APPELL-LERCH SERIES VIEWED AS MOCK THETA FUNCTIONS

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Abstract. The goal of this paper is to prove that the quotient of a first order Appell-Lerch series by a suitable theta-function can be written, near every given root of unity, as the sum of two functions, one of which has a finite limit and the other one has an asymptotic behavior like as one Jacobi's theta-function when q tends to this root. In order to simplify the exposition, we propose the definition of what we mean almost theta-type function, false theta-type function and mock theta-type function. The utilization of continued fractions and linear fractional transformations plays a central role in this paper.

Contents

Date: November 14, 2015.

Introduction

The Ramanujan's mock-theta functions have a ultimate link with Appell-Lerch series. For this nice subject, one can see the short paper [19] of K. Ono in the Notices of the AMS or, more extensively, the classical references such as [30], [11], [2], [12], [39], [32], etc .... The goal of our paper is to study the asymptotic behaviour of these series near every root of unity, in the hope of contributing later to a best possible knowledge of the world of the Ramanujan's mock-theta functions.

0.1. **Appell-Lerch series, q-difference equations and Gevrey asymptotics.** In [16], one can find the following definition of Appell-Lerch series R_{k} :

$$R_{\mathsf{k}}(z,w\,|\,\tau)=\sum_{n=-\infty}^{\infty}\frac{q^{\mathsf{k}n^2/2}\,e^{2\mathsf{k}nz\pi i}}{1-q^n\,e^{2w\pi i}}=\sum_{n=-\infty}^{\infty}\frac{e(\frac{1}{2}n^2\mathsf{k}\tau+n\mathsf{k}z)}{1-e(n\tau+w)}\,, \tag{0.1}$$

where $\mathsf{k}\in\mathbb{Z}_{>0}$, $z\in\mathbb{C}$, $w\in\mathbb{C}\setminus\big(\mathbb{Z}\oplus\tau\mathbb{Z}\big)$, $\tau\in\mathcal{H}$ and where $e(.)$ is defined by the relation $e(a)=e^{2\pi ia}$ for all $a\in\mathbb{C}$. By considering the relation

$$1-e(n\tau+w)=\frac{1-e(\mathsf{k}n\tau+\mathsf{k}w)}{1+e(n\tau+w)+...+e\big((\mathsf{k}-1)n\tau+(\mathsf{k}-1)w\big)}\,,$$

one finds from (0.1) that

$$R_{\mathsf{k}}(z,w\,|\,\tau)=\sum_{\ell=0}^{\mathsf{k}-1}e(\ell w)\,R_1(\mathsf{k}z+\ell\tau,\mathsf{k}w\,|\,\mathsf{k}\tau) \tag{0.2}$$

provided that $w\notin\frac{1}{\mathsf{k}}\,\mathbb{Z}\oplus\tau\mathbb{Z}$. This implies that the first order Appell-Lerch series $R_1(z,w\,|\,\tau)$ plays a basic role.

By direct computation, two functional equations can be easily found: on the one hand,

$$\begin{aligned}R_{\mathsf{k}}(z+\tau,w+\tau\,|\,\tau)\;&=\;\sum_{n=-\infty}^{\infty}\frac{e(\frac{1}{2}(n+1)^2\mathsf{k}\tau+n\mathsf{k}z-\frac{1}{2}\mathsf{k}\tau)}{1-e((n+1)\tau+w)}\\&=\;e(-\mathsf{k}(z+\frac{\tau}{2}))\,R_{\mathsf{k}}(z,w\;|\,\tau)\,;\end{aligned} \tag{0.3}$$

and on the other hand,

$$R_{\mathsf{k}}(z,w\,|\,\tau)-e(w)\,R_{\mathsf{k}}(z+\frac{\tau}{\mathsf{k}},w\,|\,\tau)\;=\;\sum_{n=-\infty}^{\infty}e(\frac{1}{2}n^2\mathsf{k}\tau+n\mathsf{k}z)$$

$$=\;\vartheta_3(\mathsf{k}z\,|\,\mathsf{k}\tau)\,, \tag{0.4}$$

where $\vartheta_3(z\,|\,\tau)$ denotes one of four classic theta-functions of Jacobi [20, p. 166, (76.1)]. Putting $\mathsf{k}=1$ into the relation in (0.4) yields that

$$R_1(z,w\,|\,\tau)-e(w)\,R_1(z+\tau,w\,|\,\tau)=\vartheta_3(z\,|\,\tau)\,. \tag{0.5}$$

In the above, one sees that each series $R_1(z,w\,|\,\tau)$ can be considered as *special* solution of some functional equations using difference operators of the form $z\mapsto z+\tau$, as in the classical theory of elliptic functions. In fact, in [6] and [7], Appell studied elementary decomposition of the elliptic functions of the third kind, introduced by Hermite [15]. Later, Lerch [16] studied several series with the view of writing a general elliptic function as linear combination of simple ones, where the factors may be theta functions. These series appeared also in the analytic theory of numbers and in particular in the theory of quadratic forms; see [17].

Remember that the traditional elliptic functions are all meromorphic functions in $\mathbb{C}$ that have two periods. The corresponding multiplicative model is to say that an elliptic function is simply a meromorphic function in $\mathbb{C}^*$ that is left invariant by the q-difference operator $x\mapsto qx$. As in the theory of differential equations, a q-difference equation may be singular and the structure of its analytic solutions may be made depending on whether the singular point is Fuchsian or non-Fuchsian; see [28], [26] and the references therein. It will be useful to observe that the relation of $R_1(z,w\,|\,\tau)$ in (0.5) can be transformed into a non-homogeneous and non-Fuchsian linear q-difference equation of the first order.

This is why, in this paper, we shall take the point of view of singular-irregular q-difference equations. Namely, apart from a factor of theta function, $R_1(z,w\,|\,\tau)$ is exactly the sum-function along a spiral for the q-Euler series $\hat{E}(x;q)$ ([27], [34]):

$$\hat{E}(x;q)=\sum_{n\geq 0}q^{-n(n-1)/2}\,(-x)^n\,. \tag{0.6}$$

Moreover, to this same divergent series is associated one other family of sum-functions by means of Gaussian kernel [33], which contains the well-known Mordell integrals [18]; see also [1]. Comparing these different sum-functions will furnish exponentially-small functions, that are in fact of modular-like type. In other words, modular-like transforms, exponential-scales changes or Stokes phenomena may come together. This implies the role that the singularities analysis of q-difference equation may play inside the theory of theta-like functions.

Beside, Watson has written several papers about Ramanujan's mock-theta functions, and one of the most popular may be [30]. At p. 78 of this paper, he said: *It can be proved that these expansions possess the property that (for α complex) the error due to stopping at any term never exceeds in absolute value the first term neglected; in addition, for α positive, the error is of the same sign as that term*.* The footnote here is the following: *This property... It is the fact that these expansions are asymptotic (and not terminating series) which shows that mock theta-functions are of a more complex character than ordinary theta-functions.*

This quotation shows that G. N. Watson knew well the natural role played by the asymptotic expansions for the mock-theta functions of Ramanujan. In the

following, we will make use of the theory of Gevery asymptotic expansions, whose origin can go back to G. N. Watson [29], with his famous Lemma. In fact, the theory of Gevrey asymptotic expansion is an exponential-type asymptotic analysis, and such an approach provides a framework around which the exponentially smallness appears in a natural fashion; see [8] or [25].

0.2. **Organization of the paper.** The rest of this paper is divided into five sections, the last one being included in Appendix. In §1, we shall propose the definition of what we mean theta-type function, by establishing that the classical Jacobi theta functions are really theta-type. This definition suggests how to distinguish one mock theta-type function from one false theta-type. By using all that, we will arrive at the statement of our main result of this paper, Theorem 1.3. This implies the following one:

The quotient of $R_1(z,w\,|\,\tau)$ by a suitable theta-function can be written, near every given root of unity, as the sum of two functions, one of which has a finite limit and the other one has an asymptotic behavior like as one Jacobi's theta-function when q tends to this root.

In the above, the decomposition will depend of the root of unity about what we study the asymptotic behavior. It will be important to notice that the finite part can not be uniformly bounded for all roots except a very few number of cases which are related to the half-periods. This is an essential distinction existing between one false theta function and one mock theta function, as observed by Ramanujan himself.

In §2, we will start with the analytic continuation of a Mordell integral $G(x;q)$, that is a sum-function of the divergent series $\hat{E}(x;q)$ given in (0.6). The central result of this section is Theorem 2.1, which allows one to consider this Gq-summation even for complex q of module $|q|<1$. Found by means of Laplace transform, this result implies the analytic obstruction with respect to the parameter x of this sum-function on the one hand and a generalized reciprocity law for Gauss sums on the other hand. See Theorems 2.2 and 2.5.

The sum-function $L(x,\mu;q)$ of $\hat{E}(x;q)$ will be studied in §3, and one will see that this is related to $R_1(z,w\,|\,\tau)$ via some theta-function factor. This sum-function and $G(x;q)$ yield a modular-like formula, and using this with continued fractions and linear fractional transformations allows one to get the asymptotic behavior of $L(x,\mu;q)$ at every root of unity of q. So, Subsection 3.3 and Theorem 3.6 play an important role for this section. Notice that Theorems 3.7 and 3.11 give conditions about the parameters x and μ or, equivalently, about z and w, for obtaining a false-theta type function.

In order to keep the reading smooth and easy, we will complete the proof of certain statements in §4. Finally, in §A, we will give some definitions and properties that we need in this paper in matter of Gevrey asymptotic analysis and q-Borel-Laplace summations.

1. Notations, definitions and main result

Let $\mathbb{D}$ be the open unit disc $|q|<1$, and let $\mathbb{D}^*=\mathbb{D}\setminus\{0\}$. As usual, we let $\mathcal{H}$ denote the Poincaré's half-plane $\Im z>0$. By the map $e\ :\tau\mapsto e(\tau)=e^{2\pi i\tau}$, the analytic space $\mathcal{H}$ is identified to the universal covering of $\mathbb{D}^*$. In the whole paper, $\mathbb{U}$ denotes the set of all roots of unity, that will be identified with $\mathbb{Q}\cap[0,1)$ via the relation $\mathbb{U}=e\big(\mathbb{Q}\cap[0,1)\big)$. By convention, $e(\alpha)=0$ when $\alpha=i\infty$.

In order to be able to use the linear fractional transformations for discussing the asymptotic behavior at a given root of unity, we shall extend the usual radial convergence into what we will mean *almost radial convergence*, and this will be defined in §1.1. Thus, in §1.2, we will give the definition of an almost theta-type function and that of false-theta or mock-theta in this class of functions. Some general properties of these functions, like as the algebraic structure between them and their dominant terms, will be described in §1.3.

As basic examples of theta-type functions, we will discuss Jacobi theta-functions in §1.4, that will be used later for the mock-type properties stated for Appell-Lerch series. Some functional relations will be mentioned in §1.5 for these series, before giving the main result of this paper, Theorem 1.3, in §1.6.

1.1. **Modular group, radial convergence and exponential smallness.** For any given $a \in \partial\mathbb{D}^*$, by *radially symmetric sector at a inside* $\mathbb{D}^*$ we mean any sector $V_a(d,r)$ defined for $d \in (0,\pi)$ and $r \in (0,1)$ in the following way: if $a \neq 0$,

$$V_a(d,r) = \left\{ q \in \mathbb{D}^* : |\arg(1-\frac{q}{a})| < \frac{d}{2},\ |q-a| < r \right\}; \tag{1.1}$$

otherwise, one writes

$$V_0(d,r) = \left\{ q \in \mathbb{D}^* : |\arg q| < \frac{d}{2},\ |q| < r \right\}. \tag{1.2}$$

With a view to extending the usual notion of radial convergence, we shall say that q *almost-radially tends towards* $a \in \partial\mathbb{D}^*$ and we will write $q \xrightarrow{a.r.} a$, if $q \to a$ within some radially symmetric sector at a inside $\mathbb{D}^*$.

Let $\alpha \in \partial\mathcal{H} \cup \{i\infty\}$; by *vertically symmetric sector at α in $\mathcal{H}$* we mean any sector $V_\alpha(\delta \,|\, \rho)$ defined for $\delta \in (0,\pi)$ and $\rho > 0$ as follows: if $\alpha \neq i\infty$,

$$V_\alpha(\delta \,|\, \rho) = \left\{ \tau \in \mathcal{H} : |\arg(\frac{\tau-\alpha}{i})| < \frac{\delta}{2},\ |\tau-\alpha| < \rho \right\}; \tag{1.3}$$

otherwise, one sets

$$V_{i\infty}(\delta \,|\, \rho) = \left\{ \tau \in \mathcal{H} : |\arg \frac{\tau}{i}| < \frac{\delta}{2},\ |\tau| > \rho \right\}. \tag{1.4}$$

We shall say that τ *almost-vertically tends towards* α and we will write $\tau \xrightarrow{a.v.} \alpha$, if $\tau \to \alpha$ inside some vertically symmetric sector at α in $\mathcal{H}$. See Remark A.1.

The modular group $SL(2;\mathbb{Z})$ acts naturally on $\mathcal{H}$ by the linear fractional transformation $\tau \mapsto M\tau$ for all $M \in SL(2;\mathbb{Z})$. One can find that this action is compatible with the above-introduced convergence notions in the following manner. Let $a \in \partial\mathbb{D}^*$, $\alpha \in \partial\mathcal{H} \cup \{i\infty\}$, and assume that $a = e(\alpha)$. Then the following conditions are equivalent for $q = e(\tau)$:

(1) $q \xrightarrow{a.r.} a$;
(2) $\tau \xrightarrow{a.v.} \alpha$;
(3) $M\tau \xrightarrow{a.v.} M\alpha$ for any $M \in SL(2;\mathbb{Z})$.

Moreover, if $\mathfrak{A}_a^r$ and $\mathfrak{A}_\alpha^v$ denote respectively the sheaves of germs of analytic functions in radially symmetric sector $V_a(d,r)$ and in vertically symmetric sector $V_\alpha(\delta \,|\, \rho)$, with $a = e(\alpha)$, then one can notice the following one-to-one correspondence:

$$\mathfrak{A}_a^r \ni f \ \mapsto\ f \circ e \in \mathfrak{A}_\alpha^v. \tag{1.5}$$

A function $\varphi \in \mathfrak{A}_\alpha^v$ will be called *exponentially small as* $\tau \xrightarrow{a.v.} \alpha$ and written $\varphi \in \mathfrak{E}_\alpha^{v,0}$, if there exist $\delta \in (0,\pi)$, $\rho > 0$ and $\kappa > 0$ such that $\varphi(\tau) = o(e^{-\kappa/|\tau-\alpha|})$ on $V_\alpha(\delta\,|\,\rho)$. Here and in the following, the local coordinate $\tau - \alpha$ needs to be read as $1/\tau$ if $\alpha = i\infty$.

1.2. **Definition of almost theta-type functions.** Given $a \in \partial\mathbb{D} \cup \{0\}$, we let $\mathfrak{G}_a^r$ to be the space of all $f \in \mathfrak{A}_a^r$ that has a Gevrey asymptotic expansion as $q \xrightarrow{a.r.} a$, this implies particularly that $f(q) = O(1)$ as $q \xrightarrow{a.r.} a$ for every $f \in \mathfrak{G}_a^r$. With the help of the application $f \mapsto \check{f}$ given in (1.5), one can define the sub-space $\mathfrak{G}_\alpha^v$ inside $\mathfrak{A}_\alpha^v$. See (A.3) and Remark A.2 for more details. If no confusion is possible, we will drop the upper indices r and v and simply write $\mathfrak{G}_a$ and $\mathfrak{G}_\alpha$ there.

In order to understand what might mean Ramanujan mock theta functions, we will make use of the following

Definition 1.1. *Let $q = e(\tau)$, $\tau \in \mathcal{H}$, $a = e(\alpha) \in \mathbb{U} \cup \{0\}$, and let $f \in \mathfrak{A}_a^r$.*

(1) *One says that $f(q)$ is of* **theta-type as** $q \xrightarrow{a.r.} a$ *and one writes $f \in \mathfrak{T}_a$, if for all $N \in \mathbb{Z}_{\geq 0}$, there exist $(\upsilon, \lambda) \in \mathbb{Q} \times \mathbb{R}$, a finite set $I = I_{a,N} \subset \mathbb{R}$ and a $\mathbb{C}^*$-valued map $\gamma = \gamma_{a,N}$ on I such that the following relation holds for $\tau \xrightarrow{a.v.} \alpha$:*

$$f(q) = \big(\frac{i}{\hat{\tau}}\big)^{\upsilon} e(\lambda\hat{\tau}) \big(\sum_{k\in I} \gamma(k)\, q_1^k + o(q_1^N)\big)\,, \tag{1.6}$$

where $\hat{\tau} = \tau - \alpha$ and $q_1 = e(-\frac{1}{\hat{\tau}})$.

(2) *One says that f is of* **almost theta-type as** $q \xrightarrow{a.r.} a$ *and one writes $f \in \tilde{\mathfrak{T}}_a$, if there exists $\vartheta \in \mathfrak{T}_a$ such that $f - \vartheta \in \mathfrak{G}_a$*

In (1.6) and in what follows, we make use of the following notational convention.

(1) For all $k \in \mathbb{R}$, $q_1^k = e(-\frac{k}{\hat{\tau}})$ as $q_1 = e(-\frac{1}{\hat{\tau}})$.
(2) When $a = 0$ and $\alpha = i\infty$, $\hat{\tau}$ should be read as $-\frac{1}{\tau}$, and $q = q_1$.
(3) The summation $\sum_{k\in I}(...)$ should be read as null whenever $I = \emptyset$.

Definition 1.2. *Let f be an analytic function defined in $\mathbb{D}^*$ or in the universal covering $\tilde{\mathbb{D}}^*$.*

(1) *One says that f is a* **theta-type function** *and one writes $f \in \mathfrak{T}$, if $f \in \mathfrak{T}_\zeta$ for all $\zeta \in \mathbb{U} \cup \{0\}$.*
(2) *One says that f is a* **false theta-type function** *and one writes $f \in \mathfrak{F}$, if $f \notin \mathfrak{T}$ and there exists $\vartheta \in \mathfrak{T}$ such that $f(q) - \vartheta(q) \in \mathfrak{G}_\zeta$ for all $\zeta \in \mathbb{U} \cup \{0\}$ and that, furthermore,*

$$\sup_{\zeta\in\mathbb{U}} \Big| \lim_{q \xrightarrow{a.v.} \zeta} (f(q) - \vartheta(q)) \Big| < \infty\,. \tag{1.7}$$

(3) *One says that f is a* **mock theta-type function** *and one writes $f \in \mathfrak{M}$, if $f \in \tilde{\mathfrak{T}}_\zeta$ for all $\zeta = \mathbb{U} \cup \{0\}$ and, moreover, f is not a false theta-type function.*

In his last letter to Hardy, Ramanujan emphasized the boundedness condition (1.7) that appeared for Rogers false theta-functions; see [30]. However, as most of mathematicians of his time, Ramanujan used the usual notion of asymptotic expansion instead of the Gevrey one, what would be latter initiated in [29] by Watson.

Beside, in [13, p. 98-99], one can find the definition of a *mock θ-function* and that of a *strong mock θ-function*; see also [3]. Our above definition in (3) may be seen to be situated between these two definitions. In order to avoid the confusion, we choose to call them *theta-type* instead of *theta*.

1.3. **Some general properties of almost theta-type functions.** With regard to the uniqueness of the decomposition of a theta-type function at a given point, one can notice the following

Remark 1.1. *Given f be as in* (1.6)*, if $I_{a,N} \neq \emptyset$ for some non-negative integer N, then the pair (υ, λ) is uniquely determined, independently of N. Moreover, $I_{a,N} \subset I_{a,M}$ and $\gamma_{a,M}\big|_{I_{a,N}} = \gamma_{a,N}$ for all $M > N$.*

Indeed, let $I_a = \cup_{N\geq 0} I_{a,N}$, and write $I_a = \{k_0, k_1, k_2, ...\}$ as an strictly increasing sequence. If $c_0 = \gamma(k_0)$, taking the logarithm for both sides of (1.6) gives that

$$\log f(q) = -2\pi i\, \frac{k_0}{\hat{\tau}} + \upsilon \log\big(\frac{i}{\hat{\tau}}\big) + \log c_0 + \lambda\, \hat{\tau} + o(e^{-\kappa/|\hat{\tau}|})\,, \tag{1.8}$$

where $\kappa > 0$. This implies the uniqueness of (υ, λ, c_0). The next coefficients $\gamma(k_n)$ for $n > 0$ can be determined successively by replacing in (1.8) the function f with its *k_n-th remainder* given as follows:

$$f(q) - \big(\frac{i}{\hat{\tau}}\big)^{\upsilon} e(\lambda\hat{\tau}) \big(\sum_{k\in I_a, k<k_n} \gamma(k)\, q_1^k \big)\,. \qquad \square$$

For simplify, we shall call respectively *dominant term of f at a* and *principal part of f at a*, the expressions $t_a(f)$ and $P_a(f)$ given as follows:

$$t_a(f)(q) = c_0 \big(\frac{i}{\hat{\tau}}\big)^{\upsilon} e(\lambda\hat{\tau})\, q_1^{k_0} \tag{1.9}$$

and

$$P_a(f)(q) = \big(\frac{i}{\hat{\tau}}\big)^{\upsilon} e(\lambda\hat{\tau}) \big(\sum_{k\in I_{a,0}} \gamma(k)\, q_1^k \big)\,. \tag{1.10}$$

In (1.9) in the above, $c_0 = \gamma(k_0)$ and $k_0 = \inf\{k \in I_{a,N} : N \in \mathbb{Z}_{\geq 0}\}$; by convention, $t_a(f) = 0$ if $I_{a,N} = \emptyset$ for all $N \geq 0$.

In this way, if $t_a(f) \neq 0$, (1.6) implies that

$$f(q) = t_a(f)(q) \big(1 + o(q_1^{\kappa})\big) \tag{1.11}$$

for some $\kappa > 0$. Beside, for any $I = I_{a,N}$, $N \in \mathbb{Z}_{>0}$,

$$f(q) = P_a(f)(q) + \big(\frac{i}{\hat{\tau}}\big)^{\upsilon} e(\lambda\hat{\tau}) \big(\sum_{k\in I, k>0} \gamma(k)\, q_1^k + o(q_1^N) \big)\,. \tag{1.12}$$

Consequently, f is exponentially small if, and only if, $P_a(f) = 0$.

As to the algebraic structures of $\mathfrak{T}$, one can notice the following

Proposition 1.1. (1) *The sets $\mathfrak{T}_a$ and $\mathfrak{T}$ are stable for the product between two functions.*
(2) *If $\mathfrak{T}_a^* = \{f \in \mathfrak{T}_a : t_a(f) \neq 0\}$, then $\mathfrak{T}_a^*$ constitutes a multiplicative group.*
(3) *$\mathfrak{T}$ is stable by the ramification operator $q \mapsto q^{\upsilon}$ for all $\upsilon \in \mathbb{Q}_{>0}$.*

Proof. (1) This follows directly from (1.6). Indeed, let I and J be two increasing sequences in $\mathbb{R}$ without (finite) accumulation point. Their sum

$$I + J = \{k + k' : (k, k') \in I \times J\}$$

represents also an increasing sequence that admits no finite limit in $\mathbb{R}$. This together with (1.6) imply that $fg \in \mathfrak{T}_a$ if everyone of f and g is given in $\mathfrak{T}_a$.

(2) Let $f \in \mathfrak{T}_a^*$. If one writes $f = t_a(f)\,(1 + \varphi)$ instead of (1.11), then one can easily see that $(1 + \varphi)^{-1} \in \mathfrak{T}_a$, which implies that $\frac{1}{f} \in \mathfrak{T}_a$. Thus, one gets that $\mathfrak{T}_a^*$ is a multiplicative group, in view of the assertion given in (1) for $\mathfrak{T}_a$.

(3) It suffices to observe that each ramification operator gets roots of unity moving ones from others. □

In what follows, we will see in what manner the decomposition of $f \in \tilde{\mathfrak{T}}_a$ into $f_a + \vartheta_a \in \mathfrak{G}_a + \mathfrak{T}_a$ may be unique or almost unique. We start with the following

Proposition 1.2. *Let $a = e(\alpha) \in \mathbb{U} \cup \{0\}$, $\hat{\tau} = \tau - \alpha$, and let $f \in \mathfrak{T}_a$. One has $f \in \mathfrak{G}_a$ if, and only if, $P_a(f)$ is of the form $P_a(f)(q) = c\,\hat{\tau}^n\, e(\lambda\hat{\tau})$, where $c \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{R}$.*

Proof. $f \in \mathfrak{G}_a$ implies that, as $\hat{\tau} \overset{a.v.}{\longrightarrow} 0$, one can find complex numbers a_n such that

$$f(q) = \sum_{n=0}^{M} a_n \hat{\tau}^n + O(\hat{\tau}^{M+1})$$

for any non-negative integer M. Let $n_0 = \inf\{n : a_n \neq 0\}$.

(i) If $n_0 < \infty$, taking the logarithm of both sides of this last asymptotic relation on $f(q)$ yields that

$$\log f(q) = n_0 \log \hat{\tau} + \log(a_{n_0}) + O(\hat{\tau})\,.$$

In view of (1.8), this may be possible if, and only if, $\upsilon = -n_0$ and $k_0 = 0$. In this way, one finds that $I_{a,0} = \{0\}$ and $\upsilon \in \mathbb{Z}_{\leq 0}$. By (1.10), this gives that $P_a(f)(q) = c\,\hat{\tau}^n\, e(\lambda\hat{\tau})$, with $c \in \mathbb{C}^*$, $n = -\upsilon \in \mathbb{Z}_{n \geq 0}$ and $\lambda \in \mathbb{R}$.

(ii) If $n_0 = \infty$, $f(q)$ is Gevrey flat, so this is exponentially small for $\hat{\tau} \overset{a.v.}{\longrightarrow} 0$; see [25, p. ??]. This is equivalent to say that $P_a(f) = 0$, by (1.12). □

In view of (1.12), Proposition 1.2 gives the following

Remark 1.2. *For every $f \in \mathfrak{G}_a \cap \mathfrak{T}_a$, one can find $n \in \mathbb{Z}_{\geq 0}$, $c \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and $\kappa > 0$ such that $f(q) = \hat{\tau}^n\, e(\lambda\hat{\tau})\left(c + o(e^{-\kappa/|\hat{\tau}|})\right)$ for $\hat{\tau} \overset{a.v.}{\longrightarrow} 0$.* □

With regard to writing an almost theta-type function in terms of theta-type functions, the following result will be useful.

