  $\theta_t$  corresponds to the time shift in  $\mathcal{P}$ , i.e., that, for every  $t, s \geq 0$  and  $\omega \in \Omega_0$ , one has

$$\alpha(\theta_t \omega)(s) = \alpha(\omega)(t + s).$$

However, the map  $\theta_t$  in  $(\Omega, \mathfrak{F})$  does not preserve the measure  $\mathbb{P}$  in general. Indeed, suppose that  $\mu_i = \delta_1$  for every  $i \in \underline{N}$ , where  $\delta_1$  denotes the Dirac measure at 1. In particular, a set  $E \in \mathfrak{F}$  has nonzero measure only if  $E$  contains a point  $(i_j, t_j)_{j=1}^\infty$  with  $t_j = 1$  for every  $j \in \mathbb{N}^*$ . For  $r \in \mathbb{N}^*$  and  $i_1, \dots, i_r \in \underline{N}$ , let

$$E = \{i_1\} \times \{1\} \times \dots \times \{i_r\} \times \{1\} \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus r}.$$

Then  $\mathbb{P}(E) = p_{i_1} M_{i_1 i_2} \dots M_{i_{r-1} i_r}$ , and, for  $t \geq 0$ ,  $\theta_t^{-1}(E)$  is the set of points  $(i_j^*, t_j^*)_{j=1}^\infty$  such that, setting  $s_0^* = 0$ ,  $s_k^* = \sum_{j=1}^k t_j^*$  for  $k \in \mathbb{N}^*$ , and  $n \in \mathbb{N}^*$  the unique integer such that  $t \in [s_{n-1}^*, s_n^*)$ , one has  $s_n^* - t = 1$ ,  $t_{n+j-1}^* = 1$  for  $j = 2, \dots, r$ , and  $i_{n+j-1}^* = i_j$  for  $j \in r$ . If  $t \notin \mathbb{N}$ , then  $s_n^* = t + 1 \notin \mathbb{N}$ , and thus there exists  $j \in \underline{n}$  such that  $t_j^* \neq 1$ . We have shown that, if  $t \notin \mathbb{N}$ , then, for every  $\omega = (i_j^*, t_j^*)_{j=1}^\infty \in \theta_t^{-1}(E)$ , there exists  $j \in \mathbb{N}^*$  such that  $t_j^* \neq 1$ , and thus  $\mathbb{P}(\theta_t^{-1}(E)) = 0$ , hence  $\theta_t$  does not preserve the measure  $\mathbb{P}$ .

REMARK 2.7. For some particular choices of  $\mu_1, \dots, \mu_N$ , the time-shift  $\theta_t$  may preserve  $\mathbb{P}$ , in which case the continuous-time Multiplicative Ergodic Theorem can be applied directly to (2.7). This special case falls in the framework of Li, Chen, Lam, and Mao [19]. An important particular case where  $\theta_t$  preserves  $\mathbb{P}$  is when  $\mu_1, \dots, \mu_N$  are chosen in such a way that  $\alpha$  becomes a homogeneous continuous-time Markov chain, which is the case treated, e.g., in Bolzern, Colaneri, and De Nicolao [4], and in Fang and Loparo [13]. Our stability results from Section 5 generalize the corresponding almost sure stability criteria from [4, 13, 19].

**3. Associated discrete-time system and Lyapunov exponents.** Example 2.6 shows that in general one cannot expect to obtain a random dynamical system from  $\varphi_{rc}$  in order to apply the continuous-time Oseledets' Multiplicative Ergodic Theorem. Our strategy to study the exponential behavior of  $\varphi_{rc}$  relies instead on defining a suitable discrete-time map  $\varphi_{rd}$  associated with  $\varphi_{rc}$ , in such a way that  $\varphi_{rd}$  does define a discrete-time random dynamical system — to which the discrete-time Oseledets' Multiplicative Ergodic Theorem can be applied (see, e.g., Arnold [1, Theorem 3.4.1]) — and that the exponential behavior of  $\varphi_{rc}$  and  $\varphi_{rd}$  can be compared.

**3.1. Associated discrete-time deterministic system.** In this subsection we define a discrete-time deterministic system from the continuous-time system (2.1) determined by its solution map  $\varphi_c$ .

DEFINITION 3.1. We say that an increasing sequence  $\sigma = (s_n)_{n=0}^\infty$  of nonnegative real numbers with  $s_0 = 0$  and  $\lim_{n \rightarrow \infty} s_n = \infty$  is compatible with a signal  $\alpha \in \mathcal{P}$  if  $\alpha|_{[s_n, s_{n+1})}$  is constant for every  $n \in \mathbb{N}$ , and we denote

$$\mathcal{Q} := \{(\alpha, \sigma) \in \mathcal{P} \times \mathbb{R}_+^{\mathbb{N}} \mid \sigma \text{ is compatible with } \alpha\}.$$

For  $(\alpha, \sigma) \in \mathcal{Q}$  with  $\sigma = (s_n)_{n=0}^\infty$ , we consider the difference equation

$$x_{n+1} = e^{A_{\alpha(s_n)}(s_{n+1} - s_n)} x_n. \quad (3.1)$$

System (3.1) is obtained from (2.1) by taking the values of a continuous-time solution at the discrete times  $s_n$ . The sequence  $(s_n)_{n=0}^\infty$  contains all the discontinuities of  $\alpha$

and may also contain times with trivial jumps. The solution of (3.1) associated with  $(\alpha, \sigma) \in \mathcal{Q}$  and with initial condition  $x_0 \in \mathbb{R}^d$  is denoted by  $\varphi_d(\cdot; x_0, \alpha, \sigma)$ . Notice that the solution maps  $\varphi_c$  and  $\varphi_d$  satisfy, for every  $x_0 \in \mathbb{R}^d$  and  $(\alpha, \sigma) \in \mathcal{Q}$ ,

$$\begin{aligned}\varphi_c(0; x_0, \alpha) &= x_0, \\ \varphi_c(t; x_0, \alpha) &= \Phi_{t-s_n}^{\alpha(s_n)}(\varphi_c(s_n; x_0, \alpha)), \text{ if } t \in (s_n, s_{n+1}] \text{ for some } n \in \mathbb{N},\end{aligned}\quad (3.2)$$

and

$$\begin{aligned}\varphi_d(0; x_0, \alpha, \sigma) &= x_0, \\ \varphi_d(n+1; x_0, \alpha, \sigma) &= \Phi_{s_{n+1}-s_n}^{\alpha(s_n)}(\varphi_d(n; x_0, \alpha, \sigma)), \text{ for } n \in \mathbb{N}.\end{aligned}\quad (3.3)$$

It follows immediately that, for every  $n \in \mathbb{N}$ ,

$$\varphi_c(s_n; x_0, \alpha) = \varphi_d(n; x_0, \alpha, \sigma). \quad (3.4)$$

We characterize the asymptotic behavior of systems (2.1) and (3.1) by considering the associated Lyapunov exponents defined as follows.

**DEFINITION 3.2.** *Let  $(\alpha, \sigma) \in \mathcal{Q}$  and  $x_0 \in \mathbb{R}^d \setminus \{0\}$ . The Lyapunov exponent for the continuous-time system (2.1) is*

$$\lambda_c(x_0, \alpha) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|\varphi_c(t; x_0, \alpha)\| \quad (3.5)$$

and the Lyapunov exponent for the discrete-time system (3.1) is

$$\lambda_d(x_0, \alpha, \sigma) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\varphi_d(n; x_0, \alpha, \sigma)\|. \quad (3.6)$$

The main difference between (3.5) and (3.6) lies in the terms  $\frac{1}{t}$  and  $\frac{1}{n}$ . In order to be able to compare them asymptotically, one needs an additional hypothesis.

**DEFINITION 3.3.** *Let  $(\alpha, \sigma) \in \mathcal{Q}$  with  $\sigma = (s_n)_{n=0}^\infty$ . We say that  $(\alpha, \sigma)$  is regular if the limit*

$$m(\alpha, \sigma) := \lim_{n \rightarrow \infty} \frac{s_n}{n} \quad (3.7)$$

exists and is a positive real number.

**THEOREM 3.4.** *Suppose that  $(\alpha, \sigma) \in \mathcal{Q}$  is regular. Then, for every  $x_0 \in \mathbb{R}^d \setminus \{0\}$ , the Lyapunov exponents of the continuous- and discrete-time systems (2.1) and (3.1) are related by*

$$\lambda_d(x_0, \alpha, \sigma) = m(\alpha, \sigma) \lambda_c(x_0, \alpha).$$

*Proof.* Write  $\sigma = (s_n)_{n=0}^\infty$ . Let us first show that  $\lambda_d(x_0, \alpha, \sigma) \leq m(\alpha, \sigma) \lambda_c(x_0, \alpha)$ . For every  $n \in \mathbb{N}^*$ , one has, by (3.4),

$$\frac{1}{n} \log \|\varphi_d(n; x_0, \alpha, \sigma)\| = \frac{s_n}{n} \frac{1}{s_n} \log \|\varphi_c(s_n; x_0, \alpha)\|.$$

One clearly has  $\limsup_{n \rightarrow \infty} \frac{1}{s_n} \log \|\varphi_c(s_n; x_0, \alpha)\| \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|\varphi_c(t; x_0, \alpha)\|$  and then the conclusion follows since  $\frac{s_n}{n} \rightarrow m(\alpha, \sigma)$ .



We now turn to the proof of the inequality  $\lambda_d(x_0, \alpha, \sigma) \geq m(\alpha, \sigma)\lambda_c(x_0, \alpha)$ . Let  $C, \gamma > 0$  be such that  $\|\Phi_t^i x\| \leq Ce^{\gamma t} \|x\|$  for every  $i \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ , and  $t \geq 0$ . For  $x_0 \in \mathbb{R}^d \setminus \{0\}$  and  $t > 0$ , let  $n_t \in \mathbb{N}$  be the unique integer such that  $t \in (s_{n_t}, s_{n_t+1}]$ . Then

$$\begin{aligned} \frac{1}{t} \log \|\varphi_c(t; x_0, \alpha)\| &= \frac{1}{t} \log \left\| \Phi_{t-s_{n_t}}^{\alpha(s_{n_t})}(\varphi_c(s_{n_t}; x_0, \alpha)) \right\| \\ &= \frac{1}{t} \log \left\| \Phi_{t-s_{n_t}}^{\alpha(s_{n_t})}(\varphi_d(n_t; x_0, \alpha, \sigma)) \right\| \leq \frac{\log C}{t} + \gamma \frac{t-s_{n_t}}{t} + \frac{1}{t} \log \|\varphi_d(n_t; x_0, \alpha, \sigma)\|. \end{aligned} \quad (3.8)$$

Since  $t \in (s_{n_t}, s_{n_t+1}]$ , one has

$$0 \leq \frac{t-s_{n_t}}{t} \leq \frac{s_{n_t+1}}{s_{n_t}} - 1 \xrightarrow{t \rightarrow \infty} 0, \quad (3.9)$$

where we use (3.7) to obtain that  $\frac{s_{n_t+1}}{s_{n_t}} \rightarrow 1$  as  $t \rightarrow \infty$ . We write  $\frac{1}{t} = \frac{n_t}{t} \frac{1}{n_t}$ . Since  $t \in (s_{n_t}, s_{n_t+1}]$ , one has  $\frac{n_t}{t} \in \left[ \frac{n_t}{s_{n_t+1}}, \frac{n_t}{s_{n_t}} \right)$ . Now

$$\lim_{t \rightarrow \infty} \frac{n_t}{s_{n_t}} = \frac{1}{m(\alpha, \sigma)} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{n_t}{s_{n_t+1}} = \lim_{t \rightarrow \infty} \left( \frac{n_t+1}{s_{n_t+1}} - \frac{1}{s_{n_t+1}} \right) = \frac{1}{m(\alpha, \sigma)},$$

and thus  $\frac{n_t}{t} \rightarrow \frac{1}{m(\alpha, \sigma)}$  as  $t \rightarrow \infty$ . Using this fact and inserting (3.9) into (3.8), one obtains the conclusion of the theorem by letting  $t \rightarrow \infty$ .  $\square$

**3.2. Discrete-time random dynamical system.** We have constructed, in Section 3.1, the discrete-time system (3.1) associated with the continuous-time system (2.1). In this subsection, we use (3.1) and the probabilistic setting from Section 2 to construct a random dynamical system in discrete time, to which we will apply Oseledets' Multiplicative Ergodic Theorem in Section 4. Thanks to Theorem 3.4, this will allow us also to get information on the Lyapunov exponents of the continuous-time system. In order to perform this construction, one needs to choose, for each  $\omega \in \Omega_0$ , a sequence  $\sigma$  compatible with  $\alpha(\omega)$ .

