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► **To cite this version:**

Bassem Barhoumi. An improved axisymmetric Convected Boundary Element Method formulation with uniform flow. *Mechanics & Industry*, 2015, 10.1051/meca/2016064 . hal-01232124v3

HAL Id: hal-01232124

<https://hal.science/hal-01232124v3>

Submitted on 15 Nov 2016 (v3), last revised 26 May 2017 (v4)

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An improved axisymmetric Convected Boundary Element Method formulation with uniform flow

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Abstract

This paper presents an improved form of the convected Boundary Element Method (BEM) for axisymmetric problems in a subsonic uniform flow. The proposed formulation based on the axisymmetric Convected Helmholtz Equation (CHE) and its fundamental solution that describes the sound radiation from a monopole source. The variables in the new axisymmetric boundary integral formulation can be expressed explicitly in terms of the acoustic pressure and its particular normal derivative. Also, the constant coefficients are expressed only in terms of the axisymmetric convected Green's function and its convected normal derivative. The particular and convected derivatives reduce the flow effects of the normal and the flow direction derivatives incorporated in the conventional convected boundary integral formulas. The advanced form of the axisymmetric boundary integral representation with flow is a similar form of the axisymmetric boundary element method without flow. Precisely, the two new operators significantly reduce the computational burden of the classical BEM and then becomes the CPU time of BEM without flow. The formula is verified comparing to both analytical and Finite Element Methods (FEM) of an axisymmetric infinite rigid duct in a subsonic uniform flow.

Keywords:

Axisymmetric Convected Boundary Element Method, Convected Helmholtz Equation, singular integrals, Finite Element Method, axisymmetric infinite rigid duct.

1. Introduction

The numerical studies of some acoustic problems with fluid flow require only the special methods, such as the computational techniques of the Finite Element Method (FEM) and boundary element method (BEM) for solving any axisymmetric medium in the science and engineering. The finite element method is currently the most widely used numerical method for solving the interior medium with mean flow and that the boundary element method is used for solving the exterior medium with a constant flow [1].

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The finite element method has been implemented for solving the Helmholtz equation for the acoustic wave diffraction problems without flow [2]. It has been implemented with a displacement formulation for the modelling and simulation of the fluid-structure interaction problems in which the coupling conditions are formulated [3]. A numerical technique of the FEM for the modeling of the acoustic pressure inside an axisymmetric lined duct in the presence of a constant flow was developed and presented in Ref. [4]. This method is based on the resolution of the convected Helmholtz equation in the modal representation which gives good results compared with the analytical ones. The FEM applied to determine the time-harmonic acoustic field in a 2D infinite waveguide with walls covered of an absorbing material, and in the presence of a mean flow assumed uniform far from the source [5]. The boundary conditions of the rigid and lined ducts are represented by transparent boundary conditions suitable for use in a finite element scheme which based on the Dirichlet-to-Neumann (DtN) boundary condition [5]. For particular acoustic domains, [6] has determined the acoustic propagation in a transformed domain by the Prandtl-Glauert transformation.

It is known that the finite element method has its limitations in modeling infinite domains. The unbounded medium contains the non-reflection condition which is represented by the Sommerfeld radiation condition in a medium at rest [7] or in a subsonic uniform flow [6]. Using the boundary element method (BEM), which requires a discretization of only the generator of the acoustic domain and that this Sommerfeld condition is automatically fulfilled by the fundamental solution. This makes the BEM superior to the FEM in many cases, especially for the unbounded problems, such as the wave propagation in the infinite lined ducts.

The boundary element method can be formulated in the transformed acoustic medium [8] and in the original acoustic medium [9]. In order to take advantage of both the FEM and the BEM, a coupled FEM/BEM approach has been proposed [10]. A direct collocation boundary element formulation for the calculation of acoustic propagation in a subsonic uniform mean flow has been presented in Ref. [11] in which the model is directed towards the calculation of the propagation inside and radiation from axisymmetric lined ducts. An advanced boundary element/fast Fourier transform (BE/FFT) methodology for solving axisymmetric acoustic wave scattering and radiation problems with non-axisymmetric boundary conditions has been developed in Ref. [12].

However, the main disadvantages of using boundary integral formulations in an original acoustic medium are the same as in the non-flow case [13]. In addition, the convection effects in the presence of flow substantially increase these difficulties and makes significantly more complicated the conventional integral representations [10,11]. These axisymmetric formulations are derived from the three-dimensional boundary integral formulation developed in [9] in which the convected terms due to the Green's function and its derivatives such as the normal derivative and the derivative in the flow direction.

Also, the authors in Ref. [14] have been used a reformulation of the three-dimensional convected boundary element to solve the axisymmetric acoustic problems. The reformulation based on the non-standard normal derivative similar to the non-flow normal derivative of the Green's kernel. The non-standard derivative leads to a global form of the integral formula contains another supplementary convec-

tive terms in which make difficulty the numerical implementation of the boundary element method. The acoustic variable of the classical and reformulation axisymmetric boundary integral representations in Refs. [10,11,14] in terms of the acoustic pressure as well as its normal and tangential derivatives.

The aim of this paper is to contribute to reduce the complexity of convected boundary integral formulas for axisymmetric domains in a subsonic uniform flow. Thus, a new reformulation of the boundary element method for the axisymmetric mediums (BEMA) which based on the axisymmetric convected Helmholtz equation and its fundamental solution is presented. This new axisymmetric boundary element method requires only two operators which substantially reduce the flow effects of the normal and the flow direction derivatives of the axisymmetric convected Green's function incorporated in the classical convected boundary integral formulations. Through the use of these new convected terms, the new axisymmetric BEM formula has a similar form of the axisymmetric BEM without flow. The acoustic variables in the new axisymmetric boundary integral equation can be expressed explicitly in terms of the acoustic pressure and its particular normal derivative.

The numerical implementation of BEMA is developed for the acoustic propagation in an axisymmetric rigid duct and validated by comparison with FEM method and the analytical solution in which the CPU times are presented.

