Asymptotic Consensus Without Self-Confidence

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Abstract— This paper studies asymptotic consensus in systems in which agents do not necessarily have self-confidence, i.e., may disregard their own value during execution of the update rule. We show that the prevalent hypothesis of self-confidence in many convergence results can be replaced by the existence of aperiodic cores. These are stable aperiodic subgraphs, which allow to virtually store information about an agent's value distributedly in the network. Our results are applicable to systems with message delays and memory loss.

Index Terms— asymptotic consensus, self-confidence, aperiodicity

I. INTRODUCTION

Asymptotic consensus is a phenomenon observed in certain biological, physical, and sociological systems. It is also utilized in some engineered man-made computer systems. The phenomenon consists in agents communicating in a very simple fashion to asymptotically reach agreement on a common real value. In nature, it can be observed (e.g.,[1], [2], [3], [4]) in bird flocking, firefly synchronization, synchronization of coupled oscillators, or opinion spreading. In engineering, it is used for sensor fusion, dynamic load balancing protocols, robot formation protocols, replication techniques, or rendezvous in space.

There is a very simple algorithm for asymptotic consensus that works in a large class of environments: In every computation step of a process, it updates its value to some average of all values it has received, and then sends out its new value. This simple algorithm has two remarkable properties: Firstly, it is very simple and yet manages to solve asymptotic consensus in a surprisingly large number of different environments. Secondly, it is an algorithm that can be *observed* in nature. More specifically, it serves as a widely accepted model in biology, physics, and sociology. It thus stands to reason to expect the algorithm to have a certain robustness against adverse environments. Consequently, it is used to attain approximate agreement in man-made, engineered, systems. For engineered systems, the viewpoint is not one of observing and explaining a given system, but of *analyzing* it for prediction of its future behavior or for assessing the need to improve the system. The speed of convergence in the context of asymptotic consensus is a measure for the stabilization time, or the transient phase, of the system. Obviously, the sharper the analysis of the system and its performance, the tighter it can be integrated into the timing constraints of a larger system, and hence the larger the potential performance of the larger system.

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The analysis becomes significantly harder if the communication graphs, or the weights, change over time, if communication delays are introduced, and if nodes are susceptible to certain faults. If one admits the dynamicity of the communication graph, then one has already accounted for a large class of faults, namely link faults. The addition of communication delays covers timing faults on links. A class of faults that has received considerably less attention in the literature is that of memory faults, either by memory loss or memory delays, i.e., the value read from local memory is not that of the most recent write operation. Memory delays become more probable with the advent of modern pipelined architectures and memories with weakened consistency properties. The present paper has as the goal to study of systems in which processes cannot, or do not, access their most recent value, but may read an older one or disregard it altogether. In the context of natural asymptotic consensus systems like in sociology, this phenomenon is more naturally called a *lack of self-confidence* and has its specific interest in the analysis of such systems. The paper extends a variety of convergence results known for cases with self-confidence to cases without and identifies the importance of having a certain replacement for self-confidence, which we call aperiodic cores. Self-confidence is a specific instance of this notion. Moreover, we discuss an explicit example showing the boundary between convergence and non-convergence in the context of aperiodic cores, shedding a more precise light on the frontier.

In linear algebraic terms, the study of asymptotic consensus is the study of infinite backwards products of stochastic matrices. The first convergence result for products of stochastic matrices is the Perron-Frobenius theorem, which states that the powers of an ergodic stochastic matrix converge to a rank 1 stochastic matrix. It was first generalized to a non-constant product of matrices by Wolfowitz [5] who showed that if every *finite* product of matrices of a set $\mathcal{M}$ of matrices is ergodic, then every infinite backwards product of matrices in $\mathcal{M}$ converges to a rank 1 stochastic matrix. The strict finiteness and ergodicity conditions in Wolfowitz' theorem were found to be inappropriate for many applications. Subsequently, Wolfowitz' theorem was extended in several directions (see, for example, [6], [7, Section II.G], or [8]). However, no direct generalization of Wolfowitz' theorem or the Perron-Frobenius theorem was obtained. This is due to the fact that these results all assume a strictly positive diagonal in all occurring matrices. In this sense, the results on asymptotic consensus in dynamic settings are no strict generalizations of the Perron-Frobenius theorem or Wolfowitz' theorem, precisely because of the fact that they require a strictly positive diagonal. One goal of this paper is

to remedy this deficiency; by providing convergence results for asymptotic consensus in dynamic settings without this hypothesis. Thus, our results are both strict generalizations of the Perron-Frobenius theorem and existing convergence theorems in asymptotic consensus.