A sequence  $\sigma$  that is compatible with a certain  $\alpha \in \mathcal{P}$  corresponds to a sequence of times where we observe the continuous-time solution map  $\varphi_c$  to define the discrete-time map  $\varphi_d$ . A natural choice, considering the fact that the probabilistic model from Definition 2.1 is a Markov chain, is to choose  $\sigma$  as the sequence of transition times of this chain, as follows.

**DEFINITION 3.5.** *We define the map  $\mathbf{s} : \Omega_0 \rightarrow \mathbb{R}_+^{\mathbb{N}}$  as follows: for  $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega_0$ , we set  $\mathbf{s}(\omega) = (s_n(\omega))_{n=0}^{\infty}$  with  $s_0(\omega) = 0$ ,  $s_n(\omega) = \sum_{k=1}^n t_k$  for  $n \in \mathbb{N}^*$ .*

Notice that, for every  $\omega \in \Omega_0$ ,  $\mathbf{s}(\omega)$  is compatible with  $\alpha(\omega)$ . We define the random discrete-time system  $\varphi_{rd}$  by

$$\varphi_{rd} : \begin{cases} \mathbb{N} \times \mathbb{R}^d \times \Omega_0 & \rightarrow \mathbb{R}^d \\ (n; x_0, \omega) & \mapsto \varphi_d(n; x_0, \alpha(\omega), \mathbf{s}(\omega)). \end{cases} \quad (3.10)$$

We also define the random Lyapunov exponents  $\lambda_{rc}$  and  $\lambda_{rd}$  for  $x_0 \in \mathbb{R}^d \setminus \{0\}$  and almost every  $\omega \in \Omega$  by

$$\lambda_{rc}(x_0, \omega) = \lambda_c(x_0, \alpha(\omega)), \quad \lambda_{rd}(x_0, \omega) = \lambda_d(x_0, \alpha(\omega), \mathbf{s}(\omega)). \quad (3.11)$$

A natural way to define a discrete-time metric dynamical system on  $(\Omega, \mathfrak{F}, \mathbb{P})$  is to consider the shift operator. Let  $\theta : \Omega \rightarrow \Omega$  be defined by

$$\theta((i_n, t_n)_{n=1}^\infty) = (i_{n+1}, t_{n+1})_{n=1}^\infty. \quad (3.12)$$

PROPOSITION 3.6. *The measure  $\mathbb{P}$  is invariant under  $\theta$ .*

*Proof.* It suffices to show that  $\mathbb{P}(\theta^{-1}(E)) = \mathbb{P}(E)$  for every set  $E$  of the form

$$E = \{i_1\} \times U_1 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}}$$

for some  $n \in \mathbb{N}^*$ ,  $i_1, \dots, i_n \in \underline{N}$ , and  $U_1, \dots, U_n \in \mathfrak{B}$ . For such a set  $E$ , we have

$$\theta^{-1}(E) = \bigcup_{i=1}^N \{i\} \times \mathbb{R}_+ \times \{i_1\} \times U_1 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n+1}},$$

and the previous union is disjoint, so that

$$\begin{aligned} \mathbb{P}(\theta^{-1}(E)) &= \sum_{i=1}^N p_i \mu_i(\mathbb{R}_+) M_{ii_1} \mu_{i_1}(U_1) \prod_{j=2}^n M_{i_{j-1}i_j} \mu_{i_j}(U_j) \\ &= \left( \sum_{i=1}^N p_i M_{ii_1} \right) \mu_{i_1}(U_1) \prod_{j=2}^n M_{i_{j-1}i_j} \mu_{i_j}(U_j) \\ &= p_{i_1} \mu_{i_1}(U_1) \prod_{j=2}^n M_{i_{j-1}i_j} \mu_{i_j}(U_j) = \mathbb{P}(E), \end{aligned}$$

since  $pM = p$ .  $\square$

Thanks to Proposition 3.6,  $\theta$  is a discrete-time metric dynamical system in  $(\Omega, \mathfrak{F}, \mathbb{P})$ . Moreover, since the shift operator  $\theta : \Omega \rightarrow \Omega$  satisfies  $\theta(\Omega_0) = \Omega_0$ ,  $\theta$  also defines a metric dynamical system in  $(\Omega_0, \mathfrak{F}, \mathbb{P})$  (where  $\mathfrak{F}$  and  $\mathbb{P}$  are understood to be restricted to  $\Omega_0$ ).

An important question regarding the metric dynamical system  $\theta$  in  $(\Omega, \mathfrak{F}, \mathbb{P})$  is to determine whether it is ergodic. To characterize the cases where such ergodicity holds, we start by providing the following definition.

DEFINITION 3.7. *Let  $(\Omega, \mathfrak{F})$  be the measurable space from Definition 2.1 and  $\nu \in \text{Pr}(\underline{N} \times \mathbb{R}_+)$ . We define the probability measure  $\mathbb{P}_\nu$  in  $(\Omega, \mathfrak{F})$  by requiring that, for every  $n \in \mathbb{N}^*$ ,  $i_1, \dots, i_n \in \underline{N}$ , and  $U_1, \dots, U_n \in \mathfrak{B}$ ,*

$$\begin{aligned} &\mathbb{P}_\nu \left( \{i_1\} \times U_1 \times \{i_2\} \times U_2 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}} \right) \\ &= \nu(\{i_1\} \times U_1) M_{i_1 i_2} \mu_{i_2}(U_2) \cdots M_{i_{n-1} i_n} \mu_{i_n}(U_n). \end{aligned} \quad (3.13)$$

REMARK 3.8. *If  $\nu(\{i\} \times U) = p_i \mu_i(U)$  for every  $i \in \underline{N}$  and  $U \in \mathfrak{B}$ , then  $\mathbb{P}_\nu$  coincides with the measure  $\mathbb{P}$  from Definition 2.1. Moreover, as in Proposition 2.2, for every  $\nu \in \text{Pr}(\underline{N} \times \mathbb{R}_+)$ ,  $\mathbb{P}_\nu$  is the probability measure associated with a Markov process in  $\underline{N} \times \mathbb{R}_+$  with transition probability  $P$  given by (2.2), transition operator  $T$  given by (2.4), and with initial law  $\nu$ .*

LEMMA 3.9. *The measure  $\mathbb{P}_\nu$  is invariant under the shift  $\theta$  if and only if  $\nu(\{i\} \times U) = \nu(\{i\} \times \mathbb{R}_+) \mu_i(U)$  for every  $i \in \underline{N}$ ,  $U \in \mathfrak{B}$ , and  $(\nu(\{i\} \times \mathbb{R}_+))_{i=1}^N$  is a left eigenvector of  $M$  associated with the eigenvalue 1.*

*Proof.* Notice that  $\mathbb{P}_\nu$  is invariant under  $\theta$  if and only if  $T\nu = \nu$ . Hence  $\mathbb{P}_\nu$  is invariant under  $\theta$  if and only if

$$\nu(\{j\} \times U) = \sum_{i=1}^N \nu(\{i\} \times \mathbb{R}_+) M_{ij} \mu_j(U), \quad \forall j \in \underline{N}, \forall U \in \mathfrak{B}. \quad (3.14)$$

If (3.14) holds, we apply it to  $U = \mathbb{R}_+$  to get that  $(\nu(\{i\} \times \mathbb{R}_+))_{i=1}^N$  is a left eigenvector of  $M$  associated with the eigenvalue 1, and it then follows that  $\nu(\{j\} \times U) = \nu(\{j\} \times \mathbb{R}_+) \mu_j(U)$  for every  $j \in \underline{N}$  and  $U \in \mathfrak{B}$ . The converse is immediate.  $\square$

Let

$$V = \left\{ q \in [0, 1]^N \left| \sum_{i=1}^N q_i = 1 \text{ and } qM = q \right. \right\}, \quad (3.15)$$

which is a non-empty convex subset of  $\mathbb{R}^N$ . An element  $q \in V$  is said to be *extremal* if it cannot be written as  $q = tq_1 + (1-t)q_2$  for some  $t \in (0, 1)$  and  $q_1, q_2 \in V$  with  $q_1 \neq q_2$ . With each  $q \in V$ , we associate a probability measure  $\nu_q \in \Pr(\underline{N} \times \mathbb{R}_+)$  by setting

$$\nu_q(\{i\} \times U) = q_i \mu_i(U), \quad \forall i \in \underline{N}, U \in \mathfrak{B}. \quad (3.16)$$

As a consequence of Lemma 3.9, the map  $q \mapsto \nu_q$  is a linear bijection between  $V$  and the set of all probability measures  $\nu$  in  $\underline{N} \times \mathbb{R}_+$  for which  $\mathbb{P}_\nu$  is invariant under the shift  $\theta$ . Hence, one obtains immediately from Hairer [16, Theorem 5.7] the following result.

**PROPOSITION 3.10.** *Let  $q \in V$ . Then the metric dynamical system  $\theta$  is ergodic in  $(\Omega, \mathfrak{F}, \mathbb{P}_{\nu_q})$  if and only if  $q$  is an extremal of  $V$ .*

**REMARK 3.11.** *When  $M$  is irreducible,  $V$  contains only one point  $q$  and hence  $\theta$  is ergodic for the measure  $\mathbb{P}_{\nu_q}$ .*

Now that we have defined the random discrete-time system (3.10) and provided the metric dynamical system  $\theta$ , we can show that the pair  $(\theta, \varphi_{\text{rd}})$  defines a random dynamical system.