2. An improved Axisymmetric BEM Formulation

Consider an acoustic medium Ω with a compressible fluid, inviscid, isentropic, which is characterized by the Mach flow vector \mathbf{M}_∞ , of density ρ_∞ and speed of sound c_∞ in the axisymmetric physical space (o,r,z) . In figure 1, $\partial\Omega = \Gamma_\infty \cup \Gamma$ is the generator limiting the axisymmetric medium Ω , such as Γ_∞ and Γ are the exterior and interior generators of the outgoing normal vector $\mathbf{n}_q(n_{r_q}, n_{z_q})$ and the tangential vector $\mathbf{t}_q(t_{r_q}, t_{z_q})$ at the point q , respectively. The generator $\partial\Omega$ not perturbed the streamlines and unchanged the direction of the flow.

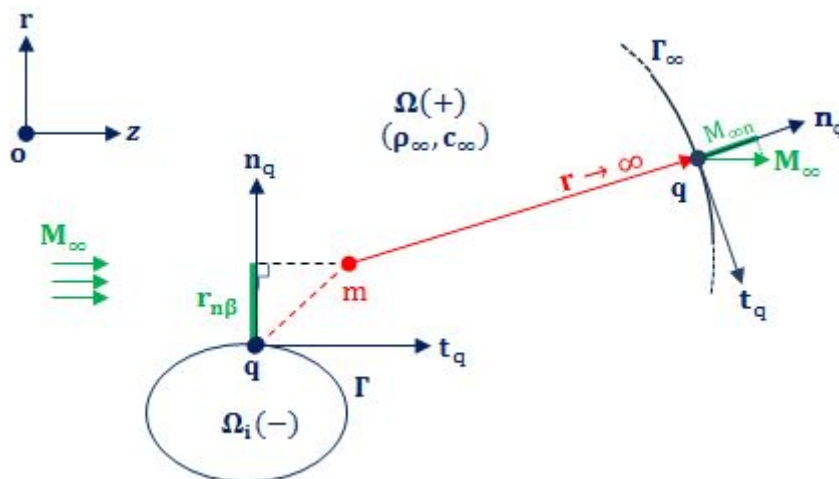


Figure 1: Axisymmetric acoustic medium.

2.1. Axisymmetric Convected Helmholtz Equation

A combination of the Axisymmetric Linearized Euler Equations can be produced the Axisymmetric Convected Helmholtz Equation in a harmonic regime ($\exp(+i\omega t)$) in which t is the time, ω is the angular frequency and i is the imaginary unit. The wave equation wearing on axisymmetric acoustic pressure p is decomposed into a non-flow part and a convection part [10]

$$[\Delta + k^2](p) - (\mathbf{M}_\infty \cdot \nabla) \left[(2ik + (\mathbf{M}_\infty \cdot \nabla))(p) \right] = 0, \quad (1)$$

where Δ and ∇ are the axisymmetric Laplace and Gradient operators. $k = \omega/c_\infty$ is the original wavenumber. The fundamental solution of Eq. (1) is the axisymmetric convected Green's function $G_0^k = \int_0^{2\pi} G^k d\beta$, which is the integration of the three-dimensional convected Green function G^k over the azimuthal angle β . The Green's kernel $G^k(m, q)$ is given for the points m and q by

$$G^k(m, q) = \exp \left[-ik^* \left(\mathbf{m} \mathbf{q} \cdot \mathbf{M}_\infty^* + R_\beta^* \right) \right] / (4\pi\alpha R_\beta^*), \quad (2)$$

where $m(r_m, z_m)$ and $q(r_q, z_q)$ are the axisymmetric source-observer points, α is the Prandtl-Glauert factor, $k^* = k/\alpha$ is the convected wavenumber, and that $\mathbf{M}_\infty^* = \mathbf{M}_\infty/\alpha$ is the convected Mach vector. The distance R_β^* is the convected radius which depends on the azimuthal order β , it's given for the points m and q by

$$R_\beta^*(m, q) = \sqrt{r^{*2} + 4r_q r_m \sin^2(\beta/2)}, \quad (3)$$

in which $r^*(m, q) = \sqrt{mq^2 + (\mathbf{m} \mathbf{q} \cdot \mathbf{M}_\infty^*)^2}$ is the axisymmetric convected radius and mq is the physical distance between source-observer. Also, the axisymmetric convected Green's function satisfies the following inhomogeneous convected Helmholtz equation

$$\begin{aligned} H \left(G_0^k(m, q) \right) &= [\Delta_q + k^2] G_0^k(m, q) \\ + (\mathbf{M}_\infty \cdot \nabla_q) \left[(2ik - (\mathbf{M}_\infty \cdot \nabla_q)) G_0^k(m, q) \right] &= -\frac{1}{2\pi r_q} \delta(q - m), \end{aligned} \quad (4)$$

in which $(\cdot)_q$ is the operator at point q and δ is the Dirac delta function such as $\delta(q - m) = \delta(r_q - r_m)\delta(z_q - z_m)$.

2.2. Axisymmetric convected Helmholtz integral equation

To obtain a boundary integral equation formulation for the acoustic problem, the Helmholtz equation Eq. (1) is multiplied by the axisymmetric convected Green's function $G_0^k(m, q)$ and its fundamental equation Eq. (4) by $p(q)$. Subtracting these two equations and integrating over the axisymmetric exterior medium Ω yields

$$p(m) = \int_{\Omega} \left[G_0^k(m, q) H(p(q)) - p(q) H \left(G_0^k(m, q) \right) \right] r_q d\Omega_q, \quad m \in \Omega \setminus \Gamma, \quad (5)$$