The rest of the paper is organized as follows: Section II introduces the model, discusses related work, and gives necessary preliminary results. The notion of aperiodic cores is defined in Section III and the first new convergence result based on this notion follows in Section IV. We generalize the definition of aperiodic cores in Section V by introducing the notion of clusterings, which is adapted to hierarchic systems with local leader agents, as they naturally appear in the reduction from non-synchronous to synchronous settings. We apply this notion in Sections VI, VII, and VIII to show quite general convergence theorems in various environments, together with upper bounds on the convergence rate where applicable. Each of our theorems is followed by a corollary in form of an already known result in the literature. We do this to facilitate finding the context in terms of classical results in which the present paper generalizes the state of the art. Section IX concludes the paper with some final remarks.

II. ASYMPTOTIC CONSENSUS

A. Computational Model

The distributed computing model in which we study asymptotic consensus is the following: There are n distinguishable agents, each agent $i \in [n] = \{1, 2, \ldots, n\}$ possessing a real state variable x_i and communicating by exchanging messages. There is a global discrete time base, referred to by nonnegative integers in $\mathbb{N} = \{0, 1, 2, \ldots\}$. At every time $t \in \mathbb{N}$, we denote the content of the agents' state variables by $x_i(t)$. The initial value of state variable x_i is $x_i(0)$. At every time $t \in \mathbb{N}$, every agent sends the content of its state variable to all other agents. Messages may be delayed and/or lost. All agents simultaneously update their state variable at all positive times $t = 1, 2, 3, \ldots$ to some weighted average value of the received values, at most one of each other agent, and its current content of its own state variable.

Since the new content of the state variable is a mean of preceding values, there exists an integer $1 \leq \Delta_{i,j}(t) \leq t$ for every $j \in [n]$ such that

$$x_i(t) = \sum_{j=1}^{n} A_{i,j}(t) \cdot x_j\big(t - \Delta_{i,j}(t)\big) \qquad (1)$$

with

$$\sum_{j=1}^{n} A_{i,j}(t) = 1 \ . \qquad (2)$$

A *configuration* is a collection of real values, one for each agent's state variable, i.e., a vector in $\mathbb{R}^n$. An *execution* of asymptotic consensus is an infinite sequence of configurations $x(t) \in \mathbb{R}^n$ following the evolution (1) for some choice of the $A_{i,j}(t)$ and the $\Delta_{i,j}(t)$. An execution *reaches asymptotic consensus* if $x(t)$ converges and all component-wise limits $\lim_{t \to \infty} x_i(t)$ are equal.

An *averaging matrix* is a matrix whose entries are all nonnegative and whose row sums are all 1. In other words, it is a row stochastic matrix. Equation (2) assures that the collection of the $A_{i,j}(t)$ is an averaging matrix for all t. A *delay matrix* for time t is a matrix of integers between 1 and t. For every t, the collection of the $\Delta_{i,j}(t)$ is a delay matrix for t. Hence an execution is determined by the initial configuration $x(0)$, the sequence of the averaging matrices $A(t)$, and the sequence of the delay matrices $\Delta(t)$. A pair consisting of a sequence of averaging matrices $A(t)$ and a sequence of matrices $\Delta(t)$ such that every $\Delta(t)$ is a delay matrix for t is referred to as a *setting*. An *environment* is a nonempty set of settings. We say that a setting or an environment reaches asymptotic consensus if all of its executions do.

An important parameter of a setting is its maximum delay. We call a setting B-*bounded* if all entries of its delay matrices are at most B. A 1-bounded setting is called *synchronous* and is determined uniquely by the sequence of averaging matrices. If the nonzero entries of the averaging matrices are lower bounded by some positive α, then we say that the setting has *minimal confidence* α. It has *self-confidence* if all diagonal entries are positive. The *communication digraph* of a stochastic matrix A in $\mathbb{R}^{n \times n}$ has node set $[n]$ and contains an edge (i, j) if and only if $A_{i,j} > 0$.