**PROPOSITION 3.12.**  *$(\theta, \varphi_{\text{rd}})$  is a discrete-time random dynamical system over  $(\Omega, \mathfrak{F}, \mathbb{P})$ .*

*Proof.* Since  $\theta$  is a discrete-time metric dynamical system over  $(\Omega, \mathfrak{F}, \mathbb{P})$ , one is only left to show that  $\varphi_{\text{rd}}$  satisfies the cocycle property

$$\varphi_{\text{rd}}(n+m; x_0, \omega) = \varphi_{\text{rd}}(n; \varphi_{\text{rd}}(m; x_0, \omega), \theta^m(\omega)), \quad \forall n, m \in \mathbb{N}, \forall x_0 \in \mathbb{R}^d, \forall \omega \in \Omega_0. \quad (3.17)$$

Let  $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega_0$ . Then it follows immediately from the definitions of  $\alpha$  and  $s$  that for  $n, m \in \mathbb{N}$ ,

$$\begin{aligned} s_n(\theta^m(\omega)) &= \sum_{k=1}^n t_{k+m} = \sum_{k=m+1}^{m+n} t_k = s_{n+m}(\omega) - s_m(\omega), \\ \alpha(\theta^m(\omega))(s_n(\theta^m(\omega))) &= i_{n+m} = \alpha(\omega)(s_{n+m}(\omega)). \end{aligned}$$

We prove (3.17) by induction on  $n$ . When  $n = 0$ , (3.17) is clearly satisfied for every  $m \in \mathbb{N}$ ,  $x_0 \in \mathbb{R}^d$ , and  $\omega \in \Omega_0$ . Suppose now that  $n \in \mathbb{N}$  is such that (3.17) is satisfied for every  $m \in \mathbb{N}$ ,  $x_0 \in \mathbb{R}^d$ , and  $\omega \in \Omega_0$ . Using (3.3), we obtain

$$\varphi_{\text{rd}}(n+1; \varphi_{\text{rd}}(m; x_0, \omega), \theta^m(\omega))$$

$$\begin{aligned}
&= \Phi_{s_{n+1}(\theta^m(\omega)) - s_n(\theta^m(\omega))}^{\alpha(\theta^m(\omega))(s_n(\theta^m(\omega)))} (\varphi_{\text{rd}}(n; \varphi_{\text{rd}}(m; x_0, \omega), \theta^m(\omega))) \\
&= \Phi_{s_{n+m+1}(\omega) - s_{n+m}(\omega)}^{\alpha(\omega)(s_{n+m}(\omega))} (\varphi_{\text{rd}}(n+m; x_0, \omega)) = \varphi_{\text{rd}}(n+m+1; x_0, \omega),
\end{aligned}$$

which concludes the proof of (3.17).  $\square$

Since our goal is to compare the asymptotic behavior of (2.7) and (3.10) using Theorem 3.4, we need to show that  $(\alpha(\omega), \mathbf{s}(\omega))$  is regular for almost every  $\omega \in \Omega$ . To do so, we first consider the structure of the matrix  $M$ , using classical notation for Markov chains (see, e.g., Seneta [26]).

**DEFINITION 3.13.** *Let  $M \in \mathcal{M}_N(\mathbb{R})$  be a stochastic matrix. For  $i, j \in \underline{N}$ , we say that  $i$  leads to  $j$  if  $i = j$  or there exist  $r \in \mathbb{N}^*$  and  $i_1, \dots, i_r \in \underline{N}$  such that  $M_{i_1 i_1} M_{i_1 i_2} \cdots M_{i_r j} > 0$ . We say that  $i$  and  $j$  communicate if  $i$  leads to  $j$  and  $j$  leads to  $i$ . This is an equivalence relation and we decompose  $\underline{N}$  into the corresponding  $R' \in \mathbb{N}^*$  equivalence classes  $C_1, \dots, C_{R'}$ . For  $i, j \in \underline{R}'$ , we say that  $C_i$  leads to  $C_j$  if there exist  $i^* \in C_i$  leading to some  $j^* \in C_j$ . A class  $C_i$  is said to be essential if it does not lead to another class, and inessential otherwise.*

At least one essential class exists. Up to a permutation in the sets of indices  $\underline{N}$  and  $\underline{R}'$ , we can assume that  $C_1 = \{1, 2, \dots, n_1\}$ ,  $C_2 = \{n_1 + 1, n_1 + 2, \dots, n_1 + n_2\}$ ,  $\dots$ ,  $C_{R'} = \{n_1 + \dots + n_{R'-1} + 1, \dots, n_1 + \dots + n_{R'}\}$  for some  $n_1, \dots, n_{R'} \in \mathbb{N}^*$ , and that  $M$  can be written as

$$M = \left( \begin{array}{cccc|c} P_1 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ 0 & 0 & P_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_R \\ \hline & & * & & Q \end{array} \right), \quad (3.18)$$

where  $R$  is the number of essential classes,  $P_i$  is the square matrix corresponding to the essential class  $C_i$  for  $i \in \underline{R}$ , and  $Q$  is the square matrix corresponding to all inessential classes. The following proposition gathers some classical properties of stochastic matrices. Its proof can be found in textbooks on the subject, such as Seneta [26, §4.2].

**PROPOSITION 3.14.** *Let  $M$  be a stochastic matrix decomposed as (3.18).*

(i). *For  $i \in \underline{R}$ ,  $P_i \in \mathcal{M}_{n_i}(\mathbb{R})$  is an irreducible stochastic matrix with a unique invariant probability  $p^i \in \mathbb{R}^{n_i}$ . We extend  $p^i$  to a vector in  $\mathbb{R}^N$  by setting to 0 its components not in  $C_i$ , and write  $p^i = (p_j^i)_{j=1}^N$ ;*

(ii). *Every probability vector  $q \in \mathbb{R}^N$  invariant under  $M$  can be decomposed as  $q = \alpha_1 p^1 + \dots + \alpha_R p^R$  for some  $\alpha_1, \dots, \alpha_R \in [0, 1]$  with  $\sum_{i=1}^R \alpha_i = 1$ .*

**REMARK 3.15.** *It follows from this proposition that the set  $V$  defined in (3.15) is the convex hull of  $\{p^1, \dots, p^R\}$ , and that  $q \in V$  is an extremal of  $V$  if and only if  $q = p^i$  for some  $i \in \underline{R}$ .*

For a probability vector  $q \in [0, 1]^N$ , we define the probability measure  $\mathbb{P}^q$  in the measurable space  $(\Omega, \mathfrak{F})$  by setting  $\mathbb{P}^q = \mathbb{P}_{\nu_q}$ , where  $\nu_q$  is defined in (3.16) and  $\mathbb{P}_{\nu_q}$  is given in Definition 3.7. Thanks to Lemma 3.9,  $\mathbb{P}^q$  is invariant under  $\theta$  if and only if  $qM = q$ . Let  $\alpha_1 q_1 + \dots + \alpha_k q_k$  be a convex combination of probability vectors  $q_1, \dots, q_k \in [0, 1]^N$ . Thanks to Definition 3.7 and (3.16), one obtains that, for every  $E \in \mathfrak{F}$ ,

$$\mathbb{P}^{\alpha_1 q_1 + \dots + \alpha_k q_k}(E) = \alpha_1 \mathbb{P}^{q_1}(E) + \dots + \alpha_k \mathbb{P}^{q_k}(E). \quad (3.19)$$

As a consequence of Proposition 3.10 and Remark 3.15, one immediately obtains the following result.

**COROLLARY 3.16.** *Let  $q \in \mathbb{R}^N$  be a probability vector with  $qM = q$  and  $p^1, \dots, p^R$  be as in Proposition 3.14(i). The map  $\theta$  is ergodic for the measure  $\mathbb{P}^q$  if and only if  $q = p^i$  for some  $i \in \underline{R}$ .*

We now provide a decomposition of the space  $\Omega$  according to the essential classes  $C_1, \dots, C_R$ . For  $i \in \underline{R}$ , we set

$$\Omega^i = \{\omega = (i_n, t_n)_{n=1}^\infty \in \Omega \mid \exists n_0 \in \mathbb{N}^* \text{ such that } i_n \in C_i \text{ for } n \geq n_0\},$$

and

$$\Omega^0 = \Omega \setminus \bigcup_{i=1}^R \Omega^i.$$

Then clearly  $\Omega = \bigcup_{i=0}^R \Omega^i$  and the union is disjoint. For  $i \in \underline{R}$ , the set  $\Omega^i$  is the set of all sequences  $(i_n, t_n)_{n=0}^\infty$  such that  $(i_n)_{n=0}^\infty$  eventually enters the class  $C_i$  and remains in this class.

**PROPOSITION 3.17.** *Let  $q = \alpha_1 p^1 + \dots + \alpha_R p^R \in V$  be as in Proposition 3.14(ii). Then, for every  $i \in \underline{R}$ ,  $\mathbb{P}^q(\Omega^i) = \alpha_i$ . In particular,  $\mathbb{P}^q(\Omega^0) = 0$ .*

*Proof.* Since the components of  $p^j$  corresponding to indices not in  $C_j$  are all zero and  $M$  has the form (3.18), one obtains that  $\mathbb{P}^{p^j} \left( (C_j \times \mathbb{R}_+)^{\mathbb{N}^*} \right) = 1$ , and thus

$$\mathbb{P}^{p^j}(\Omega^i) = \mathbb{P}^{p^j} \left( \Omega^i \cap (C_j \times \mathbb{R}_+)^{\mathbb{N}^*} \right) = \delta_{ij}, \quad (3.20)$$

where  $\delta_{ij}$  denotes the Kronecker delta. The conclusion follows immediately from (3.19) and (3.20).  $\square$

We can now prove that  $(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$  is regular for almost every  $\omega \in \Omega$ .

**PROPOSITION 3.18.** *The map  $\omega \mapsto m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$  is invariant under  $\theta$  and, for every  $i \in \underline{R}$  such that  $\mathbb{P}(\Omega^i) \neq 0$  and almost every  $\omega \in \Omega^i$ ,*

$$m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega)) = \sum_{j \in C_i} p_j^i \int_{\mathbb{R}_+} t d\mu_j(t), \quad (3.21)$$

where, for  $i \in \underline{R}$ ,  $p^i = (p_j^i)_{j=1}^N$  are the probability vectors from Proposition 3.14(i). In particular,  $(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$  is regular for almost every  $\omega \in \Omega$ .