Using the axisymmetric second Green's formula, the axisymmetric Gradient property and the axisymmetric divergence theorem (Appendix A), the axisymmetric

acoustic pressure Eq. (5) takes the following form

$$\begin{aligned}
p(\mathbf{m}) = & \int_{\partial\Omega} \left[G_0^k(\mathbf{m}, \mathbf{q}) \frac{\partial p}{\partial \mathbf{n}_q}(\mathbf{q}) - p(\mathbf{q}) \frac{\partial G_0^k}{\partial \mathbf{n}_q}(\mathbf{m}, \mathbf{q}) \right] r_q d\Gamma_q \\
& - 2ik \int_{\partial\Omega} M_{\infty n} p(\mathbf{q}) G_0^k(\mathbf{m}, \mathbf{q}) r_q d\Gamma_q \\
& - \int_{\partial\Omega} M_{\infty n} G_0^k(\mathbf{m}, \mathbf{q}) (\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) p(\mathbf{q}) \\
& + \int_{\partial\Omega} M_{\infty n} p(\mathbf{q}) (\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) G_0^k(\mathbf{m}, \mathbf{q}) r_q d\Gamma_q, \quad \mathbf{m} \in \Omega \setminus \Gamma,
\end{aligned} \tag{6}$$

where $\nabla_{\mathbf{q}} = \sigma_n \mathbf{n}_q + \sigma_t \mathbf{t}_q$ is the axisymmetric Gradient operator in which the couple $(\sigma_n = \partial/\partial n_q, \sigma_t = \partial/\partial t_q)$ designates the normal and tangential derivatives at point \mathbf{q} , respectively. $M_{\infty n} = \mathbf{M}_{\infty} \cdot \mathbf{n}_q$ is the normal Mach number.

In order to simplify the boundary integral equation Eq. (6), the normal derivative $\partial/\partial n_q$ and the flow direction derivative $(\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}})$ can be converted into a particular normal derivative d/dn_q , which is given by the following relation

$$\frac{d(\cdot)}{dn_q} = \frac{\partial(\cdot)}{\partial n_q} - M_{\infty n} (\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) (\cdot), \tag{7}$$

The particular normal derivative of the axisymmetric convected Green function $dG_0^k/dn_q = \int_0^{2\pi} dG^k/dn_q d\beta$ is the integration of the particular normal derivative of the three-dimensional Green kernel dG^k/dn_q over the azimuthal angle β . Using the normal and the flow direction derivatives of the three-dimensional Green's function in Refs. [10,11,14], one obtains the following derivative function

$$\frac{dG^k}{dn_q}(\mathbf{m}, \mathbf{q}) = -G^k(\mathbf{m}, \mathbf{q}) \left[(1 + ik^* R_{\beta}^*(\mathbf{m}, \mathbf{q})) \frac{r_{n\beta}}{R_{\beta}^{*2}(\mathbf{m}, \mathbf{q})} + ik M_{\infty n} \right], \tag{8}$$

where $r_{n\beta} = \mathbf{m} \mathbf{q}_{\text{axi}} \cdot \mathbf{n}_q$ is the normal distance and that the vector $\mathbf{m} \mathbf{q}_{\text{axi}}$ of coordinate $(r_q - r_m \cos(\beta), z_q - z_m)$. The particular normal derivative of the convected Green's function Eq. (8) is decomposed into two terms. The first term generalizes the expression of the normal derivative in a non-flow case and that the second term explains the convective influence. Substituting the particular normal derivatives Eqs. (7) and (8) in the integral formula Eq. (6) yield

$$p(\mathbf{m}) = \int_{\partial\Omega} \left[G_0^k(\mathbf{m}, \mathbf{q}) \frac{dp}{dn_q}(\mathbf{q}) - p(\mathbf{q}) \frac{d_k G_0^k}{dn_q}(\mathbf{m}, \mathbf{q}) \right] r_q d\Gamma_q, \quad \mathbf{m} \in \Omega \setminus \Gamma, \tag{9}$$

The axisymmetric boundary integral formulation Eq. (9) is decomposed into two integrals in which the acoustic variables are expressed only in terms of the acoustic field as well as its particular normal derivative which interpreting as a condition applied to the interior generator Γ . Thus, the axisymmetric BEM Eq. (9) is similar to the boundary integral equation solution for solving the axisymmetric potential flows [15] and the non-flow case (the right first term in the equation 6) [16]. From

the particular normal derivative operator Eq. (7), we obtain the convected normal derivative operator

$$\frac{d_k(\cdot)}{dn_q} = \frac{d(\cdot)}{dn_q} + 2ikM_{\infty n}(\cdot), \quad (10)$$

Substituting the particular normal derivative of the three-dimensional Green's function Eq. (8) into the convected normal derivative operator Eq. (10), one obtains

$$\frac{d_k G^k}{dn_q}(m, q) = -G^k(m, q) \left[(1 + ik^* R_\beta^*(m, q)) \frac{r_{n\beta}}{R_\beta^{*2}(m, q)} - ikM_{\infty n} \right], \quad (11)$$

The integration over the circular generator Γ_∞ given by Eq. (9), represents contributions waves which come in the interior region due to the source is very far from observer point; $q \rightarrow \infty$. Thus, this integral is null. According to the three-dimensional Green's function at infinity [17], the axisymmetric Green's kernel G_0^k is proportional to $1/\sqrt{R_\beta^*}$. Also, taking into account the term $r_{n\beta}/R_\beta^*$ is equivalent to the tangential Prandtl-Glauert factor $\beta_t^* = \beta/\sqrt{1 - M_{\infty t}^2}$, the convected normal derivative dG_0^k/dn_q of Eq. (11) is proportional to $ik(1/\sqrt{1 - M_{\infty t}^2} - M_{\infty n})/\sqrt{R_\beta^*}$. This leads to two convected radiation conditions which verified by the axisymmetric acoustic pressure and its Green's kernel

$$\lim_{r \rightarrow \infty} p(r) = 0, \quad (12)$$

$$\lim_{r \rightarrow \infty} \left\{ \sqrt{r} \left[\frac{dp}{dr}(r) + i \left(\frac{k}{\sqrt{1 - M_{\infty t}^2}} - kM_{\infty n} \right) p(r) \right] \right\} = 0, \quad (13)$$

where $M_{\infty t} = \mathbf{M}_\infty \cdot \mathbf{t}_q$ is the tangential Mach number. The first condition Eq. (12) indicates the non-reflection of waves, and the second condition Eq. (13) generalizes the classical Sommerfeld radiation condition in a uniform flow subsonic [7,8]. Substituting the non-reflection conditions into the boundary equation Eq. (9), we obtain

$$p(m) = \pm \int_\Gamma \left[G_0^k(m, q) \frac{dp}{dn_q}(q) - p(q) \frac{d_k G_0^k}{dn_q}(m, q) \right] r_q d\Gamma_q, \quad m \in \Omega \setminus \Gamma, \quad (14)$$

in which the signs (+, -) designates the exterior and interior acoustic domains. When the point m is taken on the generator at point q , the Green's kernel and its convected normal derivative Eqs. (2) and (11) contain singular and regular parts. Thus, Eq. (14) requires to study of behavior of the kernels near the singularity.