We note that not every non-synchronous setting reaches asymptotic consensus, not even with self-confidence and strongly connected bidirectional communication graphs. The following example shows this. The problem arises if the delay $\Delta_{i,i}(t)$ is strictly greater than 1, i.e., node i does not use its most recent value for the update rule. It is one of the goals of the present paper to study sufficient conditions that enable convergence even if $\Delta_{i,i}(t) > 1$ for some, or even all, i and t.

Example 1: With $n = 2$ agents, we choose the averaging matrices

$$A(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \text{for } t \geq 2$$

and the initial vector $x(0) = {}^t(0, 1)$. Thus there is self-confidence and a minimal confidence of $1/2$. For the delay matrices, we choose

$$\Delta(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \Delta(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{for } t \geq 2 \ ,$$

i.e., for times $t \geq 2$, there is a delay to itself at every agent of 2 (even though the delay to the other agent is 1). The communication graph for $t \geq 2$ is shown in Fig. 1(a). One can show that $x_1(2t) \to 1/3$ as $t \to \infty$ whereas $x_1(2t+1) \to 2/3$. Similarly, $x_2(2t) \to 2/3$ and $x_2(2t+1) \to 1/3$. That is, the system is asymptotically periodic with period 2. The issue becomes clearer when looking at the equivalent synchronous system as studied by Cao, Morse, and Anderson [9]. Its communication graph for $t \geq 2$ is depicted in Fig 1(b). This equivalent synchronous communication graph has a period of 2. We recall their reduction in more detail at the end of Section II-B.
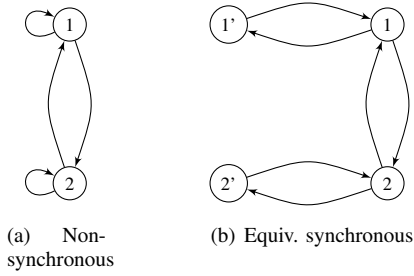
(a) Non-synchronous

(b) Equiv. synchronous

Fig. 1. Communication graphs for $t \geq 2$ in the original non-synchronous and the equivalent synchronous setting in Example 1

Fig. 2. The B copies of an agent in Cao, Morse, and Anderson's reduction

In a synchronous setting, the evolution of configurations $x(t)$ is governed by the linear recursive law

$$x(t) = A(t) \cdot x(t-1)$$

where $A(t)$ is a row stochastic matrix. Defining the product matrices $P(t) = A(t) \cdot A(t-1) \cdots A(1)$ we have $x(t) = P(t) \cdot x(0)$. In the following sections, we will also use the notation

$$P(t, s) = A(t) \cdot A(t-1) \cdots A(s+1)$$

for partial products. It is $P(t) = P(t, 0)$ for all t and $P(t, s) = I$, the identity matrix, if $t \leq s$.

B. Related Work

In this subsection, we list several convergence theorems in the literature that our results generalize. All of them suppose self-confidence.

Tsitsiklis introduced the *bounded intercommunication* assumption. It states that if an edge (i, j) appears in infinitely many communication digraphs, then is appears in one of the digraphs $G(A(t)), G(A(t+1)), \ldots, G(A(t+B-1))$ for a fixed B and all t.

Theorem 2 (Tsitsiklis [10]): A synchronous setting with the sequence of averaging matrices $A(1), A(2), \ldots$ with self-confidence and minimal confidence α reaches asymptotic consensus if the digraph G_∞ formed by the edges appearing in infinitely many communication digraphs is strongly connected and the bounded intercommunication assumption holds.

Moreau, and Hendrickx and Blondel independently showed that the bounded intercommunication assumption can be replaced by the assumption that every communication digraph is bi-directional:

Theorem 3 (Moreau [11], Hendrickx and Blondel [12]): A synchronous setting with the matrices $A(1), A(2), \ldots$ with self-confidence and minimal confidence α reaches asymptotic consensus if the digraph G_∞ as defined above is strongly connected and every communication digraph is bi-directional.

Blondel et al. generalized this result to B-bounded settings:

Theorem 4 (Blondel et al. [6]): Every B-bounded setting with averaging matrices $A(1), A(2), \ldots$ with self-confidence and minimal confidence α reaches asymptotic consensus if

the digraph G_∞ is strongly connected and every communication digraph is bi-directional.

