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# Poincaré's equations for Cosserat media : application to shells

Frederic Boyer<sup>1</sup> Federico Renda<sup>2</sup>

**Abstract** In 1901 Henri Poincaré discovered a new set of equations for mechanics. These equations are a generalization of Lagrange's equations for a system whose configuration space is a Lie group which is not necessarily commutative. Since then, this result has been extensively refined through the Lagrangian reduction theory. In the present contribution, we extend these equations from classical mechanical systems to continuous Cosserat media, i.e. media in which the usual point particles are replaced by small rigid bodies, called micro-structures. In particular, we will see how the Shell balance equations used in nonlinear structural dynamics, can be easily derived from this extension of the Poincaré's result.

## 1 Introduction

In contrast to classical continuous media where the basic constitutive element of matter is the point particle, Cosserat media are defined by small rigid bodies, called micro-structures, continuously stacked along material dimensions [1]. This fundamental difference has strong consequences on the two theories (classical vs Cosserat). In the classical theory, the geometric model of finite rotations disappears from the model, only re-appearing as a kinematic consequence of the translations (e.g. through the curl of the linear velocity field), while in the Cosserat model the rotations have a status similar to that of translations from the beginning to the end of the dynamic formulation. As a result, the Lie group structure naturally appears in the intrinsic definition of the configuration space of a Cosserat medium through the rigid transformations (in  $SO(3)$ , and more generally  $SE(3)$ ) undergone by its constitutive micro-structures. Thus, the model of Cosserat media should be recoverable from the abstract variational calculus developed by Henri Poincaré [2], known today as the Poincaré or Euler-Poincaré equations [3, 4]. These equations can be considered as a generalization of Lagrange's equations to systems whose configuration space is defined as a non commutative Lie group. As Poincaré remarked himself, they are particularly relevant when the Lagrangian of the system is left (or right) invariant by the group transformations, a property which is related to the symmetry of space (left invariance) and matter (right invariance) as Arnold and Marsden discovered later

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through the Lagrangian reduction theory [3, 4]. In this context, the right invariance has been shown to be a key concept to include the Eulerian point of view of fluid mechanics within the Euler-Poincaré's approach [5]. Arnold demonstrated that the ideal fluid is the infinite (right invariant) counterpart of the finite (left invariant) Euler and Poincaré's rigid body [6]. Another case that motivates us to shift from the finite to the infinite dimensional case is that of Cosserat media [7]. While in the case of the ideal fluid, the transformations leave in an infinite dimensional group, in the case of Cosserat media the transition toward infinite dimension is dramatically different. In this other case, in each point of a continuous material medium  $\mathcal{D}$ , a finite dimensional group  $G$  acts on a microstructure  $\mathcal{M}$ , i.e., a rigid body of infinitesimal size. Applying Poincaré's variational calculus to this context requires a shift from the basic picture of ordinary differential Poincaré equations of classical mechanical systems, to a set of partial differential equations in a field theoretical approach [8, 9]. Going back to the original aim of the Cosserat brothers [7], such an approach is developed in [8] to derive the fields equations of an unbounded Cosserat medium with no reference to the Poincaré picture. In [10], both fields equations and boundary conditions of a bounded Cosserat medium are derived in the context of Euler-Poincaré reduction. In this approach, the dynamics of a Cosserat medium are deduced from a unique Lagrangian density left invariant by the transformations of  $G$ , the transformations being parameterized by the time and the material coordinates (Lagrangian labels) of the medium  $\mathcal{D}$ . Though it may seem abstract at first, this variational calculus, which generalises Poincaré's calculus from one parametric dimension (the time axis) to several (space-time), is in fact a powerful alternative tool to Newton's laws for deriving in a blind manner the balance equations of Cosserat media. Furthermore, exploiting the intrinsic geometric nature of these media, the approach can assist in the development of numerical methods able to cope with finite rotations. In this later context, Cosserat media have been promoted in the field of the Finite Element Method, under the name of the "geometrically exact approach" by J.C. Simo and co-authors [11], [12]. However, these developments in computational mechanics have remained unrelated to the Poincaré approach. In the context of Cosserat beams, such a relation has been established in [10] for a Lagrangian density related to the non-metric space of material parameters.

The aim of the present article is to show how the same geometric picture can be extended and applied to Cosserat shells and show the relations between the resulting equations and the existing shell theory as it has been developed over the years by other means in works by Reissner [13], Green and Naghdi [14], Antman [15] among others. To that end, we will first extend the variational calculus of Poincaré to derive both the field equations and the boundary conditions of a Cosserat medium  $(\mathcal{D}, \mathcal{M})$ , with  $\mathcal{D}$  of arbitrary dimension  $p$ . Second, we will compare this set of partial differential equations when  $p = 2$  to the existing models of shell literature. This will need to extend the variational calculus of [8] and [10] to Lagrangian densities related to reference and deformed configurations of a Cosserat medium with further metric aspects playing a key role in shell theory. We will then show how these further formulations allow the so called geometri-

cally exact balance equations of hyperelastic shells in finite transformations and small strains to be recovered.

While in the case of beams, the extended Poincaré approach gives the usual geometrically exact beam equations of Reissner [16], we will see that in the case of shells, it gives a non-classical model of shells with intrinsic spin and couple stress i.e. a micropolar two-dimensional (2D) continuum [1]. We will then enhance the realism of our model by shifting from this micropolar model to that of a classical shell while stressing the role of the constitutive laws in this reduction process. In the resulting reduced dynamic model, the rotational field around the micro-structures, named "directors", is undetermined and referred as the "drilling rotation" in the shell literature [17]. This indeterminacy will be removed by imposing further kinematic constraints originally proposed in [18] and inspired from [19]. This will allow us to recover the missing angular velocity field around the shell directors from the linear velocity field on its mid-surface. This kinematic model will be then reintroduced in the reconstruction equations thus enabling us to obtain a closed formulation for classical Cosserat shells.

The article is structured as follows. Section 2 introduces all the basic definitions and statements required by the extension of the Poincaré picture to a Cosserat medium of arbitrary dimension. In section 3, we derive the Poincaré equations of a Cosserat medium. Section 4 uses these equations to give further geometric insights related to the role of duality and densities, and progressively introduces the model of stress of Cosserat media. Then, the extended Poincaré picture is first (section 5) applied to micropolar shells and second, to classical shells (section 6) with further discussion on the drilling rotation. For the purpose of illustration, the extended Poincaré equations are then applied to axisymmetric shells in section 7. Section 8 summarizes and opens perspectives for applications.

## 2 Basic statements and definitions

In this section we state the basic definitions required for the rest of the article. We invite the reader who is familiar with the geometric point of view of finite elasticity [20] to go directly to subsection 2.5. The key information of sections 2.1-4 are essentially: the definition of the configuration space and the basic kinematics of a Cosserat medium (eq. (3) and (8)), the expressions of the area element in the deformed configuration (eq. (12)), along with the figure 1 which illustrates the geometric context used in the article.

### 2.1 Definition and space of configuration of a classical medium

According to the mechanics of continuous media, a classical three-dimensional medium  $\mathcal{B}$  is a compact set of material points of Euclidean space  $\mathbb{R}^3$  labelled by 3 parameters  $\{X^i\}_{i=1,2,3}$  in a Cartesian frame  $(O, E_1, E_2, E_3)$  named material frame. A configuration of  $\mathcal{B}$  is the definition of the position  $x = x^i(X^j)e_i$  of all the

material points  $X = X^j E_j$  of  $\mathcal{B}$  in an inertial frame  $(o, e_1, e_2, e_3)$  of the ambient Euclidean space  $\mathcal{E} \cong \mathbb{R}^3$ . Formally, we define such a configuration as  $\Phi(\mathcal{B})$  (this is the set of the  $x = \Phi(X)$  for  $X$  running over  $\mathcal{B}$ ), where  $\Phi$  is a smooth invertible map from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  which preserves orientation, i.e., an element of  $\text{Diff}(\mathbb{R}^3)$ . The space of configurations of  $\mathcal{B}$  in  $\mathcal{E}$  is thus defined as:

$$C = \{\Phi(\mathcal{B}) \subset \mathcal{E}, \forall \Phi \in \text{Diff}(\mathbb{R}^3)\}, \quad (1)$$

and a motion of  $\mathcal{B}$  in  $\mathcal{E}$  is defined as a curve of configuration, i.e., a mapping:

$$t \in \mathbb{R}^+ \mapsto \Phi_t(\mathcal{B}) \subset \mathcal{E}, \quad (2)$$

where  $\Phi_t$  denotes a time-parameterized curve of  $\text{Diff}(\mathbb{R}^3)$ . Among all the possible configurations accessible to  $\mathcal{B}$ , we distinguish one of them as a reference configuration, denoted  $\Phi_o(\mathcal{B})$  in which  $\mathcal{B}$  is internally (energetically) at rest<sup>3</sup>. In practise we will assume  $(O, E_1, E_2, E_3) = (o, e_1, e_2, e_3)$  and will interchangeably speak about the "inertial" or "material frame", depending on the context. For the purpose of computation, one may consider in all subsequent developments that  $E_1 = (1, 0, 0)^T$ ,  $E_2 = (0, 1, 0)^T$ , and  $E_3 = (0, 0, 1)^T$ . Finally, note that these definitions can be extended from three, to two, and one-dimensional classical media, with a material index  $i$  running from 1 to  $n$  with  $n = 2$  and  $n = 1$  respectively. Moreover, it is worth noting that  $\{X^i\}_{i=1, \dots, n}$  defines a (material) chart on the open set  $\mathcal{B} - \partial\mathcal{B}$ , and that for topological reasons, it may be convenient to provide  $\mathcal{B}$  with an atlas of several such material charts, even if, in the following, we will consider only one of them.

## 2.2 Definition and space of configuration of a Cosserat medium

A Cosserat medium  $(\mathcal{D}, \mathcal{M})$  is a classical  $p$ -dimensional ( $p \leq 3$ ) medium  $\mathcal{D}$ , in each point, noted  $\bar{X}$ , of which, a Lie group  $G$  of rigid body mechanics ( $SO(3)$ ,  $SE(2)$ ,  $SE(3)$ ...) acts on a rigid solid of small dimensions (a "micro-solid" or "microstructure"), denoted  $\mathcal{M}$ , to generate all the possible configurations of  $(\mathcal{D}, \mathcal{M})$ . The configuration space of a Cosserat medium  $(\mathcal{D}, \mathcal{M})$  can thus be defined as the following set of parameterized maps in  $G$ :

$$\mathcal{C} = \{g : \bar{X} \in \mathcal{D} \mapsto g(\bar{X}) \in G\}. \quad (3)$$

Motions of  $(\mathcal{D}, \mathcal{M})$  in  $\mathcal{C}$  are defined as the time-parameterized curves of configuration:

$$t \in \mathbb{R}^+ \mapsto g_t(\mathcal{D}) = g(\mathcal{D}, t) \subset G. \quad (4)$$

where the notation  $g(\mathcal{D}, t)$  indicates that the space and time variables play similar roles. This basic context can be used to describe different physical situations depending on the meaning we attribute to  $\mathcal{D}$  and  $\mathcal{M}$ . For instance, a

<sup>3</sup> Here note that  $\Phi$  is not a mechanical transformation of the body in the ambient space but rather a geometric transformation or more exactly a parameterization of  $\mathcal{B}$ . The mechanical transformation between the reference and the deformed configurations is in fact defined as  $\Phi_t \circ \Phi_o^{-1}$ .

non-classical three dimensional (3D) medium constituted of three-dimensional micro-structures, also called micro-polar medium, obeys this definition if we take  $\mathcal{M} \simeq \mathcal{D} \simeq \mathbb{R}^3$ . In all the article, we will preferentially use the above definition as a reduced model to describe a classical 3D medium  $\mathcal{B}$  for which  $\mathcal{B} = \mathcal{D} \times \mathcal{M}$  where  $\mathcal{D}$  is a sub-manifold of  $\mathcal{B}$  over each point  $\bar{X}$  of which,  $\mathcal{M}$  is transformed by an element of  $G = SE(3)$ , represented by an homogeneous transformation of the general form:

$$g(\bar{X}) = \begin{pmatrix} R(\bar{X}) & r(\bar{X}) \\ 0 & 1 \end{pmatrix}, \quad (5)$$

with  $R(\bar{X}) \in SO(3)$  and  $r(\bar{X}) \in \mathbb{R}^3$  being the rotation and translation components of  $g$  respectively. Following section 2.1, the parameterization of  $\mathcal{B} = \mathcal{D} \times \mathcal{M}$  is chosen in such a manner that  $\mathcal{D}$  and  $\mathcal{M}$  are coordinatized by  $\{X^\alpha\}_{\alpha=1,2..p}$  and  $\{X^\gamma\}_{\gamma=p+1,..3}$  respectively. Hence, any point in  $\mathcal{D}$  is intrinsically defined as  $\bar{X} = (X^1, X^2, \dots, X^p)$  and the map  $e : \bar{X} \mapsto e(\bar{X}) = (\bar{X}, 0_{3-p})$ , defines an embedding from  $\mathcal{D}$  to  $\mathcal{B} \simeq \mathbb{R}^3$ . This embedding allows any configuration of  $\mathcal{D}$  in  $\mathcal{E}$  to be defined as the submanifold  $(\Phi \circ e)(\mathcal{D})$  of  $\Phi(\mathcal{B})$ . In  $\mathcal{E}$ , a configuration of the microstructure  $\mathcal{M}$  above  $\bar{X}$  will be denoted as  $\Phi(\bar{X}, \mathcal{M})$  (see figure 1). With this parametrization the reduction of  $\mathcal{B}$  into  $\mathcal{D} \times \mathcal{M}$  is motivated by considering material media as beams and shells, having dimensions along  $\mathcal{D}$  far larger than the others (along  $\mathcal{M}$ ), i.e., media for which the 3D configurations of (1) can be expanded into the following Taylor series in which  $\gamma = p + 1, ..3$ :

$$\Phi(X) = \Phi(\bar{X}, 0_{3-p}) + \frac{\partial \Phi}{\partial X_\perp^\gamma}(\bar{X}, 0_{3-p}) X_\perp^\gamma + o(\|X_\perp\|^2), \quad (6)$$

with  $X_\perp = X - e(\bar{X})$  the vector component of  $X$  along  $\mathcal{M}$ . Based on this expansion, the Cosserat based approach consists in reducing the kinematics (6) to its first order approximation with respect to  $\|X_\perp\|$  while neglecting the deformations of the material above each  $(\bar{X}, 0_{3-p})$ , a condition which defines the rigid microstructure  $\mathcal{M}$ . These approximations allow (6) to be rewritten as:

$$\Phi(X) = r(\bar{X}) + t_\gamma(\bar{X}) X_\perp^\gamma \quad (7)$$

with  $\bar{X} \mapsto r(\bar{X}) = (\Phi \circ e)(\bar{X})$  the field of position of the material points of  $\mathcal{D}$  in  $\mathcal{E}$ , and  $t_\gamma(\bar{X}) = ((\partial \Phi / \partial X_\perp^\gamma) \circ e)(\bar{X})$  a set of vectors lying in  $\Phi(\bar{X}, \mathcal{M})$ . Furthermore,  $\mathcal{M}$  being rigid, it is always possible to choose its parameterization  $\{X^\gamma\}_{\gamma=p+1,..3}$  such that  $t_\gamma = R(\bar{X}).E_\gamma$  with  $R(\bar{X}) \in SO(3)$ , and to rewrite the reduced kinematics (7) as:

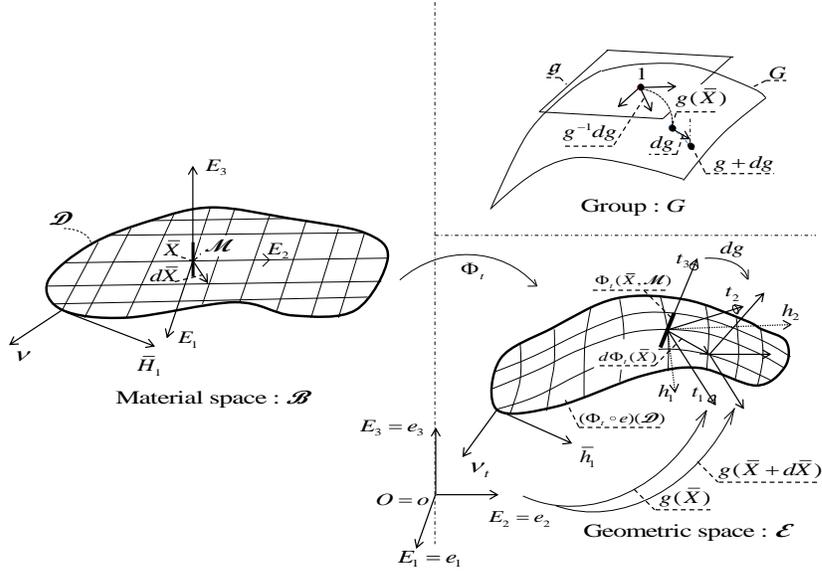
$$\Phi(X) = r(\bar{X}) + t_\gamma(\bar{X}) X_\perp^\gamma = r(\bar{X}) + R(\bar{X}).(X_\perp^\gamma E_\gamma) = r(\bar{X}) + R(\bar{X}).X_\perp, \quad (8)$$

which explicitly makes the group transformations  $g(\bar{X}) \in SE(3)$  of (3-5) appear, with  $R(\bar{X})$  defined as the two-point tensor [21]:  $R(\bar{X}) = t^i(\bar{X}) \otimes E_i$ , and:  $r(\bar{X}) = r^i(\bar{X})e_i$ . As in rigid-body mechanics, these transformations act on the material frame  $(O, E_1, E_2, E_3)$  considered as rigidly attached to  $\mathcal{M}$ . However they do not transform it into a single frame, but into a field of orthonormal mobile frames

$(t_1, t_2, t_3)(\bar{X}) = (R(\bar{X}).E_1, R(\bar{X}).E_2, R(\bar{X}).E_3)$  based in each point  $(\Phi_t \circ e)(\bar{X})$  as illustrated in figure 1. In [10], the reduced kinematics (8) are applied to beams, while in the second part of the article, they are applied to shells with  $\mathcal{D} \simeq \mathbb{R}^2$  defining the (material) reference shell's mid surface. For a shell, the microstructure  $\mathcal{M}$  models a generic rigid fiber across each point of its mid-surface  $\mathcal{D}$ , i.e. a degenerated one-dimensional rigid body named "director" in the shell's literature [14]. This is in contrast with beams, where  $\mathcal{M}$  stands for a full 3D rigid body modelling the beam cross-sections. As a result, for shells, shifting from (7) to (8), i.e. replacing  $t_3$  by  $R$  in the basic kinematics introduces an indeterminacy in the model of a classical shell which should be removed by reducing its configuration space from (3) to  $\{(r, t_3) : \bar{X} \in \mathcal{D} \mapsto (r(\bar{X}), t_3(\bar{X})) \in \mathbb{R}^3 \times S^2\}$ . However, we will initially ignore this fact and consider shells as full-Cosserat (2D) media with configuration space (3), i.e. media for which  $\mathcal{M}$  is a full (non-degenerated) rigid body to which a full 3D-orthonormal frame can be attached. This will allow us to see that  $\bar{X} \mapsto R(\bar{X})$  is univocally defined by a dynamic model obtained by applying Poincaré's picture on the configuration space (3). From a physical point of view, such a model holds for micro-polar shells, i.e. 2D-media with intrinsic kinetic spin and couple stress along  $t_3$  [1]. In a second step, we will see how the model of 2D-full Cosserat media can be adapted, and the indeterminacy on  $R$  can be removed, when considering the classical model of shell in which the frame attached to  $\mathcal{M}$  is degenerated into a single director.

### 2.3 Convected frame and co-frame

In all the article, we will use convected frames to express the tensor fields related to the mechanical state of  $\mathcal{B}$  and  $\mathcal{D}$ . A convected frame field is defined in each point of the current configuration of  $\mathcal{B}$  as the natural basis tangent to a set of material coordinate lines  $X^i = C$  ( $C \in \mathbb{R}$ ), drawn on  $\mathcal{B}$  and advected by its current deformation. Let us consider a motion as defined by (2) with  $\Phi_o = \Phi_{t=0}$ . At any time  $t$ , we may define the field of the convected frames covering the manifold  $\Phi_t(\mathcal{B})$  as a map  $\Phi_t(\mathcal{B}) \mapsto T\Phi_t(\mathcal{B})$  which assigns to any point  $\Phi_t(X) \in \mathcal{E}$ , the frame  $(g_1, g_2, g_3)(X) = ((\partial\Phi_t/\partial X^i)(X))_{i=1,2,3}$ . In this field of frame, the Euclidean metric of  $\mathcal{E}$  is defined as the fundamental tensor  $(g_i.g_j)(g^i \otimes g^j) = g_{ij}(g^i \otimes g^j)$  of determinant  $|g|$ , where  $\Phi_t(X) \mapsto (g^1, g^2, g^3)(X)$  defines a field of co-frame such that  $g^i.g_j = \delta_j^i$ . Applying the same construction to the sub-manifold  $(\Phi_t \circ e)(\mathcal{D}) \subset \Phi_t(\mathcal{B})$  allows to introduce two other fields of convected frame and co-frame whose base points lie in  $(\Phi_t \circ e)(\mathcal{D})$  respectively defined as  $(h_\alpha)_{\alpha=1,2..p}(\bar{X}) = ((\partial r/\partial X^\alpha)(\bar{X}))_{\alpha=1,2..p}$ , and  $(h^\alpha)_{\alpha=1,2..p}(\bar{X})$  such that  $h^\alpha.h_\beta = \delta_\beta^\alpha$ , with  $h = h_{\alpha\beta}(h^\alpha \otimes h^\beta)$  the fundamental metric tensor on  $(\Phi_t \circ e)(\mathcal{D})$  whose determinant is denoted by  $|h|$ . Using the expression of the reduced kinematics (7) in these definitions, it is straightforward to show that  $(g_i(\bar{X}, 0))_{i=1,2,3} = (h_1, ..h_p, t_{p+1}, ..t_3)(\bar{X})$ . and that, for a one-dimensional Cosserat medium (beam),  $h_1(X^1)$  is a tangent vector to the deformed line of the beam centroids in its material abscissa  $X^1$ , while  $(t_2, t_3)(X^1)$



**Fig. 1.** Left and right bottom: Parameterization of the deformed configuration  $\Phi_t(\mathcal{B})$  through a transformation of  $\mathcal{B}$ . Right: Kinematics in  $\mathcal{E}$  (bottom) and on  $G$  (top) of frames rigidly attached to the micro-structures.

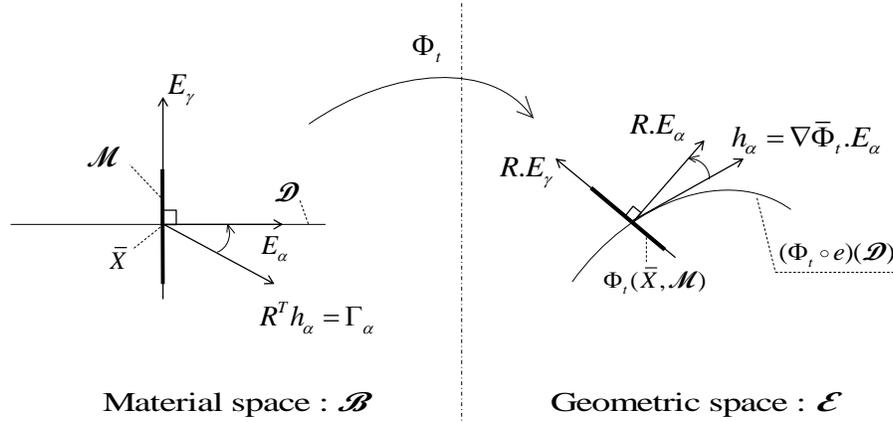
span the current configuration of  $X^1$ -cross-sections, i.e.,  $\Phi_t((X^1), \mathcal{M})$ . For a 2-dimensional Cosserat medium (a shell),  $(h_1, h_2)(X^1, X^2)$  defines a basis of the tangent planes to the shell's mid-surface in the base point of material coordinates  $(X^1, X^2)$  while  $(t_3)(X^1, X^2)$  span the current configuration of  $(X^1, X^2)$ -fibers, i.e.,  $\Phi_t((X^1, X^2), \mathcal{M})$ .

#### 2.4 Pull-back and push-forward

Our Cosserat medium is defined as  $\mathcal{B} = \mathcal{D} \times \mathcal{M}$  where  $\mathcal{M}$  and  $\mathcal{D}$  are transformed through different kinematics. This means that we can define two kinematically independent push-forward/pull-back processes, one related to the rigid transformations of  $\mathcal{M}$ , the second related to the deformations of  $\mathcal{D}$ . We now present these two processes and refer the reader to figures 1 and 2 which provide a partial illustration of the context.

**Pull-back and push-forward by the rigid transformations of  $\mathcal{M}$ :** Due to the presence of the microstructure in their basic constitutive definition, Cosserat media inherit from the geometric picture of the rigid body [22], in which orthonormal frames play a crucial role. In particular, any vector field  $(\Phi_t \circ e)(\bar{X}) \mapsto$

$v(\bar{X})$  of  $T(\Phi_t \circ e)(\mathcal{D})$  can be interpreted as a vector field of  $\mathcal{E}$ , i.e. a "spatial vector field" expressed in the field of mobile orthonormal frame according to  $v = V^i t_i$  or pulled-back in the unique material frame of  $\mathcal{B}$  through  $V = R^T . v = V^i E_i$ . Remarkably, the components of a spatial vector in its mobile frame are those of its pull-back, named the "material vector", in the material frame. Due to the orthogonality of  $R$ , the same relations apply to co-vector fields of  $T^*(\Phi_t \circ e)(\mathcal{D})$  and finally to any Euclidean tensor field tangent to  $(\Phi_t \circ e)(\mathcal{D})$ . For the purpose of illustration (see also figure 2), let us consider the case of shells for which  $p = 2$ , and consider the field of frames and co-frames defined by  $(h_1, h_2)(\bar{X})$  and  $(h^1, h^2)(\bar{X})$ . They can be transformed into  $\bar{X} \mapsto (\Gamma_1, \Gamma_2)(\bar{X}) = (R^T . h_1, R^T . h_2)(\bar{X})$  and  $\bar{X} \mapsto (\Gamma^1, \Gamma^2)(\bar{X}) = (R^T . h^1, R^T . h^2)(\bar{X})$ , which define mid-surface convected frame and co-frame fields respectively, pulled back into the material frame. Note that the change of space is identical for the two fields because  $R^{-1} = R^T$ , while duality imposes  $\Gamma_\alpha . \Gamma^\beta = \delta_\alpha^\beta$ .



**Fig. 2.** Geometric picture of frames and their push-forward/pull-back relations.

**Pull-back and push-forward by deformations of  $\mathcal{D}$ :** Due to the presence of the classical medium  $\mathcal{D}$  in the definition of the Cosserat medium  $\mathcal{D} \times \mathcal{M}$ , a second pull-back/push-forward process holds between  $T(e(\mathcal{D}))$  and  $T(\Phi_t \circ e)(\mathcal{D})$ .

Using the transformation  $\bar{\Phi}_t$ , any tensor field on  $e(\mathcal{D})$  can be pushed forward onto  $(\bar{\Phi}_t \circ e)(\mathcal{D})$ , and reciprocally pulled back from  $(\bar{\Phi}_t \circ e)(\mathcal{D})$  to  $e(\mathcal{D})$  by using the restricted linear tangent maps  $\nabla \bar{\Phi}_t = h_\alpha \otimes E^\alpha$ ,  $\nabla \bar{\Phi}_t^{-1} = E_\alpha \otimes h^\alpha$ ,  $\nabla \bar{\Phi}_t^T = E^\alpha \otimes h_\alpha$  and  $\nabla \bar{\Phi}_t^{-T} = h^\alpha \otimes E_\alpha$ . In particular, the frames and co-frames convected by the transformation are related to  $\{E_\alpha\}_{\alpha=1,..,p}$  and  $\{E^\alpha\}_{\alpha=1,..,p}$  through:

$$h_\alpha = \nabla \bar{\Phi}_t . E_\alpha, \quad E^\alpha = \nabla \bar{\Phi}_t^T . h^\alpha. \quad (9)$$

However, some geometric tangent objects as the "exterior forms" and "multi-vectors" also involve  $\sqrt{|h|}$  in this pull-back/push-forward process. This is particularly the case when considering the oriented material volume element  $dX^1 \wedge dX^2 \wedge \dots \wedge dX^p$  of  $\mathcal{D}$ , which is changed into the metric deformed volume of same orientation  $\sqrt{|h|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p$  by the transformation  $\bar{\Phi}_t \circ e$ . Similarly, the oriented material area element in any point  $\bar{X}$  of  $\mathcal{D}$  is defined as the  $p-1$  form  $dY^1 \wedge dY^2 \dots \wedge dY^{p-1}$ , where  $\{Y^\gamma\}_{\gamma=1,2,..,p-1}$  is a Cartesian chart, of natural orthonormal basis  $\{\bar{H}_\gamma = \partial/\partial Y^\gamma\}_{\gamma=1,2,..,p-1}$ , covering a material hyperplane crossing  $\bar{X}$ . This form is transformed into  $|\bar{h}|^{1/2} dY^1 \wedge dY^2 \wedge \dots \wedge dY^{p-1}$  by the deformation, with  $|\bar{h}|$  the determinant of the fundamental tensor of  $\mathcal{E}$  in the convected basis  $\{\bar{h}_\gamma = \nabla \bar{\Phi}_t . \bar{H}_\gamma = \partial r / \partial Y^\gamma\}_{\gamma=1,2,..,p-1}$ . Exploiting duality of  $(p-1)$ -forms and  $(p-1)$ -vectors, we may define the material oriented area element as the  $(p-1)$ -vector  $\bar{H}_1 \wedge \bar{H}_2 \dots \wedge \bar{H}_{p-1}$ , which can be represented by the conjugate true vector of  $T^*e(\bar{X})$  [23]:

$$\nu = \nu_\alpha E^\alpha = \frac{1}{(p-1)!} \epsilon_{\alpha\alpha_1 \dots \alpha_{p-2}\alpha_{p-1}} \left| \frac{\partial(X^{\alpha_1}, X^{\alpha_2}, \dots, X^{\alpha_{p-1}})}{\partial(Y^1, Y^2, Y^3, \dots, Y^{p-1})} \right| E^\alpha, \quad (10)$$

which defines the unit normal to the element, where  $\left| \frac{\partial(\dots)}{\partial(\dots)} \right|$  denotes a functional determinant, while for any integer  $n$ ,  $\epsilon_{i_1 i_2 \dots i_n}$  is equal to zero if two indices are identical, equal to  $+1$  (respectively  $-1$ ), if  $(i_1, i_2, \dots, i_n)$  is an even (respect. odd) permutation of  $(1, 2, \dots, n)$ . Similarly, normalizing the conjugate vector of  $\bar{h}_1 \wedge \bar{h}_2 \dots \wedge \bar{h}_{p-1}$ , defines the unit normal to the oriented deformed element as the covector:

$$\nu_t = \nu_{t,\alpha} h^\alpha = \left| \frac{h}{\bar{h}} \right|^{1/2} \frac{1}{(p-1)!} \epsilon_{\alpha\alpha_1 \dots \alpha_{p-2}\alpha_{p-1}} \left| \frac{\partial(X^{\alpha_1}, X^{\alpha_2}, \dots, X^{\alpha_{p-1}})}{\partial(Y^1, Y^2, Y^3, \dots, Y^{p-1})} \right| h^\alpha. \quad (11)$$

Note that the two vectors  $\nu$  and  $\nu_t$  whose components are related by  $\nu_{t,\alpha} = (|h|/|\bar{h}|)^{1/2} \nu_\alpha$ , contain all the information about the orientation of the original  $(p-1)$ -forms. Moreover, defining the measure (area) of the material and deformed elements as  $dS = dY^1 \dots dY^{p-1}$  and  $dS_t = |\bar{h}|^{1/2} dY^1 \dots dY^{p-1}$  respectively, where each  $dY^\gamma$  represents the components of an infinitesimal vector  $dY^\gamma \bar{H}_\gamma$  (with no summation on  $\gamma = 1, 2, \dots, p-1$ ), we also have the relation between the co-vectors  $\nu_t dS_t$  and  $\nu dS$  which will be used in the subsequent developments instead of the original  $(p-1)$  forms:

$$\nu_t dS_t = (\nu_{t,\alpha} dS_t) h^\alpha = (\sqrt{|h|} \nu_\alpha dS) \nabla \bar{\Phi}_t^{-T} . E^\alpha = \sqrt{|h|} \nabla \bar{\Phi}_t^{-T} . (\nu dS). \quad (12)$$

Taking  $dS = |\bar{H}|^{1/2}dY^1\dots dY^{p-1}$ , (12) holds for a relation between material and deformed oriented area elements of any hyper-surface of  $\mathcal{D}$  parameterically defined by  $X^\alpha(Y^\gamma)$ , with  $|\bar{H}|$  the determinant of the metric of  $\mathcal{D}$  in the natural basis  $\{\bar{H}_\gamma\}_{\gamma=1..p-1}$  covering the hyper-surface.