*Proof.* Consider the map  $f : \Omega_0 \rightarrow \mathbb{R}_+^*$  given by  $f((i_n, t_n)_{n=1}^\infty) = t_1$ . For every  $k \in \mathbb{N}$ ,  $f \circ \theta^k((i_n, t_n)_{n=1}^\infty) = t_{k+1}$ . By Birkhoff's Ergodic Theorem (see, e.g., Petersen [24, Chapter 2, Theorem 2.3]), there exists a function  $f^* \in L^1(\Omega, \mathbb{R}_+)$ , invariant under  $\theta$ , such that, for almost every  $\omega \in \Omega$ ,

$$\lim_{n \rightarrow \infty} \frac{s_n(\omega)}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \theta^k(\omega) = f^*(\omega),$$

and, moreover,  $f^*$  is the conditional expectation of  $f$  given the  $\sigma$ -algebra of invariant sets over  $\theta$ , i.e., for every set  $A \in \mathfrak{F}$  with  $\theta^{-1}(A) = A$ ,

$$\int_A f(\omega) d\mathbb{P}(\omega) = \int_A f^*(\omega) d\mathbb{P}(\omega).$$

Write  $p = \alpha_1 p^1 + \dots + \alpha_R p^R$  as in Proposition 3.14. Since, for every  $i \in \underline{R}$ , the set  $\Omega^i$  is invariant under  $\theta$ , we have

$$\int_{\Omega^i} f(\omega) d\mathbb{P}(\omega) = \int_{\Omega^i} f^*(\omega) d\mathbb{P}(\omega).$$

By Corollary 3.16,  $\theta$  is ergodic with respect to  $\mathbb{P}^{p^i}$  for every  $i \in \underline{R}$ , and thus  $f^*$  is constant  $\mathbb{P}^{p^i}$ -almost everywhere on  $\Omega$ ; since  $\mathbb{P}^{p^i}(\Omega^i) = 1$  and the restriction of  $\mathbb{P}$  to  $\Omega^i$  is precisely  $\alpha_i \mathbb{P}^{p^i}$ , one obtains that  $f^*$  is constant  $\mathbb{P}$ -almost everywhere in  $\Omega^i$ . Hence, for every  $i \in \underline{R}$  and almost every  $\tilde{\omega} \in \Omega^i$ ,

$$\alpha_i f^*(\tilde{\omega}) = \int_{\Omega^i} f(\omega) d\mathbb{P}(\omega) = \alpha_i \int_{\Omega^i} f(\omega) d\mathbb{P}^{p^i}(\omega) = \alpha_i \sum_{j \in C_i} p_j^i \int_{\mathbb{R}_+} t d\mu_j(t),$$

which proves (3.21). Since its right-hand side is a positive real number and the sets  $\Omega^i$  for which  $\mathbb{P}(\Omega^i) \neq 0$  cover  $\Omega$  except for a set of measure zero, the regularity of  $(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$  for almost every  $\omega \in \Omega$  follows.  $\square$

An immediate consequence of Theorem 3.4 and Proposition 3.18 is the following.

**THEOREM 3.19.** *For every  $x_0 \in \mathbb{R}^d \setminus \{0\}$  and almost every  $\omega \in \Omega$ , the Lyapunov exponents of the continuous- and discrete-time systems (2.7) and (3.10), given by (3.11), are related by*

$$\lambda_{\text{rd}}(x_0, \omega) = m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega)) \lambda_{\text{rc}}(x_0, \omega).$$

As a final result in this section, we prove the following proposition, which evaluates the average time spent in a certain state  $k$ .

**PROPOSITION 3.20.** *Let  $k \in \underline{N}$ . For every  $i \in \underline{R}$  such that  $\mathbb{P}(\Omega^i) \neq 0$  and almost every  $\omega \in \Omega^i$ ,*

$$\lim_{T \rightarrow \infty} \frac{\mathcal{L}\{t \in [0, T] \mid \boldsymbol{\alpha}(\omega)(t) = k\}}{T} = \frac{\mathbb{I}_{C_i}(k) p_k^i \int_{\mathbb{R}_+} t d\mu_k(t)}{\sum_{j \in C_i} p_j^i \int_{\mathbb{R}_+} t d\mu_j(t)},$$

where  $\mathcal{L}$  denotes the Lebesgue measure in  $\mathbb{R}$  and  $\mathbb{I}_{C_i}$  is the characteristic function of the set  $C_i$ .

*Proof.* Fix  $k \in \underline{N}$ . Let  $\varphi_k : \Omega \rightarrow \mathbb{R}_+$  be given by

$$\varphi_k((i_n, t_n)_{n=1}^\infty) = \begin{cases} t_1, & \text{if } i_1 = k, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by Birkhoff's Ergodic Theorem, there exists a function  $\varphi_k^* \in L^1(\Omega, \mathbb{R}_+)$  invariant under  $\theta$  such that, for almost every  $\omega \in \Omega$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_k(\theta^j \omega) = \varphi_k^*(\omega), \quad (3.22)$$

and, for every  $i \in \underline{R}$ ,

$$\int_{\Omega^i} \varphi_k(\tilde{\omega}) d\mathbb{P}(\tilde{\omega}) = \int_{\Omega^i} \varphi_k^*(\tilde{\omega}) d\mathbb{P}(\tilde{\omega}).$$

As in the proof of Proposition 3.18, one shows that, for every  $i \in \underline{R}$ ,  $\varphi_k^*$  is constant almost everywhere on  $\Omega^i$ . Writing  $p = \alpha_1 p^1 + \dots + \alpha_R p^R$  as in Proposition 3.14, we get, for  $i \in \underline{R}$  and almost every  $\omega \in \Omega^i$ ,

$$\alpha_i \varphi_k^*(\omega) = \int_{\Omega^i} \varphi_k(\tilde{\omega}) d\mathbb{P}(\tilde{\omega}) = \alpha_i \sum_{j \in C_i} p_j^i \int_{\mathbb{R}_+} t \delta_{jk} d\mu_j(t) = \alpha_i \mathbb{I}_{C_i}(k) p_k^i \int_{\mathbb{R}_+} t d\mu_k(t). \quad (3.23)$$

By definition of  $\alpha$ , for every  $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega$ ,

$$\sum_{j=0}^{n-1} \varphi_k(\theta^j \omega) = \sum_{\substack{j=1 \\ i_j=k}}^n t_j = \mathcal{L}\{t \in [0, s_n(\omega)] \mid \alpha(\omega)(t) = k\}.$$

Hence, using Proposition 3.18 and combining (3.22) and (3.23), we obtain that, for every  $i \in \underline{R}$  with  $\mathbb{P}(\Omega^i) \neq 0$  and almost every  $\omega \in \Omega^i$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathcal{L}\{t \in [0, s_n(\omega)] \mid \alpha(\omega)(t) = k\}}{s_n(\omega)} &= \lim_{n \rightarrow \infty} \frac{n}{s_n(\omega)} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_k(\theta^j \omega) \\ &= \frac{\mathbb{I}_{C_i}(k) p_k^i \int_{\mathbb{R}_+} t d\mu_k(t)}{\sum_{j \in C_i} p_j^i \int_{\mathbb{R}_+} t d\mu_j(t)}. \end{aligned} \quad (3.24)$$

Let  $\omega \in \Omega$  be such that (3.24) holds and take  $T \in \mathbb{R}_+$ . Choose  $n_T \in \mathbb{N}$  such that  $s_{n_T}(\omega) \leq T < s_{n_T+1}(\omega)$ . Then

$$\frac{1}{T} \mathcal{L}\{t \in [0, T] \mid \alpha(\omega)(t) = k\} \leq \frac{1}{s_{n_T}(\omega)} \mathcal{L}\{t \in [0, s_{n_T+1}(\omega)] \mid \alpha(\omega)(t) = k\}$$

and

$$\frac{1}{T} \mathcal{L}\{t \in [0, T] \mid \alpha(\omega)(t) = k\} \geq \frac{1}{s_{n_T+1}(\omega)} \mathcal{L}\{t \in [0, s_{n_T}(\omega)] \mid \alpha(\omega)(t) = k\}.$$

The conclusion of the proposition then follows since, by Proposition 3.18,  $\frac{s_{n+1}(\omega)}{s_n(\omega)} \rightarrow 1$  as  $n \rightarrow \infty$  for almost every  $\omega \in \Omega$ .  $\square$

**REMARK 3.21.** *The choice of the compatible sequence in this section is not unique, and one might be interested in other possible choices. The times  $s_n(\omega)$  in the sequence  $\mathbf{s}(\omega)$  correspond to the transitions of the Markov chain from Proposition 2.2. However, if some of the diagonal elements of  $M$  are non-zero, then the discrete part of the Markov chain, i.e., its component in  $\underline{N}$ , may switch from a certain state to itself. In practical situations, it may be possible to observe only switches between different states, and another possible choice for the sequence  $\mathbf{s}(\omega)$  that may be of practical interest is to consider only the times corresponding to such non-trivial switches. This can be done if  $M_{ii} \neq 1$  for every  $i \in \underline{N}$ , i.e., if the Markov chain in the discrete space  $\underline{N}$  has no absorbing states, in which case we have almost surely an infinite number of switches between different states. Defining  $\theta$  as the shift to the next different state,  $\theta$  defines a metric dynamical system if we suppose that, instead of having  $pM = p$ , we have  $p\tilde{M} = p$ , where  $\tilde{M}_{ij} = \frac{M_{ij}}{1 - M_{ii}}$  for  $i, j \in \underline{N}$  with  $i \neq j$  and  $\tilde{M}_{ii} = 0$  for  $i \in \underline{N}$ . The counterparts of the previous results can be proved in this framework with no extra difficulty.*

REMARK 3.22. *Even though we only consider in this paper the case where  $p$  is invariant under  $M$ , our results can be generalized to the case of any probability vector  $p$  by using the following three facts. First, for any probability vector  $q \in [0, 1]^N$ , the Cesàro mean of the sequence  $(qM^n)_{n=0}^\infty$ , namely  $\frac{1}{n} \sum_{j=0}^{n-1} qM^j$ , converges as  $n \rightarrow \infty$  to an invariant probability vector (see, e.g., Meyer [23, Chapter 8]). Secondly, if  $(q_n)_{n=1}^\infty$  is a sequence of probability vectors in  $[0, 1]^N$  converging to some probability vector  $q \in [0, 1]^N$ , then  $\mathbb{P}^{q_n}(E) \rightarrow \mathbb{P}^q(E)$  uniformly in  $E \in \mathfrak{F}$  (which can be shown directly from (3.13)). Finally, if  $q \in [0, 1]^N$  is a probability vector and  $q_0 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} qM^j$ , then  $\mathbb{P}^q(E) = \mathbb{P}^{q_0}(E)$  for every set  $E \in \mathfrak{F}$  invariant under  $\theta$  (which follows from the fact that  $\mathbb{P}^q(\theta^{-1}(E)) = \mathbb{P}^{qM}(E)$  for every  $E \in \mathfrak{F}$ ). With these three properties, when the probability vector  $p \in [0, 1]^N$  is not invariant under  $M$ , it can be replaced in the previous results by the invariant probability vector given by the Cesàro mean  $q = \frac{1}{n} \sum_{j=0}^{n-1} pM^j$  and the proofs can be adapted accordingly without much extra effort.*

REMARK 3.23. *The fact that systems (2.1) and (3.1) are linear has been used only in the proof of Theorem 3.4, where one uses an exponential bound on the growth of the flows  $\Phi_t^i = e^{A_i t}$ , namely that there exist constants  $C, \gamma > 0$  such that  $\|e^{A_i t}\| \leq C e^{\gamma t}$  for every  $t \geq 0$  and  $i \in \underline{N}$ . If we consider, instead of system (2.1), the nonlinear switched system*

$$\dot{x}(t) = f_{\alpha(t)}(x(t)),$$

where  $f_1, \dots, f_N$  are complete vector fields generating flows  $\Phi^1, \dots, \Phi^N$ , and modify the discrete-time system (3.1) accordingly, all the previous results remain true, with the same proofs, under the additional assumption that there exist constants  $C, \gamma > 0$  such that  $\|\Phi_t^i(x)\| \leq C e^{\gamma t} \|x\|$  for every  $t \geq 0$ ,  $i \in \underline{N}$ , and  $x \in \mathbb{R}^d$ . However, the results from the next sections do not generalize to the nonlinear framework.