Proposition 1.3. *Let $a = e(\alpha) \in \mathbb{U} \cup \{0\}$, $\hat{\tau} = \tau - \alpha$, and let $(\vartheta_1, \vartheta_2) \in \mathfrak{T}_a \times \mathfrak{T}_a$. The following conditions are equivalent.*

(1) $\vartheta_1 - \vartheta_2 \in \mathfrak{G}_a$.
(2) $P_a(\vartheta_1) - P_a(\vartheta_2) \in \mathfrak{G}_a$.
(3) *There exist $(n, d) \in \mathbb{Z}^2_{\geq 0}$, $(c_1, c_2) \in \mathbb{C}^2$, $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\kappa > 0$ such that*

$$\vartheta_1(q) - \vartheta_2(q) = \hat{\tau}^n\big(c_1 e(\lambda_1\hat{\tau}) + c_2\hat{\tau}^d e(\lambda_2\hat{\tau}) + o(e^{-\kappa/|\hat{\tau}|})\big)$$

for $\hat{\tau} \overset{a.v.}{\longrightarrow} 0$.

Proof. The conditions included in (1) and (2) are trivially equivalent. Furthermore, the implication (3) $\Rightarrow$ (1) is clear. For seeing (2) $\Rightarrow$ (3), one observes that (2) implies that $P_a(\vartheta_1) = P_a(\vartheta_2)$, for $\vartheta_1 - \vartheta_2$ is bounded near $q = a$; see (1.11). Thus, one deduces the condition in (3) by applying to both ϑ_1 and ϑ_2 the relation given in (1.12), with the help of Proposition 1.2. □

The condition in Proposition 1.3 (3) requires that the exponent υ in (1.6) is a non-positive integer for both ϑ_1 and ϑ_2. This implies immediately the following

Proposition 1.4. *Let $a = e(\alpha) \in \mathbb{U} \cup \{0\}$, $f \in \tilde{\mathfrak{T}}_a$, $(g_1, \vartheta_1) \in \mathfrak{G}_a \times \mathfrak{T}_a$, $(g_2, \vartheta_2) \in \mathfrak{G}_a \times \mathfrak{T}_a$, and assume that $f = g_1 + \vartheta_1 = g_2 + \vartheta_2$. If $P_a(\vartheta_1) \neq 0$ and the exponent υ in the expression of $P_a(\vartheta_1)$ given by (1.10) is not a non-negative integer, then $g_1 - g_2$ is exponentially small for $q \xrightarrow{a.r.} a$.* □

1.4. **First examples of almost theta-type functions.** Given $(C, \lambda) \in \mathbb{C} \times \mathbb{R}$, it is obvious to see that the function $q = e(\tau) \mapsto Ce(\lambda\tau)$ is a theta-type function. Especially, one finds that any constant function is theta-type.

Proposition 1.5. *Any theta-type function f that is assumed to be analytic at every point $\tau = \alpha \in \mathbb{Q}$ in $\mathbb{C}$ is necessarily of the form $Ce(\lambda\tau)$, where $C \in \mathbb{C}$ and $\lambda \in \mathbb{R}$.*

Proof. Choose some $\alpha \in \mathbb{Q}$ and let $a = e(\alpha)$. Unless $f \equiv 0$, the analyticity of f at $\tau = \alpha$ implies the equality $I_{a,N} = I_{a,0} = \{0\}$ holds in (1.6) for all $N > 0$. See also Remark 1.2. □

One of the most simple false theta-type functions may be $f(q) = 1+q = 1+e(\tau)$. Indeed, as $\tau \xrightarrow{a.v.} 0$, $\hat{\tau} = \tau$ and $f \in \mathfrak{G}_1$, for f is analytic there. Thus, applying Remark 1.2 yields that $f \notin \mathfrak{T}_1$, so f is not a theta-type function. This gives directly that $f \in \mathfrak{F}$. In this same manner, one can observe the following

Proposition 1.6. *Let I be a finite set of $\mathbb{R}$ possessing at least two elements, γ be a $\mathbb{C}^*$-valued map on I, and let $f(q) = \sum_{k\in I} \gamma(k)\, e(k\tau)$.*

1. *$f \in \mathfrak{F}$ if, and only if, the set $I \cap \mathbb{R}_{<0}$ admits at most one element.*
2. *$f \in \mathfrak{M}$ if, and only if, the set $I \cap \mathbb{R}_{<0}$ admits at least two elements.*

Proof. One sees that $f \in \mathfrak{T}_0 \cap \mathfrak{G}_\zeta$ for all $\zeta \in \mathbb{U}$, for f is analytic at $\tau = r$ for all $r \in \mathbb{Q}$. First, assume that $I \cap \mathbb{R}_{<0} = \emptyset$. As f is exponentially small for $\tau \xrightarrow{a.v.} i\infty$, $f \in \mathfrak{G}_0$, this implies that $f \in \mathfrak{F}$.

Next, assume that I contains one unique negative number, denoted by k_0. By letting $g = f - \gamma(k_0)\, e(k_0\tau)$ and $\vartheta = \gamma(k_0)\, e(k_0\tau)$, one finds that $f \in \mathfrak{F}$, in view of Proposition 1.5.

Finally, assume that I contains two negative numbers k_0 and k_1. By noticing that $e(k_0\tau)$ and $e(k_1\tau)$ are unbounded for $\tau \xrightarrow{a.v.} i\infty$, none of these terms may belong to the Gevrey part of the decomposition of f for $\tau \xrightarrow{a.v.} i\infty$. Beside, by Proposition 1.5, $\gamma(k_0)\, e(k_0\tau) + \gamma(k_1)\, e(k_1\tau) \notin \mathfrak{T}_\zeta$ for all $\zeta \in \mathbb{U}$. This gives that $f \in \mathfrak{M}$. □

In line with Proposition 1.6 in the above, one may consider situations in which the set I contains an infinitely many points. This will be the case for the classical Jacobi's theta-functions. We shall see that their specializations on $\mathbb{R} \oplus \tau\mathbb{R}$ are really of theta-type in the sense of Definitions 1.1 and 1.2.

For all $z \in \mathbb{C}$, we define $\theta(z\,|\,\tau)$ to be the following Jacobi theta-function:

$$\theta(z\,|\,\tau) = \sum_{n\in\mathbb{Z}} e\big(\frac{1}{2}n(n-1)\tau + nz\big)\,. \tag{1.13}$$

By comparing this with the definition of $\vartheta_k(v\,|\,\tau)$ in [20, p. 166, (76.1)] for $k = 1$, 2, 3, and 4, where q should be read as $e(\frac{1}{2}\tau)$, it follows that

$$\vartheta_1(z\,|\,\tau) \,=\, e\big(-\frac{z}{2} + \frac{\tau}{8} + \frac{1}{4}\big)\,\theta(z + \frac{1}{2}\,|\,\tau)\,, \tag{1.14}$$

$$\vartheta_2(z\,|\,\tau) \,=\, e\big(-\frac{z}{2} + \frac{\tau}{8}\big)\,\theta(z\,|\,\tau)\,, \tag{1.15}$$

$$\vartheta_3(z\,|\,\tau) \,=\, \theta(z + \frac{\tau}{2}\,|\,\tau)\,, \tag{1.16}$$

$$\vartheta_4(z\,|\,\tau) \,=\, \theta(z + \frac{\tau}{2} + \frac{1}{2}\,|\,\tau)\,. \tag{1.17}$$

See also the right-hand sides of the expression in (0.4) for ϑ_3.

Let $(p, m) \in \mathbb{Z}\times\mathbb{Z}_{>0}$ be such that $p \wedge m = 1$, and assume that $(\alpha, \beta) \in \mathbb{Z}^2$ satisfying $\beta m - \alpha p = 1$. Let $\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} \alpha & -\beta\\ m & -p\end{pmatrix}$ in [20, p. 180, (80.8)], $\hat{\tau} = \tau - \frac{p}{m}$, and observe that $m\tau - p = m\hat{\tau}$. By making use of (1.14), one obtains that

$$\theta(z + \frac{1}{2}\,|\,\tau) = \frac{\epsilon_1}{\sqrt{m}}\sqrt{\frac{i}{\hat{\tau}}}\,e\big(\frac{1}{8}(\tau' - \tau) + \frac{1}{2}(1 - \frac{1}{m\hat{\tau}})z - \frac{z^2}{2\hat{\tau}}\big)\,\theta(\frac{z}{m\hat{\tau}} + \frac{1}{2}\,|\,\tau')\,, \tag{1.18}$$

where $\epsilon_1 = \epsilon_1(\alpha, -\beta, m, -p)$ denotes an 8-th root of unity and

$$\tau' = \frac{\alpha\tau - \beta}{m\tau - p} = -\frac{1}{m^2\hat{\tau}} + \frac{\alpha}{m}\,. \tag{1.19}$$

By noticing that $\tau' - \tau = -\frac{1}{m^2\hat{\tau}} - \hat{\tau}$ and that

$$-\frac{1}{8m^2\hat{\tau}} - \frac{z}{2m\hat{\tau}} - \frac{z^2}{2\hat{\tau}} = -\frac{1}{2\hat{\tau}}\,(z + \frac{1}{2m})^2\,,$$

the relation in (1.18) can be written as follows:

$$\theta(z + \frac{1}{2}\,|\,\tau) = \frac{\epsilon_1}{\sqrt{m}}\sqrt{\frac{i}{\hat{\tau}}}\,e\big(-\frac{1}{2\hat{\tau}}\,(z + \frac{1}{2m})^2 - \frac{\hat{\tau}}{8}\big)\,\theta(\frac{z}{m\hat{\tau}} + \frac{1}{2}\,|\,\tau')\,. \tag{1.20}$$

Given $z \in \mathbb{R}_\tau = \mathbb{R}\oplus\tau\mathbb{R}$, we will write

$$\rho(z\,|\,\tau) = \begin{cases} 1 & z_1 \notin \mathbb{Z}\\ |1 - e(z_0)| & z_1 \in \mathbb{Z}\end{cases}\,. \tag{1.21}$$

It is worth noticing that $\rho(z\,|\,\tau) = 0$ if, and only if, $z \in \mathbb{Z}\oplus\tau\mathbb{Z}$.

Lemma 1.1. *Let $z = z_0 + z_1\tau$, $(z_0, z_1) \in \mathbb{R}^2$, and let $f(q) = \theta(z + \frac{1}{2}\,|\,\tau)$. Then $f \in \mathfrak{T}_0$. Moreover, if*

$$\kappa = \kappa(z\,|\,\tau, \infty) = \frac{1}{2}\,\min_{\ell\in\mathbb{Z}}\big|z_1 - \frac{1}{2} - \ell\big|^2\,, \tag{1.22}$$

one can find $c \in \mathbb{C}$ such that $|c| = \rho(z\,|\,\tau)$ and $t_0(f)(q) = c\,q^{-\frac{1}{2}(z_1 - \frac{1}{2})^2 + \kappa}$.

This follows immediately from the fact that, by definition, $\theta(z + \frac{1}{2}\,|\,\tau)$ is a sum of exponential functions with respect to τ. For the proof, see §4.1.

Theorem 1.1. *Let $f(q)=\theta(z+\frac{1}{2}\,|\,\tau)$ be as in Lemma 1.1. Then $f\in\mathfrak{T}$.*

Furthermore, let $r=\frac{p}{m}$, $m>0$, $\begin{pmatrix}\alpha & -\beta\\ m & -p\end{pmatrix}\in SL(2,\mathbb{Z})$, $\zeta=e(r)$, $\hat{\tau}=\tau-r$, $q_1=e\big(-\frac{1}{\hat{\tau}}\big)$, and let τ' as in (1.19)*. One assumes that $\tau\xrightarrow{a.v.}r$. If*

$$\kappa=\kappa(z\,|\,\tau,r)=\frac{1}{2}\min_{\ell\in\mathbb{Z}}\big|mz_0+pz_1-\frac{1}{2}-\ell\big|^2 \tag{1.23}$$

and $z'=(\alpha z_0+\beta z_1)-(mz_0+pz_1)\tau'$, one can find $c\in\mathbb{C}$ such that $|c|=\rho(z'\,|\,\tau')$ and

$$t_\zeta(f)(q)=\frac{c}{\sqrt{m}}\sqrt{\frac{i}{\hat{\tau}}}\,e\big(-\frac{1}{8}((2z_1)^2+1)\,\hat{\tau}\big)\,q_1^{\kappa/m^2}\,. \tag{1.24}$$

The strategy is to use the modular relation in (1.20) for leading any rational point $\tau=r$ into $\tau'=\infty$, in such a way one can apply Lemma 1.1. For more details on the proof, see §4.2.

Applying Proposition 1.1 (2) gives the following

Corollary 1.1. *Given u_1, ..., u_ℓ and v_1, ..., v_m two finite families of elements in $(\mathbb{R}\oplus\tau\mathbb{R})\setminus(\mathbb{Z}\oplus\tau\mathbb{Z})$, if*

$$f(q)=\frac{\theta(u_1+\frac{1}{2}\,|\,\tau)\,...\,\theta(u_\ell+\frac{1}{2}\,|\,\tau)}{\theta(v_1+\frac{1}{2}\,|\,\tau)\,...\,\theta(v_m+\frac{1}{2}\,|\,\tau)}\,,$$

then $f\in\mathfrak{T}$. □

Using the expressions of ϑ_1, ..., ϑ_4 given in (1.14) - (1.17) implies the following

Corollary 1.2. *If $z=z_0+z_1\tau$ with $(z_0,z_1)\in\mathbb{R}^2$, then the Jacobi's theta-functions ϑ_1, ϑ_2, ϑ_3 and ϑ_4 are theta-type in the sense of Definition 1.2.*

Proof. It suffices to combine Theorem 1.1 with the fact that $e(a\tau+b)\in\mathfrak{T}$ for any given $(a,b)\in\mathbb{R}^2$; see Propositions 1.5 and 1.1 (1). □

1.5. **Appell-Lerch series and functional equations.** In the literature, every Appell-Lerch series is associated with an order k as indicated in (0.1). In addition to the decomposition formula of R_k in terms of R_1 as given in (0.2), it is useful to recall the following important

Theorem 1.2 (M. Lerch [16])**.** *Let $\mathsf{k}\in\mathbb{N}^*$ and let $R_\mathsf{k}(z,w\,|\,\tau)$ be as in* (0.1)*. Then:*

$$R_\mathsf{k}(z,w\,|\,\tau)=\sum_{s=0}^{\mathsf{k}-1}C_s(z,w\,|\,\tau)\,R_1(\mathsf{k}w+\frac{\mathsf{k}+\mathsf{k}\tau}{2},\mathsf{k}z+s\tau-\frac{\mathsf{k}+\mathsf{k}\tau}{2}\,|\,\mathsf{k}\tau)\,, \tag{1.25}$$

where

$$C_s(z,w\,|\,\tau)=e\big(\mathsf{k}z+sw+(s-\frac{5\mathsf{k}}{8}+\frac{1}{8})\tau\big)\,\frac{\theta(\mathsf{k}z+(s+\frac{\mathsf{k}}{2})\tau\,|\,\mathsf{k}\tau)}{\theta(\mathsf{k}w-\frac{1}{2}\,|\,\mathsf{k}\tau)}\,. \tag{1.26}$$

Proof. This is due to M. Lerch [16, ??], who gave the following expression for $C_s(z,w)$:

$$C_s(z,w\,|\,\tau)=e\big(\mathsf{k}z+(s-\frac{\mathsf{k}}{2})w+(s-\mathsf{k}+\frac{1}{8})\tau+\frac{1}{4}\big)\,\frac{\vartheta_3(\mathsf{k}z+s\tau\,|\,\mathsf{k}\tau)}{\vartheta_1(\mathsf{k}w\,|\,\mathsf{k}\tau)}\,.$$

In view of the relations in (1.14) and (1.16), this is equivalent to the above expression stated in (1.26). □

In the whole paper, we shall consider only the case of $\mathsf{k}=1$. Write ϑ_3 in terms of θ as in (1.16), and consider the functional equation

$$\theta(z\,|\,\tau)=e(z)\,\theta(z+\tau\,|\,\tau)\,. \tag{1.27}$$

One obtains from (0.4) that

$$\frac{R_1(z,w\,|\,\tau)}{\theta(z+\frac{\tau}{2}\,|\,\tau)}-e(w-z-\frac{\tau}{2})\,\frac{R_1(z+\tau,w\,|\,\tau)}{\theta(z+\frac{3}{2}\tau\,|\,\tau)}=1\,, \tag{1.28}$$

what suggests the following

Definition 1.3. *For all $(z,w)\in\mathbb{C}^2$ such that neither w nor $w-z$ belong to $\mathbb{Z}\oplus\tau\mathbb{Z}$, define*

$$L(z,w\,|\,\tau)=\frac{R_1(w-z-\frac{\tau}{2}+\frac{1}{2},w\,|\,\tau)}{\theta(w-z+\frac{1}{2}\,|\,\tau)}\,. \tag{1.29}$$

Given $\tau\in\mathcal{H}$, let π_τ be the associated real parametrization of $\mathbb{C}$, that is the map defined from $\mathbb{R}^2$ onto $\mathbb{C}$ by $\pi_\tau(z_0,z_1)=z=z_0+z_1\tau$. In this way, it is possible to see $L(z,w\,|\,\tau)$ as a function of τ possessing four real parameters z_0, z_1, w_0 and w_1. We shall study the asymptotic behavior of this function in relation to these parameters when $\tau\xrightarrow{a.v.}r\in\mathbb{Q}$. If $\mathbb{R}_\tau=\pi_\tau(\mathbb{R}^2)$, one sees that $\mathbb{R}_\tau$ will be reduced into the real axis when $\tau=r$, so our study concerns the platitude of the parameters space $\mathbb{C}^2$ towards $\mathbb{R}^2$.

Definition 1.4. *Let π_τ as in the above, $\mathbb{R}_\tau=\pi_\tau(\mathbb{R}^2)$, $\mathbb{Z}_\tau=\pi_\tau(\mathbb{Z}^2)$, and let*

$$\Omega=(-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2})\,,\qquad \Omega_\tau=\pi_\tau(\Omega)\,. \tag{1.30}$$

We define

$$\mathbb{R}^{2,*}_{1\oplus\tau}=\big\{(z,w)\in\mathbb{R}_\tau\times\mathbb{R}_\tau\,:\,w\neq 0,\ w\neq z \bmod \mathbb{Z}_\tau\big\} \tag{1.31}$$

and

$$\Omega^{2,*}_{1\oplus\tau}=\mathbb{R}^{2,*}_{1\oplus\tau}\cap\big(\Omega_\tau\times\Omega_\tau\big)\,. \tag{1.32}$$

To any given $(\ell,m)\in\mathbb{Z}^2$ can be associated the translation $(a,b)\mapsto(a+\ell,b+m)$ on $\mathbb{R}^2$. This gives rise to the usual action of the (sub-)group $\mathbb{Z}_\tau$ on $\mathbb{R}_\tau$. By extending this action on $\mathbb{R}^{2,*}_{1\oplus\tau}$, one sees that the corresponding fundamental domain is merely $\Omega^{2,*}_{1\oplus\tau}$. With regard to the effect of this action on $L(z,w\,|\,\tau)$, we notice the following

Proposition 1.7. *The following relations hold for $(z,w)\in\mathbb{R}^{2,*}_{1\oplus\tau}$:*

$$L(z+\ell,w\,|\,\tau)=L(z,w+m+n\tau\,|\,\tau) \tag{1.33}$$

for all $(\ell,m,n)\in\mathbb{Z}^3$, and

$$L(z,w\,|\,\tau)+e(z)\,L(z-\tau,w\,|\,\tau)=1\,. \tag{1.34}$$

Proof. (1) From (0.3) (with $\mathsf{k}=1$) and (1.27), it follows that $L(z,w+\tau\,|\,\tau)=L(z,w\,|\,\tau)$. Furthermore, one sees that both R_1 and θ are left invariant when their respective arguments z and w are increased by one. This implies that $L(z+1,w\,|\,\tau)=L(z,w+1\,|\,\tau)=L(z,w\,|\,\tau)$. Thus, one gets (1.33).

(2) This is equivalent to (1.28). □

Let $n \in \mathbb{Z}_{>0}$, and let

$$P_n(z\,|\,\tau) = \sum_{k=0}^{n-1} (-1)^k\, e\big(k(z+n\tau-\frac{k-1}{2}\tau)\big)\,. \tag{1.35}$$

By iterating the functional equation in (1.34), one gets that

$$L(z+n\tau, w\,|\,\tau) = 1 - e(z+n\tau)\, L(z+(n-1)\tau\,|\,\tau) = \dots\,,$$

what gives

$$L(z+n\tau, w\,|\,\tau) = P_n(z\,|\,\tau) + (-1)^n\, e\big(n(z+\frac{n+1}{2}\tau)\big)\, L(z, w\,|\,\tau)\,. \tag{1.36}$$

Replacing z with $z-n\tau$ in both sides of (1.36) yields that

$$L(z-n\tau, w\,|\,\tau) = P_{-n}(z\,|\,\tau) + (-1)^n\, e\big(-n(z+\frac{-n+1}{2}\tau)\big)\, L(z, w\,|\,\tau)\,, \tag{1.37}$$

where

$$P_{-n}(z\,|\,\tau) = (-1)^{n+1}\, e\big(n(-z+\frac{n-1}{2}\tau)\big)\, P_n(z-n\tau\,|\,\tau)\,.$$

By using the definition of $P_n(z\,|\,\tau)$ in (1.35), one can find that

$$P_{-n}(z\,|\,\tau) = -\sum_{k=1}^{n} (-1)^k\, e\big(-k(z-n\tau-\frac{-k-1}{2}\tau)\big)\,. \tag{1.38}$$

Proposition 1.8. *Let $n \in \mathbb{Z}$, $\zeta \in \mathbb{U}$, and let $(z,w) \in \mathbb{R}^{2,*}_{1\oplus\tau}$. Then $L(z+n\tau, w\,|\,\tau) \in \tilde{\mathfrak{T}}_\zeta$ if, and only if, $L(z,w\,|\,\tau) \in \tilde{\mathfrak{T}}_\zeta$.*

Proof. Assume that $n \in \mathbb{Z}_{>0}$, and consider $L(z\pm n\tau, w\,|\,\tau)$ instead of $L(z+n\tau, w\,|\,\tau)$. By (1.35) and (1.38), both $P_\pm(z\,|\,\tau)$ are analytic for $\tau \in \mathbb{C}$, so they belong to $\mathfrak{G}_\zeta$. Moreover, by Proposition 1.5, one knows that

$$(-1)^n\, e\big(\pm n(z+\frac{\pm n+1}{2}\tau)\big) \in \mathfrak{T}_\zeta \cap \mathfrak{G}_\zeta\,.$$

Since both $\mathfrak{T}_\zeta$ and $\mathfrak{G}_\zeta$ are stable for the product of functions (see Proposition 1.1 (1)), one obtains the desired equivalence with the help of the relations in (1.36) and (1.37). □

1.6. **Main result and basic steps of the proof.** The goal of the rest of this paper is to obtain the following

Theorem 1.3. *Let $(z,w) \in \Omega^{2,*}_{1\oplus\tau}$, and consider $f(q) = L(z,w\,|\,\tau)$ be given in (1.29). Then $f \in \mathfrak{M}$ except in the following cases:*

1. *$z \in \{\frac{1}{2}, \frac{1}{2}-\frac{\tau}{2}, -\frac{\tau}{2}\}$ and $w \in \{\frac{1}{2}, \frac{1}{2}-\frac{\tau}{2}, -\frac{\tau}{2}\}$, in which case f is a constant function.*
2. *$z \in \{\frac{1}{2}, \frac{1}{2}-\frac{\tau}{2}, -\frac{\tau}{2}\}$ and $w \notin \{\frac{1}{2}, \frac{1}{2}-\frac{\tau}{2}, -\frac{\tau}{2}\}$, in which case f is a false theta-type function that is not analytic at any $\tau = r \in \mathbb{Q}$.*

As to the proof of Theorem 1.3, letting $\mathsf{k}=1$ in the middle expression of (0.1) and expanding this with respect to q implies that $R_1(z,w\,|\,\tau) \in \mathfrak{T}_0$ when $q \xrightarrow{a.r.} 0$ exponentially. By considering both Proposition 1.1 (2) and Lemma 1.1, the expression in (1.29) yields that $L(z,w\,|\,\tau) \in \mathfrak{T}_0$. To treat any root of unity instead of 0, the step to be followed is to find a modular-like formula like as (1.20) for the θ-function. We shall see that such formula can be obtained by a viewpoint of the analytic theory of q-difference equations. Indeed, to the functional equation in

(1.34) corresponds a singular q-difference that admits two *natural* solutions coming from a same formal power series. One of these solutions being $L(z,w\,|\,\tau)$, the other, denoted in the below by $G(z\,|\,\tau)$, can be expressed by means of Mordell integral. This is to say, a certain Stokes analysis will give some modular-like relation. After all that, we shall consider the analytic continuation of $G(z\,|\,\tau)$ on one hand and the linear fractional transformations on the other hand. In this way, we shall complete the proof in §3.8.

Finally, in view of the functional relations in (1.33) and Proposition 1.8, Theorem 1.3 implies the following

Corollary 1.3. *Let $(z,w)\in\mathbb{R}^{2,*}_{1\oplus\tau}$, and consider $f(q)=L(z,w\,|\,\tau)$.*

1. *If $z\notin\{\frac{1}{2},\frac{1}{2}-\frac{\tau}{2},-\frac{\tau}{2}\}$ mod $\mathbb{Z}\oplus\tau\mathbb{Z}$, then $f\in\mathfrak{M}$.*
2. *Otherwise, f is either a finite combination of $e(\lambda\tau)$ with $\lambda\geq 0$ or a false theta-type function that is not regular at any $\tau=r\in\mathbb{Q}$.* □

1.7. **Symmetry on parameters and generate cases.** Letting $\mathsf{k}=1$ and taking $-n$ instead of n in the summation formula of (0.1) yields that

$$R_1(z,w\,|\,\tau)=-e(-w)\,R_1(\tau-z,-w\,|\,\tau)\,. \tag{1.39}$$

Furthermore, letting $\mathsf{k}=1$ and $s=0$ in (1.26) gives that

$$C_0(z,w\,|\,\tau)=e(z-\frac{\tau}{2})\,\frac{\theta(z+\frac{\tau}{2}\,|\,\tau)}{\theta(w-\frac{1}{2}\,|\,\tau)}=\frac{\theta(z-\frac{\tau}{2}\,|\,\tau)}{\theta(w-\frac{1}{2}\,|\,\tau)}\,.$$

Thus, by (1.25), one finds that

$$R_1(z,w\,|\,\tau)=\frac{\theta(z-\frac{\tau}{2}\,|\,\tau)}{\theta(w-\frac{1}{2}\,|\,\tau)}\,R_1(w+\frac{\tau}{2}+\frac{1}{2},z-\frac{\tau}{2}-\frac{1}{2}\,|\,\tau)\,. \tag{1.40}$$

Proposition 1.9. *The following identities hold for $(z,w)\in\mathbb{R}^{2,*}_{1\oplus\tau}$:*

$$L(z-\tau,w\,|\,\tau)=e(-z)\,L(-z-\tau,-w\,|\,\tau) \tag{1.41}$$

and

$$L(z-\tau,w\,|\,\tau)=e(-z)\,L(-z-\tau,w-z\,|\,\tau)\,. \tag{1.42}$$

Proof. By using (1.29), one deduces respectively (1.41) from (1.39) and (1.42) from (1.40). □

The identity in (1.41) will be useful for finding the special values under the condition $2w\in\mathbb{Z}\oplus\tau\mathbb{Z}$; see §3.7. Beside, by (1.34), the relation in (1.42) is equivalent to the following one:

$$L(z-\tau,w\,|\,\tau)+L(-z,w-z\,|\,\tau)=1\,. \tag{1.43}$$

This last identity permits to interchange the conditions $w\notin\mathbb{Z}\oplus\tau\mathbb{Z}$ and $z-w\notin\mathbb{Z}\oplus\tau\mathbb{Z}$, to what the parameters z and w are assumed to be subject in Theorem 1.3.

