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HARNACK AND SHMUL'YAN PRE-ORDER RELATIONS FOR HILBERT SPACE CONTRACTIONS

CATALIN BADEA AND LAURIAN SUCIU

Abstract. We study the behavior of some classes of Hilbert space contractions with respect to the Harnack and Shmul'yan pre-order relations. We also discuss the corresponding equivalence relations, as well as an extension, recently considered by S. ter Horst, of the Shmul'yan pre-order from contractions to the operator-valued Schur class of functions.

1. Introduction

Preamble. The geometry of the unit ball $\mathcal{B}_1(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$, the C*-algebra of all bounded linear operators on Hilbert space, is still not completely understood. The aim of this paper is to study the behavior of some classes of Hilbert space contractions (i.e., elements of $\mathcal{B}_1(\mathcal{H})$) with respect to two pre-order relations, the Harnack and Shmul'yan pre-orders, and the corresponding equivalence relations. Recall here that any binary relation which is reflexive and transitive is called a pre-order [12, Definition 5.2.2]. It is well-known that given any pre-order $\prec$ on $\mathcal{B}_1(\mathcal{H})$, if one defines a binary relation $\sim$ on $\mathcal{B}_1(\mathcal{H}) \times \mathcal{B}_1(\mathcal{H})$ by $A \sim B$ if $A \prec B$ and $B \prec A$, then $\sim$ is an equivalence relation [12, Proposition 5.2.4].

The Harnack and Shmul'yan pre-orders. The Harnack pre-order and equivalence has been introduced by Ion Suciu [14, 15] in the 1970s based on some operator inequalities for Hilbert space contractions which generalize the classical Harnack inequality for positive harmonic functions in the unit disc. The equivalence classes are called the Harnack parts, and we will denote by $\Delta(T)$ the Harnack part containing the contraction T. The concept of Harnack parts, as well as the hyperbolic metric defined in [18], are the analogues in non-commutative case of the Gleason parts and metric defined in the context of function algebras. The Shmul'yan pre-order has been introduced in [13] and studied in [8] under the name O-pre-order. We denote by $\Delta_{\mathrm{Sh}}(T)$ the Shmul'yan part of T.

More specifically, we say that T is *Harnack dominated* by T' (notation $T \overset{\mathrm{H}}{\prec} T'$) if there exists a positive constant $c \geq 1$ such that for any analytic polynomial p verifying $\operatorname{Re} p(z) \geq 0$ for $|z| \leq 1$ we have

$$\operatorname{Re} p(T) \leq c \operatorname{Re} p(T'). \tag{1.1}$$

We say that T is Harnack dominated by T' *with constant* c whenever we want to emphasize the constant. Thus T and T' are *Harnack equivalent* if $T \overset{\mathrm{H}}{\prec} T'$ and $T' \overset{\mathrm{H}}{\prec} T$; we also

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say in this case that T and T' belong to the same *Harnack part*. We refer the reader to [1–4, 8, 10, 11, 14, 15, 17, 18, 21, 22] for several properties of the Harnack pre-order.

We can define the Shmul'yan domination in the more general case of the unit ball, denoted $\mathcal{B}_1(\mathcal{H}_1, \mathcal{H}_2)$, of contractions acting between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Namely, if $A, B \in \mathcal{B}_1(\mathcal{H}_1, \mathcal{H}_2)$, then according to [8] one says that A *Shmul'yan dominates* B if $B = A + D_{A^*} X D_A$ for some $X \in \mathcal{B}(\mathcal{D}_A, \mathcal{D}_{A^*})$. Here, for a contraction $T \in \mathcal{B}_1(\mathcal{H}_1, \mathcal{H}_2)$ we denote by $D_T = (I_{\mathcal{H}_1} - T^*T)^{1/2}$ the *defect operator* and by $\mathcal{D}_T = \overline{\mathcal{R}(D_T)}$ the *defect space* of T, where $I_{\mathcal{H}_1}$ is the identity operator on $\mathcal{H}_1$. Also, one says that A and B are *Shmul'yan equivalent* whenever each of them Shmul'yan dominates the other, while the corresponding equivalence classes, in $\mathcal{B}_1(\mathcal{H}_1, \mathcal{H}_2)$, are called *Shmul'yan parts*. It was proved in [8, Proposition 1.6 and Corollary 3.3] that the Shmul'yan equivalence relation in the closed unit ball of $\mathcal{B}(\mathcal{H})$ is stronger that the Harnack equivalence, in general. Also, by [8, Proposition 1.6], A and B as before are Shmul'yan equivalent if and only if $B = A + D_{B^*} Y D_A$ for some $Y \in \mathcal{B}(\mathcal{D}_A, \mathcal{D}_{B^*})$.

Recently the Shmul'yan pre-order relation has been extended by S. ter Horst [6] from contractions to operator-valued *Schur class functions*, that is to the contractive analytic functions on $\mathbb{D}$. More precisely, let $\mathcal{E}$ be a separable Hilbert space and $H^\infty(\mathcal{E})$ the Banach space of all norm bounded analytic functions on $\mathbb{D}$ with values in $\mathcal{B}(\mathcal{E})$. We denote $H^\infty = H^\infty(\mathbb{C})$. If $H_1^\infty(\mathcal{E})$ is the closed unit ball of $H^\infty(\mathcal{E})$ and $F \in H_1^\infty(\mathcal{E})$, then the associated Toeplitz operator is the contraction T_F on the vector valued Hardy space $H^2(\mathcal{E})$ defined by

$$(T_F g)(\lambda) = F(\lambda) g(\lambda) \quad (g \in H^2(\mathcal{E}), \lambda \in \mathbb{D}).$$

If $F, G \in H_1^\infty(\mathcal{E})$ one says that F *is Shmul'yan-ter Horst dominated by* G, in notation $F \overset{\infty}{\prec} G$, whenever T_F is Shmul'yan dominated by T_G. We denote by $\Delta_\infty(T)$ the equivalence class of T for the equivalence relation induced by the Shmul'yan-ter Horst relation.

Notation and basic definitions. In the sequel $T, T' \in \mathcal{B}(\mathcal{H})$ will be linear contractions acting on the complex Hilbert space $\mathcal{H}$; V acting on $\mathcal{K}$ and V' acting on $\mathcal{K}'$ will denote the *minimal isometric dilations* of T and T' respectively (see [23]). $\mathcal{N}(T)$ and $\mathcal{R}(T)$ stand for the kernel and respectively the range of T. We shall denote by $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$, or simply by I if the Hilbert space is clear from the context. In what follows $\sigma(T)$ denotes the spectrum of T. From time to time we will consider the more general case of contractions between different Hilbert spaces.

The *asymptotic limit* $S_T \in \mathcal{B}(\mathcal{H})$ of the contraction T (see, for instance, [7, Chapter 3]) is the strong limit of the sequence $\{T^{*n}T^n\}_{n\in\mathbb{N}}$. It is a positive contraction with $\|S_T\| = 1$ whenever $S_T \neq 0$. Notice that $\mathcal{H}_1 = \mathcal{N}(I - S_T) = \bigcap_{n\geq 1} \mathcal{N}(I - T^{*n}T^n)$ is the maximal invariant subspace of $\mathcal{H}$ for T on which T is an isometry, while $\mathcal{H}_u = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*})$ is the maximal reducing subspace for T on which T is unitary (see [7]). The reducing isometric part in $\mathcal{H}$ of T can be described (see [20, Proposition 2.8]) as

$$\mathcal{H}_1 = \mathcal{H} \ominus \bigvee_{n,j\geq 0} T^n (I - T^{*j}T^j)\mathcal{H} = \{h \in \mathcal{H} : V^n T^{*m} h \in \mathcal{H},\ m, n \geq 0\}.$$

We say that T is *strongly (weakly) stable* if the sequence $\{T^n\}_{n\in\mathbb{N}}$ is strongly (weakly) convergent to 0 in $\mathcal{B}(\mathcal{H})$. Also, we say that T is of *class* $C_{0\cdot}$ (respectively, $C_{\cdot 0}$) when T (T^*) is strongly stable, which means $S_T = 0$ ($S_{T^*} = 0$), while T is of *class* C_{00} if it is of class $C_{0\cdot}$ and of class $C_{\cdot 0}$. We say that T is of *class* $C_{1\cdot}$ (respectively, $C_{\cdot 1}$) if $T^n h \nrightarrow 0$ (respectively $T^{*n} h \nrightarrow 0$) for all $0 \neq h \in \mathcal{H}$. Also, T is of *class* C_{11} if both T and T^* are of class $C_{1\cdot}$.

We say that $T \in \mathcal{B}(\mathcal{H})$ is *quasi-normal* if T commutes with T^*T and that T is a *partial isometry* if $TT^*T = T$. We say that T is a *hyponormal* operator if $TT^* \leq T^*T$.

Let T and T' in $\mathcal{B}_1(\mathcal{H})$. Then (see [15], [8]) $T \overset{\mathrm{H}}{\prec} T'$ with constant c^2 if and only if there exists a (unique) operator $A \in \mathcal{B}(\mathcal{K}', \mathcal{K})$ such that $A(\mathcal{H}) \subset \mathcal{H}$, $A|_{\mathcal{H}} = I_{\mathcal{H}}$, $AV' = VA$ and $\|A\| \leq c$. We say in this case that $A \in \mathcal{B}(\mathcal{K}', \mathcal{K})$ is the *Harnack operator associated to* (T, T').

Organization of the paper. The plan of the paper is as follows. In Section 2 we show that for two commuting contractions T and T' with $T \overset{\mathrm{H}}{\prec} T'$ the $C_{1\cdot}$-part in the canonical triangulation can be fixed (Theorem 2.1) and we study the relation between the reducing unitary parts (Theorem 2.4). We also show (Theorem 2.10) that under some commutativity conditions the Harnack domination of T by T' is equivalent to Harnack equivalence of the two contractions, with their Shmul'yan equivalence and also with the existence of a point $z_0 \in \mathbb{D}$ and a $\mathcal{B}(\mathcal{H})$-valued contractive analytic function F on $\mathbb{D}$ such that $F(0) = T$ and $F(z_0) = T'$. In particular, we obtain (Corollary 2.13) that if T and T' are doubly commuting contractions on $\mathcal{H}$ such that T is quasi-normal and is Harnack dominated by T', then T and T' are Shmul'yan equivalent. In Section 3 we study the behavior of $C_{1\cdot}$-contractions and of partial isometries with respect to Harnack and Shmul'yan pre-orders. For instance, we show that Shmul'yan equivalence of two contractions can be described by their pure parts, and this is applied to Harnack and Shmul'yan parts of a partial isometry. In Section 4 we study some properties of the Shmul'yan-ter Horst relation, including a description for the equivalence class $\Delta_\infty(T)$ of a partial isometry T, as a constant function in $H^\infty(\mathcal{E})$.

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2. Commuting contractions in Harnack domination

Recall (see [23]) that the canonical triangulation of a general contraction T on $\mathcal{H}$ gives a matrix representation of T on $\mathcal{H} = \mathcal{N}(S_T) \oplus \overline{\mathcal{R}(S_T)}$, where $\mathcal{N}(S_T)$ is the maximum invariant subspace for T on which it is of class $C_{0\cdot}$, while $\overline{\mathcal{R}(S_T)}$ is the maximum invariant subspace for T^* on which T^* is of class $C_{\cdot 1}$. This latter means that the contraction $PT|_{\overline{\mathcal{R}(S_T)}}$ is of class $C_{1\cdot}$, where P is the orthogonal projection onto $\overline{\mathcal{R}(S_T)}$. The canonical triangulation is not conserved by the Harnack domination, more precisely $\mathcal{N}(S_T) \neq \mathcal{N}(S_{T'})$ in general, when $T \overset{\mathrm{H}}{\prec} T'$. However, the classes $C_{0\cdot}$, $C_{\cdot 0}$ and C_{00} are preserved in both senses by the Harnack domination (see [2, Theorem 5.5]). On the other hand, the class C_{11} (hence $C_{1\cdot}$ and $C_{\cdot 1}$) are not preserved by the Harnack equivalence (see [2, Propositions 5.9 and 5.10]).

In the following result we show that under the condition of commutation $TT' = T'T$ the $C_{1\cdot}$-part in the canonical triangulation can be fixed.

Theorem 2.1. *Let T and T' be commuting contractions on $\mathcal{H}$ such that T is Harnack dominated by T'. The following assertions hold :*

(i) If T or T' is of class $C_{1\cdot}$ (or of class $C_{\cdot 1}$) then $T = T'$.

(ii) The canonical triangulations of T and T' have the form

$$T = \begin{pmatrix} Q & R \\ 0 & W \end{pmatrix}, \quad T' = \begin{pmatrix} Q' & R' \\ 0 & W \end{pmatrix}, \tag{2.1}$$

where Q, Q' are of class $C_{0\cdot}$ on $\mathcal{N}(S_T) = \mathcal{N}(S_{T'})$ with $QQ' = Q'Q$ and $Q \overset{\mathrm{H}}{\prec} Q'$, W is of class $C_{1\cdot}$ on $\overline{\mathcal{R}(S_T)}$ such that $QR' - Q'R = (R' - R)W$.

