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The totally real algebraic integers with diameter less than 4

VALÉRIE FLAMMANG, GEORGES RHIN AND QIANG WU

Abstract

The diameter of a totally real algebraic integer α of degree d with conjugates $\alpha_1 < \alpha_2 < \dots < \alpha_d$ is $\text{diam}(\alpha) = \alpha_d - \alpha_1$. For all positive integers k and n , $\text{diam}(2 \cos(2k\pi/n))$ is less than 4. R. M. Robinson has computed, modulo integer translations, all the other totally real algebraic integers α with $\text{diam}(\alpha) < 4$ for $d \leq 8$. We have done the computations for all $d \leq 15$. We use a large family of explicit auxiliary functions related to generalized integer transfinite diameter of real intervals. They give good bounds for the coefficients of the minimal polynomial of α . For $d = 15$ we prove a recent conjecture of Capparelli et al.

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Keywords: totally real algebraic integers, diameter, auxiliary functions, integer transfinite diameter, Chebyshev polynomials

1 Introduction

Let α be an algebraic integer of degree $d \geq 2$ with minimal polynomial

$$P = X^d + b_1 X^{d-1} + \dots + b_d = \prod_{i=1}^d (X - \alpha_i).$$

The diameter of α (or of P) is $\text{diam}(\alpha) = \max_{1 \leq i, j \leq d} |\alpha_i - \alpha_j|$. The diameter is clearly invariant by integer translation, i.e. $\text{diam}(\alpha + n) = \text{diam}(\alpha)$ for any rational integer n . It is also invariant by reflection, i.e. $\text{diam}(-\alpha) = \text{diam}(\alpha)$. In this paper we will examine the case of totally real algebraic integers α , i.e. when all the conjugates of α are real (in this case, their minimal polynomials P are also called hyperbolic polynomials). Let I be a real interval $[a, b]$ containing the numbers $\alpha_1 < \alpha_2 < \dots < \alpha_d$. If I has a length $b - a > 4$ then R. M.

Robinson [7] proved that there are infinitely many numbers α with all α_i in I . If $b - a < 4$ Schur and Pólya [10], proved that there is only a finite number of algebraic integers satisfying this property.

If $\text{diam}(\alpha) < 4$ then all conjugates of $\beta_1 - 2$ or of $3 - \beta_1$, where $0 < \beta_1 < 1$ is the fractional part of α_1 , lie in the interval $H = (-2, 2.5)$. Then for any $\varepsilon > 0$, there exists only a finite number of distinct values of $\text{diam}(\alpha) < 4 - \varepsilon$. This is easy to prove, since we may cover H by a finite number of intervals of length $= 4 - \varepsilon/2$.

So, the following question naturally arises:

what happens when we only suppose that $\text{diam}(\alpha) < 4$?

It is known that, if I is a real interval of length equal to 4 with integer endpoints, for example $I = [-2, 2]$, then there exist infinitely many numbers α with all their conjugates in I . These are the numbers $2 \cos(2k\pi/n)$ with positive integers k and n . Their minimal polynomials can be deduced from cyclotomic polynomials by the change of variable $x = z + 1/z$. By Kronecker's theorem [4] these are the only algebraic integers which satisfy this property. R. M. Robinson [8] has computed all diameters less than 4 up to degree 8.

Recently Capparelli et al. [2] have computed all diameters up to degree 14. They also give lists of diameters for the degrees 15 to 17 which have been computed with heuristic conditions on the coefficients of the polynomials P . But they conjecture that these lists are complete. In this paper, we prove that their conjecture is valid for the degree 15.

We recall in Table 1 the results that we obtain (excluding the numbers of cosine type):

Table 1

Number of polynomials whose $\text{diam} < 4$ for $2 \leq d \leq 15$.														
d	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number	1	3	10	14	13	15	21	19	15	10	9	4	9	6

Our method uses the arithmetical properties of the numbers α . One of the most important tool we use, is the construction of explicit auxiliary functions which are related to generalizations of the integer transfinite diameter. These auxiliary functions are used to give bounds for s_k which is the sum of the $k - th$ powers of the conjugates of α . Then we can deduce bounds for b_k by Newton's formula:

$$s_k + s_{k-1}b_1 + \cdots + s_1b_{k-1} + kb_k = 0.$$

As explained in Section 3, up to degree 12, we built the auxiliary functions with the algorithm given by the third author [12]. For the degrees 13 to 15 we

used the important refinement introduced by the first author [3].

The paper is organized as follows. In Section 2 we give the scheme of the computation, once we are given the bounds for the s_k . In Section 3 we explain the principle of the explicit auxiliary functions and their relation with the integer transfinite diameter. We also give all kinds of auxiliary functions that we use. In Section 4 we recall Robinson's method of computation and how we use the Chebyshev polynomials of some real intervals. Some numerical results and comments are given in Section 5. In this last section, we also study polynomials whose roots satisfy a symmetry.