Touri and Nedić extended the assumption of bi-directional digraphs to digraphs that are completely reducible. Charron-Bost recently showed its extension to B-bounded settings.

Theorem 5 (Touri and Nedić [13], Charron-Bost [14]): Every B-bounded setting with averaging matrices $A(t)$ with self-confidence and 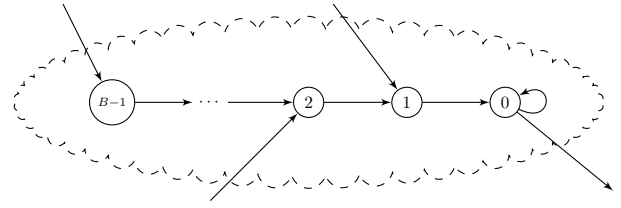minimal confidence α reaches asymptotic consensus if G_∞ is strongly connected and every communication digraph is completely reducible.

If an execution $x(t)$ reaches asymptotic consensus, one can ask the question of the speed at which this convergence occurs. Olshevsky and Tsitsiklis [15] noted that this speed tends to be exponential and have hence defined the *rate of convergence* as

$$\lim_{t \to \infty} \|x(t) - x^*\|_2^{1/t}$$

where x^* is the common limit of the agents' values. This rate is independent of the norm used.

Cao, Morse, and Anderson studied *coordinated* communication digraphs, i.e., digraphs that have a node j such that every other node has a path to j. They obtained the following result:

Theorem 6 (Cao, Morse, and Anderson [8], [9]): Every B-bounded setting with averaging matrices $A(1), A(2), \ldots$ with self-confidence and minimal confidence α reaches asymptotic consensus if every communication digraph is coordinated. Moreover, the rate of convergence is less than $1 - \alpha^{n^2}/n^2$.

To prove their result, they described a reduction of B-bounded settings to synchronous settings, albeit with B times as many agents as the original setting [9, Section 4.1]. The idea is to replicate every agent B times, but to shift the copies in time, i.e., at time t there is one copy holding the value $x_i(t)$, one $x_i(t-1)$, and so on until $x_i(t-B+1)$. This results in synchronous setting for asymptotic consensus. The replication of agents is illustrated in Fig. 2. Only the copy for the current value $x_i(t)$ has links to other agents' copies. Nonetheless, no such restriction exists for incoming edges. In the new resulting communication digraphs, even if all agents have self-loops in the original communication digraphs, not all nodes have them.

C. Dobrushin Semi-Norm for Stochastic Matrices

All stochastic matrices have 1 as an eigenvalue of maximum modulus. If the matrix is irreducible, the corresponding right-eigenspace is one-dimensional and generated by the column vector $\mathbf{1} = {}^t(1, 1, \ldots, 1)$. When studying such
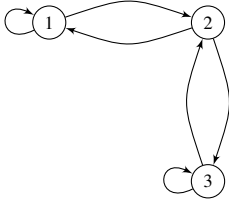
Fig. 3. Digraph $G(A)$ of matrix A

matrices, we are hence led to consider the distance of vectors to this eigenspace. Indeed, we will see that considering this distance is an appropriate tool for products of stochastic matrices.

The Dobrushin vector semi-norm on $\mathbb{R}^n$ is defined by setting $\delta(x) = \inf_{y \in \mathbb{R} \cdot \mathbf{1}} \|x - y\|_\infty$. This vector semi-norm induces the Dobrushin matrix semi-norm on $\mathbb{R}^{n \times n}$ by defining it in the operator norm fashion:

$$\delta(A) = \sup_{\substack{x \in \mathbb{R}^n \\ \delta(x) \neq 0}} \frac{\delta(Ax)}{\delta(x)}$$

Clearly, $\delta(A) = 0$ if the image of A is contained in the subspace $\mathbb{R} \cdot \mathbf{1}$.

We now give an example of a matrix whose semi-norm is strictly less than 1, but that has neither a strictly positive column nor a strictly positive diagonal. The matrix is equal to

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

and its digraph is depicted in Fig. 3. In fact, $\delta(A)$ is equal to $1/2$.

