



ONLY FOUR EULER INFINITY PRODUCTS ARE THETA-TYPE FUNCTIONS

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ONLY FOUR EULER INFINITY PRODUCTS ARE THETA-TYPE FUNCTIONS

CHANGGUI ZHANG

ABSTRACT. This paper follows our previous work [14]. A function is called theta-type when its asymptotic behavior near any root of unity is similar like as what happened for any Jacobi theta function. It will be shown that only four Euler infinite products have this property. This will be obtained by investigating the analyticity obstacle of a Laplace-type integral of the exponential generating function of Bernoulli numbers.

CONTENTS

1. Introduction	2
1.1. Ramanujan's two-variable theta functions	2
1.2. Behavior of theta-type functions at all roots of unity	4
1.3. Stokes phenomenon and analytic obstruction	6
1.4. Plan for the rest of the paper	7
2. Theta-type functions and exponential-convergent expansions	8
2.1. Preliminary commentaries on Theta-type functions	8
2.2. Case of Ramanujan's two-variables theta-function	10
2.3. Exponential-convergent behaviour of composite Euler infinite products	11
2.4. The exponential-finite behaviour of composite Jacobi Theta functions	12
3. Stokes' analysis and modular properties about Euler infinite products	14
3.1. An integral associated with the exponential generating function of Bernoulli numbers	14
3.2. A symmetrical Bernoulli integral	17
3.3. Two types of singularities for Bernoulli integral	18
3.4. Modularity found by means of Stokes' analysis on Euler products	20
4. Analytic continuation and Ramanujan's asymptotic formula	21
4.1. Branch points and analytic continuation domain of Bernoulli integral	22
4.2. Functional equations satisfied by $A(z \tau)$ and $L_2(z \tau)$	24
4.3. Non-modular term in a Ramanujan asymptotic formula	26
4.4. Equivalence between Theorem 4.2 and Theorem 1.3	27
5. Conditions for an Euler product to have an exponential-convergent expansion	29
5.1. Ramis-Sibuya's Theorem on Gevrey asymptotic expansions	29
5.2. Exponentially small terms related with Euler infinite products	30
5.3. From asymptotics to analyticity for Bernoulli integral and related functions	31

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5.4. Necessary conditions for an Euler product to have an exponential- convergent expansion	33
6. Asymptotic behaviour at an arbitrary root via continued fractions	35
6.1. Bernoulli integral and associated functions on real axis	35
6.2. Continued fractions and modular transforms	37
6.3. Concluding proofs	39
References	40

1. INTRODUCTION

In his last letter to Hardy, Ramanujan wrote that he had discovered very interesting functions that he called mock ϑ -functions. He claimed that these functions enter into mathematics as beautifully as the ordinary ϑ -functions.

However, as said in Watson's L.M.S. presidential address [9], the first three pages where Ramanujan explained what he meant by a "mock ϑ -functions" are very obscure. Therefore, Watson quoted the following comment of Hardy: *a mock ϑ -function is a function defined by a q -series convergent when $|q| < 1$, for which we can calculate asymptotic formulae, when q tends to a "rational point" $e^{2r\pi i/s}$ of the unit circle, of the same degree of precision as those furnished for the ordinary ϑ -functions by the theory of linear transformation.*

One of the most important questions concerning Ramanujan's mock ϑ -functions would be to find a natural way which might lead one to meet and realize these "modified" ϑ -functions. By writing the third order mock ϑ -functions into the form of confluent q -hypergeometric series, we shall consider certain linear q -difference equations that have a non-Fuchsian singular point in the Riemann sphere $\mathbb{C} \cup \{\infty\}$ [8]. These functional equations are like those satisfied by theta functions and their variants such as Appell-Lerch series or Mordell's integral. So to speak, one can decompose each Ramanujan's third order mock ϑ -function into simple elements among which the ordinary ϑ -functions and Euler infinity products will be appearing. This transcendental analysis will allow one to compare mock ϑ -functions with the ordinary ϑ -functions, and this general point-view will be cleared up in one series of papers, including our previous work [14] and this one.

In [14], we gave a definition of what we call *theta-type*, *false theta-type* and *mock theta-type* functions, in the direct line of the above-mentioned comment of Hardy. It is shown that every Appell-Lerch series is mock theta-type except a very few cases for which it becomes false theta-type.

In this paper, the values of x for which the associated Euler infinity product $(x; q)_\infty$ is theta-type will be determined. Like as what done by Euler Γ -function for the theory of hypergeometric functions, it is commonly recognized that the Euler infinity products play a central role in the whole theory of q -series.

In the below, we will start with Ramanujan's theta functions and outline some principal results that will be established in this paper. We conclude this introduction by a plan of the sections that follow.

1.1. Ramanujan's two-variable theta functions. Let \mathbb{D} be the open disk $|q| < 1$ in the complex plane. As in [2, Chapter 16], we denote by $f(a, b)$ the Ramanujan's

two-variable theta function defined by

$$(1.1) \quad f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$$

for $ab \in \mathbb{D}$. Given $(x, q) \in \mathbb{C}^* \times \mathbb{D}$, we let $\theta(x; q)$ be the following Jacobi theta function:

$$(1.2) \quad \theta(x; q) = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n.$$

By using the Jacobi triple product identity, it follows that, letting $q = ab$,

$$(1.3) \quad f(a, b) = \theta(a; q) = (-a; q)_{\infty} (-b; q)_{\infty} (q; q)_{\infty}.$$

Here, the expression $(x; q)_{\infty}$ denotes the Euler product defined by

$$(1.4) \quad (x; q)_{\infty} = \prod_{n \geq 0} (1 - xq^n).$$

Especially, the following functions are in the heart of the Ramanujan's theory about theta-functions and modular equations [2, Entry 22, p. 36]:

$$\phi(q) = f(q, q), \quad \psi(q) = f(q, q^3), \quad f(-q) = f(-q, -q^2).$$

By (1.3), one finds that (*loc. cit.*)

$$(1.5) \quad \phi(q) = \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty}}{(q; q^2)_{\infty} (-q^2; q^2)_{\infty}}, \quad \psi(q) = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \quad f(-q) = (q; q)_{\infty}.$$

In what follows, we will reserve the notation f for any general function under consideration. In order to avoid any confusion, we will write $\pi(q)$ instead of $f(-q)$, this is to say:

$$(1.6) \quad \pi(q) = f(-q) = (q; q)_{\infty}.$$

As in [2, Chapter 16], one adds the following Ramanujan's χ -function to complete the above list of ϕ , ψ and π :

$$(1.7) \quad \chi(q) = (-q; q^2)_{\infty} = \frac{(q^2; q^4)_{\infty}}{(q; q^2)_{\infty}} = \frac{\pi(q^2) \pi(q^2)}{\pi(q) \pi(q^4)}.$$

Here, the equality in the middle can be found by replacing (x, q) with (q, q^2) in the “duplication” formula

$$(1.8) \quad (-x; q)_{\infty} (x; q)_{\infty} = (x^2; q^2)_{\infty},$$

while the last one can be obtained by considering $(x, q, n) = (q, q, 2)$ or $(x, q, n) = (q^2, q^2, 2)$ in the “decomposition” formula

$$(1.9) \quad (x; q)_{\infty} = (x; q^n)_{\infty} (xq; q^n)_{\infty} \dots (xq^{n-1}; q^n)_{\infty}, \quad n \in \mathbb{Z}_{>0}.$$

In a similar way as what done for (1.7), one deduces from (1.5) that $\phi(q)$ and $\psi(q)$ can be expressed as follows:

$$(1.10) \quad \phi(q) = \frac{\pi(q^2)^3}{\pi(q)^2 \pi(q^4)}, \quad \psi(q) = \frac{\pi(q^2)^2}{\pi(q)}.$$

Definition 1.1. Let $f(a, b)$ and $\pi(q)$ be as in (1.1) and (1.6). We denote by $\mathbf{\Phi}(q)$ and $\mathbf{\Pi}(q)$ the respective multiplicative group of analytic functions in \mathbb{D} which is generated by the following sets:

$$\{f(-q^j, -q^k) : (j, k) \in \mathbb{Z}_{>0}^2\}, \quad \{\pi(q^k) : k \in \mathbb{Z}_{>0}\}.$$

It is clear from the above, therefore, that the Ramanujan's functions ϕ , ψ and χ belong to $\mathbf{\Pi}(q)$, as well as π itself. Furthermore, $\mathbf{\Pi}(q)$ is clearly a sub-group of $\mathbf{\Phi}(q)$. In the meanwhile, considering both (1.8) and (1.9) (with $x = q$ and $n = 2$) yields the following relations:

$$(1.11) \quad (q; q)_\infty = \pi(q), \quad (-q; q)_\infty = \frac{\pi(q^2)}{\pi(q)},$$

$$(1.12) \quad (q; q^2)_\infty = \frac{\pi(q)}{\pi(q^2)}, \quad (-q; q^2)_\infty = \frac{\pi(q^2)\pi(q^2)}{\pi(q)\pi(q^4)}.$$

Here, the first equality is exactly the same as (1.6) and the last one has already been in (1.7). It follows that the Euler products $(\pm q; q^k)_\infty$, where $k \in \{1, 2\}$, belong to both groups $\mathbf{\Pi}(q)$ and $\mathbf{\Phi}(q)$.

In this paper, we will explain how to show that, modulo any ramification operator $q \mapsto q^k$ with $k \in \mathbb{Z}_{>0}$, the above Euler products included in (1.11) and (1.12) are the *only* possible Euler products belonging to the group $\mathbf{\Phi}(q)$. Namely, we have the following

Theorem 1.1. *Let $x = x_0 q^\beta$ with $x_0 \in \mathbb{C}$, $|x_0| = 1$ and $\beta \in \mathbb{Z}_{\geq 0}$, and let ζ be any root of unity. The following conditions are equivalent.*

- (1) $x \in \{q, -q, q^2, -q^2\}$.
- (2) $(x; q^2)_\infty \in \mathbf{\Pi}(q)$.
- (3) $(x; q^2)_\infty \in \mathbf{\Phi}(q)$.
- (4) $(x; q^2)_\infty \in \mathfrak{T}_\zeta \setminus \{0\}$.
- (5) $(x; q^2)_\infty \in \mathfrak{E}_\zeta \setminus \{0\}$.

In the above, $\mathfrak{T}_\zeta \setminus \{0\}$ (resp. $\mathfrak{E}_\zeta \setminus \{0\}$) denotes the multiplicative group of germs of non-zero functions satisfying the condition stated in (1.13) (resp. in (1.16) with $f_0(t) \in t\mathbb{C}[t]$ there). The above result will be proved in §6.3, by an analytical point of view, with the help of Theorem 1.2 stated in the paragraph that follows.

1.2. Behavior of theta-type functions at all roots of unity. From now on, we will let $q = e^{2\pi i\tau}$ and $\tau \in \mathcal{H}$, where \mathcal{H} denotes the usual Poincaré's half-plane $\Im\tau > 0$. To simplify, we will write $e(\alpha) = e^{2\pi i\alpha}$ for all $\alpha \in \mathbb{C}$. Following our previous work [14], we recall that, given $\zeta = e(r)$ with $r \in \mathbb{Q}$ or $i\infty$, a function $f(q)$ is said **theta-type as $q \xrightarrow{a.r.} \zeta$** and one writes $f \in \mathfrak{T}_\zeta$, if there exist a quadruplet (v, λ, I, γ) , composed of a couple $(v, \lambda) \in \mathbb{Q} \times \mathbb{R}$, a strictly increasing sequence I without finite limit and a \mathbb{C}^* -valued map γ defined on I , such that the following relation holds for any $N \in \mathbb{Z}_{\geq 0}$ as $\tau \xrightarrow{a.v.} r$:

$$(1.13) \quad f(q) = \left(\frac{i}{\hat{\tau}}\right)^v e(\lambda\hat{\tau}) \left(\sum_{k \in I \cap (-\infty, N]} \gamma(k) q_1^k + o(q_1^N) \right).$$

Here and in the following, $\hat{\tau} = \tau - r$ and $q_1 = e(-\frac{1}{\hat{\tau}})$; if $r = i\infty$, $\hat{\tau} = -\frac{1}{\tau}$ and $q_1 = q$.

In the above, the symbol " $q \xrightarrow{a.r.} \zeta$ " (resp. " $\tau \xrightarrow{a.v.} r$ ") could be read as " q almost radially converges to ζ " (resp. " τ almost vertically converges to r "). When ζ denotes a root of unity $e(r)$, this means that q tends to ζ in some open sector of vertex at ζ and symmetrical with respect to the radius $O\zeta$ of the unit circle $|q| < 1$ (resp. τ tends to r in some open sector of vertex at r and symmetrical with respect to the vertical half straight-line $r + i\mathbb{R}^+$ in the half-plane $\Im\tau > 0$). If $\zeta = 0$

and $r = i\infty$, the above sectors are replaced by symmetrical sectors with respect to half-lines from the origin of the complex plane or some vertical unbounded strips in the Poincaré's half-plane. See (2.7) for more details for the definition of \mathfrak{T}_0 .

By convention, $f \equiv 0$ if, and only if, $I = \emptyset$ in (1.13). When $I \neq \emptyset$, letting $k = \min I$ and $c = \gamma(k)$ there implies that

$$(1.14) \quad f(q) = c \left(\frac{i}{\hat{\tau}} \right)^v e(\lambda \hat{\tau} - \frac{k}{\hat{\tau}}) (1 + f_1(q)).$$

Here, f_1 denotes some exponentially small function as $q \xrightarrow{a,r} \zeta$, this means that $f_1 = o(e^{-\kappa/|\hat{\tau}|})$ for some $\kappa > 0$.

The above relation in (1.14) suggests us to introduce a class of functions that will be said to have one **exponential**-finite expansion. Let $a = e(\alpha)$, with $\alpha \in \mathbb{R} \cup \{i\infty\} (= \partial\mathcal{H} \cup \{i\infty\})$. In the same way as what done for (1.13), we define the radially symmetric sectors at a and vertically symmetric sectors at α . Let \mathcal{A}_a^r and \mathcal{A}_α^v denote respectively the sheaf of germs of analytic functions in such symmetric sectors at a and at α . The map $f \mapsto f \circ e$ defines an one-to-one correspondence from \mathcal{A}_a^r onto \mathcal{A}_α^v .

As before, let $\hat{\tau} = \tau - \alpha$. We remember that the local coordinate $\hat{\tau}$ needs to be read as $-\frac{1}{\tau}$ if $\alpha = i\infty$. Let $\mathcal{A}_\alpha^{\leq -1}$ be the sub-sheaf of \mathcal{A}_α^v consisting of exponentially small functions for $\hat{\tau} \xrightarrow{a,v} 0$, and let $\mathcal{A}_a^{\leq -1} = \{f \in \mathcal{A}_a^r : f \circ e \in \mathcal{A}_\alpha^{\leq -1}\}$. See also (5.3) for a reformulation of $\mathcal{A}_\alpha^{\leq -1}$, in line with [5, p. 175, 1.2.4]. If $\mathbb{C}\{t\}$ denotes the ring of germs of analytic functions at $t = \alpha$, it is well-known that

$$(1.15) \quad \mathcal{A}_\alpha^{\leq -1} \cap \mathbb{C}\{\hat{\tau}\} = \{0\}.$$

Definition 1.2. Let $q = e(\tau)$, $\tau \in \mathcal{H}$, $a = e(\alpha)$, $\alpha \in \partial\mathcal{H} \cup \{i\infty\}$, let $f \in \mathcal{A}_a^r$.

- (1) One says that $f(q)$ admits an **exponential**-convergent expansion as $q \xrightarrow{a,r} a$ and one writes $f \in \mathfrak{E}_a$, if there exist $(c, v, \delta) \in \mathbb{C} \times \mathbb{Q} \times \mathbb{R}$, $f_0(t) \in t\mathbb{C}\{t\}$ and $f_1 \in \mathcal{A}_a^{\leq -1}$ such that the following relation holds for $\tau \xrightarrow{a,v} \alpha$:

$$(1.16) \quad f(q) = c \left(\frac{i}{\hat{\tau}} \right)^v e\left(\frac{\delta}{\hat{\tau}} + f_0(\hat{\tau})\right) (1 + f_1(q)).$$

- (2) If, moreover, $f_0(t) \in t\mathbb{C}[t]$ in (1.16), one will say that $f(q)$ admits an **exponential**-finite expansion as $q \xrightarrow{a,r} a$ and write $f \in \mathfrak{E}_a$.

Let \mathbb{U} be the set of all roots of unity. In view of the above definition combined with the relations stated in (1.14) and in (1.15), it is easy to see the following

Remark 1.1. (1) For any $\zeta = e(r) \in \mathbb{U} \cup \{0\}$, it follows that $\mathfrak{T}_\zeta \subset \mathfrak{E}_\zeta \subset \mathfrak{C}_\zeta$.

- (2) The data (c, v, δ, f_0) appearing in (1.16) is unique. \square

A key point in the course of the proof of Theorem 1.1 will, therefore, consist of proving that if $(x; q^2)_\infty \in \mathfrak{E}_\zeta$, then $x \in \{\pm q, \pm q^2\}$ necessarily. This will be obtained by means of the following

Theorem 1.2 (Main Theorem). Let $r \in \mathbb{Q} \cap [0, 1)$ and $\zeta = e(r) \in \mathbb{U}$. Given any germ ω of analytic function at $\tau = r$ in \mathbb{C} such that $\omega(r) \in \mathbb{R}$, one has $(\omega(\tau) | \tau)_\infty \in \mathfrak{C}_\zeta$ if, and only if, $\omega(\tau) = \frac{1}{2}(n + m\tau)$ for some $(n, m) \in \mathbb{Z}^2$.

In the above, the notation $(\cdot | \cdot)_\infty$ is defined as follows: if $x = e(z)$ and $q = e(\tau)$, we write

$$(1.17) \quad (z | \tau)_\infty = (x; q)_\infty.$$

When $r = 0$, Theorem 1.2 will be deduced, in §5.4, from the modular-like formula established by Theorem 1.3 in the below. The general case $r \in (0, 1) \cap \mathbb{Q}$ will be reduced to the previous case $r = 0$ by considering a finite number of linear fractional transformations; see §6.2.