2 Scheme of computations

Since we already know all the polynomials which have their roots in $(-2, 2)$ we only need to compute the numbers α such that α_d lies in $(2, 2.5)$. We divide this interval in 5 subintervals of length 0.1. Then all the conjugates of α_d belong to an interval $I_i = (a_i, b_i)$ of length 4.1 where $a_i = -2 + 0.1(i - 1)$ and $b_i = 2 + 0.1i$ for $1 \leq i \leq 5$.

For any such interval I_i we compute all the possible values of $s_1 = \text{trace}(\alpha)$. We give in Table 2 the bounds obtained with the auxiliary functions for the degree 15.

Table 2
Bounds of s_1 for d=15

Interval	[-2.0, 2.1]	[-1.9, 2.2]	[-1.8, 2.3]	[-1.7, 2.4]	[-1.6, 2.5]
Lower Bound	-2	-2	0	2	3
Upper Bound	4	5	7	9	10

It is noticeable that they greatly improve the naïve bounds, which are, for I_1 , -29 and 31 respectively. Then we compute all possible values of s_2 for a given value of s_1 . For the degree 15 we get 212 triples (I_i, s_1, s_2) . For all these triples we give only relations between the s_k for two values of k whose difference is equal to 1 or 2, because the auxiliary functions become less efficient when k is "large". As explained in Section 3, we give bounds for s_k for $k \geq 3$ depending on the values of s_j for $j < k$ by another method.

3 Auxiliary functions and integer transfinite diameter

3.1 Relation between auxiliary functions and integer transfinite diameter

Let I be a fixed interval I_i , i.e. $I = (a_i, b_i)$. We consider the auxiliary function

$$(3.1) \quad f(x) = x - a \log |Q(x)| \geq m \quad \text{for } x \in I_i,$$

where a is a positive real number and $Q \in \mathbb{Z}[x]$.

By summation on the α_i , we get

$$\sum_{i=1}^d f(\alpha_i) \geq dm$$

then

$$s_1 = \sum_{i=1}^d \alpha_i \geq dm + a \log \left| \prod_{i=1}^d Q(\alpha_i) \right|.$$

$\prod_{i=1}^d Q(\alpha_i)$ is the resultant of P and Q . If P does not divide the polynomial Q , then this is a nonzero integer. Therefore

$$s_1 \geq dm.$$

For an historical survey on the use of these auxiliary functions to get a lower bound for $s_1 = \text{trace}(\alpha)$ when α is totally positive see Aguirre and Peral [1]. If we take the auxiliary function as

$$f(x) = -x - a \log |Q(x)|$$

we have an upper bound for s_1 . And if we replace x by $\pm x^k$ we get bounds for s_k .

If we replace a by t/h where $h = \deg Q$, we have

$$(3.2) \quad f(x) = x - \frac{t}{h} \log |Q(x)|.$$

We search a polynomial Q in $\mathbb{Z}[x]$ such that

$$\max_{x \in I} |Q(x)|^{t/h} e^{-x} \leq e^{-m}.$$

If t is fixed (say $t = 1$), we relate this inequality to

$$t_{\mathbb{Z}, \varphi}(I) = \liminf_{\substack{h \geq 1 \\ h \rightarrow \infty}} \min_{\substack{Q \in \mathbb{Z}[x] \\ \deg Q = h}} \max_{x \in I} |Q(x)|^{1/h} \varphi(x)$$

which is the integer transfinite diameter of I with the weight $\varphi(x) = e^{-x}$.

3.2 Construction of auxiliary functions

If Q_1, Q_2, \dots, Q_J are the irreducible factors of Q in 3.1, then

$$(3.3) \quad f(x) = x - \sum_{j=1}^J e_j \log |Q_j(x)|.$$

When the polynomials Q_j are given, we use the semi-infinite linear programming to optimize f (i.e. to get the largest m). This method has been introduced into number theory by C. J. Smyth in 1984 [9]. So the main problem is to obtain a "good set" of polynomials Q_j .