(iii) $S_T = S_T^2$ if and only if $S_{T'} = S_{T'}^2$. In this case $S_T = S_{T'}$, W is an isometry and $R = R' = 0$; in addition T and T' are Harnack equivalent if and only if $Q' \overset{\mathrm{H}}{\prec} Q$.

Proof. Assume $TT' = T'T$ and that $T \overset{\mathrm{H}}{\prec} T'$. It is known from [15] or [8] that there exists an operator $B \in \mathcal{B}(l^2_{\mathbb{N}}(\mathcal{D}_{T'}), \mathcal{H})$ such that

$$T = T' + BJ'D_{T'}, \quad BS' = TB,$$

where J' is the canonical inclusion of $\mathcal{D}_{T'}$ into $l^2_{\mathbb{N}}(\mathcal{D}_{T'})$ while S' is the unilateral shift on $l^2_{\mathbb{N}}(\mathcal{D}_{T'})$. Then multiplying the first relation by T^{n-1} for $n \geq 1$ we infer

$$T^n = T^{n-1}T' + BS'^{(n-1)}J'D_{T'},$$

or equivalently

$$T^{*n} = T'^*T^{*(n-1)} + D_{T'}P_{\mathcal{H}}S'^{*(n-1)}B^*.$$

Since $S'^{*n} \to 0$ strongly we have $(T^* - T'^*)T^{*(n-1)} \to 0$ strongly on $\mathcal{H}$, and using that T and T' commute we get $S_{T^*}(T^* - T'^*) = 0$. Therefore, if T is of class $C_{\cdot 1}$, that is $\mathcal{N}(S_{T^*}) = \{0\}$, then $T = T'$. Similarly, the same conclusion holds if T is of class $C_{1\cdot}$ because $T^* \overset{\mathrm{H}}{\prec} T'^*$. When T' is of class $C_{1\cdot}$ (or $C_{\cdot 1}$) the conclusion $T = T'$ follows from the assertion (ii) of theorem proved below, because in this case $T' = W = T$ on $\mathcal{H} = \overline{\mathcal{R}(S_{T'})} = \overline{\mathcal{R}(S_T)}$. The assertion (i) is proved.

To show (ii), we first notice that $\mathcal{N}(S_T)$ is invariant for both T and T' (by the relation $TT' = T'T$). So, T and T' have triangulations on $\mathcal{H} = \mathcal{N}(S_T) \oplus \overline{\mathcal{R}(S_T)}$ of the form

$$T = \begin{pmatrix} Q & R \\ 0 & W \end{pmatrix}, \quad T' = \begin{pmatrix} Q' & R' \\ 0 & W' \end{pmatrix},$$

where $QQ' = Q'Q$, $WW' = W'W$ and $QR' + RW' = Q'R + R'W$. In addition, since the joint (closed) invariant subspaces conserve the Harnack domination, one has $Q \overset{\mathrm{H}}{\prec} Q'$ and $W^* \overset{\mathrm{H}}{\prec} W'^*$, hence $W \overset{\mathrm{H}}{\prec} W'$. Also, Q' is of class $C_{0\cdot}$ because Q is such (see [2, Theorem 5.5]). Therefore $\mathcal{N}(S_T) \subset \mathcal{N}(S_{T'})$. But $W = W'$ by (i), because W is of class $C_{1\cdot}$. This forces the equality $\mathcal{N}(S_T) = \mathcal{N}(S_{T'})$, so $\overline{\mathcal{R}(S_T)} = \overline{\mathcal{R}(S_{T'})}$. Hence the above matrix of T' is just its canonical triangulation like that of T.

For (iii) let us assume that $S_T = S_T^2$ that is S_T is an orthogonal projection onto $\mathcal{R}(S_T) = \mathcal{N}(I - S_T)$, which becomes a reducing subspace for T and T'. So $R = R' = 0$ and $T = Q \oplus W$, $T' = Q' \oplus W$, while W is an isometry on $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T'})$ because Q' is of class $C_{0\cdot}$ and $\mathcal{N}(I - S_{T'}) \subset \mathcal{R}(S_{T'})$ is the maximum invariant subspace for T' on which it is an isometry. Hence $S_{T'} = S_{T'}^2 = S_T$.

Conversely, we suppose that $S_{T'} = S_{T'}^2$. Then by (ii) we have that $W = T'|_{\mathcal{R}(S_T)}$ is an isometry. Therefore $\mathcal{N}(I - S_T) \subset \overline{\mathcal{R}(S_T)} = \mathcal{N}(I - S_{T'})$. As the reverse inclusion also holds (we use the relation $T \overset{\mathrm{H}}{\prec} T'$ and [2, Lemma 5.1]) we obtain $\mathcal{N}(I - S_T) = \mathcal{R}(S_T) = \mathcal{N}(I - S_{T'})$. We conclude that $S_T = S_T^2$ and, in this case, the relation of Harnack equivalence between T and T' reduces (by (ii)) to the relation $T' \overset{\mathrm{H}}{\prec} T$. This ends the proof. □

Remark 2.2. The matrix representation of T' in the above assertion (ii) was inferred from the canonical triangulation of T on $\mathcal{H} = \mathcal{N}(S_T) \oplus \overline{\mathcal{R}(S_T)}$. The same representation for T can be obtained by considering the canonical triangulation of T' on $\mathcal{H} = \mathcal{N}(S_{T'}) \oplus \overline{\mathcal{R}(S_{T'})}$ (as in (ii)). Indeed, in this case $\mathcal{N}(S_{T'})$ is invariant for T, and using the above assertion (i), T will have a matrix representation of the form quoted in (ii) under the last orthogonal decomposition of $\mathcal{H}$. As $Q \overset{\mathrm{H}}{\prec} Q'$ and Q' is of class $C_{0\cdot}$, Q will be also of class $C_{0\cdot}$ on $\mathcal{N}(S_{T'})$ (see [2, Theorem 5.5]). Therefore $\mathcal{N}(S_{T'}) \subset \mathcal{N}(S_T)$. But $T^* = T'^* = W^*$ on $\overline{\mathcal{R}(S_{T'})} \supset \overline{\mathcal{R}(S_T)}$ and W is of class $C_{1\cdot}$, so $\mathcal{N}(S_T) = \mathcal{N}(S_{T'})$. Hence the representation of T in (ii) is just its canonical triangulation.

Corollary 2.3. *Let T and T' be commuting contractions on $\mathcal{H}$ such that T is Harnack dominated by T'. Then $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T'})$ if and only if $\mathcal{N}(I - S_T)$ is invariant for T' (which occurs, in particular, when T' and S_T commute).*

Proof. As we have already mentioned, the inclusion $\mathcal{N}(I - S_{T'}) \subset \mathcal{N}(I - S_T)$ always holds and $T = T'$ on $\mathcal{N}(I - S_{T'})$. Now if $\mathcal{N}(I - S_T)$ is invariant for T' then the relation $T \overset{\mathrm{H}}{\prec} T'$ also works on this subspace, while by the assertion (i) we have $T = T'$ on $\mathcal{N}(I - S_T)$. So, T' is an isometry on this subspace, which imposes $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T'})$. The other implication is obvious. □

This corollary gives conditions under which the invariant isometric part of a contraction is preserved by Harnack domination. In some cases even the reducing isometric, or unitary parts can be conserved by Harnack domination, as follows from the assertion (iii) of Theorem 2.1. In this case the corresponding asymptotic limits are orthogonal projections.

If $T \overset{\mathrm{H}}{\prec} T'$ then we always have the inclusion $\mathcal{H}'_u \subset \mathcal{H}_u$ between the reducing unitary parts. However, in general, there is no relationship between the reducing isometric parts $\mathcal{H}_1$ and $\mathcal{H}'_1$. The following result examines the case of commuting contractions.

Theorem 2.4. *Let T and T' be commuting contractions on $\mathcal{H}$ such that T is Harnack dominated by T', and let $\mathcal{H}_1$ and $\mathcal{H}'_1$ be as above. Then :*

(i) $\mathcal{H}'_1 \subset \mathcal{H}_1$ and $T'_1 = T_1$ on $\mathcal{H}'_1$.

(ii) $\mathcal{H}'_1$ reduces T if $\mathcal{H}_1 \subset \mathcal{N}(D_{T'})$.

In addition, if $\mathcal{N}(I-S_{T'})$ reduces T' then $\mathcal{H}_1=\mathcal{H}_1'=\mathcal{N}(I-S_{T'})$.

(iii) If the Harnack operator associated to pair (T,T') is injective (i.e., one-to-one), then $\mathcal{H}_1'$ reduces T.

(iv) If T and T' are Harnack equivalent then $\mathcal{H}_1=\mathcal{H}_1'$.

Proof. To see the inclusion in (i) we use the above descriptions for $\mathcal{H}_1$ and $\mathcal{H}_1'$ with respect to the minimal isometric dilations V on $\mathcal{K}$ and V' on $\mathcal{K}'$ of T and T', respectively. Since

$$\mathcal{H}_1'\subset\mathcal{N}(I-S_{T'})\subset\mathcal{N}(I-S_T)\cap\mathcal{N}(D_{T'})$$

we have $T=T'$ on $\mathcal{H}_1'$. Therefore $\mathcal{H}_1'$ is invariant for T and T is an isometry on $\mathcal{H}_1'$.

Now let $h\in\mathcal{H}_1'$. Then $V'^nT'^{*m}h\in\mathcal{H}$ for $m,n\in\mathbb{N}$. Let also $A\in\mathcal{B}(\mathcal{K}',\mathcal{K})$ be the Harnack operator associated to (T,T'), that is satisfying $AV'=VA$ with $A|_{\mathcal{H}}=I_{\mathcal{H}}$. Since $TT'=T'T$ we have

$$\begin{aligned}V^nT^{*m}h &= V^nT^{*m}T'^{*m}T'^mh=V^nAT'^{*m}T^{*m}T^mh\\ &= AV'^nT'^{*m}h=V'^nT'^{*m}h\in\mathcal{H}.\end{aligned}$$

Hence $h\in\mathcal{H}_1$ and so $\mathcal{H}_1'\subset\mathcal{H}_1$. So (i) is proved and this also implies (iv).

Suppose now that $\mathcal{H}_1\subset\mathcal{N}(D_{T'})$. Then $T=T'$ on $\mathcal{H}_1$, while for $h\in\mathcal{H}_1'$ and $k\in\mathcal{H}_1\ominus\mathcal{H}_1'$ we have $T^*h\in\mathcal{H}_1$, $T'^*h\in\mathcal{H}_1'$ and

$$\langle T^*h,k\rangle=\langle h,T'k\rangle=\langle T'^*h,k\rangle=0.$$

So $T^*h\in\mathcal{H}_1'$ if $h\in\mathcal{H}_1'$, and consequently $\mathcal{H}_1'$ reduces T. In addition, since $T'=T$ is an isometry on $\mathcal{H}_1$ it follows that $\mathcal{H}_1\subset\mathcal{N}(I-S_{T'})$. Therefore, if $\mathcal{N}(I-S_{T'})$ reduces T', then one obtains $\mathcal{H}_1'=\mathcal{H}_1=\mathcal{N}(I-S_{T'})$. Thus, the statements in (ii) are proved.

For (iii) we assume that the above operator A is injective. For $h\in\mathcal{H}_1'$ and $m,n\in\mathbb{N}$ we have

$$AV'^nT'^{*m}T^*h=V^nT^*T'^{*m}h=:h_{n,m}\in\mathcal{H}.$$

We have used here that $T'^*h\in\mathcal{H}_1'\subset\mathcal{H}_1$. As $h_{n,m}=Ah_{n,m}$ it follows that $V'^nT'^{*m}T^*h=h_{n,m}$, hence $T^*h\in\mathcal{H}_1'$. Therefore $\mathcal{H}_1'$ is invariant for T^*. As $T\mathcal{H}_1'=T'\mathcal{H}_1'\subset\mathcal{H}_1'$, it follows that $\mathcal{H}_1'$ reduces T. The statement (iii) is proved, and this ends the proof. □

Remark 2.5. If T is an isometry and $T\overset{\mathrm{H}}{\prec}T'$ with $T'\neq T$ then certainly (by Theorem 2.1 (i)) one has $TT'\neq T'T$. But even under this latter condition the requirement that the Harnack operator A associated to pair (T,T') is injective (as in the above condition (iii)) is possible only when T is not an isometry. Indeed, if T is an isometry, then $A\mathcal{K}'=\mathcal{H}$ and the injectivity of A implies $\mathcal{K}'=\mathcal{K}$ (since $Ah=h$, $h\in\mathcal{H}$); so T' is an isometry and $T=T'$.