**4. Multiplicative Ergodic Theorem.** In this section, we apply the discrete-time Oseledets' Multiplicative Ergodic Theorem (see, e.g., Arnold [1, Theorem 3.4.1]) in the one-sided invertible case to system (3.10) and we use Proposition 3.18 and Theorem 3.19 to obtain that several of its conclusions also hold for the continuous-time system (2.7).

Recall that, for  $i \in \underline{N}$ , we consider  $A_i \in \mathcal{M}_d(\mathbb{R})$  and  $\Phi_t^i = e^{A_i t}$ . Let  $A : \Omega \rightarrow \mathcal{M}_d(\mathbb{R})$  be the function defined for  $\omega = (i_n, t_n)_{n=1}^\infty$  by  $A(\omega) = e^{A_{i_1} t_1}$ , so that  $\varphi_{\text{rd}}(n; x_0, \omega) = A(\theta^{n-1}\omega) \varphi_{\text{rd}}(n-1; x_0, \omega)$  for every  $x_0 \in \mathbb{R}^d$ ,  $\omega \in \Omega_0$ , and  $n \in \mathbb{N}^*$ . For  $\omega \in \Omega_0$  and  $n \in \mathbb{N}$ , we denote  $\Phi(n, \omega)$  the linear operator defined by  $\Phi(n, \omega)x = \varphi_{\text{rd}}(n; x, \omega)$  for every  $x \in \mathbb{R}^d$ , which is thus given by  $\Phi(n, \omega) = e^{A_{i_n} t_n} \dots e^{A_{i_1} t_1}$  for  $\omega = (i_j, t_j)_{j=1}^\infty \in \Omega_0$  and  $n \in \mathbb{N}^*$ .

THEOREM 4.1. *There exists an invariant measurable subset  $\widehat{\Omega} \subset \Omega$  of full  $\mathbb{P}$ -measure such that, for every  $\omega \in \widehat{\Omega}$ ,*

- (i). *the limit  $\Psi(\omega) = \lim_{n \rightarrow \infty} (\Phi(n, \omega)^T \Phi(n, \omega))^{1/2n}$  exists and is a positive definite matrix;*
- (ii). *there exist an integer  $q(\omega) \in \underline{d}$  and  $q(\omega)$  vector subspaces  $V_1(\omega), \dots, V_{q(\omega)}(\omega)$  with respective dimensions  $d_1(\omega) > \dots > d_{q(\omega)}(\omega)$  such that*

$$V_{q(\omega)}(\omega) \subset \dots \subset V_1(\omega) = \mathbb{R}^d,$$

*and  $A(\omega)V_i(\omega) = V_i(\theta(\omega))$  for every  $i \in \underline{q}(\omega)$ ;*

- (iii). *for every  $x_0 \in \mathbb{R}^d \setminus \{0\}$ , the Lyapunov exponents  $\lambda_{\text{rd}}(x_0, \omega)$  and  $\lambda_{\text{rc}}(x_0, \omega)$*



exist as limits, i.e.,

$$\begin{aligned}\lambda_{\text{rd}}(x_0, \omega) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\varphi_{\text{rd}}(n; x_0, \omega)\|, \\ \lambda_{\text{rc}}(x_0, \omega) &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \|\varphi_{\text{rc}}(t; x_0, \omega)\|;\end{aligned}$$

(iv). there exist real numbers  $\lambda_1^{\text{d}}(\omega) > \dots > \lambda_{q(\omega)}^{\text{d}}(\omega)$  and  $\lambda_1^{\text{c}}(\omega) > \dots > \lambda_{q(\omega)}^{\text{c}}(\omega)$  such that, for every  $i \in \underline{q}(\omega)$ ,

$$\lambda_{\text{rd}}(x_0, \omega) = \lambda_i^{\text{d}}(\omega) \iff \lambda_{\text{rc}}(x_0, \omega) = \lambda_i^{\text{c}}(\omega) \iff x_0 \in V_i(\omega) \setminus V_{i+1}(\omega),$$

where  $V_{q(\omega)+1}(\omega) = \{0\}$ ;

(v). the eigenvalues of  $\Psi(\omega)$  are  $e^{\lambda_1^{\text{d}}(\omega)} > \dots > e^{\lambda_{q(\omega)}^{\text{d}}(\omega)}$ ;

(vi).  $q(\theta(\omega)) = q(\omega)$  and, for  $i \in \underline{q}(\omega)$ ,  $d_i(\theta(\omega)) = d_i(\omega)$ ,  $\lambda_i^{\text{d}}(\theta(\omega)) = \lambda_i^{\text{d}}(\omega)$ , and  $\lambda_i^{\text{c}}(\theta(\omega)) = \lambda_i^{\text{c}}(\omega)$ ;

(vii). if  $\theta$  is ergodic,  $q$  is constant on  $\widehat{\Omega}$ , and so are  $d_i$ ,  $\lambda_i^{\text{d}}$ , and  $\lambda_i^{\text{c}}$  for  $i \in \underline{q}$ .

*Proof.* Let us show that Oseledets' Multiplicative Ergodic Theorem can be applied to the random dynamical system  $(\theta, \varphi_{\text{rd}})$ . Recall that there are  $C \geq 1$ ,  $\gamma > 0$  such that, for every  $i \in \underline{N}$  and  $t \in \mathbb{R}$ ,  $\|e^{A_i t}\| \leq C e^{\gamma|t|}$ . Then, for  $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega_0$ ,  $\log^+ \|A(\omega)^{\pm 1}\| \leq \log C + \gamma t_1$ , so that

$$\int_{\Omega} \log^+ \|A(\omega)^{\pm 1}\| d\mathbb{P}(\omega) \leq \log C + \gamma \sum_{i=1}^N p_i \int_{\mathbb{R}_+} t d\mu_i(t) < \infty.$$

Then Oseledets' Multiplicative Ergodic Theorem can be applied to  $(\theta, \varphi_{\text{rd}})$ , yielding all the conclusions for  $\Psi$ ,  $q$ ,  $d_i$ ,  $V_i$ ,  $\lambda_{\text{rd}}(x_0, \omega)$ , and  $\lambda_i^{\text{d}}$ . The conclusions concerning  $\lambda_{\text{rc}}(x_0, \omega)$  and  $\lambda_i^{\text{c}}(\omega)$  in (iv), (vi), and (vii) follow from Theorem 3.19, with  $\lambda_i^{\text{c}}(\omega) = \frac{\lambda_i^{\text{d}}(\omega)}{m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))}$ . One is now left to show that the Lyapunov exponent  $\lambda_{\text{rc}}(x_0, \omega)$  exists as a limit.

Notice that  $\|e^{-A_i t} x\| \leq C e^{\gamma t} \|x\|$  for every  $i \in \underline{N}$ ,  $x \in \mathbb{R}^d$  and  $t \geq 0$ , and hence  $\|e^{A_i t} x\| \geq C^{-1} e^{-\gamma t} \|x\|$ . Let  $t > 0$  and choose  $n_t \in \mathbb{N}$  such that  $t \in (s_{n_t}(\omega), s_{n_t+1}(\omega)]$ . Then, proceeding as in (3.8), one gets

$$\frac{1}{t} \log \|\varphi_{\text{rc}}(t; x_0, \omega)\| \geq -\frac{\log C}{t} - \gamma \frac{t - s_{n_t}}{t} + \frac{1}{t} \log \|\varphi_{\text{rd}}(n_t; x_0, \omega)\|.$$

Using (3.9), we thus obtain that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \|\varphi_{\text{rc}}(t; x_0, \omega)\| \geq \frac{1}{m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))} \lambda_{\text{rd}}(x_0, \omega) = \lambda_{\text{rc}}(x_0, \omega),$$

which yields the existence of the limit.  $\square$

**5. The maximal Lyapunov exponent.** We are interested in this section in the maximal Lyapunov exponents for systems (2.7) and (3.10), i.e., the real numbers  $\lambda_1^{\text{c}}(\omega)$  and  $\lambda_1^{\text{d}}(\omega)$  from Theorem 4.1(iv). We denote these numbers by  $\lambda_{\text{max}}^{\text{c}}(\omega)$  and  $\lambda_{\text{max}}^{\text{d}}(\omega)$ , respectively. Before proving the main results of this section, we state the following lemma, which shows that the Gelfand formula for the spectral radius  $\rho$  holds uniformly over compact sets of matrices. This follows from the estimates derived in Green [15, Section 3.3]. For the reader's convenience, we provide a proof.

LEMMA 5.1. *Let  $\mathcal{A} \subset \mathcal{M}_d(\mathbb{R})$  be a compact set of matrices. Then the limit*

$$\lim_{n \rightarrow \infty} \|A^n\|^{1/n} = \rho(A)$$

*is uniform over  $\mathcal{A}$ .*

*Proof.* Let  $\varepsilon > 0$  and define a continuous function  $F : \mathcal{A} \rightarrow \mathcal{M}_d(\mathbb{R})$  by

$$F(A) = \frac{1}{\rho(A) + \varepsilon} A.$$

Then  $F(\mathcal{A})$  is compact and for every  $F(A) \in F(\mathcal{A})$  its spectral radius is  $\rho(F(A)) = \frac{\rho(A)}{\rho(A) + \varepsilon} < 1$ . Fix  $A \in \mathcal{A}$ . Then (see, e.g., Horn and Johnson [18, Lemma 5.6.10]) there is a norm  $\|\cdot\|_A$  in  $\mathbb{R}^d$  with  $\|F(A)\|_A < \frac{1 + \rho(F(A))}{2}$ . Then for all  $B$  in a neighborhood  $U$  of  $A$

$$\|F(B)\|_A < \frac{1 + \rho(F(A))}{2}.$$

Since all norms on  $\mathcal{M}_d(\mathbb{R})$  are equivalent, there is  $\beta_A > 0$  such that for all  $B \in U$

$$\|F(B)^n\| \leq \beta_A \|F(B)^n\|_A \leq \beta_A \|F(B)\|_A^n \leq \beta_A \left( \frac{1 + \rho(F(A))}{2} \right)^n.$$

Then there is  $N \in \mathbb{N}^*$ , depending only on  $A$  and  $\varepsilon$ , such that for all  $n \geq N$  and all  $B \in U$ ,

$$\frac{1}{\rho(B) + \varepsilon} \|B^n\|^{1/n} = \|F(B)^n\|^{1/n} < 1,$$

implying  $\|B^n\|^{1/n} < \rho(B) + \varepsilon$ . Since this holds for every  $B$  in a neighborhood  $U$  of  $A$  and  $\|B^n\|^{1/n} \geq \rho(B)$  for every  $n \in \mathbb{N}^*$ , one obtains the uniformity of the convergence in  $U$ , and the assertion follows by compactness of  $\mathcal{A}$ .  $\square$

We can now prove our first result regarding the characterization of  $\lambda_{\max}^c$  and  $\lambda_{\max}^d$ .