Up to degree 12, we use the method given by the third author in [12] using LLL to get a good polynomial Q . For degree 13 to 15, a more elaborate algorithm was given by the first author [3]. The main idea is to get good polynomials Q_j by induction. Suppose that we have Q_1, Q_2, \dots, Q_J and an optimal f for this set of polynomials in the form 3.1. Then, for several values of the integer k , we search a polynomial $R \in \mathbb{Z}[x]$ of degree k such that

$$\sup_{x \in I} |Q(x)R(x)|^{\frac{t}{h+k}} e^{-x} \leq e^{-m}$$

that is to say

$$\sup_{x \in I} |Q(x)R(x)| e^{\frac{-x(h+k)}{t}}$$

is as small as possible. We apply LLL to the set of linear forms

$$Q(x_n)R(x_n) e^{\frac{-x_n(h+k)}{t}}$$

where the numbers x_n are suitable points in I , including the points where f has its smallest local minima. We get a polynomial R whose factors R_j are good candidates to enlarge the set of polynomials (Q_1, Q_2, \dots, Q_J) . We only keep the polynomials R_j which have a nonzero coefficient e_j in the new optimized auxiliary function f . After optimization some previous polynomials Q_j may have a zero coefficient and are removed.

We give in Table 3 the polynomials Q_j and the coefficients e_j of the auxiliary function f given in 3.3 which gives the inequality $s_1 \geq -0.179 \cdots d$ for α_d in I_1 .

Table 3
The explicit auxiliary function for the lower bound of s_1 in I_1

e_j	d_j	Coefficients of Q_j from degree 0 to d_j
0.011924266819	1	-1 1
0.096866488945	1	0 1
0.451013485389	1	1 1
0.527960449230	1	2 1
0.019566079389	2	-3 0 1
0.023273574120	2	-2 0 1
0.172825687626	2	-1 1 1
0.005291433336	3	-1 -4 0 1
0.012502796101	3	-1 -3 0 1
0.094662108579	3	-1 -2 1 1
0.003914026079	3	1 -3 0 1
0.071375556773	4	1 -4 -4 1 1
0.013261762401	4	1 -1 -4 0 1
0.021700560081	5	1 3 -3 -4 1 1
0.008611832838	5	1 5 -1 -5 0 1
0.028107777149	5	3 4 -5 -5 1 1
0.002577198250	5	3 8 -1 -6 0 1
0.002519477706	6	-5 -1 13 0 -7 0 1
0.005370851864	6	-4 3 13 -1 -7 0 1
0.010760212439	6	-1 3 6 -4 -5 1 1
0.020912491011	6	1 8 2 -9 -4 2 1 *
0.010610276065	6	1 8 8 -6 -6 1 1
0.000063033687	6	1 14 11 -8 -7 1 1
0.002043090659	8	-1 -7 -5 13 7 -10 -5 2 1 *
0.033279104801	8	-1 -1 10 23 5 -14 -6 2 1 *
0.002949394220	8	1 -4 -10 10 15 -6 -7 1 1
0.002368600679	9	1 5 -10 -20 15 21 -7 -8 1 1
0.000672281393	10	1 -1 -27 -39 25 57 3 -24 -6 3 1
0.003803608820	10	1 8 -6 -45 -12 46 19 -17 -8 2 1 *
0.015037380017	11	-1 -1 22 26 -47 -51 34 35 -10 -10 1 1 *
0.000861196960	12	-5 -14 31 84 -21 -137 -29 85 32 -22 -10 2 1 *
0.000592076853	12	-1 -1 34 83 -3 -130 -42 79 34 -21 -10 2 1 *
0.000631943824	12	-1 3 60 121 -26 -186 -45 103 40 -24 -11 2 1 *
0.005649700707	13	-1 -6 9 64 51 -109 -146 39 109 12 -31 -8 3 1*

The polynomials with * are polynomials which have complex roots.

3.3 Computation of the minimum of $f(x)$

We suppose that f is an auxiliary function defined as in 3.3. We give an algorithm to prove that f is a convex function in its domain of definition \mathbf{D} . It is sufficient to prove that the second derivative of f is positive in \mathbf{D} . By factorization of the polynomials Q_j in irreducible real factors, we see that f'' is a sum of terms of type 1: $\frac{e_j}{(x-\alpha)^2}$, where α is a real root of a polynomial Q_j and of type 2: $\frac{2e_k((x-\gamma)^2-\delta^2)}{((x-\gamma)^2+\delta^2)^2}$ where $\gamma + i\delta$ ($\delta > 0$) is a complex root of a polynomial Q_k (k may be equal to j). We suppose now that all the real roots α are taken in increasing order and that the complex roots $\gamma + i\delta$ are taken in increasing order of their real parts.

Algorithm

Step 1:

let S be a sequence of complex roots $\gamma + i\delta$ with consecutive real parts. We add all terms of type 2 related to this sequence S . Then we add to this rational function all the terms of type 1, associated to a real root α , from the greatest α less than the smallest δ of the sequence S to the smallest α greater than the greatest δ of the sequence. Let F_S be the rational function that we obtain. By Sturm's process, we compute the number of real zeros of the numerator of the function F_S which are in the domain of definition of f . If, for all sequences S , there is no zero in \mathbf{D} then we are done: f is convex. If this is not the case we use the:

Step 2:

We add to the exceptional functions F_S , which have zeros in \mathbf{D} , some terms of type 1 with real roots α which are close to the real roots already used in F_S such that now F_S has no zeros in \mathbf{D} .

