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► **To cite this version:**

Gestur <https://www.Math.Lsu.Edu/olafsson/> Olafsson, Angela Pasquale. Ramanujan's Master theorem for the hypergeometric Fourier transform on root systems. *Journal of Fourier Analysis and Applications*, 2013, 19 (6), pp.1150-1183. 10.1007/s00041-013-9290-5 . hal-01232021

HAL Id: hal-01232021

<https://hal.science/hal-01232021>

Submitted on 21 Nov 2015

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RAMANUJAN'S MASTER THEOREM FOR THE HYPERGEOMETRIC FOURIER TRANSFORM ON ROOT SYSTEMS

G. ÓLAFSSON AND A. PASQUALE

ABSTRACT. Ramanujan's Master theorem states that, under suitable conditions, the Mellin transform of an alternating power series provides an interpolation formula for the coefficients of this series. Ramanujan applied this theorem to compute several definite integrals and power series, which explains why it is referred to as the "Master Theorem". In this paper we prove an analogue of Ramanujan's Master theorem for the hypergeometric Fourier transform on root systems. This theorem generalizes to arbitrary positive multiplicity functions the results previously proven by the same authors for the spherical Fourier transform on semisimple Riemannian symmetric spaces.

INTRODUCTION

Ramanujan's First Quarterly Report [3, p. 297] contains the following formal identity, nowadays known as Ramanujan's Master theorem: if a function $f(x)$ can be expanded around $x = 0$ in a power series of the form

$$f(x) = \sum_{k=0}^{\infty} (-1)^k a(k) x^k$$

then

$$\int_0^{+\infty} x^{-\lambda-1} f(x) dx = -\frac{\pi}{\sin(\pi\lambda)} a(\lambda). \quad (0.1)$$

By replacing $a(\lambda)$ with $A(\lambda) = a(\lambda)\Gamma(\lambda + 1)$, one obtains an equivalent version of (0.1) as follows: if a function $f(x)$ can be expanded around $x = 0$ in a power series of the form

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{A(k)}{k!} x^k$$

then

$$\int_0^{+\infty} x^{-\lambda-1} f(x) dx = \Gamma(-\lambda) A(\lambda). \quad (0.2)$$

Ramanujan's presented this formula as "an instrument by which at least some of the definite integrals whose values are at present not known can be evaluated" (see [3, p. 297]). As reported by Berndt [3, p. 299], "Ramanujan was evidently quite fond of this clever, original

2010 *Mathematics Subject Classification*. Primary: 33C67; secondary: 43A32, 43A90.

Key words and phrases. Ramanujan's Master theorem, hypergeometric functions, Jacobi polynomials, spherical functions, root systems, Cherednik operators, hypergeometric Fourier transform.

The research of G. Ólafsson was supported by NSF grant DMS-1101337. A. Pasquale would like to thank the Louisiana State University, Baton Rouge, for hospitality and financial support. She also gratefully acknowledges financial support from Tufts University and travel support from the Commission de Colloques et Congrès Internationaux (CCCI).

technique, and he employed it in many contexts”. Most of the examples given by Ramanujan by applying his Master Theorem turn out to be correct. But formulas (0.1) and (0.2) cannot hold without additional assumptions, as one can easily see from the example $a(\lambda) = \sin(\pi\lambda)$. The first rigorous reformulation of Ramanujan’s Master theorem was given by Hardy in his book on Ramanujan’s work [16]. Using the residue theorem, Hardy proved that (0.1) holds for a natural class of functions a and a natural set of parameters λ .

Let \mathbf{A} , \mathbf{P} , δ be real constants so that $\mathbf{A} < \pi$ and $0 < \delta \leq 1$. Let $\mathcal{H}(\delta) = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\delta\}$. The Hardy class $\mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$ consists of all functions $a : \mathcal{H}(\delta) \rightarrow \mathbb{C}$ that are holomorphic on $\mathcal{H}(\delta)$ and satisfy the growth condition

$$|a(\lambda)| \leq C e^{-\mathbf{P}(\operatorname{Re} \lambda) + \mathbf{A}|\operatorname{Im} \lambda|}$$

for all $\lambda \in \mathcal{H}(\delta)$. Hardy’s version of Ramanujan’s Master theorem is the following, see [16, p. 189].

Theorem 0.1 (Ramanujan’s Master Theorem). *Suppose $a \in \mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$. Then:*

(a) *The power series*

$$f(x) = \sum_{k=0}^{\infty} (-1)^k a(k) x^k \tag{0.3}$$

converges for $0 < x < e^{\mathbf{P}}$ and defines a real analytic function on this domain.

(b) *Let $0 < \sigma < \delta$. For $0 < x < e^{\mathbf{P}}$ we have*

$$f(x) = \frac{1}{2\pi i} \int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{-\pi}{\sin(\pi\lambda)} a(\lambda) x^\lambda d\lambda. \tag{0.4}$$

The integral on the right hand side of (0.4) converges uniformly on compact subsets of $]0, +\infty[$ and is independent of the choice of σ .

(c) *Formula (0.1) holds for the extension of f to $]0, +\infty[$ and for all $\lambda \in \mathbb{C}$ with $0 < \operatorname{Re} \lambda < \delta$.*

The last part of Theorem 0.1 is obtained from its second part by applying Mellin’s inversion.

Ramanujan’s Master theorem has been extended to Riemannian symmetric spaces in duality by several authors. The rank-one semisimple case has been considered by Bertram in [5]. The starting point of Bertram’s extension is the following group theoretic interpretation of (0.1). The functions x^λ ($\lambda \in \mathbb{C}$) are the spherical functions on $X_G = \mathbb{R}^+$ and the x^k ($k \in \mathbb{Z}$) are the spherical functions on the torus $X_U = U(1)$. Both X_G and X_U can be realized as real forms of their complexification $X_{\mathbb{C}} = \mathbb{C}^*$. Hence (0.1) gives a relation between the compact and noncompact spherical transforms of the restrictions to X_U and X_G of a “good” function defined on $X_{\mathbb{C}}$. Bertram’s version of Ramanujan’s Master theorem was obtained by replacing the duality between $U(1)$ and \mathbb{R}^+ inside \mathbb{C}^* with the duality between symmetric spaces of the compact type $X_U = U/K$ and of noncompact type $X_G = G/K$ inside their complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. Following the same point of view, the authors of the present paper proved in [30] an analogue of Ramanujan’s Master theorem for (reductive) Riemannian symmetric spaces of arbitrary rank. Some special classes of semisimple or reductive symmetric space situations have also been considered by Bertram [4], Ding, Gross and Richard [9] and Ding [8].

In this paper we consider Ramanujan's Master theorem on root systems. It is a generalization of our result concerning semisimple Riemannian symmetric spaces. The harmonic analysis associated with root systems is developed on a complex torus $A_{\mathbb{C}} = TA$ built up from a triple $(\mathfrak{a}, \Sigma, m)$ consisting of a finite dimensional Euclidean real vector space \mathfrak{a} , a root system Σ in the dual \mathfrak{a}^* of \mathfrak{a} , and a positive multiplicity function m on Σ ; see Section 1.1 for the precise definitions. The pair (T, A) inside $A_{\mathbb{C}}$ plays the role of the pair (X_U, X_G) of Riemannian symmetric spaces in duality inside their common complexification $X_{\mathbb{C}}$. The compact and noncompact spherical Fourier transforms on X_U and X_G are respectively replaced by the Jacobi and hypergeometric Fourier transforms on T and A . Our theorem deals with (normalized) alternating series of Jacobi polynomials on T of the form

$$f(t) = \sum_{\mu \in P^+} (-1)^{|\mu|} a(\mu + \rho) \tilde{P}_{\mu}(t). \quad (0.5)$$

Here the \tilde{P}_{μ} are suitable normalizations of the Heckman-Opdam's Jacobi polynomials (see Section 2.3). They are parametrized by the set P^+ of positive restricted weights. Moreover, as in Ramanujan's Master Theorem 0.1, the coefficients $a(\mu + \rho)$ are obtained from a holomorphic function a belonging to a certain Hardy class $\mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$ associated with $(\mathfrak{a}, \Sigma, m)$ and depending on three real parameters \mathbf{A} , \mathbf{P} and δ . The function f is proved to define a W -invariant holomorphic function on a neighborhood of T in $A_{\mathbb{C}}$. It then extends holomorphically to a neighborhood of A in $A_{\mathbb{C}}$ by means of the inverse hypergeometric Fourier transform

$$f(a) = \int_{\sigma + ia^*} a(\lambda) b(\lambda) F_{\lambda}(a) \frac{d\lambda}{c(\lambda)c(-\lambda)}, \quad (0.6)$$

where $F_{\lambda}(a)$ denotes the hypergeometric function of spectral parameter λ . The analogue of Ramanujan's formula (0.1) is then obtained from (0.6) using an inversion theorem for the hypergeometric Fourier transform. The function $b(\lambda)$ appearing on the right-hand side of (0.6) is a normalizing factor and is used to produce singularities at the right points. It plays the role of the function $\frac{i}{2} \frac{1}{\sin(\pi x)}$ occurring in the classical formula by Ramanujan. We refer the reader to Theorem 5.1 for the complete statement of Ramanujan's Master theorem for the hypergeometric Fourier transform and for the unexplained notation.

Many tools needed to prove Ramanujan's Master theorem for the hypergeometric transform on root systems are known. For instance, the L^p -harmonic analysis of the hypergeometric Fourier transform on A is known for $p = 2$ from the fundamental work of Opdam [28], and the case of $1 \leq p < 2$ has been recently developed in [26]. Some results related to the geometric case of symmetric spaces need nevertheless to be extended. Among them, the study of the holomorphic extension of Jacobi series inside $A_{\mathbb{C}}$ of (multivariate) Jacobi series. This is a generalization of some results of Lassalle on holomorphic extensions of Laurent series on Riemannian symmetric spaces of the compact type; see [24]. It is presented in Section 2 below. Notice also that in [30] several objects have been introduced or studied using the classification of root multiplicities associated with semisimple Riemannian symmetric spaces. Their definition and analysis need to be generalized to the context of positive multiplicity functions. Among them we mention the function d (which in the symmetric space case is the polynomial from Weyl's dimension formula), see Section 3, and the normalizing function b occurring in Ramanujan's formula (0.6), which will be studied in Section 4. The proof of Theorem 5.1 is given in Section 6. We will not work out all details, our aim being to prove

what is necessary to apply the methods used in [30]. The final section, Section 7, is devoted to some examples.

1. NOTATION AND PRELIMINARIES

1.1. Root systems and their structure. In the following we shall work with triples $(\mathfrak{a}, \Sigma, m)$ where:

- \mathfrak{a} is finite dimensional real Euclidean vector space, with inner product $\langle \cdot, \cdot \rangle$,
- Σ is a (not necessarily reduced) root system in the dual \mathfrak{a}^* of \mathfrak{a} ,
- m is a positive multiplicity function on Σ , that is a W -invariant function $m : \Sigma \rightarrow]0, \infty[$. Here W is the Weyl group of Σ .

We shall write $m_\alpha := m(\alpha)$ for $\alpha \in \Sigma$ and extend m to \mathfrak{a}^* by setting $m_\alpha = 0$ for $\alpha \notin \Sigma$. The *rank* of the triple $(\mathfrak{a}, \Sigma, m)$ is the dimension of \mathfrak{a} , which we shall indicate by l . We remark that our notation is based on the theory of symmetric spaces and differs from the Heckman-Opdam's notation as follows: the root system R used by Heckman and Opdam is related to our root system Σ by $R = \{2\alpha : \alpha \in \Sigma\}$, and the multiplicity function k in Heckman-Opdam's work is given by $k_{2\alpha} = m_\alpha/2$. The triple $(\mathfrak{a}, \Sigma, m)$ is said to be *geometric* if there exists a Riemannian symmetric space of the noncompact type with restricted root system Σ such that m_α is the multiplicity of the root α for all $\alpha \in \Sigma$; see [2] or [21, Ch. X, Exercise F].

Let Σ^+ be a choice of positive roots in Σ . We indicate by \mathfrak{a}^+ the open Weyl chamber in \mathfrak{a} on which all elements of Σ^+ are strictly positive. We denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the set of simple roots associated with Σ^+ . A root $\alpha \in \Sigma$ is said to be *unmultipliable* if $2\alpha \notin \Sigma$. We denote by Σ_* the set of unmultipliable roots and by $\Sigma_*^+ = \Sigma_* \cap \Sigma^+$ the set of positive unmultipliable roots. We set

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \frac{1}{2} \sum_{\beta \in \Sigma_*^+} \left(\frac{m_\beta/2}{2} + m_\beta \right) \beta. \quad (1.1)$$

For every non-zero $\lambda \in \mathfrak{a}^*$, let $A_\lambda \in \mathfrak{a}$ be determined by $\lambda(H) = \langle H, A_\lambda \rangle$ for all $H \in \mathfrak{a}$, and set $H_\lambda := 2A_\lambda / \langle A_\lambda, A_\lambda \rangle$. The complexification $\mathfrak{a}_\mathbb{C} := \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ of \mathfrak{a} can be viewed as the Lie algebra of the complex torus $A_\mathbb{C} := \mathfrak{a}_\mathbb{C} / 2\pi i \{H_\alpha : \alpha \in \Sigma\}$. We write $\exp : \mathfrak{a}_\mathbb{C} \rightarrow A_\mathbb{C}$ for the exponential map, with multi-valued inverse \log . The real form $A := \exp \mathfrak{a}$ of $A_\mathbb{C}$ is an abelian subgroup of $A_\mathbb{C}$ with Lie algebra \mathfrak{a} such that $\exp : \mathfrak{a} \rightarrow A$ is a diffeomorphism. We set $A^+ := \exp \mathfrak{a}^+$. The polar decomposition of $A_\mathbb{C}$ is $A_\mathbb{C} = TA$, where $T = \exp(i\mathfrak{a})$ is a compact torus with Lie algebra $i\mathfrak{a}$.

We extend the inner product to \mathfrak{a}^* by setting $\langle \lambda, \mu \rangle = \langle A_\lambda, A_\mu \rangle$. Let $\mathfrak{a}_\mathbb{C}^*$ be the space of all \mathbb{C} -linear functionals on \mathfrak{a} . The \mathbb{C} -bilinear extension to $\mathfrak{a}_\mathbb{C}^*$ and $\mathfrak{a}_\mathbb{C}$ of the inner products $\langle \cdot, \cdot \rangle$ on \mathfrak{a}^* and \mathfrak{a} will also be denoted by $\langle \cdot, \cdot \rangle$. The action of W extends to \mathfrak{a} by duality, to $\mathfrak{a}_\mathbb{C}^*$ and $\mathfrak{a}_\mathbb{C}$ by \mathbb{C} -linearity, and to $A_\mathbb{C}$ and A by the exponential map. Moreover, W acts on functions f on any of these spaces by $(wf)(x) := f(w^{-1}x)$, $w \in W$. The set of W -invariant elements in a space of functions \mathcal{A} will be indicated by \mathcal{A}^W .