By using (0.1) with $\mathsf{k}=1$ and (1.29), one finds the following residues at $w=0$ for R_1 and L:

$$\mathrm{Res}\big(R_1(z,w\,|\,\tau):w=0\big)=\frac{i}{2\pi}$$

and

$$\mathrm{Res}\big(L(z,w\,|\,\tau):w=0\big)=\frac{i}{2\pi\,\theta(-z+\frac{1}{2}\,|\,\tau)}\,.$$

This together with (1.43) suggest us to introduce the following

Definition 1.5. (1) *For any $z \in \mathbb{C}$, we define*

$$R_1(z\,|\,\tau) = R_1^*(z,0\,|\,\tau) = \lim_{w\to 0}\big(R_1(z,w\,|\,\tau) - \frac{i}{2\pi w}\big)\,. \tag{1.44}$$

(2) *If $z \notin \mathbb{Z}\oplus\tau\mathbb{Z}$, we define*

$$L(z\,|\,\tau) = L^*(z,0\,|\,\tau) = \lim_{w\to 0}\big(L(z,w\,|\,\tau) - \frac{i}{2\pi w\,\theta(-z+\frac{1}{2}\,|\,\tau)}\big) \tag{1.45}$$

and

$$L^*(z\,|\,\tau) = L^*(z,z\,|\,\tau) = \lim_{w\to 0}\big(L(z,z+w\,|\,\tau) + \frac{i}{2\pi w\,\theta(-z+\frac{1}{2}\,|\,\tau)}\big)\,. \tag{1.46}$$

Since $\frac{1}{\theta(-z+\frac{1}{2}\,|\,\tau)}$ satisfies the homogeneous equation associated with (1.34), namely $y(z)+e(z)\,y(z-\tau)=0$, one finds that $L(z,w\,|\,\tau) - \frac{i}{2\pi w\,\theta(-z+\frac{1}{2}\,|\,\tau)}$ satisfies (1.34). This implies that

$$L(z\,|\,\tau) + e(z)\,L(z-\tau\,|\,\tau) = 1\,. \tag{1.47}$$

Similarly, one obtains that

$$L^*(z\,|\,\tau) + e(z)\,L^*(z-\tau\,|\,\tau) = 1\,. \tag{1.48}$$

Proposition 1.10. *The following relations holds for $z \notin \mathbb{Z}\oplus\tau\mathbb{Z}$:*

$$L(z\,|\,\tau) = \frac{1}{\theta(-z+\frac{1}{2}\,|\,\tau)}\,\big(R_1(-z-\frac{\tau}{2}+\frac{1}{2}\,|\,\tau) - \frac{i}{2\pi}\,\frac{\theta'(-z+\frac{1}{2}\,|\,\tau)}{\theta(-z+\frac{1}{2}\,|\,\tau)}\big)\,, \tag{1.49}$$

where $\theta'(-z+\frac{1}{2}\,|\,\tau)$ denotes the derivative of $\theta(z\,|\,\tau)$ with respective to the variable z at $-z+\frac{1}{2}$, and

$$L(z\,|\,\tau) = 1 - L^*(-z-\tau\,|\,\tau)\,. \tag{1.50}$$

Proof. (1) By observing that, as $w\to 0$,

$$R_1(w-z-\frac{\tau}{2}+\frac{1}{2},w\,|\,\tau) = R_1(-z-\frac{\tau}{2}+\frac{1}{2},w\,|\,\tau) + O(w)\,,$$

one deduces from (1.44) that

$$R_1(w-z-\frac{\tau}{2}+\frac{1}{2},w\,|\,\tau) = R_1(-z-\frac{\tau}{2}+\frac{1}{2}\,|\,\tau) + \frac{i}{2\pi\,w} + O(w)\,.$$

Thus, considering the second order Taylor series of $\theta(w-z+\frac{1}{2}\,|\,\tau)$ at $w=0$ allows one to write the relation in (1.29) into the following form:

$$L(z,w\,|\,\tau) = \frac{R_1(-z-\frac{\tau}{2}+\frac{1}{2},w\,|\,\tau)+\frac{i}{2\pi\,w}}{\theta(-z+\frac{1}{2}\,|\,\tau)}\,\big(1-\frac{\theta'(-z+\frac{1}{2}\,|\,\tau)}{\theta(-z+\frac{1}{2}\,|\,\tau)}\,w\big) + O(w)\,.$$

This implies immediately (1.49).

(2) This follows immediately from (1.43). □

In view of (1.50), one will consider only the function $L(z\,|\,\tau)$. Theorem 1.3 can be completed as follows.

Theorem 1.4. *Given $z\in\Omega_\tau\setminus\{0\}$, let $f(q)=L(z\,|\,\tau)$. Then $f\in\mathfrak{M}$ except when $z\in\{\frac{1}{2},\frac{1}{2}-\frac{\tau}{2},-\frac{\tau}{2}\}$. Moreover, the following identities hold:*

$$L(\frac{1}{2}\,|\,\tau) = 1\,,\qquad L(\frac{1}{2}-\frac{\tau}{2}\,|\,\tau) = L(-\frac{\tau}{2}\,|\,\tau) = \frac{1}{2}\,. \tag{1.51}$$

By using the functional equation in (1.47), one obtains that

$$L(z\,|\,\tau)=\sum_{k=0}^{n-1}(-1)^k e\big(kz-\frac{1}{2}k(k-1)\tau\big)+(-1)^n e\big(nz-\frac{1}{2}n(n-1)\tau\big)\,L(z-n\tau\,|\,\tau)$$

for all positive integer n. Thus, one deduces from Theorem 1.4 the following statement, which completes Corollary 1.3.

Corollary 1.4. *Given $z\in(\mathbb{R}\oplus\tau\mathbb{R})\setminus(\mathbb{Z}\oplus\tau\mathbb{Z})$, let $f(q)=L(z\,|\,\tau)$.*

1. *If $z\notin\{\frac{1}{2},\frac{1}{2}-\frac{\tau}{2},-\frac{\tau}{2}\}$ mod $\mathbb{Z}\oplus\tau\mathbb{Z}$, then $f\in\mathfrak{M}$.*
2. *Otherwise, there exists a finite numbers of pairs $(c_n,\lambda_n)\in\mathbb{R}\times\mathbb{R}_{\geq 0}$, $0\leq n\leq N$, such that $f=\sum_{k=0}^{N}c_n\,e(\lambda_n\tau)$.* □

The above Theorem 1.4 will be proved in §3.9.

2. Asymptotic behavior of Mordell integral

Let $(z,w)\in\mathbb{R}^{2,*}_{1\oplus\tau}$, and consider $L(z,w\,|\,\tau)$ in (1.29). If one writes $x=e(z)$, $u=e(w)$ and

$$L(x,u;q)=L(z,w\,|\,\tau)\,, \tag{2.1}$$

where $q=e(\tau)$, then the equation in (1.34) becomes

$$y(x)+x\,y(\frac{x}{q})=1\,. \tag{2.2}$$

This q-difference equation admits $x=0$ as singular irregular point, following [26] and [33].

Let $\hat{E}(x;q)=\sum_{n\geq 0}q^{-n(n-1)/2}\,x^n$. This is a formal solution of (2.2), and is divergent for all $x\neq 0$ in $\mathbb{C}$ because of $|q|<1$. In §3, the function $L(x,u;q)$ of (2.1) will be seen as a q-Borel sum of $\hat{E}(x;q)$ defined by (A.10); see (3.1).

In this section, we will consider the q-Borel-sum of this series defined with the help of (A.9). Indeed, applying this together with (A.6) to $\hat{E}(x;q)$ yields the following q-sum over the whole Riemann surface $\tilde{\mathbb{C}}^*$ of the logarithm:

$$G(x;q)=\int_0^\infty\frac{\omega(\xi/x;q)}{1+\xi}\,\frac{d\xi}{\xi}\,. \tag{2.3}$$

In the above, the integration path is any straight-line starting from 0 to infinity in the principal cut-plane $\mathbb{C}\setminus(-\infty,0]$. One can verify that $G(x;q)$ is solution of the q-difference equation included in (2.2), as said by Proposition A.3.

Putting $\xi=e^\sigma$ in the integral of (2.3) yields the following expression:

$$G(x;q)=\frac{2\,q^{-3/8}}{\sqrt{2\pi x\ln(1/q)}}\int_{-\infty}^{\infty}\frac{e^{(\sigma-\log(qx))^2/2\ln q}}{\cosh(\sigma/2)}\,d\sigma\,,$$

where the integration is taken along a horizontal line with a distance at most π from the real axis. Thus, the integral appearing in (2.3) is of Mordell type; see [18], [1] and [38].

Proposition 2.1. *The function $G(x;q)$ satisfies the following modular-type relation:*

$$G(\frac{x}{q};q)=2\pi i\,\omega(e^{\pi i}x;q)\,G(e^{2\pi i\frac{\log(xe^{2\pi i})}{\ln q}};e^{\frac{4\pi^2}{\ln q}})\,. \tag{2.4}$$

The proof will be given in §4.3, by considering the link between both q-Borel-sum functions $G(x;q)$ and $L(x,u;q)$; see (3.2) and its equivalent form (4.8).

Remember that in (2.3), $G(x;q)$ is defined for $x \in \tilde{\mathbb{C}}^*$ and $q \in (0,1)$. In §2.1, we shall consider the analytic continuation of $G(x;q)$ with respect to q when the "parameter" x represents a germ of analytic function of q; see Theorem 2.1. This permits to find the analytic obstruction for the composed function to become analytic at $q=1$ in $\mathbb{C}$; see Theorem 2.2. One will see that the analyticity depends of whether the parameter x belongs to the set of half-periods $\{\pm q^n : 2n \in \mathbb{Z}\}$.

In line with Theorem 2.1, we will define the function $G(z\,|\,\tau)$ and consider the asymptotic behavior of this function for $\tau \xrightarrow{a.v.} 0$ when $z \in \mathbb{R}_\tau = \mathbb{R} \oplus \tau\mathbb{R}$ and $\tau \in \mathbb{C} \setminus [0, -i\infty)$. See Definition 2.1 and Theorem 2.3 in §2.3. This last theorem together with the modular-like relation (2.4) and its equivalent form (2.30) give the asymptotic behavior of $G(z\,|\,\tau)$ for $\tau \xrightarrow{a.v.} i\infty$; see Theorem 2.4 in §2.4.

The last three subsections of this section will concern the evaluation of $G(z\,|\,\tau)$ when z represents one half-period or when τ is reduced to a rational number. Especially, we will give a proof of a law of reciprocity on some generalized Gaussian sums; see Theorem 2.5 in §2.6.

2.1. **Analytic continuation of Mordell integral.** The function $G(x;q)$ defined by (2.3) is not uniform for the argument x. Indeed, if the integration-path $[0,\infty)$ is replaced with $[0,\infty e^{-2\pi i})$ in (2.3), the corresponding integral gives $G(xe^{2\pi i};q)$. Applying the Cauchy residue theorem to this integral over the contour composed of two half straight-lines $-[0,\infty)$ $(=[\infty,0])$ and $[0,\infty e^{-2\pi i})$ yields that

$$G(xe^{2\pi i};q) = G(x;q) - 2\pi i\omega(\frac{e^{-\pi i}}{x};q)\,. \tag{2.5}$$

Let φ denote a germ of analytic function at $t=0 \in \mathbb{C}$ such that $\varphi(0) \neq 0$. We shall choose an argument of $\varphi(0)$, consider φ as an analytic function valued in $\tilde{\mathbb{C}}^*$ and then get the composed function $G(\varphi(\epsilon);e^{-\epsilon})$ for $\epsilon \sim 0^+$ in $\mathbb{R}_{>0}$. In view of (2.5), we shall assume that $\arg(\varphi(0)) \in (-\pi,\pi]$.

Theorem 2.1. *For any $R>0$, let $\nabla_R = \{\epsilon \in \tilde{\mathbb{C}}^* : |\epsilon| < R, |\arg\epsilon| < 3\pi/2\}$. Let φ to be a germ of analytic function at $t=0$ in $\mathbb{C}$ such that $\varphi(0) \neq 0$, and define*

$$\varphi(\epsilon) = G(\varphi(\epsilon);e^{-\epsilon}) \tag{2.6}$$

for all enough small $\epsilon > 0$. Then φ can be continued to be an analytic function in ∇_R for some suitable $R>0$. Moreover, one has the following properties.

1. *If $\arg(\varphi(0)) \in (-\pi,\pi]$ and $\varphi(0) \neq -1$, then $\varphi(\epsilon)$ admits a Gevrey asymptotic expansion as $\epsilon \to 0$ in ∇_R and*
$$\varphi(\epsilon) = \frac{1}{1+\varphi(0)} + O(\epsilon)\,.$$
2. *If $\varphi(0) = -1 = e^{i\pi}$, $\varphi(t) = -e^{t(\psi(t)+\frac{1}{2})}$ and*
$$\tilde{\varphi}(\epsilon) = \varphi(\epsilon) + i\sqrt{\frac{\pi}{2\epsilon}}\,e^{-\frac{\epsilon}{2}(\psi(\epsilon))^2}\,, \tag{2.7}$$
then $\tilde{\varphi}(\epsilon)$ admits a Gevrey asymptotic expansion as $\epsilon \to 0$ in ∇_R and
$$\tilde{\varphi}(\epsilon) = 1 + \varphi'(0) + O(\epsilon)\,.$$

Proof. With the help of (A.8), writing $x = e^s$ and $\xi = e^{\sigma+s-\frac{\epsilon}{2}}$ in (2.3) yields that

$$G(e^s; e^{-\epsilon}) = \frac{1}{\sqrt{2\pi\epsilon}} \int_{-\infty+ic}^{\infty+ic} \frac{e^{-\frac{\sigma^2}{2\epsilon}}}{1+e^{\sigma+s-\frac{\epsilon}{2}}}\, d\sigma\,,$$

where the real number c will be chosen in such manner that $c + \Im(s) \in (-\pi, \pi)$. Letting $s = \log(\varphi(\epsilon))$, the above integral shows that

$$\varphi(\epsilon) = \frac{1}{\sqrt{2\pi\epsilon}} \int_{-\infty+ic}^{\infty+ic} \frac{e^{-\frac{\sigma^2}{2\epsilon}}}{1+\varphi(\epsilon)\, e^{\sigma-\frac{\epsilon}{2}}}\, d\sigma\,. \tag{2.8}$$

If $R > 0$ is chosen enough small such that $\varphi(t)$ is well defined in the disc $|t| < R$, one can get an analytic function for $\epsilon \in \nabla_R$ by replacing the horizontal integration-path with oblique lines L in (2.8) as shown in the the following figure.

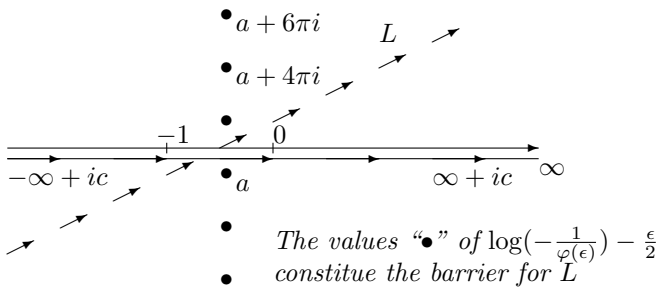

FIGURE 1. The horizontal line $(-\infty + ic, \infty + ic)$ can be replaced by an oblique line L, which is not vertical !

Especially, when $\arg(\varphi(0)) \in (-\pi, \pi]$ and $\varphi(0) \neq -1$, applying Proposition A.1 shows that φ has a Gevrey asymptotic expansion. Indeed, let $\tilde{\sigma} = \sigma^2/2$, write the integration path as union of two segments both starting from 0 to infinity on the two sides of the plan, and define

$$F(\epsilon, \tilde{\sigma}) = \frac{1}{1+\varphi(\epsilon)\, e^{\sigma-\frac{\epsilon}{2}}} + \frac{1}{1+\varphi(\epsilon)\, e^{-\sigma-\frac{\epsilon}{2}}}\,. \tag{2.9}$$

It follows that

$$\varphi(\epsilon) = \frac{1}{\sqrt{2\pi\epsilon}} \int_0^{\infty} F(\epsilon, \tilde{\sigma}) e^{-\frac{\tilde{\sigma}}{\epsilon}}\, \frac{d\tilde{\sigma}}{\sqrt{2\tilde{\sigma}}}\,, \tag{2.10}$$

where the integration-path, initially equal to $[0, +\infty)$ for $\arg(\varphi(0)) \neq \pi$, needs to be deformed with a half-circle when $\arg(\varphi(0))$ becomes π; see Figure 2 in the below.

Therefore, applying Proposition A.1 to this last integral in (2.10) yields the expansion of φ. Namely, since

$$F(0, \tilde{\sigma}) = \frac{2}{1+\varphi(0)}(1 + O(\tilde{\sigma})), \quad \int_0^{\infty} e^{-\frac{\tilde{\sigma}}{\epsilon}}\, \frac{d\tilde{\sigma}}{\sqrt{\tilde{\sigma}}} = \Gamma(\frac{1}{2})\sqrt{\epsilon} = \sqrt{\pi\epsilon}\,,$$

one finds the limit of $\varphi(\epsilon)$ for $\epsilon \to 0$.

Otherwise, suppose that $\varphi(0) = -1 = e^{\pi i}$, so that one can write $\varphi(\epsilon) = -e^{\epsilon(\psi(\epsilon)+\frac{1}{2})}$, with $\psi(\epsilon) \in \mathbb{C}\{\epsilon\}$. From (2.8), it follows that

$$\varphi(\epsilon) = \frac{e^{-\frac{1}{2}\epsilon(\psi(\epsilon))^2}}{\sqrt{2\pi\epsilon}}\, \Psi(\epsilon), \quad \Psi(\epsilon) = \int_{-\infty+ic}^{\infty+ic} \frac{e^{-\frac{\sigma^2}{2\epsilon}+\psi(\epsilon)\,\sigma}}{1-e^{\sigma}}\, d\sigma\,, \tag{2.11}$$

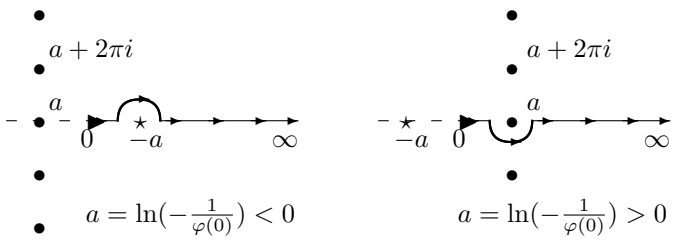

FIGURE 2. The segment $[0, +\infty)$ is deformed with a half-circle when $\arg(\varphi(0)) = \pi$ and $\varphi(0) \neq -1$!

where $c \in (-2\pi, 0)$. By making use of Residues Theorem, one can find that

$$\Big(\int_{-\infty+ic}^{\infty+ic} + \int_{\infty-ic}^{-\infty-ic}\Big) \frac{e^{-\frac{\sigma^2}{2\epsilon}+\psi(\epsilon)\,\sigma}}{1-e^{\sigma}}\, d\sigma = -2\pi i\,. \tag{2.12}$$

Let

$$H(\epsilon,\sigma) = \frac{1}{2}\Big(\frac{e^{\psi(\epsilon)\,\sigma}}{1-e^{\sigma}} + \frac{e^{-\psi(\epsilon)\,\sigma}}{1-e^{-\sigma}}\Big)\,, \quad I(\epsilon) = \int_{-\infty+ic}^{\infty+ic} H(\epsilon,\sigma)\, e^{-\frac{\sigma^2}{2\epsilon}}\, d\sigma\,. \tag{2.13}$$

By observing that

$$I(\epsilon) = \frac{1}{2}\Big(\int_{-\infty+ic}^{\infty+ic} - \int_{\infty-ic}^{-\infty-ic}\Big) \frac{e^{-\frac{\sigma^2}{2\epsilon}+\psi(\epsilon)\,\sigma}}{1-e^{\sigma}}\, d\sigma\,,$$

from (2.12) one gets the following expression for the function Ψ defined in (2.11):

$$\Psi(\epsilon) = I(\epsilon) - \pi i\,;$$

consequently, one finds that

$$\varphi(\epsilon) = \frac{e^{-\frac{1}{2}\epsilon(\psi(\epsilon))^2}}{\sqrt{2\pi\epsilon}}\,(I(\epsilon) - \pi i)\,, \quad \tilde{\varphi}(\epsilon) = \frac{e^{-\frac{1}{2}\epsilon(\psi(\epsilon))^2}}{\sqrt{2\pi\epsilon}}\, I(\epsilon)\,. \tag{2.14}$$

Moreover, the function $H(\epsilon,\sigma)$ defined by (2.13) can be continued to be an even analytic function for $\sigma \in \mathbb{C} \setminus (2\pi i\mathbb{Z}^*)$, with $H(\epsilon, 0) = \frac{1}{2} - \psi(\epsilon)$. Thus, one can choose $c = 0$ in the integral in (2.13), in such manner that one obtains the Gevrey asymptotic expansion of $I(\epsilon)$ as $\epsilon \to 0$, with

$$I(\epsilon) \sim H(0,0)\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2\epsilon}}\, d\sigma = \sqrt{2\pi\epsilon}\,\big(\frac{1}{2} - \psi(0)\big) = \sqrt{2\pi\epsilon}\,\big(1 + \varphi'(0)\big)\,.$$

This finishes the proof. □

2.2. **Analytical obstruction and half-periods.** The following theorem states for what functions φ the corresponding composite function $\varphi(\epsilon)$ defined by (2.6) represents an analytic function in a neighborhood of ϵ in $\mathbb{C}$.

Theorem 2.2. *In Theorem 2.1, the Gevrey function $\varphi(\epsilon)$ or $\tilde{\varphi}(\epsilon)$ can be continued to be analytic at $\epsilon = 0$ in $\mathbb{C}$ if, and only if, there exists some $n \in \mathbb{Z}$ such that*

$$\varphi(t) = e^{(n+\frac{1}{2})t} \quad \textit{or} \quad \varphi(t) = e^{i\pi+\frac{n}{2}t},$$

respectively.

Proof. Let

$$\nabla_R^- = \{\epsilon \in \nabla_R : \arg\epsilon \in (-\frac{3\pi}{2}, -\frac{\pi}{2})\}.$$

The function φ can be continued to be a germ of analytic function at $\epsilon = 0$ if, and only if, $\varphi(\epsilon e^{2\pi i}) = \varphi(\epsilon)$ in ∇_R^-. Consider the integral representation of $\varphi(\epsilon)$ in (2.8) and express the difference $(\varphi(\epsilon e^{2\pi i}) - \varphi(\epsilon))$ by a contour integral. By the Residues Theorem, one finds that, for $\epsilon < 0$ enough closed to zero,

$$\varphi(\epsilon e^{2\pi i}) - \varphi(\epsilon) = -2\pi i \sum_{k\geq 0} \big(e^{-\frac{(a-2k\pi i)^2}{2\epsilon}} - e^{-\frac{(a+2(k+1)\pi i)^2}{2\epsilon}}\big)\,, \tag{2.15}$$

where $a = \log(-\frac{1}{\varphi(\epsilon)}) - \frac{\epsilon}{2}$, $-\pi < \Im(a) < c$ (see Figure 1). Let $\alpha = \Re(a)$ and, for all integer k, let $\alpha_k = \Im(a) - 2k\pi$; it follows that

$$e^{-\frac{(a-2k\pi i)^2}{2\epsilon}} = e^{\frac{\alpha_k^2 - \alpha^2}{\epsilon}}\, e^{-\frac{\alpha\alpha_k}{\epsilon} i}\,.$$

Thus, by considering the dominant terms in the right-hand side of (2.15) as $\epsilon \to 0^+$, one finds that $\varphi(\epsilon e^{2\pi i}) = \varphi(\epsilon)$ if and only if

$$\alpha_k^2 = \alpha_{1-k}^2\,, \quad e^{\frac{\alpha\alpha_k}{\epsilon} i} = e^{\frac{\alpha\alpha_{1-k}}{\epsilon} i}$$

for all $k \in \mathbb{Z}_{\geq 0}$. It follows that $\alpha_0 = \pi$ and $2\alpha \in \mathbb{Z}$, which gives that $a = \pi i + \frac{n}{2}\epsilon$ for some $n \in \mathbb{Z}$.