Remark 2.6. Suppose that $T\overset{\mathrm{H}}{\prec}T'$. It is easy to see that if $\mathcal{H}_1=\mathcal{H}_1'$ then $\mathcal{H}_u=\mathcal{H}_u'$, but the converse implication is not true, in general. In fact, it was proved in [2, Example 6.2 and Proposition 6.3] that there exists a Harnack part which contains a completely non-isometric contraction and another completely non-unitary contraction which has non-trivial reducing isometric part. Moreover, such contractions can Harnack dominate a unitary operator (in this case the right bilateral shift on $l^2_{\mathbb{Z}}(\mathcal{E})$ for a Hilbert space $\mathcal{E}$). Hence, in general one has

$\mathcal{H}'_u \subset \mathcal{H}_u$ under the Harnack domination $T \overset{\mathrm{H}}{\prec} T'$ but $\mathcal{H}'_1 \nsubseteq \mathcal{H}_1$ even under the Harnack equivalence of T and T'.

Notice that even if $TT' = T'T$ with $T \overset{\mathrm{H}}{\prec} T'$ it is possible that $\mathcal{H}_u = \mathcal{H}'_u$ and $\mathcal{H}'_1 \nsubseteq \mathcal{H}_1$, for example when T is a hyponormal contraction. Indeed, it is known (see [7]) that in this case $S_{T^*} = S^2_{T^*}$, therefore $\mathcal{N}(I - S_{T^*}) \subset \mathcal{N}(I - S_T)$ and by Theorem 2.1 (iii), or Theorem 2.4 (ii), we have

$$\mathcal{H}_u = \mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_{T'^*}) = \mathcal{H}'_u.$$

Also, since $\mathcal{N}(D_{T^*}) \subset \mathcal{N}(D_T)$ this latter subspace is invariant for T, so $\mathcal{N}(D_T) = \mathcal{N}(I - S_T)$. On the other hand, by the condition $T \overset{\mathrm{H}}{\prec} T'$ we have $\mathcal{N}(D_{T'}) \subset \mathcal{N}(D_T)$, while the equality between the kernels gives $T = T'$ on $\mathcal{N}(D_T)$. In this last case T' is an isometry on $\mathcal{N}(D_T)$, which shows that the following chain of equalities hold

$$\mathcal{N}(I - S_{T'}) = \mathcal{N}(D_{T'}) = \mathcal{N}(D_T) = \mathcal{N}(I - S_T),$$

if the central one is true. This subspace is even the invariant isometric part in $\mathcal{H}$ of both T and T', and this fact happens if and only if $\mathcal{N}(D_T)$ is invariant for T'. In addition, if $\mathcal{N}(D_T)$ reduces T' then the above joint subspace is even the reducing isometric part in $\mathcal{H}$ of T' and, consequently, of T.

In the case when $\mathcal{N}(D_T)$ is invariant for T' and T^* it follows that $\mathcal{N}(D_T) = \mathcal{N}(D_{T'})$ reduces T to an isometry. So, for $h \in \mathcal{N}(D_T)$ we have $h = T'^*T'h = T'^*Th$ which gives

$$T'^*h = T'^*T^*Th = T^*T'^*Th = T^*h \in \mathcal{N}(D_T).$$

From this one infers that $\mathcal{N}(D_T)$ also reduces T', hence T and T' have the same reducing isometric part in $\mathcal{H}$ (by the previous remark).

We summarize these facts in the following

Proposition 2.7. *Let T and T' be commuting contractions on $\mathcal{H}$ such that T is hyponormal and it is Harnack dominated by T'. Then they have the same unitary part. In addition, the following are equivalent :*

(i) $\mathcal{N}(D_T)$ is invariant for T';

(ii) $\mathcal{N}(D_T) = \mathcal{N}(D_{T'})$;

(iii) T and T' have the same invariant isometric part in $\mathcal{H}$.

Furthermore, if one of these assertions holds then T and T' have the same reducing isometric part in $\mathcal{H}$ (which is even their joint invariant isometric part) if and only if $\mathcal{N}(D_T)$ reduces either T' or T.

Notice that if the above condition (ii) is satisfied then by Theorem 2.4 (ii) the subspace $\mathcal{H}'_1$ reduces T. However, in general, $\mathcal{H}'_1 \neq \mathcal{H}_1$ when $T^*(\mathcal{N}(D_T)) \nsubseteq \mathcal{N}(D_T)$ and $T'^*(\mathcal{N}(D_T)) \nsubseteq \mathcal{N}(D_T)$.

Remark 2.8. The canonical triangulations of T and T' given by Theorem 2.1 (ii) can be refined by considering the dual triangulations for their C_0.-parts Q and Q', respectively.

Indeed, as $Q \overset{\mathrm{H}}{\prec} Q'$ and $QQ' = Q'Q$ we obtain by Theorem 2.1 the matrix representations

$$Q = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}, \quad Q' = \begin{pmatrix} X & Y' \\ 0 & Z' \end{pmatrix}$$

on $\mathcal{N}(S_T) = \overline{\mathcal{R}(S_{Q^*})} \oplus \mathcal{N}(S_{Q^*})$, where X is of class C_{01}, Z and Z' are of class C_{00}, $ZZ' = Z'Z$ and $Z \overset{\mathrm{H}}{\prec} Z'$. We conclude that in the case of Harnack domination of two commuting contractions their corresponding C_{01}, C_{11} and C_{10} parts coincide. Also, the Harnack domination is preserved (in the same order) for their C_{00}-parts.

In the particular case when $S_T = S_T^2$ and $S_{Q^*} = S_{Q^*}^2$ we have (by Theorem 2.1) that $T = X \oplus Z \oplus W$, $T' = X \oplus Z' \oplus W$ under the decomposition $\mathcal{H} = \mathcal{N}(I - S_{Q^*}) \oplus \mathcal{N}(S_{Q^*}) \oplus \mathcal{N}(I - S_T)$, where X is a *coshift* (i.e. the adjoint of a shift), W is an isometry and Z, Z' are as above. The same representations for T and T' are obtained if $S_{T'} = S_{T'}^2$ and $S_{Q'^*} = S_{Q'^*}^2$.

Let us remark that in the latter cases the contractions T and T' differ only by their C_{00} parts Z and Z'. But if the defect spaces $\mathcal{D}_Z$ and $\mathcal{D}_{Z^*}$ are one dimensional, then it is known [23, vi.3, Corollary 3.7] that $Z' = f(Z)$ for some contractive analytic function f on $\mathbb{D}$. So, in this case any contraction T' which commutes with T and Harnack dominates T can be intrinsically expressed by T. In particular, it follows that in the Harnack part of a contraction T of class C_{00} with the defect space one dimensional, those contractions which commute with T are of the form $f(T)$ for some $f \in H^\infty$, $\|f\|_\infty \leq 1$. On the other hand, not any contraction $g(T)$ with $g \in H^\infty$ is Harnack equivalent with T. More precisely, the contraction T as above is even of class C_0 (see [23, vi.5, Theorem 5.2]), that is $u(T) = 0$ for some $u \in H^\infty$. Hence if $\sigma(T) \cap \mathbb{T} \neq \varnothing$ then T is not Harnack dominated by $u(T) = 0$.

Another special case is when the contraction T is quasi-normal. Recall here that this means that T commutes with T^*T. Then it is well-known that $S_T = S_T^2$ and $S_{T^*} = S_{T^*}^2$ (see [19], [20]), so $\mathcal{N}(I - S_{T^*})$ and $\mathcal{N}(I - S_T)$ are unitary and the reducing isometric parts in $\mathcal{H}$ for T and T', respectively. Hence $S_{Q^*} = 0$, and from what was remarked before and by Proposition 2.7 we obtain the following result.

Corollary 2.9. *Let T, T' be commuting contractions on $\mathcal{H}$ such that one of these is quasi-normal and T is Harnack dominated by T'. Then $T = Z \oplus W$ and $T' = Z' \oplus W$ on $\mathcal{H} = \mathcal{N}(S_T) \oplus \mathcal{N}(I - S_T)$, where Z, Z' are commuting contractions of class C_{00} on $\mathcal{N}(S_T) = \mathcal{N}(S_{T'})$ and $Z \overset{\mathrm{H}}{\prec} Z'$, while W is an isometry on the subspace*

$$\mathcal{N}(I - S_T) = \mathcal{N}(D_T) = \mathcal{N}(D_{T'}) = \mathcal{N}(I - S_{T'}).$$

Concerning the Harnack domination of a quasi-normal contraction we have the following main result.

Theorem 2.10. *Let T, T' be contractions on $\mathcal{H}$ such that T^* is quasi-normal and T commutes with T' and T'^*T', while T' commutes with TT^*. The following are equivalent :*

(i) T is Harnack dominated by T';

(ii) T is Harnack equivalent to T';

(iii) There exists an operator $B \in \mathcal{B}(l^2_{\mathbb{N}}(\mathcal{D}_{T'}), \mathcal{H})$ *such that*

$$T = T' + BJ'D_{T'}, \quad BS' = TB, \tag{2.2}$$

where J' *is the canonical inclusion of* $\mathcal{D}_{T'}$ *into* $l^2_{\mathbb{N}}(\mathcal{D}_{T'})$ *and* S' *is the natural shift on* $l^2_{\mathbb{N}}(\mathcal{D}_{T'})$*;*

(iv) T *is Shmul'yan equivalent to* T'*, that is, there exists an operator* $W \in \mathcal{B}(\mathcal{D}_{T'}, \mathcal{D}_{T^*})$ *such that*

$$T = T' + D_{T^*}WD_{T'}; \tag{2.3}$$

(v) There exists $z_0 \in \mathbb{D}$ *and a* $\mathcal{B}(\mathcal{H})$*-valued contractive analytic function* F *on* $\mathbb{D}$ *such that* $F(0) = T$ *and* $F(z_0) = T'$*.*

Proof. Let V and V' be the minimal isometric dilations on $\mathcal{K} = \mathcal{H} \oplus l^2_{\mathbb{N}}(\mathcal{D}_T)$ and $\mathcal{K}' = \mathcal{H} \oplus l^2_{\mathbb{N}}(\mathcal{D}_{T'})$ of T and T', respectively. Then the matrix representations of V and V' are given by

$$V = \begin{pmatrix} T & 0 \\ JD_T & S \end{pmatrix}, \quad V' = \begin{pmatrix} T' & 0 \\ J'D_{T'} & S' \end{pmatrix}, \tag{2.4}$$

where J and S have the same meaning like J' and S' of above, respectively. Let A be the Harnack operator associated to T and T'. It is known [15], [8] that $A|_{\mathcal{H}} = I_{\mathcal{H}}$ and $AV' = VA$. This last equality yields

$$P_{\mathcal{H}}AV' = P_{\mathcal{H}}VP_{\mathcal{H}}A = TP_{\mathcal{H}}A$$

because $V(\mathcal{K} \ominus \mathcal{H}) \subset \mathcal{K} \ominus \mathcal{H}$. On the other hand, since $V'h = T'h \oplus J'D_{T'}h$ for $h \in \mathcal{H}$ we get

$$Th = P_{\mathcal{H}}VAh = P_{\mathcal{H}}AV'h = T'h + P_{\mathcal{H}}AJ'D_{T'}h.$$

Therefore the operator $B := P_{\mathcal{H}}A|_{l^2_{\mathbb{N}}(\mathcal{D}_{T'})}$ satisfies the identities of (iii) when $T \overset{\mathrm{H}}{\prec} T'$. This shows that (i) implies (iii).

Next, the relations from (iii) give a representation of B with respect to the operator $B_0 = BJ' \in \mathcal{B}(\mathcal{D}_{T'}, \mathcal{H})$ in the form

$$\begin{aligned} Bd &= B\sum_{n=0}^{\infty} S'^n J'D_{T'}h_n = B_0D_{T'}h_0 + \sum_{n=1}^{\infty} T^n B_0 D_{T'}h_n \\ &= [B_0, TB_0, T^2B_0, ...]d, \end{aligned}$$

for $d = \{D_{T'}h_n\}_{n\in\mathbb{N}} \in l^2_{\mathbb{N}}(\mathcal{D}_{T'})$ with $h_n \in \mathcal{H}$. So, one can consider the positive operator $Z \in \mathcal{B}(\mathcal{H})$ given by

$$Z = \sum_{n=0}^{\infty} T^n B_0 B_0^* T^{*n}.$$

Since by hypothesis T commutes with T'^*T' it follows that $\mathcal{D}_{T'}$ reduces T, and we set $T_1 = T|_{\mathcal{D}_{T'}}$. As T and T' commute we have

$$TB_0D_{T'} = T(T - T') = (T - T')T = B_0D_{T'}T = B_0T_1D_{T'},$$

that is $TB_0 = B_0T_1$. Because T^* is quasi-normal, T_1^* will be quasi-normal too. Furthermore, since T' commutes with TT^* one can see, as above, that $TT^*B_0 = B_0T_1T_1^*$. So one obtains

$$\begin{aligned} Z &= B_0B_0^* + \sum_{n=1}^{\infty} T^n B_0 B_0^* T^{*n} = B_0B_0^* + TT^*B_0B_0^* + \sum_{n=2}^{\infty} B_0 T^n T^{*n} B_0^* \\ &= B_0B_0^* + TT^*B_0B_0^* + \sum_{n=2}^{\infty} B_0 (TT^*)^n B_0^* \\ &= B_0B_0^* + TT^*B_0B_0^* + \sum_{n=2}^{\infty} TT^*T^{n-1} B_0 B_0^* T^{*(n-1)} \\ &= B_0B_0^* + TT^*Z. \end{aligned}$$

This shows that $B_0B_0^* = D_{T^*}^2 Z = D_{T^*} Z D_{T^*}$ which implies that $B_0 = D_{T^*}W$ for some operator $W \in \mathcal{B}(\mathcal{D}_{T'}, \mathcal{D}_{T^*})$. This relation, together with the first identity from (iii), lead to the identity from (iv). Hence (iii) implies (iv).