1.2. The lattice of restricted weights. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $\alpha \in \Sigma$ we shall employ the notation

$$\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}. \quad (1.2)$$

We set $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda_\alpha \geq 0 \text{ for all } \alpha \in \Sigma^+\}$. The lattices P and P^+ of restricted weights, respectively positive restricted weights, are

$$P = \{\mu \in \mathfrak{a}^* : \mu_\alpha \in \mathbb{Z} \text{ for all } \alpha \in \Sigma^+\}, \quad (1.3)$$

$$P^+ = \{\mu \in \mathfrak{a}^* : \mu_\alpha \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+\}. \quad (1.4)$$

Here and in the following we shall employ the notation \mathbb{Z}^+ for the set on nonnegative integers $0, 1, 2, \dots$. According to Helgason's Theorem (see [22, Theorem 4.1, p. 535]), when $(\mathfrak{a}, \Sigma, m)$ is geometric, P^+ coincides with the set of restrictions to \mathfrak{a} of the highest weights of the finite-dimensional irreducible K -spherical representations of G .

Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the basis of \mathfrak{a}^* consisting of simple roots in Σ^+ . For $j = 1, \dots, l$ set

$$\beta_j = \begin{cases} \alpha_j & \text{if } 2\alpha_j \notin \Sigma \\ 2\alpha_j & \text{if } 2\alpha_j \in \Sigma \end{cases}. \quad (1.5)$$

Then $\{\beta_1, \dots, \beta_l\}$ is a basis of \mathfrak{a}^* consisting of simple roots in Σ_*^+ . The *fundamental restricted weights* $\omega_1, \dots, \omega_l$ are defined by the conditions

$$(\omega_j)_{\beta_k} = \frac{\langle \omega_j, \beta_k \rangle}{\langle \beta_k, \beta_k \rangle} = \delta_{jk}. \quad (1.6)$$

Then $\Pi_* = \{\omega_1, \dots, \omega_l\}$ is a basis of \mathfrak{a}^* . For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we have

$$\lambda = \sum_{j=1}^l \lambda_j \omega_j \quad \text{with} \quad \lambda_j := \lambda_{\beta_j} = \frac{\langle \lambda, \beta_j \rangle}{\langle \beta_j, \beta_j \rangle}. \quad (1.7)$$

Set

$$\rho = \sum_{j=1}^l \rho_j \omega_j. \quad (1.8)$$

Since β_j is a multiple of a simple root, we have

$$\rho_j = \frac{1}{2} \left(m_{\beta_j} + \frac{m_{\beta_j/2}}{2} \right). \quad (1.9)$$

For an arbitrary $\beta \in \Sigma_*^+$, we set

$$\tilde{\rho}_\beta = \frac{1}{2} \left(m_\beta + \frac{m_{\beta/2}}{2} \right). \quad (1.10)$$

Notice that $\tilde{\rho}_\beta = \rho_j = \rho_{\beta_j}$ if $\beta = \beta_j$, but $\tilde{\rho}_\beta \neq \rho_\beta$ in general.

In the following, we denote by $\|\cdot\|$ the W -invariant norm on \mathfrak{a} which is induced by the inner product $\langle \cdot, \cdot \rangle$. The same notation will also be employed for the corresponding norm on \mathfrak{a}^* . If $\lambda = \operatorname{Re} \lambda + i \operatorname{Im} \lambda \in \mathfrak{a}_\mathbb{C}^*$ with $\operatorname{Re} \lambda, \operatorname{Im} \lambda \in \mathfrak{a}^*$, then we set $\|\lambda\|^2 = \|\operatorname{Re} \lambda\|^2 + \|\operatorname{Im} \lambda\|^2$.

We set

$$\Omega = \max_{j=1, \dots, l} \|\omega_j\|. \quad (1.11)$$

The choice of Π_* as a basis for \mathfrak{a}^* (and $\mathfrak{a}_\mathbb{C}^*$) is related to the fact that $\mu \in P^+$ if and only if

$$\mu = \sum_{j=1}^l \mu_j \omega_j \quad \text{with } \mu_j \in \mathbb{Z}^+, j = 1, \dots, l.$$

We shall write for such an element:

$$|\mu| = \mu_1 + \cdots + \mu_l. \quad (1.12)$$

If $\mu \in P$, then the exponential $e^\mu : A_{\mathbb{C}} \rightarrow \mathbb{C}$ defined by $e^\mu(h) := e^{\mu(\log h)}$ is single valued. The e^μ with $\mu \in P$ are the algebraic characters of $A_{\mathbb{C}}$, and their \mathbb{C} -linear span coincides with the ring of regular functions $\mathbb{C}[A_{\mathbb{C}}]$ on the affine algebraic variety $A_{\mathbb{C}}$. The set $A_{\mathbb{C}}^{\text{reg}} := \{h \in A_{\mathbb{C}} : e^{2\alpha(h)} = 1 \text{ for all } \alpha \in \Sigma\}$ consists of the regular points of $A_{\mathbb{C}}$ for the action of W . The algebra $\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]$ of regular functions on $A_{\mathbb{C}}^{\text{reg}}$ is the subalgebra of the quotient field of $\mathbb{C}[A_{\mathbb{C}}]$ generated by $\mathbb{C}[A_{\mathbb{C}}]$ and by $1/(1 - e^{-2\alpha})$ for $\alpha \in \Sigma^+$.

1.3. c -functions. For $\alpha \in \Sigma^+$ define

$$c_\alpha(\lambda) = \frac{\Gamma(\lambda_\alpha + \frac{m_\alpha/2}{4})}{\Gamma(\lambda_\alpha + \frac{m_\alpha/2}{4} + \frac{m_\alpha}{2})}. \quad (1.13)$$

Set $\Gamma^*(x) = \Gamma(1 - x)^{-1}$ and let c_α^* be given by the same formula as c_α but with Γ^* in place of Γ , i.e.

$$c_\alpha^*(\lambda) = \frac{\Gamma(1 - \lambda_\alpha - \frac{m_\alpha/2}{4} - \frac{m_\alpha}{2})}{\Gamma(1 - \lambda_\alpha - \frac{m_\alpha/2}{4})}. \quad (1.14)$$

Then, by the classical relation $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}$, we obtain

$$c_\alpha^*(\lambda) = S_\alpha(\lambda)c_\alpha(\lambda) \quad (1.15)$$

where

$$S_\alpha(\lambda) = \frac{\sin(\pi(\lambda_\alpha + \frac{m_\alpha/2}{4}))}{\sin(\pi(\lambda_\alpha + \frac{m_\alpha/2}{4} + \frac{m_\alpha}{2}))}. \quad (1.16)$$

Define (see e.g. [20, (3.4.2) and (3.5.2)])

$$\tilde{c}(\lambda) = \prod_{\alpha \in \Sigma^+} c_\alpha(\lambda), \quad (1.17)$$

$$\tilde{c}^*(\lambda) = \prod_{\alpha \in \Sigma^+} c_\alpha^*(\lambda), \quad (1.18)$$

and set

$$c(\lambda) = c_{\text{HC}} \tilde{c}(\lambda), \quad (1.19)$$

$$c^*(\lambda) = c_{\text{HC}}^* \tilde{c}^*(\lambda), \quad (1.20)$$

where c_{HC} and c_{HC}^* are normalizing constants so that $c(\rho) = 1$ and $c^*(-\rho) = 1$. Recall the classical relation $\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z)\Gamma(z + \frac{1}{2})$ and that $\lambda_{\alpha/2} = 2\lambda_\alpha$. Then we can write the functions c and c^* as

$$\begin{aligned} c(\lambda) &= c_{\text{HC}} \prod_{\beta \in \Sigma_*^+} \frac{\Gamma(2\lambda_\beta)}{\Gamma(2\lambda_\beta + \frac{m_\beta/2}{4})} \frac{\Gamma(\lambda_\beta + \frac{m_\beta/2}{4})}{\Gamma(\lambda_\beta + \frac{m_\beta/2}{4} + \frac{m_\beta}{2})} \\ &= c'_{\text{HC}} \prod_{\beta \in \Sigma_*^+} \frac{2^{-2\lambda_\beta} \Gamma(2\lambda_\beta)}{\Gamma(\lambda_\beta + \frac{m_\beta/2}{4} + \frac{1}{2}) \Gamma(\lambda_\beta + \frac{m_\beta/2}{4} + \frac{m_\beta}{2})} \end{aligned} \quad (1.21)$$

and, similarly,

$$c^*(\lambda) = c_{\text{HC}}^* \prod_{\beta \in \Sigma_*^+} \frac{\Gamma(1 - 2\lambda_\beta - \frac{m_\beta/2}{2}) \Gamma(1 - \lambda_\beta - \frac{m_\beta/2}{4} - \frac{m_\beta}{2})}{\Gamma(1 - 2\lambda_\beta) \Gamma(1 - \lambda_\beta - \frac{m_\beta/2}{4})} \quad (1.22)$$

$$= (c_{\text{HC}}^*)' \prod_{\beta \in \Sigma_*^+} \frac{\Gamma(-\lambda_\beta - \frac{m_\beta/2}{4} + \frac{1}{2}) \Gamma(1 - \lambda_\beta - \frac{m_\beta/2}{4} - \frac{m_\beta}{2})}{2^{2\lambda_\beta} \Gamma(1 - 2\lambda_\beta)}. \quad (1.23)$$

In (1.21) and (1.23) we have

$$c'_{\text{HC}} = c_{\text{HC}} \prod_{\beta \in \Sigma_*^+} (\pi^{1/2} 2^{1-m_\beta/2/2}) \quad \text{and} \quad (c_{\text{HC}}^*)' = c_{\text{HC}}^* \prod_{\beta \in \Sigma_*^+} (\pi^{-1/2} 2^{-m_\beta/2/2}). \quad (1.24)$$

1.4. Heckman-Opdam's hypergeometric functions. Let $S(\mathfrak{a}_{\mathbb{C}})$ denote the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}$ considered as the space of polynomial functions on $\mathfrak{a}_{\mathbb{C}}^*$. Every $p \in S(\mathfrak{a}_{\mathbb{C}})$ defines a constant-coefficient differential operator $\partial(p)$ on $A_{\mathbb{C}}$ and on $\mathfrak{a}_{\mathbb{C}}$ such that $\partial(x)$ is the directional derivative in the direction of x for all $x \in \mathfrak{a}$. The algebra of the differential operators $\partial(p)$ with $p \in S(\mathfrak{a}_{\mathbb{C}})$ will also be indicated by $S(\mathfrak{a}_{\mathbb{C}})$. We denote by $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) := \mathbb{C}[A_{\mathbb{C}}^{\text{reg}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$ the algebra of differential operators on $A_{\mathbb{C}}$ with coefficients in $\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]$. Let L_A denote the Laplace operator on A and set

$$L_m := L_A + \sum_{\alpha \in \Sigma^+} m_\alpha \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \partial(A_\alpha). \quad (1.25)$$

Heckman and Opdam proved in [19] that, for the triple $(\mathfrak{a}, \Sigma, m)$, the commutant $\mathbb{D}(\mathfrak{a}, m, \Sigma) := \{Q \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W : L_m Q = Q_m\}$ of L_m in $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$ is a commutative algebra which plays the role the commutative algebra of the radial components on A^+ of the invariant differential operators on a Riemannian symmetric space of noncompact type. For geometric triples $(\mathfrak{a}, \Sigma, m)$ the operator L_m coincides with the radial component on A^+ of the Laplace operator on a Riemannian symmetric space of noncompact type. The algebra $\mathbb{D}(\mathfrak{a}, m, \Sigma)$ can be constructed algebraically. Indeed one has

$$\mathbb{D}(\mathfrak{a}, m, \Sigma) := \{D_p : p \in S(\mathfrak{a}_{\mathbb{C}})^W\},$$

where the differential operator D_p can be explicitly given in terms of Cherednik operators (also called trigonometric Dunkl operators); see [28] or [18].

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be fixed. The system of differential equations

$$D_p \varphi = p(\lambda) \varphi, \quad p \in S(\mathfrak{a}_{\mathbb{C}})^W, \quad (1.26)$$

is called the *hypergeometric system of differential equations with spectral parameter* λ associated with the data $(\mathfrak{a}, \Sigma, m)$. The *hypergeometric function of spectral parameter* λ is the unique analytic W -invariant function $F_\lambda(a)$ on A which satisfies the system of differential equations (1.26) and which is normalized by $F_\lambda(e) = 1$. Here $e = \exp 0$. In the geometric case, the function F_λ agrees with Harish-Chandra's (elementary) spherical function of spectral parameter λ .

Example 1.1 (The rank-one case). The rank-one case corresponds to triples $(\mathfrak{a}, \Sigma, m)$ in which \mathfrak{a} is one dimensional. Then the set Σ^+ consists at most of two elements: β and, possibly, $\beta/2$. Fix $H_0 \in \mathfrak{a}$ such that $\beta(H_0) = 1$ and normalize the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{a} so that $\langle H_0, H_0 \rangle = 1$. Then, in the notation of Section 1.1, $H_\beta/2 = A_\beta = H_0$ and

$\langle \beta, \beta \rangle = \langle A_\beta, A_\beta \rangle = 1$. We identify \mathfrak{a} and \mathfrak{a}^* with \mathbb{R} (and their complexifications $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ with \mathbb{C}) by identifying H_β and β with 1. Hence $\lambda = \lambda_\beta \beta \in \mathfrak{a}_\mathbb{C}^*$ is identified with $\lambda_\beta \in \mathbb{C}$ and $H \in \mathfrak{a}_\mathbb{C}$ with $\beta(H)/2 \in \mathbb{C}$. In the following we shall use the simplified notation $\lambda \in \mathbb{C}$ instead of $\lambda_\beta \in \mathbb{C}$. We also identify A and $\mathfrak{a} \equiv \mathbb{R}$ by means of the exponential map. The Weyl chamber \mathfrak{a}^+ coincides with the half-line $]0, +\infty[$. The Weyl group W reduces to $\{-1, 1\}$ acting on \mathbb{R} and \mathbb{C} by multiplication. The algebra $\mathbb{D}(\mathfrak{a}, \Sigma, m)$ is generated by L_m , and the hypergeometric differential system with spectral parameter $\lambda \in \mathbb{C}$ is equivalent to the single Jacobi differential equation. Heckman-Opdam's hypergeometric function F_λ coincides with the Jacobi function of the first kind

$$F_\lambda(\exp H) = {}_2F_1\left(\frac{1}{2}\left(\frac{m_\beta/2}{2} + m_\beta\right) + \lambda, \frac{1}{2}\left(\frac{m_\beta/2}{2} + m_\beta\right) - \lambda; \frac{m_\beta/2 + m_\beta + 1}{2}; -\sinh^2 \frac{\beta(H)}{2}\right).$$

Schapira proved in [31] that F_λ is real and strictly positive for $\lambda \in \mathfrak{a}^*$. Moreover

$$|F_\lambda| \leq F_{\operatorname{Re} \lambda}, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*. \quad (1.27)$$

For every $\lambda \in \mathfrak{a}_\mathbb{C}^*$ the hypergeometric function F_λ extends holomorphically as a W -invariant function on the domain $\exp(2\Omega_\pi)$ in $A_\mathbb{C}$, where

$$\Omega_\pi = \{H \in \mathfrak{a}_\mathbb{C} : |\alpha(\operatorname{Im} H)| < \pi/2 \text{ for all } \alpha \in \Sigma\}. \quad (1.28)$$

An elementary proof of this fact was given by J. Faraut at the conference ‘‘Harmonic Analysis on Complex Homogeneous Domains and Lie Groups’’, Rome, May 17-19, 2001. Faraut's argument has been reproduced in [6, p. 26].