## 2.5 Lagrangian of a Cosserat medium

On the definition (3) of the configuration space, the Lagrangian of a Cosserat medium  $\mathcal{B} = \mathcal{D} \times \mathcal{M}$  at current time  $t$  is defined as the following functional:

$$L = \int_{\mathcal{D}} \mathcal{L}_t \left( g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial X^\alpha} \right) \sqrt{|h|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p, \quad (13)$$

where  $\mathcal{L}_t \sqrt{|h|} dX^1 \wedge \dots \wedge dX^p$  is the Lagrangian volume-form of the Cosserat medium, and  $\mathcal{L}_t$  is the density of Lagrangian per unit of metric volume  $\sqrt{|h|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p$  of the current deformed configuration  $(\Phi_t \circ e)(\mathcal{D})$ . Alternatively,  $L$  can be related to the volume of reference configuration  $(\Phi_o \circ e)(\mathcal{D})$  as:

$$L = \int_{\mathcal{D}} \mathcal{L}_o \left( g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial X^\alpha} \right) \sqrt{|h_o|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p, \quad (14)$$

or directly to the volume of parametric (material) space  $\mathcal{D}$ :

$$L = \int_{\mathcal{D}} \mathcal{L} \left( g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial X^\alpha} \right) dX^1 \wedge dX^2 \wedge \dots \wedge dX^p. \quad (15)$$

In (14) (respectively (15))  $\mathcal{L}_o$ , (respect.  $\mathcal{L}$ ), is the density of the Lagrangian of  $\mathcal{B}$  per unit of metric volume  $\sqrt{|h_o|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p$  (respect. non-metric volume  $dX^1 \wedge dX^2 \wedge \dots \wedge dX^p$ ) of  $(\Phi_o \circ e)(\mathcal{D})$  (respect., of  $\mathcal{D}$ ).

## 2.6 Reduction of the Lagrangian

Physically, the Lagrangian  $L$  depends on  $\partial g / \partial t$  through the kinetic energy of  $\mathcal{B}$ , and on  $\partial g / \partial X^\alpha$  through the internal strain energy of its material that is assumed to be hyperelastic. The Lagrangian density  $\mathcal{L}$  is left invariant in the sense that substituting  $g$  by  $kg$ , in it (with  $k$  a constant transformation in  $G$  over each point of space-time  $\mathcal{D} \times \mathbb{R}^+$ ), does not change its value. Physically, this reflects the fact that both the densities of kinetic and internal strain energy are the same when observed from any frame of geometric space. The first property is a key result of rigid body mechanics [22], while the second is a consequence of the material frame indifference of mechanics of continuous media [24]. As a result, taking  $k = g^{-1}$ , allows the transformation of (13) into the reduced Lagrangian:

$$L_r = \int_{\mathcal{D}} \mathfrak{L}_t(\eta, \xi_\alpha) \sqrt{|h|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p, \quad (16)$$

where we introduced the following left invariant vector fields leaving in the Lie algebra  $\mathfrak{g}$  of  $G$  (considered as a group of matrices):

$$\eta = g^{-1} \frac{\partial g}{\partial t}, \quad \xi_\alpha = g^{-1} \frac{\partial g}{\partial X^\alpha}, \quad \alpha = 1, 2, \dots, p. \quad (17)$$

In (16),  $\mathfrak{L}_t$  defines the reduced Lagrangian volume density per unit of metric volume of deformed configuration  $(\Phi_t \circ e)(\mathcal{D})$ . Alternatively, applying the same reduction process to (14) or (15) instead of (13), permits the reduced Lagrangian densities to be defined per unit of metric volume of reference configuration  $(\Phi_o \circ e)(\mathcal{D})$  and per unit of non-metric (material) volume  $\mathcal{D}$ , denoted  $\mathfrak{L}_o$  and  $\mathfrak{L}$  respectively. As  $\mathfrak{L}_t$ , these two further densities only depend on  $(\eta, \xi_\alpha)$ . Finally equating (13), (14) and (15) allows stating the following relations between the three reduced Lagrangian densities:

$$\mathfrak{L} = \mathfrak{L}_o \sqrt{|h_o|} = \mathfrak{L}_t \sqrt{|h|}. \quad (18)$$

In the following, we will preferentially use the density  $\mathfrak{L}_t$  related to the current configuration and discuss the results using this choice in relation to  $\mathfrak{L}_o$  and  $\mathfrak{L}$ .

### 3 Poincaré's equations of Cosserat media

Following Poincaré's approach [2], a Cosserat medium  $\mathcal{B} = \mathcal{D} \times \mathcal{M}$  submitted to a set of external forces is governed by the extended Hamilton principle, which can be stated directly on the definition (3) of  $\mathcal{C}$  ( $\alpha$  running from 1 to  $p$ ), as:

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{D}} \mathfrak{L}_t(\eta, \xi_\alpha) \sqrt{|h|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p dt = - \int_{t_1}^{t_2} \delta W_{ext} dt, \quad (19)$$

for any  $\delta g = g \delta \zeta$  where  $\delta \zeta \in \mathfrak{g}$  is a field of material variation of  $g$  achieved while  $t$  and all the  $X^\alpha$  are maintained fixed and such that  $\delta \zeta(t_1) = \delta \zeta(t_2) = 0$ . In (19),  $\delta W_{ext}$  models the virtual work of external forces and can be detailed as:

$$\begin{aligned} \delta W_{ext} = & \int_{\mathcal{D}} \langle F_{ext,t}, \delta \zeta \rangle \sqrt{|h|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p + \\ & \int_{\partial \mathcal{D}} \langle \bar{F}_{ext,t}, \delta \zeta \rangle |\bar{h}|^{1/2} dY^1 \wedge dY^2 \wedge \dots \wedge dY^{p-1}, \end{aligned} \quad (20)$$

with  $\langle \dots \rangle$  the duality product in the Lie algebra  $\mathfrak{g}$ ,  $(Y^\gamma)_{\gamma=1,2,\dots,p-1}$ , a set of material coordinates covering the boundary  $\partial \mathcal{D}$ . Finally,  $F_{ext,t}$  and  $\bar{F}_{ext,t}$  are densities of external forces in  $\mathfrak{g}^*$  per unit of volume of  $(\Phi_t \circ e)(\mathcal{D})$  and  $(\Phi_t \circ e)(\partial \mathcal{D})$  respectively. Note that we consider boundaries submitted to external forces only, the case of imposed motions being easily modelled by defining two types of boundaries. Also note that for beams, i.e. one-dimensional Cosserat media, one has to take:

$$|\bar{h}|^{1/2} = \delta(\partial \mathcal{D}), \quad (21)$$

where  $\delta$  is the Dirac distribution, and (20) has to be integrated within the meaning of distributions. Now, let us invoke the constraints of variation at fixed time and material labels:

$$\delta \frac{\partial g}{\partial t} = \frac{\partial \delta g}{\partial t}, \quad \delta \frac{\partial g}{\partial X_\alpha} = \frac{\partial \delta g}{\partial X_\alpha}, \quad \text{for } \alpha = 1, 2..p. \quad (22)$$

Then inserting " $\delta g = g\delta\zeta$ " into (22) gives the following relations, as first derived by Poincaré [2], which play a key role in the variational calculus on Lie groups [25]:

$$\delta\eta = \frac{\partial \delta\zeta}{\partial t} + ad_\eta(\delta\zeta), \quad \delta\xi = \frac{\partial \delta\zeta}{\partial X_\alpha} + ad_{\xi_\alpha}(\delta\zeta), \quad (23)$$

with  $ad$  the adjoint map of  $\mathfrak{g}$  on itself. As detailed in the next section, applying the standard variational calculus to (19), with (23) running before the usual by part integration in time, and the divergence theorem in the convected basis  $\{h_\alpha\}_{\alpha=1,2..p}$  covering  $(\Phi_t \circ e)(\mathcal{D})$ , gives the Poincaré equations of a Cosserat medium in the material frame (we use summation convention on repeated indices  $\alpha$ ):

$$\begin{aligned} & \frac{1}{\sqrt{|h|}} \left( \frac{\partial}{\partial t} \left( \sqrt{|h|} \frac{\partial \mathfrak{L}_t}{\partial \eta} \right) - ad_\eta^* \left( \sqrt{|h|} \frac{\partial \mathfrak{L}_t}{\partial \eta} \right) \right) + \\ & \frac{1}{\sqrt{|h|}} \left( \frac{\partial}{\partial X^\alpha} \left( \sqrt{|h|} \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha} \right) - ad_{\xi_\alpha}^* \left( \sqrt{|h|} \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha} \right) \right) = F_{ext,t}, \\ & \left( \frac{|h|}{|\bar{h}|} \right)^{1/2} \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha} \nu_\alpha = -\bar{F}_{ext,t}, \end{aligned} \quad (24)$$

where  $ad^*$  is the co-adjoint map of  $\mathfrak{g}$  on  $\mathfrak{g}^*$ , while  $\nu = \nu_\alpha E^\alpha$  is the unit outward normal to  $e(\mathcal{D})$ , and  $\partial \mathfrak{L}_t / \partial \eta$ , and  $\partial \mathfrak{L}_t / \partial \xi_\alpha$  define the conjugate generalized moments of the medium.

These equations represent the dynamics of a Cosserat medium reduced in the dual of its Lie algebra here identified as the space of infinitesimal material (right) transformations of  $G$ . They govern the time-evolution of the material velocities  $\eta$  and when the external forces are left invariant, they can be time-integrated separately to compute the velocity field in a first step. In a second step, the motion of the medium can be reconstructed by using the so-called reconstruction equation, which can be simply stated as:

$$\forall \bar{X} \in \mathcal{D} : \frac{\partial g}{\partial t}(\bar{X}, t) = (g\eta)(\bar{X}, t). \quad (25)$$

In all cases (symmetric or not), (25) supplements (24), to give a set of time-evolution equations in the following definition of a Cosserat medium's state space:

$$\mathcal{S} = \{(g, \eta) : \bar{X} \in \mathcal{D} \mapsto (g, \eta)(\bar{X}) \in G \times \mathfrak{g}\}. \quad (26)$$

### 3.1 Proof of (24)

Equation (24) can be proved using Stokes theorem applied to differential forms or alternatively using the divergence theorem and vector analysis. We will follow the latter approach and will denote, according to the context of section 2.4,  $dX^1 dX^2 \dots dX^p = d\mathcal{D}$  and  $|\bar{H}|^{1/2} dY^1 dY^2 \dots dY^{p-1} = d\partial\mathcal{D}$  the material volumes of  $\mathcal{D}$  and  $\partial\mathcal{D}$ , which are assumed to be two manifolds consistently oriented according to the outward unit normal convention. We start from (19),  $\delta$  being achieved at fixed time and material parameters, this enables us to shift it under the integral. Then, applying usual rules on composed derivations, we can write:

$$\int_{t_1}^{t_2} \int_{\mathcal{D}} \left( \left\langle \frac{\partial \mathfrak{L}_t}{\partial \eta}, \delta \eta \right\rangle + \left\langle \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha}, \delta \xi_\alpha \right\rangle \right) \sqrt{|h|} d\mathcal{D} dt = - \int_{t_1}^{t_2} \delta W_{ext} dt. \quad (27)$$

Then invoking (23) and applying a by-part time-integration with fixed extreme times condition, allows the left hand side of (27) to be rewritten as:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\mathcal{D}} \left\langle \frac{1}{\sqrt{|h|}} ad_\eta^* \left( \sqrt{|h|} \frac{\partial \mathfrak{L}_t}{\partial \eta} \right) - \frac{1}{\sqrt{|h|}} \frac{\partial}{\partial t} \left( \sqrt{|h|} \frac{\partial \mathfrak{L}_t}{\partial \eta} \right), \delta \zeta \right\rangle \sqrt{|h|} d\mathcal{D} dt \\ & + \int_{t_1}^{t_2} \int_{\mathcal{D}} \left\langle \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha}, \frac{\partial \delta \zeta}{\partial X^\alpha} + ad_{\xi_\alpha}(\delta \zeta) \right\rangle \sqrt{|h|} d\mathcal{D} dt. \end{aligned} \quad (28)$$

Now let us remark that:

$$\begin{aligned} & \int_{\mathcal{D}} \left\langle \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha}, \frac{\partial \delta \zeta}{\partial X^\alpha} \right\rangle \sqrt{|h|} d\mathcal{D} = \\ & \int_{\mathcal{D}} \frac{\partial}{\partial X^\alpha} \left( \sqrt{|h|} \left\langle \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha}, \delta \zeta \right\rangle \right) d\mathcal{D} - \int_{\mathcal{D}} \left\langle \frac{\partial}{\partial X^\alpha} \left( \sqrt{|h|} \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha} \right), \delta \zeta \right\rangle d\mathcal{D}, \end{aligned} \quad (29)$$

whose the first right-hand-side term is merely the divergence of a vector field of contravariant components  $v^\alpha = \langle \partial \mathfrak{L}_t / \partial \xi_\alpha, \delta \zeta \rangle$  in the convected basis  $\{h_\alpha\}_{\alpha=1, \dots, p}$ . Applying the divergence theorem to this term gives:

$$\int_{\mathcal{D}} \frac{\partial}{\partial X^\alpha} \left( \sqrt{|h|} \left\langle \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha}, \delta \zeta \right\rangle \right) d\mathcal{D} = \int_{\partial\mathcal{D}} \left\langle \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha}, \delta \zeta \right\rangle \nu_{t,\alpha} |\bar{h}|^{1/2} d\partial\mathcal{D}, \quad (30)$$

where we introduce the metric volume element  $|\bar{h}|^{1/2} d\partial\mathcal{D}$  on  $(\Phi_t \circ e)(\partial\mathcal{D})$ , and  $\nu_{t,\alpha} h^\alpha$  is the unit outward normal to the tangent planes of  $(\Phi_t \circ e)(\partial\mathcal{D})$  which, from (12), is related to the outward unit normal  $\nu_\alpha E^\alpha$  by  $\nu_{t,\alpha} |\bar{h}|^{1/2} d\partial\mathcal{D} = \nu_\alpha \sqrt{|h|} d\partial\mathcal{D}$ . Then, inserting (30) into (29) and the result into (28) whose the last term is dualized, gives, with (20), a balance of two integral components, one over  $\mathcal{D}$  with metric volume  $\sqrt{|h|} d\mathcal{D}$  and the second over  $\partial\mathcal{D}$ , whose metric volume is  $|\bar{h}|^{1/2} d\partial\mathcal{D}$ . This balance being satisfied for any variation  $\delta \zeta \in \mathfrak{g}$ , it gives the set of equations (24), where due to (21),  $|\bar{h}|^{1/2} = 1$  in the case of beams.  $\square$

## 4 Geometric model of Cosserat media

The above calculus handles velocity-type vectors  $(\eta, \xi_\alpha)$  and force-type vectors  $(\partial\mathfrak{L}_t/\partial t, \partial\mathfrak{L}_t/\partial\xi_\alpha, F_{ext,t}, \bar{F}_{ext,t})$  of  $\mathfrak{g}$  and  $\mathfrak{g}^*$  respectively, which are dual of each other through the duality product  $\langle \cdot, \cdot \rangle$ . However, this calculus hides a further dimensionality involving the space-time base-manifold  $\mathcal{D} \times \mathbb{R}^+$  [8]. Though not readily apparent, this further aspect of the theory strongly structures the geometric model of Cosserat-media and especially that of internal stress. The purpose of this section is to introduce this aspect and to use it to prepare the ground for the model of Cosserat shells as it will be discussed in the further sections. To introduce this important point, we will first build on the concept of duality.