The following lemma characterizes the matrices with a Dobrushin semi-norm strictly smaller than 1. It uses the notion of a *scrambling* matrix. A stochastic matrix A is scrambling if for all indices i_1, i_2 there exists an index j such that both $A_{i_1, j} > 0$ and $A_{i_2, j} > 0$. Note that, a fortiori, A is scrambling if it has a strictly positive column. Its proof follows from the formula $\delta(A) = \max_{i_1, i_2 \in [n]} \sum_{j=1}^n (A_{i_1, j} - A_{i_2, j})_+$ for the Dobrushin matrix semi-norm where we denote by $(x)_+ = \max\{x, 0\}$ the positive part of x.

Lemma 7 ([8], [14]): Let A be a stochastic matrix. We always have $\delta(A) \leq 1$ and $\delta(A) < 1$ if and only if A is scrambling. In this case, $\delta(A) \leq 1 - \alpha$ where α is the smallest nonzero entry of A.

The next lemma shows the utility of δ to show convergence and asymptotic agreement.

Lemma 8: The sequence of backwards products $P(t)$ converges to a rank 1 stochastic matrix if and only if $\delta(P(t)) \to 0$ as $t \to \infty$.

D. Graph Interpretation of Matrix Products

Let i and j be nodes of a digraph G. A *walk* in G from i to j is a finite sequence of adjacent nodes in G that starts at i and ends at j. Its length is the number of nodes in the sequence minus one.

The following lemma characterizes positivity of entries in products of stochastic matrices solely in terms of the matrices' associated digraphs. It should be noted that, because we study backward products, the walks grow at the start node and not at the end node.

Lemma 9: Let $0 \leq s \leq t$ and $i, j \in [n]$. Then $P_{i,j}(t, s)$ is positive if and only if there exist $i_t, i_{t-1}, \ldots, i_s \in [n]$ with $i_t = i$ and $i_s = j$ such that $(i_\tau, i_{\tau-1})$ is an edge of $G(A(\tau))$ for all $s + 1 \leq \tau \leq t$.

If a strongly connected digraph is aperiodic, there exist walks of arbitrary length between all pairs of nodes as long as the length is greater or equal to a number called the *exponent* (sometimes also *index*) of the digraph. Formally, we denote the smallest T such that there is a walk from i to j of length t for all nodes i and j such that j is reachable from i in G and all $t \geq T$ by $T(G)$. Wielandt provided an upper bound on the exponent, although many more followed [16], [17], [18], [19], [20]. Wielandt's bound is the best possible upper bound in terms of only the number of nodes. If other parameters of the graph are known, however, tighter bounds exist. Since the exponent $T(G)$ appears in some of our bounds, it may be worthwhile to find a more precise bound for the specific graph appearing in a given application framework.

Theorem 10 (Wielandt [21]): Let G be a strongly connected aperiodic digraph with n nodes. Then the exponent of G is bounded by

$$T(G) \leq W(n) = \begin{cases} n^2 - 2n + 2 & \text{if } n \geq 2 \\ 0 & \text{if } n = 1 \end{cases}.$$

III. APERIODIC CORES

Classically, in asymptotic consensus, self-confidence of the agents is assumed. That is, every communication digraph contains self-loops at all nodes. This can model the fact that an agent does not ignore or forget its own previous value. We generalize the existence of self-loops, however: A missing self-loop in a specific communication digraph can model memory loss of an agent. We replace the assumption of self-loops to *aperiodic cores*, which are sub-digraphs of all of the settings' communication digraphs. They can be seen as a "distributed safety net against memory loss". In this sense, existence of self-loops is the assumption of a non-distributed safety measure against memory loss or temporary self-distrust. Their function in the proofs is similar to that of self-loops, but they are more general. A parameter that we use over and over in our results is that of the exponent of the aperiodic core. If one assumes self-loops, then H only consists of self-loops at all nodes and this parameter is equal to 0. So, in our theorem statements, if one assumes self-confidence, then $T(H) = 0$.