We shall need the following estimates of the holomorphically extended hypergeometric functions. Recall the constant Ω from (1.11).

Lemma 1.2. *There is a constant $C > 0$ so that*

$$|F_\lambda(\exp H)| \leq C e^{-\min_{w \in W} \operatorname{Im}(w\lambda(H_2)) + \max_{w \in W} \operatorname{Re}(w\lambda(H_1))} \quad (1.29)$$

for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and all $H = H_1 + iH_2 \in \overline{\Omega_\pi}$ with $H_1, H_2 \in \mathfrak{a}$. In particular:

(a) For all $\lambda \in \mathfrak{a}_+^* + i\mathfrak{a}^*$ and $H \in \mathfrak{a}$ we have

$$|F_\lambda(\exp H)| \leq C e^{\Omega \|H\| (\sum_{j=1}^l \operatorname{Re} \lambda_j)}$$

(b) For all $H \in \overline{\Omega_\pi}$ and $\lambda \in \mathfrak{a}^*$ we have

$$|F_\lambda(\exp H)| \leq C e^{\|\operatorname{Im} H\| \|\operatorname{Im} \lambda\|}.$$

Proof. (see [30, Lemma 5.1]) The estimates (1.29) are due to Opdam; see [28], Proposition 6.1(2) and Theorem 3.15. For (a), we can suppose by W -invariance that $H \in \overline{\mathfrak{a}^+}$. In this case, for $\lambda \in \mathfrak{a}_+^* + i\mathfrak{a}^*$, we have

$$0 \leq \operatorname{Re} \lambda(H) = \sum_{j=1}^l \operatorname{Re} \lambda_j \omega_j(H) \leq \Omega \|H\| \left(\sum_{j=1}^l \operatorname{Re} \lambda_j \right).$$

Part (b) follows immediately from (1.29). □

1.5. The hypergeometric Fourier transform. Let da denote a fixed normalization of the Haar measure on A . We associate with the triple $(\mathfrak{a}, \Sigma, m)$ the measure $d\mu(a) = \mu(a) da$ on A , where

$$\mu(a) = \prod_{\alpha \in \Sigma^+} |e^{\alpha(H)} - e^{-\alpha(H)}|^{m_\alpha}, \quad a = \exp(H). \quad (1.30)$$

Notice that when $(\mathfrak{a}, \Sigma, m)$ comes from a Riemannian symmetric space G/K , then $d\mu$ is the component along A of the Haar measure on G with respect to the Cartan decomposition $G = KAK$.

The *hypergeometric Fourier transform* $\mathcal{F}f = \tilde{f}$ of a sufficiently regular W -invariant functions on A is the W -invariant function on $i\mathfrak{a}^*$ defined by

$$(\mathcal{F}f)(\lambda) = \tilde{f}(\lambda) = \int_A f(a) F_{-\lambda}(a) d\mu(a). \quad (1.31)$$

The Plancherel theorem states that the hypergeometric Fourier transform \mathcal{F} is an isometry of $L^2(A, d\mu)^W$ onto $L^2(i\mathfrak{a}^*, |W|^{-1}|c(\lambda)|^{-2}d\lambda)^W$. Here $|W|$ denotes the order of the Weyl group W and c is the c -function (1.19). Moreover, \mathcal{F} has the following inversion formula, which holds for instance if $f \in L^p(A, d\mu)^W$, with $1 \leq p \leq 2$, and $\mathcal{F}f \in L^1(i\mathfrak{a}^*, |c(\lambda)|^{-2}d\lambda)^W$: for almost all $a \in A$ we have

$$f(a) = \frac{1}{|W|} \int_{i\mathfrak{a}^*} \tilde{f}(\lambda) F_\lambda(a) \frac{d\lambda}{c(\lambda)c(-\lambda)}. \quad (1.32)$$

See [26, Theorem 5.4].

Finally, we shall need the W -invariant L^p -Schwartz space isomorphism. For $1 \leq p \leq 2$, the L^p -Schwartz space $\mathcal{S}^p(A)^W$ is the set of all W -invariant C^∞ functions f on A such that for each $N \in \mathbb{N}_0$ and $q \in S(\mathfrak{a}_\mathbb{C})$,

$$\sup_{H \in \mathfrak{a}} (1 + \|H\|)^N F_0(\exp H)^{-\frac{2}{p}} |\partial(q)f(\exp H)| < \infty. \quad (1.33)$$

Notice that $C_c^\infty(A)^W \subset \mathcal{S}^p(A)^W \subset L^p(A, d\mu)^W$. Hence $\mathcal{S}^p(A)^W$ is dense in $L^p(A, d\mu)^W$. Moreover, $\mathcal{S}^p(A)^W$ is a Fréchet space with respect to the seminorms defined by the left-hand side of (1.33). Set $\epsilon_p = \frac{2}{p} - 1$. Let $C(\epsilon_p \rho)$ be the convex hull in \mathfrak{a}^* of the set $\{\epsilon_p w \rho : w \in W\}$, and let $\mathfrak{a}_{\epsilon_p}^* = C(\epsilon_p \rho) + i\mathfrak{a}^*$. Notice that, for $p = 1$, the set $\mathfrak{a}_{\epsilon_1}^* = C(\rho) + i\mathfrak{a}^*$ is precisely the set of parameters λ for which F_λ is bounded; see [26, Theorem 4.2]. Let $\mathcal{S}(\mathfrak{a}_{\epsilon_p}^*)^W$ be the set of all W -invariant functions $g : \mathfrak{a}_{\epsilon_p}^* \rightarrow \mathbb{C}$ which are holomorphic in the interior of $\mathfrak{a}_{\epsilon_p}^*$, continuous on $\mathfrak{a}_{\epsilon_p}^*$ and satisfy for all $r \in \mathbb{N}_0$ and $s \in S(\mathfrak{a}_\mathbb{C})$

$$\sup_{\lambda \in \mathfrak{a}_{\epsilon_p}^*} (1 + \|\lambda\|)^r |\partial(s)g(\lambda)| < \infty. \quad (1.34)$$

Then $\mathcal{S}(\mathfrak{a}_{\epsilon_p}^*)^W$ is a Fréchet algebra under pointwise multiplication and with the topology induced by the seminorms defined by the left-hand side of (1.34). Notice that when $p = 2$, this space reduces to the usual space of Schwartz functions on $i\mathfrak{a}^*$. The Schwartz space isomorphism theorem states that the hypergeometric Fourier transform is a topological isomorphism between $\mathcal{S}^p(A)^W$ and $\mathcal{S}(\mathfrak{a}_{\epsilon_p}^*)^W$. This theorem was proved in [31, Theorem 4.1] for the case $p = 2$ and in [26, Theorem 5.6] for the general case. (The proof is in fact an easy adaptation of Anker's method for the geometric case [1].)

2. JACOBI POLYNOMIALS AND JACOBI SERIES

2.1. Jacobi polynomials. Recall that $\mathbb{C}[A_{\mathbb{C}}]$ denotes the space of finite \mathbb{C} -linear combinations of elements e^{μ} with $\mu \in P$, and let $\mathbb{C}[A_{\mathbb{C}}]^W$ be its subspace of W -invariant elements. Set

$$\delta(m, t) := \prod_{\alpha \in \Sigma^+} |e^{i\alpha(H)} - e^{-i\alpha(H)}|^{m_{\alpha}}$$

for $t = \exp(iH)$ with $H \in \mathfrak{a}$. Define an inner product $\langle \cdot, \cdot \rangle_m$ on $\mathbb{C}[A_{\mathbb{C}}]^W$ by

$$\langle f, g \rangle_m := \int_T f(t) \overline{g(t)} \delta(m, t) dt,$$

where dt is the normalized Haar measure on T .

Recall from (1.4) the lattice P^+ of positive restricted weights. The *orbit sums*

$$M_{\mu} := \sum_{\nu \in W\mu} e^{\nu} \tag{2.1}$$

form a basis of $\mathbb{C}[A_{\mathbb{C}}]^W$ as μ varies in P^+ because each W -orbit in P intersects P^+ in exactly one point. The *Jacobi polynomial* P_{μ} is the exponential polynomial

$$P_{\mu} := \sum_{\nu \geq \mu \in P^+} c_{\mu\nu} M_{\nu} \tag{2.2}$$

where the coefficients $c_{\mu\nu}$ are defined by the following conditions, cf. [20, §1.3], or [25, §11]:

- (a) $c_{\mu\mu} = 1$,
- (b) $\langle P_{\mu}, M_{\nu} \rangle_m = 0$ for all $\nu \in P^+$ with $\nu < \mu$.

Condition (b) turns out to be equivalent to the fact that P_{μ} satisfies the 2nd order differential equation on T :

$$L_m^T P_{\mu} = -\langle \mu + 2\rho, \mu \rangle P_{\mu} \tag{2.3}$$

where

$$L_m^T = L_T + \sum_{\alpha \in \Sigma^+} m_{\alpha} \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \partial(iA_{\alpha}),$$

and L_T is the Euclidean Laplace-Beltrami operator on the compact torus T . The fact that P_{μ} is an eigenfunction of L_m^T yields recursion relations for the coefficients $c_{\mu\nu}$. One can then deduce that $c_{\mu\nu} > 0$ for all $\mu, \nu \in P^+$ with $\nu \leq \mu$. See [25, pp. 34–35].

Observe that, by definition, $P_{\mu}(t)$ extends holomorphically to $A_{\mathbb{C}}$ as a function of t . Moreover $\{P_{\mu} \mid \mu \in P^+\}$ is a basis for $\mathbb{C}[A_{\mathbb{C}}]^W$ which is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_m$ (cf. [20, Corollary 1.3.13]).

The relation between Jacobi polynomials and Heckman-Opdam's hypergeometric functions is given by the following lemma.

Lemma 2.1. *Let $\mu \in P^+$. Then for all $a \in A$*

$$F_{\mu+\rho}(a) = c(\mu + \rho) P_{\mu}(a). \tag{2.4}$$

Formula (2.4) provides a holomorphic extension of $F_{\mu+\rho}$ to $A_{\mathbb{C}}$. Moreover, for all $\mu \in P^+$ and $w \in W$, we have

$$c(w(\mu + \rho) - \rho) P_{w(\mu+\rho)-\rho} = c(\mu) P_{\mu}. \tag{2.5}$$

Proof. Formula (2.4) is [20, (4.4.1)], and (2.5) is a consequence of (2.4) and the W -invariance of F_λ in $\lambda \in \mathfrak{a}_\mathbb{C}^*$. \square

Example 2.2 (The rank-one case). We keep the notation and identifications of Example 1.1. In particular, we have the identification of P^+ with \mathbb{Z}^+ so that $\mu = n\beta \in P^+$ corresponds to $n \in \mathbb{Z}^+$. The polynomial

$$F_{n\beta+\rho}(\exp H) = {}_2F_1\left(n + m_\beta + \frac{m_\beta/2}{2}, -n; \frac{m_\beta/2 + m_\beta + 1}{2}; \frac{1 - \cos \beta(H)}{2}\right). \quad (2.6)$$

is related to the classical *Jacobi polynomial* $P_n^{(a,b)}(x)$ (see e.g. [12, 10.8(16)]) by

$$\binom{n+a}{n} F_{n\beta+\rho}(\exp H) = P_n^{(a,b)}(\cos \beta(H))$$

where

$$a = (m_\beta/2 + m_\beta - 1)/2 \quad \text{and} \quad b = (m_\beta - 1)/2.$$

As a special instance, one finds the *symmetric Jacobi polynomials*

$$X_n^{(m-1)/2}(x) = {}_2F_1\left(-n, n + m; \frac{m+1}{2}; \frac{1-x}{2}\right), \quad (2.7)$$

which are constant multiples of the Jacobi polynomials $P_n^{(a,a)}(x)$ with $a = \frac{m-1}{2}$. The polynomials (2.7) are hypergeometric functions corresponding to a rank-one reduced root system Σ with $m_\beta = m$. By selecting special values of m , one obtains specific classes of special polynomials, such as the *Legendre polynomials*

$$P_n(x) = P_n^{(0,0)}(x) = X_n^{-1/2}(x) = {}_2F_1\left(-n, n + 1; 1; \frac{1-x}{2}\right),$$

the *Tchebichef polynomials of first kind*

$$T_n(x) = X_n^{-1/2}(x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right),$$

and the *Tchebichef polynomials of second kind*

$$U_n(x) = (n+1)X_n^{1/2}(x) = (n+1) {}_2F_1\left(-n, n + 2; \frac{3}{2}; \frac{1-x}{2}\right).$$

See [12, formulas 10.10(3), 10.11(24), 10.11(25)]