1.3. Stokes phenomenon and analytic obstruction. Let us consider Theorem 1.2 with $r = 0$ and $\zeta = 1$, and let $f(q) = (\omega(\tau) | \tau)_\infty$. As $\omega(\tau)$ is a germ of analytic function at $\tau = 0$, $f(q)$ is defined only for τ near zero in \mathcal{H} . This implies that f is defined and analytic near $q = 1$ but $|q| < 1$. By (1.16), if $f \in \mathfrak{C}_1$, then one can find $(c, v, \delta, \tilde{f}_1) \in \mathbb{C} \times \mathbb{Q} \times \mathbb{R} \times \mathcal{A}_1^{\leq -1}$ such that

$$(1.18) \quad F_0(q) := c \left(\frac{\tau}{i} \right)^v e(-\frac{\delta}{\tau}) (1 + \tilde{f}_1(q)) f(q)$$

can be analytically continued into a germ of analytic function at $\tau = 0$ in \mathbb{C} , that is to say, for q in some neighborhood of $q = 1$ in \mathbb{C} . Here, it may be worthwhile to notice that \tilde{f}_1 denotes certain function defined only for $q \xrightarrow{a.r.} 1$ inside the unit disc $|q| < 1$, with $(1 + \tilde{f}_1(q))(1 + f_1(q)) = 1$ and $f_1 \in \mathcal{A}_1^{\leq -1}$.

However, by putting $z = \omega(\tau) - \frac{1}{2}$ into the formula in (1.22) in the below, we will see that the function $F_0(q)$ like as in (1.18) can be expressed by means of a Laplace-type integral; see (1.21). It is known that such integral admits an asymptotic expansion that is generally a divergent power series, and this yields the so-called Stokes phenomenon. The corresponding function becomes analytic in some neighborhood of $\tau = 0$ only when the Stokes multiplier is trivial, which gives the necessary condition stated by Theorem 1.2 for f to belong in \mathfrak{C}_1 . All this will be explained in Sections 3 and 5.

In order to get an expression as in (1.18) for any Euler infinite product, we let

$$(1.19) \quad \mathbf{B}(t) = \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2},$$

what is an exponential generating function of Bernoulli numbers; see (3.1). Given $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$, define

$$(1.20) \quad V_d = V_d^+ = \{\tau \in \mathbb{C} : \Re(\tau e^{id}) > 0\}, \quad H_d = (V_d - \frac{1}{2}) \cap (-V_d + \frac{1}{2}),$$

and consider the integral

$$(1.21) \quad B_d(z | \tau) = \int_0^{\infty e^{id}} \frac{e^{zu} - e^{-zu}}{e^{u/2} - e^{-u/2}} \mathbf{B}(\tau u) \frac{du}{u}$$

for $(z, \tau) \in H_d \times V_d$. One can notice that H_d represents the strip delimited by two straight lines passing respectively through the points of affix $\pm \frac{1}{2}$ and remaining all perpendicular with the direction of argument $-d$. See Figure 1.

The following result will play an important role for the study of the asymptotic behavior of an Euler infinite product.

Theorem 1.3. *Let $\Omega = \cup_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})} H_d \times V_d$ and let $B(z | \tau)$ be the analytic continuation in Ω of the function $B_d(z | \tau)$ given by (1.21) for $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The following relation holds for all $(z, \tau) \in \Omega$ with $\tau \in \mathcal{H}$:*

$$(1.22) \quad (z + \frac{1}{2} | \tau)_\infty = A(z | \tau) e^{L_2(z | \tau) + B(z | \tau)} \left(\frac{z - 1/2}{\tau} \middle| -\frac{1}{\tau} \right)_\infty,$$

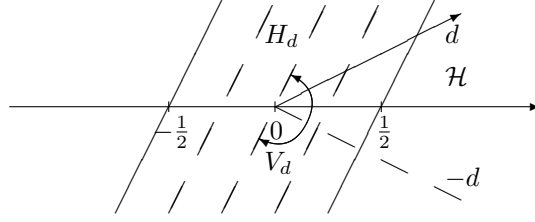


FIGURE 1. The domain H_d represents the intersection of the half-plane $V_d - \frac{1}{2}$ with its opposite half-plane $-V_d + \frac{1}{2}$

where

$$A(z | \tau) = e\left(-\frac{\tau}{24}\right) \sqrt{1 + e(z)}$$

and

$$(1.23) \quad L_2(z | \tau) = \frac{\text{Li}_2(-e(z))}{2\pi i \tau} = \frac{i}{2\pi \tau} \int_0^1 \log(1 + e(z)t) \frac{dt}{t}$$

In (1.23) in the above, \log means the principal branch of the complex logarithm function; Li_2 denotes the traditional dilogarithm function as considered in [11]. The proof of the above theorem will be given in §3.4.

In [13], we found a modular-type relation for Euler infinite product $(x; q)_\infty$, that is intimately linked with an asymptotic formula of Ramanujan. See Theorem 4.2 together with [3, p. 265 Entry 6 and p. 268 Entry 6']. In §4.4, we will show that this modular-type relation is equivalent to the statement given by Theorem 1.3.

1.4. Plan for the rest of the paper. This will be divided into five sections and the paper will be concluded by the proof of Theorems 1.1 and 1.2 in §6.3. Schematically, Sections 3 and 4 constitute one part that can be read together and independently of the rest of the paper, but they will be used in the sections that follow. Here are some key-stages of what follows.

In §2, we will start with some general remarks about Theta-type functions. Special properties of the Ramanujan's two-variables theta function $f(a, b)$ will be found in §2.2 for $a = a_0 q^\alpha$ and $b = b_0 q^\beta$. Thus, one can get, in Theorem 2.2, a family of Euler infinite products that have an exponential-finite expansion at a given root of unity. This gives a proof for one *simple* part of Theorem 1.2. After that, we will make use of the θ -modular formula to find that all composite Jacobi theta functions satisfy Definition 1.2; see Theorem 2.4.

Sections 3 and 4 are mainly devoted to two integrals related to the exponential generating function of Bernoulli numbers; see (3.2) and (1.21). These integrals can be read as of Laplace-type, and they depend of the choice of the path starting from 0 to infinity. Two types of singularities are contained there: fix and movable singularities. Theorem 3.1 says that the two *branches* of the function defined by the first integral have a difference that can be expressed in terms of Euler infinite products.

In §3.2, by considering the integral in (1.21) as a symmetrical form of the integral in (3.2), we shall propose an analysis in line with the so-called *Stokes' phenomenon*. This gives an expression relating any Euler infinite product into its *modular* counterpart; see Theorem 1.3. In a previous work [13] (see also [12]), we obtained a similar expression that makes an analytic sense to an asymptotic formula of Ramanujan. In

order to clarify the relation between this result and the above-mentioned Theorem 1.3, we consider, in §4, the analytic continuation of all these functions used there. Finally, it will be shown that these two expressions are equivalent; see Theorem 4.3.

It may be of interest to notice that the approaches used in our previous paper [13] allow one to unify eta and theta-modular relations and other modular-type relations for Lambert series. But in the present paper, Theorem 1.3 is established with the help of classical modular formulas.

The remaining two sections are putting behind the *hard* part of the proof of Theorem 1.2. In §5, we shall give the necessary condition to an Euler infinite product to have an exponential-convergent expansion at $q = 1$, *i.e.* $\tau = 0$. The crucial point in our strategy is to express this condition in terms of the analyticity (obstacle) of the functions introduced above with Bernoulli integrals. Fortunately, one can make use of Ramis-Sibuya's Theorem to find that all these functions have in general a divergent but Gevrey asymptotic expansion; see Theorems 5.2, 5.3 and 5.4. In this way, one obtains a few family of Euler infinite products that belong to \mathfrak{C}_1 , as stated by Theorem 5.1.

The lastest section will permit to pass any root of unity to 1 by means of modular transforms based upon continued fractions. For doing this, we shall investigate, in §6.1, the degenerate cases of Bernoulli integrals when τ goes to the real axis. This is to say that the general elliptic curve will be replaced by a circle, with only a period. Thanks to Theorems 6.2 and 6.3, one knows that, while τ tends to a real point from \mathcal{H} , the related Bernoulli integrals remain analytic and one can iterated modular transforms until the last τ -variable goes to zero. This is done, in §6.2, by representing any rational number into continued fraction. In 6.3, we outline a summary about previously-stated results related to Theorem 1.2 and complete, therefore, the proof of Theorem 1.1 with the help of Theorem 1.2.

2. THETA-TYPE FUNCTIONS AND EXPONENTIAL-CONVERGENT EXPANSIONS

Let $(z|\tau)_\infty$ be as in (1.17). For any $n \in \mathbb{Z}_{\geq 0}$, it follows that

$$(2.1) \quad (z|\tau)_\infty = (z|\tau)_n (z + n\tau|\tau)_\infty,$$

where $(z|\tau)_0 = 1$ and $(z|\tau)_n = (1 - e(z)) \dots (1 - e(z + (n-1)\tau))$ for $n \geq 1$. This is equivalent to say that

$$(x;q)_\infty = (x;q)_n (xq^n|q)_\infty,$$

where $(x;q)_n = (z|\tau)_n$. Similarly like as in (1.17), define

$$(2.2) \quad \theta(z|\tau) = \theta(x;q)$$

for $x = e(z)$ and $q = e(\tau)$.

2.1. Preliminary commentaries on Theta-type functions. As before, let \mathbb{U} denote the set of roots of unity. We start with commentaries about the real meaning of the relation given in (1.13) as $q \xrightarrow{a.r.} 0$. First of all, given $\zeta \in \mathbb{U}$, we will say that “ $q \xrightarrow{a.r.} 0$ ” along the direction ζ if there exists $(\epsilon, R) \in (0, \pi) \times \mathbb{R}_{>0}$ such that $q \rightarrow 0$ in the open sector $V(\zeta; \epsilon, R)$ defined in the punctuated plane $\mathbb{C} \setminus \{0\}$ as follows:

$$(2.3) \quad V(\zeta; \epsilon, R) = \{q \in \mathbb{C}^* : |\arg \frac{q}{\zeta}| < \epsilon, |q| < R\}.$$

When $V = V(\zeta; \epsilon, R)$, we let $\mathfrak{T}(V)$ denote the set of analytic functions satisfying the condition in (1.13) for $q \rightarrow 0$ in V . Consider $f \in \mathfrak{T}(V)$; since $\hat{\tau} = -\frac{1}{\tau}$ and $\tau = \frac{1}{2\pi i} \log q$, one obtains that

$$(2.4) \quad f(q) = (2\pi \log \frac{1}{q})^v e^{\lambda'/\log q} \left(\sum_{k \in I \cap (-\infty, N]} \gamma(k) q^k + o(q^N) \right),$$

where $\lambda' = 4\pi^2 \lambda$. In view of the fact that $e^{\lambda'/\log q} = 1 + o(q^N)$ for all $N \in \mathbb{Z}_{\geq 0}$ as $|q| \rightarrow 0$, it follows that the exponent λ' in (2.4) can be arbitrarily chosen. Furthermore, in the whole theory of q -series including mock-theta functions, the functions are generally analytic at $q = 0$ in the unit disc. For this reason, it will be convenient to take simply $\lambda = \lambda' = 0$. In this way, (1.13) will be read as follows for $q \rightarrow 0$ in V :

$$(2.5) \quad f(q) = (2\pi \log \frac{1}{q})^v \left(\sum_{k \in I \cap (-\infty, N]} \gamma(k) q^k + o(q^N) \right).$$

Similarly, (1.16) needs to be interpreted in the following fashion, with $f_0 = 0$:

$$(2.6) \quad f(q) = c (2\pi \log \frac{1}{q})^v q^{-\delta} (1 + o(q^\kappa)),$$

where $\kappa > 0$. This implies that $\mathfrak{E}(V) = \mathfrak{E}(V)$. Moreover, one can notice that (2.6) remains in accordance with (1.15); see Remark 1.1 (2).

The “global sheaves” located at zero for all directions can be defined as follows:

$$(2.7) \quad X_0 = \cup_{(\epsilon, R) \in \mathbb{R}_{>0}^2} (\cap_{\zeta \in \mathbb{U}} X(V(\zeta; \epsilon, R)))$$

for $X \in \{\mathfrak{T}, \mathfrak{E}, \mathfrak{C}\}$. It is easy to see the following

Remark 2.1. (1) One has $\cup_{n \in \mathbb{Z}_{\geq 0}} q^{-n} \mathbb{C}\{q\} \subset \mathfrak{T}_0 \subset \mathfrak{E}_0 = \mathfrak{C}_0$.
 (2) One has $\mathbb{C}(q) \subset \mathfrak{E}_\zeta$ and $\mathbb{C}(q) \cap \mathfrak{E}_\zeta = \mathbb{C}(q) \cap \mathfrak{T}_\zeta = \mathbb{C}$ for all $\zeta \in \mathbb{U}$.

In the above, $\cup_{n \in \mathbb{Z}_{\geq 0}} q^{-n} \mathbb{C}\{q\}$ is the fraction field of the ring of the germs of analytic functions at $q = 0$, which is simply the set of all meromorphic functions at the origin. To see the second assertion of this remark, one can easily observe that all meromorphic function at $q = \zeta \in \mathbb{U}$ belongs to \mathfrak{E}_ζ . By considering the logarithm of any non zero $f(q) \in \mathbb{C}(q)$, one obtains that $\mathbb{C}(q) \cap \mathfrak{E}_\zeta = \mathbb{C}(q) \cap \mathfrak{T}_\zeta = \mathbb{C}$ for all $\zeta \in \mathbb{U}$.

We will call *ramification operator of order $\nu \neq 0$* the application ρ_ν that sends each function $f(q)$ into $\rho_\nu f(q) = f(q^\nu)$. When $\zeta \in \mathbb{U}$, we write

$$(2.8) \quad \rho_{\nu, \zeta} : f(q) \mapsto \rho_{\nu, \zeta} f(q) = f(\zeta q^\nu).$$

It follows that $\rho_\nu = \rho_{\nu, 1}$.

Given $X \in \{\mathfrak{T}, \mathfrak{E}, \mathfrak{C}\}$, define

$$(2.9) \quad X = \cap_{\zeta \in \mathbb{U} \cup \{0\}} X_\zeta, \quad X^* = X \setminus \{0\}.$$

It may be useful to observe that $X^* = \cap_{\zeta \in \mathbb{U} \cup \{0\}} X_\zeta^*$ where $X_\zeta^* = X_\zeta \setminus \{0\}$, due to the analytic character of the functions included in each set under study.

Lemma 2.1. Let $X \in \{\mathfrak{T}, \mathfrak{E}, \mathfrak{C}\}$.

- (1) X is stable under any given operator $\rho_{\nu, \zeta}$ with $(\nu, \zeta) \in \mathbb{Q}_{>0} \times \mathbb{U}$.
- (2) Given $\zeta \in \mathbb{U} \cup \{0\}$, X_ζ^* constitutes a multiplicative group.

Proof. (1) Using (2.8), one finds that $X_\zeta = \rho_{\nu, \zeta_1}(X_{\zeta_2})$ for $(\zeta, \zeta_1, \zeta_2, \nu) \in \mathbb{U}^3 \times \mathbb{Q}_{>0}$ such that $\zeta_1 \zeta_2' = \zeta$. Thus, one obtains that

$$\rho_{\nu, \zeta}(\cap_{\zeta_2 \in \mathbb{U}} X_{\zeta_2}) = \cap_{\zeta_2 \in \mathbb{U}} X_{\zeta \zeta_2'} = \cap_{\zeta' \in \mathbb{U}} X_{\zeta'}.$$

This implies the invariance of X by $\rho_{\nu, \zeta}$, for $\rho_{\nu, \zeta} X_0 = X_0$ (see (2.7)).

(2) When $\zeta \in \mathbb{U}$, this comes directly from the respective definition of \mathfrak{T}_ζ , \mathfrak{E}_ζ and \mathfrak{C}_ζ in (1.13) and (1.16) with $f_0 \in t\mathbb{C}\{t\}$ or $t \in t\mathbb{C}[t]$. For $\zeta = 0$, see (2.5) – (2.7). This finishes the proof. \square

2.2. Case of Ramanujan's two-variables theta-function. Let $f(a, b)$ be the Ramanujan's two-variables theta-function given by (1.1).

Theorem 2.1. *Let $(\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{Q}^4$ such that $\alpha_1 + \beta_1 \in \mathbb{Q}_{>0}$, and let $a = e(\alpha_0 + \alpha_1\tau)$ and $b = e(\beta_0 + \beta_1\tau)$. Then $f(a, b) \in \mathfrak{T}$.*

Proof. Let $\theta(z|\tau)$ be as in (2.2), and let $\nu = \alpha_1 + \beta_1$. From (1.3), one deduces that $f(a, b) = \theta(\alpha_0 + \alpha_1\tau|\tau')$ with $\tau' = \alpha_0 + \beta_0 + \nu\tau$. Letting $\zeta = e(\alpha_0 + \alpha_1)$, one can find suitable $(\lambda, \mu) \in \mathbb{Q}^2$ such that

$$(2.10) \quad f(a, b) = \rho_{\nu, \zeta} g(q), \quad g(q) = \theta(\lambda + \mu\tau|\tau).$$

By [14, Theorem 1.1], one knows that $\theta(z|\tau) \in \mathfrak{T}$ for any given $z \in \mathbb{R} \oplus \mathbb{R}\tau$. So, considering both Lemma 2.1 (1) and equality (2.10) gives that $f(a, b) \in \mathfrak{T}$. \square

Consequently, one obtains easily the following

Corollary 2.1. (1) *One has $\Pi(q) \subset \Phi(q) \subset \mathfrak{T}^* \subset \mathfrak{T}_\zeta^*$ for all $\zeta \in \mathbb{U}$.*

(2) *If $x \in \{\pm q, \pm q^2\}$, then $(x; q^2)_\infty \in \mathfrak{E}^*$.*

Proof. (1) It suffices to remember that $\pi(q) = f(-q) = f(q, q^2)$ as stated in (1.5).

(2) By (1.11) and (1.12), it follows that $(x; q^2) \in \Pi(q)$, this together with Remark 1.1 (1) implies the wanted result. \square

We shall prove the part “if” of Theorem 1.2 in the below.

Theorem 2.2. *Assume $\omega(\tau) = \frac{1}{2}(n + m\tau)$, where $n \in \{0, 1\}$ and $m \in \mathbb{Z}$.*

(1) *One has $(\omega(\tau)|\tau)_\infty \in \mathfrak{C}$.*

(2) *Moreover, $(\omega(\tau)|\tau)_\infty \in \mathfrak{E}_\zeta^*$ for some $\zeta \in \mathbb{U}$ if, and only if, $\omega(\tau) \in \{\tau, \frac{\tau}{2}, \frac{1}{2} + \tau, \frac{1}{2} + \frac{\tau}{2}\}$.*

Proof. Let $\Delta_\tau = \{\tau, \frac{\tau}{2}, \frac{1}{2} + \tau, \frac{1}{2} + \frac{\tau}{2}\}$. By applying Corollary 2.1 (2) and replacing q^2 with q there, one obtains from Lemma 2.1 (2) that, for $z \in \Delta_\tau$, $(z|\tau)_\infty \in \mathfrak{E}^*$, so $(z|\tau)_\infty \in \mathfrak{E}_\zeta^*$ for all $\zeta \in \mathbb{U}$.