### 4.1 Duality in Cosserat Media

For the purpose of simplicity, we consider in this section a material Lagrangian density  $\mathfrak{L}$  and will reintroduce  $\mathfrak{L}_o$  and  $\mathfrak{L}_t$  in the next section. Moreover, we will write  $\mathfrak{L}(\eta, \xi_\alpha) = \mathfrak{T}(\eta) - \mathfrak{U}(\xi_\alpha)$ , with  $\mathfrak{T}$  and  $\mathfrak{U}$ , the density of kinetic and internal elastic potential energy per unit of material volume respectively. Let us first remark that in the Poincaré-Cosserat picture,  $\eta$  and  $\xi_{\alpha=1,2\dots p}$  are not only vector fields in the Lie algebra but also the components of a unique field of 1-form on space-time, with value in the Lie algebra  $\mathfrak{g}$  of  $G$  [10]. Endowing  $\mathfrak{g}$  with a basis  $\{l_1, l_2 \dots l_n\}$ , such a field, here generically noted as  $\mathcal{Y}$ , is defined as  $\mathcal{Y} : (\bar{X}, t) \in \mathcal{D} \times \mathbb{R}^+ \mapsto \mathcal{Y}(\bar{X}, t) \in \mathfrak{g} \otimes \wedge T^*(\mathcal{D} \times \mathbb{R}^+)$ , and may be detailed as:

$$\mathcal{Y}(\bar{X}, t) = \mathcal{Y}_0^j l_j \otimes dt + \mathcal{Y}_\beta^j l_j \otimes dX^\beta, \quad (31)$$

where 0 denotes the coordinate-index along time axis, i.e.,  $X^0 = t$ , while in the case of the left-invariant fields of (17),  $\mathcal{Y}_0^j l_j = \eta$  and  $\mathcal{Y}_\beta^j l_j = \xi_\beta$ . Similarly, the generalized momenta  $\partial\mathfrak{L}/\partial\eta$  and  $\partial\mathfrak{L}/\partial\xi_{\alpha=1,2\dots p}$  geometrically define the components of a field in the dual of the space of  $\mathcal{Y}$ , i.e. a unique vector field on space-time with components in the dual of the Lie algebra of  $G$  of basis  $\{\omega^1, \omega^2, \dots \omega^n\}$ . Such a field, generically defined as  $\Lambda : (\bar{X}, t) \in \mathcal{D} \times \mathbb{R}^+ \mapsto \Lambda \in \mathfrak{g}^* \otimes T(\mathcal{D} \times \mathbb{R}^+)$  is detailed as:

$$\Lambda(\bar{X}, t) = \Lambda_i^0 \omega^i \otimes \partial/\partial t + \Lambda_i^\alpha \omega^i \otimes \partial/\partial X^\alpha, \quad (32)$$

where in the case of our generalized momenta, we have  $\Lambda_i^0 \omega^i = \partial\mathfrak{L}/\partial\eta = \partial\mathfrak{T}/\partial\eta$  and  $\Lambda_i^\alpha \omega^i = -\partial\mathfrak{L}/\partial\xi_\alpha = \partial\mathfrak{U}/\partial\xi_\alpha$ . With these definitions, in each point of space-time, any  $\Lambda$  linearly acts on any  $\mathcal{Y}$  according to the following (double) duality product denoted  $\langle \cdot, \cdot \rangle$ :

$$\langle \Lambda, \mathcal{Y} \rangle = \Lambda_i^0 \mathcal{Y}_0^j \langle \omega^i, l_j \rangle (\partial/\partial t, dt) + \Lambda_i^\alpha \mathcal{Y}_\beta^j \langle \omega^i, l_j \rangle (\partial/\partial X^\alpha, dX^\beta),$$

with  $\langle \omega^i, l_j \rangle = \delta_j^i$ , while  $\langle \cdot, \cdot \rangle$  is another duality product requiring considering vectors  $v$  of  $T(\mathcal{D} \times \mathbb{R}^+)$  as linear functional acting on 1-forms  $\omega$  of  $T^*(\mathcal{D} \times \mathbb{R}^+)$  according to  $v(\omega) = \omega(v) = (\omega, v) = (v, \omega)$ . With these considerations, and since  $\langle \partial/\partial X^i, dX^j \rangle = \delta_i^j$  (with  $X^0 = t$ ), we simply have:

$$\langle \Lambda, \mathcal{Y} \rangle = \Lambda_i^0 \mathcal{Y}_0^i + \Lambda_i^\alpha \mathcal{Y}_\alpha^i = \langle \Lambda^0, \mathcal{Y}_0 \rangle + \langle \Lambda^\alpha, \mathcal{Y}_\alpha \rangle = (\Lambda_i, \mathcal{Y}^i), \quad (33)$$

which appears at the very beginning of the above variational calculus in the virtual work of (27) and in all its consequences. In the subsequent developments, the vectors  $\Upsilon_0 = \Upsilon_0^i l_i$  and  $\Upsilon_\alpha = \Upsilon_\alpha^i l_i$  are velocity-type vectors, or in rigid body mechanics' terminology, are "twists" of  $\mathfrak{g}$ . On the dual side,  $\Lambda^0 = \Lambda_i^0 \omega^i$  and  $\Lambda^\alpha = \Lambda_i^\alpha \omega^i$  are force-type vectors or "wrenches" of  $\mathfrak{g}^*$ . Moreover, from the above context, the components of the two fields (31) and (32) in their respective basis of  $\mathfrak{g}$  and  $\mathfrak{g}^*$ , can be written as:  $\Upsilon^i = \Upsilon_0^i dt + \Upsilon_\alpha^i dX^\alpha$  and  $\Lambda_i = \Lambda_i^0 \partial/\partial t + \Lambda_i^\alpha \partial/\partial X^\alpha$  which define 1-forms acting on  $T(\mathcal{D} \times \mathbb{R}^+)$  and  $T^*(\mathcal{D} \times \mathbb{R}^+)$  respectively. Kinetically, the  $\Upsilon^i$  twist components model the space-time variations of the rigid microstructure  $\mathcal{M}$ -configuration (in  $G$ ) in any point of  $\mathcal{D} \times \mathbb{R}^+$  when shifting along any direction of  $\mathcal{D} \times \mathbb{R}^+$ , while the  $\Lambda_i$  wrench components model the kinetic momentum of each copy of  $\mathcal{M}$  above  $\mathcal{D}$  and the stress exerted on it. We will detail further these relationships in section 4.3 after introducing the role of densities and volume forms in the next subsection.

## 4.2 Volume forms and densities

All the terms of the equilibrium described in (24) define densities of wrench in  $\mathfrak{g}^*$  related to the metric volume of  $(\Phi_t \circ e)(\mathcal{D})$  (24-top), and  $(\Phi_t \circ e)(\partial\mathcal{D})$  (24-bottom). In particular, the external volume forces involved in (20), are intrinsically defined as the volume-form field on  $\mathcal{D}$  with values in  $\mathfrak{g}^*$ :

$$F_{ext,t} \sqrt{|h|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p, \quad (34)$$

while the external boundary forces (exerted on  $\partial\mathcal{D}$ ) are defined by the volume-form field on the boundaries of  $\mathcal{D}$  with value in  $\mathfrak{g}^*$ :

$$\bar{F}_{ext,t} |\bar{h}|^{1/2} dY^1 \wedge dY^2 \wedge \dots \wedge dY^{p-1}. \quad (35)$$

Similarly, any  $\Lambda$  defined as in (32) is in fact the unique component of a volume-form field  $\Lambda dX^1 \wedge dX^2 \wedge \dots \wedge dX^p$ , with values in  $\mathfrak{g}^* \otimes T(\mathcal{D} \times \mathbb{R}^+)$  which transforms into:

$$\Lambda_t \sqrt{|h|} dX^1 \wedge dX^2 \wedge \dots \wedge dX^p \quad (36)$$

on the deformed configuration. This is apparent in the following expressions of the total mechanical energy (kinetic + elastic)  $H \triangleq \int_{\mathcal{D}} (\mathfrak{T}(\eta) + \mathfrak{U}(\xi_\alpha)) d\mathcal{D}$  of a Cosserat medium undergoing small strains:

$$H = \int_{\mathcal{D}} (\langle \Lambda, \Upsilon \rangle) d\mathcal{D} = \int_{\mathcal{D}} (\langle \Lambda_o, \Upsilon \rangle) \sqrt{|h_o|} d\mathcal{D} = \int_{\mathcal{D}} (\langle \Lambda_t, \Upsilon \rangle) \sqrt{|h|} d\mathcal{D}, \quad (37)$$

which completes the intrinsic model of conjugate momenta of (24) as density fields of tensors of  $\mathfrak{g}^* \otimes T(\mathcal{D} \times \mathbb{R}^+)$  per unit of volume of material ( $\Lambda$ ), metric reference ( $\Lambda_o$ ), and metric deformed ( $\Lambda_t$ ) volume, the three densities being related by  $\Lambda = \sqrt{|h_o|} \Lambda_o = \sqrt{|h|} \Lambda_t$ . In summary, the external forces (34) and the generalized momenta (36), define volume-densities which behave like the Lagrangian

densities  $\mathfrak{L}$ ,  $\mathfrak{L}_o$  and  $\mathfrak{L}_t$ , i.e. obey the following relations, similar to (18), when shifting from the material to the reference and current configuration volumes:

$$\begin{aligned} F_{ext} &= F_{ext,o} \sqrt{|h_o|} = F_{ext,t} \sqrt{|h|}, \\ \frac{\partial \mathfrak{L}}{\partial \eta} &= \frac{\partial \mathfrak{L}_o}{\partial \eta} \sqrt{|h_o|} = \frac{\partial \mathfrak{L}_t}{\partial \eta} \sqrt{|h|} \quad , \quad \frac{\partial \mathfrak{L}}{\partial \xi_\alpha} = \frac{\partial \mathfrak{L}_o}{\partial \xi_\alpha} \sqrt{|h_o|} = \frac{\partial \mathfrak{L}_t}{\partial \xi_\alpha} \sqrt{|h|}. \end{aligned} \quad (38)$$

In the same way, the external surface forces (35) define volume-densities on the boundaries of  $\mathcal{D}$  which behave as:

$$\overline{F}_{ext} |\overline{H}|^{1/2} = \overline{F}_{ext,o} |\overline{h}_o|^{1/2} = \overline{F}_{ext,t} |\overline{h}|^{1/2}. \quad (39)$$

In (38) and (39),  $(F_{ext,o}, \overline{F}_{ext,o})$  and  $(F_{ext,t}, \overline{F}_{ext,t})$  denote the densities of the external wrenches exerted on  $\mathcal{B}$ , per unit of metric-volume of  $((\Phi_o \circ e)(\mathcal{D}), \partial(\Phi_o \circ e)(\partial\mathcal{D}))$ , and per unit of material volume of  $(\mathcal{D}, \partial\mathcal{D})$  respectively. Similarly, as this will be detailed in the next section,  $\partial \mathfrak{L}_o / \partial \xi_\alpha$  and  $\partial \mathfrak{L} / \partial \xi_\alpha$  (respectively  $\partial \mathfrak{L}_o / \partial \eta$  and  $\partial \mathfrak{L} / \partial \eta$ ) are the opposite of the densities of internal stress wrench (respectively the densities of kinetic momentum wrench) per unit of metric volume of  $(\Phi_o \circ e)(\mathcal{D})$  and of material volume of  $\mathcal{D}$ . Re-starting the variational calculus of section 3 with Lagrangian and external forces densities per unit of metric-volume of  $((\Phi_o \circ e)(\mathcal{D}), (\Phi_o \circ e)(\partial\mathcal{D}))$  gives a new set of balance equations in terms of densities per unit of volume of  $(\Phi_o \circ e)(\mathcal{D})$  deduced from (24) by simply changing  $|\overline{h}|, |h|, \mathfrak{L}_t, F_{ext,t}, \overline{F}_{ext,t}$  into  $|\overline{h}_o|, |h_o|, \mathfrak{L}_o, F_{ext,o}, \overline{F}_{ext,o}$  respectively. Alternatively, using Lagrangian densities per unit of material volume of  $(\mathcal{D}, \partial\mathcal{D})$ , gives a third set of equations deduced from (24) by changing  $|\overline{h}|, |h|, \mathfrak{L}_t, F_{ext,t}$  and  $\overline{F}_{ext,t}$  into  $|\overline{H}|, |h| = 1, \mathfrak{L}, F_{ext}, \overline{F}_{ext}$ . Remarkably, these three sets of equations can be deduced from a single one by applying the rules (38). It is worth noting that the stress tensor of (32) being a density of stress-tensor per unit of volume, in all the subsequent developments, it will be indexed (as  $\mathfrak{L}$ ) by  $t, o$  or nothing depending upon whether we consider the stress per unit of current, reference, or material volume of  $\mathcal{D}$ .

### 4.3 The case of $G = SE(3)$ : a model of Cosserat stress

To shift from the abstract model of conjugate momenta introduced in the above subsection to the stress' model of mechanics of continuous media, we use the Euclidean structure of  $\mathcal{D} \simeq \mathbb{R}^p$  along with specific isomorphisms between the Lie algebra and the action space of  $G$  that for preparing the application to shells, we will reconsider to be  $SE(3)$  as defined in (5). Thus, using the Euclidean metric of  $\mathcal{D} \simeq \mathbb{R}^p$  allows pairing the 1-forms  $dX^\beta$  with the co-vectors  $\nabla(X^\beta) = E^\beta$ . Geometrically, this basis of co-vectors coincides with the unit vectors normal to the level sets  $X^\beta = C$  (where  $C \in \mathbb{R}$ ) in  $e(\mathcal{D})$ . As a result, because we also have  $\partial/\partial X^\alpha = E_\alpha$ , the two fields (31) and (32) can be identified to the two tensor fields:

$$\mathcal{Y} = \eta^i l_i \otimes dt + \xi_\alpha^i l_i \otimes E^\alpha, \quad \Lambda = (\Lambda_i^0 \omega^i \otimes \partial/\partial t + \Lambda_i^\alpha \omega^i \otimes E_\alpha). \quad (40)$$

For the usual Lie groups of rigid-body mechanics (e.g.  $G = SO(3), SE(2), SE(3)$ ), the Lie algebra basis and its dual can be identified through specific isomorphisms to the basis of the space on which the group transformations act. In particular, for  $G = SE(3)$ , we can identify  $\mathfrak{g} = se(3)$  with  $\mathbb{R}^6$  (i.e. two copies of  $\mathbb{R}^3$ ) endowed with the following 6-dimensional cross-product  $\star$  which defines the adjoint operator of (24) between any two vectors  $\eta$  and  $\eta'$ , both belonging to  $se(3) \simeq \mathbb{R}^6$  [26]:

$$\eta \star \eta' = \begin{pmatrix} \Omega \\ V \end{pmatrix} \star \begin{pmatrix} \Omega' \\ V' \end{pmatrix} = \begin{pmatrix} \Omega \times \Omega' \\ \Omega \times V' - \Omega' \times V \end{pmatrix} = ad_\eta(\eta'), \quad (41)$$

where  $\times$  denotes the usual cross product in  $\mathbb{R}^3$ . With this definition of  $se(3)$ , the basis  $\{l_i\}_{i=1,2,3}$  can be changed into  $\{(E_i, 0), (0, E_i)\}_{i=1,2,3}$  with 0 and  $E_i$  the zero and basis vectors of  $\mathcal{B} \simeq \mathbb{R}^3$ . Using the metric of  $\mathbb{R}^6$  allows us to go further by identifying  $se(3)^*$  to  $se(3)$ . In particular, if  $\{E_i\}_{i=1,2,3}$  is an orthonormal basis, as this is the case here, we can indifferently take  $\{l_i\}_{i=1,2,3} \simeq \{(E^i, 0), (0, E^i)\}_{i=1,2,3}$  and  $\{\omega^i\}_{i=1,2,3} \simeq \{(E_i, 0), (0, E_i)\}_{i=1,2,3}$  and detail each of the terms of (40) in the following tensorial forms:

$$\mathcal{T} = \begin{pmatrix} \Omega_i E^i \\ V_i E^i \end{pmatrix} \otimes dt + \begin{pmatrix} K_{i\alpha} E^i \\ \Gamma_{i\alpha} E^i \end{pmatrix} \otimes E^\alpha, \quad \mathcal{A} = \begin{pmatrix} \Sigma^i E_i \\ P^i E_i \end{pmatrix} \otimes \frac{\partial}{\partial t} + \begin{pmatrix} M^{i\alpha} E_i \\ N^{i\alpha} E_i \end{pmatrix} \otimes E_\alpha. \quad (42)$$