Proposition 2.3. *For $\mu \in P^+$, the Jacobi polynomial $P_\mu(a)$ is real valued and positive for $a \in A$. For all $h = ta$ with $t \in T$, $a \in A$, we have $|P_\mu(h)| \leq P_\mu(a)$. Moreover, for all $a = \exp(H) \in \overline{A^+}$,*

$$e^{\mu(H)} \leq P_\mu(a) \leq c(\mu + \rho)^{-1} e^{\mu(H)}. \quad (2.8)$$

Similarly, the hypergeometric function $F_{\mu+\rho}(a)$ is real valued and positive for $a \in A$. For all $h = ta$ with $t \in T$, $a \in A$, we have $|F_{\mu+\rho}(h)| \leq F_{\mu+\rho}(a)$. Furthermore, for all $a = \exp(H) \in \overline{A^+}$,

$$c(\mu + \rho) e^{\mu(H)} \leq F_{\mu+\rho}(a) \leq e^{\mu(H)}. \quad (2.9)$$

Proof. Notice first that $c(\mu + \rho) > 0$. Recall the defining formula (2.2) for P_μ and that $c_{\mu\nu} > 0$ for all $\nu < \mu$ and $c_{\mu\mu} = 1$. By (2.1), we have $M_\nu(e) = |W\nu| = |W|/|W_\nu|$, where $W_\nu = \{w \in W : w\nu = \nu\}$. Hence (2.4) and (2.2) yield

$$c(\mu + \rho) \sum_{\mu \geq \nu \in P^+} c_{\mu\nu} \frac{|W|}{|W_\nu|} = c(\mu + \rho) P_\mu(e) = F_{\mu+\rho}(e) = 1.$$

Moreover, $w\nu(H) \leq \nu(H) \leq \mu(H)$ for all $w \in W$, $\nu \in P^+$ with $\nu \leq \mu$ and $H \in \overline{\mathfrak{a}^+}$. So for $a = \exp(H) \in A^+$ we have

$$0 \leq M_\nu(a) \leq \frac{|W|}{|W_\nu|} e^{\nu(H)} \leq \frac{|W|}{|W_\nu|} e^{\mu(H)}.$$

Therefore

$$P_\mu(a) \leq \sum_{\mu \geq \nu \in P^+} c_{\mu\nu} \frac{|W|}{|W_\nu|} e^{\mu(H)} = c(\mu + \rho)^{-1} e^{\mu(H)}.$$

Finally,

$$P_\mu(a) = \sum_{\mu \geq \nu \in P^+} c_{\mu\nu} \sum_{\eta \in W\nu} e^{\eta(H)} \geq c_{\mu\mu} e^{\mu(H)} = e^{\mu(H)}.$$

This proves the first part of the proposition. The results for $F_{\mu+\rho}$ are then an immediate consequence of (2.4). \square

Remark 2.4. By using the non-symmetric hypergeometric functions, Schapira proved, more generally, that $F_\lambda(a)$ is real valued and positive for all $\lambda \in \mathfrak{a}^*$ and $a \in A$. See [31, Lemma 3.1 and Corollary 3.1]. The estimate (2.9) is classical for the special case of spherical functions on Riemannian symmetric spaces of the compact type. See e.g. [13, Proposition IV.5.2].

2.2. The norm of the Jacobi polynomials. Let \tilde{c} and \tilde{c}^* be as in Section 1.3. Set $\|f\|_m = \langle f, f \rangle_m^{1/2}$ for the L^2 -norm of $f \in \mathbb{C}[A_{\mathbb{C}}]^W$. The L^2 -norm of P_μ has been computed by Opdam. It is given by

$$\|P_\mu\|_m^2 = |W| \frac{\tilde{c}^*(-\mu - \rho)}{\tilde{c}(\mu + \rho)} \tag{2.10}$$

$$= |W| \prod_{\alpha \in \Sigma^+} \frac{\Gamma(\mu_\alpha + \rho_\alpha + \frac{m_\alpha/2 + m_\alpha}{4})}{\Gamma(\mu_\alpha + \rho_\alpha + \frac{m_\alpha/2}{4})} \frac{\Gamma(1 + \mu_\alpha + \rho_\alpha - \frac{m_\alpha/2}{4} - \frac{m_\alpha}{2})}{\Gamma(1 + \mu_\alpha + \rho_\alpha - \frac{m_\alpha/2}{4})}. \tag{2.11}$$

See [27] or [20, Theorem 3.5.5]. Moreover, by [20, (3.5.14)],

$$|W| = \prod_{\alpha \in \Sigma^+} \frac{(\rho_\alpha + \frac{m_\alpha/2}{4} + \frac{m_\alpha}{2})}{(\rho_\alpha + \frac{m_\alpha/2}{4})}. \tag{2.12}$$

Formula (2.10) for $\mu = 0$ yields

$$I_\delta := \int_T |\delta(m, t)| dt = \|P_0\|_m^2 = |W| \frac{\tilde{c}^*(-\rho)}{\tilde{c}(\rho)}. \tag{2.13}$$

It follows from (2.4) and (2.10) that the L^2 -norm of $F_{\mu+\rho}$ is

$$\|F_{\mu+\rho}\|_m^2 = |W| c_{\text{HC}}^2 \tilde{c}^*(-\mu - \rho) \tilde{c}(\mu + \rho) \tag{2.14}$$

$$= I_\delta \frac{\tilde{c}^*(-\mu - \rho) \tilde{c}(\mu + \rho)}{\tilde{c}^*(-\rho) \tilde{c}(\rho)} \tag{2.15}$$

$$= I_\delta c^*(-\mu - \rho) c(\mu + \rho). \tag{2.16}$$

Let $d(\mu)$ be the function on P^+ defined by means of Vretare's formula, see [23, Theorem 9.10, p. 321]:

$$d(\mu) = \frac{c(\lambda - \mu) c(-\lambda + \mu)}{c(\lambda) c(-\lambda)} \Big|_{\lambda=\mu+\rho} = \frac{1}{c(\mu + \rho)} \frac{c(-\lambda + \mu)}{c(-\lambda)} \Big|_{\lambda=\mu+\rho},$$

where we have used that $c(\rho) = 1$. Notice that, by (1.15),

$$\frac{c(-\lambda + \mu)}{c(-\lambda)} = \prod_{\alpha \in \Sigma^+} \frac{S_\alpha(-\lambda)}{S_\alpha(-\lambda + \mu)} \frac{c^*(-\lambda + \mu)}{c^*(-\lambda)} = \frac{c^*(-\lambda + \mu)}{c^*(-\lambda)}$$

since $S_\alpha(-\lambda) = S_\alpha(-\lambda + \mu)$ for $\mu_\alpha \in \mathbb{Z}$. Hence, since $c^*(-\rho) = 1$, we have

$$d(\mu) = \frac{1}{c(\mu + \rho)c^*(-\mu - \rho)}. \quad (2.17)$$

Formula (2.16) becomes therefore

$$\|F_{\mu+\rho}\|_m^2 = I_\delta \frac{1}{d(\mu)}. \quad (2.18)$$

Remark 2.5. In the case of Riemannian symmetric spaces of the compact type U/K , (2.18) reduces to the classical formula (consequence of Schur's orthogonality relations) relating the L^2 -norm of the spherical function $\psi_\mu = F_{\mu+\rho}$ on U/K to the dimension $d(\mu)$ of the corresponding spherical representation, see for instance [13, p. 146]. Indeed, let du and dt be respectively the invariant probability measures on U and T . Then, for every K -biinvariant continuous function f on U , we have $\int_U f(u) du = C \int_T f(t) |\delta(m, t)| dt$ where $C = [\int_T |\delta(m, t)| dt]^{-1} = I_\delta^{-1}$. Hence (2.18) gives

$$\int_U |\psi_\mu(u)|^2 du = I_\delta^{-1} \int_T |\psi_\mu(t)|^2 |\delta(m, t)| dt = I_\delta^{-1} \|\psi_\mu\|_m^2 = d(\mu)^{-1}.$$

2.3. The Jacobi transform and Jacobi series. The (normalized) Jacobi transform of $f \in L^2(T)^W$ is the function $\widehat{f} : P^+ \rightarrow \mathbb{C}$ defined by

$$\widehat{f}(\mu) := I_\delta^{-1} \langle f, P_\mu \rangle_m = I_\delta^{-1} \int_T f(t) P_\mu(t^{-1}) \delta(m, t) dt.$$

Here we have used the property that $\overline{P_\mu(t)} = P_\mu(t^{-1})$ for $t \in T$. Observe that, by (2.4),

$$c(\rho + \mu) \widehat{f}(\mu) = I_\delta^{-1} \langle f, F_{\mu+\rho} \rangle_m. \quad (2.19)$$

The inversion formula is given by the *Jacobi series*

$$f = I_\delta \sum_{\mu \in P^+} \widehat{f}(\mu) \frac{P_\mu}{\|P_\mu\|_m^2} \quad (2.20)$$

$$= \sum_{\mu \in P^+} d(\mu) c(\mu + \rho) \widehat{f}(\mu) F_{\mu+\rho} \quad (2.21)$$

with convergence in the sense of L^2 . In (2.21) we have used (2.4) and (2.18).

Set $B = \{H \in \mathfrak{a} : \|H\| < 1\}$. For $\varepsilon > 0$ the U -invariant domain in $A_\mathbb{C}$ defined by $D_\varepsilon = T \exp(\varepsilon B)$ is a W -invariant neighborhood of T in $A_\mathbb{C}$. In the following lemma we study the holomorphic extension of Jacobi series. It generalizes a result proven by Lassalle in the geometric case; see [24] or [13, Proposition V.2.3].

Lemma 2.6. *Let $G : P^+ \rightarrow \mathbb{C}$ and let $\varepsilon > 0$. The Jacobi series*

$$\sum_{\mu \in P^+} d(\mu) G(\mu) F_{\mu+\rho}(x) \quad (2.22)$$

converges normally on compact subsets of D_ε if and only if there is a constant $C > 0$ and so that

$$|G(\mu)| \leq C e^{-\varepsilon \|\mu\|} \quad (2.23)$$

for all $\mu \in P^+$. In this case, its sum is a W -invariant holomorphic function on D_ε .

Proof. Suppose that G satisfies (2.23). It is enough to prove the normal convergence of (2.22) on compact subsets of the form $\overline{D_r} = T \exp(r\overline{B})$ where $\overline{B} = \{H \in \mathfrak{a} : \|H\| \leq 1\}$ and $0 < r < \varepsilon$. Let $h = t \exp(rH)$ with $H \in \overline{B} \cap \overline{\mathfrak{a}^+}$. By Proposition 2.3 we have

$$|F_{\mu+\rho}(h)| \leq F_{\mu+\rho}(\exp(rH)) \leq e^{r\mu(H)} \leq e^{r\|\mu\|}.$$

This estimate extends to $\overline{D_r}$ by W -invariance. Hence

$$|d(\mu)G(\mu)F_{\mu+\rho}(h)| \leq d(\mu)e^{-(\varepsilon-r)\|\mu\|},$$

which implies the convergence of (2.22) since $d(\mu)$ has polynomial growth.

Conversely, suppose that the series (2.22) converges normally on the compact subsets of D_ε . Let $0 < \varepsilon' < \delta < \varepsilon$. By the normal convergence of (2.22) on D_δ , we have for all $H \in \exp(\delta B) \subset D_\delta$:

$$1 \geq |d(\mu)G(\mu)F_{\mu+\rho}(\exp H)| = \left| \frac{G(\mu)}{c^*(-\mu - \rho)} \right| P_\mu(\exp H).$$

Hence, by (2.8), for all $H \in \exp(\delta B)$,

$$|G(\mu)| \leq |c^*(-\mu - \rho)| e^{\mu(H)}.$$

Taking $H = -(\delta/\|\mu\|)A_\mu$, we obtain that

$$|G(\mu)| \leq |c^*(-\mu - \rho)| e^{-\delta\|\mu\|} \leq C e^{-\varepsilon'\|\mu\|}$$

where C is a constant independent of μ and ε' . This implies (2.23). \square

3. THE FUNCTION d

We extend $d(\mu)$, $\mu \in P^+$, to a meromorphic function on $\mathfrak{a}_\mathbb{C}^*$ by means of (2.17):

$$d(\lambda) = \frac{1}{c(\lambda + \rho)c^*(-\lambda - \rho)} \quad (3.1)$$

When $(\mathfrak{a}, \Sigma, m)$ is geometric, d coincides with the polynomial given by Weyl's dimension formula (written in terms of restricted roots). The polynomial nature of d is precised by Lemma 3.1 below. In the following, we set

$$\Pi(\lambda) = \prod_{\beta \in \Sigma_*^+} \lambda_\beta. \quad (3.2)$$

Recall also the notation $\tilde{\rho}_\beta = \frac{1}{2}(\frac{m_\beta/2}{2} + m_\beta)$ and the constant L_β from (4.8).

Lemma 3.1. (a) *We have*

$$\begin{aligned} d(\lambda - \rho) &= \frac{1}{c(\lambda)c^*(-\lambda)} \\ &= C_d \Pi(\lambda) \prod_{\beta \in \Sigma_*^+} \frac{\Gamma(\lambda_\beta + \frac{m_\beta/2}{4} + \frac{1}{2})\Gamma(\lambda_\beta + \tilde{\rho}_\beta)}{\Gamma(\lambda_\beta - \frac{m_\beta/2}{4} + \frac{1}{2})\Gamma(\lambda_\beta - \tilde{\rho}_\beta + 1)} \end{aligned} \quad (3.3)$$

where

$$C_d = \tilde{c}(\rho)\tilde{c}^*(-\rho) \prod_{\beta \in \Sigma_*^+} 2^{m_{\beta/2}}. \quad (3.4)$$

(b) If $(m_{\beta/2})/2 \in \mathbb{Z}^+$ and $0 \neq m_\beta \in \mathbb{Z}^+$, then $d(\lambda)$ is a polynomial in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Explicitly,

$$d(\lambda - \rho) = C_d \prod_{\beta \in \Sigma_*^+} \left(\lambda_\beta \prod_{k=0}^{\frac{m_{\beta/2}}{2}-1} [\lambda_\beta - (\frac{m_{\beta/2}}{4} - \frac{1}{2}) + k] \prod_{k=0}^{2\tilde{\rho}_\beta-2} [\lambda_\beta - (\tilde{\rho}_\beta - 1) + k] \right). \quad (3.5)$$

Here we adopt the convention that empty products are equal to 1.

(c) The function $d(\lambda)$ has polynomial growth. More precisely, set

$$M = \sum_{\beta \in \Sigma_*^+} (m_{\beta/2} + m_\beta). \quad (3.6)$$

Then, for every positive $\varepsilon > 0$, there are constants C_0, C'_0 and C''_0 so that

$$|d(\lambda - \rho)| \leq C_0 \prod_{\beta \in \Sigma_*^+} (1 + |\lambda_\beta|)^{m_{\beta/2} + m_\beta} \leq C'_0 (1 + \|\lambda\|)^M \leq C''_0 \prod_{j=1}^l (1 + |\lambda_j|)^M \quad (3.7)$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\max_{\beta \in \Sigma_*^+} |\arg(\lambda_\beta)| \leq \pi - \varepsilon$ and for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re} \lambda_\beta \geq -L_\beta + \varepsilon$ for all $\beta \in \Sigma_*^+$.