Now, assume that $\omega(\tau) = \frac{1}{2}(n + m\tau)$, where $n \in \{0, 1\}$ and $m \in \mathbb{Z}$. One can find a unique $z \in \Delta_\tau$ such that $\omega(\tau) = z + m'\tau$ with some $m' \in \mathbb{Z}$. Let

$$g(q) = \frac{(\omega(\tau)|\tau)_\infty}{(z|\tau)_\infty}.$$

By using (2.1), one finds that $g(q) \in \mathbb{C}(\sqrt{q})$.

(1) One remembers that $(z|\tau)_\infty \in \mathfrak{E}^* \subset \mathfrak{E}^*$. Since $g(q)$ represents a rational function of the variable \sqrt{q} , one obtains from Lemma 2.1 (2) and (3) that $g(q) \in \mathfrak{C}$, so $(\omega(\tau)|\tau)_\infty \in \mathfrak{C}$.

(2) By using Lemma 2.1 (2) and Remark 2.1 (2), one finds that $g(q) \in \mathfrak{E}_\zeta$ for some $\zeta \in \mathbb{U}$ if, and only if, $g(q)$ is a constant function. In this way, one gets that

$(\omega(\tau) | \tau)_\infty \in \mathfrak{E}_\zeta$ if, and only if, $\omega(\tau) \in \Delta_\tau$, which completes the proof of Theorem 2.2. \square

2.3. Exponential-convergent behaviour of composite Euler infinite products. Thanks to Euler [1, p. 490, Corollary 10.2.2 (b)], one can write that, for all $x \in \mathbb{C}$,

$$(2.11) \quad (x; q)_\infty = \sum_{n \geq 0} \frac{q^{n(n-1)/2}}{(q; q)_n} (-x)^n.$$

Lemma 2.2. *Given $\omega \in \mathbb{C}\{\tau\}$ and $\delta \in \mathbb{C}$, if $\omega(0) > 0$, then $(-\frac{\omega(\tau)}{\tau} | \delta - \frac{1}{\tau})_\infty \in \mathfrak{E}_1$.*

Proof. By using (2.11), one gets that

$$(-\frac{\omega(\tau)}{\tau} | \delta - \frac{1}{\tau})_\infty = 1 + \sum_{n \geq 1} \frac{(-1)^n e(-\frac{n\omega(\tau)}{\tau})}{(\delta - \frac{1}{\tau} | \delta - \frac{1}{\tau})_n} e(\frac{n(n-1)}{2}(\delta - \frac{1}{\tau})),$$

where $(\cdot | \cdot)_n$ is defined as in (2.1). Since $e(\delta - \frac{\nu}{\tau}) \in \mathcal{A}_0^{\leq -1}$ for any $\nu > 0$, it follows that, when $\omega(0) > 0$,

$$\begin{aligned} (-\frac{\omega(\tau)}{\tau} | \delta - \frac{1}{\tau})_\infty &= 1 - \frac{e(-\frac{\omega(\tau)}{\tau})}{1 - e(\delta - \frac{1}{\tau})} \mod \mathcal{A}_0^{\leq -1} \\ &= 1 \mod \mathcal{A}_0^{\leq -1}. \end{aligned}$$

This finishes the proof. \square

Proposition 2.1. *Let $\omega \in \mathbb{C}\{\tau\}$, $\delta \in \mathbb{C}$, and let*

$$f(q) = (-\frac{\omega(\tau)}{\tau} | \delta - \frac{1}{\tau})_\infty$$

for all $q = e(\tau)$ with $\tau \xrightarrow{a.v.} 0$ in \mathcal{H} . One supposes that $\omega(0) \in \mathbb{R}$.

- (1) *One has $f \in \mathfrak{E}_1$.*
- (2) *One has $f \in \mathfrak{E}_1$ in each of the following cases:*
 - (a) $\omega(0) > 0$.
 - (b) $\omega \in \mathbb{C}[\tau]$ and $\omega(0) \in \mathbb{R}_{<0} \setminus \mathbb{Z}$.
 - (c) $\omega = a + b\tau \in \mathbb{Z}_{\leq 0} \oplus (\mathbb{C} \setminus \mathbb{Z})\tau$.

Proof. Let n be the smallest non-negative integer such that $n = n_\omega > -\omega(0)$. For simplify, we define

$$\omega_\infty(\tau) = \omega(\tau) + n(1 - \delta\tau), \quad f_\infty(q) = (-\frac{\omega_\infty(\tau)}{\tau} | \delta - \frac{1}{\tau})_\infty$$

and

$$f_k(q) = 1 - e(k\delta - \frac{\omega(\tau) + k}{\tau}), \quad k = 0, \dots, n-1.$$

Thus, one can write

$$(2.12) \quad \begin{aligned} f(q) &= (-\frac{\omega(\tau)}{\tau} | \delta - \frac{1}{\tau})_n (-\frac{\omega_\infty(\tau)}{\tau} | \delta - \frac{1}{\tau})_\infty \\ &= f_0(q) f_1(q) \dots f_{n-1}(q) f_\infty(q). \end{aligned}$$

Since $\omega_\infty(0) > 0$, Lemma 2.2 implies that $f_\infty \in \mathfrak{E}_1 \subset \mathfrak{E}_1$. In view of Lemma 2.1 (1), one needs only to show that the statements of Proposition 2.1 remain true with each f_k instead of f .

If $w(0) + k < 0$, one writes

$$1 - e\left(-\frac{\omega(\tau) + k}{\tau}\right) = -e\left(-\frac{\omega(\tau) + k}{\tau}\right) \left(1 - e\left(\frac{\omega(\tau) + k}{\tau}\right)\right).$$

This gives that

$$(2.13) \quad \log(f_k(q)) = -2\pi i \frac{\omega(\tau) + k}{\tau} \pmod{\mathbb{C} \oplus \mathcal{A}_0^{\leq -1}}.$$

Otherwise, if $\omega(0) + k = 0$, one obtains that $\omega(0) \in \mathbb{Z}_{\leq 0}$ and $k = n_\omega = -\omega(0)$. When $f_k \neq 0$, it follows that $\omega(\tau) \neq \ell\tau$ for any $\ell \in \mathbb{Z}$. In this case, one finds that

$$(2.14) \quad \log(f_k(q)) = \log\left(1 - e\left(-\frac{\omega(\tau) - \omega(0)}{\tau}\right)\right) = \nu \log \tau \pmod{\mathbb{C}\{\tau\}},$$

where $\nu \in \mathbb{Z}_{\geq 0}$.

(1) By considering both relations in (2.13) and (2.14), it follows that $f_k \in \mathfrak{E}_1$. This implies the first assertion of Proposition 2.1.

(2) First of all, if $\omega(0) > 0$, one notices that $n_\omega \leq 0$ and the relation in (2.12) becomes $f = f_\infty$. Thus, $f \in \mathfrak{E}_1$ as stated in (2) (a).

When $\omega(0) \in \mathbb{R}_{<0} \setminus \mathbb{Z}$, it follows that $\omega(0) + k < 0$ for all integer $k \in [0, n)$. Thus, the relation in (2.13) implies that, as $\omega \in \mathbb{C}[\tau]$, $f_k \in \mathfrak{E}_1$. This gives the assertion in (2) (b).

Finally, if $\omega(0) \in \mathbb{Z}_{\leq 0}$ and $k = -\omega(0)$, considering the relation in (2.14) yields that $f_k \in \mathfrak{E}_1$ if, and only if, $\omega(\tau) - \omega(0) \in (\mathbb{C} \setminus \mathbb{Z})\tau$. This, together with (2.13), gives (2) (c) and then completes the proof of Proposition 2.1. \square

2.4. The exponential-finite behaviour of composite Jacobi Theta functions. Let $\theta(z|\tau)$ be as in (2.2). We shall understand in what manner the composite function of θ with a real analytic function behaves in terms of Definition 1.2.

Theorem 2.3. *Given any real-valued analytic function ω on \mathbb{R} , $\theta(\omega(\tau)|\tau) \in \mathfrak{E}_\zeta$ for all $\zeta \in \mathbb{U}$.*

The traditional four Jacobi theta-functions can be defined in terms of θ as follows:

$$(2.15) \quad \vartheta_1(z|\tau) = e\left(-\frac{z}{2} + \frac{\tau}{8} + \frac{1}{4}\right) \theta\left(z + \frac{1}{2}|\tau\right), \quad \vartheta_2(z|\tau) = e\left(-\frac{z}{2} + \frac{\tau}{8}\right) \theta(z|\tau),$$

$$(2.16) \quad \vartheta_3(z|\tau) = \theta\left(z + \frac{\tau}{2}|\tau\right), \quad \vartheta_4(z|\tau) = \theta\left(z + \frac{\tau}{2} + \frac{1}{2}|\tau\right).$$

For the definition of the above ϑ -functions, we are referred to [10, p. 464 & p. 487], where πz and $\pi\tau$ need to be read as z and τ , respectively.

By taking into account Lemma 2.1 (2), it is easy to see that Theorem 2.3 is equivalent to the following

Theorem 2.4. *All the statements of Theorem 2.3 remain true if the θ -function is replaced with anyone of the Jacobi ϑ -functions $\vartheta_j(z|\tau)$, $1 \leq j \leq 4$.*

The proof of Theorem 2.3 will be based upon both the theta-modular formula in (2.18) and Proposition 2.2 in the below. Indeed, let $\zeta = e(\frac{p}{m})$, where $(p, m) \in \mathbb{Z} \times \mathbb{Z}_{>0}$ and $p \wedge m = 1$; by convention, we write $0 \wedge 1 = 1$. We fix α, β in \mathbb{Z} in a

such manner that $\beta m - \alpha p = 1$; in other words, $M =: \begin{pmatrix} \alpha & -\beta \\ m & -p \end{pmatrix} \in SL(2; \mathbb{Z})$. As before, $\hat{\tau} = \tau - \frac{p}{m}$; it will be convenient to define

$$(2.17) \quad \tau' = M\tau = \frac{\alpha\tau - \beta}{m\tau - p} = -\frac{1}{m^2\hat{\tau}} + \frac{\alpha}{m}.$$

Lemma 2.3. *Let $p \in \mathbb{Z}$, $m \in \mathbb{Z}_{>0}$ such that $p \wedge m = 1$, and let τ' be as in (2.17). Then:*

$$(2.18) \quad \theta(z + \frac{1}{2} | \tau) = \frac{\epsilon_1}{\sqrt{m}} \sqrt{\frac{i}{\hat{\tau}}} e(-\frac{1}{2\hat{\tau}}(z + \frac{1}{2m})^2 + \frac{z}{2} - \frac{\hat{\tau}}{8}) \theta(\frac{z}{m\hat{\tau}} + \frac{1}{2} | \tau'),$$

where $\epsilon_1 = \epsilon_1(\alpha, -\beta, m, -p)$ denotes an 8-th root of unity.

Proof. This follows from [6, Theorem (2), p. 179]. See also [14, (1.20)]. \square

Proposition 2.2. *Given $\delta \in \mathbb{C}$ and $\omega(\tau) \in \mathbb{C}\{\tau\}$ with $\omega(0) \in \mathbb{R}$, one has $\theta(-\frac{\omega(\tau)}{\tau} | \delta - \frac{1}{\tau}) \in \mathfrak{E}_1$. Furthermore, when $\omega(0) \in \mathbb{R} \setminus \mathbb{Z}$ and $\omega \in \mathbb{C}[\tau]$, this function belongs to \mathfrak{E}_1 .*

Proof. Let $\tau' = \delta - \frac{1}{\tau}$. By the triple product formula (1.3), one can write

$$\theta(-\frac{\omega(\tau)}{\tau} | \delta - \frac{1}{\tau}) = f_0(q) f_1(q) f_2(q),$$

where

$$f_0(q) = (\tau' | \tau')_\infty, \quad f_1(q) = (-\frac{\omega(\tau)}{\tau} + \frac{1}{2} | \tau')_\infty, \quad f_2(q) = (\tau' + \frac{\omega(\tau)}{\tau} + \frac{1}{2} | \tau')_\infty.$$

One knows that $f_0(q) \in \mathfrak{T}$, so $f_0(q) \in \mathfrak{E}_1 \subset \mathfrak{E}_1$. In view of Lemma 2.1 (1), Proposition 2.2 follows by applying Proposition 2.1 to each of $f_1(q)$ and $f_2(q)$. \square

Proof of Theorem 2.3. Consider $\zeta = e(\frac{p}{m})$, with (p, m) as in Lemma 2.3. By using the relation $\tau = \hat{\tau} + \frac{p}{m}$, one finds that

$$(2.19) \quad \omega(\tau) + \frac{1}{2m} - \frac{1}{2} = \mu + \hat{\tau} F(\hat{\tau}),$$

where

$$(2.20) \quad \mu = \omega(\frac{p}{m}) + \frac{1}{2m} - \frac{1}{2}, \quad F(\hat{\tau}) = \frac{\omega(\hat{\tau} + p/m) - \omega(p/m)}{\hat{\tau}}.$$

Thus, it follows that

$$-\frac{1}{2\hat{\tau}} \left(\omega(\tau) + \frac{1}{2m} - \frac{1}{2} \right)^2 + \frac{1}{2} \left(\omega(\tau) - \frac{1}{2} \right) - \frac{\hat{\tau}}{8} = -\frac{\lambda_0}{\hat{\tau}} + c_0 + \varphi_0(\hat{\tau}),$$

where

$$\lambda_0 = \frac{\mu^2}{2}, \quad c_0 = \frac{1}{2} \left(\mu - \frac{1}{2m} \right) - \mu F(0) = -\frac{1}{4m} + \mu \left(\frac{1}{2} - \omega'(\frac{p}{m}) \right)$$

and

$$\varphi_0(\hat{\tau}) = -\mu (F(\hat{\tau}) - F(0)) - \frac{\hat{\tau} F(\hat{\tau})^2}{2} \in \hat{\tau} \mathbb{R}\{\hat{\tau}\}.$$

Replacing z with $\omega(\tau) - \frac{1}{2}$ in the modular relation (2.18) yields that

$$(2.21) \quad f(q) = C_1 \sqrt{\frac{i}{\hat{\tau}}} e(-\frac{\lambda_0}{\hat{\tau}} + \varphi_0(\hat{\tau})) f_1(q),$$

where C_1 and f_1 are given as follows:

$$C_1 = \frac{\epsilon_1 e(c_0)}{\sqrt{m}}, \quad f_1(q) = \theta\left(\frac{\omega(\tau) - 1/2}{m\hat{\tau}} + \frac{1}{2} \mid \tau'\right).$$

By using (2.17), one can write f_1 in the following manner:

$$(2.22) \quad f_1(q) = \theta\left(\frac{\omega_1(m^2\hat{\tau})}{m^2\hat{\tau}} \mid \frac{\alpha}{m} - \frac{1}{m^2\hat{\tau}}\right),$$

where

$$(2.23) \quad \omega_1(t) = m\omega\left(\frac{t+mp}{m^2}\right) + \frac{t-m}{2} \in \mathbb{R}\{t\}.$$

By using the following identification:

$$\tau \leftrightarrow m^2\hat{\tau}, \quad \omega \leftrightarrow \omega_1, \quad \delta \leftrightarrow -\frac{\alpha}{m},$$

one can apply Proposition 2.2 (1) to the above function f_1 defined in (2.22). This implies that $f_1 \in \mathfrak{C}_\zeta$. \square

3. STOKES' ANALYSIS AND MODULAR PROPERTIES ABOUT EULER INFINITE PRODUCTS

This section prepares §4 and §5, and aims at establishing an integral expression of the Euler infinite product $(x; q)_\infty$ by means of the so-called *Stokes' analysis*. Indeed, we shall show that Euler infinite product $(x; q)_\infty$ is closely related with an integral involving the exponential generating function of the Bernoulli numbers; see (1.21) and Definition 3.2. This integral contains two species of singularities, one of whom is fixed and the other one is mobile. By analyzing all these singularities, a formula will be found to make the link between $(x; q)_\infty$ and its *modular* counterpart $(x_1; q_1)_\infty$, in such a way that the modular part is considered as exponential small-remainder.

The principal result of this section is Theorem 1.3, that will play a central role in the proof of Theorem 1.2.

3.1. An integral associated with the exponential generating function of Bernoulli numbers. Following [1, Definition 1.2.3, p. 12], we let B_{2n} to denote the Bernoulli numbers for positive integers n :

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad \dots$$

Let \mathbf{B} be the function defined by (1.19). One can notice that \mathbf{B} represents the exponential generating function of (B_{2n}) as follows (*cit. loc.*):

$$(3.1) \quad \mathbf{B}(t) = \sum_{n \geq 1} \frac{B_{2n}}{(2n)!} t^{2n-1}.$$

In what follows, we shall consider the following integral:

$$(3.2) \quad b_d(z \mid \tau) = \int_0^{\infty e^{id}} \frac{e^{-zu} - 1}{e^u - 1} \mathbf{B}(\tau u) \frac{du}{u},$$

where d is a given number in $(-\frac{\pi}{2}, \frac{\pi}{2})$, z and τ are complex numbers such that the associated integral b_d converges.

As $\mathbf{B}(t) = O(t)$ at $t = 0$ in \mathbb{C} , the above integral in (3.2) converges at $u = 0$ for all values of (z, τ) in $\mathbb{C} \times \mathbb{C}$. With regard to the convergence at infinity, we define

$$(3.3) \quad V_d^+ = \{\tau \in \mathbb{C} : \Re(\tau e^{id}) > 0\}, \quad V_d^- = \{\tau \in \mathbb{C} : \Re(\tau e^{id}) < 0\}$$

and

$$(3.4) \quad U_d = \{z \in \mathbb{C} : z + 1 \in V_d^+\}.$$

By using (1.19), one finds that $\mathbf{B}(t) \rightarrow \pm 1/2$ when $\Re t \rightarrow \pm\infty$. Therefore, $b_d(z|\tau)$ defines two analytic functions, one in $U_d \times V_d^+$ and the other in $U_d \times V_d^-$.

Geometrically, U_d represents the half-plane containing the point at origin and delimited by the straight-line $-1 + e^{-d+\frac{\pi}{2}}\mathbb{R}$ while V_d^\pm are half-planes separated by the straight line $e^{i(-d+\frac{\pi}{2})}\mathbb{R}$. One can find that the interval $(-1, \infty)$ belongs to U_d for every argument $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$. See Figure 2.