2.3. Singular integrals

A classical procedure can be used to isolate the singularity problems of the axisymmetric Green's function and its convected derivative. In fact, the singularities can be isolated by decomposing the axisymmetric convected Green's function G_0^k into

a singular static part $G_0^0(k=0)$ and regular part g_0^k depending on the wavenumber. Then, the axisymmetric Green's function takes the following form

$$G_0^k(m, q) = G_0^0(m, q) + g_0^k(m, q), \quad (15)$$

Thus, the convected normal derivative is given by

$$\frac{d_k G_0^k}{dn_q}(m, q) = \frac{d_k G_0^0}{dn_q}(m, q) + \frac{d_k g_0^k}{dn_q}(m, q), \quad (16)$$

Because the difference between the singular and regular behavior, the singular functions in Eqs. (15) and (16) are evaluated analytically and that the regular parts are evaluated numerically with standard Gauss-Legendre quadrature of high order. In fact, the static axisymmetric Green's function is given by [10,11]

$$G_0^0(m, q) = \frac{K(\tau)}{\pi\alpha\tau^*}, \quad (17)$$

Using the normal and the flow direction derivatives of the static Green's kernel in Ref. [10], one obtains the following static convected derivative

$$\frac{d_k G_0^0}{dn_q}(m, q) = -\frac{1}{\pi\alpha\tau^*} \left[\left(\frac{r_{n\pi/2}}{r^{*2}} - \frac{n_{r_q}}{2r_q} \right) E(\tau) + \left(\frac{n_{r_q}}{2r_q} - ikM_{\infty n} \right) K(\tau) \right], \quad (18)$$

where $K(\tau)$ and $E(\tau)$ are the complete elliptic integrals of the first and second kinds, the parameters τ and τ^* are given by

$$\tau^2 \tau^{*2} = 4r_q r_m, \tau^* = \sqrt{r^{*2} + 4r_q r_m}, \quad (19)$$

According to the complete elliptic integrals at $r \rightarrow 0$ [17], the static Green's function exhibits a logarithmic behavior near the singularity, as follows

$$G_0^0(m, q) \sim -\frac{1}{2\pi\alpha} \frac{\text{Log}(r^*)}{2\pi r_q}, r \rightarrow 0, \quad (20)$$

Thus, the static convected normal derivative Eq. (18) near the singularity is characterized by

$$\frac{d_k G_0^0}{dn_q}(m, q) \sim -\frac{1}{2\pi\alpha} \frac{1}{2\pi r_q} \frac{r_{n0}}{r^{*2}}, r \rightarrow 0, \quad (21)$$

Since $r_{n0}/r^* = O(r^*)$, then the singularity of the static convected function $d_k G_0^k/dn_q$ is thus only apparent. Following the exclusion procedure [18,19] to isolate the singularity problems in the boundary formula Eq. (14), which can be rewritten as

$$\begin{aligned} p(m) = & \pm \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \left[G_0^k(m, q) \frac{dp}{dn_q}(q) - p(q) \frac{d_k G_0^k}{dn_q}(m, q) \right] r_q d\Gamma_q \\ & \pm \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon^\mp} \left[G_0^k(m, q) \frac{dp}{dn_q}(q) - p(q) \frac{d_k G_0^k}{dn_q}(m, q) \right] r_q d\Gamma_q, \end{aligned} \quad (22)$$

where s_ε is the disk of center m and of radius $\varepsilon = r$ bounded by the circle $\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-$ and $\Gamma = \lim_{\varepsilon \rightarrow 0} (\Gamma_\varepsilon \cup \Gamma_\varepsilon^-)$ for the exterior domain and that $\Gamma = \lim_{\varepsilon \rightarrow 0} (\Gamma_\varepsilon \cup \Gamma_\varepsilon^+)$ for the interior domain (see figure 2).

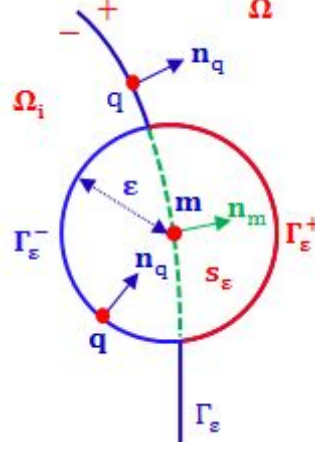


Figure 2: Analytical regular method.

Substituting Eqs. (20) and (21) in Eq. (22), we get a new original form of the integral boundary representation with uniform flow wearing on the axisymmetric acoustic pressure at any point m

$$c^\pm(m) p(m) = \pm \int_{\Gamma} G_0^k(m, q) \frac{dp}{dn_q}(q) r_q d\Gamma_q \mp \int_{\Gamma} p(q) \frac{d_k G_0^k}{dn_q}(m, q) r_q d\Gamma_q, m \in \Omega, \quad (23)$$

where $c^\pm(m)$ is the free term at any point m , which is given by

$$c^\pm(m) = 1 - \frac{\theta^\mp(m)}{2\pi} = 1 \pm \int_{\Gamma} -\frac{1}{2\pi\beta} \frac{\mathbf{mq} \cdot \mathbf{n}_q}{r^{*2}(m, q)} d\Gamma_q, \quad (24)$$

in which $\theta^\pm(m)$ is the convected angle. When the point m in the exterior and interior domains, the convected angle $\theta^\pm(m) = 0$ and that the coefficient $c^\pm(m) = 1$. If m is a regular point of the generator, $\theta^\pm(m) = \pi$ and that $c^\pm(m) = 0.5$.