Proof. Part (a) is an immediate consequence of (1.21) and (1.23).

Recall that, if $0 \neq 2a \in \mathbb{Z}^+$, then

$$\frac{\Gamma(\lambda_\beta + a)}{\Gamma(\lambda_\beta - a + 1)} = \frac{\Gamma(\lambda_\beta - a + 1 + (2a - 1))}{\Gamma(\lambda_\beta - a + 1)} = \prod_{k=0}^{2(a-1)} (\lambda_\beta - a + 1 + k).$$

This proves (b) by taking $a = \frac{m_{\beta/2}}{4} + \frac{1}{2}$ and $a = \tilde{\rho}_\beta$.

Finally, to prove (c), observe first that $|\lambda_\beta| \leq \|\lambda\| \|\beta\|^{-1}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\beta \in \Sigma_*^+$. Hence, if $a > 0$, then there is a constant $K \geq 0$ so that $(1 + |\lambda_\alpha|)^a \leq K(1 + \|\lambda\|)^a$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Similarly, since $\lambda = \sum_{j=1}^l \lambda_j \omega_j$, there is a constant $K_1 > 0$ so that $(1 + \|\lambda\|) \leq K_1 \prod_{j=1}^l (1 + |\lambda_j|)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Recall that

$$\lim_{|z| \rightarrow +\infty} \frac{\Gamma(z + a)}{\Gamma(z)} e^{-a \log z} = 1 \quad \text{for } |\arg z| \leq \pi - \varepsilon, \quad (3.8)$$

where $a \in \mathbb{C}$ is fixed, \log is the principal part of the logarithm, and $\varepsilon > 0$; see e.g. [32, Ch. IV, p. 151]. Let now $a > 0$. Then for fixed constants $C_1 > 1$ and $\varepsilon > 0$ there are constants $N_1 > 0$ and $C_2 \geq C_1$ (also depending on a) so that

$$\left| \frac{\Gamma(z + a)}{\Gamma(z - a + 1)} \right| \leq C_1 (1 + |z - a + 1|)^{2a-1} \leq C_2 (1 + |z|)^{2a-1}$$

for all z with $|z| \geq N_1$ and $|\arg z| \leq \pi - \varepsilon$. The function $\frac{\Gamma(z+a)}{\Gamma(z-a+1)}$ is holomorphic on $\operatorname{Re} z > -a$, so bounded on compact subsets of this domain. It is in particular bounded on

$$\{z \in \mathbb{C} : |z| \leq N_1 \text{ and } |\arg z| \leq \pi - \varepsilon\}$$

and (supposing without loss of generality that $\varepsilon < \pi/2$ so that the region is compact) on

$$\{z \in \mathbb{C} : \operatorname{Re} z \geq -a + \varepsilon \text{ and } \pi \geq |\arg z| \geq \pi - \varepsilon\}.$$

It follows that there is a constant $C_3 \geq C_2$ so that

$$\left| \frac{\Gamma(z+a)}{\Gamma(z-a+1)} \right| \leq C_3(1+|z|)^{2a-1}$$

for all $z \in \mathbb{C}$ with $|\arg z| \leq \pi - \varepsilon$ and for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq -a + \varepsilon$.

Choose now $a = \frac{m_{\beta/2}}{4} + \frac{1}{2}$. We get

$$\left| \frac{\Gamma(\lambda_{\beta} + \frac{m_{\beta/2}}{4} + \frac{1}{2})}{\Gamma(\lambda_{\beta} - \frac{m_{\beta/2}}{4} + \frac{1}{2})} \right| \leq C'_3(1+|\lambda_{\beta}|)^{m_{\beta/2}/2}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $|\arg(\lambda_{\beta})| \leq \pi - \varepsilon$ and for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re} \lambda_{\beta} \geq -(\frac{m_{\beta/2}}{4} + \frac{1}{2}) + \varepsilon$.

Choose then $a = \tilde{\rho}_{\beta}$. In this case, we obtain

$$\left| \frac{\lambda_{\beta} \Gamma(\lambda_{\beta} + \tilde{\rho}_{\beta})}{\Gamma(\lambda_{\beta} - \tilde{\rho}_{\beta} + 1)} \right| \leq C''_3(1+|\lambda_{\beta}|)^{m_{\beta/2}/2+m_{\beta}-1} |\lambda_{\beta}| \leq C''_3(1+|\lambda_{\beta}|)^{m_{\beta/2}/2+m_{\beta}}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $|\arg(\lambda_{\beta})| \leq \pi - \varepsilon$ and for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re} \lambda_{\beta} \geq -\tilde{\rho}_{\beta} + \varepsilon$. These estimates yield the claim for $d(\lambda - \rho)$. \square

Remark 3.2. By the infinitesimal classification of Riemannian symmetric spaces (see [2] or [21, Ch. X, Exercise F]), the condition in (b) is satisfied by all geometric triples $(\mathfrak{a}, \Sigma, m)$. This condition is also satisfied if Σ is reduced and $m_{\beta} \in \mathbb{Z}$, for instance in the so-called even multiplicity case [29].

4. THE FUNCTION b

Let b be the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ defined by the equality ¹

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = 2^{-l} d(\lambda - \rho) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \quad (4.1)$$

Then

$$b(\lambda) = 2^{-l} \frac{c(-\lambda)}{c^*(-\lambda)} \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))}. \quad (4.2)$$

Recall the functions $S_{\beta/2}$ and S_{β} from (1.16). We have for all $\beta \in \Sigma_{*}^{+}$:

$$\begin{aligned} S_{\beta/2}(\lambda) S_{\beta}(\lambda) &= \frac{\sin(2\pi\lambda_{\beta})}{\sin(\pi(2\lambda_{\beta} + \frac{m_{\beta/2}}{2}))} \frac{\sin(\pi(\lambda_{\beta} + \frac{m_{\beta/2}}{4}))}{\sin(\pi(\lambda_{\beta} + \tilde{\rho}_{\beta}))} \\ &= \frac{\sin(\pi\lambda_{\beta}) \cos(\pi\lambda_{\beta})}{\cos(\pi(\lambda_{\beta} + \frac{m_{\beta/2}}{4})) \sin(\pi(\lambda_{\beta} + \tilde{\rho}_{\beta}))}. \end{aligned}$$

It follows then from (1.15) that

$$b(\lambda) = C_b \prod_{\beta \in \Sigma_{*}^{+}} \frac{\cos(\pi(\lambda_{\beta} - \frac{m_{\beta/2}}{4}))}{\cos(\pi\lambda_{\beta}) \sin(\pi\lambda_{\beta})} \prod_{\beta \in \Sigma_{*}^{+} \setminus \{\beta_1, \dots, \beta_l\}} \sin(\pi(\lambda_{\beta} - \tilde{\rho}_{\beta})). \quad (4.3)$$

¹The constant $(i/2)^l$ of [30, formulas (37) and (53)] should be corrected as 2^{-l} .

where

$$C_b = 2^{-l} \frac{c_{\text{HC}}}{c_{\text{HC}}^*} = 2^{-l} \frac{\tilde{c}^*(-\rho)}{\tilde{c}(\rho)}. \quad (4.4)$$

Remark 4.1. By classification (see e.g. [21, Ch. X, Exercice F.4]), the parity of the geometric root multiplicities are distinguished in four different cases. They are reported in the following table together with the corresponding parity of $\tilde{\rho}_\beta$:

	$\frac{m_\beta/2}{2}$	m_β	$\tilde{\rho}_\beta = \frac{1}{2}(\frac{m_\beta/2}{2} + m_\beta)$
(a)	0	$\in 2\mathbb{Z}$	$\in \mathbb{Z}$
(b)	0	$\in 2\mathbb{Z} + 1$	$\in \mathbb{Z} + \frac{1}{2}$
(c)	$\in 2\mathbb{Z}$	$\in 2\mathbb{Z} + 1$	$\in \mathbb{Z} + \frac{1}{2}$
(d)	$\in 2\mathbb{Z} + 1$	$\in 2\mathbb{Z} + 1$	$\in \mathbb{Z}$

The corresponding values of $b(\lambda)$, which have been determined in [30, Remark 4.5], are

$$b(\lambda) = C'_b \left(\prod_{\substack{\beta \in \Sigma_*^+ \setminus \{\beta_1, \dots, \beta_l\} \\ \text{cases (b) or (c)}}} \cot(\pi \lambda_\beta) \right) \left(\prod_{\substack{\beta \in \Sigma_*^+ \setminus \{\beta_1, \dots, \beta_l\} \\ \text{case (d)}}} \tan(\pi \lambda_\beta) \right) \\ \times \left(\prod_{\substack{j \in \{1, \dots, l\} \\ \text{cases (a), (b) or (c)}}} \frac{1}{\sin(\pi \lambda_j)} \right) \left(\prod_{\substack{j \in \{1, \dots, l\} \\ \text{case (d)}}} \frac{1}{\cos(\pi \lambda_j)} \right) \quad (4.5)$$

where $C'_b = \pm C_b$ and the sign depends on the parity of the multiplicities.

Lemma 4.2. *Set*

$$T_\Pi := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : |\operatorname{Re} \lambda_\beta| < 1/2 \text{ for all } \beta \in \Sigma_*^+\}. \quad (4.6)$$

Let $\Pi(\lambda)$ be as in (3.2). Then $\Pi(\lambda)b(\lambda)$ is holomorphic on T_Π .

Proof. This is immediate from (4.5). \square

Remark 4.3. According to the possible values of m , the function $\Pi(\lambda)b(\lambda)$ might be holomorphic on a larger tube domain. See [30, Corollary 4.6] for the geometric case situation.

Lemma 4.4. *Let C_d be the constant defined in (3.4). The function*

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = 2^{-l} d(\lambda - \rho) \prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \\ = C_d \Pi(\lambda) \left(\prod_{\beta \in \Sigma_*^+} \frac{\Gamma(\lambda_\beta + \frac{m_\beta/2}{4} + \frac{1}{2}) \Gamma(\lambda_\beta + \tilde{\rho}_\beta)}{\Gamma(\lambda_\beta - \frac{m_\beta/2}{4} + \frac{1}{2}) \Gamma(\lambda_\beta - \tilde{\rho}_\beta + 1)} \right) \left(\prod_{j=1}^l \frac{1}{\sin(\pi(\lambda_j - \rho_j))} \right) \quad (4.7)$$

is meromorphic on $\mathfrak{a}_\mathbb{C}^*$. Its possible singularities are along the following hyperplanes:

$$\mathcal{H}_{1,\beta,k} := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda_\beta = -\tilde{\rho}_\beta - k\} \quad \text{with } \beta \in \Sigma_*^+, k \in \mathbb{Z}^+, \\ \mathcal{H}_{2,\beta,k} := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda_\beta = -\frac{m_\beta/2}{4} - \frac{1}{2} - k\} \quad \text{with } \beta \in \Sigma_*^+, k \in \mathbb{Z}^+, \\ \mathcal{H}_{j,k} := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda_j = \rho_j + k\} \quad \text{with } j = 1, \dots, l, k \in \mathbb{Z}^+ \quad (\text{simple poles}).$$

The occurrence and the order of the singularities along $\mathcal{H}_{1,\beta,k}$ and $\mathcal{H}_{2,\beta,k}$ depend on the root β (according to whether $\beta = \beta_j$ for some $j = 1, \dots, l$ or not) and on the multiplicities of $\beta/2$ and β . The situation is summarized in Tables 1 and 2.

Proof. The explicit formula for in Part (a) is an immediate consequence of (3.3). The hyperplanes $\mathcal{H}_{1,\beta,k}$ and $\mathcal{H}_{2,\beta,k}$ correspond respectively to the singularities of $\Gamma(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{1}{2})$ and $\Gamma(\lambda_\beta + \tilde{\rho}_\beta)$, whereas $\mathcal{H}_{j,k}$ corresponds to those of $[\Gamma(\lambda_j - \rho_j + 1) \sin(\pi(\lambda_j - \rho_j))]^{-1}$ (Recall that $\lambda_j = \lambda_{\beta_j}$ and $\rho_j = \tilde{\rho}_{\beta_j}$).

Some of the singularities along the hyperplanes $\mathcal{H}_{1,\beta,k}$ and $\mathcal{H}_{2,\beta,k}$ are cancelled by zeros coming from the gamma functions in the denominator of (4.7). More precisely, when $\beta = \beta_j$ (Table 1), the zeros of $\Gamma(\lambda_\beta - \frac{m_{\beta/2}}{4} + \frac{1}{2})^{-1}$ remove the singularities along $\mathcal{H}_{1,\beta,k}$ in case (1) and those along $\mathcal{H}_{2,\beta,k}$ in case (2). When $\beta \neq \beta_j$ (Table 2), the zeros of $\Gamma(\lambda_\beta - \frac{m_{\beta/2}}{4} + \frac{1}{2})^{-1}$ also remove the singularities along $\mathcal{H}_{1,\beta,k}$ in cases (1) and (2) and those along $\mathcal{H}_{2,\beta,k}$ in case (3). Moreover, because of the absence of the sin-functions at the denominator for $\beta \neq \beta_j$, additional cancellations occur in this case: the zeros of $\Gamma(\lambda_\beta - \tilde{\rho}_\beta + 1)^{-1}$ remove the singularities along $\mathcal{H}_{1,\beta,k}$ in case (3) and along $\mathcal{H}_{2,\beta,k}$ in cases (1) and (6). \square

In the last column of Tables 1 and 2 we have reported the first negative value l_β such that $\lambda_\beta = l_\beta$ is a singular hyperplane of $\frac{b(\lambda)}{c(\lambda)c(-\lambda)}$. Notice that l_β is not W -invariant since b is not W -invariant. On the elements of Σ_*^+ inside a single W -orbit, there are nevertheless only at most two values $l_1 \leq l_2 < 0$ for l_β , and $l_2 = l_{\beta_j}$ where β_j is an element of the basis Π_* belonging to this W -orbit. Notice also that the geometric case is contained in case (1), i.e. $\frac{m_{\beta/2}}{2} \in \mathbb{Z}$, of the two tables. To measure the largest W -invariant tube domain around $i\mathfrak{a}^*$ on which $\frac{b(\lambda)}{c(\lambda)c(-\lambda)}$ is holomorphic, we introduce the following constants.

For $\beta \in \Sigma_*^+$ define

$$L_\beta := -l_{\beta_j} \quad \text{if } \beta_j \in W\beta \cap \Pi_* \quad (4.8)$$

Hence $L_\beta \in \{\tilde{\rho}_\beta, \frac{m_{\beta/2}}{4} + \frac{1}{2}\}$.