Beside, when $\varphi(0) = -1$, we consider the integral representation of $I(\epsilon)$ given by (2.13). By writing the function $H(\epsilon, \sigma)$ in the following form:

$$H(\epsilon, \sigma) = \frac{1 - e^{(1-2\psi(\epsilon))\sigma}}{2(1 - e^{\sigma})}\, e^{\psi(\epsilon)\sigma}\,,$$

it follows that $H(\epsilon, \sigma)$ is analytic for all $\sigma \in \mathbb{C}$ if, and only if $2\psi(\epsilon) \in \mathbb{Z}$; thus, in view of (2.14), one finds the condition for $\tilde{\varphi}(\epsilon)$ to be analytic at $\epsilon = 0$ in $\mathbb{C}$. □

The choice $\arg(\varphi(0)) \in (-\pi, \pi]$ in Theorem 2.1 can be replaced with the opposite choice $\arg(\varphi(0)) \in [-\pi, \pi)$. Indeed, the change $\xi \mapsto \frac{1}{\xi}$ of the integration-variable in (2.3) implies that

$$G(\frac{x}{q}; q) = \frac{1}{x}\, G(\frac{1}{qx}; q)\,. \tag{2.16}$$

Remark 2.1. *Let $\varphi(t)$ be as given in Theorem 2.1 and let*

$$\varphi_1(t) = \frac{e^{2t}}{\varphi(t)}\,, \quad \varphi_1(\epsilon) = G(\varphi_1(\epsilon); e^{-\epsilon})\,.$$

1. *If $\varphi(0) \neq -1$ and $\arg((\varphi(0)) \in (-\pi, \pi]$, then $\arg(\varphi_1(0)) \in [-\pi, \pi)$ and $\varphi_1(\epsilon)$ admits a Gevrey asymptotic expansion as $\epsilon \to 0$ in some sector ∇_R.*
2. *If $\varphi(0) = -1 = e^{\pi i}$, $\varphi(t) = -e^{(\psi(t) - \frac{1}{2})t}$ and*

$$\tilde{\varphi}_1(\epsilon) = G(\varphi_1(\epsilon); e^{-\epsilon}) - i\sqrt{\frac{\pi}{2\epsilon}}\, e^{-\frac{\epsilon}{2}\,(\psi(\epsilon)-1)(\psi(\epsilon)+3)}\,,$$

then $\varphi_1(0) = e^{-\pi i}$ and $\tilde{\varphi}_1(\epsilon)$ admits a Gevrey asymptotic expansion as $\epsilon \to 0$ in some sector ∇_R.

Furthermore, if $\hat{\varphi}(\epsilon)$ and $\hat{\varphi}_1(\epsilon)$ are the asymptotic expansion of φ and φ_1 or $\tilde{\varphi}$ and $\tilde{\varphi}_1$ respectively, the following identity holds in the algebra $\mathbb{C}[[\epsilon]]$ of formal power series of the variable ϵ:

$$\hat{\varphi}_1(\epsilon) = e^{-\epsilon}\,\varphi(\epsilon)\,\hat{\varphi}(\epsilon)\,. \tag{2.17}$$

Proof. The relation in (2.17) follows immediately from (2.16). We omit the details. □

Thus, for any given germ of analytic function $\varphi(t)$ at $t = 0$ in $\mathbb{C}$ such that $\varphi(0) \neq -1$, the asymptotic expansion of $G(\varphi(\epsilon); e^{-\epsilon})$ can be found by considering either $\arg\big(\varphi(0)\big) \in (-\pi, \pi]$ or $\arg\big(\varphi(0)\big) \in [-\pi, \pi)$. However, by (2.5), one can notice that, when $\varphi(0) \in (-\infty, 0)$, the choice of $\arg((\varphi(0)) = \pi$ or $-\pi$ yields one plat function with respect to ϵ; in other words, the difference between the corresponding functions becomes infinitely small as $\epsilon = -\log q \to 0$.

2.3. **Asymptotic behavior of Mordell integral at unity.** Theorem 2.1 implies that, if $\varphi(t)$ is defined and analytic in the whole complex plane, then the corresponding function $G(\varphi(\epsilon); e^{-\epsilon})$ can be continued to be an analytic function in ∇_R, with $R = \infty$. Thus, for any given $(z_0, z_1) \in \mathbb{C}^2$, letting $\epsilon = -2\pi i\tau$ and $\varphi(\epsilon) = e(z_0 + z_1\tau)$ yields that the function $\tau \mapsto G(e(z_0 + z_1\tau); e(\tau))$ can be continued to be analytic for all $\tau \in \hat{\mathcal{H}}$, where

$$\hat{\mathcal{H}} = \{\tau \in \tilde{\mathbb{C}}^* : -\pi < \arg\tau < 2\pi\}.$$

Here, the Poincaré's half-plane $\mathcal{H}$ is identified to be a subset of $\tilde{\mathcal{H}}$.

In the rest of this paper, we will make use of the following notation:

$$\tilde{\mathcal{H}} = \ \{\tau \in \mathbb{C}^* : -\frac{\pi}{2} < \arg\tau < \frac{\pi}{3}\} \quad \big(= \mathbb{C} \setminus [0, -i\infty)\big)\,. \tag{2.18}$$

It is worth noticing that $\tilde{\mathcal{H}}$ contains the real axis excepted the point at origin.

Definition 2.1. *If $z = z_0 + z_1\tau$, with $(z_0, z_1) \in \mathbb{C}^2$ and $\tau \in \mathcal{H}$ or, more generally, $\tau \in \tilde{\mathcal{H}}$, we define*

$$G(z\,|\,\tau) = G(e(z); e(\tau)), \quad \omega(z\,|\,\tau) = \omega(e(z); e(\tau)). \tag{2.19}$$

Especially, by the definition of $\omega(u; q)$ in (A.8), one can obtain that

$$\omega(z\,|\,\tau) = \frac{1}{2\pi}\sqrt{\frac{i}{\tau}}\,e\big(\frac{1}{2\tau}\,(z - \frac{\tau}{2})^2\big)\,. \tag{2.20}$$

It follows that

$$\omega(z + 1\,|\,\tau) = -e\big(\frac{z + \frac{1}{2}}{\tau}\big)\,\omega(z\,|\,\tau)$$

and

$$\omega(z + \tau) = e(z)\,\omega(z\,|\,\tau) = \omega(-z\,|\,\tau)\,. \tag{2.21}$$

The relation in (2.5) can be read as follows:

$$G(z + 1\,|\,\tau) = G(z\,|\,\tau) - 2\pi i\,\omega(-z + \frac{1}{2}\,|\,\tau)\,. \tag{2.22}$$

This suggests us to consider only the case of $\Re(z_0) \in (-\frac{1}{2}, \frac{1}{2}]$.

Theorem 2.3. *Given $z = z_0 + z_1\tau$ with $(z_0, z_1) \in \mathbb{C}^2$, let $G(z\,|\,\tau)$ as in (2.19), and let $\tilde{\mathcal{H}}$ as in (2.18). Then $\tau \mapsto G(z\,|\,\tau)$ represents an analytic function for $\tau \in \tilde{\mathcal{H}}$ possessing the following asymptotic behavior as $\tau \to 0$ there.*

(1) *If* $\Re(z_0) \in (-\frac{1}{2}, \frac{1}{2}]$ *and* $z_0 \neq \frac{1}{2}$, *then* $G(z\,|\,\tau)$ *admits a Gevrey asymptotic expansion as* $\tau \to 0$ *in* $\tilde{\mathcal{H}}$, *with*

$$\lim_{\tau \xrightarrow{a.v.} 0} G(z\,|\,\tau) = \frac{1}{1+e(z_0)}\,. \tag{2.23}$$

(2) *If* $z_0 = \frac{1}{2}$, *then the function* $G(z\,|\,\tau)$ *can be put into the following form:*

$$G(z\,|\,\tau) = -\frac{i}{2}\sqrt{\frac{i}{\tau}}\, e\big(\frac{\tau}{2}(z_1+\frac{1}{2})^2\big) + g(z_1\,|\,\tau)\,, \tag{2.24}$$

where $g(z_1\,|\,\tau)$ *admits a Gevrey asymptotic expansion as* $\tau \to 0$ *in* $\tilde{\mathcal{H}}$, *with*

$$\lim_{\tau \xrightarrow{a.v.} 0} g(z_1\,|\,\tau) = 1 + z_1\,.$$

Proof. It suffices to apply Theorem 2.1, by observing that

$$\epsilon = -2\pi i\tau, \quad \varphi(t) = e(z_0)\, e^{-z_1 t}, \quad \varphi(0) = e(z_0), \quad \psi(t) = -\frac{1}{2} - z_1.$$

□

2.4. **Asymptotic behavior of Mordell integral at origin.** To every $q = e(\tau)$ with $\tau \in \mathcal{H}$ will be associated the pair (τ_1, q_1) as follows:

$$q_1 = e(\tau_1), \quad \tau_1 = -\frac{1}{\tau}\,. \tag{2.25}$$

It is obvious to see that $\tau \xrightarrow{a.v.} i\infty$ in $\mathcal{H}$ if, and only if, $\tau_1 \xrightarrow{a.v.} 0$ there; in this case, q goes to 0 exponentially rapidly.

From the functional equation of $G(x;q)$ in (2.2), one finds that

$$G(z\,|\,\tau) = 1 + e(z+\frac{1}{2})\, G(z-\tau\,|\,\tau)\,; \tag{2.26}$$

thus, one can consider only the case of $\Re(z_1) \in (-\frac{1}{2}, \frac{1}{2}]$.

Theorem 2.4. *Let* $z = z_0 + z_1\tau$, *with* z_0, $z_1 \in \mathbb{C}$. *One has the following properties.*

(1) *If* $\Re(z_1) \in (-\frac{1}{2}, \frac{1}{2}]$ *and* $z_1 \neq \frac{1}{2}$, *then* $G(z\,|\,\tau)$ *can be put into the following form:*

$$G(z\,|\,\tau) = 1 - i\sqrt{\frac{i}{\tau}}\, e\big(\frac{(z-\frac{1}{2}+\frac{\tau}{2})^2}{2\tau}\big)\, \varphi(\tau)\,, \tag{2.27}$$

where $\varphi(\tau)$ *admits a Gevrey asymptotic expansion as* $\tau \xrightarrow{a.v.} i\infty$ *in* $\mathcal{H}$ *with*

$$\lim_{\tau \xrightarrow{a.v.} i\infty} \varphi(\tau) = \frac{e(z_1)}{1+e(z_1)}\,. \tag{2.28}$$

(2) *If* $z_1 = \frac{1}{2}$, *then* $G(z\,|\,\tau)$ *can be put into the following form:*

$$G(z\,|\,\tau) = 1 - \frac{1}{2}\, e(z_0+\frac{\tau}{2}) - i\sqrt{\frac{i}{\tau}}\, e\big(\frac{(z_0-\frac{1}{2}+\tau)^2}{2\tau}\big)\,\big(1 - g(-z_0\,|\,-\frac{1}{\tau})\big), \tag{2.29}$$

where g *is the function defined in Theorem 2.3 (2) by* (2.24).

Proof. We shall apply Theorem 2.3 together with the modular-like relation given in (2.4). By making use of the notation in (2.25), one can write (2.4) into the following equivalent form:

$$G(z-\tau\,|\,\tau) = 2\pi i\, \omega(z+\frac{1}{2}\,|\,\tau)\, G(-(z+1)\tau_1\,|\,\tau_1)\,. \tag{2.30}$$

As $z = z_0 + z_1\tau$, from (2.21) it follows that

$$G(z\,|\,\tau) = 2\pi i\,\omega(-z-\frac{1}{2}\,|\,\tau)\,G(z_1+1-(z_0+1)\tau_1\,|\,\tau_1)\,. \tag{2.31}$$

By using both (2.26) and (2.22) with q_1 instead of q, one finds successively that

$$\begin{aligned} G(z_1+1-(z_0+1)\tau_1\,|\,\tau_1) &= e(-z_1+z_0\tau_1)\,\big(1-G(z_1+1-z_0\tau_1\,|\,\tau_1)\big)\\ &= e(-z_1+z_0\tau_1)\,\big(1-G(z_1-z_0\tau_1\,|\,\tau_1)\\ &\qquad +2\pi i\omega(-\frac{1}{2}-z_1+z_0\tau_1\,|\,\tau_1)\big)\,. \end{aligned}$$

A direct computation shows that

$$\omega(-z-\frac{1}{2}\,|\,\tau)\,e(-z_1+z_0\tau_1) = -\omega(-z+\frac{1}{2}\,|\,\tau)$$

and that

$$\omega(-z+\frac{1}{2}\,|\,\tau)\,\omega(-\frac{1}{2}-z_1+z_0\tau_1\,|\,\tau_1) = \frac{1}{4\pi^2}\,.$$

Thus, the relation in (2.31) can be expressed as follows:

$$\begin{aligned} G(z\,|\,\tau) &= 1-2\pi i\,\omega(-z+\frac{1}{2}\,|\,\tau)\,\big(1-G(z_1-z_0\tau_1\,|\,\tau_1)\big)\\ &= 1-i\,\sqrt{\frac{i}{\tau}}\,e\big(\frac{(z-\frac{1}{2}+\frac{\tau}{2})^2}{2\tau}\big)\,\big(1-G(z_1-z_0\tau_1\,|\,\tau_1)\big)\,. \end{aligned} \tag{2.32}$$

When $\Re(z_1) \in (-\frac{1}{2},\frac{1}{2}]$ and $z_1 \neq \frac{1}{2}$, one writes $\varphi(\tau) = 1-G(z_1-z_0\tau_1\,|\,\tau_1)$ to obtain (2.27) from (2.32). The relation in (2.28) follows from (2.23) of Theorem 2.3 (2), in view of the fact that $\tau \xrightarrow{a.v.} i\infty$ iff $\tau_1 \xrightarrow{a.v.} 0$.

When $z_1 = \frac{1}{2}$, (2.24) with $\tau_1 = -\frac{1}{\tau}$ implies that

$$G(\frac{1}{2}-z_0\tau_1\,|\,\tau_1) = -\frac{i}{2}\,\sqrt{\frac{\tau}{i}}\,e\big(-\frac{(z_0-\frac{1}{2})^2}{2\tau}\big)+g(-z_0\,|\,-\frac{1}{\tau})$$

which allows one to deduce (2.29) from (2.32). □

2.5. **Values of Mordell integral at half-periods.** We shall consider some special values of the function $g(z_1\,|\,\tau)$ introduced in the above by (2.24). Letting $z = \frac{1}{2}+z_1\tau$ in (2.26) gives

$$-e(z_1\tau)\,G(z-\tau)+G(z\,|\,\tau) = 1,$$

that together with (2.24) imply that

$$g(z_1\,|\,\tau) = 1+e(z_1\tau)\,g(z_1-1\,|\,\tau)\,. \tag{2.33}$$

When $z_1 = 0$ or $\frac{1}{2}$, one will find that

$$g(0\,|\,\tau) = 1, \quad g(\frac{1}{2}\,|\,\tau) = 1+\frac{1}{2}\,e(\frac{\tau}{2})\,, \tag{2.34}$$

by considering the following

Remark 2.2. *The following relations hold for all $\tau \in \mathcal{H}$:*

$$G(-\frac{\tau}{2}\,|\,\tau) = \frac{1}{2}\,, \tag{2.35}$$

$$G(\pm\frac{1}{2}\,|\,\tau) = 1\mp\frac{i}{2}\,\sqrt{\frac{i}{\tau}}\,e(\frac{\tau}{8})\,, \tag{2.36}$$

and

$$G(\pm\frac{1}{2}+\frac{\tau}{2}\,|\,\tau)=1+\frac{1}{2}\,e(\frac{\tau}{2})\mp\frac{i}{2}\,\sqrt{\frac{i}{\tau}}\,e(\frac{\tau}{2})\,. \tag{2.37}$$

Proof. Since $G(-\frac{\tau}{2}\,|\,\tau)=G(\frac{1}{\sqrt{q}};q)$, putting $\varphi=\frac{\epsilon}{2}$ into (2.9) yields that$F(\epsilon,\tilde{\sigma})=1$, so (2.12) gives (2.35). Another way to obtain that is to use (2.16) and the functional equation (2.26).

Putting $z=e(-\frac{1}{2})$ in (2.22) yields that

$$G(\frac{1}{2}\,|\,\tau)=G(-\frac{1}{2}\,|\,\tau)-2\pi i\,\omega(0\,|\,\tau)\,.$$

By gathering (2.26) together with (2.16), one finds that

$$G(e^{-\pi i}x;q)=1+x-xG(\frac{e^{\pi i}}{x};q)\,. \tag{2.38}$$

Applying this to $x=1=e(0)$ gives that

$$G(\frac{1}{2}\,|\,\tau)+G(-\frac{1}{2}\,|\,\tau)=2\,,$$

which implies the values given in (2.36) for $G(\pm\frac{1}{2}\,|\,\tau)$.

Putting $x=e(\frac{\tau}{2})=\sqrt{q}$ in the relation (2.38) yields that

$$G(-\frac{1}{2}+\frac{\tau}{2}\,|\,\tau)=1+e(\frac{\tau}{2})-\sqrt{q}\,G(\frac{e^{\pi i}}{\sqrt{q}};q)=2+e(\frac{\tau}{2})-G(\frac{1}{2}+\frac{\tau}{2}\,|\,\tau)\,,$$

where the last implication is obtained with the help of the functional equation (2.26). Thus, by combining this with (2.22) for $z=\frac{\tau}{2}$, one obtains that

$$G(\pm\frac{1}{2}+\frac{\tau}{2}\,|\,\tau)=1+\frac{1}{2}\,e(\frac{\tau}{2})\mp\pi i\,\omega(-\frac{\tau}{2}\,|\,\tau)\,,$$

which are equivalent to (2.37). □

By (2.24) and (2.33), one finds that, for any integer n, every function $g(\frac{n}{2}\,|\,\tau)$ can be expressed as polynomials of $e(\pm\frac{\tau}{2})$. Furthermore, by taking into account the functional equations in (2.26) and (2.22), the above formulas in (2.35), (2.36) and (2.37) allow one to obtain explicit expressions for the functions $G(k+(n+\frac{1}{2})\tau\,|\,\tau)$ and $G(k+\frac{1}{2}+\frac{n}{2}\tau\,|\,\tau)$ when $k,\,n\in\mathbb{Z}$. This shows that, as stated in Theorem 2.2, the corresponding functions $\varphi(\epsilon)$ and $\tilde{\varphi}(\epsilon)$ are all analytic at $\epsilon=0$ in $\mathbb{C}$, with $\epsilon=-2\pi i\tau$.

2.6. **Reciprocity law for generalized Gauss sums.** In order to evaluate $G(z\,|\,\tau)$ for $\tau\in\mathbb{Q}\setminus\{0\}$, we let $\zeta=e(r)$, $r=\frac{p}{m}\in(0,1)$, $(p,m)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}$, $p\wedge m=1$. If $x=e(z)$, we define

$$h(x;\zeta)=h(z\,|\,r)=\sum_{n=0}^{m-1}x^n\,\zeta^{n(n-1)/2} \tag{2.39}$$

and

$$d(x;\zeta)=d(z\,|\,r)=\sum_{n=0}^{p-1}e\big(\frac{m}{2p}(n+\frac{1}{2}+\frac{p}{2m}+z)^2\big)\,. \tag{2.40}$$

By expanding

$$(n+\frac{1}{2}+\frac{p}{2m}+z)^2=(\frac{1}{2}+\frac{p}{2m}+z)^2+(2+\frac{p}{m}+2z)n+n(n-1)\,,$$

it follows that

$$d(x;\zeta) = e^{\frac{m\pi i}{p}(\frac{1}{2}+\frac{p}{2m}+z)^2}\, h(x_1;\zeta_1)\,, \tag{2.41}$$

where $\zeta_1 = e(\frac{m}{p})$ and $x_1 = -\zeta_1\, e(\frac{mz}{p})$.

The goal of this section is to express the value of $G(z\,|\,r)$ in terms of $h(z\,|\,r)$.

Lemma 2.1. *Let $r = \frac{p}{m}$ and $d(z\,|\,r)$ as in the above. For any given $z \in \mathbb{C}$,*

$$G(z+p\,|\,r) = G(z\,|\,r) + e(-\frac{1}{8})\sqrt{\frac{m}{p}}\, d(z\,|\,r)\,. \tag{2.42}$$

Proof. Let $q = \zeta = e(r)$. By (2.22), it follows that, for all integer n,

$$G(z+n+1\,|\,r) = G(z+n\,|\,r) - i\sqrt{\frac{i}{r}}\, e\big(\frac{1}{2r}(\frac{1}{2}+z+n+\frac{r}{2})^2\big)\,,$$

what together with (2.40) imply (2.42), by noticing that $-i\sqrt{i} = e(-\frac{1}{8})$. □

Theorem 2.5. *The following identities hold for $G(z\,|\,r)$, where $r = \frac{p}{m} \in (0,1)$, $p \wedge m = 1$.*

(1) *If $mz + \frac{1}{2} \notin \mathbb{Z}$, then:*

$$G(z\,|\,r) = \frac{h(z+\frac{1}{2}\,|\,-r) - e(-\frac{1}{8})\sqrt{\frac{m}{p}}\, d(z\,|\,r)}{1+e(mz)}\,. \tag{2.43}$$

(2) *If $mz = \ell + \frac{1}{2}$ and $\ell \in \mathbb{Z}$, then:*

$$h(z+\frac{1}{2}\,|\,-r) = e(-\frac{1}{8})\sqrt{\frac{m}{p}}\, d(z\,|\,r) \tag{2.44}$$

and

$$G(z\,|\,r) = \frac{\frac{\partial}{\partial z}h(z+\frac{1}{2}\,|\,-r) - e(-\frac{1}{8})\sqrt{\frac{m}{p}}\,\frac{\partial}{\partial z}d(z\,|\,r)}{-2\pi i m}\,. \tag{2.45}$$

Proof. Putting $\tau = r$ into the functional equation in (2.26) yields that

$$\begin{aligned} G(z+mr\,|\,r) &= 1 - e\big(z+mr\big)\, G(z+(m-1)r\,|\,r) \\ &= 1 - e\big(z+mr\big)\,\big(1 - e(z+(m-1))\, G(z+(m-2)r\,|\,r)\big) \\ &= \overset{\cdots}{} \\ &= \sum_{n=0}^{m-1} (-1)^n\, e\big(nz - \frac{1}{2}n(n-1)r\big) - e(mz)\, G(z\,|\,r)\,. \end{aligned}$$

Thus, by using the function $h(x;\zeta)$ defined in (2.39), one gets that

$$G(z+p\,|\,r) = -e(mz)\, G(z\,|\,r) + h(z+\frac{1}{2}\,|\,-r)\,. \tag{2.46}$$

(1) When $e(mz) \neq -1$, gathering (2.46) with (2.42) yields (2.43).

(2) When $e(mz) = -1$, it follows that the relations in (2.46) and (2.42) must be the same, so one finds the equality in (2.44). Furthermore, by taking the limit $z \to z' = \frac{1}{2m} + \frac{\ell}{m}$ in (2.43) gives immediately (2.45). □

Remark 2.3. *The identity in* (2.44) *is the so-called* reciprocity theorem for generalized Gauss sums*; see [9, p. 13, Theorem 1.2.2].*

2.7. **Values of Mordell integral at real half-periods.** In order to simplify, we will write

$$h_1(z\,|\,\zeta)=\frac{1}{2\pi i}\,\frac{\partial}{\partial z}h(z\,|\,r),\quad d_1(z\,|\,\zeta)=\frac{1}{2\pi i}\,\frac{\partial}{\partial z}d(z\,|\,r)\,. \tag{2.47}$$

Thus, (2.45) can be read as follows:

$$G(z\,|\,r)=-\frac{1}{m}\big(h_1(z+\frac{1}{2}\,|\,-r)-e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d_1(z\,|\,r)\big)\,. \tag{2.48}$$

Remark 2.4. (1) *If p is even, then:*

$$h(-\frac{r}{2}+\frac{1}{2}\,|\,-r)-e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d(-\frac{r}{2}\,|\,r)=1\,. \tag{2.49}$$

(2) *If p is odd, then:*

$$h(-\frac{r}{2}+\frac{1}{2}\,|\,-r)=e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d(-\frac{r}{2}\,|\,r) \tag{2.50}$$

and

$$h_1(-\frac{r}{2}+\frac{1}{2}\,|\,-r)-e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d_1(-\frac{r}{2}\,|\,r)=-\frac{m}{2}\,. \tag{2.51}$$

(3) *If m is even, then:*

$$h(0\,|\,-r)-e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d(\pm\frac{1}{2}\,|\,r)=2\mp e\big(\frac{1}{8}(3+\frac{p}{m})\big)\,\sqrt{\frac{m}{p}} \tag{2.52}$$

and

$$h(\frac{r}{2}\,|\,-r)-e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d(\pm\frac{1}{2}+\frac{r}{2}\,|\,r)=2+e(\frac{p}{2m})\,\big(1\mp e(\frac{3}{8})\big)\,\sqrt{\frac{m}{p}}\,. \tag{2.53}$$

(4) *If m is odd, then:*

$$h(0\,|\,-r)=e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d(\pm\frac{1}{2}\,|\,r)\,, \tag{2.54}$$

$$h(\frac{r}{2}\,|\,-r)=e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d(\pm\frac{1}{2}+\frac{r}{2}\,|\,r)\,, \tag{2.55}$$

$$h_1(0\,|\,-r)-e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d_1(\pm\frac{1}{2}\,|\,r)=-\frac{m}{2}\,\big(2\mp\,e\big(\frac{1}{8}(3+\frac{p}{m})\big)\,\sqrt{\frac{m}{p}}\big)\,, \tag{2.56}$$

and

$$h_1(\frac{r}{2}\,|\,-r)-e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d_1(\pm\frac{1}{2}+\frac{r}{2}\,|\,r)=-\frac{m}{2}\,\big(2+e(\frac{p}{2m})\,\big(1\mp e(\frac{3}{8})\,\sqrt{\frac{m}{p}}\big)\big)\,. \tag{2.57}$$

Proof. This follows from Remark 2.2, with the help of Theorem 2.5 and (2.48). Moreover, one sees that some of these formulas can be found by elementary method as in [9, p. 44, 9 and 10]. □

Remark 2.5. *When $z=0$, there is no close-form for $G(0\,|\,r)$ but one can make use of the following relation:*

$$G(0\,|\,r)=\frac{1}{2}\,\big(h(\frac{1}{2}\,|\,-r)-e(-\frac{1}{8})\,\sqrt{\frac{m}{p}}\,d(0\,|\,r)\big)\,. \tag{2.58}$$

3. Asymptotic bahavior of Appell-Lerch series

As in [34] and [27], we will call a q-spiral any set of the form $\{aq^n : n \in \mathbb{Z}\}$, where $a \in \mathbb{C}^*$. This set will be denoted as $aq^{\mathbb{Z}}$ or $[a;q]$. By following (A.6) and (A.10), to every given $u \in \mathbb{C}^* \setminus [1;q]$ can be associated the q-Borel sum of $\hat{E}(x;q)$ along the q-spiral $[-u;q]$. Let $\mathcal{S}^{[-u;q]}\hat{E}$ denote this sum-function, this is to say:

$$\mathcal{S}^{[-u;q]}\hat{E}(x;q) = \sum_{\xi\in[-u;q]} \frac{1}{1+\xi}\,\frac{1}{\theta(\frac{\xi}{x};q)}\,,$$

where $\theta(t;q) = \sum_{n\in\mathbb{Z}} q^{n(n-1)/2}t^n$. By the functional relation

$$\theta(q^n t;q) = t^{-n}q^{-n(n-1)/2}\theta(t;q) \qquad (n\in\mathbb{Z}),$$

it follows that

$$\mathcal{S}^{[-u;q]}\hat{E}(x;q) = \sum_{n\in\mathbb{Z}} \frac{1}{1-uq^n}\,\frac{1}{\theta(-\frac{u}{x};q)}\,(-\frac{u}{x})^n\,q^{n(n-1)/2}\,.$$

Comparing this with the definition of $L(x,u;q)$ in (2.1) and the expression of $L(z,w\,|\,\tau)$ in (1.29) gives that

$$L(x,u;q) = \mathcal{S}^{[-u;q]}\hat{E}(x;q)\,. \tag{3.1}$$

In other words, $L(x,u;q)$ is the q-Borel sum of $\hat{E}(x;q)$ along the spiral $[-u;q]$.