To prove that (iv) implies (v) we firstly deduce from (iv) that $T^*T = T^*T' + T^*D_{T^*}WD_{T'}$. Thus

$$\begin{aligned} D_T^2 &= I - T^*T' - T^*D_{T^*}WD_{T'} = \mathrm{Re}(I - T^*T') - \mathrm{Re}(T^*D_{T^*}WD_{T'}) \\ &= \frac{1}{2}(D_T^2 + D_{T'}^2 + (T-T')^*(T-T')) - \mathrm{Re}(D_T T^* W D_{T'}) \\ &\geq \frac{1}{2}(D_T^2 + D_{T'}^2) - \mathrm{Re}(D_{T'}W^*TD_T). \end{aligned}$$

So, for $h \in \mathcal{H}$ with $\|h\| = 1$ we have

$$\|D_{T'}h\| \leq \|D_Th\|^2 + 2\|W\|\|D_Th\| \leq (1 + 2\|W\|)\|D_Th\|$$

which gives $D_{T'} = YD_T$ for some operator $Y \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T'})$.

On the other hand, from the identity (2.3) for the pair (T, T') one obtains a similar identity for the pair (T'^*, T^*), therefore as above one has $D_{T^*} = Y_*D_{T'^*}$ for some operator $Y_* \in \mathcal{B}(\mathcal{D}_{T'^*}, \mathcal{D}_{T^*})$. This yields $T^* - T'^* = D_{T'}W^*D_{T^*} = D_{T'}W^*Y_*D_{T'^*}$, or $T = T' + D_{T'^*}W^*D_{T'}$, where $W_* = Y_*^*W$. Using this last equality we get

$$\begin{aligned} I - T'^*T &= D_{T'}^2 + T'^*(T' - T) = D_{T'}^2 - T'^*D_{T'^*}W_*D_{T'} \\ &= D_{T'}(I - T'^*W_*)D_{T'} = D_{T'}(I - T'^*W_*)YD_T. \end{aligned}$$

Let $z_0 \in \mathbb{D}$ be such that the operator $X = (1 - |z_0|^2)^{1/2}(I - T'^*W_*)Y$ is a contraction from $\mathcal{D}_T$ into $\mathcal{D}_{T'}$. Then we have

$$I - T^*T' = (1 - |z_0|^2)^{-1/2} D_T X^* D_{T'},$$

which ensures that the following operator matrix on $\mathcal{H} \oplus \mathcal{H}$ is positive definite:

$$\begin{pmatrix} D_T^2 & I - T^*T' \\ I - T'^*T & \frac{D_{T'}^2}{1-|z_0|^2} \end{pmatrix} \geq 0.$$

But this is the classical Pick condition for the operator Nevanlinna-Pick interpolation problem for the data $\{0, z_0\}$ and $\{T, T'\}$, which means that there exists a $\mathcal{B}(\mathcal{H})$-valued contractive

analytic function F on $\mathbb{D}$ such that $F(0) = T$ and $F(z_0) = T'$ (see [5], [16]). So, (iv) implies (v).

It was proved in [17, Theorem 4] (see also Remark 2.12 below) that the range of a contractive analytic function is contained in a Harnack part, hence the contractions T and T' satisfying (v) are Harnack equivalent. Therefore (v) implies (ii), and (ii) obviously implies (i). This ends the proof. □

Because for two contractions T and T' one has $T \overset{\mathrm{H}}{\prec} T'$ if and only if $T^* \overset{\mathrm{H}}{\prec} T'^*$, a dual version of Theorem 2.10 holds when T is quasi-normal and the corresponding commuting relations are satisfied, that is $TT' = T'T$, $TT'T'^* = T'T'^*T$, $T'T^*T = T^*TT'$. In this case the statements of (iii), (iv) and (v) can be also formulated in terms of T^* and T'^* instead of T and T', respectively.

Remark 2.11. Using the same notation as in the above proof, if $T \overset{\mathrm{H}}{\prec} T'$ then the intertwining relation $AV' = VA$ leads to four relations between the entries of the matrices of V and V' in (2.4) and of A, this latter operator having the form

$$A = \begin{pmatrix} I & B \\ 0 & C \end{pmatrix} : \mathcal{H} \oplus l^2_{\mathbb{N}}(\mathcal{D}_{T'}) \to \mathcal{H} \oplus l^2_{\mathbb{N}}(\mathcal{D}_T), \tag{2.5}$$

with B as above and $C \in \mathcal{B}(l^2_{\mathbb{N}}(\mathcal{D}_{T'}), l^2_{\mathbb{N}}(\mathcal{D}_T))$. More precisely, we have the identities (2.2) as well as the following ones:

$$CJ'D_{T'} = JD_T, \quad CS' = SC + JD_TB. \tag{2.6}$$

Theorem 2.10 asserts that only the relations (2.2) are essential in this case and, in addition, by the statement (ii) the operators C and A are invertible.

In particular, from the first relation in (2.6) it follows that $D_T = C_0 D_{T'}$ for some invertible operator $C_0 \in \mathcal{B}(\mathcal{D}_{T'}, \mathcal{D}_T)$. Thus, the identity (2.3) is equivalent in this case with everyone of the following two relations :

$$T = T' + D_{T^*}W_0D_T, \quad T = T' + D_{T'^*}W_1D_{T'}. \tag{2.7}$$

Here $W_0 = WC_0^{-1} \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$, and $W_1 = C_1^*W \in \mathcal{B}(\mathcal{D}_{T'}, \mathcal{D}_{T'^*})$, where as above $C_1 \in \mathcal{B}(\mathcal{D}_{T'^*}, \mathcal{D}_{T^*})$ is an invertible operator such that $D_{T^*} = C_1 D_{T'^*}$. We have used here that $T^* \overset{\mathrm{H}}{\sim} T'^*$ by Theorem 2.10.

Remark 2.12. By [16, Theorem 4.1] and the proof of [17, Theorem 4], any Shmul'yan part is a union of ranges of $\mathcal{B}(\mathcal{H})$-valued contractive analytic functions on $\mathbb{D}$. Thus, Theorem 2.10 asserts that, under some commuting conditions, a contraction T' which Harnack dominates a quasi-normal T belongs together with T to the range of an analytic functions on $\mathbb{D}$, which is contained in the Shmul'yan part of T.

A consequence of Theorem 2.10 is the following result, which is an effective extension of a result from [1] concerning the Harnack domination between two normal contractions, and which was obtained in [1] in a different way.

Corollary 2.13. *Let T and T' be doubly commuting contractions on $\mathcal{H}$ such that T is quasi-normal and it is Harnack dominated by T'. Then T and T' are Shmul'yan equivalent.*

Notice that this corollary is not true if the role of T and T' is interchanged, that is when T' is quasi-normal and T is not. For example, the null contraction Harnack dominates any contraction T with $\sigma(T) \subset \mathbb{D}$ and $\|T\| = 1$, but these are not Harnack equivalent (see [4]).

3. Non-stable contractions and partial isometries in Harnack domination and Shmul'yan equivalence

We have seen in Theorem 2.1 that a contraction of class $C_{1\cdot}$ (or $C_{\cdot 1}$, C_{11}) cannot be in Harnack domination with another, different, commuting contraction. Without the commutation condition even unitary operators can be Harnack dominated by completely nonunitary contractions which do not belong to the classes $C_{1\cdot}$ or $C_{\cdot 1}$. Also, the Harnack part of some contractions of class C_{11} can contain contractions which are not of the same class, for example partial isometries T with S_T a non-trivial orthogonal projection (see [2, Example 6.5]). This also shows that the class of contractions whose asymptotic limit is an orthogonal projection is not preserved by the Harnack equivalence, in general.

We present now other general facts about $C_{1\cdot}$-contractions or partial isometries with respect to the Harnack pre-order.

Proposition 3.1. *Let T and T' be contractions on $\mathcal{H}$ such that T is Harnack dominated by T'. The following statements hold :*

(i) If T' is of class $C_{1\cdot}$ then either T is injective, or $\mathcal{N}(T)$ is not invariant for T'.

(ii) If $S_{T'}$ is an orthogonal projection then T and T' have matrix representations on $\mathcal{H} = \mathcal{N}(I - S_{T'}) \oplus \mathcal{N}(S_{T'})$ of the form

$$T = \begin{pmatrix} V & R \\ 0 & Q \end{pmatrix}, \quad T' = V \oplus Q', \tag{3.1}$$

where V is an isometry with $V^*R = 0$ and Q, Q' are of class $C_{0\cdot}$ such that $Q \overset{\mathrm{H}}{\prec} Q'$. In addition, $\mathcal{N}(S_{T'})$ is invariant for T if and only if $S_T = S_{T'}$. In particular, this happens when V is unitary.

Proof. (i). Assume that T' is a $C_{1\cdot}$-contraction, and that $\mathcal{N}(T)$ is invariant for T'. Since $T \overset{\mathrm{H}}{\prec} T'$ there exists by [1, Lemma 1] a constant $c \geq 1$ such that $\|T' - \frac{1}{c}T\| \leq 1 - \frac{1}{c}$. So, for $h \in \mathcal{N}(T)$ we have $\|T'h\| \leq \frac{c-1}{c}\|h\|$, and by recurrence (as $T'h \in \mathcal{N}(T)$) one obtains

$$\|T'^n h\| \leq \left(\frac{c-1}{c}\right)^n \|h\|$$

for every integer $n \geq 1$. This yields $S_{T'}h = 0$ hence $h = 0$, because $\mathcal{N}(S_{T'}) = \{0\}$. Therefore $\mathcal{N}(T)$ it is not invariant for T' whenever $\mathcal{N}(T) \neq \{0\}$. This proves (i).

(ii). Assume that $S_{T'} = S_{T'}^2$. So $T' = V \oplus Q'$ on $\mathcal{H} = \mathcal{H}' \oplus \mathcal{N}(S_{T'})$, where V is an isometry on $\mathcal{H}' = \mathcal{N}(I - S_{T'})$ and Q' of class $C_{0\cdot}$ on $\mathcal{N}(S_{T'})$. Since $T \overset{\mathrm{H}}{\prec} T'$ we have $T = T' = V$ on

$\mathcal{H}'$, therefore $\mathcal{H}'$ is invariant for T. In addition, denoting $Q^* = T^*|_{\mathcal{N}(S_{T'})}$ we have $Q \overset{\mathrm{H}}{\prec} Q'$ and Q like Q' are of class $C_{0\cdot}$ [2, Theorem 5.5]. Hence T can be represented as in (3.1), the condition $V^*R = 0$ being assured by the fact that V is an isometry and $T^*T \leq I$.

The invariance of $\mathcal{N}(S_{T'})$ to T means $R = 0$ in (3.1). In this case $T = V \oplus Q$, which yields $S_T = I \oplus 0 = S_{T'}$. The converse assertion is trivial, that is $S_T = S_{T'}$ implies $\mathcal{N}(S_{T'}) = \mathcal{N}(S_T)$ and it is invariant for T. When V is unitary we have $R = 0$ from $V^*R = 0$. This ends the proof. □

Remark 3.2. Let T, T' be as in Proposition 3.1 (ii) such that T'^* is injective (in particular of class $C_{1\cdot}$). Then the isometry V in (3.1) is unitary. In general, when V is unitary and $T \neq T'$ then T cannot be of class $C_{1\cdot}$. But if V is a shift and $Q = Q'$ is a coshift then it is possible that $T \neq T'$ (that is $R \neq 0$ in (3.1)), T is of class C_{11}, and T, T' are in the same Harnack part [2, Example 6.5 and Proposition 6.6]. This fact also shows that the assumption of injectivity of T'^* (which holds in particular when T' is of class $C_{\cdot 1}$) is essential to obtain a unitary operator V.

We summarize these facts in the following

Corollary 3.3. *If T is a nonunitary contraction of class $C_{1\cdot}$ then $S_{T'} \neq S_{T'}^2$ for every contraction T' which Harnack dominates T such that T'^* is injective.*

In the context of hyponormality one obtains the following corollary.