The two tensor fields  $\mathcal{T}$  and  $\mathcal{A}$  not only act on each other through the duality product ( $\langle \cdot, \cdot \rangle$ ), but also operate on the base manifold  $T(\mathcal{D} \times \mathbb{R}^+)$  and its dual respectively. In particular, the time (0)-components of (42) operate as follows on the vectors and co-vectors of the time axis:

$$\begin{pmatrix} \Omega \\ V \end{pmatrix} \otimes dt(\partial/\partial t) = \begin{pmatrix} \Omega \\ V \end{pmatrix}, \quad \begin{pmatrix} \Sigma \\ P \end{pmatrix} \otimes (\partial/\partial t)(dt) = \begin{pmatrix} \Sigma \\ P \end{pmatrix}. \quad (43)$$

In the same way, the space ( $\alpha$ )-components of (42) operate on the vectors  $dX^\beta E_\beta$  and the co-vectors  $dS\nu_\beta E^\beta = dS\nu$  of (12) respectively, as follows:

$$\begin{pmatrix} K_\alpha \\ \Gamma_\alpha \end{pmatrix} \otimes E^\alpha \cdot (dX^\beta E_\beta) = \begin{pmatrix} K_\beta \\ \Gamma_\beta \end{pmatrix} dX^\beta, \quad \begin{pmatrix} M^\alpha \\ N^\alpha \end{pmatrix} \otimes E_\alpha \cdot (dS\nu_\beta E^\beta) = \begin{pmatrix} M^\beta \\ N^\beta \end{pmatrix} \nu_\beta dS. \quad (44)$$

The first operation (from left to right) of (43) and (44), gives the rate of rotation  $R^T \cdot dR$  (of axis) and displacement  $R^T \cdot dr$  (of the base point) of the orthonormal mobile spatial frames  $(R(\bar{X}, t) \cdot E_i) = (t_i)_{i=1,2,3}(\bar{X}, t)$  due to a small time variation (43), and a small displacement  $dX^\beta E_\beta$  along  $\mathcal{D}$  (44) (see figure 1). In particular, note that the vectors  $\Gamma_\alpha$  are those defined in section 2.4 as the pulled back of the convected basis  $\{h_\alpha\}_{\alpha=1,2,3}$  in the material frame. The second operation (on the right) of (43) gives the density of kinetic wrench per unit of material volume of  $\mathcal{D}$  at time  $t$ . The second operation of (44) models the material resultant ( $dN_{int}$ ), and the material moment ( $dM_{int}$ ) of internal contact forces exerted across an oriented material surface element  $dS\nu = dS\nu_\beta E^\beta$  of  $\mathcal{D}$  by the piece of material toward which  $\nu$  points, onto its complement part (see figure 3 in the case of a shell). This allows us to introduce the tensor field of Cosserat stress

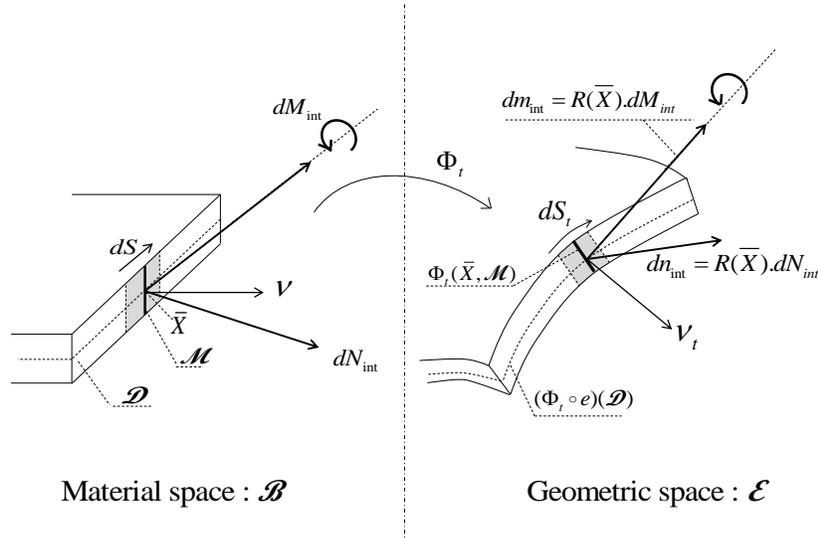
whose components are related to the space  $(\alpha)$ -conjugate momenta of (24) as follows:

$$\begin{aligned} \begin{pmatrix} M^\alpha \\ N^\alpha \end{pmatrix} \otimes E_\alpha &= \sqrt{|h_o|} \begin{pmatrix} M_o^\alpha \\ N_o^\alpha \end{pmatrix} \otimes E_\alpha = \sqrt{|h_t|} \begin{pmatrix} M_t^\alpha \\ N_t^\alpha \end{pmatrix} \otimes E_\alpha = \\ &= \frac{\partial \mathfrak{U}}{\partial \xi_\alpha} \otimes E_\alpha = \sqrt{|h_o|} \frac{\partial \mathfrak{U}_o}{\partial \xi_\alpha} \otimes E_\alpha = \sqrt{|h|} \frac{\partial \mathfrak{U}_t}{\partial \xi_\alpha} \otimes E_\alpha, \end{aligned} \quad (45)$$

with  $\mathfrak{U}_o$  and  $\mathfrak{U}_t$  the reduced densities of internal elastic energy related to the reference and deformed configuration respectively. Using this tensor allows the second operation of (44) to be rewritten as:

$$\frac{d}{dS} \begin{pmatrix} M_{int} \\ N_{int} \end{pmatrix} = \sqrt{|h|} \begin{pmatrix} M_t^\alpha \\ N_t^\alpha \end{pmatrix} \otimes E_\alpha \cdot \nu = \sqrt{|h|} \begin{pmatrix} M_t^\beta \\ N_t^\beta \end{pmatrix} \nu_\beta = \sqrt{|h|} \frac{\partial \mathfrak{U}_t}{\partial \xi_\beta} \nu_\beta, \quad (46)$$

which represents the density of internal contact wrench exerted across an oriented material surface element of normal  $\nu$ , per unit of its material (non metric) area. Introducing  $dS_t = |\bar{h}|^{1/2} dS$  (see section 2.4) in (46), gives the same density, but



**Fig. 3.** Picture of the stress in Cosserat shells:  $dm_{int}$  and  $dn_{int}$  define the spatial wrench of internal contact forces exerted through the element  $dS$  from right to left.

related to the metric deformed area:

$$\frac{d}{dS_t} \begin{pmatrix} M_{int} \\ N_{int} \end{pmatrix} = \left( \frac{|h|}{|\bar{h}|} \right)^{1/2} \begin{pmatrix} M_t^\beta \\ N_t^\beta \end{pmatrix} \nu_\beta = \left( \frac{|h|}{|\bar{h}|} \right)^{1/2} \frac{\partial \mathcal{U}_t}{\partial \xi_\beta} \nu_\beta. \quad (47)$$

We can now precisely define the physical meaning of the conjugate momentums appearing in (24). To that end, it suffices to consider surface elements which correspond to the oriented coordinate surfaces  $X^\beta = C$ , such that  $\nu = E^\beta$ . In this case, (46) shows that  $\sqrt{|h|} \frac{\partial \mathcal{U}_t}{\partial \xi_\beta}$ , is the density of internal contact wrench exerted across the coordinate surface, by the piece of material  $X^\beta > C$  on the piece  $X^\beta < C$ , measured per unit of material area  $dS_\beta \triangleq dX^1 \dots dX^{\beta-1} dX^{\beta+1} \dots dX^p$ . Moreover, from (47) we can introduce a similar density but related to the metric area of the deformed element, as this is usually done for shells [15]:

$$\begin{pmatrix} M_t^{*\beta} \\ N_t^{*\beta} \end{pmatrix} \triangleq \left( \frac{|h|}{|\bar{h}|} \right)^{1/2} \begin{pmatrix} M_t^\beta \\ N_t^\beta \end{pmatrix}. \quad (48)$$

Note that (48) is consistent with the boundary conditions of (24) in which,  $\bar{F}_{ext,t}$  is the density of external forces per unit of metric volume of  $(\Phi_t \circ e)(\partial \mathcal{D})$ . Finally, reconsidering the balance equations (24), and substituting  $\partial \mathcal{U}_t / \partial \xi_\beta = (M_t^{\beta T}, N_t^{\beta T})^T$  by their expressions (48) in terms of  $(M_t^{*\beta T}, N_t^{*\beta T})^T$  gives an equivalent set of balance equations in terms of internal contact wrench per unit of metric area of the deformed coordinate surfaces. In the same manner, referring the stress densities to the metric area of the coordinate surfaces of the reference configuration  $(\Phi_o \circ e)(\mathcal{D})$  (denoted  $M_o^{*\beta}$  and  $N_o^{*\beta}$ ), as is done in [15] for shells, requires the substitution of  $\partial \mathcal{U}_o / \partial \xi_\beta = (N_o^{\beta T}, M_o^{\beta T})^T$  with:

$$M_o^\beta = \left( \frac{|\bar{h}_o|}{|h_o|} \right)^{1/2} M_o^{*\beta}, \quad N_o^\beta = \left( \frac{|\bar{h}_o|}{|h_o|} \right)^{1/2} N_o^{*\beta}, \quad (49)$$

in the reference configuration-related Poincaré equations.

#### 4.4 From material to spatial stress tensors

In the exposition above, the  $\alpha$ -components of (42) define the two pairs of material tensors:  $K \triangleq K_\alpha E^\alpha = K_{i\alpha} E^i \otimes E^\alpha$ ,  $\Gamma \triangleq \Gamma_\alpha E^\alpha = \Gamma_{i\alpha} E^i \otimes E^\alpha$ , and:  $\sqrt{|h|} M_t \triangleq \sqrt{|h|} M_t^\alpha \otimes E_\alpha = \sqrt{|h|} M_t^{i\alpha} E_i \otimes E_\alpha$ ,  $\sqrt{|h|} N_t \triangleq \sqrt{|h|} N_t^\alpha \otimes E_\alpha = \sqrt{|h|} N_t^{i\alpha} E_i \otimes E_\alpha$ . From these basic tensors, we can construct others more commonly used in shell literature. In particular, the left tensorial factors of the first pair can be pushed forward from the material to the microstructure frames according to:

$$K_{i\alpha} t^i \otimes E^\alpha = \left( \frac{\partial R}{\partial X^\alpha} \cdot R^T \right)^\vee \otimes E^\alpha \triangleq k, \quad (50)$$

$$\Gamma_{i\alpha} t^i \otimes E^\alpha = \frac{\partial r}{\partial X^\alpha} \otimes E^\alpha \triangleq \gamma, \quad (51)$$

where  $\vee$  changes a skew symmetric tensor of  $\mathbb{R}^3 \otimes \mathbb{R}^3$  into its axial vector in  $\mathbb{R}^3$ . In the same way, we have for the second pair:

$$\sqrt{|h|}M_t^{i\alpha}t_i \otimes E_\alpha = \sqrt{|h|}m_t^\alpha \otimes E_\alpha \triangleq \sqrt{|h|}m_t, \quad (52)$$

$$\sqrt{|h|}N_t^{i\alpha}t_i \otimes E_\alpha = \sqrt{|h|}n_t^\alpha \otimes E_\alpha \triangleq \sqrt{|h|}n_t, \quad (53)$$

where  $m_t^\alpha = R.M_t^\alpha$  and  $n_t^\alpha = R.n_t^\alpha$  are spatial forces and couples. We recognize in (51) the expression of the gradient of the transformation restricted to  $\mathcal{D}$ , i.e.,  $\nabla\bar{\Phi} = h_\alpha \otimes E^\alpha$ , while (50) represents the gradient of the rotation of the microstructures. In the same line, we recognize in (53), the expression of the first Piola-Kirchhoff stress tensor  $P$ , here restricted to  $\mathcal{D}$ , i.e., it is a tensor which acts on  $\nu dS$  to give the resultant of internal contact forces on the deformed configuration denoted  $dn_{int}$ , while (52) is its rotational counterpart, which acts on  $\nu dS$  to give the resultant of internal contact couples on the deformed configuration noted  $dm_{int}$ . This context is summarized in figure 3 in the case of shells. The two pairs of tensors:  $(k, \gamma)$  and  $(\sqrt{|h|}m_t, \sqrt{|h|}n_t)$ , and the two others:  $(K, \Gamma)$  and  $(\sqrt{|h|}M_t, \sqrt{|h|}N_t)$ , are dual of each other, as are the first Piola-Kirchhoff stress tensor and the gradient of transformation in the case of 3D media [24]. Furthermore (anticipating on section 6), for a classical (not micropolar) shell, using the general Cosserat Kinematics (8), we have the following expression of the virtual work of the internal forces exerted inside  $\mathcal{B}$ :

$$\delta W_{int} = \int_{\mathcal{B}} P : \nabla\delta\Phi_t d\mathcal{B} = \int_{\mathcal{D}} (M_t : \delta K + N_t : \delta\Gamma)\sqrt{|h|}d\mathcal{D}, \quad (54)$$

where ":" denotes the double contracted product<sup>4</sup>. Note that the second tensorial factor of the two-point stress tensors (52) and (53), can be pushed forward on the deformed configuration by  $\nabla\bar{\Phi}_t^T$ . Then, using  $\nu_{t,\alpha} = (|h|/|\bar{h}|)^{1/2}\nu_\alpha$  (see section 2.4), allows (47) to be rewritten on the deformed configuration as:

$$\frac{d}{dS_t} \begin{pmatrix} m_{int} \\ n_{int} \end{pmatrix} = \left( \begin{pmatrix} R.M_t^\alpha \\ R.N_t^\alpha \end{pmatrix} \otimes h_\alpha \right) \cdot (\nu_{t,\beta} h^\beta) \triangleq \left( \begin{pmatrix} m_t^\alpha \\ n_t^\alpha \end{pmatrix} \otimes h_\alpha \right) \cdot \nu_t, \quad (55)$$

which makes the two (Eulerian) stress tensors  $\tilde{m}_t \triangleq M^{i\alpha}t_i \otimes h_\alpha = m_t^\alpha \otimes h_\alpha$  and  $\tilde{n}_t \triangleq N^{i\alpha}t_i \otimes h_\alpha = n_t^\alpha \otimes h_\alpha$  appear. These two tensors are entirely defined on the deformed configuration and were introduced for shells in [27].

<sup>4</sup> Note that (54) can be computed by first remarking that  $\mathcal{B}$  being classical, its Cauchy stress tensor  $\sigma = \sigma^{ij}g_i \otimes g_j$  is symmetric and  $P : \nabla\delta\Phi_t d\mathcal{B} = \frac{1}{2}\sigma^{ij}\delta g_{ij}|g|^{1/2} d\mathcal{B}$ . Then, introducing the Cosserat kinematics (8) into  $g_i = \partial\Phi_t/\partial X^i$ ,  $i = 1, 2, 3$ , transforms the  $g_{ij} = g_i \cdot g_j$  terms into functions of  $T_\alpha$  and  $K_\alpha$  only (left invariance), and the  $\delta g_{ij}$  terms into linear forms of the  $\delta\Gamma_\alpha$  and  $\delta K_\alpha$ . These latter factors, once integrated along  $\mathcal{M}$ , define the internal Cosserat wrenches  $(M_t^{\alpha T}, N_t^{\alpha T})^T$  as functions of the Cauchy stress tensor on  $\mathcal{B}$ .

## 5 Application to full Cosserat (micropolar) shells

In this section we consider the case of shells having an intrinsic spin and couple stress around their directors, or "micropolar" shells. In this nominal context, the full Cosserat-Poincaré construction applies. In section 6, we will show how this model can be adapted to classical shells, i.e. shells with no intrinsic spin nor couple stress around their directors.