2.4. Numerical implementation

The numerical implementation of the axisymmetric boundary integral equation Eq. (23) for a body of arbitrary shape is obtained by discretizing the generator Γ of the body with N one-dimensional quadratic isoparametric elements, according to the following convergence criterion

$$n_e h_e \leq \lambda \alpha^2, \quad (25)$$

where n is the number of finite elements for a wavelength λ and h_e is the diameter of the finite element. Then, by using the quadratic shape functions, the acoustic

pressure p and its normal derivative σ_n of any point q on the generator Γ are assumed to be given in terms of the nodal coordinates. Thus, the tangential derivative in the local coordinates system along the generator Γ at point q is given by [11,20]

$$\sigma_t(q) = \sigma_\xi(q) / J^i[\xi(q)], \quad (26)$$

with $-1 \leq \xi \leq 1$ is the local coordinate and J^i is the Jacobien of reference-local transformation. Using the collocation method so that the point m coincides with the nodes q_i of the discretized generator Γ , Eq. (14) can be written in complex matrix form with Robin condition

$$[\mathbf{A}] \mathbf{p} = [\mathbf{B}_n] \sigma_n + [\mathbf{B}_c] \mathbf{p}_c, \quad (27)$$

where \mathbf{p} , \mathbf{p}_c and σ_n are the nodal vectors containing the values of the two-dimensional acoustic pressure p , its Dirichlet acoustic pressure p_c and its Neumann normal derivative σ_n at each node, respectively, whereas $[\mathbf{A}]$ and $[\mathbf{B}_n]$ are the axisymmetric acoustic matrices with the elements of matrix $[\mathbf{A}]$ contain the convected angles and elementary integrals related to the axisymmetric convected Green's function G_0^k and its convected normal derivative $d_k G_0^k / dn_q$, while the elements of matrix $[\mathbf{B}_n]$ is composed only by the elementary integrals containing the integrands of axisymmetric Green's function and its convected derivative. $[\mathbf{B}_c]$ is the acoustic matrix containing the integrands G_0^k and $d_k G_0^k / dn_q$.

3. Axisymmetric Finite Element Method

The axisymmetric Finite Element Method is based on the weak variational formulation, which can be obtained by multiplying the axisymmetric wave equation Eq. (1) by a test function p^* (the axisymmetric acoustic pressure p^* satisfies the axisymmetric convected Helmholtz equation Eq. (1)). Then, integrating the result over the axisymmetric interior medium Ω_i , one obtains

$$\int_{\Omega_i} p^* \Delta p d\Omega_i + k^2 \int_{\Omega_i} p^* p d\Omega_i - 2ikM_\infty \int_{\Omega_i} p^* \frac{\partial p}{\partial z} d\Omega_i - M_\infty^2 \int_{\Omega_i} p^* \frac{\partial^2 p}{\partial z^2} d\Omega_i, \quad (28)$$

Using the axisymmetric second Green's formula, one obtains

$$\int_{\Omega_i} [p^* \Delta p] rd\Omega_i = - \int_{\Omega_i} \nabla p^* \nabla p rd\Omega_i + \int_{\Gamma_i} p^* \frac{\partial p}{\partial n_{\Gamma_i}} rd\Gamma_i, \quad (29)$$

$$\int_{\Omega_i} \left[p^* \frac{\partial^2 p}{\partial z^2} \right] rd\Omega_i = - \int_{\Omega_i} \frac{\partial p^*}{\partial z} \frac{\partial p}{\partial z} rd\Omega_i + \int_{\Gamma_i} p^* \frac{\partial p}{\partial n_z} rd\Gamma_i, \quad (30)$$

$$2 \int_{\Omega_i} p^* \frac{\partial p}{\partial z} rd\Omega_i = \int_{\Omega_i} p^* \frac{\partial p}{\partial z} rd\Omega_i - \int_{\Omega_i} \frac{\partial p^*}{\partial z} p rd\Omega_i + \int_{\Gamma_i} p^* p n_z rd\Gamma_i, \quad (31)$$

Substituting Eqs. (29)-(31) into Eq. (28), we obtain a developed form of the weak variational formulation in Refs. [4,21]

$$\begin{aligned}
& - \int_{\Omega_i} \nabla p^* \nabla p r d\Omega_i + k^2 \int_{\Omega_i} p^* p r d\Omega_i + ikM_\infty \int_{\Omega_i} p^* \frac{\partial p}{\partial z} r d\Omega_i \\
& - ikM_\infty \int_{\Omega_i} \frac{\partial p^*}{\partial z} p r d\Omega_i + M_\infty^2 \int_{\Omega_i} \frac{\partial p^*}{\partial z} \frac{\partial p}{\partial z} r d\Omega_i \\
& + \int_{\Gamma_i} p^* \frac{\partial p}{\partial n_{\Gamma_i}} r d\Gamma_i + ikM_\infty \int_{\Gamma_i} p^* p n_z r d\Gamma_i - M_\infty^2 \int_{\Gamma_i} p^* \frac{\partial p}{\partial n_z} r d\Gamma_i = 0,
\end{aligned} \tag{32}$$

The same classical numerical discretization in Ref. [4], the elementary integrals of the numerical variational formulation Eq. (32) over the reference triangular or interval elements \hat{T} take the following form

$$\int_{\hat{T}} f * P d\hat{T}, \tag{33}$$

in which f and P are the regular polynomials of order n ($n < 3$) containing the shape functions. Then, these integrals are evaluated analytically [22]. Thus, the solve equation is a similar form of the complex matrix form Eq. (27) in which $[\mathbf{A}]$, $[\mathbf{B}_n]$ and $[\mathbf{B}_c]$ are the axisymmetric acoustic matrices with the various integrals in Eq. (32).