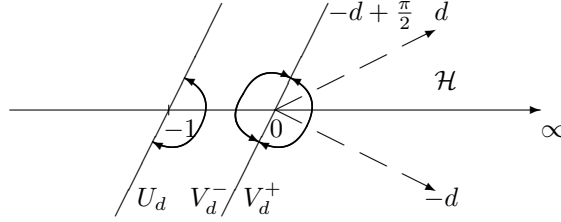


FIGURE 2. Half-planes U_d , V_d^- and V_d^+

Let

$$(3.5) \quad \mathcal{W}^+ = \cup_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})} U_d \times V_d^+, \quad \mathcal{W}^- = \cup_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})} U_d \times V_d^-.$$

Since

$$\cup_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})} V_d^+ = \mathbb{C} \setminus (-\infty, 0], \quad \cup_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})} V_d^- = \mathbb{C} \setminus [0, \infty)$$

it follows that

$$(3.6) \quad (-1, \infty) \times (\mathbb{C} \setminus (-\infty, 0]) \subset \mathcal{W}^+, \quad (-1, \infty) \times (\mathbb{C} \setminus [0, \infty)) \subset \mathcal{W}^-.$$

Definition 3.1. We call Bernoulli integral any expression $b_d(z|\tau)$ as given in (3.2) and we will define $b^+(z|\tau)$ and $b^-(z|\tau)$ respectively in \mathcal{W}^+ and \mathcal{W}^- by applying the analytic continuation procedure to $b_d(z|\tau)$ from $U_d \times V_d^+$ and $U_d \times V_d^-$ as d runs through $(-\frac{\pi}{2}, \frac{\pi}{2})$.

If no confusion is possible, we will write $b(z|\tau)$ instead of $b^+(z|\tau)$.

We shall make use of the following result to express the difference $b^+(z|\tau) - b^-(z|\tau)$ in their common domain $\mathcal{W}^+ \cap \mathcal{W}^-$.

Lemma 3.1. If $\tau \in \mathcal{H}$ and $z \in \mathcal{H}$, then:

$$(3.7) \quad \sum_{n \geq 1} \frac{1}{n} \frac{e(nz)}{1 - e(n\tau)} = -\log((z|\tau)_\infty)$$

Proof. This follows from [2, p. 36, (21.1)]. \square

By (3.6), one finds that $((-1, \infty) \times (\mathbb{C} \setminus \mathbb{R})) \subset \mathcal{W}^+ \cap \mathcal{W}^-$. In what follows, we will write $\mathbb{C} \setminus \mathbb{R} = \mathcal{H} \cup (-\mathcal{H})$.

Theorem 3.1. Let $(z, \tau) \in \mathcal{W}^+ \cap \mathcal{W}^-$. The following assertions hold.

(1) If $\tau \in \mathcal{H}$, then:

$$(3.8) \quad b^+(z|\tau) - b^-(z|\tau) = -\log \frac{(-(z+1)/\tau | -1/\tau)_\infty}{(-1/\tau | -1/\tau)_\infty}.$$

(2) If $\tau \in \mathcal{H}^- = (-\mathcal{H})$, then:

$$(3.9) \quad b^+(z|\tau) - b^-(z|\tau) = -\log \frac{((z+1)/\tau | 1/\tau)_\infty}{(1/\tau | 1/\tau)_\infty}.$$

Proof. (1) By the standard argument of analytical continuation, it suffices to prove (3.8) for $(z, \tau) \in (-1, \infty) \times \mathcal{H}$. Thus, one chooses $d_1 \in (-\frac{\pi}{2}, 0)$ and $d_2 \in (0, \frac{\pi}{2})$ such that τ belongs both $V_{d_1}^+$ and $V_{d_2}^-$. The contour integral in (3.2) allows one to write that

$$(3.10) \quad \begin{aligned} b^+(z|\tau) - b^-(z|\tau) &= B_{d_1}^+(z|\tau) - B_{d_2}^-(z|\tau) \\ &= \left(\int_0^{\infty e^{id_1}} - \int_0^{\infty e^{id_2}} \right) \frac{e^{-zu} - 1}{e^u - 1} \mathbf{B}(\tau u) \frac{du}{u}. \end{aligned}$$

Since both d_1 and d_2 belong to $(-\frac{\pi}{2}, \frac{\pi}{2})$, the two half straight-lines used in the contour-integral (3.10) are separated in the u -plane by the half straight-line ℓ_τ defined as follows:

$$\ell_\tau = \{u \in \mathbb{C}^* : \Re(\tau u) = 0, \Re u > 0\};$$

see Figure 3.

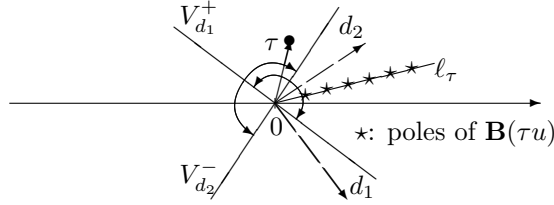


FIGURE 3. τ belongs to the common domain $V_{d_1}^+ \cap V_{d_2}^-$ while the directions d_1 and d_2 are separated by the half-line ℓ_τ

By observing that the function $\mathbf{B}(\tau u)$ admits simple poles $u = 2n\pi i/\tau$ ($n \in \mathbb{Z}_{>0}$) on the line ℓ_τ , applying Residues Theorem to the second member of (3.10) yields that

$$(3.11) \quad \begin{aligned} b^+(z|\tau) - b^-(z|\tau) &= \sum_{n \geq 1} \frac{1}{n} \frac{e(-nz/\tau) - 1}{e(n/\tau) - 1} \\ &= \sum_{n \geq 1} \frac{1}{n} \frac{e(-n(z+1)/\tau)}{1 - e(-n/\tau)} - \sum_{n \geq 1} \frac{1}{n} \frac{e(-n/\tau)}{1 - e(-n/\tau)}. \end{aligned}$$

By using the relation in (3.7), the above expression in (3.11) implies that

$$b^+(z|\tau) - b^-(z|\tau) = -\log\left(\left(-\frac{z+1}{\tau} \mid -\frac{1}{\tau}\right)_\infty\right) + \log\left(\left(-\frac{1}{\tau} \mid -\frac{1}{\tau}\right)_\infty\right),$$

so that one obtains (3.8).

(2) When $\tau \in \mathcal{H}^-$, the above proof can be adopted as follows: choose $d_1 \in (0, \frac{\pi}{2})$ and $d_2 \in (-\frac{\pi}{2}, 0)$, and observe that the simple poles of $\mathbf{B}(\tau u)$ to which Resides

Theorem is applied become $u = -2n\pi i/\tau$ ($n \in \mathbb{Z}_{>0}$). An direct calculation implies finally (3.9), what ends the proof of Theorem 3.1. \square

3.2. A symmetrical Bernoulli integral. Consider the integral given in (1.21) for $B_d(z|\tau)$, where $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$. One can observe two species of singularities for this integral:

- (1) The fixed singularities over the imaginary axis $i\mathbb{R}$, due to the simple zeros of the denominator $e^{u/2} - e^{-u/2}$.
- (2) The mobile or parametric singularities that depend of the value of τ , due to the simple poles of the generating function $\mathbf{B}(\tau u)$.

The choice of the direction argument d needs to avoid the first singularities; once a suitable argument d given, the analytic continuation on the corresponding sectors V_d^\pm is made up of an analysis about the second family of singularities. See Figure 4 in the below, which is to be compared with Figure 3.

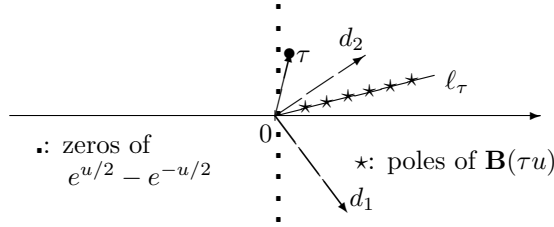


FIGURE 4. Fixed singularities on the imaginary axis and mobile singularities depending of τ viewed in the u -plane

We shall explore these singularities to find *global* relations about the function $B(z|\tau)$ considered in Theorem 1.3. One will see that each of both types of singularities allows one to get a functional equation, as stated in Theorems 3.2 and 3.3 respectively. Our approach is inspired by the so-called *Stokes' analysis* for the analytic theory of ordinary differential equations; see [?].

First of all, comparing this integral $B_d(z|\tau)$ with $b_d(z|\tau)$ in (3.2) yields

$$(3.12) \quad B_d(z|\tau) = b_d(-z - \frac{1}{2}|\tau) - b_d(z - \frac{1}{2}|\tau).$$

One deduces that $B_d(z|\tau)$ is well-defined for all (z, τ) such that $\tau \in V_d^\pm$ and $\pm z + 1/2 \in V_d^+$ or, equivalently, for $(z, \tau) \in H_d \times V_d^\pm$. See (1.20) for the definition of H_d and V_d , where $V_d = V_d^+$.

In this way, one obtains two analytic functions whose domains of definition are respectively $H_d \times V_d^+$ and $H_d \times V_d^-$; these functions will be denoted by $B_d^+(z|\tau)$ and $B_d^-(z|\tau)$.

Lemma 3.2. *The following relations hold for all $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$:*

$$(3.13) \quad B_d^+(z|\tau) = B_{d+\pi}^+(z|-\tau) = -B_{d+\pi}^-(z|\tau)$$

and

$$(3.14) \quad B_d^-(z|\tau) = B_{d+\pi}^-(z|-\tau) = -B_{d+\pi}^+(z|\tau).$$

Proof. By (3.3) and (1.20), it follows that $V_{d+\pi}^+ = V_d^- = -V_d^+$ and $H_{d+\pi} = H_d$ for all $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, one deduces (3.13) and (3.14) directly from the identity $B(-t) = -B(t)$. \square

The symmetry stated in Lemma 3.2 allows us to deal with only the case of $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$ most of the time. Thus, as what happened for the functions $b^\pm(z|\tau)$, the families $(B_d^+)_{d \in I}$ and $(B_d^-)_{d \in I}$, where $I = (-\frac{\pi}{2}, \frac{\pi}{2})$, yields two *global* functions by the analytic continuation procedure.

Definition 3.2. *Let*

$$(3.15) \quad \Omega^+ = \cup_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})} H_d \times V_d^+, \quad \Omega^- = \cup_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})} H_d \times V_d^-.$$

We will denote by $B^+(z|\tau)$ and $B^-(z|\tau)$ the respective function defined in Ω^+ and Ω^- by the corresponding family $(B_d^+)_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})}$ and $(B_d^-)_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})}$. If there is no possible confusion, we will simply write $B(z|\tau)$ instead of $B^+(z|\tau)$.

By (3.12), it follows that

$$(3.16) \quad B^\pm(z|\tau) = b^\pm(-z - \frac{1}{2}|\tau) - b^\pm(z - \frac{1}{2}|\tau).$$

3.3. Two types of singularities for Bernoulli integral. Let us start with the fixed singularities contained in the integral (1.21) that defines $B_d(z|\tau)$. If one writes

$$V^\pm = \cup_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})} V_d^\pm,$$

one can easily see that

$$V^+ = \mathbb{C} \setminus (-\infty, 0], \quad V^- = \mathbb{C} \setminus [0, +\infty).$$

So, the common domain $V^+ \cap V^-$ contains two components as follows:

$$(3.17) \quad V^+ \cap V^- = \mathcal{H} \cup (-\mathcal{H}).$$

Theorem 3.2. *The following relation holds for all $(z, \tau) \in \Omega^+ \cap \Omega^-$ such that $\tau \in \mathcal{H}$:*

$$(3.18) \quad B^+(z|\tau) + B^-(z|\tau) = -\pi iz - \frac{\beta(z)}{2\pi i \tau} + \gamma(z|\tau),$$

where $\beta(z)$ and $\gamma(z|\tau)$ are given as follows:

$$(3.19) \quad \beta(z) = \frac{\pi^2}{6} - 2\pi^2 z^2 + 2\text{Li}_2(-e(z))$$

and

$$(3.20) \quad \gamma(z|\tau) = \log \frac{(z + 1/2|\tau)_\infty}{(-z + 1/2|\tau)_\infty}.$$

Proof. Let $\Delta = (-1/2, 1/2) \times \mathcal{H}$; by (1.20) and (3.15), one can observe that $\Delta \subset (\Omega^+ \cap \Omega^-)$. By taking into account the standard argument of analytic continuation, we shall only prove Theorem 3.2 in Δ .

Let $(z, \tau) \in \Delta$. By (3.17), $\tau \in V^+ \cap V^-$, thus both $B^+(z|\tau)$ and $B^-(z|\tau)$ can be expressed by means of the integral in (1.21) as follows:

$$(3.21) \quad B^+(z|\tau) = B_{d_1}^+(z|\tau), \quad B^-(z|\tau) = B_{d_2}^-(z|\tau),$$

where $d_1 \in (-\frac{\pi}{2}, 0)$ and $d_2 \in (0, \frac{\pi}{2})$. By (3.14), it follows that

$$B_{d_2}^-(z|\tau) = -B_{d_2'}^+(z|\tau), \quad d_2' = d_2 + \pi \in (\pi, \frac{3\pi}{2}).$$

This implies that

$$B^+(z|\tau) + B^-(z|\tau) = B_{d_1}^+(z|\tau) - B_{d_2'}^+(z|\tau)$$

$$= \left(\int_0^{\infty e^{id_1}} - \int_0^{\infty e^{id'_2}} \right) \frac{e^{zu} - e^{-zu}}{e^{u/2} - e^{-u/2}} \mathbf{B}(\tau u) \frac{du}{u}.$$

In the above, d_1 can be chosen enough near to $-\frac{\pi}{2}$ while d_2 can be infinitely closed to $\frac{\pi}{2}$, so that d_1 and d'_2 are “infinitely near” one to other but are separated by the axis $(0, -i\infty)$. See Figure 5 in the below.

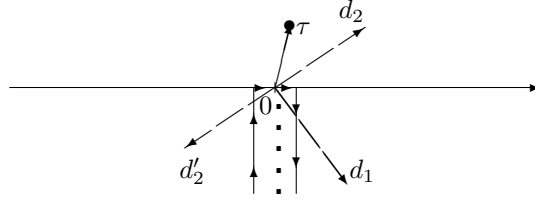


FIGURE 5. The integral-contour rotates around the negative imaginary-axis

Therefore, by Residues Theorem, one finds that

$$\begin{aligned} B^+(z|\tau) + B^-(z|\tau) &= -2\pi i \sum_{n \geq 1} \frac{e(-zn) - e(zn)}{(-1)^n} \mathbf{B}(-2\pi i n \tau) \frac{1}{-2n\pi i} \\ (3.22) \quad &= \sum_{n \geq 1} (-1)^n \frac{e(-zn) - e(zn)}{n} \mathbf{B}(-2\pi i n \tau). \end{aligned}$$

By using the definition of \mathbf{B} in (1.19), one can notice that

$$\mathbf{B}(-2\pi i n \tau) = \frac{1}{2} + \frac{1}{2\pi i \tau n} + \frac{1}{e(-\tau n) - 1}.$$

Since $\Im \tau > 0$, $e(-\tau n)$ becomes exponentially large as $n \rightarrow +\infty$; thus, the series in the right hand side of (3.22) converges and can be expressed as follows:

$$(3.23) \quad B^+(z|\tau) + B^-(z|\tau) = \frac{\alpha(z)}{2} - \frac{\beta_0(z)}{2\pi i \tau} + \gamma_0(z|\tau),$$

where $\alpha(z)$, $\beta_0(z)$ and $\gamma(z|\tau)$ are defined in the following manner:

$$\alpha(z) = \log(1 + e(z)) - \log(1 + e(-z)),$$

$$\beta_0(z) = \text{Li}_2(-e(z)) - \text{Li}_2(-e(-z))$$

and

$$\gamma_0(z|\tau) = \sum_{n \geq 1} \frac{(-1)^n}{n} \frac{e(-zn) - e(zn)}{e(-\tau n) - 1}.$$

On the one hand, because of $z \in (-1/2, 1/2)$, it is easy to see that $\alpha(z) = 2\pi i z$. On the other hand, by using the following relation [11, §2]:

$$\text{Li}_2\left(-\frac{1}{x}\right) = -\text{Li}_2(-x) - \frac{\pi^2}{6} - \frac{1}{2} \log^2 x,$$

one finds that $\beta_0(z)$ is exactly the same as $\beta(z)$ in (3.19).

Moreover, with the help of the relation in (3.7), one finds that the last term $\gamma_0(z|\tau)$ in (3.23) can be expressed as follows:

$$\gamma_0(z|\tau) = \log \frac{(z + \tau + 1/2|\tau)_\infty}{(-z + \tau + 1/2|\tau)_\infty} = -\alpha(z) + \gamma(z|\tau),$$

where $\gamma(z|\tau)$ is as given in (3.20). Thus, one deduces (3.18) from (3.23) and this finishes the proof of Theorem 3.2. \square

We return to the second specie of singularities coming from the function $\mathbf{B}(\tau u)$: this can be seen as a typical situation where Stokes phenomenon occurs.

Theorem 3.3. *The following assertions hold for all $(z, \tau) \in \Omega^+ \cap \Omega^-$.*

(1) *If $\tau \in \mathcal{H}$, then:*

$$(3.24) \quad B^+(z|\tau) - B^-(z|\tau) = \log \frac{((-z - 1/2)/\tau| - 1/\tau)_\infty}{((z - 1/2/\tau| - 1/\tau)_\infty}.$$

(2) *If $\tau \in \mathcal{H}^- = (-\mathcal{H})$, then:*

$$(3.25) \quad B^+(z|\tau) - B^-(z|\tau) = \log \frac{((z + 1/2)/\tau| 1/\tau)_\infty}{((-z + 1/2/\tau| 1/\tau)_\infty}.$$

Proof. This follows immediately from (3.16) together with (3.8) and (3.9). \square

When $\tau \xrightarrow{a.v.} 0$ in \mathcal{H} , $e(-1/\tau)$ is exponentially small et it follows that the right-hand side of (3.24) represents an infinitely small function. This implies that both $B^+(z|\tau)$ and $B^-(z|\tau)$ have the same asymptotic expansion as $\tau \xrightarrow{a.v.} 0$ in their respective domain of definition; see §5.1.

3.4. Modularity found by means of Stokes' analysis on Euler products.

The strategy is to express the function $B(z|\tau)$ by making use of both Theorem 3.2 and Theorem 3.3. The proof will be completed with the help of the classical θ and η -modular formulas.