In §3.1 in the below, we will recall a modular-like relation for $L(z,w\,|\,\tau)$ that can be found by a viewpoint of Stokes analysis on sum-functions of divergent power series. By means of this formula, we will deduce, in §3.2, that $L(z,w\,|\,\tau)$ is of almost theta-type for $q \xrightarrow{a.r.} 1$; see Theorem 3.5. In order to extend this approach to any root of unity, we will introduce, like in the classical theory of elliptic functions, a family of linear fractional transformations associated to any given root of unity in §3.3.

From §3.4, we shall consider the asymptotic behavior of $L(z,w\,|\,\tau)$ when q tends to any non-trivial root of unity. By Theorem 3.6, we shall show that this function can be decomposed by a natural manner into a Gevrey function plus a theta-type function. This key result will be proved at the end of §3.5. In §3.6, we shall see that, even the Gevrey part of $L(z,w\,|\,\tau)$ at each root $\zeta = e(\frac{p}{m})$ has a finite limit, this limit becomes increasing as $m\to\infty$ except for very few special values of z. These values are non-zero half-periods, for what the associated function $L(z,w\,|\,\tau)$ is constant or false-theta; see Theorem 3.9 in §3.7. Finally, in §3.8, we will complete the proof of Theorem 1.3.

3.1. **Modularity viewed from Stokes phenomenon.**

The above sum-function $L(x,u;q)$ is the unique solution of the q-difference equation in (2.2) that is analytic for $x\in\mathbb{C}^*\setminus[u;q]$ and admits a simple pole at each point of the spiral $[u;q]$. Beside, the Mordell integral $G(x;q)$ of (2.3) is also a solution of (2.2) and a sum-function of the q-Euler series $\hat{E}(x;q)$. Thus $L(x,u;q)-G(x;q)$ satisfies the homogeneous equation $y(x)+xy(\frac{x}{q}) = 0$ and is infinitely flat when $x\to 0$. This is a Stokes phenomenon as currently observed in the analytic theory of ODE. In this way, one can find the following

Theorem 3.1. *([38, Theorem 1.2]) Let $\tilde{\mathbb{C}}^*$ be the Riemann surface of logarithm, and let $\pi : \tilde{\mathbb{C}}^* \mapsto \mathbb{C}^*$ be the associated natural map. The following relation holds*

for all $u \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$ *and* $x \in \tilde{\mathbb{C}}^*$ *with* $\pi(x) \in \mathbb{C}^* \setminus (uq^{\mathbb{Z}})$*:*

$$L(x,u;q) = G(xe^{-2\pi i};q) - 2\pi i\,\omega(\frac{e^{\pi i}}{x};q)\,L(e^{2\pi i\frac{\log x}{\log q}}, e^{2\pi i\frac{\log u}{\log q}}; e^{\frac{4\pi^2}{\log q}}). \tag{3.2}$$

Proof. Putting $\alpha = -\pi$ in [38, Theorem 1.2], one can find that $f_\alpha(e^{-\pi i}\sqrt{q}\,x, \sqrt{q}\,) = G(x;q)$, which implies (3.2), for $g_\lambda(-\sqrt{q}\,x, \sqrt{q}\,) = L(x,\lambda;q)$. □

The above relation in (3.2) will play an important role for what follows. Indeed, this is a modular-like relation, for $L(x,u;q)$ is expressed by means of the modular variable $(x_1,u_1,q_1) = \big(e(\frac{\log x}{\log q}), e(\frac{\log u}{\log q}), e(-\frac{2\pi i}{\log q})\big)$.

Theorem 3.2. *For all* $(z,w) \in \mathbb{R}^{2,*}_{1\oplus\tau}$*,*

$$L(z,w\,|\,\tau) = G(z\,|\,\tau) + C(z\,|\,\tau)\,\big(L(\frac{z}{\tau},\frac{w}{\tau}\,|\,-\frac{1}{\tau}) - 1\big)\,, \tag{3.3}$$

where

$$C(z\,|\,\tau) = -2\pi i\,\omega(\frac{e^{\pi i}}{e(z)};q) = -2\pi i\,\omega(-z+\frac{1}{2}\,|\,\tau)\,. \tag{3.4}$$

Proof. Using (2.1), (2.19) and (3.4) allows one to write the relation in (3.2) into the following form:

$$L(z,w\,|\,\tau) = G(z-1\,|\,\tau) + C(z\,|\,\tau)\,L(\frac{z}{\tau},\frac{w}{\tau}\,|\,-\frac{1}{\tau})\,. \tag{3.5}$$

Thus, one reduces (3.3) from the functional equation in (2.22) for $G(z\,|\,\tau)$. □

By considering the definition of $w(z\,|\,\tau)$ in (2.20) and the classical modular formula for $\theta(x;q)$ [4, p. 626, (D.4.2)], the factor $C(z\,|\,\tau)$ defined in the above by (3.4) has the following alternative expressions:

$$C(z\,|\,\tau) = -i\sqrt{\frac{i}{\tau}}\,e\big(\frac{(z+\frac{\tau}{2}-\frac{1}{2})^2}{2\tau}\big) = \frac{1}{\tau}\,\frac{\theta(-\frac{z}{\tau}+\frac{1}{2}\,|\,-\frac{1}{\tau})}{\theta(-z+\frac{1}{2}\,|\,\tau)}\,. \tag{3.6}$$

Remind that the second expression requests that $z \notin \mathbb{Z}\oplus\tau\mathbb{Z}$.

Beside, from the definition of $L(z,w\,|\,\tau)$ in (1.29), it follows that

$$R_1(z,w\,|\,\tau) = \theta(z+\frac{\tau}{2}\,|\,\tau)\,L(-z+w-\frac{\tau}{2}+\frac{1}{2},w\,|\,\tau)\,. \tag{3.7}$$

Theorem 3.3. *The following relation holds for all* $z \in \mathbb{C}$ *and* $w \in \mathbb{C}\setminus(\mathbb{Z}\oplus\tau\mathbb{Z})$*:*

$$R_1(z,w\,|\,\tau) = \theta(z+\frac{\tau}{2}\,|\,\tau)\,G(-z+w-\frac{\tau}{2}-\frac{1}{2}\,|\,\tau) + C(z,w\,|\,\tau)\,R_1(\frac{z}{\tau},\frac{w}{\tau}\,|\,-\frac{1}{\tau})\,, \tag{3.8}$$

where

$$C(z,w\,|\,\tau) = \frac{1}{\tau}\,\frac{C(-z+w-\frac{\tau}{2}+\frac{1}{2}\,|\,\tau)}{C(-z-\frac{\tau}{2}+\frac{1}{2}\,|\,\tau)}\,. \tag{3.9}$$

Proof. This follows directly from the relation in (3.5). □

3.2. **Almost theta-type properties at origin and unity.** By Lemma 1.1, one knows that $\theta(z\,|\,\tau) \in \mathfrak{T}_0$ for all $z \in \mathbb{R}_\tau = \mathbb{R} \oplus \tau\mathbb{R}$. In a similar way, one can prove the following

Lemma 3.1. *Given $(u, v) \in \mathbb{R}_\tau \times \mathbb{R}_\tau$, one has $R_1(u, v\,|\,\tau) \in \mathfrak{T}_0$ provided that $v \notin \mathbb{Z} \oplus \tau\mathbb{Z}$.*

This follows directly from the definition of R_1 in (0.1) with $\mathsf{k} = 1$. For the proof, see §4.4. By considering Proposition 1.1 (2) and both Lemmas 1.1 and 3.1, one gets immediately the following

Proposition 3.1. *Given $(z, w) \in \mathbb{R}^{2,*}_{1\oplus\tau}$, it follows that $L(z, w\,|\,\tau) \in \mathfrak{T}_0$.* □

This can be easily generalized as follows.

Remark 3.1. *Let I be a finite set of real numbers and γ be a $\mathbb{C}^*$-valued function on I. If*

$$f(q) = L(z, w\,|\,\tau) + \sum_{k\in I} \gamma(k)\, e(k\tau)\,,$$

then $f \in \mathfrak{T}_0$ provided that $(z, w) \in \mathbb{R}^{2,*}_{1\oplus\tau}$. □

We shall give the dominant term of $L(z, w\,|\,\tau)$ for $\tau \xrightarrow{a.v.} i\infty$. As often as possible, we will denote by $\varnothing$ for any function that is exponentially small.

Proposition 3.2. *Let $u = u_0 + u_1\tau$, $v = v_0 + v_1\tau$, with u_j, $v_j \in \mathbb{R}$, $(v_0, v_1) \notin \mathbb{Z}^2$, and let $f(q) = R_1(u, v\,|\,\tau)$. One assumes that $u_1 \in (-\frac{1}{2}, \frac{3}{2})$ and $\tau \xrightarrow{a.v.} i\infty$ in $\mathcal{H}$*

(1) *If $v_1 \in (-1, 0)$ and $u + v \neq -\frac{\tau}{2} \mod \mathbb{Z}$, then:*

$$f(q) = \big(e(u + \frac{\tau}{2}) - e(-v)\big)(1 + \varnothing)\,. \tag{3.10}$$

(2) *If $v_1 \in (-1, 0)$ and $u + v = -\frac{\tau}{2} \mod \mathbb{Z}$, then:*

$$f(q) = \begin{cases} \big(e(\tau) - e(-2v)\big)\,(1 + \varnothing) & 2v + \tau \notin \mathbb{Z} \\ 0 & 2v + \tau \in \mathbb{Z} \end{cases}\,. \tag{3.11}$$

(3) *If $v_1 = 0$, then $f(q) = \frac{1}{1-e(v_0)}\,(1 + \varnothing)$.*

The proof can be done by direct calculations; see §4.5. Proposition 3.2 and Lemma 1.1 give the following

Theorem 3.4. *Let $(z, w) \in \mathbb{R}^{2,*}_{1\oplus\tau}$, $z = z_0 + z_1\tau$, $w = w_0 + w_1\tau$, and let $f(q) = 1 - L(z, w\,|\,\tau)$. One assumes that $\tau \xrightarrow{a.v.} i\infty$. If either $w_1 < z_1 \leq \frac{1}{2}$ and $w_1 \in (-\frac{1}{2}, 0]$ or $z_1 \in [-\frac{1}{2}, \frac{1}{2}]$ and $w_1 \in (0, \frac{1}{2})$, then $P_0(f) = 0$.*

The proof will be given in §4.6. Indeed, one can find $c \in \mathbb{C}^*$ and $\kappa > 0$, depending of (z_0, z_1, w_0, w_1), such that $t_0(f)(q) = c\, e\big(\kappa\tau\big)$ in each of the cases involved in Theorem 3.4.

Remark 3.2. *Given $(z, w) \in \Omega^{2,*}_{1\oplus\tau}$, let $f(q) = L(z, w\,|\,\tau)$. If either $w_1 > 0$ or $w_1 \leq 0$ but $z_1 > w_1$, then $t_0(f)(q) = 1$.*

This follows from Theorem 3.4, for $L(z, w\,|\,\tau) = 1 - f_1(q)$ and $P_0(f_1) = 0$. □

Theorem 3.5. *Given $(z, w) \in \Omega^{2,*}_{1\oplus\tau}$, the function $L(z, w\,|\,\tau)$ belongs to $\tilde{\mathfrak{T}}_1$.*

Proof. Let $f(q) = L(z, w \,|\, \tau)$. When $q \overset{a.r.}{\longrightarrow} 1$, it follows that $\hat{\tau} = \tau \overset{a.v.}{\longrightarrow} 0$, $-\frac{1}{\tau} \overset{a.v.}{\longrightarrow} i\infty$ and that $q_1 = e(-\frac{1}{\tau}) \overset{a.r.}{\longrightarrow} 0$ exponentially. By Remark 3.1, it follows that $L(\frac{z}{\tau}, \frac{w}{\tau} \mid -\frac{1}{\tau}) - 1 \in \mathfrak{T}_1$. Thus, by considering the fact that $C(z \,|\, \tau) \in \mathfrak{T}$, applying Proposition 1.1 (2) to the last term of the relation in (3.3) implies that $f(q) - G(z \,|\, \tau) \in \mathfrak{T}_1$.

On the one hand, Theorem 2.3 (1) gives that $G(z \,|\, \tau) \in \mathfrak{G}_1$ provided that $z = z_0 + z_1\tau$, $z_0 \in (-\frac{1}{2}, \frac{1}{2})$. Under this last condition, one obtains that $f \in \tilde{\mathfrak{T}}_1$, in view of Definition 1.1 (1).

On the other hand, assume $z = \frac{1}{2} + z_1\tau$, apply (2.24) of Theorem 2.3, and make use of the first expression of $C(z \,|\, \tau)$ in (3.6). It follows that

$$G(\frac{1}{2} + z_1\tau \,|\, \tau) - C(\frac{1}{2} + z_1\tau \,|\, \tau) = g(z_1 \,|\, \tau)\,, \tag{3.12}$$

where g admits a Gevrey expansion for $\tau \overset{a.v.}{\longrightarrow} 0$ in $\mathcal{H}$ as stated by Theorem 2.3 (2). Thus, (3.3) becomes

$$f(q) = g(z_1 \,|\, \tau) + C(z \,|\, \tau)\, L(\frac{z}{\tau}, \frac{w}{\tau} \mid -\frac{1}{\tau})\,,$$

what gives that $f \in \tilde{\mathfrak{T}}_1$. □

3.3. **Continued fractions and modular transforms.** In the above, the asymptotic behavior of $L(z, w \,|\, \tau)$ stated by Theorem 3.5 for $q \overset{a.r.}{\longrightarrow} 1$ or $\tau \overset{a.v.}{\longrightarrow} 0$ was found by means of the modular-like relation in (3.3). It is worth noticing that $L(z, w \,|\, \tau)$ is left invariant if one increases or decreases by an integer anyone of the entries z, w and τ. This idea applied to both sides of (3.3) leads one to make use of the continued fractions to reach any root of unity for q, as in the classical theory of theta-functions.

As usual, let $[a]$ and $\{a\}$ denote the integral and fractional parts of any real a:

$$[a] \in \mathbb{Z} \cap (a-1, a]\,; \qquad \{a\} = a - [a] \in [0, 1)\,. \tag{3.13}$$

Given $r \in (0,1) \cap \mathbb{Q}$, we define $r_0 = r$ and, for $n \geq 0$,

$$r_{n+1} = \{\frac{-1}{r_n}\}, \qquad m_n = -[\frac{-1}{r_n}]\,. \tag{3.14}$$

Let ν to denote the positive integer such that $r_{\nu-1} \neq 0$ and $r_\nu = 0$, this is to say, $r_{\nu-1}$ becomes the inverse of certain positive integer, $\frac{1}{m_{\nu-1}}$. By convention, we will write $m_\nu = 0$. By using the standard notation $[., ., ...]$ [5, ??], r can be represented by the following continued fraction:

$$r = \Big[0, m_0, -m_1, ..., (-1)^\nu m_\nu\Big]. \tag{3.15}$$

First, to the sequence (r_n, m_n) of (3.14) will be associated a family of linear fractional transformations $(\tau \mapsto \tau_n)$ as follows:

$$\tau_0 = \tau, \qquad \tau_{n+1} = \frac{-1}{\tau_n} + m_n \quad (n \in \mathbb{Z} \cap [0, \nu])\,. \tag{3.16}$$

Letting $M_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M_n = \begin{pmatrix} m_n & -1 \\ 1 & 0 \end{pmatrix}$ for $n \geq 0$, one finds that $\tau_{n+1} = M_n(\tau_n)$, this means that τ_{n+1} is the Möbius transformation of τ_n associated with $M_n \in SL(2, \mathbb{Z})$ on $\mathcal{H}$. In this way, one gets that, for all integer n in $[0, \nu+1]$:

$$\tau_n = \bar{M}_n(\tau_0) = \bar{M}_n(\tau), \quad \bar{M}_n = M_{n-1}...M_{-1}\,. \tag{3.17}$$

Lemma 3.2. *Let $r = \frac{p}{m} \in (0,1) \cap \mathbb{Q}$, $(p,m) \in \mathbb{Z}^2_{>0}$, $p \wedge m = 1$, and let τ_n as in* (3.16) *or in* (3.17). *Then:*

$$\tau \xrightarrow{a.v.} r \qquad \Longleftrightarrow \qquad \tau_n \xrightarrow{a.v.} r_n \qquad \Longleftrightarrow \qquad \tau_\nu \xrightarrow{a.v.} 0 \tag{3.18}$$

where $0 < n < \nu$, and

$$\tau_0\, \tau_1 \ldots \tau_\nu = m(\tau - r)\,. \tag{3.19}$$

Moreover, there exists $\alpha \in \mathbb{Z}$ such that $m \wedge \alpha = 1$ and

$$\tau_{\nu+1} = -\frac{1}{m^2(\tau - r)} + \frac{\alpha}{m}\,. \tag{3.20}$$

Proof. Comparing (3.14) with (3.16) yields directly the relations of equivalence included in (3.18).

Let $A(\tau) = \tau_0\, \tau_1 \ldots \tau_\nu$. By the definition of τ_n given in (3.17), let

$$\tau_n = \frac{\alpha_n \tau + \beta_n}{\gamma_n \tau + \delta_n}\,, \qquad \bar{M}_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \in SL(2,\mathbb{Z})\,. \tag{3.21}$$

The relation $\bar{M}_{n+1} = M_n\, \bar{M}_n$ implies that

$$(\alpha_{n+1}, \beta_{n+1}) = m_n(\alpha_n, \beta_n) - (\gamma_n, \delta_n)\,, \qquad (\gamma_{n+1}, \delta_{n+1}) = (\alpha_n, \beta_n)\,.$$

Thus, one finds that $A(\tau) = \alpha_\nu \tau + \beta_\nu$. By (??), it follows that $\alpha_\nu = m$ and $\beta_\nu = -p$, which gives the equality in (3.19).

Furthermore, the foregoing consideration on (α_ν, β_ν) gives that $\bar{M}_{\nu+1}$ is of the following form:

$$\bar{M}_{\nu+1} = \begin{pmatrix} \alpha & -\beta \\ m & -p \end{pmatrix} \in SL(2,\mathbb{Z})\,. \tag{3.22}$$

In this way, (3.21) with $n = \nu + 1$ implies the expression of $\tau_{\nu+1}$ in (3.20). □

Secondly, given $z = z_0 + z_1\tau \in \Omega_\tau$, we define the sequence $(\hat{z}_n)$ for $0 \le n \le \nu + 1$ as follows: $\hat{z}_n = z_{n,0} + z_{n,1}\tau_n \in \Omega_{\tau_n}$ with

$$\begin{pmatrix} z_{n+1,0} \\ -z_{n+1,1} \end{pmatrix} = M_n \begin{pmatrix} z_{n,0} \\ -z_{n,1} \end{pmatrix} + \begin{pmatrix} \delta \\ 0 \end{pmatrix}\,, \tag{3.23}$$

where $\delta \in \mathbb{Z}$. In the above, $\hat{z}_0 = z$, so $z_{0,k} = z_k$ for $k = 0$ or 1. It can be easily seen that

$$\hat{z}_{n+1} = \frac{\hat{z}_n}{\tau_n} \pmod{\mathbb{Z}}\,, \quad \begin{pmatrix} z_{n,0} \\ -z_{n,1} \end{pmatrix} = \bar{M}_n \begin{pmatrix} z_0 \\ -z_1 \end{pmatrix} + \begin{pmatrix} \delta_{n,0} \\ \delta_{n,1} \end{pmatrix}\,, \tag{3.24}$$

where $\delta_{n,0}$, $\delta_{n,1} \in \mathbb{Z}$.

Lemma 3.3. *The sequence $(\hat{z}_n)$ defined in* (3.23) *has the following properties.*

1. *$\hat{z}_n = 0$ if, and only if, $\hat{z}_{n+1} = 0$.*
2. *$\hat{z}_n \in \{\frac{1}{2}, -\frac{1}{2}\tau_n, \frac{1}{2} - \frac{1}{2}\tau_n\}$ if, and only if, $\hat{z}_{n+1} \in \{\frac{1}{2}, -\frac{1}{2}\tau_{n+1}, \frac{1}{2} - \frac{1}{2}\tau_{n+1}\}$.*
3. *$\kappa(z \,|\, \tau, r) = \kappa(\hat{z}_n \,|\, \tau_n, r_n) = \kappa(\hat{z}_{\nu+1} \,|\, \tau_{\nu+1}, \infty)$, where $\kappa(z \,|\, \tau, \infty)$ and $\kappa(z \,|\, \tau, r)$ are defined in* (1.22) *and in* (1.23).

Proof. The relations of equivalence stated in (1) and (2) follow directly from the second relation in (3.24).

(3) For $n \le \nu$, let $r_n = \frac{p_n}{d_n}$, $(p_n, d_n) \in \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{>0}$, $p_n \wedge d_n = 1$. By (3.14), it follows that $-\frac{d_n}{p_n} = \frac{p_{n+1}}{d_{n+1}} - m_n$, thus

$$d_{n+1} = p_n\,, \quad p_{n+1} = m_n p_n - d_n\,. \tag{3.25}$$

By the second relation in (3.24), $z_{n+1,0} = m_n z_{n,0} + z_{n,1} + \delta$ and $z_{n+1,1} = -z_{n,0}$, where $\delta \in \mathbb{Z}$. Thus, $d_{n+1}z_{n+1,0} + p_{n+1}z_{n+1,1} = d_n z_{n,0} + p_n z_{n,1} + \delta p_n$, what implies $\kappa(\hat{z}_{n+1} \,|\, \tau_{n+1}, r_{n+1}) = \kappa(\hat{z}_n \,|\, \tau_n, r_n)$ for $n < \nu$, by (1.23). By combining this together with (1.22), one obtains the same equality for $n = \nu$, with $d_{\nu+1} = p_\nu = 0$ and $p_{\nu+1} = -d_n = -1$. □

Letting $n = \nu + 1$ in (3.24) gives the following

Remark 3.3. *Let α, β be as in* (3.22). *Then:*

$$\hat{z}_{\nu+1} = (\alpha z_0 + \beta z_1) - (m z_0 + p z_1)\tau_{\nu+1} \mod (\mathbb{Z} \oplus \tau_{\nu+1}\mathbb{Z}) . \tag{3.26}$$

In what follows, we consider two examples for illustrating the general results established in the above.

Remark 3.4. *For $r = \frac{1}{m}$, where $m \in \mathbb{Z}_{\geq 2}$, it follows that $m_0 = m$, $\nu = 1$,*

$$\tau_1 = -\frac{1}{\tau} + m , \qquad \tau_2 = -\frac{1}{m^2\hat{\tau}} - \frac{1}{m} ,$$

where $\hat{\tau} = \tau - r$. Thus, the definition of $(\hat{z}_n)$ in (3.23) *becomes: $\hat{z}_0 = z = z_0 + z_1\tau$, $\hat{z}_1 = z_{1,0} + z_{1,1}\tau_1 = (m z_0 + z_1 + \delta) - z_0\tau_1$ and*

$$\hat{z}_2 = z_{2,0} + z_{2,1}\tau_2 = -z_0 + \delta' - (m z_0 + z_1 + \delta)\tau_2 , \tag{3.27}$$

where δ and δ' denote some suitable integers.

Remark 3.5. *Let $p \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{Z}_{\geq 2}$ and $m = kp - 1$. For $r = \frac{p}{m}$, $m_0 = k$, $r_1 = \frac{1}{p}$, $m_1 = p$, $\nu = 2$,*

$$\tau_1 = -\frac{1}{\tau} + k , \quad \tau_2 = -\frac{1}{k^2(\tau - \frac{1}{k})} + p - \frac{1}{k} , \quad \tau_3 = -\frac{1}{m^2\hat{\tau}} - \frac{k}{m} .$$

By (3.23), *$\hat{z}_0 = z = z_0 + z_1\tau$, $\hat{z}_1 = (k z_0 + z_1 + \delta) - z_0\tau_1$, $\hat{z}_2 = m z_0 + p z_1 + \delta' - (k z_0 + z_1 + \delta)\tau_2$ and*

$$\hat{z}_3 = z_{3,0} + z_{3,1}\tau_3 = -k z_0 - z_1 + \delta'' - (m z_0 + p z_1 + \delta')\tau_3 . \tag{3.28}$$

In the above, δ, δ' and δ'' belong to $\mathbb{Z}$.