### 5.1 Full Cosserat shells kinematics

In the case of shells, the Cosserat micro-structure  $\mathcal{M}$  models rigid fibers (supported by  $E_3$ ), traversing the shell's material mid surface  $\mathcal{D}$  (supported by  $(E_1, E_2)$ ). They are labelled by  $(X^1, X^2)$ , two parameters which define a set of coordinates on  $\mathcal{D}$ . The configuration space of one such rigid microstructure, say that labelled by  $(X^1, X^2)$ , is the set of transformations  $g(X^1, X^2) \in SE(3)$ , and the whole configuration space of the shell is the space of the  $(X^1, X^2)$ -parameterized surfaces in  $SE(3)$  :

$$C := \{g : (X^1, X^2) \in [0, 1]^2 \mapsto g(X^1, X^2) \in SE(3)\}. \quad (56)$$

Here, with no restriction,  $(X^1, X^2)$  are normalized on  $\mathcal{D}$ . Note that this definition appears in the following Reissner shells kinematics [13], which instantiates the general kinematics (8) for  $p = 2$ :

$$\Phi(X) = r(X^1, X^2) + R(X^1, X^2).(X^3 E_3). \quad (57)$$

Let us recall that  $(O, E_1, E_2, E_3)$  interchangeably defines the material or inertial frame, to which  $r$  and  $R$  are related. With this definition of the shell configuration space, the left invariant fields of the general construction are:

$$\eta = g^{-1} \frac{\partial g}{\partial t} = \begin{pmatrix} \Omega & V \\ 0 & 0 \end{pmatrix}, \quad \xi_\alpha = g^{-1} \frac{\partial g}{\partial X^\alpha} = \begin{pmatrix} K_\alpha & \Gamma_\alpha \\ 0 & 0 \end{pmatrix}, \quad \alpha = 1, 2. \quad (58)$$

where  $\Omega(X^1, X^2)$  and  $V(X^1, X^2)$  denote the linear and angular velocities of the  $(X^1, X^2)$ -rigid fiber pulled back in the material frame. Similarly,  $(K_1, K_2)$  and  $(\Gamma_1, \Gamma_2)$  are in the same frame and stand for the  $X^1$  and  $X^2$ -rates of rotation and position of the mobile frames  $(t_i)_{i=1,2,3}(X^1, X^2)$ . According to (31), all the vector fields of (58) define the components of the unique 1-form:

$$\eta^i l_i \otimes dt + \xi_\alpha^i l_i \otimes dX^\alpha = \begin{pmatrix} \Omega \\ V \end{pmatrix} \otimes dt + \begin{pmatrix} K_\alpha \\ \Gamma_\alpha \end{pmatrix} \otimes dX^\alpha, \quad (59)$$

which maps tangent vectors of  $T(\mathcal{D} \times \mathbb{R}^+)$  to space-time rate vectors of  $\mathfrak{g}$ .

### 5.2 Poincaré picture for full-Cosserat shells

Based on the previous kinematics and the frame indifference of finite elasticity [20], we consider a reduced shell's Lagrangian volume form of the form:

$$(\mathfrak{T}_t(\eta) - \mathfrak{U}_t(\xi_1, \xi_2)) \sqrt{|h|} dX^1 \wedge dX^2 \in \wedge T_X^*(\mathcal{D}), \quad (60)$$

where  $\sqrt{|\hbar|}\mathfrak{T}_t$  and  $\sqrt{|\hbar|}\mathfrak{U}_t$  denote the (left-reduced) kinetic and elastic potential energy densities, with  $\sqrt{|\hbar|}\mathfrak{L}_t = \sqrt{|\hbar|}(\mathfrak{T}_t - \mathfrak{U}_t)$  the reduced density of Lagrangian, all these densities being measured per unit of material volume. Introducing (60) in the general Cosserat equations (24) with  $p = 2$  makes the fields of densities of wrenches per unit of metric area of deformed mid-surface shell appear:

$$\frac{\partial \mathfrak{T}_t}{\partial \eta} = \begin{pmatrix} \partial \mathfrak{T}_t / \partial \Omega \\ \partial \mathfrak{T}_t / \partial V \end{pmatrix} = \begin{pmatrix} \Sigma_t \\ P_t \end{pmatrix}, \quad \frac{\partial \mathfrak{U}_t}{\partial \xi_\alpha} = \begin{pmatrix} \partial \mathfrak{U}_t / \partial K_\alpha \\ \partial \mathfrak{U}_t / \partial \Gamma_\alpha \end{pmatrix} = \begin{pmatrix} M_t^\alpha \\ N_t^\alpha \end{pmatrix}. \quad (61)$$

Note that (61) defines a generalized (space-time) constitutive law of the full Cosserat model of hyperelastic shells. Furthermore, according to (32) and (42), (61) geometrically defines the unique tensor density field per unit of material volume  $\sqrt{|\hbar|}A_t$ , with:

$$A_t = \left( \frac{\partial \mathfrak{T}_t}{\partial \eta^i} \right) \omega^i \otimes \frac{\partial}{\partial t} + \left( \frac{\partial \mathfrak{U}_t}{\partial \xi_\alpha^i} \right) \omega^i \otimes \frac{\partial}{\partial X^\alpha} = \begin{pmatrix} \Sigma_t \\ P_t \end{pmatrix} \otimes \frac{\partial}{\partial t} + \begin{pmatrix} M_t^\alpha \\ N_t^\alpha \end{pmatrix} \otimes E_\alpha. \quad (62)$$

Assuming that the material mid-surface  $\mathcal{D}$  crosses the microstructure in its geometric center which coincides with its mass center, and positioning the material frame in this point, the shell's kinetic energy density per unit of deformed mid-surface area is:

$$\mathfrak{T}_t(\eta) = \frac{1}{2} \langle (\Omega, V), \begin{pmatrix} \bar{J}_t \Omega \\ \bar{\rho}_t V \end{pmatrix} \rangle, \quad (63)$$

with  $\bar{\rho}_t$  and  $\bar{J}_t$ , the mass and the material angular inertia density per unit of volume of  $(\Phi_t \circ e)(D)$  defined as  $\bar{\rho}_t = \bar{\rho} / \sqrt{h}$  and  $\bar{J}_t = \bar{J} / \sqrt{h}$ , with  $\bar{\rho}$  and  $\bar{J} = \bar{J}_\perp (E_1 \otimes E^1 + E_2 \otimes E^2) + \bar{J}_\parallel E_3 \otimes E^3$ , the mass and the inertia tensor of the director  $\mathcal{M}$ . With (63), the density of kinetic wrench per unit of current mid-surface area is:

$$\frac{\partial \mathfrak{T}_t}{\partial \eta} = \begin{pmatrix} \Sigma_t \\ P_t \end{pmatrix} = \begin{pmatrix} \bar{J}_t \Omega \\ \bar{\rho}_t V \end{pmatrix}. \quad (64)$$

It is worth noting here that  $\mathcal{M}$  is assumed to have an intrinsic spin  $\bar{J}_\parallel \Omega_3$  about its axis, an assumption which will be relaxed in the classical shell model in section 7.

### 5.3 Model of stress for full Cosserat shells

According to the Cosserat stress picture in section 4, the  $X^\alpha$ -terms of  $\sqrt{|\hbar|}A_t$ , with  $A_t$  detailed as in (62), define two densities of tensors per unit of material mid-surface area:  $\sqrt{|\hbar|}N_t = \sqrt{|\hbar|}N_t^\alpha \otimes E_\alpha = \sqrt{|\hbar|}N_t^{i\alpha} E_i \otimes E_\alpha$  and  $\sqrt{|\hbar|}M_t = \sqrt{|\hbar|}M_t^\alpha \otimes E_\alpha = \sqrt{|\hbar|}M_t^{i\alpha} E_i \otimes E_\alpha$ , which operate as follows on co-vectors of  $T^*(e(\mathcal{D}))$ , here standing for the material surface elements  $\nu dS$ , with  $\nu$  the unit normal to any material line cutting  $\mathcal{D}$ :

$$\begin{aligned} (\sqrt{|\hbar|}N_t^{i\beta} E_i \otimes E_\beta) \cdot (\nu_\alpha E^\alpha dS) &= \sqrt{|\hbar|}N_t^{i\alpha} E_i \nu_\alpha dS = \sqrt{|\hbar|}N_t^\alpha \nu_\alpha dS, \\ (\sqrt{|\hbar|}M_t^{i\beta} E_i \otimes E_\beta) \cdot (\nu_\alpha E^\alpha dS) &= \sqrt{|\hbar|}M_t^{i\alpha} E_i \nu_\alpha dS = \sqrt{|\hbar|}M_t^\alpha \nu_\alpha dS. \end{aligned} \quad (65)$$

From (46),  $\sqrt{|\hbar|}N_t^\alpha\nu_\alpha dS$  and  $\sqrt{|\hbar|}M_t^\alpha\nu_\alpha dS$  represent the resultant of internal contact forces and couples exerted on the material cross-sectional line of normal  $\nu_\alpha E^\alpha$  per unit of its material length. In particular,  $\sqrt{|\hbar|}N_t^1$  and  $\sqrt{|\hbar|}M_t^1$  (respectively  $\sqrt{|\hbar|}N_t^2$  and  $\sqrt{|\hbar|}M_t^2$ ) represent the resultant and the momentum of internal stress forces exerted across the shell's transverse section  $X^1 = C$  (respectively  $X^2 = C$ ) per unit of coordinate length  $X^2$  (respectively  $X^1$ ) by the piece of material  $X^1 > C$  (respectively  $X^2 > C$ ) onto the contiguous piece  $X^1 < C$  (resp.  $X^2 < C$ ). One can partition them according to:

$$\begin{aligned}\sqrt{|\hbar|}N_t^{i\alpha}E_i \otimes E_\beta &= \sqrt{|\hbar|}N_t^{\alpha\beta}E_\alpha \otimes E_\beta + \sqrt{|\hbar|}N_t^{3\beta}E_3 \otimes E_\beta, \\ \sqrt{|\hbar|}M_t^{i\alpha}E_i \otimes E_\beta &= \sqrt{|\hbar|}M_t^{\alpha\beta}E_\alpha \otimes E_\beta + \sqrt{|\hbar|}M_t^{3\beta}E_3 \otimes E_\beta.\end{aligned}\quad (66)$$

Physically,  $\sqrt{|\hbar|}N_t^{1\alpha}$  and  $\sqrt{|\hbar|}N_t^{2\alpha}$  (resp.  $\sqrt{|\hbar|}M_t^{1\alpha}$  and  $\sqrt{|\hbar|}M_t^{2\alpha}$ ) model the resultant stress (respectively couple stress) exerted perpendicularly to the microstructure  $\mathcal{M}$ , while  $\sqrt{|\hbar|}N_t^{3\alpha}$  is the transverse shearing resultant stress aligned with  $\mathcal{M}$ . Finally, it is worth noting here that in the classical shell model that will be introduced later, the microstructure frames are replaced by single vectors or "directors", and the two components  $\sqrt{|\hbar|}M_t^{3\alpha}$ , which represent couple stress aligned with  $\mathcal{M}$ , are zero. Moreover, in the same model, the stress couples  $\sqrt{|\hbar|}M_t^\alpha$  are often replaced by the director stress couples  $\sqrt{|\hbar|}\widehat{M}_t^\alpha$  defined by  $M_t^\alpha = E_3 \times \widehat{M}_t^\alpha$  [11].

#### 5.4 Geometrically exact dynamic balance of full-Cosserat shells

Introducing (58) and (62) into (24) with the expression of  $ad^*$  on  $se(3) \cong \mathbb{R}^6$  (see equation (41)), gives the geometrically exact dynamic equations of a Cosserat shell:

$$\begin{aligned}\frac{\partial P_t}{\partial t} + \Omega \times P_t &= \frac{1}{\sqrt{|\hbar|}} \left( \frac{\partial \sqrt{|\hbar|}N_t^\alpha}{\partial X^\alpha} + K_\alpha \times \sqrt{|\hbar|}N_t^\alpha \right) + N_{ext,t}, \\ \frac{\partial \Sigma_t}{\partial t} + \Omega \times \Sigma_t &= \frac{1}{\sqrt{|\hbar|}} \left( \frac{\partial \sqrt{|\hbar|}M_t^\alpha}{\partial X^\alpha} + K_\alpha \times \sqrt{|\hbar|}M_t^\alpha + \Gamma_\alpha \times \sqrt{|\hbar|}N_t^\alpha \right) + M_{ext,t},\end{aligned}\quad (67)$$

which, once pushed-forward from the material frame to the microstructures' frame (see section 2.4), can be rewritten in the form:

$$\begin{aligned}\frac{\partial p_t}{\partial t} &= \frac{1}{\sqrt{|\hbar|}} \left( \frac{\partial \sqrt{|\hbar|}n_t^\alpha}{\partial X^\alpha} \right) + n_{ext,t}, \\ \frac{\partial \sigma_t}{\partial t} &= \frac{1}{\sqrt{|\hbar|}} \left( \frac{\partial \sqrt{|\hbar|}m_t^\alpha}{\partial X^\alpha} \right) + \frac{\partial r}{\partial X^\alpha} \times n_t^\alpha + m_{ext,t},\end{aligned}\quad (68)$$

where small characters denote spatial tensors, i.e.,  $\sigma_t = R.\Sigma_t$ ,  $p_t = R.P_t$ ,  $n_t^\alpha = R.M_t^\alpha$ ,  $m_t^\alpha = R.M_t^\alpha$ ,  $n_{ext,t} = R.N_{ext,t}$ , and  $m_{ext,t} = R.M_{ext,t}$ , while  $\partial r / \partial X^\alpha = R.\Gamma_\alpha$ . These equations are well known in the shell literature where they are

derived from Newton's law either by directly using the Cosserat model [28], [27], or indirectly from 3D elasticity [11]. In this case,  $n_t^\alpha$  and  $m_t^\alpha$  are constructed by normalizing the Cauchy stress resultant which is exerted across each shell section  $X^\alpha = C$  of the deformed configuration, with respect to the deformed mid surface area. In (68), all the terms are densities related to the metric volume element  $\sqrt{|h|} dX^1 \wedge dX^2$  of  $(\Phi_t \circ e)(\mathcal{D})$  and can be equivalently related to  $\sqrt{|h_o|} dX^1 \wedge dX^2$ , the metric volume element of  $(\Phi_o \circ e)(\mathcal{D})$ , if we start with a density of Lagrangian (14) related to the reference configuration  $(\Phi_o \circ e)(\mathcal{D})$ . In this case, using (38), we can rewrite (68) in the following alternative form:

$$\begin{aligned} \frac{\partial p_o}{\partial t} &= \frac{1}{\sqrt{|h_o|}} \left( \frac{\partial \sqrt{|h_o|} n_o^\alpha}{\partial X^\alpha} \right) + n_{ext,o}, \\ \frac{\partial \sigma_o}{\partial t} &= \frac{1}{\sqrt{|h_o|}} \left( \frac{\partial \sqrt{|h_o|} m_o^\alpha}{\partial X^\alpha} \right) + \frac{\partial r}{\partial X^\alpha} \times n_o^\alpha + m_{ext,o}. \end{aligned} \quad (69)$$

As for (68), this second set of balance equations can be deduced from Newton's laws and 3D elasticity where  $n_o^\alpha$  and  $m_o^\alpha$  now stand for the normalized Kirchhoff resultants exerted across each shell section  $X^\alpha = C$  of the reference configuration [29]. Finally, these balance equations represent the left-reduced shell dynamics in the dual of the Lie algebra of  $SE(3)$ . They have to be supplemented with the following reconstruction equations for shells:

$$\forall (X^1, X^2) \in [0, 1]^2 : \frac{\partial}{\partial t} \begin{pmatrix} R r \\ 0 \ 1 \end{pmatrix} = \begin{pmatrix} R r \\ 0 \ 1 \end{pmatrix} \begin{pmatrix} \Omega V \\ 0 \ 0 \end{pmatrix}, \quad (70)$$

which stands for a kinematic model supplementing (67) into a set of time-evolution equations in the state space of the shell initialized with  $\bar{X} \mapsto (g_o, \eta_o)(\bar{X})$  at  $t = 0$ .

## 6 From the full Cosserat to the classical shell model

Despite its elegance, the above construction cannot be directly applied to the classical shell model. It would hold for a 2-D micropolar medium whose the micro-structure  $\mathcal{M}$  has a full rank angular dynamics i.e., has a non negligible intrinsic spin, external couple and couple stress along  $\mathcal{M}$ , here modelled respectively by the components  $\Sigma_t^3 = (\bar{J}_\parallel / \sqrt{h}) \Omega_3$ ,  $M_{ext,t}^3$ ,  $M_t^{31}$  and  $M_t^{32}$  of (67). However, the classical shell model has no such features and the full Cosserat model must be modified to remove these artifacts for classical shells. To achieve this, we apply the usual procedure leading to the symmetry of the Cauchy stress tensor of a classical (non-Cosserat) 3D medium to our full-Cosserat shell. We will first define a set of reduced strain measures adapted to the classical model, and by duality will introduce a reduced model of the stress, related to these strains through hyperelastic constitutive laws that force the symmetry of two tensor stress fields modelling the stress state in the shell's mid-surface. As in the three-dimensional case [24], these symmetry conditions will be proved to be

equivalent to the angular balance equations along the microstructures which in the classical model are directors, i.e. degenerated rigid bodies with no intrinsic spin nor couple stress along them.