Then only the singular hyperplanes $\mathcal{H}_{j,k}$, with $j = 1, \dots, l$ and $k \in \mathbb{Z}^+$, intersect the region

$$L_\Sigma := \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \operatorname{Re} \lambda_\beta > -L_\beta \text{ for all } \beta \in \Sigma_*^+\} \supset \overline{\mathfrak{a}_+^*} + i\mathfrak{a}^*. \quad (4.9)$$

Corollary 4.5. *For all $w \in W$ the function $\frac{b(w\lambda)}{c(\lambda)c(-\lambda)}$ is holomorphic in the tube domain*

$$T_\Sigma = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : |\operatorname{Re} \lambda_\beta| < L_\beta \text{ for all } \beta \in \Sigma_*^+\} \quad (4.10)$$

Remark 4.6. In the geometric case we have $L_\beta = \tilde{\rho}_\beta$ for all $\beta \in \Sigma_*^+$. Observe that if $L_\beta = \tilde{\rho}_\beta$ for all $\beta \in \Sigma_*^+$, then the base of the tube T_Σ is $C(\rho)^0$, the interior of the convex hull of $\{w\rho : w \in W\}$ in \mathfrak{a}^* . In this case, $T_\Sigma = C(\rho)^0 + i\mathfrak{a}^*$ is the interior of the domain in $\mathfrak{a}_\mathbb{C}^*$ in which all hypergeometric functions F_λ are bounded, see [26, Theorem 4.2].

4.1. Tube domains in $\mathfrak{a}_\mathbb{C}^*$. Recall the constants L_β introduced in (4.8). Let m_L be the positive multiplicity function on Σ defined for $\beta \in \Sigma^*$ by

$$(m_L)_{\beta/2} := m_{\beta/2} \quad (4.11)$$

$$(m_L)_\beta := \begin{cases} m_\beta & \text{if } L_\beta = \tilde{\rho}_\beta \\ \frac{1}{2} & \text{if } L_\beta = \frac{m_{\beta/2}}{4} + \frac{1}{2}. \end{cases} \quad (4.12)$$

The corresponding ρ -function is

$$\rho_L = \frac{1}{2} \sum_{\beta \in \Sigma_*^+} \left[\frac{(m_L)_{\beta/2}}{2} + (m_L)_\beta \right] \beta = \frac{1}{2} \sum_{\beta \in \Sigma_*^+} L_\beta \beta. \quad (4.13)$$

Hence $L_\beta = (\widetilde{\rho}_L)_\beta$ in the notation of (1.10).

For $\delta > 0$, we consider the following tube domains in $\mathfrak{a}_\mathbb{C}^*$ around the imaginary axis:

$$T_\delta = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : |\operatorname{Re} \lambda_\beta| < \delta L_\beta \text{ for all } \beta \in \Sigma_*^+\}, \quad (4.14)$$

$$T'_\delta = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : |\operatorname{Re} \lambda_j| < \delta L_{\beta_j} \text{ for all } j = 1, \dots, l\}, \quad (4.15)$$

$$T''_\delta = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \operatorname{Re} \lambda_j < \delta L_{\beta_j} \text{ for all } j = 1, \dots, l\}. \quad (4.16)$$

The following lemma is standard. A proof can be found for instance in [30, Lemma 1.2].

Lemma 4.7. *Let w_0 be the longest element of W . Then*

$$T'_\delta = T''_\delta \cap w_0(T''_\delta) \quad (4.17)$$

and

$$T_\delta = \bigcap_{w \in W} w(T'_\delta) = \bigcap_{w \in W} w(T''_\delta). \quad (4.18)$$

In particular, T_δ is the largest W -invariant tube domain contained in T'_δ . Moreover,

$$T_\delta = C(\delta \rho_L)^0 + i\mathfrak{a}^* \quad (4.19)$$

where $C(\nu)$ is the convex hull of the W -orbit $\{w\nu : w \in W\}$ of $\nu \in \mathfrak{a}^*$ and $C(\nu)^0$ is its interior.

Remark 4.8. Notice that $T_1 = T_\Sigma$ is the tube domain introduced in Corollary 4.5.

5. STATEMENT OF RAMANUJAN'S MASTER THEOREM FOR ROOT SYSTEMS

Let $A_\mathbb{C} = AT$ be the complex torus associated with a triple $(\mathfrak{a}, \Sigma, m)$ as in Section 1.1, and let L_β , with $\beta \in \Sigma_*^+$, be defined by (4.8). Let $\mathbf{A}, \mathbf{P}, \delta$ be constants so that $\mathbf{A} < \pi$, $\mathbf{P} > 0$ and $0 < \delta \leq 1$, and define

$$\mathcal{H}(\delta) = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \operatorname{Re} \lambda_\beta > -\delta \widetilde{\rho}_\beta \text{ for all } \beta \in \Sigma_*^+\}. \quad (5.1)$$

The *Hardy class* $\mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$ consists of the functions $a : \mathcal{H}(\delta) \rightarrow \mathbb{C}$ that are holomorphic on $\mathcal{H}(\delta)$ and so that

$$|a(\lambda)| \leq C \prod_{j=1}^l e^{-\mathbf{P}(\operatorname{Re} \lambda_j) + \mathbf{A}|\operatorname{Im} \lambda_j|} \quad (5.2)$$

for some constant $C \geq 0$ and for all $\lambda \in \mathcal{H}(\delta)$.

Theorem 5.1 (Ramanujan's Master Theorem for root systems). *Let d and b be the meromorphic functions on $\mathfrak{a}_\mathbb{C}^*$ defined by (3.1) and (4.2), respectively. Suppose $a \in \mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$.*

(1) *Let Ω be as in (1.11). Then the alternating normalized Jacobi series*

$$f(t) = \sum_{\mu \in P^+} (-1)^{|\mu|} d(\mu) a(\mu + \rho) F_{\mu+\rho}(t) \quad (5.3)$$

converges normally on compact subsets of $D_{\mathbf{P}/\Omega} = T \exp((\mathbf{P}/\Omega)B)$ where $B = \{H \in \mathfrak{a} : \|H\| < 1\}$ is the open unit ball in \mathfrak{a} . Its sum is a W -invariant holomorphic function on the neighborhood $D_{\mathbf{P}/\Omega}$ of T in $A_{\mathbb{C}}$.

- (2) Let T_{δ} be the tube domain in (4.14) and let $\sigma \in T_{\delta} \cap \mathfrak{a}^*$. Then for $x = \exp H \in A$ with $\|H\| < \mathbf{P}/\Omega$, we have

$$f(x) = \frac{1}{|W|} \int_{\sigma + i\mathfrak{a}^*} \left(\sum_{w \in W} a(w\lambda)b(w\lambda) \right) F_{\lambda}(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}. \quad (5.4)$$

The integral on the right-hand side of (5.4) is independent of the choice of σ . It converges uniformly on compact subsets of A and extends to a holomorphic W -invariant function on a neighborhood of A in $A_{\mathbb{C}}$.

- (3) The extension of f to A satisfies

$$\int_A |f(x)|^2 d\mu(x) = \frac{1}{|W|} \int_{i\mathfrak{a}^*} \left| \sum_{w \in W} a(w\lambda)b(w\lambda) \right|^2 \frac{d\lambda}{|c(\lambda)|^2}.$$

Moreover,

$$\int_A f(x)F_{-\lambda}(x) d\mu(x) = \sum_{w \in W} a(w\lambda)b(w\lambda) \quad (5.5)$$

for all $\lambda \in T_{\delta} \cap T_{\Pi}$. More precisely, the integral on the left-hand side of (5.5) converges in L^2 -sense and absolutely on $i\mathfrak{a}^*$. It defines a W -invariant holomorphic function on a W -invariant tube domain around $i\mathfrak{a}^*$, and (5.5) extends as an identity between holomorphic functions on $T_{\delta} \cap T_{\Pi}$.

Remark 5.2. As in the classical case (0.2) or in the case of semisimple Riemannian symmetric spaces in [30], there is an equivalent formulation of Ramanujan's Master theorem for root systems using the gamma function. This version, as well as some immediate consequences, can be easily obtained, as in [30, Remark 2.6], from Theorem 5.1.

6. PROOF OF THEOREM 5.1

The proof of Ramanujan's Master Theorem 5.1 for root systems follows the same pattern used in the proof of the corresponding theorem for semisimple Riemannian symmetric spaces [30, Theorem 2.1]. The first statement is an application of Lemma 2.6. The crucial step for the remaining parts, and this is the content of this section, consists in extending the necessary estimates from [30] to the setting of positive multiplicity functions.

Observe first that there is a constant $K > 0$ so that

$$|\sin(\pi(\lambda_j - \rho_j))|^{-1} \leq K e^{-\pi|\operatorname{Im} \lambda_j|} \quad (6.1)$$

for $|\operatorname{Im} \lambda_j| \geq 1$ or for $\operatorname{Re} \lambda_j = \rho_j + N + 1/2$ with $N \in \mathbb{Z}^+$.

The second part of Theorem 5.1 is proven by computing the integral on the right-hand side of (5.4) by separately applying the Residue Theorem to each variable λ_j . The term of parameter $\mu + \rho$ in the alternating normalized Jacobi series appears as the result of taking residues (for $j = 1, 2, \dots, l$) in λ_j at the singularity $\rho_j + \mu_j$ corresponding to the singular hyperplane \mathcal{H}_{j, μ_j} from Lemma 4.4. The residues are computed using (4.1). The following lemma contains the estimates needed to apply the Residue Theorem. It is an easy generalization of [30, Lemma 5.3], and its proof is omitted.

Lemma 6.1. (a) Let N be a positive integer and let M be as in (3.6). Let $\lambda = \sum_{j=1}^l \lambda_j \omega_j \in \mathfrak{a}_{\mathbb{C}}^*$ with $|\operatorname{Im} \lambda_j| \geq 1$ or $\operatorname{Re} \lambda_j = \rho_j + N + 1/2$ or $\operatorname{Re} \lambda_j = 0$ for all $j = 1, \dots, l$. Then there is a positive constant C_1 , independent of N , so that

$$\left| \frac{b(\lambda)}{c(\lambda)c(-\lambda)} \right| \leq C_1 \prod_{j=1}^l [(1 + |\lambda_j|)^M e^{-\pi |\operatorname{Im} \lambda_j|}].$$

(b) Set

$$B = \left\{ \lambda = \sum_{j=1}^l \lambda_j \omega_j \in \mathfrak{a}_+^* + i\mathfrak{a}^* : |\operatorname{Im} \lambda_j| \geq 1 \text{ or } \operatorname{Re} \lambda_j \in (\rho_j + \mathbb{Z}^+ + 1/2) \cup \{0\} \text{ for all } j = 1, \dots, l \right\}. \quad (6.2)$$

Let $a \in \mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$. Then there is a constant $C_2 > 0$ so that for all $\lambda \in B$ and $H \in \overline{\mathfrak{a}^+}$ we have

$$\left| \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} F_{\lambda}(\exp H) \right| \leq C_2 \prod_{j=1}^l [(1 + |\lambda_j|)^M e^{(\mathbf{A}-\pi) |\operatorname{Im} \lambda_j| + (\|H\| \Omega - \mathbf{P}) \operatorname{Re} \lambda_j}]. \quad (6.3)$$

The estimates in the next lemma allow us to prove that the integral on the right-hand side of (5.4) is independent of the choice of $\sigma \in T_{\delta} \cap \mathfrak{a}^*$.

Lemma 6.2. Let $0 < \delta \leq 1$ and let T_{δ} be the tube domain from (4.14). Let M be the constant defined in (3.6).

(a) There is a constant $C_{\delta} > 0$ so that

$$\left| \frac{b(\lambda)}{c(\lambda)c(-\lambda)} \right| \leq C_{\delta} (1 + \|\lambda\|)^M e^{-\pi (\sum_{j=1}^l |\operatorname{Im} \lambda_j|)} \quad (6.4)$$

for all $\lambda \in T_{\delta}$.

(b) Let $a \in \mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$. For every $R > 0$ and every integer $N \geq 0$ there is a constant $C_{R,N,\delta} > 0$ so that for all $\lambda \in T_{\delta}$ and $H \in \mathfrak{a}$ with $\|H\| < R$, we have

$$\left| \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} F_{\lambda}(\exp H) \right| \leq C_{R,N,\delta} (1 + \|\lambda\|)^{-N} \quad (6.5)$$

Consequently,

$$\left| \left(\sum_{w \in W} a(w\lambda)b(w\lambda) \right) \frac{F_{\lambda}(\exp H)}{c(\lambda)c(-\lambda)} \right| \leq C_{R,N,\delta} |W| (1 + \|\lambda\|)^{-N}. \quad (6.6)$$

Proof. By Corollary 4.5, for all $w \in W$, the function $\frac{b(w\lambda)}{c(\lambda)c(-\lambda)}$ is holomorphic on the W -invariant tube domain T_{δ} . Set

$$p_j(\lambda) = (\lambda_j - (\rho_j - 1))(\lambda_j - (\rho_j - 2)) \cdots (\lambda_j - (\rho_j - h_j))$$

where $h_j \in \mathbb{Z}^+$ is chosen so that $2\rho_j - 1 \leq h_j < 2\rho_j$. Then, by (4.7),

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = C_d \Pi(\lambda) \left(\prod_{\beta \in \Sigma_*^+} \frac{\Gamma(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{1}{2})}{\Gamma(\lambda_\beta - \frac{m_{\beta/2}}{4} + \frac{1}{2})} \right) \times \left(\prod_{\beta \in \Sigma_*^+ \setminus \{\beta_1, \dots, \beta_l\}} \frac{\Gamma(\lambda_\beta + \tilde{\rho}_\beta)}{\Gamma(\lambda_\beta - \tilde{\rho}_\beta + 1)} \right) \left(\prod_{j=1}^l \frac{\Gamma(\lambda_j + \rho_j)}{\Gamma(\lambda_j - (\rho_j - h_j) + 1)} \frac{p_j(\lambda)}{\sin(\pi(\lambda_j - \rho_j))} \right) \quad (6.7)$$

For fixed $\eta \in]0, 1[$, the function $\frac{z}{\sin(\pi z)}$ is bounded on $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq 1, |\operatorname{Re} z| \leq \eta\}$. By (6.1), we conclude that there is a constant $C'_\delta > 0$ so that for any fixed $j = 1, \dots, l$ and every $\lambda = \sum_h \lambda_h \omega_h$ with $|\operatorname{Re} \lambda_j| \leq \delta L_{\beta_j}$ and arbitrary $\lambda_h \in \mathbb{C}$ with $h \neq j$, we have

$$\left| \frac{p_j(\lambda)}{\sin(\pi(\lambda_j - \rho_j))} \right| \leq C'_\delta (1 + |\lambda_j|)^{\deg p_j} e^{-\pi |\operatorname{Im} \lambda_j|}.$$

Part (a) then follows from these estimates and (6.7).