3.4. **Modular-like formula at an arbitrary root of unity.** Let $r \in \mathbb{Q} \cap (0, 1)$, and keep the corresponding construction of (r_n, m_n) and τ_n in (3.14) and in (3.16), respectively. Let z and w be given in Ω_τ, consider the sequence $(\hat{z}_n)$ of (3.23), and define $(\hat{w}_n)$ in the same fashion as what was done for $(\hat{z}_n)$. By Lemma 3.3 (1), one can easily see that $w \notin \{0, z\}$ if, and only if, $\hat{w}_n \notin \{0, \hat{z}_n\}$, where $n = 1$, ..., $\nu + 1$. This gives that

$$(z, w) \in \Omega^{2,*}_{1\oplus\tau} \iff (\hat{z}_n, \hat{w}_n) \in \Omega^{2,*}_{1\oplus\tau_n} \ (1 \leq n \leq \nu + 1). \tag{3.29}$$

This relation in (3.29) in the above implies that once $L(z, w \,|\, \tau)$ is well-defined, the functions $L(\hat{z}_n, \hat{w}_n \,|\, \tau_n)$ continue to be well-defined till $n = \nu + 1$. By using the first relation in (3.24) and iterating the modular-like formula in (3.3), one finds that

$$L(z, w \,|\, \tau) = g_n(z \,|\, \tau) + C_n(z \,|\, \tau)\, L(\hat{z}_{n+1}, \hat{w}_{n+1} \,|\, \tau_{n+1}) , \tag{3.30}$$

where g_n and C_n are determined recursively as follows. We let $g_{-1}(z \,|\, \tau) = 0$ and $C_{-1}(z \,|\, \tau) = 1$; for $n \in [0, \nu] \cap \mathbb{Z}$,

$$C_n(z \,|\, \tau) = C_{n-1}(z \,|\, \tau)\, C(\hat{z}_n \,|\, \tau_n) \tag{3.31}$$

and

$$g_n(z\,|\,\tau) = g_{n-1}(z\,|\,\tau) + C_{n-1}(z\,|\,\tau)\,\big(G(\hat{z}_n\,|\,\tau_n) - 1\big)\,. \tag{3.32}$$

Remark 3.6. *For $n \in [0,\nu)\cap\mathbb{Z}$, both $C_n(z\,|\,\tau)$ and $g_n(z\,|\,\tau)$ are analytic at $\tau = r$ in the cut-plane $\mathbb{C}\setminus[0,-i\infty)$. Moreover, it follows that*

$$\Big|C_{\nu-1}(z\,|\,\tau)_{|_{\tau=r}}\Big| = \sqrt{m}\,. \tag{3.33}$$

Indeed, from (3.31), one gets that

$$C_n(z\,|\,\tau) = \prod_{k=0}^{n} C(\hat{z}_k\,|\,\tau_k)\,. \tag{3.34}$$

Combining this together with the first expression of $C(z\,|\,\tau)$ in (3.6) yields that

$$C_n(z\,|\,\tau) \,=\, (-i)^{n+1}\,\prod_{k=0}^{n}\sqrt{\frac{i}{\tau_k}}\,e\big(\frac{(2\hat{z}_k+\tau_k-1)^2}{8\tau_k}\big)\,. \tag{3.35}$$

By (3.18), it follows that, as $\tau \overset{a.v.}{\longrightarrow} r$, $\tau_k \overset{a.v.}{\longrightarrow} r_k \in (0,1)$ for $k \le n < \nu$. Thus, each *new* variable τ_k describes a neighborhood of r_k in $\mathbb{C}\setminus[0,-i\infty)$ when τ remains in an enough small open-disc centered at r in this cut-plane. In this way, one finds that $\tau \mapsto C_n(z\,|\,\tau)$ is analytic at $\tau = r$. This implies also the analyticity of g_n, for each $G(\hat{z}_k\,|\,\tau_k)$ is analytic for $\tau_k \sim r_k \neq 0$ as stated by Theorem 2.3.

By combining (3.19) with (3.20), it follows that

$$\tau_0...\tau_{\nu-1} = \frac{m\,\hat{\tau}}{\tau_\nu} = -m\,\hat{\tau}\,\tau_{\nu+1} = \frac{1}{m} - \alpha\,\hat{\tau}\,.$$

Thus, letting $\tau \overset{a.v.}{\longrightarrow} r$ yields that $r_0...r_{\nu-1} = \frac{1}{m}$. One deduces immediately (3.33) from (3.35). □

Letting $n = \nu$ into (3.30) yields that

$$L(z,w\,|\,\tau) = g_\nu(z\,|\,\tau) + C_\nu(z\,|\,\tau)\,L(\hat{z}_{\nu+1},\hat{w}_{\nu+1}\,|\,\tau_{\nu+1})\,. \tag{3.36}$$

This relation extends the modular-like formula of (3.3) to the case of $r \in (0,1)\cap\mathbb{Q}$. In order to identify this parameter r, we let

$$C(z\,|\,\tau,r) = C_\nu(z\,|\,\tau) = \prod_{k=0}^{\nu} C(\hat{z}_k\,|\,\tau_k)\,, \tag{3.37}$$

$$g(z\,|\,\tau,r) = g_\nu(z\,|\,\tau) - \varsigma\,C(z\,|\,\tau,r) \tag{3.38}$$

and

$$\vartheta(z,w\,|\,\tau,r) = C(z\,|\,\tau,r)\,\big(L(\hat{z}_{\nu+1},\hat{w}_{\nu+1}\,|\,\tau_{\nu+1}) + \varsigma\big)\,. \tag{3.39}$$

Here, we set $\varsigma = 0$ or 1 when $z_{\nu,0} \neq \frac{1}{2}$ or $z_{\nu,0} = \frac{1}{2}$. By (3.24) and (3.26), it follows that $z_{\nu,0} = -z_{\nu+1,1} = mz_0 + pz_1 \mod \mathbb{Z}$. Thus, $\varsigma = 0$ or 1 when $mz_0 + pz_1 + \frac{1}{2} \neq 0$ or $= 0 \mod \mathbb{Z}$.

Consequently, (3.36) can be read as follows:

$$L(z,w\,|\,\tau) = g(z\,|\,\tau,r) + \vartheta(z,w\,|\,\tau,r)\,. \tag{3.40}$$

Theorem 3.6. *Given $(z,w) \in \Omega^{2,*}_{1\oplus\tau}$ and $r \in \mathbb{Q}\cap(0,1)$, one has $g(z\,|\,\tau,r) \in \mathfrak{G}_\zeta$, $\vartheta(z,w\,|\,\tau,r) \in \mathfrak{T}_\zeta$ and $L(z,w\,|\,\tau) \in \widetilde{\mathfrak{T}}_\zeta$ for $\zeta = e(r)$.*

The proof will be given in §3.5 in the below.

3.5. **Almost theta property at an arbitrary root of unity.** We begin with the following

Lemma 3.4. *Let $z \in \Omega_\tau$ and let $C(z\,|\,\tau, r)$ be in* (3.37).

(1) *If $z \neq 0$, then:*

$$C(z\,|\,\tau, r) = \frac{\theta(-\hat{z}_{\nu+1} + \frac{1}{2}\,|\,\tau_{\nu+1})}{m(\tau - r)\,\theta(-z + \frac{1}{2}\,|\,\tau)}\,. \tag{3.41}$$

(2) *If $z = 0$, then there exists an 8-th root of unity, ζ_8, such that*

$$C(0\,|\,\tau, r) = \frac{\zeta_8}{\sqrt{m}}\sqrt{\frac{i}{\tau - r}}\,e\big(\frac{1}{8}(\tau - \tau_{\nu+1})\big)\,. \tag{3.42}$$

Proof. (1) By Lemma 3.3 (1), none of $\hat{z}_n$ is null, so one can make use of the second expression of $C(z\,|\,\tau)$ in (3.6). Thus, by considering the definition of τ_k in (3.16) and the first relation in (3.24), (3.34) with $n = \nu$ becomes:

$$\begin{aligned} C(z\,|\,\tau, r) \;&=\; \frac{1}{\tau_0\tau_1 ... \tau_\nu}\,\prod_{k=0}^{\nu}\frac{\theta(-\frac{\hat{z}_k}{\tau_k} + \frac{1}{2}\,|\,-\frac{1}{\tau_k})}{\theta(-\hat{z}_k + \frac{1}{2}\,|\,\tau_k)} \\ &=\; \frac{1}{\tau_0\tau_1 ... \tau_\nu}\,\prod_{k=0}^{\nu}\frac{\theta(-\hat{z}_{k+1} + \frac{1}{2}\,|\,\tau_{k+1})}{\theta(-\hat{z}_k + \frac{1}{2}\,|\,\tau_k)}\,. \end{aligned}$$

Consequently, from (3.19) one deduces the expression of $C(z\,|\,\tau, r)$ in (3.41).

(2) When $z = 0$, it follows from Lemma 3.3 (1) that $\hat{z}_k = 0$ for k till $\nu + 1$. Putting $\hat{z}_k = 0$ and $n = \nu$ in (3.35) yields that

$$C(z\,|\,\tau, r) \;=\; (-1)^{\nu+1}\,\sqrt{\frac{i^{\nu+1}}{\tau_0\tau_1 ... \tau_\nu}}\,e\big(\frac{1}{8}\,A_\nu(\tau)\big)\,,$$

where

$$A_\nu(\tau) = \sum_{k=0}^{\nu}(\tau_k + \frac{1}{\tau_k}) = \sum_{k=0}^{\nu+1}(\tau_k - \tau_{k+1} + m_k) = \tau - \tau_{\nu+1} + \sum_{k=0}^{\nu} m_k\,;$$

see (3.14) and (3.16) for the integers m_k. Finally, setting

$$\zeta_8 = -e\big(\frac{1}{8}\,(5\nu + \sum_{k=0}^{\nu} m_k)\big) \tag{3.43}$$

gives (3.42), with the help of (3.19). This achieves the proof. □

Comparing (1.19) with (3.20) yields that $\tau' = \tau_{\nu+1}$, *i.e.*:

$$\tau_{\nu+1} = \tau' = -\frac{1}{m^2\hat{\tau}} + \frac{\alpha}{m}\,, \qquad \alpha \wedge m = 1. \tag{3.44}$$

In this way, Lemma 3.4 implies the following

Proposition 3.3. *Given $z \in \Omega_\tau$, one has $C(z\,|\,\tau, r) \in \mathfrak{T}_\zeta$.*

Proof. Let $f(q) = C(z\,|\,\tau, r) \in \mathfrak{T}_\zeta$. First, assume $z \neq 0$ and let

$$f_1 = \theta(-\hat{z}_{\nu+1} + \frac{1}{2}\,|\,\tau_{\nu+1}), \qquad f_2(q) = m(\tau - r)\,\theta(-z + \frac{1}{2}\,|\,\tau). \tag{3.45}$$

The relation in (3.41) gives that $f(q) = \frac{f_1(q)}{f_2(q)}$. By Theorem 1.1, one knows that both f_1, $f_2 \in \mathfrak{T}_\zeta$ and that $t_\zeta(f_2) \neq 0$, so Proposition 1.1 (2) implies that $f \in \mathfrak{T}_\zeta$.

Next, assume that $z = 0$. By (3.44) and the fact that $\tau = \hat{\tau} + r$, (3.42) becomes:

$$C(0 \,|\, \tau, r) = \frac{c'}{\sqrt{m}} \sqrt{\frac{i}{\hat{\tau}}}\, e\big(\frac{1}{8}(\hat{\tau} + \frac{1}{m^2 \hat{\tau}})\big)\,, \tag{3.46}$$

where $c' \in \mathbb{U}$. This implies $C(0 \,|\, \tau, r) \in \mathfrak{T}_\zeta$. □

Proof of Theorem 3.6. Let $f(q) = L(z, w \,|\, \tau)$. We shall make use of the decomposition of $f(q)$ given in (3.40) and show that $g(z \,|\, \tau, r) \in \mathfrak{G}_\zeta$ and $\vartheta(z, w \,|\, \tau, r) \in \mathfrak{T}_\zeta$.

By letting $n = \nu$ into (3.32), it follows that

$$g_\nu(z \,|\, \tau) = g_{\nu-1}(z \,|\, \tau) + C_{\nu-1}(z \,|\, \tau)\, \big(G(\hat{z}_\nu \,|\, \tau_\nu) - 1\big)\,. \tag{3.47}$$

By Remark 3.6, both $g_{\nu-1}(z \,|\, \tau)$ and $C_{\nu-1}(z \,|\, \tau)$ are analytic at $\tau = r$. As $\hat{\tau} \overset{a.v.}{\longrightarrow} 0$, one knows that $\tau_\nu \overset{a.v.}{\longrightarrow} 0$; see Lemma 3.2. Following Theorem 2.3, we shall distinguish two cases: $\hat{z}_\nu = z_{\nu,0} + z_{\nu,1}\tau_\nu$ with $z_{\nu,0} \in (-\frac{1}{2}, \frac{1}{2})$ or $z_{\nu,0} = \frac{1}{2}$.

First, assume $z_{\nu,0} \in (-\frac{1}{2}, \frac{1}{2})$; it follows that $g(z \,|\, \tau, r) = g_\nu(z \,|\, \tau)$ and $\varsigma = 0$ in (3.38). On the one hand, by Theorem 2.3 (1), $G(\hat{z}_\nu \,|\, \tau_\nu)$ has a Gevrey asymptotic expansion for $\tau_\nu \overset{a.v.}{\longrightarrow} 0$ or, equivalently, for $\hat{\tau} \overset{a.v.}{\longrightarrow} 0$. As $\tau \mapsto \tau_\nu = \frac{\alpha_\nu \tau + \beta_\nu}{\gamma_\nu \tau + \varsigma_\nu}$ is a Möbius transformation, this asymptotic expansion can be expressed in terms of the variable $\hat{\tau}$ near 0, thus $G(\hat{z}_\nu \,|\, \tau_\nu) \in \mathfrak{G}_\zeta$. By considering Proposition A.2, one finds that $g(z \,|\, \tau, r) \in \mathfrak{G}_\zeta$. On the other hand, by writing $\tau_{\nu+1}$ as in (3.44), one deduces from Proposition 3.1 that $L(\hat{z}_{\nu+1}, \hat{w}_{\nu+1} \,|\, \tau_{\nu+1}) \in \mathfrak{T}_\zeta$. Therefore, by combining Proposition 1.1 (1) with Proposition 3.3, letting $\varsigma = 0$ in (3.39) implies that $\vartheta(z, w \,|\, \tau, r) \in \mathfrak{T}_\zeta$.

Next, assume $z_{\nu,0} = \frac{1}{2}$; it follows that $\varsigma = 1$ in both (3.38) and (3.39). Similarly to the relation in (3.12) in the second part of the proof of Theorem 3.5, one finds that

$$G(\hat{z}_\nu \,|\, \tau_\nu) = C(\frac{1}{2} + z_{\nu,1}\tau_\nu \,|\, \tau_\nu) + g(z_{\nu,1} \,|\, \tau_\nu)\,.$$

Thus, comparing (3.36) with (3.40) gives

$$g(z \,|\, \tau, r) = g_{\nu-1}(z \,|\, \tau) + C_{\nu-1}(z \,|\, \tau)\, \big(g(z_{\nu,1} \,|\, \tau_\nu) - 1\big)\,. \tag{3.48}$$

Applying Theorem 2.3 (2) implies that $g(z_{\nu,1} \,|\, \tau_\nu) \in \mathfrak{G}_\zeta$. In this way, one can see that $g(z \,|\, \tau, r) \in \mathfrak{G}_\zeta$. Moreover, by Remark 3.1, $L(\hat{z}_{\nu+1}, \hat{w}_{\nu+1} \,|\, \tau_{\nu+1}) + 1 \in \mathfrak{T}_\zeta$, this implies that $\vartheta(z, w \,|\, \tau, r) \in \mathfrak{T}_\zeta$ in (3.39). Thus, one obtains Theorem 3.6. □

Remark 3.7. *Let $r = \frac{p}{m}$, $\zeta = e(r)$, $\hat{\tau} = \tau - r$, $q_1 = e(-\frac{1}{\hat{\tau}})$, and let $f(q) = C(z \,|\, \tau, r)$ be as in* (3.37)*. If*

$$\kappa = \frac{1}{2}\,(mz_0 + pz_1 - \delta - \frac{1}{2})^2 \tag{3.49}$$

with $\delta \in \mathbb{Z} \cap (mz_0 + pz_1 - \frac{1}{2}, mz_0 + pz_1 + \frac{1}{2}]$, then there exists $c \in \mathbb{C}^*$ such that

$$t_\zeta(f)(q) = \frac{c}{\sqrt{m}} \sqrt{\frac{i}{\hat{\tau}}}\, e\big(\frac{1}{8}((2z_1)^2 + 1)\,\hat{\tau}\big)\, q_1^{-\kappa/m^2}\,. \tag{3.50}$$

Moreover, $|c| = \frac{\rho(\hat{z}_{\nu+1} \,|\, \tau_{\nu+1})}{\rho(z \,|\, \tau)}$ or $|c| = 1$ if $z \neq 0$ or $z = 0$, respectively.

For the definition of $\rho(z \,|\, \tau)$, see (1.21). It is easy to observe that (3.46) gives, in fact, (3.50) for $z = 0$. Thus, assume $z \neq 0$ and let f_1 and f_2 be as in (3.45).

By (3.44), one knows that $q_1 = e(m^2\tau_{\nu+1})$. In view of the relation $t_\zeta(f) = \frac{t_\zeta(f_1)}{t_\zeta(f_2)}$, applying (1.22) to f_1 and (1.23) to f_2 yields (3.50) with

$$\kappa = \frac{1}{2}\,(z_{\nu+1,1} + \frac{1}{2})^2 - \kappa(\hat{z}_{\nu+1}\,|\,\tau_{\nu+1},\infty) + \kappa(z\,|\,\tau,r)\,.$$

By Lemma 3.3 (3), $\kappa(\hat{z}_{\nu+1}\,|\,\tau_{\nu+1},\infty) = \kappa(z\,|\,\tau,r)$. So, the expression of $\hat{z}_{\nu+1}$ in (3.26) gives immediately the relation stated for κ in (3.49). □

Remark 3.8. *Let r, ζ, $\hat{\tau}$, q_1 be as in Remark 3.7, and let $f(q) = \vartheta(z,w\,|\,\tau,r)$ be as in* (3.39)*. One can find $c \in \mathbb{C}^*$ and $\kappa > 0$ such that*

$$t_\zeta(f)(q) = c\,\sqrt{\frac{i}{\hat{\tau}}}\,e\big(\frac{1}{8}((2z_1)^2+1)\hat{\tau}\big)\,q_1^{-\kappa} \tag{3.51}$$

provided that either (i) $w_{\nu+1,1} > 0$ or (ii) $w_{\nu+1,1} \le 0$ but $z_{\nu+1,1} > w_{\nu+1,1}$.

Indeed, let $f_1(q) = C(z\,|\,\tau,r)$ and $f_2(q) = L(\hat{z}_{\nu+1},\hat{w}_{\nu+1}\,|\,\tau) + \varsigma$. The relation in (3.39) says that $f = f_1 f_2$. By Remark 3.2, $t_0(f_2)(q) = 1+\varsigma \neq 0$ in each of both cases (i) and (ii) in the above. Using (3.50) gives $t_0(f_1)$, which together with the relation $t_\zeta(f) = t_\zeta(f_1)\,t_\zeta(f_2)$ yield the expression stated in (3.51). □

3.6. **One necessary and sufficient condition for the boundedness of limits.** By Theorem 3.6, one knows that $g(z\,|\,\tau,r) \in \mathfrak{G}_\zeta$, where $\zeta = e(r)$. So, for all $z \in \Omega_\tau$ and $r \in \mathbb{Q}\cap(0,1)$, we define

$$\gamma(z,r) = \lim_{\tau \overset{a.v.}{\longrightarrow} r} g(z\,|\,\tau,r)\,. \tag{3.52}$$

Let $r = \frac{p}{m}$, $m \in \mathbb{Z}_{>0}$ and $p \wedge m = 1$. By Theorem 2.3 and Remark 3.6, it follows from (3.47) and (3.48) that either

$$\gamma(z,\frac{p}{m}) = g_{\nu-1}(z_0+\frac{p}{m}z_1\,|\,\frac{p}{m}) - \frac{e(z_{\nu,0})}{1+e(z_{\nu,0})}\,C_{\nu-1}(z_0+\frac{p}{m}z_1\,|\,\frac{p}{m})\,, \tag{3.53}$$

where $z_{\nu,0} \neq \frac{1}{2}$, or

$$\gamma(z,\frac{p}{m}) = g_{\nu-1}(z_0+\frac{p}{m}z_1\,|\,\frac{p}{m}) + z_{\nu,1}\,C_{\nu-1}(z_0+\frac{p}{m}z_1\,|\,\frac{p}{m}) \tag{3.54}$$

when $z_{\nu,0} = \frac{1}{2}$. In the above, $|C_{\nu-1}(z_0+\frac{p}{m}z_1\,|\,\frac{p}{m})| = \sqrt{m}$, by (3.33). This implies that, in general, $\gamma(z,\frac{p}{m})$ *is unbounded as* $m \to \infty$. See Theorems 3.7 and 3.11 in the below.

In order to simplify the exposition, we shall consider the case of $r = \frac{1}{m}$, $m \in \mathbb{Z}_{\ge 2}$, for which case both τ_n and $\hat{z}_n$ have been given in Remark 3.4; see (3.27).

Theorem 3.7. *Let $\mathbf{M} \subset \mathbb{Z}_{\ge 2}$ be such that $\#\mathbf{M} = \infty$. The following conditions are equivalent.*

(1) $\sup_{m\in\mathbf{M}} |\gamma(z,\frac{1}{m})| < \infty$.
(2) $z \in \{\frac{1}{2}, \frac{1}{2} - \frac{1}{2}\tau, -\frac{1}{2}\tau\}$.

The proof will be given at the end of this section. For doing that, it will be convenient to introduce the following notation:

$$\mathbf{E}(z) = \mathbf{E}(z\,|\,\tau) = \{m \in \mathbb{Z}_{\ge 2} : mz_0 + z_1 = \frac{1}{2} \mod \mathbb{Z}\} \tag{3.55}$$

and

$$\mathbf{N}(z) = \mathbf{N}(z\,|\,\tau) = \mathbb{Z}_{\ge 2} \setminus \mathbf{E}(z\,|\,\tau)\,. \tag{3.56}$$

Remark 3.9. (1) *$\#\mathbf{E}(z) = \infty$ if, and only if, $\#\mathbf{E}(z) \geq 2$. If this is the case, then $z \in \mathbb{Q} \oplus \tau\mathbb{Q}$.*
(2) *When $z_0 = 0$, either $\mathbf{E}(z) = \emptyset$ or $\mathbf{N}(z) = \emptyset$. This last equality holds if, and only if, $z = -\frac{1}{2}\tau$.*
(3) *When $z \neq -\frac{1}{2}\tau$, $\#\mathbf{N}(z) = \infty$.*
(4) *When $z_0 = \frac{1}{2}$, $\#\mathbf{E}(z) = \infty$ if, and only if, $z_1 \in \{0, -\frac{1}{2}\}$.*

Indeed, if both m and $m+d$ belong to $\mathbf{E}(z)$, where $d > 0$, then $dz_0 \in \mathbb{Z}$, thus one gets that $z_0 \in \mathbb{Q}$ and $z_1 \in \mathbb{Q}$. Furthermore, one sees that $m + nd \in \mathbf{E}(z)$ for all $n \in \mathbb{Z}_{\geq 1}$, what implies the assertion stated in (1). The other assertions can be directly seen from (3.55) and (3.56). □

In what follows, we shall assume that $z \in \Omega_\tau \setminus \{-\frac{1}{2}\tau\}$. By Remark 3.9 (3), $\mathbf{N}(z)$ contains an infinitely many integers.

Proposition 3.4. *Let $z = z_0 + z_1\tau \in \Omega_\tau \setminus \{-\frac{1}{2}\tau\}$. We assume that $m \to +\infty$ in $\mathbf{N}(z)$.*
(1) *If $z_0 \neq \frac{1}{2}$, then:*

$$\gamma(z, \frac{1}{m}) = \frac{U(z,m)}{1+e(z_1+mz_0)}\sqrt{m} - \frac{e(z_0)}{1+e(z_0)} + O(\frac{1}{m}), \tag{3.57}$$

where $|U(z,m)| = 1$.
(2) *If $z_0 = \frac{1}{2}$, then:*

$$\gamma(z, \frac{1}{m}) = A(z,m)\, e\big(\frac{1}{2m}(z_1+\frac{1}{2})^2 + \frac{7}{8}\big)\sqrt{m} + z_1 + O(\frac{1}{m}), \tag{3.58}$$

where

$$A(z,m) = \frac{1}{1+(-1)^m e(z_1)} - \frac{1}{2}.$$

Proof. By the definition of $\mathbf{N}(z)$ in (3.56), it follows that $z_{1,0} \neq \frac{1}{2}$. Thus, letting $p = 1$ and $\nu = 1$ in (3.53) gives that

$$\gamma(z, \frac{1}{m}) = G(z_0 + \frac{z_1}{m} \,|\, \frac{1}{m}) - 1 - \frac{e(mz_0+z_1)}{1+e(mz_0+z_1)} C(z_0 + \frac{z_1}{m} \,|\, \frac{1}{m}). \tag{3.59}$$

(1) First, assume that $z_0 \neq \frac{1}{2}$. By Theorem 2.3 (1), it follows that, as $m \to +\infty$,

$$G(z_0 + \frac{z_1}{m} \,|\, \frac{1}{m}) = \frac{1}{1+e(z_0)} + O(\frac{1}{m}). \tag{3.60}$$

This together with (3.59) and (3.33) (with $\nu = 1$) imply (3.57).
(2) Now, assume that $z_0 = \frac{1}{2}$. Letting $\tau = \frac{1}{m}$ in (2.24) gives that

$$\begin{aligned} G(\frac{1}{2} + \frac{z_1}{m} \,|\, \frac{1}{m}) &= -\frac{i}{2}\sqrt{\frac{i}{1/m}}\, e\big(\frac{1}{2m}(z_1+\frac{1}{2})^2\big) + g(z_1 \,|\, \frac{1}{m}) \\ &= -\frac{1}{2}\, C(z_0 + \frac{z_1}{m} \,|\, \frac{1}{m}) + g(z_1 \,|\, \frac{1}{m}). \end{aligned} \tag{3.61}$$

By Theorem 2.3 (2), one knows that $g(z_1 \,|\, \frac{1}{m}) = 1 + z_1 + O(\frac{1}{m})$ for $m \to +\infty$. Thus, putting together (3.59) and (3.61) yields (3.58). □

Proposition 3.5. *Let $z = z_0 + z_1\tau \in \Omega_\tau$. One supposes that $\#\mathbf{E}(z) = \infty$ and one assumes that $m \to +\infty$ in $\mathbf{E}(z)$.*

(1) *If* $z_0 \neq \frac{1}{2}$, *then:*

$$\gamma(z, \frac{1}{m}) = V(z,m)\,\sqrt{m} - \frac{e(z_0)}{1+e(z_0)} + O(\frac{1}{m})\,, \tag{3.62}$$

where $|V(z,m)| = |z_0|$.

(2) *If* $z_0 = \frac{1}{2}$, *then* $\gamma(z, \frac{1}{m}) = z_1 + O(\frac{1}{m})$.

Proof. By (3.55), it follows that $z_{1,0} = \frac{1}{2}$. Thus, due to (3.54), (3.59) needs to be modified as follows:

$$\gamma(z\,|\,\frac{1}{m}) = G(z_0 + \frac{z_1}{m}\,|\,\frac{1}{m}) - 1 - z_0\,C(z_0 + \frac{z_1}{m}\,|\,\frac{1}{m})\,, \tag{3.63}$$

where $-z_0 = z_{1,1}$, by Remark 3.4. This allows one to complete the proof by using (3.60), (3.61) and (3.33) (with $\nu = 1$). We omit the details. □

Proof of Theorem 3.7. Let $\mathbf{M}_0 = \mathbf{M} \cap \mathbf{N}(z)$ and $\mathbf{M}_1 = \mathbf{M} \cap \mathbf{E}(z)$. By hypothesis, $\#\mathbf{M} = \infty$, so one can find at least one of $\mathbf{M}_0$ and $\mathbf{M}_1$ that contains an infinitely many integers.