### 6.1 Strains of the Classical shell model

According to the general Poincaré-Cosserat picture, any set of strain measures properly invariant under rigid transformations, should depend on the left invariant fields  $\xi_1$  and  $\xi_2$  of (59) only. This basic fact can be easily verified in the case of classical shells with no couple stress along the directors. Using the field of frame and co-frame  $\bar{X} \mapsto (h_1, h_2)(\bar{X})$  and  $\bar{X} \mapsto (h^1, h^2)(\bar{X})$  on  $(\Phi_t \circ e)(\mathcal{D})$  as they are defined in section 2.3, one can parameterize the strain state of a classical shell with the following set of tensor fields [11]:

$$\epsilon = \epsilon_{\alpha\beta} h^\alpha \otimes h^\beta = \frac{1}{2} (h_{\alpha \cdot} h_\beta - h_\alpha^o \cdot h_\beta^o) h^\alpha \otimes h^\beta, \quad (71)$$

$$\rho = \rho_{\alpha\beta} h^\alpha \otimes h^\beta = \left( h_\alpha \cdot \frac{\partial t_3}{\partial X^\beta} - h_\alpha^o \cdot \frac{\partial t_3^o}{\partial X^\beta} \right) h^\alpha \otimes h^\beta, \quad (72)$$

$$\tau = \tau_\alpha h^\alpha = \left( t_3 \cdot \frac{\partial r}{\partial X^\alpha} - t_3^o \cdot \frac{\partial r_o}{\partial X^\alpha} \right) h^\alpha, \quad (73)$$

where  $\tau_\alpha$ ,  $\rho_{\alpha\beta}$  and  $\epsilon_{\alpha\beta}$  ( $\alpha, \beta = 1, 2$ ) measure the transverse shearing, the curvature and the membrane stretching and shearing of the shell in the two material directions  $X^1$  and  $X^2$ , with respect to the reference configuration. This configuration is throughout the article distinguished with the upper index  $o$ . From section 2.4, we also have  $(h_1, h_2)(\bar{X}) = (R \cdot \Gamma_1, R \cdot \Gamma_2)(\bar{X})$ ,  $(h^1, h^2)(\bar{X}) = (R \cdot \Gamma^1, R \cdot \Gamma^2)(\bar{X})$  and  $t_3 = R \cdot E_3$ , which once introduced into (71-73) give the following alternative expressions of the strain tensors pulled back in the material space by applying  $\nabla \bar{\Phi}_t$  and  $\nabla \bar{\Phi}_t^T$ :

$$\epsilon_{\alpha\beta} E^\alpha \otimes E^\beta = \frac{1}{2} [\Gamma_\alpha \cdot \Gamma_\beta - \Gamma_\alpha^o \cdot \Gamma_\beta^o] E^\alpha \otimes E^\beta, \quad (74)$$

$$\rho_{\alpha\beta} E^\alpha \otimes E^\beta = [E_3 \cdot (K_\alpha \times \Gamma_\beta - K_\alpha^o \times \Gamma_\beta^o)] E^\alpha \otimes E^\beta, \quad (75)$$

$$\tau_\alpha E^\alpha = [E_3 \cdot (\Gamma_\alpha - \Gamma_\alpha^o)] E^\alpha. \quad (76)$$

Finally, as expected the strain fields (74), (75), and (76) only depend on the left invariant fields  $\xi_\alpha = (K_\alpha^T, \Gamma_\alpha^T)^T$  of (59).

### 6.2 Constitutive equations for classical shells

In the simplest case, where the shell is made of an isotropic hyperelastic material, the internal potential energy can be defined as the following stored energy function [11]:

$$\mathfrak{U}_t = \bar{\rho}_t \psi(\epsilon_{\alpha\beta}, \rho_{\alpha\beta}, \tau_\alpha), \quad (77)$$

where  $\psi$  is a point-wise function on  $\mathcal{D}$  of the strains (74-76) and  $\mathfrak{U}_t$  is a density per unit of deformed volume of  $\mathcal{D}$ <sup>5</sup>. From (77), one can derive the properly

<sup>5</sup> In particular,  $\bar{\rho}_t \sqrt{h} \psi = \bar{\rho} \psi$ , defines the same density, but per unit of material volume

invariant hyperelastic constitutive relations:

$$\mathcal{N}_t^{\alpha\beta} = \bar{\rho}_t \frac{\partial \psi}{\partial \epsilon_{\alpha\beta}}, \quad \mathcal{M}_t^{\alpha\beta} = \bar{\rho}_t \frac{\partial \psi}{\partial \rho_{\alpha\beta}}, \quad \mathcal{Q}_t^\alpha = \bar{\rho}_t \frac{\partial \psi}{\partial \tau_\alpha}, \quad (78)$$

where  $\sqrt{|h|}\mathcal{N}_t = \sqrt{|h|}\mathcal{N}_t^{\alpha\beta} E_\alpha \otimes E_\beta$ ,  $\sqrt{|h|}\mathcal{M}_t = \sqrt{|h|}\mathcal{M}_t^{\alpha\beta} E_\alpha \otimes E_\beta$  and  $\sqrt{|h|}\mathcal{Q}_t = \sqrt{|h|}\mathcal{Q}_t^\alpha E_\alpha$  define a set of stress tensors, called "effective" in [11], which are entirely defined as the dual of the strain fields (74-76) according to the following expression of the virtual work of the internal forces exerted inside  $\mathcal{B}$  [11]:

$$\delta W_{int} = \int_{\mathcal{D}} (\mathcal{N}_t^{\alpha\beta} \delta \epsilon_{\alpha\beta} + \mathcal{M}_t^{\alpha\beta} \delta \rho_{\alpha\beta} + \mathcal{Q}_t^\alpha \delta \tau_\alpha) \sqrt{|h|} dX^1 \wedge dX^2. \quad (79)$$

Defined in this way, the effective stress tensors inherit of the symmetry (w.r.t. the  $(\alpha, \beta)$  indices) of the strain measures (74-76) and satisfy by construction  $\mathcal{M}^{\alpha\beta} = \mathcal{M}^{\beta\alpha}$ ,  $\mathcal{N}^{\alpha\beta} = \mathcal{N}^{\beta\alpha}$  ( $\alpha, \beta = 1, 2$ ), a restriction which will be discussed further in section 7.4. In particular, for small strains (but finite transformations), (77) can be defined as the following quadratic form of the strains (74-76):

$$\mathfrak{U}_t(\xi) = \frac{1}{2} H_{m,t}^{\alpha\beta\lambda\mu} \epsilon_{\alpha\beta} \epsilon_{\lambda\mu} + \frac{1}{2} H_{b,t}^{\alpha\beta\lambda\mu} \rho_{\alpha\beta} \rho_{\lambda\mu} + \frac{1}{2} H_{s,t}^{\alpha\beta} \tau_\alpha \tau_\beta, \quad (80)$$

where  $H_{m,t}$ ,  $H_{b,t}$  and  $H_{s,t}$  are constitutive reduced Hooke-like tensors respectively related to the shell's membrane, bending and shearing state [14]. With (80), the constitutive equations of classical shells can be stated in the so called reduced Hooke's law:

$$\mathcal{N}_t^{\alpha\beta} = H_{m,t}^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu}, \quad \mathcal{M}_t^{\alpha\beta} = H_{b,t}^{\alpha\beta\lambda\mu} \rho_{\lambda\mu}, \quad \mathcal{Q}_t^\alpha = H_{s,t}^{\alpha\beta} \tau_\beta. \quad (81)$$

Finally, in contrast to the case of beams [10], the generalized momenta  $\partial \mathfrak{U}_t / \partial \xi_\alpha$  of the Poincaré-Cosserat construction cannot be directly identified to these (effective) stresses, but are rather related to them through the relation:

$$\frac{\partial \mathfrak{U}_t}{\partial \xi_\gamma} = \bar{\rho}_t \left( \frac{\partial \psi}{\partial \epsilon_{\alpha\beta}} \right) \left( \frac{\partial \epsilon_{\alpha\beta}}{\partial \xi_\gamma} \right) + \bar{\rho}_t \left( \frac{\partial \psi}{\partial \rho_{\alpha\beta}} \right) \left( \frac{\partial \rho_{\alpha\beta}}{\partial \xi_\gamma} \right) + \bar{\rho}_t \left( \frac{\partial \psi}{\partial \tau_\alpha} \right) \left( \frac{\partial \tau_\alpha}{\partial \xi_\gamma} \right). \quad (82)$$

Then, using the definitions of effective strains (74-76) and stress (78), allows detailing (82) as:

$$\frac{\partial \mathfrak{U}_t}{\partial \xi_\alpha} = \begin{pmatrix} M_t^\alpha \\ N_t^\alpha \end{pmatrix} = \begin{pmatrix} \mathcal{M}_t^{\alpha\beta} (E_3 \times \Gamma_\beta) \\ \mathcal{M}_t^{\beta\alpha} (K_\beta \times E_3) + \mathcal{N}_t^{\alpha\beta} \Gamma_\beta + \mathcal{Q}_t^\alpha E_3 \end{pmatrix}. \quad (83)$$

Finally, by substituting (74-76) into (78) or (81) and the result into (83), we obtain a constitutive law for classical shells which relates the left invariant fields of (58) to their dual (61).

### 6.3 Reduction of the internal stress state when shifting from the full Cosserat to the classical shell model

The aim of this subsection is to show how the above constitutive law (78) completely takes charge the process of degenerating the micro-solid  $\mathcal{M}$  into directors, i.e. one-dimensional rigid bodies. To achieve this reduction process, we can take inspiration from the reduction of the angular balance equations of a 3D Cosserat (micropolar) medium into those of a classical (non-Cosserat) one. We remove  $M_t^{31}$  and  $M_t^{32}$  in (67) along with the kinetic momenta  $\Sigma_t^3 = (\bar{J}_\parallel / \sqrt{h}) \Omega_3$  and external couples  $M_{ext,t}^3$  around the directors  $t_3$ . This simplification changes the sixth row of (67) into:

$$E_3 \cdot (\Gamma_\alpha \times N_t^\alpha + (K_\alpha \times E_3) \times (M_t^\alpha \times E_3)) = 0, \quad (84)$$

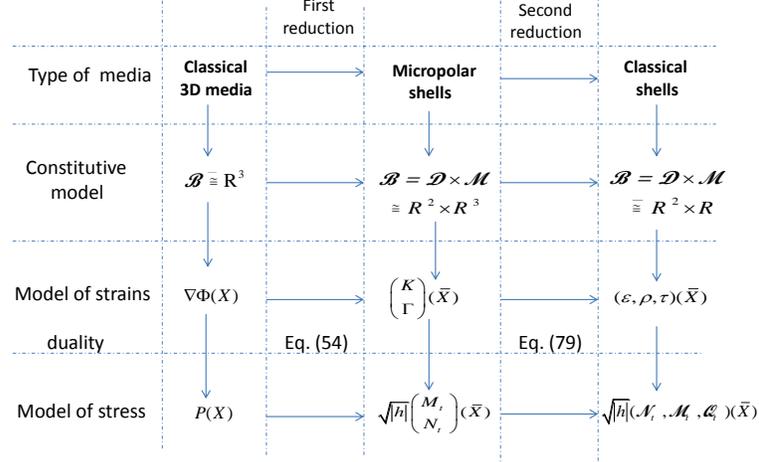
which models the (degenerated) angular dynamics about the directors with no intrinsic spin and couple stress. Expression (84) stands for a static constitutive constraint on the internal stress of the classical shell model. Moreover, inserting (83) into (84) and achieving simple vector algebra allows changing (84) into the following constitutive constraint on the effective stress tensor components:

$$\mathcal{N}_t^{\alpha\beta} (\Gamma_\alpha \times \Gamma_\beta) \cdot E_3 = 0, \quad (85)$$

which obliges the effective stress tensor  $\sqrt{|h|} \mathcal{N}_t = \sqrt{|h|} \mathcal{N}_t^{\alpha\beta} E_\alpha \otimes E_\beta$  to be symmetric, as the Cauchy stress tensor field is in a classical (not Cosserat) 3D medium [24]. Thus, the constitutive law (78) imposing defacto the symmetry condition (85), it implicitly forces the constitutive constraint (84), and the degenerated angular balance along the directors, from which it derives. Moreover, (78) also imposes the symmetry of the couple stress tensor  $\sqrt{|h|} \mathcal{M}_t = \sqrt{|h|} \mathcal{M}_t^{\alpha\beta} E_\alpha \otimes E_\beta$ , a further restriction which is not required by (84). Finally, the internal stress state dimension has been reduced from the 12 components of the Cosserat-stress tensors ( $\sqrt{|h|} M_t^\alpha$ ,  $\sqrt{|h|} N_t^\alpha$ ) to the 8 components of the effective stress fields  $\sqrt{|h|} \mathcal{N}_t^{\alpha\beta}$ ,  $\sqrt{|h|} \mathcal{M}_t^{\alpha\beta}$  and  $\sqrt{|h|} \mathcal{Q}_t^\alpha$ . For each dimension of  $\mathcal{M}$  which is degenerated,  $p$  couple stress components are removed. This first simplification forces a second one through the angular momentum balance which leads to the removal of the  $(p^2 - p)/2 = p(p - 1)/2$  symmetric components of the stress tensor. In the 3D case,  $p = 3$ , and this process reduces the stress state from the  $6 \cdot p = 18$  components of the Cauchy stress and couple stress tensors to the  $18 - 3 \cdot 3 - 3(3 - 1)/2 = 6$  independent components of the Cauchy-stress tensor. In the 2D case, we have  $p = 2$  and the same process reduces the stress state from the  $6 \cdot p = 12$  components of the full Cosserat model to  $12 - 1 \cdot 2 - 2(2 - 1)/2 = 9$  and finally, with the couple stress tensor symmetry, to the 8 independent components of the effective stress tensors and vectors. This reduction process is schematized in figure 4.