To prove (b), observe first that T_δ is a W -invariant subset of $\mathcal{H}(\delta)$. Hence

$$\left(\sum_{w \in W} a(w\lambda) b(w\lambda) \right) \frac{F_\lambda(\exp H)}{c(\lambda)c(-\lambda)}$$

is holomorphic on T_δ for every fixed $a \in \mathcal{H}(A, P, \delta)$. Let $R > 0$ be fixed. By (1.27) there is a constant $C_{R,\delta} > 0$ so that

$$|F_\lambda(\exp H)| \leq C_{R,\delta} \quad (6.8)$$

for all $\lambda \in T_\delta$ and $H \in \mathfrak{a}$ with $\|H\| \leq R$. (In fact, since $T_\delta \subset T_1$, one knows from [26, Theorem 4.2] that $C_{R,\delta}$ can be chosen to be equal to 1.) Together with Part (a), (6.8) yields that there is a constant $C'_{R,\delta} > 0$ so that

$$\left| \frac{a(\lambda)b(\lambda)}{c(\lambda)c(-\lambda)} F_\lambda(\exp H) \right| \leq C'_{R,\delta} (1 + \|\lambda\|)^M e^{(\mathbf{A} - \pi) \sum_{j=1}^l |\operatorname{Im} \lambda_j|}.$$

This implies (6.5) as $\mathbf{A} < \pi$. □

The last group of estimates we need shows that if $a \in \mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$, then there is $\varepsilon \in]0, 1[$ so that the function

$$\tilde{a}(\lambda) = \sum_{w \in W} a(w\lambda) b(w\lambda) \quad (6.9)$$

belongs to the W -invariant Schwartz space $\mathcal{S}(\mathfrak{a}_\varepsilon^*)^W$ defined in Section 1.5.

For $0 \leq \eta < 1/2$ set

$$T_{\Pi,\eta} = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : |\operatorname{Re} \lambda_\beta| < \frac{1}{2} - \eta \text{ for all } \beta \in \Sigma_*^+\}. \quad (6.10)$$

So $T_{\Pi,0} = T_\Pi$ is the tube domain on which $\Pi(\lambda)b(\lambda)$ is holomorphic; see Corollary 4.5.

Lemma 6.3. *Set $s = |\Sigma_*^+|$.*

(a) *Let $0 < \eta < 1/2$. Then there is a constant $C_\eta > 0$ so that*

$$|\Pi(\lambda)b(\lambda)| \leq C_\eta (1 + \|\lambda\|)^s e^{-\pi (\sum_{j=1}^l |\operatorname{Im} \lambda_j|)}$$

for all $\lambda \in T_{\Pi,\eta}$.

- (b) Let $a \in \mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$ and set $\tilde{a}(\lambda) = \sum_{w \in W} a(w\lambda)b(w\lambda)$. Then \tilde{a} is holomorphic in $T_{\Pi} \cap T_{\delta}$. Moreover, let $0 < \eta < \min\{1/2, \delta\}$. Then there are positive constants $C_{\eta, a} > 0$ and C_0 so that

$$|\tilde{a}(\lambda)| \leq C_{\eta, a}(1 + \|\lambda\|)^s e^{(\mathbf{A} - \pi)C_0 \|\operatorname{Im} \lambda\|}$$

for all $\lambda \in T_{\Pi, \eta} \cap T_{\delta - \eta}$.

- (c) Let $0 < \eta < \min\{1/2, \delta\}$ and set

$$\gamma = \min \left\{ \delta - \eta, \left(\frac{1}{2} - \eta\right) \min_{\beta \in \Sigma_*^+} L_{\beta}^{-1} \right\}.$$

Then $T_{\gamma} \subset T_{\Pi, \eta} \cap T_{\delta - \eta}$. Moreover, let $0 < \varepsilon < \gamma$. Then $\tilde{a} \in \mathcal{S}(\mathfrak{a}_{\varepsilon}^*)^W$, the W -invariant Schwartz space on the tube domain T_{ε} .

Proof. Since $\Pi(\lambda)b(\lambda)$ is bounded on $T_{\Pi, \eta}$, the proof of the estimate in (a) follows the same argument used in part (a) of Lemma 6.2.

To prove part (b), notice first that, by Corollary 4.5, on T_{Π} the function $b(\lambda)$ has at most simple poles on hyperplanes of the form $\lambda_{\beta} = 0$ with $\beta \in \Sigma_*^+$. The same property holds on $T_{\Pi} \cap T_{\delta}$ for $b(w\lambda)a(w\lambda)$, with $w \in W$, and hence for $\tilde{a}(\lambda)$. But $\tilde{a}(\lambda)$, as a W -invariant function, cannot admit first order singularities on root hyperplanes through the origin. Thus \tilde{a} is holomorphic on $T_{\Pi} \cap T_{\delta}$.

Let $0 < \eta < \eta' < \min\{1/2, \delta\}$. Choose $C_0 > 0$ so that $\sum_{j=1}^l |\operatorname{Im} \lambda_j| \leq C \|\operatorname{Im} \lambda\|$ for all $\lambda \in \mathfrak{a}^*$. By (a), there is a constant $C_{\eta'} > 0$ so that

$$|\Pi(\lambda)b(\lambda)a(\lambda)| \leq C_{\eta'}(1 + \|\lambda\|)^s e^{(\mathbf{A} - \pi)\|\operatorname{Im} \lambda\|_1} \leq C_{\eta'}(1 + \|\lambda\|)^s e^{(\mathbf{A} - \pi)C_0 \|\operatorname{Im} \lambda\|}$$

for all $\lambda \in T_{\Pi, \eta'} \cap T_{\delta - \eta'}$. The required estimate for \tilde{a} on $T_{\Pi, \eta} \cap T_{\delta - \eta}$ follows then as in [30, Lemma 5.6(b)].

To show that $T_{\gamma} \subset T_{\Pi, \eta} \cap T_{\delta - \eta}$, notice that $\gamma \leq \delta - \eta$ and that if $\lambda \in T_{\gamma}$, then $|\operatorname{Re} \lambda_{\beta}| < \gamma L_{\beta} \leq \frac{1}{2} - \eta$.

The property that $\tilde{a} \in \mathcal{S}(\mathfrak{a}_{\varepsilon}^*)^W$ for $0 < \varepsilon < \gamma$ can be proven as in [30, Lemma 5.6(c)], using (b) and Cauchy's estimates. \square

Lemma 6.3 allows us to apply the inversion formula (1.32) to \tilde{a} and conclude that $\mathcal{F}^{-1}\tilde{a} \in \mathcal{S}^p(A)^W \subset (L^p \cap L^2)(A, d\mu)^W$ where $p = 2/(\varepsilon + 1) \in]1, 2[$. The function \tilde{a} is therefore the hypergeometric Fourier transform of $\mathcal{F}^{-1}\tilde{a}$ and the equality in part (c) of Theorem 5.1, initially valid on T_{ε} , extends holomorphically to $T_{\Pi} \cap T_{\delta}$.

We leave the reader to follow the proof of [30, Theorem 2.1, Section 6], to fill in the missing details of the proof of Theorem 5.1.

7. EXAMPLES

Let $(\mathfrak{a}, \Sigma, m)$ be a triple of rank one with Σ of type A_1 . Hence $\Sigma^+ = \{\beta\}$ for a unique root β . To simplify notation, we shall write m instead of m_{β} . Recall from Example 2.2 the identification of the polynomials $F_{n\beta+\rho}(t)$, where $t = \exp H \in T$, with the symmetric Jacobi polynomials $X_n^{(m-1)/2}(x) = {}_2F_1(-n, n+m; (m+1)/2; (1-x)/2)$ with $x = \cos \beta(H)$. By (3.1),

$$d(n\beta) = \frac{1}{c(n\beta + \rho)c^*(-n\beta - \rho)} = C_d \left(n + \frac{m}{2}\right) \frac{\Gamma(n+m)}{\Gamma(n+1)},$$

where $C_d = \tilde{c}(\rho)\tilde{c}^*(-\rho) = 2/\Gamma(m+1)$. We identify $\mathfrak{a} \equiv \mathfrak{a}^* \equiv \mathbb{C}$ as in Example 2.2. This means that $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $H \in \mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \oplus \mathfrak{t}$ are respectively identified with $\lambda_\beta \in \mathbb{C}$ and $\beta(H)/2 \in \mathbb{C}$.

Let $0 \leq \mathbf{A} < \pi$ and $\mathbf{P} > 0$ be fixed, and set $a(\lambda) = e^{-(\mathbf{P}+i\mathbf{A})\lambda}$. Then

$$|a(\lambda)| = e^{-\mathbf{P} \operatorname{Re} \lambda + \mathbf{A} \operatorname{Im} \lambda} \leq e^{-\mathbf{P} \operatorname{Re} \lambda + \mathbf{A} |\operatorname{Im} \lambda|}$$

for all $\lambda \in \mathfrak{a}^* \equiv \mathbb{C}$. Hence $a \in \mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$ for every value of $m > 0$ and for all $\delta \in]0, 1]$.

Consider the alternating normalized Jacobi series (5.3) for $t = \exp H \in T$

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} (-1)^n d(n\beta) a(n\beta + \rho) F_{n\beta+\rho}(t) \\ &= C_d e^{-(\mathbf{P}+i\mathbf{A})m/2} \sum_{n=0}^{\infty} (-1)^n \left(n + \frac{m}{2}\right) \frac{\Gamma(n+m)}{\Gamma(n+1)} e^{-(\mathbf{P}+i\mathbf{A})n} X_n^{(m-1)/2}(\cos \beta(H)). \end{aligned}$$

According to [7, (10)], one has for $|\tau| < 1$

$$\sum_{n=0}^{\infty} \left(n + \frac{m}{2}\right) \frac{\Gamma(n+m)}{\Gamma(n+1)} X_n^{(m-1)/2}(x) \tau^n = \frac{\Gamma(m+1)}{2} G(m, x, \tau) \quad (7.1)$$

where

$$G(m, x, \tau) = \frac{1 - \tau^2}{(1 - 2\tau x + \tau^2)^{1 + \frac{m}{2}}} \quad (7.2)$$

Thus for $t = \exp H \in T$

$$f(t) = e^{-(\mathbf{P}+i\mathbf{A})\frac{m}{2}} G(m, \cos \beta(H), -e^{-(\mathbf{P}+i\mathbf{A})}). \quad (7.3)$$

The first part of Ramanujan's Master theorem 5.1 states that the normalized series (7.1) converges normally to f on compact subsets of $\{z = i\beta(H)/2 \in \mathbb{C} : |\operatorname{Re} z| < \mathbf{P}/2\}$. Special instances of the above formulas, involving Legendre or Tchebishef polynomials, are obtained by selecting particular values of m ; see Example 2.2.

Since Σ is reduced, we have $b(\lambda) = C_b[\sin(\pi\lambda)]^{-1}$, where

$$C_b = \frac{1}{2} \frac{\tilde{c}^*(-\rho)}{\tilde{c}(\rho)} = \frac{1}{2} \frac{\Gamma(m)}{\Gamma(\frac{m}{2})\Gamma(1 + \frac{m}{2})}$$

is as in (4.4). Hence

$$\sum_{w \in W} a(w\lambda) b(w\lambda) = 2C_b \frac{\sinh((\mathbf{P} + i\mathbf{A})\lambda)}{\sin(\pi\lambda)}. \quad (7.4)$$

Set $u = \beta(H)$ for $H \in \mathfrak{a}$. Then

$$f(\exp H) = e^{-(\mathbf{P}+i\mathbf{A})\frac{m}{2}} G(m, \cosh u, -e^{-(\mathbf{P}+i\mathbf{A})}) = \frac{2^{-\frac{m}{2}} \sinh(\mathbf{P} + i\mathbf{A})}{(\cosh u + \cosh(\mathbf{P} + i\mathbf{A}))^{\frac{m}{2}+1}}. \quad (7.5)$$

The second part of Ramanujan's Master theorem says that for $u \in] - \mathbf{P}, \mathbf{P}[$ and $\sigma \in] - m/2, m/2[$ we have

$$\frac{2^{-\frac{m}{2}} \sinh(\mathbf{P} + i\mathbf{A})}{(\cosh u + \cosh(\mathbf{P} + i\mathbf{A}))^{\frac{m}{2}+1}} = C_b \int_{\sigma+i\mathbb{R}} \frac{\sinh((\mathbf{P} + i\mathbf{A})\lambda)}{\sin(\pi\lambda)} {}_2F_1\left(\frac{m+\lambda}{2}, \frac{m-\lambda}{2}; \frac{m+1}{2}; -\sinh^2 u\right) \frac{d\lambda}{c(\lambda)c(-\lambda)}, \quad (7.6)$$

where the integral on the right-hand side extends as an even homomorphic function on a neighborhood of \mathbb{R} inside \mathbb{C} . Notice that the left-hand side is holomorphic near \mathbb{R} . The equality (7.6) therefore holds for all $u \in \mathbb{R}$. In (7.6) we have used the formula for $F_\lambda(x)$ given in Example 1.1 together with the duplication formula

$${}_2F_1(a, b; a + b + 1/2; 4z(1 - z)) = {}_2F_1(2a, 2b; a + b + 1/2; z). \quad (7.7)$$

See [11, 2.1.5(27)]. (Recall also that the c -function appearing under the integral sign in fact depends on the parameter $m > 0$.)