4. Application: Axisymmetric rigid duct

We consider an infinite axisymmetric rigid duct of truncated domain Ω of radius $R = 0.1\text{m}$, of length $L = 0.5\text{m}$, of symmetry axis (oz), in a subsonic uniform flow of the Mach vector $\mathbf{M}_\infty = M_\infty \mathbf{e}_z$ in the (r - z) meridian plane. The truncated medium is limited by the rigid wall Γ_R of the normal vector \mathbf{n}_{Γ_R} and by two generators, the left boundary Γ_- at $z = 0$ and the right boundary Γ_+ at $z = L$ of normal vectors \mathbf{n}_{Γ_-} and \mathbf{n}_{Γ_+} , respectively (see Figure 3).

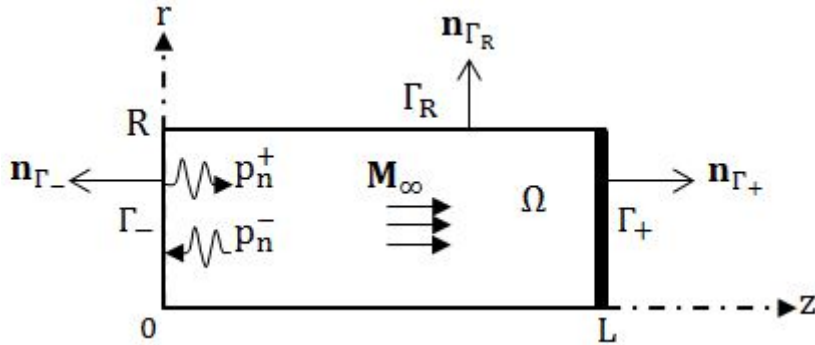


Figure 3: Axisymmetric rigid duct.

The acoustic pressure in an axisymmetric rigid duct is [3]

$$p_n^\pm = A_n J_0(k_{r,n} r) e^{ik_{z,n}^\pm z}, \tag{34}$$

in which (0) is the azimuthal order and (n) is the radial order, A_n is the axisymmetric amplitude of order n equal to $\frac{\sqrt{2}}{R} \frac{1}{\sqrt{J_0^2(k_{r,n}R) - J_1^2(k_{r,n}R)}}$, J_0 is the Bessel function of the first kind and that $k_{z,n}^\pm$ and $k_{r,n}$ are the axial and radial wavenumbers (\pm indicates the propagation mode in the increased (+) and decreased (-) z-direction) which satisfies the following relations

$$\left. \frac{\partial J_0(k_{r,n}r)}{\partial r} \right|_{r=R} = 0, \quad (35)$$

$$k_{z,n}^\pm = \frac{-kM_\infty \pm \sqrt{k^2 - \alpha^2 k_{r,n}^2}}{\alpha^2}, \quad (36)$$

Since the term as the function of tangential derivative σ_t over the axisymmetric generator duct is zero, then the new radiation condition is given by the following particular derivative σ at right and left generators Γ_\pm

$$\sigma_\pm = \pm i \alpha^2 k_{z,n}^\pm p \quad (37)$$

We note $k_n^\pm = \alpha^2 k_{z,n}^\pm$ is the new wavenumber in which its module reduces the module of the classical wavenumber $k_{z,n}^\pm$ for the axisymmetric excitations.

4.1. Numerical methodologies

- The two-dimensional rigid duct is discretized by triangular elements structured and regular with three nodes isoparametric of two straight perpendiculars.
- The axial and radial wavenumbers satisfies the convergence criterion of the discretization Eq. (25) in which the number of triangles is 1652 elements (see figure 4).
- The evaluation of the acoustic pressure at point m in an interior axisymmetric rigid duct Ω given by Eq. (23) is numerically where $\Gamma = \Gamma_- \cup \Gamma_R \cup \Gamma_+$ is the generator of this duct, and taking into account that the free term at the corner points $m = \Gamma_- \cap \Gamma_R$ and $m = \Gamma_R \cap \Gamma_+$ is $c^-(m) = 3/4$.

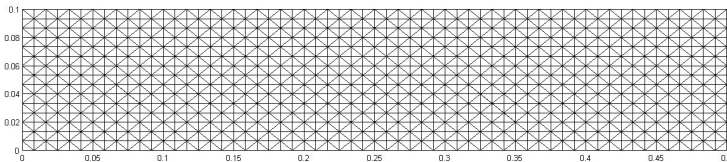


Figure 4: Axisymmetric mesh.

4.2. Numerical results

We apply BEMA, FEMA techniques and the analytical solution (S.Anal) associated to the axisymmetric acoustic pressure which propagated in the increased

z-direction of axisymmetric excitation (p_n^+/A_n). Figures 5 and 6 give the cartographies of the real parts $\text{Re}(P_n^+/A_n)$ for different values of the Mach vector by S.BEMA, S.FEMA and S.Anal methods.