THEOREM 5.2. *For almost every  $\omega \in \Omega$ , we have*

$$\lambda_{\max}^d(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\Phi(n, \omega)\|. \quad (5.1)$$

*If  $\theta$  is ergodic, then  $\lambda_{\max}^d$  is constant almost everywhere and its constant value satisfies*

$$\lambda_{\max}^d \leq \inf_{n \in \mathbb{N}^*} \frac{1}{n} \int_{\Omega} \log \|\Phi(n, \omega)\| d\mathbb{P}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \log \|\Phi(n, \omega)\| d\mathbb{P}(\omega). \quad (5.2)$$

*Proof.* Notice that (5.1) and (5.2) do not depend on the norm in  $\mathcal{M}_d(\mathbb{R})$ . We fix in this proof the norm induced by the Euclidean norm in  $\mathbb{R}^d$ , given by  $\|A\| = \sqrt{\rho(A^T A)}$ . Notice that, in this case,  $\|A^T A\| = \sqrt{\rho((A^T A)^2)} = \rho(A^T A) = \|A\|^2$ .

By Theorem 4.1(v),  $e^{\lambda_{\max}^d(\omega)}$  is the spectral radius  $\rho(\Psi(\omega))$  of  $\Psi(\omega)$ . Using the continuity of the spectral radius and Theorem 4.1(i), one then gets that

$$e^{\lambda_{\max}^d(\omega)} = \lim_{n \rightarrow \infty} \rho \left[ (\Phi(n, \omega)^T \Phi(n, \omega))^{1/2n} \right].$$

By Gelfand's Formula for the spectral radius,

$$e^{\lambda_{\max}^d(\omega)} = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \left\| (\Phi(n, \omega)^T \Phi(n, \omega))^{k/2n} \right\|^{1/k}. \quad (5.3)$$

The sequence of matrices  $\left( (\Phi(n, \omega)^T \Phi(n, \omega))^{1/2n} \right)_{n=1}^{\infty}$  converges to  $\Psi(\omega)$ , hence this sequence is bounded in  $\mathcal{M}_d(\mathbb{R})$ . By Lemma 5.1, the limit in Gelfand's Formula is uniform, which shows that one can take the limit in (5.3) along the line  $k = 2n$  to obtain

$$e^{\lambda_{\max}^d(\omega)} = \lim_{n \rightarrow \infty} \left\| \Phi(n, \omega)^T \Phi(n, \omega) \right\|^{1/2n} = \lim_{n \rightarrow \infty} \left\| \Phi(n, \omega) \right\|^{1/n}.$$

Hence (5.1) follows by taking the logarithm.

If  $\theta$  is ergodic, then, by Theorem 4.1(vii),  $\lambda_{\max}^d$  is constant almost everywhere. Let  $m \in \mathbb{N}^*$ . By (5.1), for almost every  $\omega \in \Omega$ ,

$$\lambda_{\max}^d = \lim_{n \rightarrow \infty} \frac{1}{nm} \log \left\| \Phi(nm, \omega) \right\|. \quad (5.4)$$

One has  $\Phi(nm, \omega) = \Phi(m, \theta^{(n-1)m}\omega) \cdots \Phi(m, \theta^m\omega) \Phi(m, \omega)$ , and thus

$$\frac{1}{nm} \log \left\| \Phi(nm, \omega) \right\| \leq \frac{1}{nm} \sum_{k=0}^{n-1} \log \left\| \Phi(m, \theta^{mk}\omega) \right\|. \quad (5.5)$$

Since  $\theta^m$  preserves  $\mathbb{P}$  and  $\log \left\| \Phi(m, \cdot) \right\| \in L^1(\Omega, \mathbb{R})$ , Birkhoff's Ergodic Theorem shows that

$$\lim_{n \rightarrow \infty} \frac{1}{nm} \sum_{k=0}^{n-1} \log \left\| \Phi(m, \theta^{mk}\omega) \right\| = \frac{1}{m} \int_{\Omega} \log \left\| \Phi(m, \omega) \right\| d\mathbb{P}(\omega). \quad (5.6)$$

Combining (5.4), (5.5), and (5.6), one obtains the inequality in (5.2). The sequence  $\left( \int_{\Omega} \log \left\| \Phi(n, \omega) \right\| d\mathbb{P}(\omega) \right)_n$  is subadditive, since  $\Phi(n+m, \omega) = \Phi(m, \theta^n\omega) \Phi(n, \omega)$  for  $n, m \in \mathbb{N}$  and  $\theta$  preserves  $\mathbb{P}$ . This subadditivity implies that the equality in (5.2) holds.  $\square$

Under some extra assumptions on the probability measures  $\mu_i$ ,  $i \in \underline{N}$ , one obtains that the inequality in (5.2) is actually an equality.

**THEOREM 5.3.** *Suppose that  $\theta$  is ergodic and that there exists  $r > 1$  such that, for every  $i \in \underline{N}$ ,  $\int_{(0, \infty)} t^r d\mu_i(t) < \infty$ . Then  $\lambda_{\max}^d$  is constant almost everywhere and given by*

$$\lambda_{\max}^d = \inf_{n \in \mathbb{N}^*} \frac{1}{n} \int_{\Omega} \log \left\| \Phi(n, \omega) \right\| d\mathbb{P}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \log \left\| \Phi(n, \omega) \right\| d\mathbb{P}(\omega).$$

*Proof.* One clearly has, using (5.1), that

$$\lambda_{\max}^d = \int_{\Omega} \lambda_{\max}^d(\omega) d\mathbb{P}(\omega) = \int_{\Omega} \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| \Phi(n, \omega) \right\| d\mathbb{P}(\omega).$$

The theorem is proved if we show one can exchange the limit and the integral in the above expression, which we do by applying Vitali's convergence theorem (see, e.g., Rudin [25, Chapter 6]). We are thus left to show that the sequence of functions  $\left( \frac{1}{n} \log \left\| \Phi(n, \cdot) \right\| \right)_{n=1}^{\infty}$  is uniformly integrable, i.e., for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, for every  $E \in \mathfrak{F}$  with  $\mathbb{P}(E) < \delta$ , one has  $\frac{1}{n} \left| \int_E \log \left\| \Phi(n, \omega) \right\| d\mathbb{P}(\omega) \right| < \varepsilon$ .

For  $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega_0$  and  $n \in \mathbb{N}^*$ , one has  $\Phi(n, \omega) = e^{A_{i_n} t_n} \cdots e^{A_{i_1} t_1}$ . Let  $C, \gamma > 0$  be such that  $\left\| e^{A_i t} \right\| \leq C e^{\gamma t}$  for every  $i \in \underline{N}$  and  $t \geq 0$ . Then

$$\log \left\| \Phi(n, \omega) \right\| \leq n \log C + \gamma \sum_{j=1}^n t_j = n \log C + \gamma s_n(\omega),$$

where  $\mathbf{s}(\omega) = (s_n(\omega))_{n=0}^\infty$ . Hence, it suffices to show that the sequence  $(\frac{s_n}{n})_{n=1}^\infty$  is uniformly integrable.

For  $n \in \mathbb{N}^*$  and  $E \in \mathfrak{F}$ , we have, by Hölder's inequality,

$$\int_E \frac{s_n(\omega)}{n} d\mathbb{P}(\omega) = \frac{1}{n} \sum_{j=1}^n \int_E t_j d\mathbb{P}(\omega) \leq \frac{1}{n} \sum_{j=1}^n \left( \int_\Omega t_j^r d\mathbb{P}(\omega) \right)^{\frac{1}{r}} \mathbb{P}(E)^{\frac{1}{r'}} \leq K^{\frac{1}{r}} \mathbb{P}(E)^{\frac{1}{r'}}, \quad (5.7)$$

where  $\frac{1}{r} + \frac{1}{r'} = 1$  and  $K = \max_{i \in \underline{N}} \int_{(0, \infty)} t^r d\mu_i(t) < \infty$ . Equation (5.7) establishes the uniform integrability of  $(\frac{s_n}{n})_{n=1}^\infty$ , which yields the result.  $\square$

As an immediate consequence of Proposition 3.18, Theorem 3.19, Theorem 5.2, and Theorem 5.3, we obtain the following result.

**COROLLARY 5.4.** *Suppose that  $\theta$  is ergodic. Then  $\lambda_{\max}^c$  and  $\lambda_{\max}^d$  are constants almost everywhere satisfying*

$$\lambda_{\max}^d \leq \inf_{n \in \mathbb{N}^*} \frac{1}{n} \int_\Omega \log \|\Phi(n, \omega)\| d\mathbb{P}(\omega), \quad (5.8)$$

$$\lambda_{\max}^c = \frac{\lambda_{\max}^d}{\sum_{i=1}^N p_i \int_{\mathbb{R}_+} t d\mu_i(t)}.$$

In particular, if

$$\text{there exists } n \in \mathbb{N}^* \text{ such that } \int_\Omega \log \|\Phi(n, \omega)\| d\mathbb{P}(\omega) < 0, \quad (5.9)$$

then systems (2.7) and (3.10) are almost surely exponentially stable.

If we have further that there exists  $r > 1$  such that  $\int_{\mathbb{R}_+} t^r d\mu_i(t) < \infty$  for every  $i \in \underline{N}$ , then (5.8) is an equality and (5.9) is equivalent to the almost sure exponential stability of (3.10) and to the almost sure exponential stability of (2.7).

**6. Application to stabilization of control systems with arbitrary decay rate.** In this section, we consider the linear control system

$$\dot{x}(t) = Ax(t) + B_{\alpha(t)} u_{\alpha(t)}(t), \quad (6.1)$$

where  $x(t) \in \mathbb{R}^d$ ,  $A \in \mathcal{M}_d(\mathbb{R})$ ,  $\alpha : \mathbb{R}_+ \rightarrow \underline{N}$  belongs to the class  $\mathcal{P}$  of right continuous, piecewise constant switching signals, and, for  $j \in \underline{N}$ ,  $u_j(t) \in \mathbb{R}^{m_j}$  for some positive integer  $m_j$  and  $B_j \in \mathcal{M}_{d, m_j}(\mathbb{R})$ . System (6.1) is a switched control system with dynamics given by the  $N$  equations  $\dot{x} = Ax + B_j u_j$ ,  $j \in \underline{N}$ .