If we take for instance $\mathbf{A} = \sigma = 0$, then (7.6) yields

$$\frac{2^{-\frac{m}{2}} \sinh \mathbf{P}}{(\cosh u + \cosh \mathbf{P})^{\frac{m}{2}+1}} = C_b \int_{i\mathbb{R}} \frac{\sinh(\mathbf{P}\lambda)}{\sin(\pi\lambda)} {}_2F_1\left(\frac{m+\lambda}{2}, \frac{m-\lambda}{2}; \frac{m+1}{2}; -\sinh^2 u\right) \frac{d\lambda}{c(\lambda)c(-\lambda)},$$

that is (since the Lebesgue measure on $i\mathbb{R}$ is $d\lambda$, $i\lambda \in i\mathbb{R}$)

$$\frac{2^{-\frac{m}{2}+1} \sinh \mathbf{P}}{(\cosh u + \cosh \mathbf{P})^{\frac{m}{2}+1}} = \frac{\Gamma(m)}{\Gamma(\frac{m}{2})\Gamma(1 + \frac{m}{2})} \int_{-\infty}^{+\infty} \frac{\sin(\mathbf{P}\lambda)}{\sinh(\pi\lambda)} {}_2F_1\left(\frac{m+\lambda}{2}, \frac{m-\lambda}{2}; \frac{m+1}{2}; -\sinh^2 u\right) \frac{d\lambda}{c(i\lambda)c(-i\lambda)}. \quad (7.8)$$

Suppose moreover that $u = 0$. Since $1 + \cosh \mathbf{P} = 2 \cosh^2(\mathbf{P}/2)$, we obtain from (7.8)

$$2^{-m+1} [\cosh(\mathbf{P}/2)]^{-m} \tanh(\mathbf{P}/2) = \frac{\Gamma(m)}{\Gamma(\frac{m}{2})\Gamma(1 + \frac{m}{2})} \int_{-\infty}^{+\infty} \frac{\sin(\mathbf{P}\lambda)}{\sinh(\pi\lambda)} \frac{d\lambda}{c(i\lambda)c(-i\lambda)}. \quad (7.9)$$

In the case $m = 2$, (7.9) gives

$$\int_{-\infty}^{+\infty} \frac{\sin(\mathbf{P}\lambda)}{\sinh(\pi\lambda)} \lambda^2 d\lambda = \frac{1}{2} [\cosh(\mathbf{P}/2)]^{-2} \tanh(\mathbf{P}/2). \quad (7.10)$$

Notice that (7.10) can also be computed from the classical integral formula

$$\int_{-\infty}^{+\infty} \frac{\sin(\mathbf{P}x)}{\sinh(\pi x)} dx = \frac{1 - e^{-\mathbf{P}}}{1 + e^{\mathbf{P}}} = \tanh(\mathbf{P}/2), \quad (7.11)$$

see e.g. [10, (861.61)]. In fact,

$$\int_{-\infty}^{+\infty} \frac{\sin(\mathbf{P}\lambda)}{\sinh(\pi\lambda)} \lambda^2 d\lambda = -\frac{d^2}{d\mathbf{P}^2} \int_{-\infty}^{+\infty} \frac{\sin(\mathbf{P}\lambda)}{\sinh(\pi\lambda)} d\lambda = -\frac{d^2 \tanh(\mathbf{P}/2)}{d\mathbf{P}^2}.$$

For $m = 1$, (7.9) yields

$$\int_{-\infty}^{+\infty} \frac{\sin(\mathbf{P}\lambda)}{\sinh(\pi\lambda)} \lambda \tanh(\pi\lambda) d\lambda = [\cosh(\mathbf{P}/2)]^{-1} \tanh(\mathbf{P}/2), \quad (7.12)$$

which is again a classical integral formula (see e.g. [10, (861.81)]).

Finally, the Plancherel formula in the last part of the theorem proves that

$$\begin{aligned} \sinh^2(\mathbf{P} + i\mathbf{A}) \int_0^\infty \frac{\sinh^m u}{(\cosh u + \cosh(\mathbf{P} + i\mathbf{A}))^{m+2}} du \\ = \left[\frac{\Gamma(m)}{2\Gamma(\frac{m}{2})\Gamma(\frac{m}{2} + 1)} \right]^2 \int_0^\infty \frac{|\sin((\mathbf{P} + i\mathbf{A})\lambda)|^2}{\sinh^2(\pi\lambda)} \frac{d\lambda}{c(\lambda)c(-\lambda)}. \end{aligned} \quad (7.13)$$

Moreover, according to (5.5),

$$\begin{aligned} 2^{\frac{m}{2}+1} \sinh^2(\mathbf{P} + i\mathbf{A}) \int_0^\infty \frac{{}_2F_1\left(\frac{m+\lambda}{2}, \frac{m-\lambda}{2}; \frac{m+1}{2}; -\sinh^2 u\right) \sinh^m u}{(\cosh u + \cosh(\mathbf{P} + i\mathbf{A}))^{\frac{m}{2}+1}} du = \\ \frac{\Gamma(m)}{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2} + 1)} \frac{\sinh((\mathbf{P} + i\mathbf{A})\lambda)}{\sin(\pi\lambda)} \end{aligned} \quad (7.14)$$

for all $\lambda \in \mathbb{C}$ with $|\operatorname{Re} \lambda| < \min\{1/2, m/2\}$.

For $\mathbf{A} = 0$, the right-hand side of (7.13) can be computed using the integral formula [15, 3.516(3)]:

$$\int_0^\infty \frac{\sinh^m u}{(\cosh \mathbf{P} + \cosh u)^{m+2}} du = \pi^{-\frac{1}{2}} 2^{\frac{m}{2}} e^{-i\frac{m}{2}\pi} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m + 2)} (\coth^2 \mathbf{P} - 1) Q_{2+m/2}^{m/2}(\coth \mathbf{P}),$$

where

$$Q_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1) \Gamma(1/2)}{2^{\nu+1} \Gamma(\nu + 3/2)} (z^2 - 1)^{\mu/2} z^{-\mu-\nu-1} {}_2F_1\left(\frac{\nu + \mu + 2}{2}, \frac{\nu + \mu + 1}{2}; \nu + \frac{3}{2}; z^{-2}\right)$$

is the associated Legendre function of second kind.

As a special case of (7.14) for $\mathbf{A} = 0$ and $m = 2$, we notice the formula

$$\frac{1}{\lambda} \int_0^{+\infty} \frac{\sinh(\lambda u) \tanh u}{(\cosh \mathbf{P} + \cosh u)^2} du = \frac{1}{4 \sinh \mathbf{P}} \frac{\sinh(\mathbf{P}\lambda)}{\sin(\pi\lambda)}, \quad (7.15)$$

which is a consequence of [11, 2.11(2) and 2.8(12)] because

$${}_2F_1\left(1 + \frac{\lambda}{2}, 1 - \frac{\lambda}{2}; \frac{3}{2}; -\sinh^2 u\right) = {}_2F_1\left(\frac{1}{2} + \frac{\lambda}{4}, \frac{1}{2} - \frac{\lambda}{4}; \frac{3}{2}; -\sinh^2(2u)\right) = \frac{2 \sinh(\lambda u)}{\lambda \sinh(2u)}.$$

REFERENCES

- [1] Anker, J.-P.: The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan. *J. Funct. Anal.* **96** (1991), no. 2, 331–349.
- [2] Araki, S. I. : On root systems and an infinitesimal classification of irreducible symmetric spaces. *J. Math. Osaka City Univ.* **13** (1962), 1–34.
- [3] Berndt, B.: *Ramanujan's Notebooks, Part I*, Springer-Verlag, New York, 1985.
- [4] Bertram, W.: Généralisation d'une formule de Ramanujan dans le cadre de la transformation de Fourier sphérique associée à la complexification d'un espace symétrique compact. *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), no. 11, 1161–1166
- [5] ———: Ramanujan's master theorem and duality of symmetric spaces. *J. Funct. Anal.* **148** (1997), no. 1, 117–151.
- [6] Branson, T. , G. Ólafsson and A. Pasquale: The Paley-Wiener theorem for the Jacobi transform and the local Huygens' principle for root systems with even multiplicities. *Indag. Math. (N.S.)* **16** (2005), no. 3–4, 429–442.

- [7] Cowgill, A. P.: On the summability of a certain class of series of Jacobi polynomials. *Bull. Amer. Math. Soc.* **41** (1935), 541–549.
- [8] Ding, H.: Ramanujan’s master theorem for Hermitian symmetric spaces. *Ramanujan J.* **1** (1997), no. 1, 35–52.
- [9] Ding, H., K. I. Gross and D. St. P. Richards: Ramanujan’s master theorem for symmetric cones. *Pacific J. Math.* **175** (1996), no. 2, 447–490.
- [10] Dwight, H. B.: *Tables of integrals and other mathematical data*, The Macmillan Company, 1961.
- [11] Erdélyi, A. et al.: *Higher transcendental functions*, volume 1, McGraw-Hill, New York, 1953.
- [12] Erdélyi, A. et al.: *Higher transcendental functions*, volume 2, McGraw-Hill, New York, 1953.
- [13] Faraut, J.: Espaces hilbertiens invariants de fonctions holomorphes. In: J. Faraut, F. Rouvière, M. Vergne, *Analyse sur les groupes de Lie et théorie des représentations*, Séminaires et Congrès **7** (2003), 101–167.
- [14] Gangolli, R. and V. S. Varadarajan: *Harmonic analysis of spherical functions on real reductive groups*. Springer-Verlag, Berlin, 1988.
- [15] Gradshteyn, I. S. and I. M. Ryzhik: *Table of Integrals, Series, and Products*, 7th edition, edited by A. Jeffrey and D. Zwillinger, Academic Press, 2007.
- [16] Hardy, G. H.: *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work*. Chelsea Publishing Company, 1991.
- [17] Heckman, G. J.: Root systems and hypergeometric functions. II. *Compositio Math.* **64**, no. 3, 353–373.
- [18] ———: Dunkl operators. *Astérisque* **245** (1997), Exp. No. 828, 4, 223–246. Séminaire Bourbaki, Vol. 1996/97.
- [19] Heckman, G. J. and E. M. Opdam: Root systems and hypergeometric functions. I. *Compositio Math.* **64** (1987), no. 3, 329–352.
- [20] Heckman, G. and H. Schlichtkrull: *Harmonic analysis and special functions on symmetric spaces*. Academic Press, 1994.
- [21] Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, New York, 1978.
- [22] Helgason, S.: *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*. Pure and Applied Mathematics, 113. Academic Press, Inc., Orlando, FL, 1984.
- [23] ———: *Geometric Analysis on Symmetric Spaces*. American Mathematical Society, Providence, 1994.
- [24] Lassalle, M.: Séries de Laurent des fonctions holomorphes dans la complexification d’un espace symétrique compact, *Ann. Sci. Ec. Norm. Sup.* **11** (1978), 167–210.
- [25] Macdonald, I. G.: Orthogonal polynomials associated with root systems. *Sém. Lothar. Combin.* **45** (2000/01), Art. B45a, 40 pp. (electronic)
- [26] Narayanan, E. K., A. Pasquale, and S. Pusti: Asymptotics of Harish-Chandra expansions, bounded hypergeometric functions associated with root systems, and applications, 30 pages, arXiv:1201.3891
- [27] Opdam, E. M.: Some applications of hypergeometric shift operators. *Invent. Math.* **98** (1989), 1–18.
- [28] Opdam, E. M.: Harmonic analysis for certain representations of graded Hecke algebras. *Acta math.* **175** (1995), 75–12.
- [29] Ólafsson, G. and Pasquale, A.: A Paley-Wiener theorem for the Θ -hypergeometric transform: the even multiplicity case, *J. Math. Pures Appl.* **83** (2004), no. 7, 869–927.
- [30] Ólafsson, G. and A. Pasquale: Ramanujan’s Master Theorem for Riemannian symmetric spaces, *J. Funct. Anal.* (2012), <http://dx.doi.org/10.1016/j.jfa.212.3.006>
- [31] Schapira, B.: Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel. *Geom. Funct. Anal.* **18** (2008), no. 1, 222–250.
- [32] Titchmarsh, E. C.: *The theory of functions*, 2nd edition, Oxford, 1939.

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case	multiplicities	$\mathcal{H}_{1,\beta,k}$	$\mathcal{H}_{2,\beta,k}$	l_β
(1)	$\frac{m_\beta/2}{2} \in \mathbb{Z}$	no sing	simple poles	$-\tilde{\rho}_\beta$
(2)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_{\beta/2} + m_\beta \in 2\mathbb{Z} + 1$	simple poles	no sing	$-\left(\frac{m_\beta/2}{4} + \frac{1}{2}\right)$
(3)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_\beta \in 2\mathbb{Z} + 3$	simple poles for $k = 0, \dots, \frac{m_\beta-3}{2}$	double poles	$-\left(\frac{m_\beta/2}{4} + \frac{1}{2}\right)$
(4)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_\beta = 1$	double poles ($\mathcal{H}_{1,\beta,k} = \mathcal{H}_{2,\beta,k}$)		$-\left(\frac{m_\beta/2}{4} + \frac{1}{2}\right)$
(5a)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_{\beta/2} + m_\beta \notin 2\mathbb{Z} + 1, m_\beta \notin 2\mathbb{Z} + 1, m_\beta > 1$	simple poles	simple poles	$-\left(\frac{m_\beta/2}{4} + \frac{1}{2}\right)$
(5b)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_{\beta/2} + m_\beta \notin 2\mathbb{Z} + 1, m_\beta \notin 2\mathbb{Z} + 1, m_\beta \leq 1$	simple poles	simple poles	$-\tilde{\rho}_\beta$

TABLE 1. Case $\beta = \beta_j$

case	multiplicities	$\mathcal{H}_{1,\beta,k}$	$\mathcal{H}_{2,\beta,k}$	l_β
(1)	$\frac{m_\beta/2}{2} \in \mathbb{Z}, m_\beta \in \mathbb{Z}$	no sing	no sing	--
(2)	$\frac{m_\beta/2}{2} \in \mathbb{Z}, m_\beta \notin \mathbb{Z}$	no sing	simple poles	$-\tilde{\rho}_\beta$
(3)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_{\beta/2} + m_\beta \in 2\mathbb{Z} + 1$	no sing	no sing	--
(4)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_\beta \in 2\mathbb{Z} + 3$	simple poles for $k = 0, \dots, \frac{m_\beta-3}{2}$	double poles	$-\left(\frac{m_\beta/2}{4} + \frac{1}{2}\right)$
(5)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_\beta = 1$	double poles ($\mathcal{H}_{1,\beta,k} = \mathcal{H}_{2,\beta,k}$)		$-\left(\frac{m_\beta/2}{4} + \frac{1}{2}\right)$
(6)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_{\beta/2} + m_\beta \notin 2\mathbb{Z} + 1, m_\beta \notin 2\mathbb{Z} + 1, 2\tilde{\rho}_\beta \in \mathbb{Z}$	simple poles	no sing	$-\left(\frac{m_\beta/2}{4} + \frac{1}{2}\right)$
(7a)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_{\beta/2} + m_\beta \notin 2\mathbb{Z} + 1, m_\beta \notin 2\mathbb{Z} + 1, 2\tilde{\rho}_\beta \notin \mathbb{Z}, m_\beta > 1$	simple poles	simple poles	$-\left(\frac{m_\beta/2}{4} + \frac{1}{2}\right)$
(7b)	$\frac{m_\beta/2}{2} \notin \mathbb{Z}, m_{\beta/2} + m_\beta \notin 2\mathbb{Z} + 1, m_\beta \notin 2\mathbb{Z} + 1, 2\tilde{\rho}_\beta \notin \mathbb{Z}, m_\beta \leq 1$	simple poles	simple poles	$-\tilde{\rho}_\beta$

TABLE 2. Case $\beta \neq \beta_j$