We call a node j in a digraph G a *leader* of another node i if G contains a path from i to j. A digraph is *j-coordinated* if j is a leader of every node. In this case, node j is called a *leader* of G. A digraph is *coordinated* if it is j-coordinated for some j. If j is a node of a digraph G, we say that G is *j-aperiodic* if j's strongly connected component in G is primitive. A digraph H is a *core* of a sequence $G_1, G_2, \ldots$
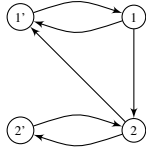
Fig. 4.   Variant of Example 1 that converges

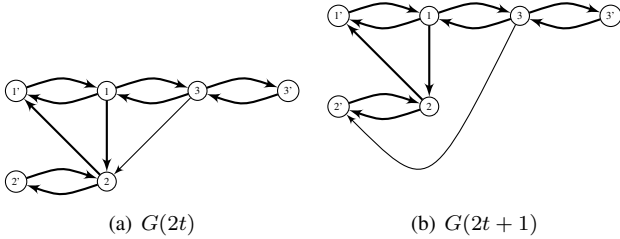(a) $G(2t)$ (b) $G(2t+1)$

Fig. 5.   Equivalent synchronous communication graphs that alternate in time

of digraphs if H is a sub-digraph of every G_t. A core is *spanning* if it is a spanning sub-digraph of all digraphs in the sequence, i.e., includes all nodes and no node is isolated.

IV. COORDINATED APERIODIC CORES

We start with assuming that there is a core that is coordinated and leader-aperiodic. The assumption of a core in particular applies if the communication digraph is constant. We hence get a direct generalization of the constant ergodic case:

Theorem 11: A synchronous setting with averaging matrices $A(t)$ with spanning core H and minimal confidence α reaches asymptotic consensus if there exists some agent j_0 such that H is j_0-coordinated and j_0-aperiodic. Moreover, the rate of convergence is at most $1 - \alpha^{T(H)}/T(H)$.

We remark that Theorem 11 in particular shows that the setting of Example 1 reaches asymptotic consensus if we change the delay $\Delta_{2,1}(t) = 2$, i.e., *increase* the message delay from agent 1 to agent 2, for $t \geq 2$. Indeed, the resulting equivalent synchronous setting has an aperiodic core from time $t = 2$ on, as is shown in Fig. 4. Note that, as the resulting stochastic matrix for the synchronous system is ergodic and constant, that also the Perron-Frobenius theorem shows convergence to asymptotic consensus in this case. However, embedding this structure into a slightly larger but simple system of 3 agents, as in Fig. 5 (the aperiodic core is almost the whole graph and is shown in bold; only a single edge changes continuously over time) shows the need the generalization that Theorem 11 provides.

V. CLUSTERINGS

We pair the idea of the distributed safety net in form of an aperiodic core with the notion of *clusters*, which have a leader that is the sole agent of the cluster to regard values of agents other than the cluster's. We will prove that it is not necessary for every agent to be contained in an aperiodic component, but only for the cluster leaders.

A digraph is a *cluster* with leader l if the digraph is l-coordinated. A *clustering* $\mathcal{C}$ is a collection of node-disjoint
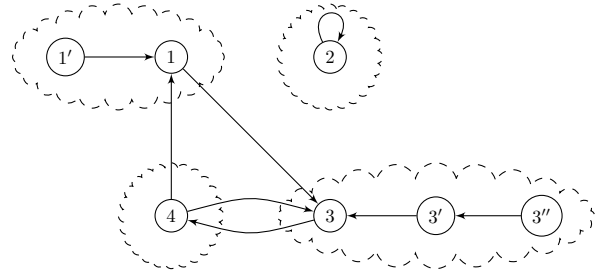

Fig. 6. $\mathcal{C}$-aperiodic digraph with leaders $1, 2, 3, 4$

clusters $C_1, C_2, \ldots, C_m$ together with respective leaders $l_1, l_2, \ldots, l_m$. A digraph is $\mathcal{C}$-*aperiodic* if every cluster C_j is a sub-digraph, every node is contained in some cluster, and it is l-aperiodic for every leader l_j of $\mathcal{C}$. Fig. 6 shows an example of a $\mathcal{C}$-aperiodic digraph.

A digraph *obeys* a clustering $\mathcal{C}$ if the only edges leaving a cluster are the leader's. Given a digraph that obeys clustering $\mathcal{C}$, the corresponding *cluster digraph* is the digraph when collapsing all clusters of $\mathcal{C}$ to single node.