Proof of Theorem 1.3. By putting together the relations in (3.18) and (3.24), one finds that

$$(3.26) \quad 2B(z|\tau) = 2B^+(z|\tau) = -\pi iz - \frac{\beta(z)}{2\pi i\tau} + \delta(z|\tau),$$

where $\beta(z)$ is as given in (3.19) and

$$\delta(z|\tau) = \log \frac{(z + 1/2|\tau)_\infty}{(-z + 1/2|\tau)_\infty} + \log \frac{((-z - 1/2)/\tau| - 1/\tau)_\infty}{((z - 1/2/\tau| - 1/\tau)_\infty}.$$

We shall use the θ -modular formula to simplify the expression of $\delta(z|\tau)$, observing that

$$(3.27) \quad \delta(z|\tau) = \log \frac{(z + 1/2|\tau)_\infty}{((z - 1/2/\tau| - 1/\tau)_\infty} + \log \frac{((-z - 1/2)/\tau| - 1/\tau)_\infty}{(-z + 1/2|\tau)_\infty}.$$

If $p = 0$, $m = 1$ and z is replaced with $z - 1/2$ in the general θ -modular relation (2.18), one gets that $\tau' = \tau$, $\tilde{\tau} = -1/\tau$ and

$$\theta(z + \frac{1}{2}|\tau) = \sqrt{\frac{i}{\tau}} e(-\frac{1}{2\tau}(z - \frac{\tau}{2})^2) \theta(\frac{z - 1/2}{\tau} + \frac{1}{2} | -\frac{1}{\tau}).$$

Thus, by the triple product formula (1.3), it follows that

$$(3.28) \quad \frac{(-z + 1/2 | \tau)_\infty}{((z - 1/2)/\tau | -1/\tau)_\infty} = C(z | \tau) \frac{((z - 1/2)/\tau | -1/\tau)_\infty}{(z + 1/2 | \tau)_\infty},$$

where

$$C(z | \tau) = \sqrt{\frac{i}{\tau}} (1 + e(-z)) e\left(-\frac{1}{2\tau}(z - \frac{\tau}{2})^2\right) \frac{(-1/\tau | -1/\tau)_\infty}{(\tau | \tau)_\infty}.$$

Moreover, from the classical η -modular relation, one obtains that

$$\sqrt{\frac{i}{\tau}} \frac{(-1/\tau | -1/\tau)_\infty}{(\tau | \tau)_\infty} = e((\tau + \frac{1}{\tau})/24).$$

Therefore, the coefficient $C(z | \tau)$ in (3.28) can be read as follows:

$$(3.29) \quad C(z | \tau) = (1 + e(-z)) e\left(-\frac{1}{2\tau}(z - \frac{\tau}{2})^2\right) e((\tau + \frac{1}{\tau})/24).$$

By gathering together (3.27) and (3.28), one finds that

$$\delta(z | \tau) = 2 \log \frac{(z + 1/2 | \tau)_\infty}{((z - 1/2)/\tau | -1/\tau)_\infty} - \log(C(z | \tau)).$$

Finally, the equality in (3.26) implies that

$$B(z | \tau) = -\frac{\pi iz}{2} - \frac{\beta(z)}{4\pi i \tau} + \log \frac{(z + 1/2 | \tau)_\infty}{((z - 1/2)/\tau | -1/\tau)_\infty} - \frac{1}{2} \log(C(z | \tau)).$$

So, one finds that

$$(3.30) \quad \log \frac{(z + 1/2 | \tau)_\infty}{((z - 1/2)/\tau | -1/\tau)_\infty} = B(z | \tau) + \frac{\pi iz}{2} + \frac{\beta(z)}{4\pi i \tau} + \frac{1}{2} \log(C(z | \tau)).$$

An direct calculation using the definition of $\beta(z)$ in (3.19) and that of $C(z | \tau)$ in (3.29) shows that the above equation in (3.30) yields (1.22), thus one finishes the proof of Theorem 1.3. \square

4. ANALYTIC CONTINUATION AND RAMANUJAN'S ASYMPTOTIC FORMULA

The goal of this section consists to bring about as much light as possible about the set of singularities of the function $B(z | \tau)$ in the complex z -plane each time when τ is fixed in the upper half-plane \mathcal{H} . This will be done either by using the functional equation deduced from integral representation (1.21) or by putting analytic continuation process of $A(z | \tau)$ and $L_2(z | \tau)$ into the relation (1.22) of Theorem 1.3. See Theorem 4.1 in §4.1 and Proposition 4.1 in §4.2.

In our previous works [12] and [13], we found a modular-type relation for Euler infinite product $(x; q)_\infty$, that is intimately linked with an asymptotic formula of Ramanujan. Compare Theorem 4.2 in the below with [3, p. 265 Entry 6 and p. 268 Entry 6']. In §4.3, we will deal with some properties of the key-term $P(z | \tau)$ appearing in Theorem 4.2. It will be shown that this function is closely linked with both $b^+(z | \tau)$ and $B(z | \tau)$ considered in the previous section. We will conclude this section by Theorem 4.3, which states that Theorems 1.3 and 4.2 are really equivalent.

4.1. Branch points and analytic continuation domain of Bernoulli integral. Consider the analytic continuation of the function B appearing in (1.22). Unless expressly stated otherwise, τ will be fixed in \mathcal{H} and our analysis will be *a priori* done about the variable z .

As $B(z|\tau) = B^+(z|\tau)$ and this last one is related with $b^+(z|\tau)$ by (3.16), we shall start with the analytic continuation of $b^+(z|\tau)$. By the definition of \mathcal{W}^+ in (3.5), it follows that for all given $\tau \in \mathcal{H}$, with $\arg \tau = \delta \in (0, \pi)$, the relation $(z, \tau) \in \mathcal{W}^+$ means exactly $z \in V_d^+$ for some suitable $d \in (-\frac{\pi}{2}, \frac{\pi}{2} - \delta)$. Thus, one obtains the following equivalence:

$$(4.1) \quad (z, \tau) \in \mathcal{W}^+ \iff z \in \mathcal{H} \cup Z_\tau,$$

where Z_τ is the half-plane associated with τ in the following manner:

$$Z_\tau = \{z \in \mathbb{C} : \Im \frac{z+1}{\tau} < 0\}.$$

One may notice that if $z \in Z_\tau$, then $z + z_0 \in Z_\tau$ for all positive real z_0 . See Figure 6 in the below.

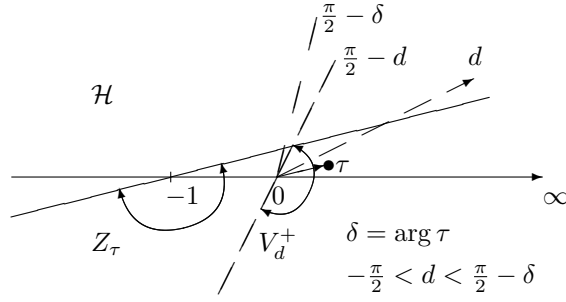


FIGURE 6. The half-plane Z_τ contains both the point τ and the segment $(-1, \infty)$

Lemma 4.1. *Let $(z, \tau) \in \mathcal{W}^+$, $b(z|\tau) = b^+(z|\tau)$ as in Definition 3.1 and let $s = z/\tau$. One supposes that $\tau \in \mathcal{H}$. If $(z-1, \tau) \in \mathcal{W}^+$, then $s \notin (-\infty, 0]$ and*

$$(4.2) \quad b(z-1|\tau) - b(z|\tau) = \log \Gamma(s) - (s - \frac{1}{2}) \log s + s - \frac{1}{2} \log 2\pi.$$

Proof. Since $(z-1, \tau) \in \mathcal{W}^+$, relation (4.1) implies that either $z \in \mathcal{H} \cup (0, \infty)$ or $z \in \mathcal{H}^-$ but $\Im(z/\tau) < 0$, so that $s \notin (-\infty, 0]$.

By observing that

$$b(z-1|\tau) - b(z|\tau) = \int_0^{\infty e^{id}} \mathbf{B}(t) e^{-zt} \frac{dt}{t},$$

relation (4.2) follows immediately from Binet's formula [1, Theorem 1.6.3 (i), p. 28] on $\log \Gamma(x)$:

$$(4.3) \quad \log \Gamma(x) = (x - \frac{1}{2}) \log x - x + \frac{1}{2} \log 2\pi + \int_0^{\infty e^{id}} \mathbf{B}(t) e^{-xt} \frac{dt}{t}.$$

Here, first of all, one supposes that $\Re x > 0$, so the integration path is the half-axis $(0, +\infty)$. By using an open interval $(0, \infty e^{id})$ in the half-plane $\Re t > 0$, this integral representation can then be valid for all $x \in \mathbb{C} \setminus (-\infty, 0]$. \square

It should be noted that the above relation (4.2) is valid for any $(z, \tau) \in \mathcal{H} \times \mathcal{H}$. If $z \notin \mathcal{H}$, we have to avoid the poles of the Gamma function, and the right-hand side of (4.2) continues to be well-defined over the Riemann surface of \log while $s \notin \mathbb{Z}_{\leq 0}$. Thus, Lemma 4.1 allows one to make the analytic continuation of the function b^+ at (z, τ) provided that $(z+n)/\tau \notin \mathbb{Z}_{\leq 0}$ for all $n \in \mathbb{Z}_{\geq 1}$. In other words, one obtains the following

Remark 4.1. For any fixed $\tau \in \mathcal{H}$, $b(z|\tau)$ can be continued into an analytic function on the universal covering of $\mathbb{C} \setminus \Delta_\tau$, where

$$(4.4) \quad \Delta_\tau = \mathbb{Z}_{\leq -1} \oplus \tau \mathbb{Z}_{\leq 0}.$$

In what follows, to each $\tau \in \mathcal{H}$ will be associated the half-lattice \mathcal{L}_τ in the following manner:

$$(4.5) \quad \mathcal{L}_\tau = ((\mathbb{Z}_{\leq 0} - \frac{1}{2}) \oplus \tau \mathbb{Z}_{\leq 0}) \cup ((\mathbb{Z}_{\geq 0} + \frac{1}{2}) \oplus \tau \mathbb{Z}_{\geq 0}).$$

See Figure 7 in the below.

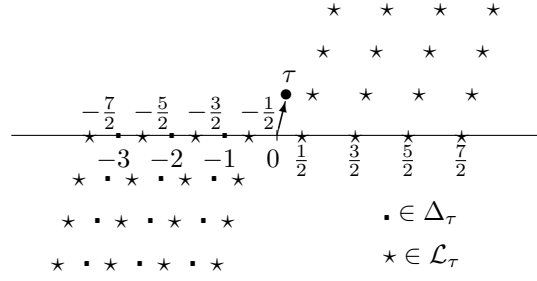


FIGURE 7. The set Δ_τ contains all negative integers while all half-integers belong to \mathcal{L}_τ

Thus, one can state the following

Theorem 4.1. Let $\tau \in \mathcal{H}$ and let \mathcal{L}_τ be the half-lattice given as in (4.5). The function $B(z|\tau)$ can be extended into an analytic function over the universal covering of $\mathbb{C} \setminus \mathcal{L}_\tau$ in such a manner that the following relations hold.

(1) If $(z + 1/2)/\tau \in \mathcal{H}$, then:

$$(4.6) \quad B(z + 1|\tau) = B(z|\tau) - \log\left(1 - e\left(\frac{z + 1/2}{\tau}\right)\right).$$

(2) If $(z + 1/2)/\tau \in \mathcal{H}^-$, then:

$$(4.7) \quad B(z + 1|\tau) = B(z|\tau) - \log\left(1 - e\left(-\frac{z + 1/2}{\tau}\right)\right).$$

Proof. Since $B(z|\tau)$ is linked with $b(\pm z - 1/2|\tau)$ by (3.16), the analytic continuation of $B(z|\tau)$ follows the same way as what done for $b(z|\tau)$ in Lemma 4.1. Particularly, by using (4.4), one obtains the expression in (4.5) about the set \mathcal{L}_τ of singularities of $B(z|\tau)$.

In order to prove (4.6) and (4.7), one makes use of (3.16) together with (4.2). In this way, it follows that

$$B(z + 1|\tau) - B(z|\tau) = (b(-z - \frac{3}{2}|\tau) - b(-z - \frac{1}{2}|\tau))$$

$$\begin{aligned}
& + \left(b\left(z - \frac{1}{2} \mid \tau\right) - b\left(z + \frac{1}{2} \mid \tau\right) \right) \\
(4.8) \quad & = I\left(-\frac{z + 1/2}{\tau}\right) + I\left(\frac{z + 1/2}{\tau}\right),
\end{aligned}$$

where $I(s)$ denotes the right-hand side of (3.16) for $s \in \mathbb{C} \setminus (-\infty, 0]$, that is to say,

$$(4.9) \quad I(s) = \log \Gamma(s) - \left(s - \frac{1}{2}\right) \log s + s - \frac{1}{2} \log 2\pi.$$

(1) One supposes that $s \in \mathcal{H}$; as $-s \in (\mathbb{C} \setminus (-\infty, 0])$, one writes $-s = e^{-\pi i} s$. By using Euler's reflection formula, one finds that

$$\Gamma(s)\Gamma(-s) = \frac{1}{-s} \frac{\pi}{\sin \pi s} = \frac{e^{i\pi}}{s} \frac{\pi}{\sin \pi s},$$

which implies that

$$\begin{aligned}
I(s) + I(-s) &= \log \Gamma(s) + \log \Gamma(-s) - \pi i s - \frac{\pi i}{2} + \log s - \log 2\pi \\
&= \log \frac{1}{2 \sin \pi s} - \pi i s + \frac{\pi i}{2} \\
(4.10) \quad &= -\log(1 - e(s)).
\end{aligned}$$

Here, for the last implication, one has used the following relations:

$$2 \sin \pi s = \frac{e(s/2) - e(-s/2)}{i} = \frac{e(-s/2)}{-i} (1 - e(s)).$$

By (4.8), putting $s = (z + 1/2)/\tau$ into (4.10) yields finally (4.6).

(2) By an argument of the symmetry, one can easily see that if $s \in \mathcal{H}^-$, then:

$$I(s) + I(-s) = -\log(1 - e(-s)),$$

and this allows one to end the proof of Theorem 4.1. \square

4.2. Functional equations satisfied by $A(z \mid \tau)$ and $L_2(z \mid \tau)$. Another way to prove Theorem 4.1 consists of making use (1.22) together with the following fact.

Proposition 4.1. *Let $A(z \mid \tau)$ and $L_2(z \mid \tau)$ be as given in Theorem 1.3. The following assertions hold for all $\tau \in \mathcal{H}$.*

(1) *If $z \in \mathcal{H}$, then*

$$(4.11) \quad A(z + 1 \mid \tau) = A(z \mid \tau)$$

and

$$(4.12) \quad L_2(z + 1 \mid \tau) = L_2(z \mid \tau).$$

(2) *If $z \in \mathcal{H}^- = (-\mathcal{H})$, then:*

$$(4.13) \quad A(z + 1 \mid \tau) = e^{\pi i} A(z \mid \tau)$$

and

$$(4.14) \quad L_2(z + 1 \mid \tau) = L_2(z \mid \tau) - \frac{2\pi i}{\tau} \left(z + \frac{1}{2}\right).$$

Proof. (1) Let $z \in \mathcal{H}$ and let

$$x = x(t) = 1 + e(z + t)$$

for $t \in [0, 1]$. Since

$$|e(z)| = |e(z + t)| < 1,$$

it follows that $\Re(x(t)) > 0$ when $t \in [0, 1]$; in other words, $\sqrt{x(t)}$ does not have monodromy as t goes from 0 to 1, so one obtains the relation in (4.11) for $A(z | \tau)$.

The relation in (4.12) comes from the fact that, for $t \in [0, 1]$, $e(z + t)$ belongs to the unit disc, in which the dilogarithm is well-defined and analytic.

(2) By writing

$$1 + e(z) = e(z)(1 + e(-z)),$$

one finds that the following identity holds for all $z \in \mathcal{H}^-$:

$$(4.15) \quad A(z | \tau) = e\left(\frac{z}{2}\right) A(-z | \tau).$$

Thus one gets (4.13), by combining (4.15) together with (4.11).

The relation in (4.14) follows from the monodromy property of the dilogarithm $\text{Li}_2(x)$ at $x = 1$. Indeed, let $z \in \mathcal{H}^-$ and define

$$x = x(t) = -e(z + t)$$

for $t \in [0, 1]$. Since $\Re(z) < 0$, x will describe a circle of radius > 1 when t runs through $[0, 1]$. Thus, $\text{Li}_2(x(1))$ can be expressed as the analytic continuation of $\text{Li}_2(x(0))$ along a circle centered at $x = 1$ and of radius < 1 in the anti clock-wise direction.

By using the following relation between $\text{Li}_2(x_0)$ and $\text{Li}_2(1 - x_0)$ [11, §2]:

$$\text{Li}_2(1 - x_0) = -\text{Li}_2(x_0) + \frac{\pi^2}{6} - \log x_0 \log(1 - x_0),$$

one finds that the monodromy of Li_2 around $x = 1$ can be expressed as follows:

$$(4.16) \quad \text{Li}_2(1 + x_0 e^{2\pi i}) = \text{Li}_2(1 + x_0) - 2\pi i \log(1 + x_0).$$

Thus, one obtains that

$$(4.17) \quad \text{Li}_2(-e(z + 1)) = \text{Li}_2(-e(z)) - 2\pi i \log(-e(z)).$$

If $z \rightarrow 1/2$ from \mathcal{H}^- , this last equation becomes

$$\text{Li}_2(-e(-\frac{1}{2})) = \text{Li}_2(-e(-\frac{1}{2})) - 2\pi i \log(-e(-\frac{1}{2}));$$

hence, (4.17) can be read as follows:

$$\text{Li}_2(-e(z + 1)) = \text{Li}_2(-e(z)) - 2\pi i \log(e(z + \frac{1}{2})).$$

With the definition of L_2 in (1.23) one deduces immediately the expected relation in (4.14). \square

It should be noted that, in all the relations stated in Proposition 4.1, the analytic continuation is performed in such a way that the continuation-path linking z to $z + 1$ remains entirely either in the upper plane \mathcal{H} or in its opposite side. Over the real axis, $A(z | \tau)$ vanishes for all half-integers, or more precisely, this function has branch points there as well as the function $L_2(z | \tau)$. So, one arrives at the following

Remark 4.2. *The above relations in Proposition 4.1 remain valid even for $z \in \mathbb{R} \setminus (\mathbb{Z} + 1/2)$, provided that one adopts the following convention: Assertion (1) or (2) will be applied in case jumping the ramification point situated between z and $z + 1$ is made in the upper or lower plane \mathcal{H} or \mathcal{H}^- .*

Accordingly, replacing z with $z + 1$ in (1.22) and applying Proposition 4.1 allows one to find the relations in (4.6) and (4.7) of Theorem 1.3. We omit the details.