Assume first that $z_0 \neq \frac{1}{2}$. By considering both (3.57) and (3.62), one obtains that $\sup_{m\in\mathbf{M}} |\gamma(z, \frac{1}{m})| < \infty$ implies that $\#\mathbf{M}_0 < \infty$, $\#\mathbf{M}_1 = \infty$ and that $z_0 = 0$. Furthermore, in view of Remark 3.9 (2), this is possible only if $z = -\frac{1}{2}\tau$.

Now, consider the case of $z_0 = \frac{1}{2}$. On the one hand, if $\#\mathbf{M}_0 = \infty$, (3.58) implies that $\sup_{m\in\mathbf{M}_0} |\gamma(z, \frac{1}{m})| < \infty$ if, and only if, $A(z,m) = 0$ for $m \in \mathbf{M}_0$. Thus, one finds that $(-1)^m = e(z_1)$, so $z_1 \in \{0, -\frac{1}{2}\}$. On the other hand, in view of Remark 3.9 (4), the case (2) of Proposition 3.5 in the above occurs only for $z = \frac{1}{2}$ and $z = \frac{1}{2} - \frac{1}{2}\tau$. This achieves the proof. □

3.7. **False theta-type functions and half-periods.** One knows that $\mathbb{Z}_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ is the period set for the elliptic functions associated with double periods 1 and τ. Assume $w \in \Omega_\tau$ such that $2w \in \mathbb{Z}_\tau$ and $w \neq 0$. This means exactly that $w \in \Delta_\tau$ if one lets

$$\Delta_\tau = \{\frac{1}{2}, -\frac{\tau}{2}, \frac{1}{2} - \frac{\tau}{2}\}\,. \tag{3.64}$$

Since $L(z, -w\,|\,\tau) = L(z, w\,|\,\tau)$, putting $z = \frac{1}{2}$ in the "reflection" relation (1.41) yields that $L(\frac{1}{2} - \tau, w\,|\,\tau) = 0$. Thus, $L(\frac{1}{2}, w\,|\,\tau) = 1$, by (2.26). At the same time, it follows from (1.41) and (2.26) that

$$L(-z-\tau, w\,|\,\tau) = e(z)\,L(z-\tau, w\,|\,\tau) = 1 - L(z, w\,|\,\tau)$$

One obtains that $L(z, w\,|\,\tau) = \frac{1}{2}$ when $2z = -\tau \mod \mathbb{Z}$. This gives the following

Remark 3.10. *The following statements hold.*

(1) *If* $z = \frac{1}{2}$ *and* $w \in \{-\frac{\tau}{2}, \frac{1}{2} - \frac{\tau}{2}\}$, *then* $L(\frac{1}{2}, w\,|\,\tau) = 1$.

(2) *If* $z \in \{-\frac{\tau}{2}, \frac{1}{2} - \frac{\tau}{2}\}$ *and* $w \in \Delta_\tau \setminus \{z\}$, *then* $L(z, w\,|\,\tau) = \frac{1}{2}$. □

The set Δ_τ defined in (3.64) contains the non-zero half-periods of the fundamental domain Ω_τ. One can also notice that Remark 3.10 is similar to Remark 2.2 for $G(z\,|\,\tau)$. The following result is certainly known in the classical theory of elliptic functions or q-series, although one can rediscover it from [10, Theorem 3.13 & Remark 3.14].

Theorem 3.8. *Let $w,\, w' \in \mathbb{C} \setminus \mathbb{Z} \oplus \tau\mathbb{Z}$. Then the following identity holds for all $z \in \mathbb{C}$ such that neither $z - w$ nor $z - w'$ belong to $\mathbb{Z} \oplus \tau\mathbb{Z}$:*

$$L(z, w \,|\, \tau) - L(z, w' \,|\, \tau) = -\frac{(\tau \,|\, \tau)_\infty^3 \, \theta(w - w' + \frac{1}{2}, w + w' - z + \frac{1}{2} \,|\, \tau)}{\theta(w + \frac{1}{2}, w - z + \frac{1}{2}, -w' + \frac{1}{2}, w' - z + \frac{1}{2} \,|\, \tau)} \,. \tag{3.65}$$

□

In the right-hand side of (3.65), the factor $(\tau \,|\, \tau)_\infty$ is defined as follows: for $x = e(z)$ and $z \in \mathbb{C}$,

$$(z \,|\, \tau)_\infty = (x; q)_\infty = \prod_{n \geq 0} (1 - xq^n) \,. \tag{3.66}$$

By putting together this last theorem with Remark 3.10, one arrives at the following

Theorem 3.9. *The following assertions hold.*

(1) *If $w \in \mathbb{C} \setminus (\frac{1}{2}\mathbb{Z}) \oplus \tau\mathbb{Z}$, then:*

$$L(\frac{1}{2}, w \,|\, \tau) = 1 - \frac{(\tau \,|\, \tau)_\infty^3 \, \theta(w + \frac{\tau}{2} + \frac{1}{2}, w - \frac{\tau}{2} \,|\, \tau)}{\theta(w, w + \frac{1}{2}, \frac{\tau}{2} + \frac{1}{2}, -\frac{\tau}{2} \,|\, \tau)} \,.$$

(2) *If $w \in \mathbb{C} \setminus \mathbb{Z} \oplus \frac{\tau}{2}\mathbb{Z}$, then:*

$$L(-\frac{\tau}{2}, w \,|\, \tau) = \frac{1}{2} - \frac{(\tau \,|\, \tau)_\infty^3 \, \theta(w, w + \frac{\tau}{2} \,|\, \tau)}{\theta(w + \frac{1}{2}, w + \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{2}, 0 \,|\, \tau)} \,.$$

(3) *If $w \in \mathbb{C} \setminus \mathbb{Z} \oplus \tau\mathbb{Z}$ such that $w - \frac{1}{2} - \frac{1}{\tau} \notin \mathbb{Z} \oplus \tau\mathbb{Z}$, then:*

$$L(\frac{1}{2} - \frac{\tau}{2}, w \,|\, \tau) = \frac{1}{2} - \frac{(\tau \,|\, \tau)_\infty^3 \, \theta(w, w + \frac{\tau}{2} + \frac{1}{2} \,|\, \tau)}{\theta(w + \frac{1}{2}, w + \frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}, 0 \,|\, \tau)} \,.$$

Proof. This follows from Theorem 3.8 combined with Remark 3.10, by letting respectively $(z, w') = (\frac{1}{2}, -\frac{\tau}{2})$, $(-\frac{\tau}{2}, \frac{1}{2})$ and $(\frac{1}{2} - \frac{\tau}{2}, \frac{1}{2})$ in (3.65). □

The triple product identity [4, p. 497, Theorem 10.4.1] says that

$$\theta(z + \frac{1}{2} \,|\, \tau) = (\tau \,|\, \tau)_\infty \, (z \,|\, \tau)_\infty \, (\tau - z \,|\, \tau)_\infty \,. \tag{3.67}$$

Replacing (z, τ) with $(\tau, 3\tau)$ in this formula yields that

$$\theta(\tau + \frac{1}{2} \,|\, 3\tau) = (3\tau \,|\, 3\tau)_\infty \, (\tau \,|\, 3\tau)_\infty \, (2\tau \,|\, 3\tau)_\infty = (\tau \,|\, \tau)_\infty \,.$$

Thus, applying both Theorem 1.1 and Proposition 1.1 (3) gives the following

Remark 3.11. $(\tau \,|\, \tau)_\infty \in \mathfrak{T}$. □

Another way to get this is to use the classic Dedekind η-modular formula; see [20, p. 145, (67.4)]. By combing this last Remark 3.11 with Theorem 3.9, Proposition 1.1 (2) and Corollary 1.1, one obtains the following

Corollary 3.1. *Let $(z, w) \in \Omega_{1 \oplus \tau}^{2,*}$, and let $f(q) = L(z, w \,|\, \tau)$.*

(1) *If both z and w belong to Δ_τ, then f is constant.*
(2) *If $z \in \Delta_\tau$ and $w \notin \Delta_\tau$, then f is a false theta-type function, i.e.: $f \in \mathfrak{F}$.* □

3.8. **Mock theta-type properties and the proof of Theorem 1.3.** Let $(z,w)\in \Omega^{2,*}_{1\oplus\tau}$, and consider $f(q)=L(z,w\,|\,\tau)$. We start with the following

Theorem 3.10. *Let $\vartheta\in\mathfrak{T}$, and define $g(q)=f(q)-\vartheta(q)$. One assumes that $g\in\mathfrak{G}_\zeta$ for all $\zeta\in\mathbb{U}$. If $z\notin\Delta_\tau$, then one can find $U\subset\mathbb{U}$ such that*

$$\sup\Big\{\lim_{q\overset{a.r.}{\longrightarrow}\zeta}\big|g(q)\big|\,:\,\zeta\in U\Big\}=\infty\,. \tag{3.68}$$

In order to find such set U of roots of unity satisfying (3.68), we shall make use of the following

Proposition 3.6. *Let $r\in(0,1)\cap\mathbb{Q}$, $\zeta=e(r)$, and let $\gamma(z,r)$ be as in (3.52). Let (g,ϑ) be as in Theorem 3.10. If $w_{\nu+1,1}>0$ or if $w_{\nu+1,1}\leq 0$ but $z_{\nu+1,1}>w_{\nu+1,1}$, then:*

$$\lim_{q\overset{a.r.}{\longrightarrow}\zeta} g(q)=\gamma(z,r)\,. \tag{3.69}$$

Proof. Let $\tilde{g}(q)=g(z\,|\,\tau,r)$ and $\tilde{\vartheta}(q)=\vartheta(z,w\,|\,\tau,r)$. The identity in (3.40) says that $f=\tilde{g}+\tilde{\vartheta}$. By Theorem 3.6, one knows that $\tilde{g}\in\mathfrak{G}_\zeta$ and $\tilde{\vartheta}\in\mathfrak{T}_\zeta$. Moreover, Remark 3.8 implies that $P_\zeta(\tilde{\vartheta})\neq 0$ when $w_{\nu+1,1}>0$ or when $w_{\nu+1,1}\leq 0$ but $z_{\nu+1,1}>w_{\nu+1,1}$. Thus, applying Proposition 1.4 gives that g is, in fact, exponentially close to $\tilde{g}$ as q tends to ζ radially. This implies (3.69). □

In what follows, we shall consider two species of roots of unity:

(i) $\zeta=e(\frac{1}{m})$ with $m\in\mathbb{Z}_{\geq 2}$.
(ii) $\zeta=e(\frac{p}{m})$ with $m=kp-1$, $(k,p)\in\mathbb{Z}_{\geq 2}$.

Notice that the behavior of $\gamma(z,r)$ in the first case (i) has been mentioned by Theorem 3.7. Before giving the proof of Theorem 3.10, we discuss the second case (ii) to get a statement similar to Theorem 3.7.

Given $z=z_0+z_1\tau\in\Omega_\tau\setminus\Delta_\tau$, choose $k\in\mathbb{Z}_{\geq 2}$ such that $kz_0+z_1\neq\frac{1}{2}\bmod\mathbb{Z}$, and consider $r=\frac{p}{m}$ with $m=kp-1$, $p\in\mathbb{Z}_{\geq 2}$. Similarly to the notation $\mathbf{E}(z)$ and $\mathbf{N}(z)$ introduced in (3.55) and (3.56), we define:

$$\mathbf{E}'(z)=\mathbf{E}'(z\,|\,\tau,k)=\{p\in\mathbb{Z}_{\geq 2}:(kp-1)z_0+pz_1=\frac{1}{2}\ \bmod\ \mathbb{Z}\} \tag{3.70}$$

and

$$\mathbf{N}'(z)=\mathbf{N}'(z\,|\,\tau,k)=\mathbb{Z}_{\geq 2}\setminus\mathbf{E}'(z\,|\,\tau,k)\,. \tag{3.71}$$

By Remark 3.5, one obtains that $z_{1,0}=kz_0+z_1+\delta\in(-\frac{1}{2},\frac{1}{2})$, where $\delta\in\mathbb{Z}$.

Proposition 3.7. *Let z, r, p, m and k be as in the above.*

(1) *If $p\in\mathbf{N}'(z)$, then there exist U_1 and $U_2\in\mathbb{C}$ such that $|U_1|=|U_2|=1$ and*

$$\gamma(z,r)=G(z_0+z_1r\,|\,r)-1+\frac{U_1}{\sqrt{r}}\big(G(z_{1,0}-\frac{z_0}{p}\,|\,\frac{1}{p})-1\big)+\frac{U_2\sqrt{m}}{1+e(mz_0+pz_1)}\,. \tag{3.72}$$

(2) *If $p\in\mathbf{E}'(z)$, then there exist U_1 and $U_2\in\mathbb{C}$ such that $|U_1|=|U_2|=1$ and*

$$\gamma(z,r)=G(z_0+z_1r\,|\,r)-1+\frac{U_1}{\sqrt{r}}\big(G(z_{1,0}-\frac{z_0}{p}\,|\,\frac{1}{p})-1\big)+z_{1,0}\,U_2\sqrt{m}\,. \tag{3.73}$$

Proof. This can be proved by using the same method as for Propositions 3.4 and 3.5. Indeed, by Remark 3.5, one sees that $\nu = 2$ in both (3.53) and (3.54). Thus, one obtains finally both (3.72) and (3.73) by using the definition of g_1 in (3.32) and the module of C_1 stated in (3.33). □

Theorem 3.11. *For any $\mathbf{M}' \subset \mathbb{Z}_{\geq 2}$ such that $\#\mathbf{M}' = \infty$, one has*

$$\sup_{p \in \mathbf{M}'} \left|\gamma(z, \frac{p}{kp-1})\right| = \infty\,. \tag{3.74}$$

Proof. This follows from Proposition 3.7. Indeed, when $p \to \infty$, $r \to \frac{1}{k}$. Theorem 2.3 (1) gives that $G(z_{1,0} - \frac{z_0}{p} \mid \frac{1}{p}) \to \frac{1}{1+e(z_{1,0})}$. Thus, from (3.72) it follows that

$$\gamma(z,r) = G(z_0 + \frac{z_1}{k} \mid \frac{1}{k}) - 1 - \frac{U_1'\sqrt{k}}{1+e(kz_0+z_1)} + \frac{U_2\sqrt{m}}{1+e(mz_0+pz_1)} + O(\frac{1}{p})\,,$$

where $U_1' = U_1\, e(kz_0 + z_1)$. A similar formula can be found by using (3.73). By hypothesis, $z \notin \Delta_\tau$, this implies that if $p \in \mathbf{E}'(z)$, then $z_{1,0} \neq 0$. So, one gets (3.74), for $\sqrt{m} \to \infty$ as $p \to \infty$. □

Proof of Theorem 3.10. For any $a \in \mathbb{R}$, we will denote by $\overline{(a)}$ the number in $(-\frac{1}{2}, \frac{1}{2}]$ that equals to a modulo $\mathbb{Z}$. It is useful to notice that, as $z \in \Omega_\tau$, $\overline{(z_1)} = z_1$ unless $z_1 = -\frac{1}{2}$.

First, consider the case of $r = \frac{1}{m}$ with $m \in \mathbb{Z}_{\geq 2}$, and let $\mathbf{M} = \mathbf{M}_- \cup \mathbf{M}_+$ with

$$\mathbf{M}_- = \Big\{ m \in \mathbb{Z}_{\geq 2} : \overline{(mw_0 + w_1)} \in (-\frac{1}{2}, 0) \Big\}$$

and

$$\mathbf{M}_+ = \Big\{ m \in \mathbb{Z}_{\geq 2} : \overline{(mz_0 + z_1)} < \overline{(mw_0 + w_1)} \in [0, \frac{1}{2}] \Big\}\,.$$

One finds that $\#\mathbf{M}_- = \infty$ except when $w_0 = 0$ and $\overline{(w_1)} \in [0, \frac{1}{2}]$ or when $w_0 = \frac{1}{2}$ and $w_1 \in \{-\frac{1}{2}, 0\}$. For this last case with $w_0 = \frac{1}{2}$, one sees that $\#\mathbf{M}_+ = \infty$, due to fact that $z \neq -\frac{1}{2}\tau$. It follows that $\#\mathbf{M} = \infty$ except when the following conditions hold:

$$z_0 = w_0 = 0\,, \qquad z_1 > \overline{(w_1)} \in [0, \frac{1}{2}]\,. \tag{3.75}$$

Furthermore, Proposition 3.6 and Remark 3.4 imply that

$$\lim_{q \overset{a.r.}{\longrightarrow} e(\frac{1}{m})} g(q) = \gamma(z, \frac{1}{m})$$

for all $m \in \mathbf{M}$. Thus, Theorem 3.7 gives (3.68) for $U = \{e(\frac{1}{m}) : m \in \mathbf{M}\}$ except when (3.75) is satisfied.

Assume then $(z,w) \in \Omega^{2,*}_{1\oplus\tau}$ be such as in (3.75). Since $z \notin \Delta_\tau$ and $w \neq 0$, it follows that $z_1 \neq -\frac{1}{2}$ and $w_1 \neq 0$. Let $k \in \mathbb{Z}_{\geq 2}$, and define $\mathbf{M}' = \mathbf{M}'_- \cup \mathbf{M}'_+$ with

$$\mathbf{M}'_- = \Big\{ p \in \mathbb{Z}_{\geq 2} : \overline{(pw_1)} \in (-\frac{1}{2}, 0) \Big\}$$

and

$$\mathbf{M}'_+ = \Big\{ p \in \mathbb{Z}_{\geq 2} : \overline{(pz_1)} < \overline{(pw_1)} \in [0, \frac{1}{2}] \Big\}\,.$$

Applying Proposition 3.6 and Remark 3.5 yields that, letting $m = kp - 1$,

$$\lim_{q \overset{a.r.}{\longrightarrow} e(\frac{p}{m})} g(q) = \gamma(z, \frac{p}{m})$$

for all $p \in \mathbf{M}'$. One observes that $\#\mathbf{M}'_- = \infty$ except when $w_1 = -\frac{1}{2}$, and this last case remains impossible because of (3.75). Thus, $\#\mathbf{M}' = \infty$. Consequently, if one lets

$$U = \Big\{ e(\frac{p}{kp-1}) : p \in \mathbf{M}' \Big\},$$

one gets (3.68) with the help of Theorem 3.11. This achieves the proof of Theorem 3.10. □

Proof of Theorem 1.3. By combing Theorem 3.6 with Corollary 3.1, it remains only to prove that $f \notin \mathfrak{F}$ when $z \notin \Delta_\tau$. Thus, in view of (1.7), one deduces Theorem 1.3 from Theorem 3.10. This achieves the proof of Theorem 1.3. □

3.9. **Proof of Theorem 1.4.** Let $z = z_0 + z_1\tau$, $z_0 \in (-\frac{1}{2}, \frac{1}{2}]$, $z_1 \in [-\frac{1}{2}, \frac{1}{2})$, and let $f(q) = L(z\,|\,\tau)$ be given by (1.45). Firstly, by (1.44), one finds that, for all $u \in \mathbb{C}$:

$$R_1(u\,|\,\tau) = \sum_{n\in\mathbb{Z}^*} \frac{e(\frac{1}{2}n^2\tau + nu)}{1 - e(n\tau)} + \frac{1}{2}\,. \tag{3.76}$$

By taking the logarithmic derivative with respect to z both sides of the Jacobi's triple product (3.67), one gets that, for $u \neq \frac{1}{2} \bmod \mathbb{Z} \oplus \tau\mathbb{Z}$:

$$\frac{\theta'(u\,|\,\tau)}{\theta(u\,|\,\tau)} = 2\pi i \sum_{n\geq 0} \Big(\frac{e(u+n\tau)}{1+e(u+n\tau)} - \frac{e\big(-u+(n+1)\tau\big)}{1+e\big(-u+(n+1)\tau\big)} \Big)\,. \tag{3.77}$$

Therefore, it follows that

$$R_1(-z - \frac{\tau}{2} + \frac{1}{2}) - \frac{i}{2\pi}\,\frac{\theta'(-z+\frac{1}{2}\,|\,\tau)}{\theta(-z+\frac{1}{2}\,|\,\tau)} \in \mathfrak{T}_0\,. \tag{3.78}$$

Beside, one knows from Lemma 1.1 that $\theta(-z+\frac{1}{2}\,|\,\tau) \in \mathfrak{T}_0$, Thus, by considering Proposition 1.1 (2), the expression of $f(q)$ in (1.49) together with (3.78) imply that $f \in \mathfrak{T}_0$.

Secondly, one notices that the modular-like relation in (3.3) is reduced into the following form:

$$L(z\,|\,\tau) = G(z\,|\,\tau) + C(z\,|\,\tau)\,\big(L(\frac{z}{\tau}\,|\,-\frac{1}{\tau}) - 1\big)\,, \tag{3.79}$$

where $G(z\,|\,\tau)$ and $C(z\,|\,\tau)$ are the same as in Theorem 3.2. Indeed, if one writes

$$f(z,w\,|\,\tau) = L(z,w\,|\,\tau) - \frac{i}{2\pi w\,\theta(-z+\frac{1}{2}\,|\,\tau)}\,,$$

then the formula in (3.3) together with the second expression of $C(z\,|\,\tau)$ in (3.6) give that

$$f(z,w\,|\,\tau) = G(z\,|\,\tau) + C(z\,|\,\tau)\,\big(f(\frac{z}{\tau},\frac{w}{\tau}\,|\,-\frac{1}{\tau}) - 1\big)\,.$$

Thus, taking the limit $w \to 0$ in this last relation yields immediately (3.79). Moreover, by using both (3.79) and Theorem 2.3, one finds that $f \in \tilde{\mathfrak{T}}_1$.

Thirdly, given $r \in (0,1) \cap \mathbb{Q}$, one makes use of the sequence $(r_j, \tau_j, \hat{z}_j)_{0\leq j\leq \nu+1}$ defined in §3.3, where $r_0 = r$, $r_\nu = 0$ and $r_{\nu+1} = \infty$. In addition to the functions $C(z\,|\,\tau, r)$ and $g(z\,|\,\tau, r)$ introduced in (3.37) and (3.38), one sets

$$\vartheta(z\,|\,\tau, r) = C(z\,|\,\tau, r)\,\big(L(\hat{z}_{\nu+1}\,|\,\tau_{\nu+1}) + \varsigma\big)\,, \tag{3.80}$$

where ς takes either 0 or 1 in the same manner as in (3.39). Therefore, iterating several times the above relation in (3.79) yields the following identity, which is similar to (3.40):

$$L(z\,|\,\tau) = g(z\,|\,\tau, r) + \vartheta(z\,|\,\tau, r)\,. \tag{3.81}$$

By the same manner as what done for the proof of Theorem 3.6, this last relation in (3.81) shows that $f \in \tilde{\mathfrak{T}}_\zeta$, where $\zeta = e(r)$.

Fourthly, let $\Delta_\tau = \{\frac{1}{2}, \frac{1}{2}(1-\tau), -\frac{1}{2}\tau\}$ as in (3.64). By using both Theorems 3.7 and 3.10, one finds that if $f \in \mathfrak{M}$, then $z \notin \Delta_\tau$.

Finally, it remains only to consider the value of $L(z\,|\,\tau)$ when z belongs to the set Δ_τ. Since

$$\theta(-z+\tau+\frac{1}{2}\,|\,\tau) = \theta(z+\frac{1}{2}\,|\,\tau) = -e(z)\,\theta(z+\tau+\frac{1}{2}\,|\,\tau),$$

it follows from (1.41) that

$$\begin{aligned} &L(z-\tau, w\,|\,\tau) - \frac{i}{2\pi w\,\theta(-z+\tau+\frac{1}{2}\,|\,\tau)} \\ &= e(-z)\,\big(L(-z-\tau, -w\,|\,\tau) + \frac{i}{2\pi w\,\theta(z+\tau+\frac{1}{2}\,|\,\tau)}\big)\,. \end{aligned}$$

By letting $w \to 0$ in this last equality, one obtains that

$$L(z-\tau\,|\,\tau) = e(-z)\,L(-z-\tau\,|\,\tau)\,. \tag{3.82}$$

It is easy to see that $L(z+\frac{1}{2}\,|\,\tau) = L(z-\frac{1}{2}\,|\,\tau)$. By gathering (1.47) and (3.82), it follows that

$$L(z\,|\,\tau) + L(-z-\tau\,|\,\tau) = 1\,, \tag{3.83}$$

which gives the value of $L(\frac{1}{2}-\frac{\tau}{2}\,|\,\tau)$ and $L(-\frac{\tau}{2}\,|\,\tau)$. Furthermore, from (1.47) one finds that $L(-z-\tau\,|\,\tau) = e(z)\,\big(1 - L(-z\,|\,\tau)\big)$, so (3.83) becomes $L(z\,|\,\tau) - e(z)\,L(-z\,|\,\tau) = 1 - e(z)$, which gives the value of $L(\frac{1}{2}\,|\,\tau)$, as stated in (1.51). In this manner, one completes the proof of Theorem 1.4. □

4. Complementary proofs

In this section, we will consider the statements mentioned in the previous sections for which we have not made the proof. One common point for these statements is that they are somewhat easy to be understood but their proof may be technically hard and/or long. Especially, this is the case for Proposition 3.2 in §4.5 and Theorem 3.4 in §4.6.

4.1. **Proof of Lemma 1.1.** One assumes that $q \xrightarrow{a.r.} 0$, so $\tau \xrightarrow{a.v.} i\infty$. By the second item of the notational convention stated after Definition 1.1, $\hat{\tau} = \tau$ and $q = q_1$.

Let

$$\rho(n) = \rho_{z_1}(n) = \frac{1}{2}n(n-1) + nz_1 \tag{4.1}$$

for all $n \in \mathbb{Z}$, and let $J = \rho(\mathbb{Z})$. Define the $\mathbb{C}$-valued map $\tilde{\gamma}$ on J as follows:

$$\tilde{\gamma}(k) = \sum_{\rho(n)=k} (-1)^n\, e\big(nz_0\big) \qquad (k \in J)\,. \tag{4.2}$$

Write $J^* = \{k \in J : \tilde{\gamma}(k) \neq 0\}$, and set $I_N = J^* \cap (-\infty, N]$ for all non-negative integer N. Thus, letting $I = I_N$, $\gamma = \tilde{\gamma}\big|_I$ and $\upsilon = \lambda = 0$ gives the relation in (1.6) for $f = \theta(z + \frac{1}{2}\,|\,\tau)$, and this implies that $f \in \mathfrak{T}_0$.