Our main motivation to consider (6.1) comes from the analysis of linear systems of the form

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t) \quad (6.2)$$

with persistently exciting signals  $\alpha$  (see, e.g., Chaillet, Chitour, Loría, and Sigalotti [5], Chitour, Colonius, and Sigalotti [8], Chitour, Mazanti, and Sigalotti [9], Chitour and Sigalotti [10], Mazanti, Chitour, and Sigalotti [22]). A (deterministic) measurable signal  $\alpha : \mathbb{R}_+ \rightarrow [0, 1]$  is called a *persistently exciting (PE)  $(T, \mu)$ -signal* with constants  $T \geq \mu > 0$  if, for every  $t \geq 0$

$$\int_t^{t+T} \alpha(s) ds \geq \mu. \quad (6.3)$$

In this framework, one is interested in stabilizing the system by a linear feedback  $u = Kx$  with  $K$  depending on  $A, B, T, \mu$  but chosen uniformly with respect to the  $(T, \mu)$ -signal  $\alpha$ . It is also of interest to determine the decay rates that can be achieved by such feedback laws  $K$ . In particular, [10, Proposition 4.5] shows that there are (two dimensional) controllable systems for which the achievable decay rates are bounded below, even when we consider only PE signals  $\alpha$  taking values in  $\{0, 1\}$  instead of  $[0, 1]$ . Our main result, Theorem 6.1, implies that, in the probabilistic setting defined below, one can get arbitrarily small (almost sure) decay rates for the generalization (6.1) of system (6.2), which is in contrast to the situation for persistently excited systems. An explanation for this fact is that the probability of having a signal  $\alpha$  with very fast switching for an infinitely long time, such as the signals used in the proof of [10, Proposition 4.5], is zero, and hence such signals do not interfere with the behavior of the (random) maximal Lyapunov exponent.

Let  $M \in \mathcal{M}_N(\mathbb{R})$  be an irreducible stochastic matrix,  $p$  be its unique invariant probability vector,  $\mu_1, \dots, \mu_N$  be probability measures on  $\mathbb{R}_+^*$  with its Borel  $\sigma$ -algebra, and consider the probability space  $(\Omega, \mathfrak{F}, \mathbb{P})$  from Definition 2.1. We consider system (6.1) in a probabilistic setting by taking random signals  $\alpha(\omega)$  as in Definition 2.5, i.e., the random control system  $\dot{x}(t) = Ax(t) + B_{\alpha(\omega)(t)}u_{\alpha(\omega)(t)}(t)$ . The problem treated in this section is the arbitrary rate stabilizability of this system by linear feedback laws  $u_j = K_jx$ ,  $j \in \underline{N}$ . More precisely, consider the closed-loop random switched system

$$\dot{x}(t) = (A + B_{\alpha(\omega)(t)}K_{\alpha(\omega)(t)})x(t). \quad (6.4)$$

We wish to know if, given  $\lambda \in \mathbb{R}$ , there exist matrices  $K_j \in \mathcal{M}_{m_j, d}(\mathbb{R})$ ,  $j \in \underline{N}$ , such that the maximal Lyapunov exponent  $\lambda_{\max}^c$  of the continuous-time system (6.4), defined as in Section 5, satisfies  $\lambda_{\max}^c(\omega) \leq \lambda$  for almost every  $\omega \in \Omega$ . Notice that, since we assume that  $M$  is irreducible, the discrete-time metric dynamical system  $\theta$  defined in (3.12) is ergodic (see Remark 3.11), and hence, by Corollary 5.4,  $\lambda_{\max}^c$  is constant almost everywhere in  $\Omega$ .

For  $j \in \underline{N}$ , let

$$\mathbf{V}_j = \text{Range}(B_j \quad AB_j \quad \dots \quad A^{d-1}B_j). \quad (6.5)$$

Notice that, by Cayley–Hamilton theorem, for every  $n \in \mathbb{N}$ , all columns of  $A^n B_j$  belong to  $\mathbf{V}_j$ . Some of the spaces  $\mathbf{V}_j$  may have dimension zero.

**THEOREM 6.1.** *Let  $A \in \mathcal{M}_d(\mathbb{R})$ ,  $B_j \in \mathcal{M}_{d, m_j}(\mathbb{R})$  for  $j \in \underline{N}$  and some  $m_j \in \mathbb{N}^*$ , and suppose that the spaces  $\mathbf{V}_1, \dots, \mathbf{V}_N$  defined in (6.5) satisfy  $\mathbf{V}_1 \oplus \dots \oplus \mathbf{V}_N = \mathbb{R}^d$ . Then, for every  $\lambda \in \mathbb{R}$ , there exist matrices  $K_j \in \mathcal{M}_{m_j, d}(\mathbb{R})$ ,  $j \in \underline{N}$ , such that the maximal Lyapunov exponent  $\lambda_{\max}^c$  of the closed-loop random switched system (6.4) satisfies  $\lambda_{\max}^c(\omega) \leq \lambda$  for almost every  $\omega \in \Omega$ .*

*Proof.* For  $j \in \underline{N}$ , let  $n_j = \dim \mathbf{V}_j$ . Up to a linear change of variables in  $\mathbb{R}^d$ , we can suppose that  $\mathbf{V}_1 = \{e_1, \dots, e_{n_1}\}$ ,  $\mathbf{V}_2 = \{e_{n_1+1}, \dots, e_{n_1+n_2}\}$ ,  $\dots$ ,  $\mathbf{V}_N = \{e_{n_1+\dots+n_{N-1}+1}, \dots, e_{n_1+\dots+n_N}\}$ . In this case, for  $j \in \underline{N}$ , the matrices  $A$  and  $B_j$  have the block structure

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & A_N \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b_j \\ \vdots \\ 0 \end{pmatrix}, \quad (6.6)$$

with  $A_j \in \mathcal{M}_{n_j}(\mathbb{R})$  and  $b_j \in \mathcal{M}_{n_j, m_j}(\mathbb{R})$ . Whenever  $n_j \neq 0$ , it follows immediately from the definition of  $V_j$  that the pair  $(A_j, b_j)$  is controllable. Denoting by  $P_j = (e_{n_1+\dots+n_{j-1}+1}, \dots, e_{n_1+\dots+n_j})^T \in \mathcal{M}_{n_j, d}(\mathbb{R})$ , we have that  $b_j = P_j B_j$  and  $A_j = P_j A P_j^T$ .

Let  $C \geq 1$ ,  $\beta > 0$  be such that, for every  $j \in \underline{N}$  and every  $t \geq 0$ ,  $\|e^{A_j t}\| \leq C e^{\beta t}$ . Thanks to Cheng, Guo, Lin, and Wang [7, Proposition 2.1], we may assume that  $C$  is chosen large enough such that the following property holds: there exists  $L \in \mathbb{N}^*$  such that, for every  $\gamma \geq 1$  and  $j \in \underline{N}$ , there exists a matrix  $k_j \in \mathcal{M}_{m_j, n_j}(\mathbb{R})$  with

$$\left\| e^{(A_j + b_j k_j)t} \right\| \leq C \gamma^L e^{-\gamma t}, \quad \forall t \in \mathbb{R}_+. \quad (6.7)$$

Let  $K_j = k_j P_j \in \mathcal{M}_{m_j, d}(\mathbb{R})$ . With this choice of feedback laws, we have

$$A + B_j K_j = \begin{pmatrix} A_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_j + b_j k_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & A_N \end{pmatrix},$$

and thus, for every  $t \in \mathbb{R}$ ,

$$e^{(A + B_j K_j)t} = \begin{pmatrix} e^{A_1 t} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & e^{A_2 t} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{(A_j + b_j k_j)t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & e^{A_N t} \end{pmatrix}.$$

Since  $M$  is irreducible and  $p$  is invariant under  $M$ , we have  $p_j > 0$  for every  $j \in \underline{N}$ . The irreducibility of  $M$  also provides the existence of  $r \geq N$  and  $(i_1^*, \dots, i_r^*) \in \underline{N}^r$  such that  $\{i_1^*, \dots, i_r^*\} = \underline{N}$  and  $M_{i_1^* i_2^*} \cdots M_{i_{r-1}^* i_r^*} > 0$ . In order to apply Corollary 5.4, consider

$$\begin{aligned} \int_{\Omega} \log \|\Phi(r, \omega)\| d\mathbb{P}(\omega) &= \sum_{(i_1, \dots, i_r) \in \underline{N}^r} p_{i_1} M_{i_1 i_2} \cdots M_{i_{r-1} i_r} \\ &\cdot \int_{(0, \infty)^r} \log \left\| e^{(A + B_{i_r} K_{i_r})t_r} \cdots e^{(A + B_{i_1} K_{i_1})t_1} \right\| d\mu_{i_1}(t_1) \cdots d\mu_{i_r}(t_r). \end{aligned} \quad (6.8)$$

Since  $\sum_{j=1}^N P_j^T P_j = \text{Id}_d$  and  $P_j e^{(A + B_i K_i)t} P_k^T = 0$  if  $j \neq k$ , we have, for every  $(i_1, \dots, i_r) \in \underline{N}^r$  and  $(t_1, \dots, t_r) \in \mathbb{R}_+^r$ ,

$$\begin{aligned} &e^{(A + B_{i_r} K_{i_r})t_r} \cdots e^{(A + B_{i_1} K_{i_1})t_1} \\ &= \left( \sum_{j_r=1}^N P_{j_r}^T P_{j_r} \right) e^{(A + B_{i_r} K_{i_r})t_r} \cdots \left( \sum_{j_1=1}^N P_{j_1}^T P_{j_1} \right) e^{(A + B_{i_1} K_{i_1})t_1} \left( \sum_{j_0=1}^N P_{j_0}^T P_{j_0} \right) \\ &= \sum_{j=1}^N P_j^T P_j e^{(A + B_{i_r} K_{i_r})t_r} \cdots P_j^T P_j e^{(A + B_{i_1} K_{i_1})t_1} P_j^T P_j. \end{aligned}$$

$$= \sum_{j=1}^N P_j^T e^{(A_j + \delta_{j i_r} b_j k_j) t_r} \dots e^{(A_j + \delta_{j i_1} b_j k_j) t_1} P_j. \quad (6.9)$$

Since, for every  $j \in \underline{N}$  and  $t \geq 0$ , we have  $\|e^{A_j t}\| \leq C e^{\beta t}$  and  $\|e^{(A_j + b_j k_j) t}\| \leq C \gamma^L e^{-\gamma t}$ , we get, for every  $(i_1, \dots, i_r) \in \underline{N}^r$  and  $(t_1, \dots, t_r) \in \mathbb{R}_+^r$ ,

$$\left\| e^{(A + B_{i_r} K_{i_r}) t_r} \dots e^{(A + B_{i_1} K_{i_1}) t_1} \right\| \leq N C^r \gamma^{rL} e^{\beta \sum_{j=1}^r t_j}. \quad (6.10)$$

When  $(i_1, \dots, i_r) = (i_1^*, \dots, i_r^*)$ , we can obtain a sharper bound than (6.10). For  $j \in \underline{N}$ , denote by  $N(j)$  the nonempty set of all indices  $k \in \underline{r}$  such that  $i_k^* = j$ , and denote by  $n(j) \in \mathbb{N}^*$  the number of elements in  $N(j)$ . Then

$$\left\| P_j^T e^{(A_j + \delta_{j i_r^*} b_j k_j) t_r} \dots e^{(A_j + \delta_{j i_1^*} b_j k_j) t_1} P_j \right\| \leq C^r \gamma^{n(j)L} e^{-\gamma \sum_{k \in N(j)} t_k} e^{\beta \sum_{k \in \underline{r} \setminus N(j)} t_k},$$

which shows, using (6.9), that

$$\begin{aligned} \left\| e^{(A + B_{i_r^*} K_{i_r^*}) t_r} \dots e^{(A + B_{i_1^*} K_{i_1^*}) t_1} \right\| &\leq \sum_{j=1}^N C^r \gamma^{n(j)L} e^{-\gamma \sum_{k \in N(j)} t_k} e^{\beta \sum_{k \in \underline{r} \setminus N(j)} t_k} \\ &\leq N C^r \gamma^{rL} e^{-\gamma \min_{k \in \underline{r}} t_k} e^{r\beta \max_{k \in \underline{r}} t_k}. \end{aligned} \quad (6.11)$$