4.3. Non-modular term in a Ramanujan asymptotic formula. As in [12] and [13], let \mathcal{U} denote the domain defined in $\mathbb{C} \times \mathcal{H}$ by the relation

$$(4.18) \quad \mathcal{U} = \cup_{\delta \in (0, \pi)} \mathbb{C}_\delta \times \mathcal{H}_\delta.$$

Here, \mathcal{H}_δ is the open sector $\{\tau \in \mathcal{H} : \arg \tau \in (0, \delta)\}$ and \mathbb{C}_δ is the domain in \mathbb{C} containing the interval $(-1, 1)$ and whose complementary equals to the closure of the union $(1 + \mathcal{H}_\delta) \cup (-1 - \mathcal{H}_\delta)$. See Figure 8 in the below.

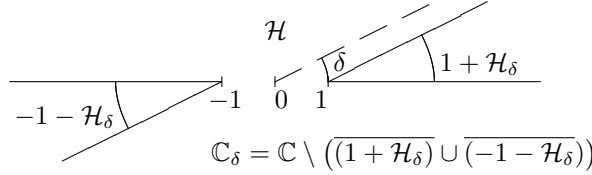


FIGURE 8. The domain \mathbb{C}_δ contains $(-1, 1)$ and is symmetrical with respect to 0

The following equivalent relations hold for any $(z, \tau) \in \mathbb{C} \times \mathcal{H}$:

$$(4.19) \quad (z, \tau) \in \mathcal{U} \iff (-z, \tau) \in \mathcal{U} \iff \left(\frac{z}{\tau}, -\frac{1}{\tau}\right) \in \mathcal{U}.$$

Theorem 4.2 ([12, Theorem 3.2], [13, Theorem 2.9]). *Let $(z, \tau) \in \mathcal{U}$ and let $s = z/\tau$. If $s \notin (-\infty, 0]$ and $\log s$ denotes the principal branch of the logarithm, then:*

$$(4.20) \quad (z|\tau)_\infty = \frac{\sqrt{2\pi s(1 - e(z))}}{\Gamma(s+1)} e(-\frac{\tau}{24}) e^{s(\log s - 1) + \frac{\text{Li}_2(e(z))}{2\pi i \tau} + P(z|\tau)} \left(\frac{z-1}{\tau} \middle| -\frac{1}{\tau}\right)_\infty,$$

where Γ denotes the Euler Gamma function, Li_2 denotes the dilogarithm, and where $P(z|\tau)$ denotes the function defined in \mathcal{U} by the analytic continuation procedure from the following integral:

$$(4.21) \quad P^d(z|\tau) = \int_0^{\infty e^{id}} \frac{\sin(\frac{zt}{\tau})}{e^{it/\tau} - 1} \left(\cot \frac{t}{2} - \frac{2}{t}\right) \frac{dt}{t},$$

with $-\pi < d < 0$.

In the following, we shall discuss the link between $P(z|\tau)$ and $b(z|\tau)$. In view of the fact that

$$\cot \frac{t}{2} - \frac{2}{t} = \frac{e^{it/2} + e^{-it/2}}{e^{it/2} - e^{-it/2}} i - \frac{2}{t} = 2i \left(\frac{1}{e^{it} - 1} + \frac{1}{2} - \frac{1}{it} \right),$$

it follows that

$$\cot \frac{t}{2} - \frac{2}{t} = 2i \mathbf{B}(it),$$

where \mathbf{B} is the generating function given in (1.19). Thus, replacing the integration path $(0, \infty e^{id})$ with $(0, i\infty e^{id})$ in the integral in (4.21) yields that

$$(4.22) \quad P^d(z|\tau) = \int_0^{\infty e^{id'}} \frac{e^{zt/\tau} - e^{-zt/\tau}}{e^{t/\tau} - 1} \mathbf{B}(t) \frac{dt}{t},$$

with $d' = d + \frac{\pi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$. For all (z, τ) in \mathcal{U} , it follows that

$$|\Re(z \frac{e^{id'}}{\tau})| < \Re(\frac{e^{id'}}{\tau}),$$

so that P^d yields an analytic function $P(z|\tau)$ in the domain \mathcal{U} of $\mathbb{C} \times \mathcal{H}$.

Comparing the integral in (4.22) with (3.2) allows one to observe immediately the following

Remark 4.3. *Let \mathcal{W}^+ and $b(z|\tau) = b^+(z|\tau)$ be as in Definition 3.1. The function $P(z|\tau)$ can be expressed as follows:*

$$(4.23) \quad P(z|\tau) = -b(z|\tau) + b(-z|\tau).$$

Accordingly, it follows that

$$(4.24) \quad \mathcal{U} = \{(z, \tau) \in \mathcal{W}^+ : \tau \in \mathcal{H}, (-z, \tau) \in \mathcal{W}^+\}.$$

By Remark 4.1 and the above equation in (4.23), one can observe that, for every given $\tau \in \mathcal{H}$, $P(z|\tau)$ can be extended into an analytic function on the universal covering of $\mathbb{C} \setminus \Delta_\tau^{\text{sym}}$, where Δ_τ^{sym} denotes the symmetrization of Δ_τ , that is to say,

$$(4.25) \quad \Delta_\tau^{\text{sym}} = \Delta_\tau \cup (-\Delta_\tau) = (\mathbb{Z}_{\leq -1} \oplus \tau \mathbb{Z}_{\leq 0}) \cup (\mathbb{Z}_{\geq 1} \oplus \tau \mathbb{Z}_{\geq 0}).$$

See (4.4) for the definition of Δ_τ and Figure 7 for the illustration of Δ_τ^{sym} . Furthermore, by gathering together (4.23) with (4.3) and (4.9), it follows that

$$(4.26) \quad P(z+1|\tau) - P(z|\tau) = I(-s) - I(s + \frac{1}{\tau}),$$

where $s = z/\tau$ and $I(s)$ denotes the function defined by (4.9) with the help of Binet's formula (4.3).

4.4. Equivalence between Theorem 4.2 and Theorem 1.3. In the following, we will express the function $B(z|\tau)$ in terms of $P(z|\tau)$ to find our previous Theorem 1.3 from Theorem 4.2, and vice versa.

Let $\Omega = \Omega^+$ be the definition domain of $B(z|\tau)$ as given in Definition 3.2, and let $(z, \tau) \in \Omega \cap (\mathbb{C} \times \mathcal{H})$. Then, $(z, \tau) \in H_d \times V_d^+$ for some $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if, and only if, letting $d' = \frac{\pi}{2} - d$, all of the following conditions are satisfied:

$$\Im(\frac{z-1/2}{e^{id'}}) < 0, \quad \Im(\frac{z+1/2}{e^{id'}}) > 0, \quad 0 < \arg \tau < d'.$$

Thus, one arrives at the following

Remark 4.4. *Let $\Omega = \Omega^+$ be as given in (3.15). The domain $\Omega \cap (\mathbb{C} \times \mathcal{H})$ is strictly included in the above \mathcal{U} defined by (4.18), particularly with the following relation:*

$$(4.27) \quad \Omega \cap (\mathbb{C} \times \mathcal{H}) = \{(z + \frac{1}{2}, \tau) \in \mathcal{U} : \Im(\frac{z+1/2}{\tau}) < 0, \Im(\frac{z-1/2}{\tau}) > 0\}.$$

Let \mathcal{L}_τ be the set of singularities of $B(z|\tau)$ as given in (4.5). Comparing this with the definition of Δ_τ^{sym} in (4.25) yields that

$$(4.28) \quad \mathcal{L}_\tau = \left(\frac{1}{2} + \Delta_\tau^{\text{sym}}\right) \cup \left(\frac{1}{2} \oplus \tau\mathbb{Z}_{\geq 0}\right).$$

This goes back to Remark 4.4, including particularly the relation stated in (4.27). See Figure 9 in the below and compare this with Figure 7.

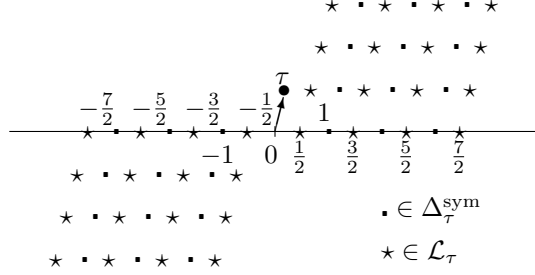


FIGURE 9. The set Δ_τ^{sym} contains all non-zero integers while all half-integers belong to the half-lattice \mathcal{L}_τ

Moreover, by observing that

$$\frac{e^{zt/\tau} - e^{-zt/\tau}}{e^{t/\tau} - 1} = \frac{e^{(z-1/2)t/\tau} - e^{-(z-1/2)t/\tau}}{e^{t/2\tau} - e^{-t/2\tau}} + e^{-zt/\tau},$$

one obtains that

$$(4.29) \quad B(z|\tau) = P\left(z + \frac{1}{2}|\tau\right) - I\left(\frac{z+1/2}{\tau}\right).$$

This is the key-point that permits one to see in what manner both Theorems 1.3 and 4.2 are really equivalent.

Theorem 4.3. *Theorem 4.2 implies Theorem 1.3, and vice versa.*

Proof. In view of the above consideration about the respective domain of definition of $B(z|\tau)$ and $P(z|\tau)$, and by the argument of analytic continuation, one needs only to prove Theorem 4.3 for $z \in (-1/2, 1/2)$ and $\tau \in \mathcal{H}$. Thus, by replacing z with $z + 1/2$, the relation in (4.20) takes the following form:

$$(4.30) \quad \left(z + \frac{1}{2}|\tau\right)_\infty = A(z|\tau) D(z|\tau) e^{L_2(z|\tau)} \left(\frac{z-1/2}{\tau} \middle| -\frac{1}{\tau}\right)_\infty,$$

where $A(z|\tau)$ and $L_2(z|\tau)$ are the same as in (1.22) of Theorem 1.3 and where we write

$$(4.31) \quad D(z|\tau) = \sqrt{\frac{2\pi}{s'}} \frac{1}{\Gamma(s')} e^{s'(\log s' - 1) + P(z+1/2|\tau)}$$

together with

$$s' = \frac{z+1/2}{\tau} = s + \frac{1}{2\tau}.$$

If one takes the logarithm for both sides of the equation in (4.31), using the function $I(s)$ defined by the expression in (4.9) gives that

$$(4.32) \quad \log D(z|\tau) = P\left(z + \frac{1}{2}|\tau\right) - I\left(\frac{z+1/2}{\tau}\right).$$

Finally, by (4.29), it turns out that

$$\log D(z | \tau) = B(z | \tau).$$

In other words, the equation in (4.30) allows (4.20) to imply (1.22), and vice versa. By this way, one ends the proof of Theorem 4.3. \square

5. CONDITIONS FOR AN EULER PRODUCT TO HAVE AN EXPONENTIAL-CONVERGENT EXPANSION

In this section, we shall establish the inverse of Theorem 2.2 (1) with $\zeta = 1$ as follows.

Theorem 5.1. *Given $\omega \in \mathbb{C}\{\tau\}$ such that $\omega(0) \in \mathbb{R}$, one has $f \in \mathfrak{C}_1$ only if $\omega(\tau) = \frac{1}{2}(n + m\tau)$ for some integers $n, m \in \mathbb{Z}$.*

The strategy we shall elaborate in this paragraph comes from the following observation: Replacing z by $\omega(\tau)$ in both Theorems 1.3 and 4.2 allows one to obtain, modulo an exponentially small remainder, the analyticity obstruction of $f \circ e$ by the Stokes's co-cycles on B^\pm or P . In this way, one finds the very rare cases where $f(q)$ can be exponentially-convergent in the sense of Definition 1.2. This approach is likely to be linked with Ecalle's theory of resurgent functions.

So, in what follows, the approach we will make use of is of analytical nature and it is natural to consider $\omega(\tau)$ in the ring of germs of analytic functions at $\tau = 0$. In order to be able to treat *globally* the corresponding functions B^\pm and P , we will be led to the supplementary condition that requires the reality of $\omega(\tau)$ at 0.

In §5.1, we shall recall Ramis-Sibuya's Theorem on Gevrey asymptotic expansions and deduce from that a result that will be applied to the previous functions B^\pm and P ; see Corollary 5.1. In §5.2, we shall deal with exponential smallness of terms with $-1/\tau$.

Paragraph 5.3 can be viewed as the heart of this section, in which we are led to the equivalent conditions for the functions B and P to be analytic. See Theorems 5.3 and 5.4. In view of all that, we end this section by a proof of Theorem 5.1.

5.1. Ramis-Sibuya's Theorem on Gevrey asymptotic expansions. Let $x_0 \in \mathbb{C}$ and let \mathcal{R}_{x_0} be the Riemann surface of the function $x \mapsto \log(x - x_0)$; let $I = (\alpha_1, \alpha_2) \subset \mathbb{R}$ and let $R > 0$. We let $V_{x_0}(I; R)$ denote the sector of vertex at x_0 in \mathcal{R}_{x_0} , with opening in I and radius R , that is to say:

$$(5.1) \quad V_{x_0}(I; R) = \{x_0 + re^{i\alpha} : \alpha \in I, r \in (0, R)\}.$$

We will adopt the following terminologies for any given sector $V = V_{x_0}(I; R)$:

- (1) A *germ of sector at x_0 in V* means any domain of the form $V_{x_0}(I; \rho)$ such that $\rho < R$.
- (2) A *proper sub-sector of V* means any domain of the form $V_{x_0}(J; \rho)$ such that $\bar{J} \subset I$ and $\rho < R$.

If the length of the open interval I is smaller than or equal to 2π , any sector $V_{x_0}(I; R)$ is not overlapped in $\tilde{\mathbb{C}}^*$; in this case, one will consider $V_{x_0}(I; R)$ as **a sector in \mathbb{C}** . When $x_0 = 0$, we will remove the sub-index 0 and write simply $V(I; R)$ instead of $V_0(I; R)$.

Let $V = V(I; R)$ be a sector in \mathbb{C} at 0. By definition ([5], [7], ...), a given function f defined and analytic in V is said to *have a power series $\sum_{n \geq 0} a_n x^n$, $a_n \in \mathbb{C}$, as Gevrey or exponential asymptotic expansion at 0 in V* if for any proper

sub-sector $U = V(J; \rho)$, one can find $C > 0$ and $A > 0$ such that the following estimates hold for all $n \in \mathbb{Z}_{\geq 0}$:

$$(5.2) \quad \sup_{x \in U} \left| (f(x) - \sum_{m=0}^{n-1} a_m x^m) x^{-n} \right| \leq C A^n n!.$$

As typical example, the Borel-sum function of a given divergent series, if exists, admits this series as Gevrey asymptotic expansion. A Gevrey type asymptotic expansion is also called *exponential asymptotic expansion*, due to the following fact.

Remark 5.1. [5, p. 175, Théorème 1.2.4.1 1)] *A function f admits the identically null series as Gevrey asymptotic expansion at 0 in V if, and only if, f is exponentially small there, that means that, for all proper sub-sector U in V , there exists $C > 0$ and $\kappa > 0$ such that, for all $x \in U$, $|f(x)| \leq C e^{-\kappa/|x|}$.*

For any given sector $V = V_{x_0}(I; R)$, we will denote by $\mathcal{A}^{\leq -1}(V)$ the space of all exponentially small functions as $x \rightarrow x_0$ in V . It will be useful to observe that the space $\mathcal{A}_\alpha^{\leq -1}$ introduced in §1.2 can be written as follows:

$$(5.3) \quad \mathcal{A}_\alpha^{\leq -1} = \bigcup_{I \ni \frac{\pi}{2}, R > 0} \mathcal{A}^{\leq -1}(V_\alpha(I; R)).$$

Remark 5.1 gives a necessary condition for two functions to have a same Gevrey asymptotic expansion. In practice, one may want make sure a family of solutions found for a given problem have a same Gevrey asymptotic expansion. The following result shows the exponential smallness between each other of the family can constitute a sufficient condition to the existence of such expansion provided that a sufficient large number of functions are in our disposal.

Theorem 5.2. [5, p. 176, Théorème 1.3.2.1] *Let V_1, \dots, V_m, V_{m+1} be a family of open sectors at 0 in \mathbb{C} such that $V_{m+1} = V_1$, $V_j \cap V_{j+1} \neq \emptyset$ for $1 \leq j \leq m$ and that the whole union $\cup_{j=1}^m V_j$ contains a neighborhood of 0 in \mathbb{C} . For every j , let f_j be a given analytic and bounded function in V_j . If*

$$f_{j+1} = f_j \mod \mathcal{A}^{\leq -1}(V_j \cap V_{j+1}),$$

then all f_j 's admit the same Gevrey asymptotic expansion at 0.

The above result is currently called *Ramis-Sibuya's Theorem*.

5.2. Exponentially small terms related with Euler infinite products. In what follows, we will identify the upper half-plane \mathcal{H} as the sector $V(I; R)$ with $I = (0, \pi)$, $R = \infty$, and denote by $\mathcal{A}^{\leq -1}(\mathcal{H})$ the space of all analytic functions in \mathcal{H} possessing an exponential decrease towards 0 as $\tau \xrightarrow{a.v.} 0$. In this way, one can reformulated Lemma 2.2 as follows.

Proposition 5.1. *Let $\omega \in \mathbb{C}\{\tau\}$, $\delta \in \mathbb{C}$, and let*

$$\psi(\tau) = \log\left(\left(-\frac{\omega(\tau)}{\tau} \mid \delta - \frac{1}{\tau}\right)_\infty\right)$$

for all τ enough near 0 in \mathcal{H} . If $\omega(0) > 0$, then $\psi \in \mathcal{A}^{\leq -1}(\mathcal{H})$.

Proof. One can take the same argument as that used in the proof of Lemma 2.2. \square

In order to study the asymptotic expansion of our previous functions $P(z | \tau)$, $B(z | \tau)$ and $b^\pm(z | \tau)$ introduced in §3 and §4, we give the following *simplified* version of Theorem 5.2.

Corollary 5.1. *Let $R > 0$ and let I_1 and I_2 be open intervals such that*

$$[-\epsilon, \pi - \epsilon] \subset I_1 \subset (-\pi, \pi), \quad [\pi - \epsilon, 2\pi - \epsilon] \subset I_2 \subset (0, 2\pi),$$

where ϵ denoted a fixed number in $(0, \pi)$. Let

$$V_1 = V(I_1; R), \quad V_2 = V(I_2; R),$$

and consider two analytic and bounded functions f_1 and f_2 defined respectively in V_1 and V_2 .

If $f_1 - f_2 \in \mathcal{A}^{\leq -1}(V_1 \cap V_2)$, then f_1 and f_2 have the same Gevrey asymptotic expansion and, moreover, the following statements are equivalent.