Corollary 3.4. *If T is a contraction of class $C_{1\cdot}$ (or $C_{\cdot 1}$) which is Harnack dominated by a cohyponormal (hyponormal) contraction T' then $T = T'$ and it is unitary.*

Proof. When T' is cohyponormal one can apply the last assertion of (ii) from above, because $S_{T'} = S_{T'}^2$ and T' is unitary on $\mathcal{N}(I - S_{T'})$. The case when T is hyponormal follows then by duality. □

Corollary 3.5. *The Harnack part of a nonunitary hyponormal contraction $T = U \oplus Q$ on $\mathcal{H} = \mathcal{N}(I - S_{T^*}) \oplus \mathcal{N}(S_{T^*})$ where U is unitary and Q is of class $C_{\cdot 0}$ consists of all contractions $T' = U \oplus Q'$ where Q' is in the Harnack part of Q.*

In this corollary T' cannot be of class $C_{\cdot 1}$ and $S_{T'} \neq S_{T'}^2$, in general. But the Harnack part of a hyponormal contraction can contain only contractions which either are of class C_{10}, or have the asymptotic limits orthogonal projections, as we shall see below.

Example 3.6. Consider, for $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, the weighted shift $T(\lambda)$ on $l^2_{\mathbb{N}}(\mathcal{E})$, $\mathcal{E}$ being a complex Hilbert space, given by

$$T(\lambda)(e_0, e_1, ...) = (0, \lambda e_0, e_1, ...)$$

for $e = (e_n)_{n \in \mathbb{N}} \in l^2_{\mathbb{N}}(\mathcal{E})$. Then its adjoint is given by $T(\lambda)^* e = (\overline{\lambda} e_1, e_2, ...)$, and for $n \geq 1$ we have

$$T(\lambda)^{*n} T(\lambda)^n e = (|\lambda|^2 e_0, e_1, ...) = S_{T(\lambda)} e.$$

Clearly, $T(\lambda)$ is hyponormal, and for $\lambda \neq 0$, $S_{T(\lambda)}$ is invertible. Therefore $T(\lambda)$ is of class $C_{1\cdot}$ and $T(\lambda)^*$ is of class $C_{0\cdot}$. Moreover, $T(\lambda)$ are all Harnack equivalent for $|\lambda| < 1$, which follows from Theorem 3.7 below. Also, $T(\lambda)$ Harnack dominates the isometries $T(\mu)$ with $|\mu| = 1$ (see [2, Example 6.1]).

On the other hand, $T = T(0)$ has the property that $S_T = S_T^2$. We have

$$\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T(\lambda)}) = \{(e_n)_{n\in\mathbb{N}} \in l^2_{\mathbb{N}}(\mathcal{E}) : e_0 = 0\}$$

and

$$\mathcal{N}(S_T) = \mathcal{N}(T) = \mathcal{N}(I - S_{T(\lambda)})^{\perp} = \{(e_0, 0, ...) : e_0 \in \mathcal{E}\}$$

for $0 < |\lambda| < 1$. In fact, $T = S \oplus 0$ on $l^2_{\mathbb{N}}(\mathcal{E}) = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$ where S is a shift on $\mathcal{N}(I - S_T)$, hence T is quasi-normal. But this orthogonal decomposition of $l^2_{\mathbb{N}}(\mathcal{E})$ is not reducing for the C_{10}-contractions $T(\lambda)$ with $\lambda \neq 0$, even in the case $|\lambda| = 1$ when $S_{T(\lambda)} = I$. Therefore, in Proposition 3.1 (ii) the condition $S_T = S_T^2$ does not imply that $\mathcal{N}(S_{T'})$ is reducing for T, in general.

Now let $T' \neq T(\lambda)$ be another contraction on $l^2_{\mathbb{N}}(\mathcal{E})$ belonging to the Harnack part of T. Then $\mathcal{N}(I - S_{T'}) = \mathcal{N}(I - S_T)$, hence

$$\mathcal{N}(S_{T'}) \subset \mathcal{N}(I - S_{T'})^{\perp} = \mathcal{N}(S_T).$$

So, either $\mathcal{N}(S_{T'}) = \{0\}$ that is T' is of class C_{10} like T, or $\mathcal{N}(S_{T'}) \neq \{0\}$ and then $\mathcal{N}(S_{T'}) \neq \mathcal{N}(S_T)$ when $\dim\mathcal{E} > 1$ and $S_{T'} \neq S_{T'}^2$. But in this case $T'^*|_{\mathcal{N}(S_T)} \overset{\mathrm{H}}{\sim} 0$, therefore

$$\|P_{\mathcal{N}(S_T)} T'|_{\mathcal{N}(S_T)}\| < 1.$$

When $\mathcal{E} = \mathbb{C}$, if $\mathcal{N}(S_{T'}) \neq \{0\}$ then $\mathcal{N}(S_{T'}) = \mathcal{N}(S_T)$, so $S_{T'} = S_{T'}^2$. In this latter case, any contraction T' in the Harnack part of T is either a C_{10}-contraction, or $T' = S \oplus Q'$ with $S = T|_{\mathcal{N}(I-S_T)}$ an isometry and $\|Q'\| < 1$ on $\mathcal{N}(S_T)$, hence with $S_{T'} = S_T$ an orthogonal projection.

Notice also that in this Harnack part only $T = T(0)$ is a quasi-normal partial isometry, the other contractions being either hyponormal (even subnormal) of class C_{10} (as $T(\lambda)$ with $0 < |\lambda| < 1$), or which do not belong to these classes (as T' before).

Let us also notice that the reducing isometric part is not preserved by the Harnack equivalence, in general ($\mathcal{N}(I - S_T)$ being only invariant for $T(\lambda)$ when $\lambda \neq 0$).

In the above example one can easily see that the Harnack part of $T(0)$ coincides with the Shmul'yan part of $T(0)$. This always happens for any Harnack part containing a partial isometry, which follows from a more general result on Shmul'yan parts proved below, showing that Shmul'yan equivalence of two contractions can be characterized in terms of their pure parts. Recall here (see [5, Ch. V]) that T admits a unique decomposition of the form

$$T = U \oplus Q : \mathcal{H} = \mathcal{N}(D_T) \oplus \mathcal{D}_T \to \mathcal{H} = \mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*},$$

where $U \in \mathcal{B}(\mathcal{N}(D_T), \mathcal{N}(D_{T^*}))$ is unitary, and $Q \in \mathcal{N}(\mathcal{D}_T, \mathcal{D}_{T^*})$ is a *pure contraction*, that is $\|Qh\| < \|h\|$ for all nonzero $h \in \mathcal{H}$.

Theorem 3.7. *Let $T, T' \in \mathcal{B}_1(\mathcal{H})$ having the decompositions $T = U \oplus Q$ and $T' = U' \oplus Q'$ into the unitary and pure parts. If $T_0 = 0 \oplus Q$ and $T_0' = 0 \oplus Q'$ then the following are equivalent :*

(i) T and T' are Shmul'yan equivalent in $\mathcal{B}_1(\mathcal{H})$;

(ii) $U = U'$ and T_0 and T_0' are Shmul'yan equivalent in $\mathcal{B}_1(\mathcal{H})$;

(iii) $U = U'$ and Q and Q' are Shmul'yan equivalent in $\mathcal{B}_1(\mathcal{D}_T, \mathcal{D}_{T^*})$.

Proof. Assume that T and T' are Shmul'yan equivalent. Then $\mathcal{D}_T = \mathcal{D}_{T'}$, $\mathcal{N}(D_T) = \mathcal{N}(D_{T'})$, $\mathcal{D}_{T^*} = \mathcal{D}_{T'^*}$, $\mathcal{N}(D_{T^*}) = \mathcal{N}(D_{T'^*})$, and so $U = U'$. Hence the representations $T = U \oplus Q$ and $T' = U \oplus Q'$ act from $\mathcal{H} = \mathcal{N}(D_T) \oplus \mathcal{D}_T$ into $\mathcal{H} = \mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*}$, and the same hold for T_0 and T_0'. Also, $D_T, D_{T_0} \in \mathcal{B}(\mathcal{N}(D_T) \oplus \mathcal{D}_T)$ and $D_{T^*}, D_{T_0^*} \in \mathcal{B}(\mathcal{N}(D_{T^*}), \mathcal{D}_{T^*})$ are given by

$$D_T = 0 \oplus D_Q, \quad D_{T_0} = I \oplus D_Q, \quad D_{T^*} = 0 \oplus D_{Q^*}, \quad D_{T_0^*} = I \oplus D_{Q^*},$$

and similarly for $D_{T'}, D_{T_0'}, D_{T'^*}$ and $D_{T_0'^*}$. Because Q, Q^*, Q' and Q'^* are pure contractions we have

$$\mathcal{D}_Q = \mathcal{D}_{Q'} = \mathcal{D}_T, \quad \mathcal{D}_{Q^*} = \mathcal{D}_{Q'^*} = \mathcal{D}_{T^*},$$

and for the corresponding defect spaces we have

$$\mathcal{D}_{T_0} = \mathcal{D}_{T_0'} = \mathcal{H} = \mathcal{N}(D_T) \oplus \mathcal{D}_T, \quad \mathcal{D}_{T_0^*} = \mathcal{D}_{T_0'^*} = \mathcal{H} = \mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*}.$$

Now by the Shmul'yan domination of T' by T there exists an operator $Z \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ such that $T' = T + D_{T^*} Z D_T$. If we identify $\mathcal{D}_T \simeq \{0\} \oplus \mathcal{D}_T$, $\mathcal{D}_{T^*} \simeq \{0\} \oplus \mathcal{D}_{T^*}$ and D_T, D_{T^*} as above, we infer from the previous relation between T and T' that $Q' = Q + D_{Q^*} Z D_Q$, which means that Q Shmul'yan dominates Q' in $\mathcal{B}_1(\mathcal{D}_T, \mathcal{D}_{T^*})$ with the bounded operator Z from $\mathcal{D}_Q$ into $\mathcal{D}_{Q^*}$. By symmetry one has that Q and Q' are Shmul'yan equivalent, hence (i) implies (iii).

Assuming now (iii) we have $\mathcal{D}_T = \mathcal{D}_Q = \mathcal{D}_{Q'} = \mathcal{D}_{T'}$, $\mathcal{D}_{T^*} = \mathcal{D}_{Q^*} = \mathcal{D}_{Q'^*} = \mathcal{D}_{T'^*}$ (Q, Q' as Q^*, Q'^* being pure), and $Q' = Q + D_{Q^*} Z D_Q$ with $Z \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$. This relation leads to

$$T_0' = T_0 + (0 \oplus D_{Q^*})(0 \oplus Z)(0 \oplus D_Q) = T_0 + D_{T_0^*} Z_0 D_{T_0},$$

where $Z_0 = 0 \oplus Z : \mathcal{D}_{T_0} = \mathcal{N}(D_T) \oplus \mathcal{D}_T \to \mathcal{D}_{T_0^*} = \mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*}$. So T_0 Shmul'yan dominates T_0', and by symmetry they are Shmuly'an equivalent. Hence (iii) implies (ii).

Finally, we assume $U = U'$ and that T_0, T_0' are Shmul'yan equivalent. So $\mathcal{D}_T = \mathcal{D}_{T'}$, $\mathcal{D}_{T^*} = \mathcal{D}_{T'^*}$ and we have $T_0' = T_0 + D_{T_0^*} Z_0 D_{T_0}$ for some operator $Z_0 \in \mathcal{B}(\mathcal{D}_{T_0}, \mathcal{D}_{T_0^*}) = \mathcal{B}(\mathcal{H})$. Then writing the matrix representation of Z_0 in the form

$$Z_0 = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} : \mathcal{N}(D_T) \oplus \mathcal{D}_T \to \mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*},$$

from the previous equality one obtains $Z_1 = 0$, $Z_2 D_Q = 0$, $D_{Q^*} Z_3 = 0$ and $Q' = Q + D_{Q^*} Z_4 D_Q$. As D_Q and D_{Q^*} are injective (Q, Q^* being pure) one has $Z_2 = 0$, $Z_3 = 0$; therefore $Z_0 = 0 \oplus Z_4$. Putting $Z := Z_0|_{\{0\} \oplus \mathcal{D}_T}$ we get $T' = T + D_{T^*} Z D_T$, which shows that T Shmul'yan dominates T'. By symmetry T and T' are Shmul'yan equivalent, which ends the proof. $\square$

A first important application refers to partial isometries.