A natural example of these clusterings occurs in the reduction of B-bounded settings with self-confidence to synchronous ones (see Fig. 2), for which $T(H) = B - 1$. If we do not assume self-confidence in B-bounded settings, then asymptotic consensus is not necessarily reached, even if the averaging matrices are constant and ergodic. By proving results on cluster-aperiodic cores in synchronous settings, we are hence also proving results on B-bounded settings with self-confidence.

VI. DYNAMIC COORDINATED COMMUNICATION DIGRAPHS

We now prove that asymptotic consensus is also reached if there is no coordinated core, but that coordination at every time step suffices.

Theorem 12: A synchronous setting with averaging matrices $A(1), A(2), \ldots$ with a $\mathcal{C}$-aperiodic spanning core H and minimal confidence α reaches asymptotic consensus if every communication digraph obeys clustering $\mathcal{C}$ and is coordinated. Moreover, the rate of convergence is at most

$$1 - \alpha^{(n-1)^2(T(H)+1)}/(n-1)^2(T(H)+1)$$

where n is the number of clusters in $\mathcal{C}$.

Corollary 13: A B-bounded setting with averaging matrices $A(1), A(2), \ldots$ with self-confidence and minimal confidence α reaches asymptotic consensus if every communication digraph is coordinated. Moreover, the rate of convergence is at most $1 - \alpha^{(n-1)^2 B}/(n-1)^2 B$.

Corollary 13, without the explicit bound on the rate of convergence is included in Theorem 6.

VII. DYNAMIC COMMUNICATION DIGRAPHS WITH FIXED LEADER

In this subsection, we assume a *fixed* leader in every communication digraph and are able to show a tighter bound

on the rate of convergence. The case of strongly connected communication digraphs is a special case.

Theorem 14: A synchronous setting with averaging matrices $A(1), A(2), \ldots$ with a $\mathcal{C}$-aperiodic spanning core H and minimal confidence α reaches asymptotic consensus if

- every communication digraph obeys clustering $\mathcal{C}$ and
- there is an agent j_0 such that every communication digraph is j_0-coordinated.

Moreover, the rate of convergence is at most

$$1 - \alpha^{(n-1)(T(H)+1)}/(n-1)(T(H)+1) \qquad (3)$$

where n is the number of clusters in $\mathcal{C}$.

Corollary 15: A B-bounded setting with averaging matrices $A(1), A(2), \ldots$ with self-confidence and minimal confidence α reaches asymptotic consensus if there is an agent j_0 such that every communication digraph is j_0-coordinated. Moreover, the rate of convergence is at most $1 - \alpha^{(n-1)B}/(n-1)B$.

Corollary 15, without the explicit bound on the rate of convergence is included in Theorem 6.

VIII. Completely Reducible Communication Digraphs

We now show that one can replace the assumption of coordination by the assumption of completely reducibility at every time step and eventual weak connectivity.

Theorem 16: A synchronous setting with averaging matrices $A(1), A(2), \ldots$ with a $\mathcal{C}$-aperiodic spanning core H and minimal confidence α reaches asymptotic consensus if

- every communication digraph obeys clustering $\mathcal{C}$,
- all cluster communication digraphs are completely reducible, and
- the digraph G_∞ formed by all edges that appear in infinitely many cluster communication digraphs is weakly connected.

Corollary 17: A B-bounded setting with averaging matrices $A(1), A(2), \ldots$ with self-confidence and minimal confidence α reaches asymptotic consensus if every communication digraph is completely reducible and the digraph G_∞ of edges that appear in infinitely many communication digraphs is weakly connected.

Corollary 17 for synchronous settings is Theorem 5.

IX. Conclusion

The paper introduced the novel notion of aperiodic cores and showed that the prevalent hypothesis of self-confidence can be replaced by the hypothesis of the existence of an aperiodic core in a large variety of convergence results for asymptotic consensus in dynamic settings. In particular, we discussed and explored the case of non-synchronous environments, for which we gave an explicit example of a 2-bounded system with 2 agents that could not be handled by existing convergence theorems. We also highlighted the need to be careful in these matters by showing that a small variant of the example does not reach asymptotic consensus (and does not even converge). In a linear algebraic view, our results are strict generalizations of the Perron-Frobenius theorem, which was not the case for most results on asymptotic consensus in the literature, as they require self-confidence.

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