- (1) *One of the functions f_1 and f_2 can be continued into an analytic function at 0 in \mathbb{C} .*
- (2) *Both f_1 and f_2 can be continued into an analytic function at 0 in \mathbb{C} .*
- (3) *$f_1 \equiv f_2$ in $V_1 \cap V_2$.*

Proof. The existence of Gevrey asymptotic expansion for f_1 and f_2 follows immediately from Theorem 5.2.

Let $\hat{f} = \sum_{n \geq 0} a_n x^n$ be the common asymptotic expansion of f_1 and f_2 . Since the length of I_1 and that of I_2 are larger than π , one finds that f_1 and f_2 are the respective Borel-sum functions of \hat{f} in V_1 and V_2 . Thus, the above statement in (1) implies that \hat{f} is really a convergent series, so that their two Borel-sums are equal each other. In this way, one obtains that (1) implies all other statements.

On the other hand, if the statement in (3) is true, then both f_1 and f_2 equal to a same analytic and bounded function in the punctuated disc $\{0 < |x| < R\}$. By Riemann removable singularities Theorem, one finds the statements (1) and (2). \square

5.3. From asymptotics to analyticity for Bernoulli integral and related functions. Let us come back to the functions $b(z|\tau)$, $B(z|\tau)$ and $P(z|\tau)$ given respectively in Definition 3.1, Definition 3.2 and Theorem 4.2. By Theorem 2.2, one can find that all these functions are analytic at $\tau = 0$ when z takes these special values $\omega(\tau)$ given there. We shall prove that this is exclusively the only case.

Proposition 5.2. *Let $\omega \in \mathbb{C}\{\tau\}$ and let $I = (-\pi, \pi)$. If $\omega(0) > -1$, then there exists $R > 0$ such that the composite functions $b(\omega(\tau)|\tau)$ is well-defined and analytic in $V(I; R)$ and is bounded in every proper sub-sector of $V(I; R)$.*

Proof. For all $\tau \in \mathbb{C}^*$, let D_τ be the sector containing 0 and bounded by $(-\infty, -1] \cup [-1, -1 - \infty\tau)$, where $[-1, -1 - \infty\tau)$ denotes the half straight-line starting from -1 to ∞ with the direction $-\tau$. By combining (3.3) together with (3.4), one can find that, for all fixed $\tau \in \mathbb{C} \setminus (-\infty, 0]$, the function $b(z|\tau)$ is defined and analytic for $z \in D_\tau$; see the definition in (4.4) and Figure 7 for the set Δ_τ of singularities of $b(z|\tau)$ in the case $\tau \in \mathcal{H}$.

If $\omega(\tau) > -1$, one can easily see that $\omega(\tau)$ belongs to this half-plane D_τ when $|\tau|$ is enough small with $\tau \notin \mathbb{R}^-$. This implies that $b(\omega(\tau)|\tau)$ is well-defined and analytic in some sector $V(I; R)$, $R > 0$.

The boundedness of this function over any proper sub-sector comes from direct estimates done for (3.2). \square

In a similar way, one can find that the statement of Proposition 5.2 remains true if $b(z|\tau)$ ($= b^+(z|\tau)$) and I are replaced with $b^-(z|\tau)$ and $(0, 2\pi)$ respectively. Thus, one obtains the following

Theorem 5.3. *Let $\omega \in \mathbb{C}\{\tau\}$ and let $b(z|\tau)$ as in Definition 3.1. If $\omega(0) > -1$, then there exists $R > 0$ such that $b(\omega(\tau)|\tau)$ admits a Gevrey asymptotic expansion in $V(I; R)$ with $I = (-\pi, \pi)$.*

Moreover, $b(\omega(\tau)|\tau)$ can be continued into an analytic function at $\tau = 0$ if, and only if, $\omega(\tau) = m\tau$ for some $m \in \mathbb{Z}$.

Proof. Let $R > 0$ be as in Proposition 5.2 and write

$$V_1 = V((-\pi, \pi); R), \quad V_2 = V((0, 2\pi); R).$$

Define

$$f_1(\tau) = b(\omega(\tau)|\tau) = b^+(\omega(\tau)|\tau), \quad f_2(\tau) = b^-(\omega(\tau)|\tau)$$

for $\tau \in V_1$ and V_2 respectively. By putting $z = \omega(\tau)$ into both relations (3.8) and (3.9) of Theorem 3.1, it follows that

$$(5.4) \quad f_1(\tau) - f_2(\tau) = -\log \frac{(-(\omega(\tau) + 1)/\tau | -1/\tau)_\infty}{(-1/\tau | -1/\tau)_\infty}$$

if $\tau \in \mathcal{H} \cap V_1 \cap V_2$, and that

$$(5.5) \quad f_1(\tau) - f_2(\tau) = -\log \frac{((\omega(\tau) + 1)/\tau | 1/\tau)_\infty}{(1/\tau | 1/\tau)_\infty}$$

if $\tau \in \mathcal{H}^- \cap V_1 \cap V_2$.

One observes that $V_1 \cap V_2 \cap \mathbb{R} = \emptyset$. Therefore, by considering Proposition 5.1, relation (5.4) together with (5.5) imply that $f_1(\tau) - f_2(\tau)$ is exponentially small in the common domain $V_1 \cap V_2$. This allows us to apply Corollary 5.1 to get particularly the common Gevrey asymptotic expansion of both f_1 and f_2 .

Furthermore, Corollary 5.1 implies that f_1 can be extended into an analytic function at $\tau = 0$ in \mathbb{C} if, and only if,

$$\left(-\frac{\omega(\tau) + 1}{\tau} \mid -\frac{1}{\tau}\right)_\infty = \left(-\frac{1}{\tau} \mid -\frac{1}{\tau}\right)_\infty$$

for all $\tau \in \mathcal{H}$, or equivalently,

$$-\frac{\omega(\tau) + 1}{\tau} = -\frac{1}{\tau} \pmod{\mathbb{Z}}.$$

In this way, one finds the necessary and sufficient condition $\omega(\tau) \in \tau\mathbb{Z}$ in order to have an analytic function $b(\omega(\tau)|\tau)$ at $\tau = 0$ in \mathbb{C} . This ends the proof of Theorem 5.3. \square

In a similar way as in Theorem 5.3, one finds the following result about $B(\omega(\tau)|\tau)$ and $P(\omega(\tau)|\tau)$.

Theorem 5.4. *Let $\omega \in \mathbb{C}\{\tau\}$, $I = (-\pi, \pi)$, and let $B(z|\tau)$ and $P(z|\tau)$ be as in Definition 3.2 and Theorem 4.2. Then, there exists $R > 0$ such that the composite functions $B(\omega(\tau)|\tau)$ and $P(\omega(\tau)|\tau)$ represent each an analytic function in the sector $V = V(I; R)$ at 0 in \mathbb{C} and admits a Gevrey asymptotic expansion as $\tau \rightarrow 0$ in V provided that $\omega(0) \in (-1/2, 1/2)$ and $\omega(0) \in (-1, 1)$ respectively.*

Furthermore, the following conditions are equivalent.

- (1) $B(\omega(\tau)|\tau)$ can be continued into an analytic function at $\tau = 0$.
- (2) $P(\omega(\tau)|\tau)$ can be continued into an analytic function at $\tau = 0$.
- (3) $\omega(\tau) = m\tau/2$ for some $m \in \mathbb{Z}$.

Proof. By using (3.16) and (4.23), both $B(\omega(\tau) | \tau)$ and $P(\omega(\tau) | \tau)$ can be expressed in terms of $b(\omega(\tau) | \tau)$ as follows:

$$B(\omega(\tau) | \tau) = b(-\omega(\tau) - \frac{1}{2} | \tau) - b(\omega(\tau) - \frac{1}{2} | \tau)$$

and

$$P(\omega(\tau) | \tau) = -b(\omega(\tau) | \tau) + b(-\omega(\tau) | \tau).$$

Thus, Theorem 5.3 implies the existence of the sector $V(I; R)$ in what $B(\omega(\tau) | \tau)$ and $P(\omega(\tau) | \tau)$ are analytic and have a Gevrey asymptotic expansion as $\tau \rightarrow 0$ when $\omega(0) \in (-1/2, 1/2)$ and $\omega(0) \in (-1, 1)$ respectively.

In order to see that the conditions (1) and (3) are equivalent, one makes use of the relations in (3.24), (3.25), observing that

$$z \in \frac{\tau}{2} \mathbb{Z} \iff \frac{z + 1/2}{\tau} = \frac{-z + 1/2}{\tau} \pmod{\mathbb{Z}}.$$

Finally, combining the expression of $P(z | \tau)$ in (4.23) with the Stokes's relations (3.8) and (3.9) allows one to obtain the following equation: for all $\tau \in \mathcal{H}$,

$$P(z | \tau) - P^-(z | \tau) = \log \frac{(-(z + 1)/\tau | -1/\tau)_\infty}{-(-z + 1)/\tau | -1/\tau)_\infty},$$

where $P^-(z | \tau)$ denotes the function defined by (4.22) with $d \in (0, \pi)$. In this way, a similar reasoning implies that both statements (3) and (2) are equivalent. \square

5.4. Necessary conditions for an Euler product to have an exponential-convergent expansion.

Proof of Theorem 5.1. By using (2.1), one can suppose, without any loss of generality, that $\omega(0) \in [-1/2, 1/2)$.

The rest of the proof will be composed of two parts, relatively independent, each of which will be devoted to one case of the value of ω at 0.

Case (1): $\omega(0) \in (-1/2, 1/2)$. By putting $z = \omega(\tau)$ into the expression given in (1.22) by Theorem 1.3 yields that

$$(5.6) \quad \check{f}(\tau) = \frac{1}{2\pi i} \log f(q) = A(\tau) + L(\tau) + \frac{1}{2\pi i} B(\omega(\tau) | \tau) + R(\tau),$$

where one introduces the following notation:

$$A(\tau) = \frac{1}{2\pi i} \log \left(e\left(-\frac{\tau}{24}\right) \sqrt{1 + e(\omega(\tau))} \right) = -\frac{\tau}{24} + \frac{1}{4\pi i} \log(1 + e(\omega(\tau))),$$

$$L(\tau) = -\frac{\text{Li}_2(-\omega(\tau))}{4\pi^2 \tau}, \quad R(\tau) = \frac{1}{2\pi i} \log\left(\frac{\omega(\tau) - 1/2}{\tau} \middle| -\frac{1}{\tau}\right)_\infty.$$

On the one hand, as $\omega(0) - 1/2 < 0$, relation Lemma 2.2 implies that $R(\tau) \in \mathcal{A}_0^{\leq -1}$. On the other hand, it is easy to see that

$$A(\tau) \in \mathbb{C}\{\tau\}, \quad L(\tau) = -\frac{\text{Li}_2(-\omega(0))}{4\pi^2 \tau} \pmod{\mathbb{C}\{\tau\}}.$$

Thus, comparing (1.16) ($\alpha = 0$) with (5.6) yields that, as $f \in \mathfrak{C}_1$:

$$(5.7) \quad B(\omega(\tau) | \tau) = \nu \log \tau + \frac{\lambda}{\tau} + \varphi(\tau) + \psi(\tau),$$

where $\nu, \lambda \in \mathbb{C}$, $\varphi \in \mathbb{C}\{\tau\}$ and $\psi \in \mathcal{A}_0^{\leq -1}$.

By Theorem 5.4, one finds that $B(\omega(\tau) | \tau)$ has a Gevrey asymptotic expansion as $\tau \rightarrow 0$ in some sector $V = V(I; R)$, where $I = (-\pi, \pi)$ and $R > 0$. One finds that

$\nu = 0$ and $\lambda = 0$ in (5.7). Furthermore, the germ of exponentially small function ψ will be bounded in any proper sub-sector of V . As the openness of V is larger than π , a classical argument such as Phragmen-Lindeloff Theorem implies that $\psi \equiv 0$; see also [4] for this matter.

Consequently, all terms excepted φ are necessarily null in the expected expression of $B(\omega(\tau) | \tau)$ in (5.7). This together with Theorem 5.4 (3) imply that $\omega(\tau) = \frac{m}{2} \tau$ with $m \in \mathbb{Z}$.

Case (2): $\omega(0) = -1/2$. Let $\omega_1(\tau) = \omega(\tau) + 1/2$. If $\omega_1(\tau) = m\tau$ with $m \in \mathbb{Z}_{\leq 0}$, then $f \equiv 0$ and $f \in \mathfrak{C}_1$. Suppose then that $\omega_1(\tau) \neq m\tau$ for any $m \in \mathbb{Z}_{\leq 0}$. In view of (2.1) and Lemma 2.1 (1), one can suppose that

$$\omega_1(\tau) = \mu\tau + O(\tau^2) \quad (\tau \rightarrow 0 \in \mathbb{C}),$$

where $\mu > 0$.

Putting $z = \omega_1(\tau)$ in (4.20) of Theorem 4.2 gives that

$$f(q) = \frac{\sqrt{2\pi s(1 - e(\omega_1(\tau)))}}{\Gamma(s+1)} e(-\frac{\tau}{24}) e^{s(\log s - 1) + L(\tau) + P(\omega_1(\tau) | \tau)} R_1(\tau),$$

so that

$$(5.8) \quad \check{f}(\tau) = \frac{1}{4\pi i} \log \frac{\tau}{i} + G(\tau) + A(\tau) + \frac{1}{2\pi i} L(\tau) + \frac{1}{2\pi i} P(\omega_1(\tau) | \tau) + R(\tau).$$

In the above, $s = \omega_1(\tau)/\tau$,

$$G(\tau) = \frac{s}{2\pi i} (\log s - 1) - \frac{\log \Gamma(s+1)}{2\pi i} + \frac{\log 2\pi}{2\pi i},$$

$$A(\tau) = -\frac{\tau}{24} + \frac{1}{4\pi i} \log \frac{e(\omega_1(\tau)) - 1}{2\pi i \omega_1(\tau)},$$

$$L(\tau) = \frac{\text{Li}_2(e(\omega_1(\tau)))}{2\pi i \tau} + \log(\omega_1(\tau))$$

and

$$R(\tau) = \frac{1}{2\pi i} \log R_1(\tau), \quad R_1(\tau) = \left(\frac{\omega_1(\tau) - 1}{\tau} \mid -\frac{1}{\tau} \right)_\infty.$$

Since $\omega_1(\tau)/\tau \rightarrow \mu > 0$ for $\tau \rightarrow 0 \in \mathbb{C}$, it follows that $G \in \mathbb{C}\{\tau\}$. And this is the same for the function $A(\tau)$. Furthermore, by Lemma 2.2, one gets that $R \in \mathcal{A}_0^{\leq -1}$.

In order to compare (1.16) with the expression in (5.8), we shall consider $L(\tau)$ and $P(\omega_1(\tau) | \tau)$. When τ makes a complete rotation along a circle around $\tau = 0$, the image $X = e(\omega_1(\tau))$ forms a circle around $X = 1$. By (4.16), one finds that

$$\text{Li}_2(e(\omega_1(\tau e^{2\pi i}))) - \text{Li}_2(e(\omega_1(\tau))) = -2\pi i \log(e(\omega_1(\tau))),$$

so that

$$(5.9) \quad L(\tau e^{2\pi i}) - L(\tau) = -2\pi i \left(\frac{\omega_1(\tau)}{\tau} - 1 \right).$$

Thus, if one defines

$$(5.10) \quad \tilde{L}(\tau) = L(\tau) - \left(\frac{\omega_1(\tau)}{\tau} - 1 \right) \log \tau,$$

then \tilde{L} is a function without monodromy at $\tau = 0$.

Let

$$(5.11) \quad P^*(\tau) = \frac{\omega_1(\tau) - (\mu - 1)\tau}{\tau} \log \tau + P(\omega_1(\tau) | \tau).$$

It follows that

$$L(\tau) + P(\omega_1(\tau) | \tau) = \tilde{L}(\tau) + P^*(\tau).$$

By comparing (5.8) with (1.16) with the help of (5.9), (5.10) and the above analysis for $G(\tau)$, $A(\tau)$ and $R(\tau)$, one finds that, if $f \in \mathfrak{C}_1$, then $P^*(\tau) \bmod \mathcal{A}_0^{\leq -1}$ can be continued into an analytic function at $\tau = 0$. This is possible only when both condition in the below are fulfilled: (i) $\omega_1(\tau) = \alpha\tau$ for some $\alpha \in \mathbb{C}$ and, (ii) $P(\omega_1(\tau) | \tau) \bmod \mathcal{A}_0^{\leq -1}$ can be continued into an analytic function at $\tau = 0$. By applying Theorem 5.4 as what done for $B(\omega(\tau) | \tau)$ in the above, one obtains finally that $\omega_1(\tau) = \frac{1}{2}m\tau$, $m \in \mathbb{Z}$.

In summary, in both cases (1) and (2), one finds that $f \in \mathfrak{C}_1$ implies that $\omega(0) = \frac{1}{2}m\tau + \epsilon$, with $\epsilon = 0$ or $-\frac{1}{2}$. This ends the proof of Theorem 5.1. \square

In the above, one can notice that the function \tilde{L} given in (5.10) is analytic at $\tau = 0$ in \mathbb{C} . At the same time, if $\omega_1(\tau) \neq \mu\tau$, the function $P^*(\tau)$ defined in (5.11) would contain two essentially different types of singularities, one being of log-type, the other being Gevrey-type.

6. ASYMPTOTIC BEHAVIOUR AT AN ARBITRARY ROOT VIA CONTINUED FRACTIONS

By taking into account Theorem 5.1 together with Theorem 2.2 (1), Theorem 1.2 will be established in full with the help the following

Theorem 6.1. *Let $r \in \mathbb{Q} \cap (0, 1)$, $\zeta = e(r)$, and let $\omega(\tau)$ be a germ of analytic function at $\tau = r$ in \mathbb{C} such that $\omega(r) \in \mathbb{R}$. One has $f(q) = (\omega(\tau) | \tau)_\infty \in \mathfrak{C}_\zeta$ if, and only if, there exist $n, m \in \mathbb{Z}$ such that $\omega(\tau) = \frac{1}{2}(n + m\tau)$.*

The key-point that permits us to pass an arbitrary rational value r to the case of $r = 0$ consists of using continued fractions relative to r ; see Theorem 6.4. In fact, one will observe, in §6.1, that the above-introduced functions B and P are analytic at each generic non-zero rational point.