Proposition 3.8. *The Harnack part $\Delta(T)$ of a partial isometry $T \in \mathcal{B}_1(\mathcal{H})$ reduces to its Shmul'yan part. In the Harnack part $\Delta(T)$ only T is a partial isometry. Moreover, if $T = U \oplus 0$ from $\mathcal{N}(D_T) \oplus \mathcal{N}(T)$ into $\mathcal{N}(D_{T^*}) \oplus \mathcal{N}(T^*)$, where $U \in \mathcal{B}(\mathcal{N}(D_T), \mathcal{N}(D_{T^*}))$ is unitary, then*

$$\begin{aligned} \Delta(T) &= \{T' = U \oplus Z : Z \in \mathcal{B}(\mathcal{N}(T), \mathcal{N}(T^*)),\ \|Z\| < 1\} \\ &= \{T' = T + D_{T^*} Z D_T : Z \in \mathcal{B}(\mathcal{N}(T), \mathcal{N}(T^*)),\ \|Z\| < 1\}. \end{aligned} \tag{3.2}$$

Proof. Let $T' \in \Delta(T)$. Then

$$\mathcal{N}(D_{T'}) = \mathcal{N}(D_T) = \mathcal{R}(T^*), \quad \mathcal{D}_{T'} = \mathcal{D}_T = \mathcal{N}(T),$$

and similarly when T', T are replaced by T'^*, T^* respectively. Also $T' = T = U$ on $\mathcal{N}(D_T)$, therefore $T' = U \oplus Z$ from $\mathcal{H} = \mathcal{N}(D_T) \oplus \mathcal{N}(T)$ into $\mathcal{H} = \mathcal{N}(D_{T^*}) \oplus \mathcal{N}(T^*)$ where Z is the pure part of T'. But by the Harnack domination of T by T' one has for all $h \in \mathcal{N}(T)$

$$\|T'h\|^2 = \|(T - T')h\|^2 \leq c\|D_{T'}h\|^2$$

with some constant $c \geq 1$, that is

$$\|T'h\|^2 \leq \frac{c-1}{c+1}\|h\|^2.$$

As $Z = P_{\mathcal{N}(T^*)} T'|_{\mathcal{N}(T)}$ it follows that $\|Z\| < 1$. So $\Delta(T)$ is contained in the set of the right hand side of the first equality in (3.2), denoted here Δ_0.

Next, if $T'_0 = 0 \oplus Z$ for $T' = U \oplus Z \in \Delta_0$ then $\|T'_0\| < 1$, and T'_0 belongs to the Shmul'yan part of the null operator on $\mathcal{H}$. But by Theorem 3.7 this means that T' belongs to the Shmul'yan part $\Delta_{\mathrm{Sh}}(T)$ of T. Therefore $\Delta(T) \subset \Delta_0 \subset \Delta_{\mathrm{Sh}}(T)$ whence we get $\Delta(T) = \Delta_0 = \Delta_{\mathrm{Sh}}(T)$, that is the former equality in (3.2).

Clearly, this equality also gives an inclusion for the second equality in (3.2), since $D_T = 0 \oplus I$ on $\mathcal{H} = \mathcal{N}(D_T) \oplus \mathcal{N}(T)$ and $D_{T^*} = 0 \oplus I$ on $\mathcal{H} = \mathcal{N}(D_{T^*}) \oplus \mathcal{N}(T^*)$.

Assume now that $T' = T + D_{T^*} Z D_T$ with $Z \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ and $\|Z\| < 1$. This implies $T'|_{\mathcal{N}(D_T)} = T|_{\mathcal{N}(D_T)} = U$ and $T'|_{\mathcal{N}(T)} = Z$, D_{T^*} being the orthogonal projection on $\mathcal{N}(T^*)$. So $T' = U \oplus Z \in \Delta(T)$ by using the first equality in (3.2), which shows that the second equality in (3.2) occurs.

We finally remark from (3.2) that only T is a partial isometry in $\Delta(T)$. This ends the proof. □

Other characterizations for contractions which are Harnack equivalent with a partial isometry are included in the following

Proposition 3.9. *The following statements are equivalent for a contraction T on $\mathcal{H}$:*

(i) T belongs to a Harnack part of a partial isometry;

(ii) The range of the defect operator D_T is closed;

(iii) $\{(T^*T)^n\}_{n\in\mathbb{N}}$ uniformly converges in $\mathcal{B}(\mathcal{H})$ to the orthogonal projection on $\mathcal{N}(D_T)$.

Proof. Let $W = U \oplus 0$ be a partial isometry. If T is in the same Harnack part as W, then by (3.2) we have $T = U \oplus Z$ with $\|Z\| < 1$. So $D_T = 0 \oplus D_Z$ and D_Z is invertible in $\mathcal{B}(\mathcal{D}_T)$, while

$$(T^*T)^n = I \oplus (Z^*Z)^n \to I \oplus 0 = P_{\mathcal{N}(D_T)},$$

the convergence being uniform in $\mathcal{B}(\mathcal{H})$. Hence (i) implies (iii).

Now (iii) implies that the Cesàro means of T^*T also converge uniformly to $P_{\mathcal{N}(D_T)}$. Using the uniform ergodic theorem of M. Lin [9], this implies that $D_T\mathcal{H}$ is closed. So (iii) implies (ii).

Finally, if $D_T\mathcal{H}$ is closed then, by using the representation $T = T_u \oplus T_p$ into the unitary and pure parts, we infer that $D_{T_p}\mathcal{D}_T$ is also closed. As D_{T_p} is injective one has $1 \notin \sigma(T_p^*T_p)$ which implies that $\|T_p\| < 1$. Then by Proposition 3.8 it follows that T belongs to the Harnack part of the partial isometry $T_u \oplus 0$. Thus (ii) implies (i). □

Corollary 3.10. *Any contraction T on $\mathcal{H}$ for which T^*T has only eigenvalues in its spectrum belong to the Harnack part of a partial isometry.*

This corollary shows, in particular, that any compact contraction belongs to the Harnack (or Shmul'yan) part of a partial isometry. Hence in a finite dimensional Hilbert space any Harnack part of contractions contains (only) a partial isometry. On the other hand, the spectral assumption in Corollary 3.10 strictly includes the class of compact contractions, in infinite dimensional case; for example $T = \lambda I$ with $|\lambda| < 1$ is not compact but satisfies this assumption.

We record now another special case which completes Theorem 2.10.

Proposition 3.11. *The following statements are equivalent for a quasi-normal contraction T:*

(i) T belongs to the Harnack part of a partial isometry;

(ii) The C_{00}-part of T is a strict contraction on $\mathcal{N}(S_T)$;

(iii) A power T^{n_0} with $n_0 > 1$ belongs to the Harnack part of a partial isometry;

(iv) The sequence $\{T^{*n}T^n\}$ uniformly converges in $\mathcal{B}(\mathcal{H})$ to $S_T = P_{\mathcal{N}(D_T)}$.

If one of these equivalent assertions occurs, then T belongs to the Harnack part of $V \oplus 0$ on $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$, where $V = T|_{\mathcal{N}(I-S_T)}$.

Proof. The equivalence $(i) \Leftrightarrow (iv)$ is immediate by Proposition 3.9 (iii), because $(T^*T)^n = T^{*n}T^n$ for $n \geq 1$ when T is quasi-normal.

We write now $T = V \oplus Q$ on $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$, where V is an isometry and Q of class C_{00}. Notice that $\mathcal{N}(S_T) \subset \mathcal{N}(S_{T^*})$ and both subspaces reduce T and that Q is a quasi-normal operator. Thus, the convergence (iv) means $\|(Q^*Q)^n\| \to 0$, that is $\sigma(Q^*Q) \subset \mathbb{D}$, or equivalently $\|Q\| < 1$. Hence the assertions (ii) and (iv) are equivalent.

Condition (ii) implies (iii) for every integer $n_0 > 1$. Conversely, if (iii) holds for some $n_0 > 1$ then by applying (ii) to T^{n_0} one has $\|Q^{n_0}\| < 1$ which gives $\sigma(Q) \subset \mathbb{D}$, and finally $\|Q\| < 1$, that is (ii). Consequently (ii) and (iii) are equivalent.

Clearly, if $\|Q\| < 1$ and $Q_0 = JQ \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$, where J is the natural embedding of $\mathcal{D}_T = \mathcal{N}(S_T)$ into $\mathcal{D}_{T^*} = \mathcal{N}(S_{T^*})$, then $\|Q_0\| < 1$. Notice that in this case one has $\mathcal{D}_T \subset \mathcal{D}_{T^*}$. So putting $W = V \oplus 0$ we have $D_W = 0 \oplus I$ and $D_{W^*} = P_{\mathcal{N}(V^*)} \oplus I$ on $\mathcal{H} = \mathcal{N}(D_T) \oplus \mathcal{D}_T$. Since

$$\mathcal{N}(D_{W^*}) = \mathcal{R}(W) = \mathcal{R}(V) = \mathcal{N}(D_{T^*}), \quad \mathcal{D}_{W^*} = \mathcal{N}(V^*) \oplus \mathcal{D}_T = \mathcal{D}_{T^*},$$

one can write $D_{W^*} = 0 \oplus I$ on $\mathcal{H} = \mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*}$. So we get

$$T = W + (0 \oplus Q) = W + D_{W^*} Q_0 D_W,$$

which yields, using (3.2), that T is in the Harnack part of W. This ends the proof. □

Concerning the statement (iv) in Proposition 3.11 it is known that the sequence $\{T^{*n}T^n\}$ always converges strongly to S_T, so (iv) says that the convergence is even uniform whenever T is Harnack equivalent with a partial isometry.

It is also clear that the uniform convergence in (iv), for a normaloid contraction T with $S_T = S_T^2$, ensures that the Harnack part $\Delta(T)$ contains a partial isometry. The converse assertion is not true in general, for example for $T = C \oplus 0$ on $\mathcal{H} \oplus \mathcal{H}$ where C is a coshift (the adjoint of a shift).

Next, we want to see, in a more general context, when a contraction T is Harnack equivalent with a partial isometry, by using the matrix representation (3.1) under the decomposition $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \overline{\mathcal{R}(I - S_T)}$. Here we know that $V = T|_{\mathcal{N}(I-S_T)}$ is an isometry, but we have no information on the contractions R from $\overline{\mathcal{R}(I - S_T)}$ into $\mathcal{N}(I - S_T)$ and Q acting on $\overline{\mathcal{R}(I - S_T)}$. A first assertion on Q is given in the following preliminary result.

Lemma 3.12. *Let T be a contraction having the representation* (3.1) *on $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \overline{\mathcal{R}(I - S_T)}$, with V, R, Q as above. The following are equivalent :*

(i) $\mathcal{N}(D_T)$ is an invariant subspace for T;

(ii) $\mathcal{N}(D_{T^*}) \subset \mathcal{N}(D_T)$;

(iii) Q is a pure contraction on $\overline{\mathcal{R}(I - S_T)}$.

Proof. Assuming (i) we have $\mathcal{N}(D_T) = \mathcal{N}(I - S_T)$ and $\overline{\mathcal{R}(I - S_T)} = \mathcal{D}_T$. As $T|_{\mathcal{D}_T}$ is a pure contraction from $\mathcal{D}_T$ into $\mathcal{D}_{T^*}$ it follows that $Q = P_{\mathcal{D}_T} T|_{\mathcal{D}_T}$ is pure on $\mathcal{D}_T$. So (i) implies (iii). Now (iii) says that $Q^* = T^*|_{\overline{\mathcal{R}(I-S_T)}}$ is a pure contraction, therefore $\overline{\mathcal{R}(I - S_T)} \subset \mathcal{D}_{T^*}$ which yields $\mathcal{N}(D_{T^*}) \subset \mathcal{N}(I - S_T) \subset \mathcal{N}(D_T)$. Hence (iii) implies (ii), and obviously (ii) implies (i). □

Assuming the above equivalent assertions $(i) - (iii)$ one can consider the matrix representation of T on $\mathcal{H} = \mathcal{N}(D_T) \oplus \mathcal{D}_T$ with the contractions V, R, Q as above, which can be related to the decomposition $T = U \oplus \widetilde{Q}$ into the unitary operator $U \in \mathcal{B}(\mathcal{N}(D_T), \mathcal{N}(D_{T^*}))$ and the pure contraction $\widetilde{Q} \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$. Since $\mathcal{D}_T \subset \mathcal{D}_{T^*}$ in this case, it follows that

$$V = J_1 U, \quad \widetilde{Q} = \begin{pmatrix} R \\ Q \end{pmatrix} : \mathcal{D}_T \to \mathcal{D}_{T^*} = \mathcal{D}_T^{\perp} \oplus \mathcal{D}_T,$$

where J_1 is the natural embedding of $\mathcal{N}(D_{T^*})$ into $\mathcal{N}(D_T)$. Thus, by Proposition 3.8 and Lemma 3.12 one obtains the following

Corollary 3.13. *Let T be a contraction on $\mathcal{H}$ and R, Q be the contractions given by*

$$R = P_{\mathcal{N}(I-S_T)}T|_{\overline{\mathcal{R}(I-S_T)}}, \quad Q = P_{\overline{\mathcal{R}(I-S_T)}}T|_{\overline{\mathcal{R}(I-S_T)}}.$$

The following are equivalent :

(i) The Harnack part of T contains a partial isometry and Q is a pure contraction;

(ii) $\|R^*R + Q^*Q\| < 1$.

Clearly, the above condition (ii) is stronger than $\|R\| < 1$ and $\|Q\| < 1$, while (i) does not imply (ii) without the requirement that Q is pure, that is $\mathcal{N}(D_{T^*}) \subset \mathcal{N}(D_T)$. This last condition holds for a large class of contractions including hyponormal operators.