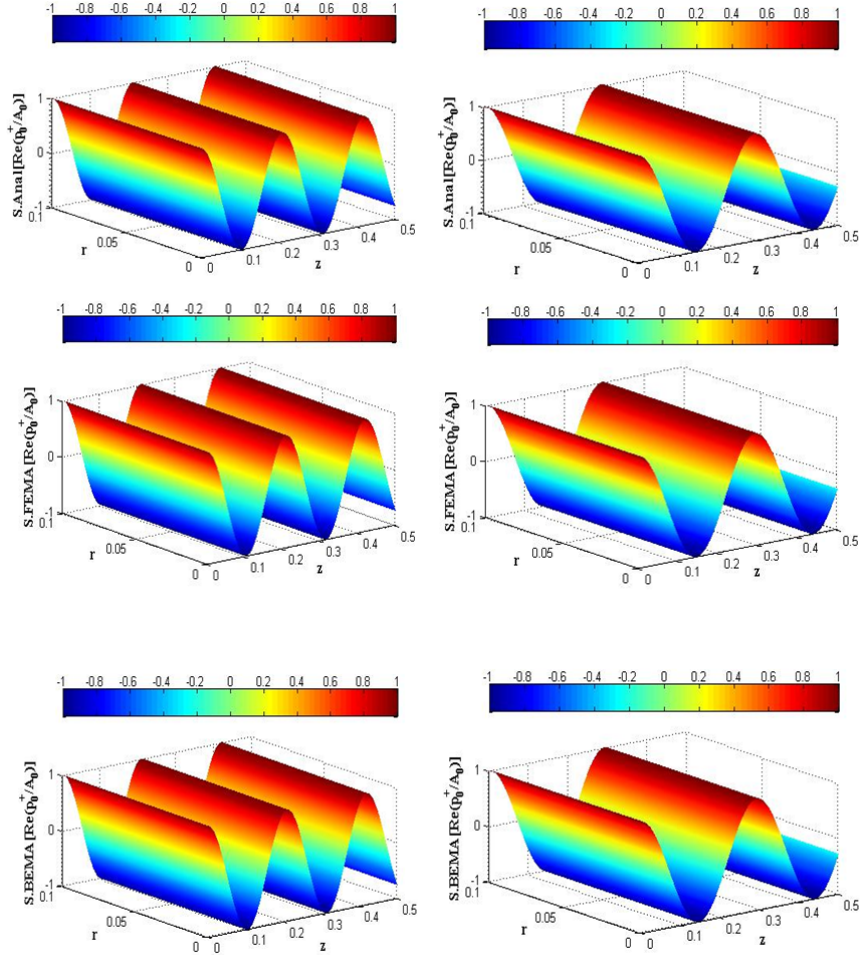


Figure 5: Real part of the axisymmetric acoustic pressure $\text{Re}(P_0^+/A_0)$ by S.Anal (top), FEMA (middle) and BEMA (bottom) for $\mathbf{M}_\infty = \mathbf{0}$ (left) and $\mathbf{M}_\infty = 0.4\mathbf{e}_z$ (right) with $k = 30$.

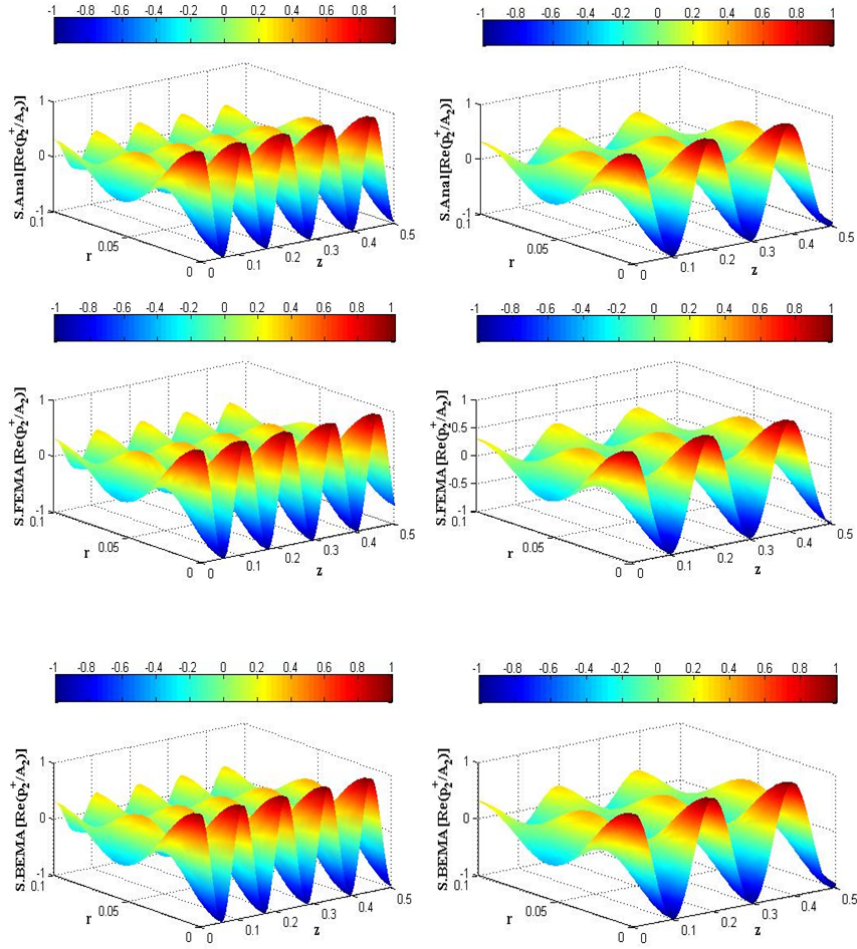


Figure 6: Real part of the axisymmetric acoustic pressure $\text{Re}(P_2^+/A_2)$ by S.Anal (top), FEMA (middle) and BEMA (bottom) for $\mathbf{M}_\infty = \mathbf{0}$ (left) and $\mathbf{M}_\infty = 0.4\mathbf{e}_z$ (right) with $k = 90$.

Figures 5 and 6 are presented the real parts of the axisymmetric acoustic pressure $\text{Re}(P_n^+/A_n)$ by FEMA and BEMA which are in very good concord with analytical solution of relative error given by the L_2 norm. For FEMA, the relative error is less than 0.5% and for the axisymmetric boundary integral method, the error is less than 0.8%.

We observe that for the initial mode (0), the axisymmetric acoustic pressure is uniform along the radial and axial waves. Also, when the subsonic uniform flow is increased, the wavelength increases, which is explained by the offset of the wave in the positive z -direction (see Figure 5).

In the other mode, the axisymmetric acoustic pressure along the radial and axial axis is non-uniform in the presence and absence of the flow. Indeed, in the upstream region, the acoustic field is radially amplified to be greater than that obtained in the downstream region in which we observe the Sommerfeld radiation condition Eq. (12), and we notice also that the uniform flow is proportional to the wavelength (see Figure 6).