### 6.4 Kinematic reconstruction for the classical shell model

The price paid for this reduction is that the time-evolution of the angular velocity  $\Omega_3$  in (67) becomes dynamically indeterminate, which compromises the



**Fig. 4.** Reduction process for classical shells: The first reduction is based on Cosserat kinematics (8) and leads to the micropolar model with drill indeterminacy, the second is based on the constitutive law (77), and leads to a closed model once supplemented with a kinematic reconstruction of the drill.

kinematic reconstruction (70) on  $G = SE(3)$ . To circumvent this issue and recover a kinematic model of  $\Omega_3$  consistent with the classical shell model, one can fix the rotation around the directors or "drilling rotation", to ensure that the full field  $R$  matches the rotational part of the polar decomposition of the gradient of the transformation (57) restricted to the mid-surface shell. The restricted gradient has been defined as  $\nabla\bar{\Phi} = h_\alpha \otimes E^\alpha$  (see section 2.4). Inspired by [19], this approach has been originally applied to the restricted gradient of transformations  $\bar{F} \triangleq h_\alpha \otimes h_\sigma^\alpha$  in the context of the finite element method for geometrically exact shells [18]. In our context, it consists of fixing the drilling rotation about the directors at each instant, by imposing the symmetry of the tensor  $(R^T \cdot \nabla\bar{\Phi})(\bar{X}) = \bar{U}(\bar{X})$ , where  $\bar{U}$  is a symmetric tensor on the shell mid-surface measuring the stretching and shearing between the material mid surface and its deformed configuration. As a result, imposing the symmetry of  $\bar{U}$  on the mid-surface is equivalent to force for  $\alpha, \beta = 1, 2$ :

$$E_\alpha \cdot \bar{U} \cdot E_\beta = E_\beta \cdot \bar{U} \cdot E_\alpha. \quad (86)$$

But since we also have:

$$E_\alpha \cdot (R^T \cdot \nabla\bar{\Phi}) \cdot E_\beta = E_\alpha \cdot (R^T \cdot (h_\gamma \otimes E^\gamma)) \cdot E_\beta = E_\alpha \cdot (R^T \cdot h_\beta) = E_\alpha \cdot \Gamma_\beta, \quad (87)$$

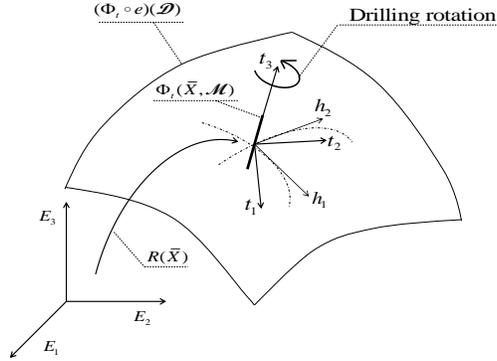
the dynamically undetermined drilling rotation is fixed by forcing the following point-wise geometric constraint on the shell mid-surface:

$$E_1 \cdot \Gamma_2 = E_2 \cdot \Gamma_1. \quad (88)$$

Time-differentiating (88) gives the following expression of the missing angular field, with  $Tr(K) = K_{11} + K_{22}$  and summations on  $\alpha$ :

$$\Omega_3 = \frac{1}{\Gamma_{11} + \Gamma_{22}} \left[ \left( \frac{\partial V_2}{\partial X^1} - \frac{\partial V_1}{\partial X^2} \right) - Tr(K)V_3 + K_{3\alpha}V_\alpha + \Gamma_{3\alpha}\Omega_\alpha \right]. \quad (89)$$

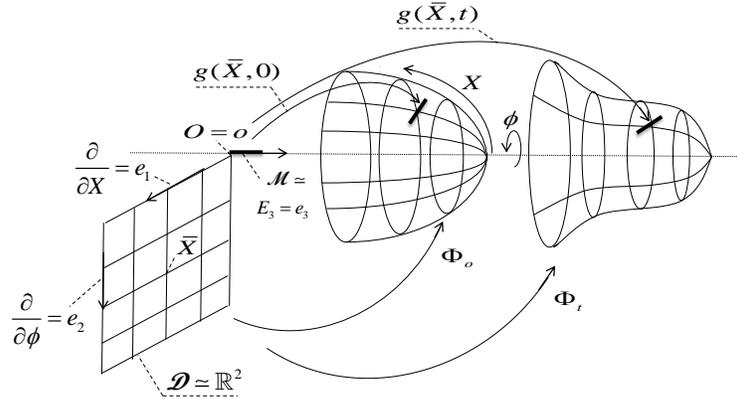
It is worth noting here that (89) is the shell counterpart of the angular velocity field defined as the skew-symmetric part of the 3D velocity gradient in the classical (not Cosserat) 3D media. As an illustration, consider the case of a rigid plate, where  $K_\alpha = 0$ ,  $\Gamma_{3\alpha} = 0$ , and  $\Gamma_{11} = \Gamma_{22} = 1$  are imposed in (89). As expected,  $\Omega_3 = (1/2)(\partial V_2/\partial X^1 - \partial V_1/\partial X^2)$ , i.e.,  $\Omega_3$  is equal to the half curl of the linear velocity field on the rigid plate mid-surface. Finally, once inserted in (67) and (70), the kinematic model (89) of  $\Omega_3$  completes the degenerated dynamics which now govern the time evolution of  $V$  and  $\Omega_\alpha$ ,  $\alpha = 1, 2$ . It is worth noting here that as this model of the drilling rotation is only kinematic, and not dynamic, the reference configuration plays no role other than fixing the initial conditions of the state-space equations through the field of frame  $\bar{X} \mapsto R_o(\bar{X})$ , which support the directors at  $t = 0$ . In particular, the symmetric tensor  $\bar{U}$  of the above construction does not match with the stretching tensor related to the mechanical states in the reference and deformed configuration.



**Fig. 5.** Sketch of the drilling rotation in classical Cosserat shells.

## 7 Illustrative example: application to axisymmetric shells

In this section we illustrate the previous general picture by applying it to an axisymmetric shell  $\mathcal{B}$  with a mid-surface diffeomorphic to a disc as pictured in figure 6. The shell undergoes a net translation and axisymmetric shape deformations along the  $(o, e_3)$  direction of an inertial frame  $(o, e_1, e_2, e_3)$  with no rotation around it. According to the problem symmetry, the inertial frame  $(o, e_1, e_2, e_3)$  is endowed with a chart of cylindrical coordinates  $(r, \phi, z)$  of local orthonormal basis  $(e_r, e_\phi, e_3)$ . The material space  $\mathcal{B}$ , of material frame  $(O, E_1, E_2, E_3) = (o, e_1, e_2, e_3)$ , is identified to  $\mathcal{D} \times \mathcal{M}$ , with  $\mathcal{D}$  the shell's material mid surface supported by  $(E_1, E_2)$ , and  $\mathcal{M}$  its director supported by  $E_3$ , and crossed by  $\mathcal{D}$ , in its center ( $l$  and  $j$  denote the length and the perpendicular angular geometric inertia moment of  $\mathcal{M}$ ). The reference configuration is  $\Phi_o(\mathcal{B})$ . Its symmetry axis



**Fig. 6.** Application of the construction to an axisymmetric shell.

is  $(O, e_3)$ , and the cylindrical coordinates of its points are denoted  $(r_o, z_o, \phi_o)$ . In this context, the open set  $\mathcal{D} - \{\partial\mathcal{D} \cup O\}$  is covered with the material chart  $\{X^1, X^2\} = \{X, \phi\}$  of natural basis  $\{E_1, E_2\} = \{\partial/\partial X, \partial/\partial \phi\}$ , where  $X$  is the metric length along the meridians of  $(\Phi_o \circ e)(\mathcal{D})$ . In any configuration  $\Phi(\mathcal{B})$  of the shell, any cross section fiber crossing  $\mathcal{D}$  in  $(X, \phi)$ , is supported by the third unit

vector of a director frame  $(t_1, t_2, t_3)(X, \phi)$  deduced from  $(O, E_1, E_2, E_3)$  through a transformation of  $SE(3)$  of the form:

$$g(X, \phi) = \begin{pmatrix} \exp(\phi \hat{e}_3) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp(-\theta \hat{e}_\phi) & r e_r + z e_3 \\ 0 & 1 \end{pmatrix}, \quad (90)$$

with  $\theta$ , the angle parameterizing the local rotation of the director  $t_3$  with respect to  $e_3$ . Using (90) in (58), gives with  $se(3) \cong \mathbb{R}^6$ , the three left invariant fields:

$$\eta = \begin{pmatrix} 0 \\ \Omega_2 \\ 0 \\ V_1 \\ 0 \\ V_3 \end{pmatrix}, \xi_X = \begin{pmatrix} 0 \\ K_{2X} \\ 0 \\ \Gamma_{1X} \\ 0 \\ \Gamma_{3X} \end{pmatrix} = \begin{pmatrix} 0 \\ -\theta' \\ 0 \\ r' \cos \theta + z' \sin \theta \\ 0 \\ z' \cos \theta - r' \sin \theta \end{pmatrix}, \xi_\phi = \begin{pmatrix} K_{1\phi} \\ 0 \\ K_{3\phi} \\ 0 \\ \Gamma_{2\phi} \\ 0 \end{pmatrix} = \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \\ 0 \\ r \\ 0 \end{pmatrix},$$

where ' denotes  $\partial./\partial X$ . These expressions define  $\Gamma_\alpha$  and  $K_\alpha$  ( $\alpha = X, \phi$ ), which once inserted in (74-76), give the expressions of (effective) strain components:

$$\begin{pmatrix} \tau_X \\ \tau_\phi \end{pmatrix} = \begin{pmatrix} \Gamma_{3X} \\ 0 \end{pmatrix}, \begin{pmatrix} \epsilon_{XX} & \epsilon_{X\phi} \\ \epsilon_{\phi X} & \epsilon_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \Gamma_{1X}^2 + \Gamma_{3X}^2 - 1 & 0 \\ 0 & \Gamma_{2\phi}^2 - r_o^2 \end{pmatrix}, \\ \begin{pmatrix} \rho_{XX} & \rho_{X\phi} \\ \rho_{\phi X} & \rho_{\phi\phi} \end{pmatrix} = \begin{pmatrix} K_{2X}^o \Gamma_{1X}^o - K_{2X} \Gamma_{1X} & 0 \\ 0 & K_{1\phi} \Gamma_{2\phi} - r_o K_{1\phi}^o \end{pmatrix}, \quad (91)$$

where we assume no transverse shearing in the reference configuration. Then, introducing the Cosserat and effective stress of the general construction, and remarking that the axisymmetry imposes:  $\mathcal{N}_t^{X\phi} = \mathcal{N}_t^{\phi X} = 0$ ,  $\mathcal{M}_t^{X\phi} = \mathcal{M}_t^{\phi X} = 0$  and  $\mathcal{Q}_t^\phi = 0$ , (83) allows writing the relations between the Cosserat and the effective stress tensors components, all related to the reference configuration, as:

$$\begin{pmatrix} 0 \\ M_o^{2X} \\ 0 \\ N_o^{1X} \\ 0 \\ N_o^{3X} \end{pmatrix} = \begin{pmatrix} 0 \\ \Gamma_{1X} \mathcal{M}_o^{XX} \\ 0 \\ K_{1\phi} \mathcal{M}_o^{X\phi} + \Gamma_{1X} \mathcal{N}_o^{XX} \\ 0 \\ \Gamma_{3X} \mathcal{N}_o^{XX} + \mathcal{Q}_o^X \end{pmatrix}, \begin{pmatrix} M_o^{1\phi} \\ 0 \\ 0 \\ 0 \\ N_o^{2\phi} \\ 0 \end{pmatrix} = \begin{pmatrix} -\Gamma_{2\phi} \mathcal{M}_o^{\phi\phi} \\ 0 \\ 0 \\ 0 \\ \Gamma_{2\phi} \mathcal{N}_o^{\phi\phi} - K_{2X} \mathcal{M}_o^{\phi\phi} \\ 0 \end{pmatrix}.$$

Applying the material Poincaré-Cosserat equations (67) related to the reference configuration  $(\Phi_o \circ e)(\mathcal{D})$  to our shell, with the expressions of  $\eta$ ,  $\xi_X$ ,  $\xi_\phi$  above,  $\sqrt{|h_o|} = (\Gamma_{2\phi}^o (\Gamma_{1X}^o)^2 + (\Gamma_{3X}^o)^2)^{1/2} = r_o$ , and no dependency of the Lagrangian density with respect to  $\phi$ , gives the three following scalar equations :

$$\begin{aligned} \rho_o l \left( \frac{\partial V_1}{\partial t} - V_3 \Omega_2 \right) &= \frac{1}{r_o} \frac{\partial r_o N_o^{1X}}{\partial X} + K_{2X} N_o^{3X} - K_{3\phi} N_o^{2\phi} + N_{ext,o}^1, \\ \rho_o l \left( \frac{\partial V_2}{\partial t} + V_1 \Omega_2 \right) &= \frac{1}{r_o} \frac{\partial r_o N_o^{3X}}{\partial X} - K_{2X} N_o^{1X} + K_{1\phi} N_o^{2\phi} + N_{ext,o}^2, \\ \rho_o j \frac{\partial \Omega_2}{\partial t} &= \frac{1}{r_o} \frac{\partial r_o M_o^{2X}}{\partial X} - \Gamma_{1X} N_o^{3X} + \Gamma_{3X} N_o^{1X} + K_{3\phi} M_o^{1\phi} + M_{ext,o}^2. \end{aligned}$$

Pushing forward these equations in the field of mobile director frames  $(X, \phi) \mapsto (t_1, t_2, t_3)(X, \phi)$  with  $t_i(X, \phi) = R(X, \phi).E_i$  and remarking that  $\partial t_2/\partial \phi = -e_r$ , gives two equations (one vectorial and one scalar) which represent the spatial Poincaré equations (68) for an axisymmetric shell:

$$\begin{aligned}\rho_o r_o l \left( \frac{\partial v}{\partial t} \right) &= \frac{\partial}{\partial X} [r_o (N_o^{1X} t_1 + N_o^{3X} t_3)] - r_o (N_o^{2\phi} e_r - n_{ext,o}), \\ \rho_o r_o j \left( \frac{\partial \Omega_2}{\partial t} \right) &= \frac{\partial}{\partial X} [r_o M_o^{2X}] + r_o (K_{3\phi} M_o^{1\phi} + \Gamma_{3X} N_o^{1X} - \Gamma_{1X} N_o^{3X} + M_{ext,o}^2),\end{aligned}\tag{92}$$

where  $v = R.V$ . Moreover, introducing the densities of internal wrench per unit of metric length of the material coordinate lines on the reference configuration (49), we find in the present case, i.e., with  $|h_{o,XX}| = 1$ ,  $|h_{o,\phi\phi}| = r_o^2$ :

$$M_o^X = (r_o/r_o) M_o^{*X}, \quad N_o^X = (r_o/r_o) N_o^{*X}, \quad M_o^\phi = (1/r_o) M_o^{*\phi}, \quad N_o^\phi = (1/r_o) N_o^{*\phi}.$$

Once inserted in (92), these relations give an alternative form of balance equations in terms of metric densities of stress:

$$\begin{aligned}\rho_o r_o l \frac{\partial v}{\partial t} &= \frac{\partial}{\partial X} [r_o (N_o^{*1X} t_1 + N_o^{*3X} t_3)] - N_o^{*2\phi} e_r + r_o n_{ext,o}, \\ \rho_o r_o j \frac{\partial \Omega_2}{\partial t} &= \frac{\partial}{\partial X} [r_o M_o^{*2X}] + K_{3\phi} M_o^{*1\phi} + r_o (\Gamma_{3X} N_o^{*1X} - \Gamma_{1X} N_o^{*3X} + M_{ext,o}^2),\end{aligned}$$

which are the classical equations of axisymmetric shells as given in [15], where they are expressed in terms of director stress couples. These balance equations, which involve densities of wrenches per unit of metric volume of  $(\Phi_o \circ e)(\mathcal{D})$ , have to be supplemented with the expressions of  $\eta$ ,  $\xi_X$  and  $\xi_\phi$ , the constitutive equations in terms of effective stresses (81), the reconstruction equation (70) with their initial conditions, and a model of external forces  $(N_{ext,o}^1, N_{ext,o}^2, M_{ext,o}^2)$ .

## 8 Conclusion

In this article we proposed a general picture which allowed the partial differential equations and the boundary conditions of a Cosserat medium, i.e., a set of small rigid bodies continuously staked along one or several material dimensions, to be derived. This is achieved through a variational calculus from a unique Lagrangian density and its symmetry group. The approach is systematic and requires no phenomenological input. It is based on an extension of the Poincaré equations from classical mechanical systems to field theory. As with the classical Poincaré equations, these equations are stated in the dual of the Lie algebra of the symmetry group of the Lagrangian. As a result, they are a set of first order dynamics equations governing the medium's velocities, i.e., the components of the infinitesimal right (material) or left (spatial) transformations of the group's Lie algebra. When the system is fully symmetric, they can be time-integrated autonomously in a first step, and in a second step, the resulting velocity field allows the recovery of the motion of the medium's transformations on the group,

through reconstruction equations. Remarkably, these general equations allow the recovery of the usual formulations of non-linear shells theories in both the reference and current configuration, and in the material and spatial setting. More precisely, this may be achieved from a unique set of covariant partial differential equations leading to all the formulations deduced from each other through some transformations which change densities related to the material, reference and deformed configurations, and others which change material into spatial infinitesimal transformations. While the approach gives a fully determinate dynamic model of micro-polar shells, it has been reduced further to obtain a model of classical shells with no couple stress, nor spin about their directors. This reduction process is based on the use of constitutive laws imposing the symmetry of a set of effective stress defined in the shell's mid surface. Due to the indeterminacy of the drilling rotation about the directors, the shell motion reconstruction requires a kinematic model of the missing rotation deduced from the polar decomposition of the gradient of the mid-surface transformation. This model will be used to study the hydrodynamic forces exerted on swimming cephalopods in the future. These animals have an open soft cavity, a mantle, that they contract and dilate for jet propulsion, an object which is of great interest to the field of soft bio-robotics [30]. We also plan to model an octopus modelled with an axisymmetric Cosserat shell prolonged with eight arms modelled by Cosserat beams.

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