4. Connections with Shmul'yan-ter Horst domination of Schur class functions

Recall that the condition (ii) in Lemma 3.12 has been used to described the Harnack equivalence of a contraction T to a partial isometry in terms of contractions R and Q coming from the matrix of T with respect to the decomposition $\mathcal{H} = \mathcal{N}(D_T) \oplus \mathcal{D}_T$. The Shmul'yan equivalence of arbitrary two contractions can be described by their pure parts when the dual condition of (ii) (that is, with T being interchanged with T^*) is assumed, as follows.

Theorem 4.1. *Let T be a contraction on $\mathcal{H}$ for which $\mathcal{N}(D_T) \subset \mathcal{N}(D_{T^*})$ and having the decomposition $T = U \oplus Q$ into the unitary and pure parts. Then a contraction T' belongs to the Shmul'yan part of T if and only if $T' = U \oplus Q'$ relative to the same orthogonal decompositions of $\mathcal{H}$ as for T, where Q' is pure such that JQ and JQ' are in the same Shmul'yan part in $\mathcal{B}_1(\mathcal{D}_T)$, J being the natural embedding of $\mathcal{D}_{T^*}$ into $\mathcal{D}_T$.*

Proof. Assume that T and T' are in the same Shmul'yan part. By Theorem 3.7 they have the decompositions $T = U \oplus Q$, $T' = U \oplus Q'$ with U unitary between $\mathcal{N}(D_T) = \mathcal{N}(D_{T'})$ and $\mathcal{N}(D_{T^*}) = \mathcal{N}(D_{T'^*})$, and Q, Q' pure from $\mathcal{D}_T$ into $\mathcal{D}_{T^*}$. If $J \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{D}_T)$ is given by $Jh = h$, $h \in \mathcal{D}_{T^*}$ and $P = J^*$ is the projection of $\mathcal{D}_T$ to $\mathcal{D}_{T^*}$, then JQ and Q^*P are pure contractions as Q, Q^*, and

$$D_{JQ} = D_Q, \quad \mathcal{D}_{JQ} = \mathcal{D}_{Q^*P} = \mathcal{D}_Q = \mathcal{D}_T, \quad \mathcal{D}_{Q^*} = \mathcal{D}_{T^*}.$$

Now Q and Q' are Shmul'yan equivalent, by Theorem 3.7, so there is an operator $Z \in \mathcal{B}(\mathcal{D}_Q, \mathcal{D}_{Q^*}) = \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ such that

$$Q' = Q + (I - QQ^*)^{1/2} Z (I - Q^*Q)^{1/2}.$$

It is easy to see that the isometry J intertwines the operators $I - QQ^*$ and $I - JQQ^*P$, that is $J(I - QQ^*) = (I - JQQ^*P)J$. Therefore we also have $JD_{Q^*} = D_{Q^*P}J$. Since $D_{JQ} = D_Q$ we get

$$JQ' = JQ + D_{Q^*P}JZD_{JQ} = JQ + D_{Q^*P}Z_0D_{JQ},$$

where $Z_0 = JZ \in \mathcal{B}(\mathcal{D}_T) = \mathcal{B}(\mathcal{D}_{JQ}, \mathcal{D}_{Q^*P})$. This means that JQ' is Shmul'yan dominated by JQ, and by symmetry we have that JQ and JQ' are Shmul'yan equivalent.

Conversely, let us assume that $T' = U \oplus Q'$ as above and that the contractions JQ and JQ' are Shmul'yan equivalent in $\mathcal{B}_1(\mathcal{D}_T)$. Then there exists $Z_0 \in \mathcal{B}(\mathcal{D}_{JQ}, \mathcal{D}_{Q^*P}) = \mathcal{B}(\mathcal{D}_T)$ with $JQ' = JQ + \mathcal{D}_{Q^*P} Z_0 D_{JQ}$. As $JP \in \mathcal{B}(\mathcal{D}_T)$ is the orthogonal projection on $\mathcal{D}_{T^*} \subset \mathcal{D}_T$ and $(I - JQQ^*P)(I - JP) = I - JP$, we obtain as above that

$$\begin{aligned} JQ' &= JQ + (I - JQQ^*P)^{1/2}(JP + I - JP)Z_0 D_{JQ} \\ &= JQ + J(I - QQ^*)^{1/2} P Z_0 D_{JQ} + (I - JP) Z_0 D_{JQ}. \end{aligned}$$

Also, as J is an isometry from $\mathcal{D}_{T^*}$ into $\mathcal{D}_T$, by multiplying to the left side with $J^* = P$ we deduce that

$$Q' = Q + (I - QQ^*)^{1/2} Z (I - Q^*Q)^{1/2},$$

where $Z = PZ_0 \in \mathcal{B}(\mathcal{D}_T) = \mathcal{B}(\mathcal{D}_Q, \mathcal{D}_{Q^*})$. This means that Q' is Shmul'yan dominated by Q, and by symmetry one obtains the reverse domination of Q by Q'. Hence Q and Q' are Shmul'yan equivalent in $\mathcal{B}_1(\mathcal{D}_T, \mathcal{D}_{T^*})$, which by Theorem 3.7 shows that T and T' are Shmul'yan equivalent in $\mathcal{B}_1(\mathcal{H})$. This ends the proof. □

Corollary 4.2. *Let T be a contraction on $\mathcal{H}$ with $\mathcal{N}(D_T) \subset \mathcal{N}(D_{T^*})$ and having the form $T = U \oplus Q$ as above. Then the Shmul'yan part $\Delta_{\mathrm{Sh}}(T)$ of T is*

$$\Delta_{\mathrm{Sh}}(T) = \{T' = U \oplus PZ \in \mathcal{B}(\mathcal{H}) : \|T'\| \leq 1,\ Z \in \Delta_{\mathrm{Sh}}(JQ)\}, \tag{4.1}$$

where P is the projection of $\mathcal{D}_T$ onto $\mathcal{D}_{T^*}$.

Moreover, the relation $\phi(T') = Z$ for $T' = U \oplus PZ$ defines an injective (i.e., one-to-one) mapping of $\Delta_{\mathrm{Sh}}(T)$ onto $\Delta_{\mathrm{Sh}}(JQ)$.

Proof. Let $T' \in \Delta_{\mathrm{Sh}}(T)$. By the previous theorem $T' = U \oplus Q'$ with $JQ' \in \Delta_{\mathrm{Sh}}(JQ)$. Letting $Z = JQ'$ one has $PZ = Q'$, therefore $T' = U \oplus PZ$ with $Z \in \Delta_{\mathrm{Sh}}(JQ)$. Conversely, let $Z \in \Delta_{\mathrm{Sh}}(JQ)$. Then $Z = JQ + D_{Q^*J} X D_{JQ}$ for some operator $X \in \mathcal{B}(\mathcal{D}_{JQ}, \mathcal{D}_{Q^*P}) = \mathcal{B}(\mathcal{D}_T)$. Using the intertwining relation $JD_{Q^*} = D_{Q^*P} J$ quoted in the previous proof, this gives

$$PZ = Q + PD_{Q^*J} X D_{JQ} = Q + D_{Q^*} P X D_Q.$$

So PZ is Shmul'yan dominated by Q, and by symmetry these contractions are Shmul'yan equivalent in $\mathcal{B}_1(\mathcal{D}_T, \mathcal{D}_{T^*})$. Then by Theorem 3.7 we have that the contraction $T' = U + PZ$ belongs to $\Delta_{\mathrm{Sh}}(T)$, and we get the structure of $\Delta_{\mathrm{Sh}}(T)$ from (4.1). The last assertion of corollary is now obvious. □

Remark 4.3. The condition $\mathcal{N}(D_T) \subset \mathcal{N}(D_{T^*})$, which means that $\mathcal{N}(D_{T^*})$ is invariant for T^*, or $\mathcal{N}(D_{T^*}) = \mathcal{N}(I - S_{T^*})$, covers a large class of non-normaloid contractions which include the cohyponormal operators. A cohyponormal operator T can be decomposed as $T = W \oplus Q_0$ on $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$ with W unitary and Q of class $C_{0\cdot}$. In fact, if we compare this decomposition of T with $T = U \oplus Q$ from $\mathcal{N}(D_T) \oplus \mathcal{D}_T$ into $\mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*}$ we have $W = U|_{\mathcal{N}(I - S_T)}$ and $Q = Q_0|_{\mathcal{D}_T}$. So $T\mathcal{D}_T = Q\mathcal{D}_T \subset \mathcal{D}_{T^*} \subset \mathcal{D}_T$ and $Q_1 := T|_{\mathcal{D}_T} = JQ \in \mathcal{B}_1(\mathcal{D}_T)$ is a pure contraction of class $C_{0\cdot}$. Then from Proposition 3.8 and Proposition 3.9 we infer that T belongs to the Harnack part of a partial isometry if and only if $\|Q_1\| < 1$.

In fact, this last condition is equivalent with that of Corollary 3.13 (ii) for the hyponormal contraction T^*.

Concerning the Shmul'yan-ter Horst relation we have the following result

Theorem 4.4. *Let T be a partial isometry on the separable Hilbert space $\mathcal{E}$. A contraction T' on $\mathcal{E}$ belongs to the Harnack part $\Delta(T)$ of T if and only if there exist $z_0 \in \mathbb{D}$ and $F \in H_1^\infty(\mathcal{E})$ such that $F(0) = T$ and $F(z_0) = T'$. In this case we have $F(\mathbb{D}) \subset \Delta(T)$. The operator T, viewed as a constant function in $H_1^\infty(\mathcal{E})$, Shmul'yan-ter Horst dominates any function $G \in H_1^\infty(\mathcal{E})$ with $G(\mathbb{D}) \subset \Delta(T)$.*

Proof. Assume $T' \in \Delta(T)$. By (3.2) we have $T' = T + D_{T^*}ZD_T = U \oplus Z$ with $Z \in \mathcal{B}(\mathcal{N}(T), \mathcal{N}(T^*))$, $\|Z\| < 1$ and U a unitary operator from $\mathcal{N}(D_T)$ onto $\mathcal{N}(D_{T^*})$. Since $I - Z^*Z$ is invertible in $\mathcal{B}(\mathcal{N}(T))$ we have $D_T = 0 \oplus I = 0 \oplus D_Z^{-1}D_Z = (0 \oplus D_Z^{-1})D_{T'}$. Therefore one obtains

$$T' = T + D_{T^*}[0 \oplus ZD_Z^{-1}]D_{T'} = T + D_{T^*}R'D_{T'}.$$

Hence $T = T' + D_{T^*}RD_{T'}$, with $R = -R' = -ZD_Z^{-1}$. We used here that $D_{T^*} = P_{\mathcal{N}(T^*)}$. By using this relation between T and T' one can prove as in the proof of Theorem 2.10 (implication $(iv) \Rightarrow (v)$) that there exist $z_0 \in \mathbb{D}$ and $F \in H_1^\infty(\mathcal{E})$ with $F(0) = T$ and $F(z_0) = T'$. This forces the inclusion $F(\mathbb{D}) \subset \Delta(T)$, $F(\mathbb{D})$ being necessarily contained in a Shmul'yan part (as it was mentioned in Remark 2.12), which coincides with the Harnack part $\Delta(T)$ by Proposition 3.8.

The last remark also proves that if $T = F(0)$ and $T' = F(z_0)$ for some $z_0 \in \mathbb{D}$ and F as above, then $T' \in F(\mathbb{D}) \subset \Delta(T)$. In addition, in this case the function F is given by the formula (see [5, Ch. XIII, 3])

$$F(\lambda) = T + \lambda D_{T^*}F_1(\lambda)[I + \lambda T^*F_1(\lambda)]^{-1}D_T, \quad \lambda \in \mathbb{D},$$

where $F_1 \in H_1^\infty(\mathcal{D}_T, \mathcal{D}_{T^*})$. Since T is a partial isometry one has $\mathcal{D}_{T^*} = \mathcal{N}(T^*)$ so $T^*F_1(\lambda) = 0$ for $\lambda \in \mathbb{D}$. Therefore, we have in this case

$$F(\lambda) = T + \lambda D_{T^*}F_1(\lambda)D_T, \quad \lambda \in \mathbb{D}, \tag{4.2}$$

with $\sup_{\lambda\in\mathbb{D}} \|\lambda F_1(\lambda)\| \le 1$. By applying [6, Theorem 2.6] this implies that F is dominated by the constant function T in $H_1^\infty(\mathcal{E})$.