Let

$$\begin{aligned} E_0 &= \max_{i \in \underline{N}} \int_{(0, \infty)} t d\mu_i(t), \\ E_{\min} &= \int_{(0, \infty)^r} \min_{k \in \underline{r}} t_k d\mu_{i_1^*}(t_1) \dots d\mu_{i_r^*}(t_r) > 0, \\ E_{\max} &= \int_{(0, \infty)^r} \max_{k \in \underline{r}} t_k d\mu_{i_1^*}(t_1) \dots d\mu_{i_r^*}(t_r) < \infty. \end{aligned}$$

Then, combining (6.10) and (6.11), we obtain from (6.8) that

$$\begin{aligned} \int_{\Omega} \log \|\Phi(r, \omega)\| d\mathbb{P}(\omega) &\leq N^r (\log(NC^r) + rL \log \gamma + r\beta E_0) \\ &+ p_{i_1^*} M_{i_1^* i_2^*} \dots M_{i_{r-1}^* i_r^*} (\log(NC^r) + rL \log \gamma - \gamma E_{\min} + r\beta E_{\max}). \end{aligned} \quad (6.12)$$

The right-hand side of (6.12) tends to  $-\infty$  as  $\gamma \rightarrow \infty$ , which can be achieved by (6.7). Hence it follows from Corollary 5.4 that the maximal Lyapunov exponent of (6.4) can be made arbitrarily small.  $\square$

**REMARK 6.2.** *By writing the matrices  $A$  and  $B_j$ ,  $j \in \underline{N}$ , in the form (6.6), system (6.1) can be seen as  $N$  independent control systems such that, at each time, only one of them is controlled, while the others follow their uncontrolled dynamics.*

**REMARK 6.3.** *In order to establish a more precise link between Theorem 6.1 and the case of deterministic persistently excited systems treated in [5, 8, 9, 10, 22], consider the case of (6.1) with  $\alpha(t) \in \{0, 1\}$ ,  $B_0 = 0$ ,  $B_1 = B$ , and  $(A, B)$  controllable. Moreover, in order to simplify, we assume that, in the probabilistic model of  $\alpha$ , trivial switches do not occur, which amounts to choosing*

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with unique invariant probability vector  $p = (\frac{1}{2}, \frac{1}{2})$ . In general, such signals  $\alpha(\omega)$  cannot be persistently exciting. In fact, suppose that  $\mu_0$  satisfies  $\mu_0((0, T]) < 1$  for every  $T > 0$ . Then

$$\mathbb{P}\{\omega \in \Omega \mid \exists T \geq \mu > 0 \text{ such that } \alpha(\omega) \text{ is a PE } (T, \mu)\text{-signal}\} = 0. \quad (6.13)$$

Indeed, since a  $(T, \mu)$ -signal is also a  $(T', \mu')$ -signal for every  $T' \geq T$  and  $0 < \mu' \leq \mu$ , we have

$$\begin{aligned} & \{\omega \in \Omega \mid \exists T \geq \mu > 0 \text{ such that } \alpha(\omega) \text{ is a PE } (T, \mu)\text{-signal}\} \\ &= \bigcup_{T>0} \bigcup_{\mu \in (0, T]} \{\omega \in \Omega \mid \alpha(\omega) \text{ is a PE } (T, \mu)\text{-signal}\} \\ &= \bigcup_{T \in \mathbb{N}^*} \bigcup_{\frac{1}{\mu} \in \mathbb{N}^*} \{\omega \in \Omega \mid \alpha(\omega) \text{ is a PE } (T, \mu)\text{-signal}\}. \end{aligned}$$

If  $\alpha$  is a PE  $(T, \mu)$ -signal, the PE condition implies that  $\alpha$  cannot remain zero during time intervals longer than  $T - \mu$ , and thus

$$\begin{aligned} & \{\omega \in \Omega \mid \alpha(\omega) \text{ is a PE } (T, \mu)\text{-signal}\} \\ & \subset \{\omega = (i_n, t_n)_{n=1}^\infty \in \Omega \mid \forall n \in \mathbb{N}^*, i_n = 0 \implies t_n \leq T - \mu\}. \end{aligned} \quad (6.14)$$

Since  $i_n$  takes the value 0 infinitely many times for almost every  $\omega \in \Omega$  and  $\mu_0((0, T - \mu]) < 1$ , the right-hand side of (6.14) has measure zero, and thus (6.13) holds.

However, one can link the random signals  $\alpha(\omega)$  with a weaker, asymptotic notion of persistence of excitation. A (deterministic) measurable signal  $\alpha : \mathbb{R}_+ \rightarrow [0, 1]$  is said to be asymptotically persistently exciting with constant  $\rho > 0$  if

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \alpha(s) ds \geq \rho.$$

It follows easily from (6.3) that every persistently exciting  $(T, \mu)$ -signal is also asymptotically persistently exciting with constant  $\rho = \frac{\mu}{T}$ . Proposition 3.20 implies that, for almost every  $\omega \in \Omega$ ,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \alpha(\omega)(s) ds = \frac{\int_{\mathbb{R}_+} t d\mu_1(t)}{\int_{\mathbb{R}_+} t d\mu_0(t) + \int_{\mathbb{R}_+} t d\mu_1(t)},$$

and thus, in particular, almost every signal  $\alpha(\omega)$  is asymptotically persistently exciting with constant  $\rho = \frac{\int_{\mathbb{R}_+} t d\mu_1(t)}{\int_{\mathbb{R}_+} t d\mu_0(t) + \int_{\mathbb{R}_+} t d\mu_1(t)} > 0$ .

## REFERENCES

- [1] LUDWIG ARNOLD, *Random Dynamical Systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
- [2] YURI BAKHTIN AND TOBIAS HURTH, *Invariant densities for dynamical systems with random switching*, *Nonlinearity*, 25 (2012), pp. 2937–2952.
- [3] MICHEL BENAÏM, STÉPHANE LE BORGNE, FLORENT MALRIEU, AND PIERRE-ANDRÉ ZITT, *Qualitative properties of certain piecewise deterministic Markov processes*, *Ann. Inst. Henri Poincaré Probab. Stat.*, 51 (2015), pp. 1040–1075.
- [4] P. BOLZERN, P. COLANERI, AND G. DE NICOLAO, *On almost sure stability of continuous-time Markov jump linear systems*, *Automatica J. IFAC*, 42 (2006), pp. 983–988.



- [5] ANTOINE CHAILLET, YACINE CHITOUR, ANTONIO LORÍA, AND MARIO SIGALOTTI, *Uniform stabilization for linear systems with persistency of excitation: the neutrally stable and the double integrator cases*, Math. Control Signals Systems, 20 (2008), pp. 135–156.
- [6] DEBASISH CHATTERJEE AND DANIEL LIBERZON, *On stability of randomly switched nonlinear systems*, IEEE Trans. Automat. Control, 52 (2007), pp. 2390–2394.
- [7] DAIZHAN CHENG, LEI GUO, YUANDAN LIN, AND YUAN WANG, *A note on overshoot estimation in pole placements*, J. Control Theory Appl., 2 (2004), pp. 161–164.
- [8] YACINE CHITOUR, FRITZ COLONIUS, AND MARIO SIGALOTTI, *Growth rates for persistently excited linear systems*, Math. Control Signals Systems, 26 (2014), pp. 589–616.
- [9] YACINE CHITOUR, GUILHERME MAZANTI, AND MARIO SIGALOTTI, *Stabilization of persistently excited linear systems*, in Hybrid Systems with Constraints, Jamal Daafouz, Sophie Tarbouriech, and Mario Sigalotti, eds., Wiley-ISTE, London, UK, 2013, ch. 4.
- [10] YACINE CHITOUR AND MARIO SIGALOTTI, *On the stabilization of persistently excited linear systems*, SIAM J. Control Optim., 48 (2010), pp. 4032–4055.
- [11] OSWALDO L. V. COSTA, MARCELO D. FRAGOSO, AND MARCOS G. TODOROV, *Continuous-Time Markov Jump Linear Systems*, Probability and its Applications (New York), Springer, Heidelberg, 2013.
- [12] M. H. A. DAVIS, *Piecewise-deterministic Markov processes: a general class of nondiffusion stochastic models*, J. Roy. Statist. Soc. Ser. B, 46 (1984), pp. 353–388.
- [13] YUGUANG FANG AND KENNETH A. LOPARO, *Stabilization of continuous-time jump linear systems*, IEEE Trans. Automat. Control, 47 (2002), pp. 1590–1603.
- [14] XIANGBO FENG, KENNETH A. LOPARO, YUANDONG JI, AND HOWARD JAY CHIZECK, *Stochastic stability properties of jump linear systems*, IEEE Trans. Automat. Control, 37 (1992), pp. 38–53.
- [15] JOHN JAMES GREEN, *Uniform Convergence to the Spectral Radius and Some Related Properties in Banach Algebras*, PhD thesis, University of Sheffield, 1996.
- [16] MARTIN HAIRER, *Ergodic properties of Markov processes*. Lecture notes from the University of Warwick, Spring 2006.
- [17] PAUL R. HALMOS, *Measure Theory*, vol. 18 of Graduate Texts in Mathematics, Springer-Verlag, 1974.
- [18] ROGER A. HORN AND CHARLES R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, second ed., 2013.
- [19] CHANYING LI, MICHAEL Z. Q. CHEN, JAMES LAM, AND XUERONG MAO, *On exponential almost sure stability of random jump systems*, IEEE Trans. Automat. Control, 57 (2012), pp. 3064–3077.
- [20] DANIEL LIBERZON, *Switching in Systems and Control*, Birkhäuser Boston, 1 ed., 2003.
- [21] HAI LIN AND PANOS J. ANTSAKLIS, *Stability and stabilizability of switched linear systems: a survey of recent results*, IEEE Trans. Automat. Control, 54 (2009), pp. 308–322.
- [22] GUILHERME MAZANTI, YACINE CHITOUR, AND MARIO SIGALOTTI, *Stabilization of two-dimensional persistently excited linear control systems with arbitrary rate of convergence*, SIAM J. Control Optim., 51 (2013), pp. 801–823.
- [23] CARL MEYER, *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
- [24] KARL PETERSEN, *Ergodic Theory*, vol. 2 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1989. Corrected reprint of the 1983 original.
- [25] WALTER RUDIN, *Real and Complex Analysis*, McGraw-Hill Book Co., New York, New York, third ed., 1987.
- [26] EUGENE SENETA, *Non-Negative Matrices and Markov Chains*, Springer Series in Statistics, Springer, New York, 2006. Revised reprint of the second edition [Springer-Verlag, New York, 1981].
- [27] ROBERT SHORTEN, FABIAN WIRTH, OLIVER MASON, KAI WULFF, AND CHRISTOPHER KING, *Stability criteria for switched and hybrid systems*, SIAM Rev., 49 (2007), pp. 545–592.