In addition, the CPU times of the Numerical boundary element method, the

finite element method and that the classical boundary element method in Ref. [14] with mean flow are obtained by a machine 2.93 GHz using MATLAB-ACOUSTIC codes, which are given by the following table 1

Mach vector	Presented BEM	Classical BEM	Presented FEM
$\mathbf{M}_\infty = \mathbf{0}$	2.71 min	3.08 min	8.45 min
$\mathbf{M}_\infty = 0.4\mathbf{e}_z$	2.73 min	6.92 min	13.86 min

Table 1: The CPU times of BEM, Classical BEM and FEM.

A comparison between the CPU times of FE and BE methods in table 1, show that the axisymmetric Convected boundary element formulation Eq. (23) reduces 80% of the computation time of Finite element method Eq. (32).

Compared to the classical axisymmetric boundary integral formulae, the proposed method reduces a CPU time equal to 60% of the classical axisymmetric BEM. Also, the relative error for the classical axisymmetric boundary integral equation is less than 2%.

We observe that when the number of the discretization increase in term of the numerical implementation of the axisymmetric boundary element method with mean flow, the gain time between BEM-FEM and BEM-Classical BEM increase.

For the BEM case, we observe that the numerical execution time for the non-flow case is the numerical execution time for the flow case. This is because the improved form of the axisymmetric boundary integral representation Eq. (23) with the mean flow is similar to the boundary element method without flow.

5. Conclusion and prospects

In this work, an improved form of the axisymmetric Helmholtz integral equation and conventional finite element method for an axisymmetric medium in a subsonic uniform flow have been developed and presented. The axisymmetric convected boundary element method is derived from two new operators such as the convected and particular derivatives, which substantially reduces the complexity due to the presence of flow in the conventional formulations. The formulation is derived to be easy to implement as a numerical tool for computational codes of the axisymmetric acoustic medium in a mean flow.

Also, the improved boundary integral representation significantly reduces the computational burden of the conventional axisymmetric finite element method. Precisely, for the rigid duct case, the axisymmetric BEM require three Matrices in terms of integrals over the generator and that axisymmetric FEM require six assembly matrices in terms of integrals over the medium and its generator. The proposed methodologies can be applied to simplified the modal boundary element method and its derivatives [11,23].

Finally, we think that the method could be extended for the analysis of physical phenomena in other noise pollution systems such as compressors, aircraft engines,

and ventilation systems that contain different materials. It is hoped that this could have a significant impact in design optimization of flow duct systems in the industry sector.

APPENDIX A. AXISYMMETRIC ACOUSTIC PROPERTIES

The axisymmetric Green's identity formula as a function of the test function p^* is given by

$$\begin{aligned} & \int_{\Omega} (\Delta_{\mathbf{q}} p^*(\mathbf{q}) p(\mathbf{q}) - \Delta_{\mathbf{q}} p(\mathbf{q}) p^*(\mathbf{q})) d\Omega_{\mathbf{q}} \\ &= \int_{\partial\Omega} [p^*(\mathbf{q}) \frac{\partial p}{\partial \mathbf{n}_{\mathbf{q}}}(\mathbf{q}) - p(\mathbf{q}) \frac{\partial p^*}{\partial \mathbf{n}_{\mathbf{q}}}(\mathbf{q})] r_{\mathbf{q}} d\Gamma_{\mathbf{q}}, \end{aligned} \quad (\text{A.1})$$

where $\partial/\partial \mathbf{n}_{\mathbf{q}}$ is the normal derivative and that $\Delta_{\mathbf{q}}$ is the axisymmetric Laplace operator in an acoustics problem r - z

$$\Delta_{\mathbf{q}}(\cdot) = \frac{\partial^2}{\partial r_{\mathbf{q}}^2}(\cdot) + \frac{1}{r_{\mathbf{q}}} \frac{\partial}{\partial r_{\mathbf{q}}}(\cdot) + \frac{\partial^2}{\partial z_{\mathbf{q}}^2}(\cdot), \quad (\text{A.2})$$

Using the axisymmetric Green's identity formula Eq. (A.1) for the axisymmetric Greens function $G_0^k(\mathbf{m}, \mathbf{q})$, yields

$$\begin{aligned} & \int_{\Omega} (\Delta_{\mathbf{q}} G_0^k(\mathbf{m}, \mathbf{q}) p(\mathbf{q}) - \Delta_{\mathbf{q}} p(\mathbf{q}) G_0^k(\mathbf{m}, \mathbf{q})) d\Omega_{\mathbf{q}} \\ &= \int_{\partial\Omega} \left[G_0^k(\mathbf{m}, \mathbf{q}) \frac{\partial p}{\partial \mathbf{n}_{\mathbf{q}}}(\mathbf{q}) - p(\mathbf{q}) \frac{\partial G_0^k}{\partial \mathbf{n}_{\mathbf{q}}}(\mathbf{m}, \mathbf{q}) \right] r_{\mathbf{q}} d\Gamma_{\mathbf{q}}, \end{aligned} \quad (\text{A.3})$$

The axisymmetric Gradient property is given by

$$\begin{aligned} & G_0^k(\mathbf{m}, \mathbf{q}) (\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) p(\mathbf{q}) + p(\mathbf{q}) (\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) G_0^k(\mathbf{m}, \mathbf{q}) \\ &= (\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) \left(p(\mathbf{q}) G_0^k(\mathbf{m}, \mathbf{q}) \right), \end{aligned} \quad (\text{A.4})$$

And that the axisymmetric Divergence theorem is given by

$$\begin{aligned} & (p(\mathbf{q}) \mathbf{M}_{\infty}) \cdot \nabla_{\mathbf{q}} \left((\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) G_0^k(\mathbf{m}, \mathbf{q}) \right) \\ &= \nabla_{\mathbf{q}} \cdot \left[\left((\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) G_0^k(\mathbf{m}, \mathbf{q}) \right) p(\mathbf{q}) \mathbf{M}_{\infty} \right] \\ &- \left((\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) G_0^k(\mathbf{m}, \mathbf{q}) \right) \left((\mathbf{M}_{\infty} \cdot \nabla_{\mathbf{q}}) p(\mathbf{q}) \right), \end{aligned} \quad (\text{A.5})$$

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