6.1. Bernoulli integral and associated functions on real axis. We will discuss the degenerate case $\tau \in \mathbb{R}_{>0}$ for each of the functions $b(z | \tau)$, $B(z | \tau)$ and $P(z | \tau)$ considered in Sections §3, §4 and §5. In what follows, we will make use of the following notations: on the one hand,

$$(6.1) \quad \epsilon \in (0, \frac{\pi}{2}), \quad W_\epsilon = V((-\epsilon, \epsilon); \infty), \quad W_\epsilon^c = \mathbb{C} \setminus \bar{W}_\epsilon;$$

on the other hand, the letter r will always denote some given positive number.

First of all, we consider the function $b(z | \tau)$. By Definition 3.1 and Remark 4.1, $b(z | \tau)$ is well-defined and analytic in the domain $(-W_\epsilon^c - 1) \times W_\epsilon$, where

$$-W_\epsilon^c - 1 = \mathbb{C} \setminus (-\bar{W}_\epsilon - 1).$$

See Figure 10 in the below.

One observes that, if $\epsilon \rightarrow 0^+$, then:

$$W_\epsilon \rightarrow (0, \infty), \quad -W_\epsilon^c - 1 \rightarrow \mathbb{C} \setminus (-\infty, -1].$$

Let Δ_τ be as given in (4.4) for all $\tau \in \mathcal{H}$. By replacing τ with r , we will continue to write

$$(6.2) \quad \Delta_r = \{n + mr : n \in \mathbb{Z}_{\leq -1}, m \in \mathbb{Z}_{\leq 0}\}.$$

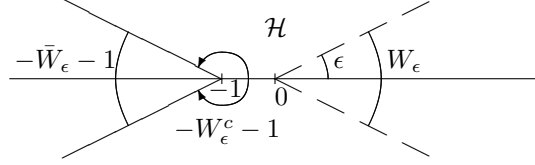


FIGURE 10. $b(z|\tau)$ is analytic for $z \in (-W_\epsilon^c - 1)$ and $\tau \in W_\epsilon$

If $z_0 \in (-\infty, -1] \setminus \Delta_r$, we define

$$(6.3) \quad \check{b}(z_0|r) = ([-z_0] - 1 - 2 \sum_{k=1}^{[-z_0]-1} \{\frac{z_0+k}{r}\}) \pi i.$$

where, for any $\alpha \in \mathbb{R}$, $[\alpha]$ denotes the integral part of α and $\{\alpha\}$ is the corresponding fractional part, that is $\alpha - [\alpha]$.

Theorem 6.2. *Let $r \in \mathbb{R}_{>0}$ and let Δ_r, \check{b} be as in the above. Then $b(z|r)$ is analytic in $\mathbb{C} \setminus (-\infty, -1]$ and can be continued to be an analytic function over the universal covering of $\mathbb{C} \setminus \Delta_r$ such that the following relation holds for all $z_0 \in (-\infty, -1] \setminus \Delta_r$:*

$$(6.4) \quad \lim_{\epsilon \rightarrow 0^+} (b(z_0 + i\epsilon|r) - b(z_0 - i\epsilon|r)) = \check{b}(z_0|r).$$

Proof. Let $D(z_0)$ denote the expression in the left-hand side of (6.4). By putting $\tau = r$ into (4.2), one finds that, if $z \notin (-\infty, 0]$, then:

$$b(z-1|r) = b(z|r) + I(s),$$

where $s = z/r$ and $I(s)$ is as given in (4.9). Thus, one can write

$$D(z_0 - 1) = D(z_0) + \lim_{\epsilon \rightarrow 0^+} (I(\frac{z_0}{r} + \epsilon i) - I(\frac{z_0}{r} - \epsilon i))$$

for all $z_0 \in (-\infty, 0] \setminus \Delta_r$. By using (4.9), one gets that

$$(6.5) \quad D(z_0 - 1) = D(z_0) + (1 - \frac{2z_0}{r})\pi i + \lim_{\epsilon \rightarrow 0^+} (\log \Gamma(\frac{z_0}{r} + \epsilon i) - \log \Gamma(\frac{z_0}{r} - \epsilon i)).$$

Let n be any negative integer, say, $n = -m$, $m > 0$, and let $s \in (n, n+1)$. From the relation

$$\Gamma(x) = \frac{\Gamma(x+m)}{(x)_m}$$

and the fact that $\log \Gamma(x+m)$ is well-defined and analytic for $\Re(x) > n$, it follows that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} (\log \Gamma(s + \epsilon i) - \log \Gamma(s - \epsilon i)) &= - \lim_{\epsilon \rightarrow 0^+} (\log(s + \epsilon i)_m - \log(s - \epsilon i)_m) \\ &= 2\pi i n. \end{aligned}$$

Therefore, (6.5) yields that

$$(6.6) \quad D(z_0 - 1) = D(z_0) + 2(n - \frac{z_0}{r})\pi i + \pi i.$$

Letting $z_0 = (n + \delta_0)r$ with $n_0 \in \mathbb{Z}_{<0}$ and $\delta_0 \in (0, 1)$, the above expression in (6.6) becomes

$$D(z_0 - 1) - D(z_0) = -2\pi i \delta_0 + \pi i.$$

By replacing z_0 with $z_0 + 1$ and iterating this process, one obtains the finite sequences (n_k) and (δ_k) associated with the pair (z_0, r) in the following manner:

$$z_0 + k = (n_k + \delta_k)r, \quad n_k \in \mathbb{Z}_{<0}$$

with $\delta_k = \delta_k(z_0 | r) = \{\frac{z_0 + k}{r}\}$. Since $D(z_0 + k) = 0$ for $k > -1 - z_0$, one finds that

$$D(z_0 - 1) = -2(\delta_0 + \dots + \delta_{[-z_0]-1})\pi i + [-z_0]\pi i.$$

Replacing z_0 with $z_0 + 1$ in this last relation gives $D(z_0) = \check{b}(z_0 | r)$, where \check{b} is as given in (6.3), so one obtains the expected relation (6.4) and Theorem 6.2. \square

Let $\epsilon \in (0, \frac{\pi}{2})$ and let W_ϵ, W_ϵ^c be as in (6.1); see Figure 10. By using (3.16), one finds that $B(z | \tau)$ is analytic in the domain $((1/2 + W_\epsilon^c) \cap (-1/2 - W_\epsilon^c)) \times W_\epsilon$. Similarly, from (4.23) one obtains that $P(z | \tau)$ can be continued to be analytic in the domain $((-1 - W_\epsilon^c) \cap (1 + W_\epsilon^c)) \times W_\epsilon$. See Figure 11.

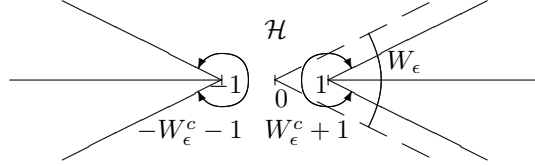


FIGURE 11. $P(z | \tau)$ is analytic for $z \in (-W_\epsilon^c - 1) \cap (W_\epsilon^c + 1)$ and $\tau \in W_\epsilon$

By letting $\epsilon \rightarrow 0^+$, one sees that, for any $r > 0$, $B(z | r)$ is analytic for all $z \in \mathbb{C} \setminus (-\infty, -1/2) \cup (1/2, \infty)$ while $P(z | r)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)$.

Theorem 6.3. *Let $r > 0$, let Δ_r and \check{b} be as in (6.2) and (6.3).*

- (1) *The function $B(z | r)$ can be continued to be analytic in the universal covering of $\mathbb{C} \setminus ((1/2 + \Delta_r) \cup (-1/2 - \Delta_r))$, and the following relations hold for all $z_0 \in \mathbb{R} \setminus ((1/2 + \Delta_r) \cup (-1/2 - \Delta_r))$:*

$$(6.7) \quad \lim_{\epsilon \rightarrow 0^+} (B(z_0 + i\epsilon | r) - B(z_0 - i\epsilon | r)) = -\check{b}(-|z_0| - \frac{1}{2} | r).$$

- (2) *The function $P(z | r)$ can be continued to be analytic in the universal covering of $\mathbb{C} \setminus (\Delta_r \cup (-\Delta_r))$, in such a way that, for all $z_0 \in \mathbb{R} \setminus (\Delta_r \cup (-\Delta_r))$,*

$$(6.8) \quad \lim_{\epsilon \rightarrow 0^+} (P(z_0 + i\epsilon | r) - P(z_0 - i\epsilon | r)) = -\check{b}(-|z_0| | r).$$

Proof. This follows directly from Theorem 6.2 together with relations (3.16) and (4.23). \square

6.2. Continued fractions and modular transforms. Let $(p, m) \in \mathbb{Z} \times \mathbb{Z}_{>0}$, $p \wedge m = 1$, $r = \frac{p}{m}$, and let $\zeta = e(r)$. We will consider the asymptotic behavior of the Euler product $(z | \tau)_\infty$ when $\tau \xrightarrow{a.v.} \frac{p}{m}$, or equivalently, when $q \xrightarrow{a.r.} \zeta = e(\frac{p}{m})$. The case $p = 0$ has just been previously treated in Section §5; without any loss of generality, one will assume in what follows that $\frac{p}{m} \in \mathbb{Q} \cap (0, 1)$.

As in (6.3), we will let $[\alpha]$ and $\{\alpha\}$ to denote respectively the integral and fractional part of any given real α . The strategy that will be put into place consists of making use of continued fractions to reduce the general case $\tau \xrightarrow{a.v.} \frac{p}{m}$ into the known case $\tau \xrightarrow{a.v.} 0$. This is possible because of the following

Lemma 6.1. *Let $r \in (0, 1)$, $\zeta = e(r)$, $\zeta_1 = e(\{-\frac{1}{r}\})$, and let $\omega \in \mathbb{C}\{\tau - r\}$. Consider*

$$f(q) = (\omega(\tau) | \tau)_\infty, \quad g(q_1) = (\omega_1(\tau_1) | \tau_1)_\infty,$$

where

$$q = e(\tau), \quad \tau_1 = -\frac{1}{\tau} + \left[-\frac{1}{r}\right], \quad q_1 = e(\tau_1)$$

and

$$\omega_1(\tau_1) = \left(\tau_1 - \left[-\frac{1}{r}\right]\right) \left(1 - \omega\left(-\frac{1}{\tau_1 - [-1/r]}\right)\right).$$

Then $f \in \mathfrak{C}_\zeta$ if, and only if, $g \in \mathfrak{C}_{\zeta_1}$.

Proof. Let $d \in \mathbb{Z}$ and let $T(z, \tau)$ to be given as follows:

$$(6.9) \quad (z | \tau)_\infty = T(z, \tau) \left(\frac{z-1}{\tau} | -\frac{1}{\tau} + d \right)_\infty.$$

By (1.22) and (4.20), $T(z, \tau)$ can be expressed by means of each one of both functions $B(z - 1/2 | \tau)$ and $P(z | \tau)$ in the following manner:

$$(6.10) \quad T(z, \tau) = A\left(z - \frac{1}{2} | \tau\right) e^{L_2(z-1/2 | \tau) + B(z-1/2 | \tau)}$$

and

$$(6.11) \quad T(z, \tau) = \frac{\sqrt{2\pi s(1-e(z))}}{\Gamma(s+1)} e\left(-\frac{\tau}{24}\right) e^{s(\log s - 1) + \frac{\text{Li}_2(e(z))}{2\pi i \tau} + P(z | \tau)}$$

Putting $z = \omega(\tau)$, $d = [-r]$ and $\tau = -1/(\tau_1 - d)$ in (6.9) gives

$$f(q) = \tilde{T}(q) g(q_1), \quad \tilde{T}(e(\tau)) = T(\omega(\tau), \tau).$$

Thanks to (2.1), one can suppose that $\omega(r) > 0$, so that in (6.11), $s \rightarrow \omega(r)/r \neq 0$ as $\tau \xrightarrow{a.v.} r > 0$. By means of Theorem 6.3, one can find $\nu \in \mathbb{C}$ such that

$$(6.12) \quad \log \tilde{T}(\tau) = \nu \log(\tau - r) \pmod{\mathbb{C}\{\tau - r\}}.$$

From (6.12), one obtains that $\tilde{T} \in \mathfrak{C}_\zeta$. By Lemma 2.1 (1), it follows that $f \in \mathfrak{C}_\zeta$ if, and only if, $f/\tilde{T} \in \mathfrak{C}_\zeta$, which is equivalent to say that $g \in \mathfrak{C}_{\zeta_1}$. \square

The above operation $(q, \tau, \omega) \mapsto (q_1, \tau_1, \omega_1)$ considered in Lemma 6.1 allows one to link two limit rationals r and r_1 as follows. For any given $r \in \mathbb{Q} \cap (0, 1)$, one can iterate the process

$$(6.13) \quad (r, z, \tau) \longrightarrow \left(\left\{ -\frac{1}{r} \right\}, \frac{z-1}{\tau}, \left[-\frac{1}{r} \right] - \frac{1}{\tau} \right)$$

and, after a finite number of steps, one arrives at the case where $\tau \xrightarrow{a.v.} 0$. This iteration procedure requires to write r into continued fraction. Thus, to any given $r = \frac{p}{m} \in \mathbb{Q} \cap (0, 1)$ will be associated the sequences $r_j \in \mathbb{Q} \cap [0, 1)$ and $d_j \in \mathbb{Z}_{>0}$ in the following manner:

$$(6.14) \quad r_0 = \frac{p}{m}, \quad d_0 = 0; \quad r_j = \left\{ -\frac{1}{r_{j-1}} \right\}, \quad d_j = \left[-\frac{1}{r} \right] \quad (1 \leq j \leq \nu),$$

where ν denotes the smallest index such that $r_\nu = 0$, i.e. $1/r_{\nu-1} \in \mathbb{Z}_{>0}$. With the standard notation for the continued fractions, one can notice that

$$(6.15) \quad r = [0, d_1, -d_2, \dots, (-1)^{\nu-1} d_\nu] = \frac{1}{d_1 - |} \frac{1}{d_2 - |} \cdots \frac{1}{|d_\nu}.$$

Let r be as in (6.15) and let $(z, \tau) \in \mathbb{C} \times \mathcal{H}$. In the same fashion as in (6.13), we introduce the r -depending sequence $(z_j, \tau_j)_{0 \leq j \leq \nu}$ as follows: $\tau_0 = \tau$, $z_0 = z$;

$$(6.16) \quad z_j = \frac{z_{j-1} - 1}{\tau_{j-1}}, \quad \tau_j = d_j - \frac{1}{\tau_{j-1}}.$$

Moreover, for any given $\omega \in \mathbb{C}\{\tau - r\}$, we will write

$$(6.17) \quad \omega_0(\tau_0) = \omega(\tau), \quad \omega_j(\tau_j) = \frac{\omega_{j-1}(\tau_{j-1}) - 1}{\tau_{j-1}},$$

where τ_j is linked with τ_{j-1} as in (6.16). If we let $\tau = \tau_0 \xrightarrow{a.v.} r$ in \mathcal{H} , then $\tau_j \xrightarrow{a.v.} r_j$ in \mathcal{H} , particularly with $\tau_\nu \xrightarrow{a.v.} 0$; thus one can observe that $\omega_j \in \mathbb{C}\{\tau_j - r_j\}$ for all positive integer $j \leq \nu$. Furthermore, it is easy to see that $\tau_j \in \mathcal{H}$, with

$$\tau_j = M_j \tau_{j-1}, \quad M_j = \begin{pmatrix} d_j & -1 \\ 1 & 0 \end{pmatrix} \in SL(2; \mathbb{Z}).$$

Thus, one can find that

$$\tau_\nu = M \tau, \quad M = M_\nu \dots M_1 \in SL(2; \mathbb{Z}).$$

Theorem 6.4. *Let ν, r be as in (6.15), $\omega \in \mathbb{C}\{\tau - r\}$, and let τ_j, ω_j be as in (6.16), (6.17), respectively. Let $\zeta_j = e(r_j)$ and $q_j = e(\tau_j)$ for j from 0 to ν . Consider*

$$f(q) = (\omega(\tau)|\tau)_\infty, \quad f_j(q_j) = (\omega_\nu(\tau_j)|\tau_j)_\infty$$

for τ and τ_j enough closed to r and r_j in \mathcal{H} respectively. The following conditions are equivalent.

- (1) $f \in \mathfrak{E}_\zeta$.
- (2) $f_j \in \mathfrak{E}_{\zeta_j}$ for all j from 1 till ν .
- (3) $\omega(\tau) = \frac{1}{2}(\ell\tau + n)$ for some $(\ell, n) \in \mathbb{Z}^2$.

Proof. The equivalence between (1) and (2) follows directly from Lemma 6.1.

(2) \Rightarrow (3): One considers $j = \nu$ in Condition (2), with $r_\nu = 0$, $\zeta_\nu = 1$. Thus, applying Theorem 5.1 to f_ν implies that $\omega_\nu(\tau_\nu) = \frac{1}{2}(n_\nu + m_\nu\tau_\nu)$ for some suitable $(n_\nu, m_\nu) \in \mathbb{Z}^2$. Moreover, by (6.17) it follows that

$$\omega_j(\tau_j) = \frac{1}{2}(a_j\tau_j + b_j) \iff \omega_{j-1}(\tau_{j-1}) = \frac{1}{2}(a_{j-1}\tau_{j-1} + b_{j-1}),$$

where $a_{j-1} = a_j d_j + b_j$, $b_{j-1} = -a_j$. As $d_j \in \mathbb{Z}$, one finds that

$$(a_j, b_j) \in \mathbb{Z}^2 \iff (a_{j-1}, b_{j-1}) \in \mathbb{Z}^2.$$

This gives the expression wanted in Condition (3) for $\omega(\tau)$.

(3) \Rightarrow (2): This follows directly from Theorem 2.2 (1). \square

6.3. Concluding proofs. On the one hand, it is clear that Theorem 6.4 contains Theorem 6.1. On the other hand, one gets Theorem 1.2 by combining Theorem 2.2 (1), Theorem 5.1 and Theorem 6.1.

Proof of Theorem 1.1. Corollary 2.1 gives that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) for Theorem 1.1. To obtain the last implication (5) \Rightarrow (1), one supposes that $(x; q^2)_\infty \in \mathfrak{E}_\zeta$, so $(x; q^2)_\infty \in \mathfrak{E}_\zeta$. By applying Theorem 1.2 with q^2 instead of q , one gets that $x = e(z)$ and $z = \frac{n}{2} + \frac{m}{2}\tau'$ for some $(n, m) \in \mathbb{Z}^2$, where $\tau' = 2\tau$. Considering Theorem 2.2 (2) with τ' in the place of τ yields that $z \in \{\tau', \frac{1}{2} + \tau', \frac{1}{2}\tau', \frac{1}{2} + \frac{1}{2}\tau'\}$, so $x \in \{q, -q, q^2, -q^2\}$. This ends the proof of Theorem 1.1. \square

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