N.B: for the function given in Table 3, it is sufficient to use the step 1 of the algorithm.

Then it is easy to compute the local minima of f within two consecutive real roots by the downhill simplex algorithm [6].

3.4 Other kinds of auxiliary functions

With

$$f(x) = x^2 - e_0x - \sum_{j=1}^J e_j \log |Q_j(x)| \geq m$$

where e_0 is real, we get $s_2 - e_0s_1 \geq dm$. So, if $s_1 = \sigma_1$, then $s_2 \geq e_0\sigma_1 + dm$. We optimize the linear form $e_0\sigma_1 + dm$ to get a lower bound of s_2 when $s_1 = \sigma_1$. If we replace x^2 by $-x^2$, we get an upper bound for s_2 . As we explained in the introduction we give relations between the numbers s_k . Let us give an example for the relation between s_4 and s_2 . We use the auxiliary function

$$f(x) = x^4 - e_0x^2 - \sum_{j=1}^J e_j \log |Q_j(x)| \geq m$$

where e_0 is real and $m \geq 0$. Then we get $s_4 \geq e_0s_2$ and this relation does not depend on the degree d . We maximize e_0 and stop the optimization when $m \geq 0.01$. For the upper bound $x^4 - e_0x^2$ is replaced by $-x^4 + e_0x^2$ and we minimize e_0 .

3.5 Chebyshev polynomials

Let T_k the Chebyshev polynomial of degree k of the interval (a,b) . This is the monic polynomial whose sup norm is the smaller on (a,b) . They may be defined by the relations

$T_1 = x - A$, $T_2 = x^2 - 2Ax + A^2 - 2B^2$ and $T_k = (x - A)T_{k-1} - B^2T_{k-2}$ for $k > 2$ where $A = (a + b)/2$ and $B = (b - a)/4$. Then $\max_{a \leq x \leq b} |T_k(x)| = 2B^k$ for $k \geq 1$.

Then we have the relation $|\sum_{1 \leq i \leq d} T_k(\alpha_i)| \leq 2dB^k$ and this gives a lower and an upper bound for s_k depending on the known values of s_j for $0 \leq j \leq k - 1$.

4 Robinson's method

The coefficients of the minimal polynomial P of α are computed by induction. In our case I_i , b_1 and b_2 are fixed. We suppose that we already know b_1, b_2, \dots, b_k and want to compute the bounds for b_{k+1} . Let Q_{k+1} be the following polynomial

$$Q_{k+1} = \frac{1}{(d - k + 1)!} \left(\frac{d}{dx}\right)^{d-k+1} P = c_0x^{k+1} + c_1x^k + \dots + c_kx + a$$

where c_j is an integer multiple of b_j , and a is an unknown integer (which will be b_{k+1}). $\tilde{Q}_{k+1} = Q_{k+1} - a$ is a well known polynomial. In I_i \tilde{Q}_{k+1} has k local extrema at the roots of its derivative. These roots r_1, r_2, \dots, r_k have been computed before. We take $r_0 = a_i$ and $r_{k+1} = b_i$. We compute the values of $\tilde{Q}_{k+1}(r_i)$ for $0 \leq i \leq k + 1$. The admissible integers a are those for which Q_{k+1} has all its minima negative and all its maxima positive. For each a we

compute the roots of Q_{k+1} with the Newton-Raphson method starting at the points $(r_i + r_{i+1})/2$ for $0 \leq i \leq k$. When the diameter of Q_{k+1} is less than 4, we replace k by $k + 1$. When $k = d$, we eliminate with Pari [5] the reducible polynomials.

5 Numerical results

5.1 Symmetry

Some polynomials P of even degree $2d$ have their roots which are pairwise symmetric with respect to $1/2$. We say that P is symmetric. If we put in P $y = -x^2 + x + 2$, we get a polynomial Q of degree d whose roots lie in the interval $(-1.75, 2.25)$. The polynomial Q may be of cosine type. The reverse transformation will not always give an irreducible polynomial. We give on the Web site [11] all pairs P, Q . There are 17 such pairs. For the degree 16, Capparelli et al. give 3 polynomials and 2 of them are symmetric.

So, we may reformulate the classical question:

Is there infinitely many α with $\text{diam}(\alpha) < 4$ and if it is so how many are symmetric?

5.2 The computations

All the computations have been done with Pascal on a PC. For the degree 14 the computing time was 34 days and for the degree 15 it took 440 days.

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