Let now $G \in H_1^\infty(\mathcal{E})$ be an arbitrary function with the range $G(\mathbb{D})$ contained in $\Delta(T)$. Let $\lambda \in \mathbb{D}$. Then by the above assertion for the above contractions T and $G(\lambda)$ there exist $z_\lambda \in \mathbb{D}$ and $K_\lambda \in H_1^\infty(\mathcal{E})$ such that $K_\lambda(0) = T$ and $K_\lambda(z_\lambda) = G(\lambda)$. As above, there is a function $F_\lambda \in H_1^\infty(\mathcal{D}_T, \mathcal{D}_{T^*})$ with

$$K_\lambda(z) = T + D_{T^*}F_\lambda(z)D_T, \quad z \in \mathbb{D}.$$

For $z = z_\lambda$ this yields

$$G(\lambda) = T + D_{T^*}F_\lambda(z_\lambda)D_T = T + D_{T^*}Q_\lambda D_T, \quad \lambda \in \mathbb{D}. \tag{4.3}$$

Here $Q_\lambda := F_\lambda(z_\lambda) \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ is such that $\|Q_\lambda\| \le 1$ for $\lambda \in \mathbb{D}$.

Notice that the contraction Q_λ is uniquely defined by λ, $G(\lambda)$, z_λ and $K_\lambda \in H_1^\infty(\mathcal{E})$ as above. Indeed, assume that for $\lambda \in \mathbb{D}$ there are $z'_\lambda \in \mathbb{D}$ and $K'_\lambda \in H_1^\infty$ such that $K'_\lambda(0) = T$ and $K'_\lambda(z'_\lambda) = G(\lambda)$. Then K'_λ has the form of F in (4.2), hence there exists $F'_\lambda \in H_1^\infty(\mathcal{E})$ such that

$$K'_\lambda(z) = T + D_{T^*} F'_\lambda(z) D_T, \quad z \in \mathbb{D},$$

which also gives for $z = z'_\lambda$

$$G(\lambda) = T + D_{T^*} F'_\lambda(z'_\lambda) D_T, \quad \lambda \in \mathbb{D}.$$

From this expression and that in (4.3) of G we infer

$$D_{T^*}(Q_\lambda - F'_\lambda(z'_\lambda)) D_T = 0, \quad \lambda \in \mathbb{D},$$

that is $F'_\lambda(z'_\lambda) = Q_\lambda = F_\lambda(z_\lambda)$, $\lambda \in \mathbb{D}$. Hence the mapping $\widetilde{Q}$ from $\mathbb{D}$ into $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ defined by the formula

$$\widetilde{Q}(\lambda) = Q_\lambda, \quad \lambda \in \mathbb{D},$$

is well-defined and bounded on $\mathbb{D}$. Then by [6, Theorem 0.2] it follows that the function G satisfying the relation (4.3) is dominated by the constant function T in $H_1^\infty(\mathcal{E})$. This ends the proof. □

Concerning the last assertion in Theorem 4.4 we remark that T is not dominated by $G \in H_1^\infty(\mathcal{E})$ with $G(\mathbb{D}) \subset \Delta(T)$, in general, that is the corresponding part of T in $H_1^\infty(\mathcal{E})$ does not contain all those functions G. For example, when $\mathcal{E} = \mathbb{C}$, the identity function $u(\lambda) = \lambda$ ($\lambda \in \mathbb{D}$) satisfies $u(\mathbb{D}) \subset \mathbb{D}$ (the Harnack part of the null operator $0_\mathbb{C}$ is the set of all strict contractions [4]), and $\|u\|_\infty = 1$. But only functions $f \in H^\infty$ with $\|f\|_\infty < 1$ dominate $0_\mathbb{C}$.

Consider now the equivalence class $\Delta_\infty(T)$ in the sense of Shmul'yan-ter Horst of a partial isometry T, as a constant function in $H_1^\infty(\mathcal{E})$. The following result provides a description of $\Delta_\infty(T)$ which is similar to that from Proposition 3.8.

Theorem 4.5. *If T is a partial isometry on a separable Hilbert space $\mathcal{E}$ then*

$$\Delta_\infty(T) = \{F(\cdot) = T + D_{T^*} F_0(\cdot) D_T \in H_1^\infty(\mathcal{E}) : F_0 \in H_1^\infty(\mathcal{D}_T, \mathcal{D}_{T^*}),\ \|F_0\|_\infty < 1\}. \tag{4.4}$$

Proof. Let $F \in H_1^\infty(\mathcal{E}) \cap \Delta_\infty(T)$, T being a partial isometry on $\mathcal{E}$. Then by [6, Theorem 2.6] there exists a norm bounded function $F_0(\cdot)$ on $\mathbb{D}$ with values in $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ such that

$$F(\lambda) = T + D_{T^*} F_0(\lambda) D_T, \quad \lambda \in \mathbb{D}.$$

Since $F(\lambda) \in \Delta(T)$ we have by (3.2)

$$F_0(\lambda) = P_* F(\lambda)|_{\mathcal{D}_T}, \quad \lambda \in \mathbb{D},$$

where P_* is the projection of $\mathcal{E}$ onto $\mathcal{D}_{T^*}$. In particular, $F_0 \in H_1^\infty(\mathcal{D}_T, \mathcal{D}_{T^*})$. Identifying canonically $\mathcal{D}_T \sim \{0\} \oplus \mathcal{D}_T$ and $\mathcal{D}_{T^*} \sim \{0\} \oplus \mathcal{D}_{T^*}$ into $\mathcal{E} = \mathcal{N}(D_T) \oplus \mathcal{D}_T$ and $\mathcal{E} = \mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*}$ respectively, one can consider the function $F_1 \in H_1^\infty(\mathcal{E})$ having the representation $F_1(\lambda) = 0 \oplus F_0(\lambda)$ from $\mathcal{N}(D_T) \oplus \mathcal{D}_T$ into $\mathcal{N}(D_{T^*}) \oplus \mathcal{D}_{T^*}$, for $\lambda \in \mathbb{D}$. Obviously, one has $F_1 \overset{\infty}{\prec} 0$ in $H_1^\infty(\mathcal{E})$.

On the other hand, as $T \overset{\infty}{\prec} F$ in $H_1^\infty(\mathcal{E})$ we have as above

$$T = F(\lambda) + D_{F(\lambda)^*} F_0'(\lambda) D_{F(\lambda)}, \quad \lambda \in \mathbb{D}$$

for some function $F_0' \in H_1^\infty(\mathcal{D}_T, \mathcal{D}_{T^*})$. We have used here that $\mathcal{D}_{F(\lambda)} = \mathcal{D}_T$ and $\mathcal{D}_{F(\lambda)^*} = \mathcal{D}_{T^*}$ for any $\lambda \in \mathbb{D}$. Therefore we get

$$\begin{aligned} F_0(\lambda) &= P_*(F(\lambda) - T)|_{\mathcal{D}_T} = -P_* D_{F(\lambda)^*} F_0'(\lambda) D_{F(\lambda)}|_{\mathcal{D}_T} \\ &= -D_{F_0(\lambda)^*} F_0'(\lambda) D_{F_0(\lambda)}, \end{aligned}$$

for $\lambda \in \mathbb{D}$. In other words we have

$$0 = F_0(\lambda) + D_{F_0(\lambda)^*} F_0'(\lambda) D_{F_0(\lambda)}, \quad \lambda \in \mathbb{D}.$$

By the canonical identifications $\mathcal{D}_T \sim \{0\} \oplus \mathcal{D}_T$ and $\mathcal{D}_{T^*} \sim \{0\} \oplus \mathcal{D}_{T^*}$ this relation leads to

$$0 = F_1(\lambda) + D_{F_1(\lambda)^*} F_0'(\lambda) D_{F_1(\lambda)}, \quad \lambda \in \mathbb{D}.$$

But this means that $0 \overset{\infty}{\prec} F_1$ in $H_1^\infty(\mathcal{E})$, and finally that F_1 is equivalent in the Shmul'yan-Horst sense with the null function in $H_1^\infty(\mathcal{E})$. Then by [6, Corollary 2.7 (ii)] it follows that $\|F_1\| < 1$ which means $\|F_0\|_\infty < 1$. Thus, an inclusion for the equality (4.4) is proved.

The converse inclusion in (4.4) is immediate. Indeed, let $F \in H_1^\infty(\mathcal{E})$ having the above form with respect to T and with a function $F_0 \in H_1^\infty(\mathcal{D}_T, \mathcal{D}_{T^*})$ such that $\|F_0\|_\infty < 1$. Then also $\|F_1\|_\infty < 1$, F_1 being as above, and by the same result in [6] we have that F_1 is in the equivalence class of the null function in $H_1^\infty(\mathcal{E})$. Since $F(\lambda)|_{\mathcal{N}(D_T)} = T|_{\mathcal{N}(D_T)}$, $\lambda \in \mathbb{D}$, the function F belongs to the equivalence class of T in $H_1^\infty(\mathcal{E})$. This ends the proof. □

According to the terminology of [16] we conclude from (3.2) and (4.4) that the Shmul'yan parts and Shmul'yan-ter Horst parts of partial isometries are *centred quasi-balls* in $\mathcal{B}_1(\mathcal{E})$, and respectively in $H_1^\infty(\mathcal{E})$. This means that they have the form (3.2) with $\|Z\| < \varepsilon$, respectively the form (4.4) with $\|F_0\|_\infty < \varepsilon$, for some $\varepsilon > 0$. We do not have a characterization of all such parts which are centered quasi-balls.

References

[1] T. Ando, I. Suciu and D. Timotin, *Characterization of some Harnack parts of contractions*, J. Operator Theory, 2 (1979), 233–245.

[2] C. Badea, L. Suciu and D. Timotin, *Classes of contractions and Harnack domination*, Preprint `arXiv:1505.01972`

[3] G. Cassier, N. Suciu, *Mapping theorems and Harnack domination for ρ-contractions*, Indiana Univ. Math. J. 55 (2006), no. 2, 483–523.

[4] C. Foias, *On Harnack parts of contractions*, Rev. Roumaine Math. Pures Appl. 19 (1974), 315–318.

[5] C Foias, A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Birkhäuser, Basel-Boston-Berlin, 1990.

[6] S. ter Horst, *A Pre-order and an equivalence relation on Schur class functions and their invariance under linear fractional transformations*, Preprint `arXiv:1408.0523`.

[7] C. S. Kubrusly, *An introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Boston, 1997.

[8] V. A. Khatskevich, Yu. L. Shmul'yan, V. S. Shul'man, *Preorder and equivalences in the operator sphere*, Sibirsk. Mat. Zh. 32 (3) (1991) (in Russian); English transl. : Siberian Math. J. 32 (3) (1991), 496–506.

[9] M. Lin, *On the uniform ergodic theorem*, Proc. Amer. Math. Soc. 43 (1974), 337–340.

[10] G. Popescu, *Noncommutative hyperbolic geometry on the unit ball of $B(H)^n$*, J. Funct. Anal. 256 (2009), no. 12, 4030–4070.

[11] G. Popescu, *Hyperbolic geometry on noncommutative balls*, Doc. Math. 14 (2009), 595–651.

[12] B. Schroeder, *Ordered sets: an introduction*, Springer, New York, 2003.

[13] Yu.L. Shmul'yan, *Generalized fractional-linear transformations of operator spheres* , Sibirsk. Mat. Zh. 21 (1980), no. 5, 114–131 (Russian); translation in Siberian Math. J. 21 (1991), no. 3, 496–506 (1992).

[14] I. Suciu, *Harnack inequalities for a functional calculus*, in : Hilbert Space Operators and Operator Algebra (Proc. Intern. Conf., Tihany, 1970), Colloq. Proc. Math. Soc. Janos Bolyai 5, North-Holland, Amsterdam, 1972, 499–511.

[15] I. Suciu, *Analytic relations between functional models for contractions*, Acta Sci Math. (Szeged) 33 (1973), 359–365.

[16] I. Suciu, *The Kobayashi Distance between two Contractions*, Operator Theory : Advances and Applications, Vol. 61, 1993, 189–200.

[17] I. Suciu, *Analytic formulas for the hyperbolic distance between two contractions*, Ann. Polon. Math. 66 (1997), 239–252.

[18] I. Suciu, I. Valuşescu, *On the hyperbolic metric on Harnack parts*, Studia Math. 55 (1975/76), no. 1, 97–109.

[19] L. Suciu, *Some invariant subspaces for A-contractions and applications*, Extracta Math., 21 (3) (2006), 221–247.

[20] L. Suciu, *Maximum A-isometric part of an A-contraction and applications*, Israel J. Math., vol. 174, (2009), 419–442.

[21] L. Suciu, N. Suciu, *Harnack domination for contractions and related intertwining operators*, Acta Sci. Math. (Szeged) 72 (2006), no. 1-2, 319–343.

[22] N. Suciu, *On Harnack domination of contractions*, Bull. Acad. Polon. Sci. Ser. Sci. Math. 27 (1979), 467–471.

[23] B. Sz.-Nagy, C. Foias, H. Bercovici, L. Kérchy, *Harmonic analysis of operators on Hilbert space. Second edition. Revised and enlarged edition*, Springer, New York, 2010.

Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
E-mail address: badea@math.univ-lille1.fr

Department of Mathematics, "Lucian Blaga" University of Sibiu, Dr. Ion Raţiu 5-7, Sibiu, 550012, Romania
E-mail address: laurians2002@yahoo.com