Let $[z_1]$ and $\{z_1\}$ be respectively the integral part and fractional part of z_1; see (3.13). Putting $n = -[z_1]$ into the functional relation

$$\theta(z + \frac{1}{2} + n\tau\,|\,\tau) = e\big(-n(z+\frac{1}{2}) - \frac{1}{2}n(n-1)\tau\big)\,\theta(z+\frac{1}{2}\,|\,\tau)$$

yields that

$$\begin{aligned} f(q) \;&=\; e\big(n(z+\frac{1}{2}) + \frac{1}{2}n(n-1)\tau\big)\,\theta(z'+\frac{1}{2}\,|\,\tau) \\ &=\; e\big(k(z_1)\,\tau - [z_1](z_0+\frac{1}{2})\big)\,\theta(z'+\frac{1}{2}\,|\,\tau)\,. \end{aligned} \tag{4.3}$$

In the above, $z' = z_0 + \{z_1\}\tau$ and

$$k(z_1) = -\frac{1}{2}[z_1]\,\big(z_1 - 1 + \{z_1\}\big)\,. \tag{4.4}$$

As $\{z_1\} \in [0,1)$, it is easy to see that the dominant term of $\theta(z'+\frac{1}{2}\,|\,\tau)$ for $\tau \xrightarrow{a.v.} i\infty$ equals to 1 or $1 - e(z_0)$ if $\{z_1\} \neq 0$ or $\{z_1\} = 0$. Replacing $[z_1]$ with $z_1 - \{z_1\}$ in (4.4) gives $k(z_1) = -\frac{1}{2}(z_1 - \frac{1}{2})^2 + \kappa$, where $\kappa = \frac{1}{2}(\{z_1\} - \frac{1}{2})^2$. This implies Lemma 1.1, with (1.22) and (1.21).

4.2. **Proof of Theorem 1.1.** Let $f(q) = \theta(z + \frac{1}{2}\,|\,\tau)$. Thanks to Lemma 1.1, it suffices to prove that $f \in \mathfrak{T}_\zeta$ for all $\zeta = e(r)$, where $r \in \mathbb{Q} \cap [0,1)$. Without loss of generality, one can suppose that $r = \frac{p}{m}$, with $p \in \mathbb{Z}$, $m \in \mathbb{Z}_{>0}$ and $p \wedge m = 1$.

Consider (1.20), set

$$f_1(q) = \frac{\epsilon_1}{\sqrt{m}}\sqrt{\frac{i}{\hat{\tau}}}\,e\big(-\frac{1}{2\hat{\tau}}\,(z+\frac{1}{2m})^2 - \frac{\hat{\tau}}{8}\big)\,, \qquad f_2(q) = \theta(\frac{z}{m\hat{\tau}} + \frac{1}{2}\,|\,\tau')\,,$$

and write $f = f_1 f_2$. As $\tau \xrightarrow{a.v.} r$, it follows that $\hat{\tau} \xrightarrow{a.v.} 0$ and $\tau' \xrightarrow{a.r.} i\infty$; see (1.19).

On the one hand, by using $\tau = \hat{\tau} + \frac{p}{m}$, one writes $z = (z_0 + \frac{p}{m}z_1) + z_1\hat{\tau}$, which gives that

$$(z+\frac{1}{2m})^2 = (z_0 + \frac{p}{m}z_1 + \frac{1}{2m})^2 + 2z_1(z_0 + \frac{p}{m}z_1 + \frac{1}{2m})\,\hat{\tau} + z_1^2\,\hat{\tau}^2\,.$$

Thus, if one lets

$$\lambda = -\frac{z_1^2}{2} - \frac{1}{8}\,, \qquad k = \frac{1}{2}\,(z_0 + \frac{p}{m}z_1 + \frac{1}{2m})^2 \tag{4.5}$$

and

$$c_1 = \frac{\epsilon_1}{\sqrt{m}}\,e\big(-z_1(z_0 + \frac{p}{m}z_1 + \frac{1}{2m})\big)\,,$$

then one gets that ($q_1 = e(-\frac{1}{\hat{\tau}})$)

$$f_1(q) = c_1\sqrt{\frac{i}{\hat{\tau}}}\,e\big(\lambda\hat{\tau}\big)\,q_1^k\,, \tag{4.6}$$

which implies that $f_1 \in \mathfrak{T}_\zeta$, by (1.6).

On the other hand, by (1.19), one finds that

$$\frac{z}{m\hat{\tau}} = \frac{z_1}{m} + \frac{z_0 + \frac{p}{m}z_1}{m\hat{\tau}} = \frac{z_1}{m} + (mz_0 + pz_1)(\frac{\alpha}{m} - \tau')\,.$$

In this way, one gets that

$$f_2(q) = \theta(z_0' + z_1'\tau' + \frac{1}{2}\,|\,\tau')\,, \quad z_0' = \frac{z_1 - \alpha z_1'}{m}\,, \quad z_1' = -mz_0 - pz_1\,. \tag{4.7}$$

By applying Lemma 1.1 to this theta-factor f_2 relative to τ' in (4.7), one obtains this is theta-type as $\tau' \xrightarrow{a.r.} i\infty$, so $f_2 \in \mathfrak{T}_\zeta$. Therefore, Proposition 1.1 (1) implies that $f \in \mathfrak{T}_\zeta$, for $f = f_1 f_2$.

In order to get $t_\zeta(f)$, we let $q' = e(\tau')$. From the relation in (1.19), one deduces that $q' = \sqrt[m^2]{q_1}\, e(\frac{\alpha}{m})$. Since $\alpha p + 1 = \beta m$, it follows that $z_0' = \alpha z_0 + \beta z_1$ in (4.7). Replacing then (τ, q, z) with (τ', q', z') in Lemma 1.1 gives $t_\zeta(f_2)$ in terms of $z' = z_0' + z_1'\tau'$. Furthermore, comparing the expression of k given in (4.5) with that of z_1' in (4.7), one finds that $k = \frac{1}{m^2}\,(z_1' - \frac{1}{2})^2$, which gives the relations in (1.24) and (1.23). This completes the proof.

4.3. **Proof of Proposition 2.1.** We shall consider only the equivalent form (2.30) of (2.4). By (2.22), one can write (3.5) into the following form:

$$L(z, w\,|\,\tau) = G(z\,|\,\tau) + 2\pi i\,\omega(-z + \frac{1}{2}\,|\,\tau)\,L^*(z, w\,|\,\tau)\,, \tag{4.8}$$

where

$$L^*(z, w\,|\,\tau) = e(\frac{z}{\tau})\,L(\frac{z+1}{\tau}, \frac{w}{\tau}\,|\,-\frac{1}{\tau})\,. \tag{4.9}$$

Replacing (z, w, τ) with $(\frac{z+1}{\tau}, \frac{w}{\tau}, -\frac{1}{\tau})$ in (4.8) yields that the function L^* of (4.9) becomes:

$$L^*(z, w\,|\,\tau) = e\big(\frac{z}{\tau}\big)\,\Big(G\big(\big(\frac{z+1}{\tau}\,|\,-\frac{1}{\tau}\big) + 2\pi i\,\omega\big(-\frac{z+1}{\tau} + \frac{1}{2}\,|\,-\frac{1}{\tau}\big)L^{**}\Big)\,.$$

Here, in view of (4.9), (1.41) and (2.26), one can express L^{**} as follows:

$$\begin{aligned} L^{**} &= e(-z)\,L(-z-\tau, -w\,|\,\tau) \\ &= L(z-\tau, w\,|\,\tau) \\ &= e(-z)\,\big(1 - L(z, w\,|\,\tau)\big)\,. \end{aligned}$$

By observing that

$$\omega(-z + \frac{1}{2}\,|\,\tau)\,\omega(-\frac{z+1}{\tau} + \frac{1}{2}\,|\,-\frac{1}{\tau}) = \frac{e(z - \frac{z}{\tau})}{4\pi^2}\,,$$

one deduces from (4.8) that

$$L(z, w\,|\,\tau) = G(z\,|\,\tau) + 2\pi i\,\omega(-z + \frac{1}{2}\,|\,\tau)\,e(\frac{z}{\tau})\,G(\frac{z+1}{\tau}\,|\,-\frac{1}{\tau}) - \big(1 - L(z, w\,|\,\tau)\big).$$

This is to say:

$$1 - G(z\,|\,\tau) = 2\pi i\,\omega(-z + \frac{1}{2}\,|\,\tau)\,e(\frac{z}{\tau})\,G(\frac{z+1}{\tau}\,|\,-\frac{1}{\tau})\,.$$

By (2.26), one knows that $1 - G(z\,|\,\tau) = e(z)\,G(z - \tau\,|\,\tau)$. Thus, one finds the expected relation in (2.30) by using the following identity:

$$\omega(-z + \frac{1}{2}\,|\,\tau)\,e(\frac{z}{\tau}) = \omega(z + \frac{1}{2}\,|\,\tau)\,e(z)\,.$$

4.4. **Proof of Lemma 3.1.** Consider the expression of $R_1(z,w\,|\,\tau)$ obtained by letting $\mathsf{k}=1$ in (0.1), replace z with $u=u_0+u_1\tau$ and w with $v=v_0+v_1\tau$, and define

$$A_n=\frac{e(\frac{1}{2}n^2\tau+nu)}{1-e(n\tau+v)}=\frac{e(\frac{1}{2}n^2\tau+nu_1\tau+nu_0)}{1-e(n\tau+v_1\tau+v_0)} \tag{4.10}$$

for $n\in\mathbb{Z}$. As $\tau\xrightarrow{a.v.}i\infty$, it follows that

$$A_n=\sum_{\ell\geq 0}e(\frac{1}{2}n^2\tau+(nu_1+n\ell+\ell v_1)\tau+nu_0+\ell v_0) \tag{4.11}$$

for $n>-v_1$, and

$$A_n=-\sum_{\ell\geq 1}e(\frac{1}{2}n^2\tau+(nu_1-n\ell-\ell v_1)\tau+nu_0-\ell v_0) \tag{4.12}$$

for $n<-v_1$. Thus, when $v_1\notin\mathbb{Z}$, by letting $I=I_+\cup I_-$ with

$$I_+=\cup_{\ell\geq 0}\big\{\frac{1}{2}n^2+nu_1+(n+v_1)\ell: n>-v_1,\big\}$$

and

$$I_-=\cup_{\ell\geq 1}\big\{\frac{1}{2}n^2+nu_1-(n+v_1)\ell: n<-v_1\big\},$$

one finds that $I\cap(-\infty,N]$ is finite for any given $N\in\mathbb{Z}_{\geq 0}$. This implies that $R_1(u,v\,|\,\tau)\in\mathfrak{T}_0$. When $v_1\in\mathbb{Z}$ but $v_0\notin\mathbb{Z}$, one adds A_{-v_1} to the above exponential expansions of A_n's and this completes the proof.

4.5. **Proof of Proposition 3.2.** Let A_n be as in (4.10). Since $e(\alpha\tau)=\varnothing$ for $\alpha>0$, from (4.11) and (4.12) one gets that

$$n+v_1>0\quad\Rightarrow\quad A_n=e\big(n(\frac{n}{2}+u_1)\tau+nu_0\big)\,(1+\varnothing) \tag{4.13}$$

and

$$n+v_1<0\quad\Rightarrow\quad A_n=-e\big((\frac{n(n-2)}{2}+nu_1-v_1)\tau+nu_0-v_0\big)\,(1+\varnothing)\,. \tag{4.14}$$

(1) Assume that $v_1\in(-1,0)$. The relation in (4.13) implies that for $n\geq 1$,

$$\frac{A_{n+1}}{A_n}=e\big((n-1)\tau+(\frac{3}{2}+u_1)\tau+u_0\big)\,(1+\varnothing)\,. \tag{4.15}$$

Similarly, from (4.14) one gets that

$$\frac{A_{n-1}}{A_n}=e\big(-n\tau+(\frac{3}{2}-u_1)\tau-u_0\big)\,(1+\varnothing) \tag{4.16}$$

for $n\leq 0$. As $u_1\in(-\frac{1}{2},\frac{3}{2})$, it follows that $A_{n+1}=A_n\,\varnothing$ and $A_{-n}=A_{-n+1}\,\varnothing$ for all $n\geq 1$. Thus, one reduces that

$$f(q)=A_0\,\big(1+\varnothing\big)+A_1\,\big(1+\varnothing\big)\,. \tag{4.17}$$

By letting $n=1$ and $n=0$ in (4.13) and (4.14) respectively, one gets that

$$A_1=e\big(\frac{\tau}{2}+u\big)\,\big(1+\varnothing\big)\,,\quad A_0=-e\big(-v\big)\,\big(1+\varnothing\big)\,. \tag{4.18}$$

By hypothesis, $u+v\neq-\frac{\tau}{2}\mod\mathbb{Z}$, this implies $e\big(\frac{\tau}{2}+u\big)-e\big(-v\big)\neq 0$. Moreover, letting $\rho=\min(-v_1,u_1+\frac{1}{2})$, one finds that

$$e\big(\frac{\tau}{2}+u\big)-e\big(-v\big)=c\,e(\rho\tau)\,(1+\varnothing)\,.$$

In the above, $c = 1$ or $e(u_0) - e(-v_0)$ or -1 when $u_1 + \frac{1}{2} + v_1 < 0$ or $= 0$ or > 0, respectively. In this way, using both relations in (4.17) and (4.18) yields the wanted asymptotic relation in (3.10) for $f(q)$.

(2) When $v_1 \in (-1, 0)$ and $u + v = -\frac{\tau}{2} \mod \mathbb{Z}$, it follows that $u_1 = -\frac{1}{2} - v_1$, thus $u_1 \in (-\frac{1}{2}, \frac{1}{2})$. Let $\rho = \min(\frac{3}{2} - u_1, \frac{3}{2} + u_1)$; one has $\rho > 1$. Therefore, (4.15) and (4.16) imply that

$$A_{n+1} = A_1\, e(c\tau)\, \varnothing, \qquad A_{-n} = A_0\, e(c\tau)\, \varnothing \tag{4.19}$$

for $n \geq 1$ and $c \in (1, \rho)$. Keep the relation in (4.17), replace (4.18) with $A_0 = \frac{1}{1-e(v)}$ and $A_1 = \frac{e(-v)}{1-e(\tau+v)}$, and observe that

$$A_0 + A_1 = \frac{e(-v) - e(\tau + v)}{\big(1 - e(v)\big)\big(1 - e(\tau + v)\big)} = \big(e(\tau) - e(-2v)\big)\,(1 + \varnothing)$$

provided that $2v \neq -\tau \mod \mathbb{Z}$. The first relation in (3.11) is thus reduced from (4.19), for $\min(1, -2v_1) < c$.

If $2v = -\tau \mod \mathbb{Z}$, a direct computation shows that $A_{n+1} + A_{-n} = 0$ for all $n \in \mathbb{Z}_{\geq 0}$, what gives $f(q) = 0$.

(3) When $v_1 = 0$, one obtains that $A_n = \varnothing$ excepted $n = 0$. This completes the proof of Proposition 3.2.

4.6. **Proof of Theorem 3.4.** By using the functional equation in (2.26), the relation in (1.41) can be written into the following form:

$$L(-z - \tau, w \,|\, \tau) = 1 - L(z, -w \,|\, \tau)$$

or, equivalently:

$$f(q) = e(z)\, f_-(q)\,, \qquad f_-(q) = 1 - L(-z, -w \,|\, \tau)\,. \tag{4.20}$$

Assume that $w_1 \in [-\frac{1}{2}, \frac{1}{2}]$ and $z_1 \in [-\frac{1}{2}, \frac{1}{2}]$. Let $u = -z + w + \frac{\tau}{2}$, write $u = u_0 + u_1\tau$, and observe that $u_1 \in [-\frac{1}{2}, \frac{3}{2}]$. By applying (2.26) with (1.29), it follows that

$$L(z, w \,|\, \tau) = 1 - e(z)\, L(z - \tau, w \,|\, \tau) = 1 - e(z)\, \frac{R_1(u + \frac{1}{2}, w \,|\, \tau)}{\theta(u + \frac{\tau}{2} + \frac{1}{2} \,|\, \tau)}\,,$$

This gives the following expression of $f(q)$:

$$f(q) = e(z)\, \frac{R_1(u + \frac{1}{2}, w \,|\, \tau)}{\theta(u + \frac{\tau}{2} + \frac{1}{2} \,|\, \tau)}\,. \tag{4.21}$$

In view of (4.3) and Lemma 1.1, one can notice that

$$\theta(u + \frac{\tau}{2} + \frac{1}{2} \,|\, \tau) = 1 + \varnothing\,, \quad 1 - e(u_0) + \varnothing\,, \quad -e\big(\frac{\tau}{2} - u\big)\,(1 + \varnothing) \tag{4.22}$$

when $u_1 \in [-\frac{1}{2}, \frac{1}{2})$ or $= \frac{1}{2}$ or $\in (\frac{1}{2}, \frac{3}{2})$, respectively.

Case of $w_1 \in (-\frac{1}{2}, 0)$. This gives that $z_1 - w_1 \in (-\frac{1}{2}, 1)$ and $u_1 \in (-\frac{1}{2}, 1)$. Apply Proposition 3.2 (1) to $R_1(u + \frac{1}{2}, w \,|\, \tau)$, and remind that $u + \frac{1}{2} + w = -\frac{\tau}{2} \Leftrightarrow z = 2w + \tau + \frac{1}{2}$. It follows that, if $z \neq 2w + \tau + \frac{1}{2} \mod \mathbb{Z}$, then:

$$\begin{aligned} e(z)\, R_1(u + \frac{1}{2}, w \,|\, \tau) &= e(z)\, \big(e(-z + w + \tau + \frac{1}{2}) - e(-w)\big)\,(1 + \varnothing) \\ &= -\big(e(w + \tau) + e(z - w)\big)\,(1 + \varnothing)\,. \end{aligned} \tag{4.23}$$

(1a) If $z_1 < w_1$, then $u_1 \in (\frac{1}{2}, 1)$ and combing (4.21) together with (4.22) and (4.23) yields that

$$t_0(f)(q) = 1\,. \tag{4.24}$$

(1b) If $z_1 = w_1$ but $z_0 \neq w_0$, then $u_1 = \frac{1}{2}$ and, by the same manner, one deduces that

$$t_0(f)(q) = \frac{e(z_0 - w_0)}{e(w_0 - z_0) - 1}\,. \tag{4.25}$$

(1c) If $z_1 > w_1$, then $u_1 \in (-\frac{1}{2}, \frac{1}{2})$ and applying (4.23) together with (4.22) implies that

$$t_0(f)(q) = -e(z-w), \quad -\big(1 + e(2w_0 - z_0)\big)\, e(z-w), \quad -e(w+\tau) \tag{4.26}$$

when $z_1 < 2w_1 + 1$, $z_1 = 2w_1 + 1$ or $z_1 > 2w_1 + 1$, respectively.

If $z = 2w + \tau + \frac{1}{2} \neq \frac{1}{2} \mod \mathbb{Z}$, then $z_1 = 2w_1 + 1$. Since $2w_1 > -1$, Proposition 3.2 (2) implies that

$$e(z)\, R_1(u + \frac{1}{2}, w \,|\, \tau) = -e(z - 2w)\,(1 + \varnothing) = e(\tau)\,(1 + \varnothing)\,.$$

By observing that $u_1 \in (0, \frac{1}{2})$, from (4.22) one deduces that

$$t_0(f)(q) = e(\tau)\,. \tag{4.27}$$

If $z = 2w + \tau + \frac{1}{2} = \frac{1}{2} \mod \mathbb{Z}$, one finds that $f(z, w \,|\, \tau) = 0$.

Case of $w_1 = 0$. It follows that $w_0 \notin \mathbb{Z}$ and $u_1 = -z_1 + \frac{1}{2} \in [0, 1]$. Applying Proposition 3.2 (3) and relation (4.22) yields that

$$t_0(f)(q) = \begin{cases} \dfrac{e(z)}{1 - e(w_0)} & z_1 \in (0, \frac{1}{2}] \\[2ex] \dfrac{e(z_0)}{\big(1 - e(w_0)\big)\big(1 - e(-z_0 + w_0)\big)} & z_1 = 0 \\[2ex] -\dfrac{e(w_0)}{1 - e(w_0)} & z_1 \in [-\frac{1}{2}, 0) \end{cases}\,. \tag{4.28}$$

Case of $w_1 \in (0, \frac{1}{2})$. By (4.20), it follows that $t_0(f) = e(z)\, t_0(f_-)$. Replacing (z, w) with $(-z, -w)$ in (4.24) (4.25) and (4.26) gives the following results.

(2a) If $z_1 > w_1$, then:

$$t_0(f)(q) = e(z)\,. \tag{4.29}$$

(2b) If $z_1 = w_1$ but $z_0 \neq w_0$, then:

$$t_0(f)(q) = \frac{e(w)}{e(z_0 - w_0) - 1}\,. \tag{4.30}$$

(2c) If $z_1 < w_1$, then:

$$t_0(f)(q) = -e(w), \quad -\big(1 + e(z_0 - 2w_0)\big)\, e(w), \quad -e(z - w + \tau) \tag{4.31}$$

when $z_1 > 2w_1 - 1$, $z_1 = 2w_1 - 1$ or $z_1 < 2w_1 - 1$, respectively.

Finally, one gets the statement of Theorem 3.4 by putting together (4.26), (4.27), (4.28), (4.29), (4.30) and (4.31).

Appendix A. Gevrey asymptotic expansion and q-analogs of Borel-Laplace summation

To help with reading the *Gevrey and q-Gevrey analysis* part of this paper, we will recall, in §A.1, the definition of a Gevrey type asymptotic expansion and some results in relation with Laplace transform and exponential decay. This is very closed to the classic Borel-Laplace summation. So, in §A.2, we will discuss two q-analogs of this summation, which are used in §2 and §3 of this paper.

A.1. **Some elements of Gevrey asymptotic analysis.** Let $x_0 \in \mathbb{C}$, $I = (\alpha, \beta) \subset \mathbb{R}$, $\alpha < \beta$, $R \in (0, \infty]$, and let $\widetilde{\mathbb{C}^*_{x_0}}$ be the universal covering of the punctuated plane $\mathbb{C} \setminus \{x_0\}$. One denotes by $V(x_0; I, R)$ the open sector of vertex x_0 given by

$$V(x_0; I, R) = \{x \in \widetilde{\mathbb{C}^*_{x_0}} : \arg(x - x_0) \in I, |x - x_0| < R\}. \tag{A.1}$$

Remark A.1. *Let $V_a(d,r)$, $V_0(d,r)$, $V_\alpha(\delta \,|\, \rho)$ and $V_{i\infty}(\delta \,|\, \rho)$ be as in (1.1) – (1.4), and let $\tilde{a} = \arg(-a)$, with $\tilde{0} = 0$. It follows that, for any $a \in \partial\mathbb{D}^*$ and $\alpha \in \partial\mathcal{H}$:*

$$V_a(d,r) = V\big(a; (\tilde{a} - \frac{d}{2}, \tilde{a} + \frac{d}{2}), r\big), \quad V_\alpha(\delta \,|\, \rho) = V\big(\alpha; (\frac{\pi - d}{2}, \frac{\pi + d}{2}), \rho\big).$$

Furthermore, $V_{i\infty}(\delta \,|\, \rho) = V_0(\delta \,|\, \infty) \setminus \overline{V_0(\delta \,|\, \rho)}$. □

In what follows, *Gevrey asymptotic expansion* should be read as Gevrey asymptotic expansion of the first order in [25, p. 57, Definition 1.2].

Definition A.1. *Let $V = V(x_0; I, R)$. A given analytic function f in V is said to have a Gevrey asymptotic expansion as $x \to x_0$ in V and will be denoted by $f \in \mathfrak{G}(V)$, if there exists a complex coefficients power series $\sum_{n\geq 0} a_n (x - x_0)^n$ with the following property: for any $J \Subset I$ and $\rho \in (0, R)$, one can find $C > 0$ and $A > 0$ such that*

$$\big| f(x) - \sum_{n=0}^{N-1} a_n (x - x_0)^n \big| \leq C\, A^N\, N!\, |x - x_0|^N$$

for all $(x, N) \in V(x_0; J, \rho) \times \mathbb{N}$.

Let $\mathbb{C}[[x - x_0]]$ be the space of power series of variable $(x - x_0)$ and let T denote the $\mathbb{C}$-linear map that associates to every $f \in \mathfrak{G}(V)$ its asymptotic expansion in $\mathbb{C}[[x - x_0]]$. By [25, p. 60, Proposition 1.2], it follows that $\ker T = \mathfrak{E}^0(V)$, where $\mathfrak{E}^0(V)$ denotes the space of exponentially small functions in V. This implies that, if $T(f_1) = T(f_2)$, then f_1 is exponentially close to f_2 as $x \to x_0$ in V.

Proposition A.1. *Let $d \in \mathbb{R}$, $\delta > 0$, and let $U = V\big(0; (d - \delta, d + \delta), \infty\big)$. Let $\varphi = \xi^{-\upsilon} \sum_{n\geq 0} \alpha_n \xi^n$ represent an analytic function in U such that $|\varphi(\xi)| \leq C\, |\xi|^{-\upsilon}\, e^{B|\xi|}$ for all $\xi \in U$, where $\upsilon \in [0, 1)$, $C > 0$ and $B > 0$. Consider its Laplace transform:*

$$f(x) = \int_0^{\infty e^{id}} \varphi(t)\, e^{-\xi/x}\, d\xi. \tag{A.2}$$

Then f defines an analytic function in V such that $x^\upsilon f(x)$ admits the power series $\sum_{n\geq 0} \alpha_n\, \Gamma(n + 1 - \upsilon)\, x^{n+1}$ as Gevrey asymptotic expansion for $x \to 0$ in V, where

$$V = V\big(0; I, \frac{1}{B}\big), \qquad I = (d - \delta - \frac{\pi}{2}, d + \delta + \frac{\pi}{2}).$$

Proof. This follows from a direct computation using $\Gamma(a+1) = \int_0^\infty e^{-t}\, t^a\, dt$, where $\Re(a) > 0$. See [25, Proposition 1.5] for a similar statement. □

Given $(x_0, d) \in \mathbb{C} \times \mathbb{R}$, let $\mathfrak{G}_{x_0,d}$ denote the sheaf of functions having a Gevrey asymptotic expansion as $x \to x_0$ in some sector of vertex x_0 containing the segment $(x_0, x_0 + Re^{id})$ for enough small $R > 0$, *i.e.*

$$\mathfrak{G}_{x_0,d} = \bigcup_{I \ni d, R>0} \mathfrak{G}\big(V(x_0; I, R)\big)\,. \tag{A.3}$$

Proposition A.2. *Every $\mathfrak{G}_{x_0,d}$ constitutes a sheaf of rings containing as sub-ring the space of germs of analytic functions at $x = x_0$ in $\mathbb{C}$.*

Proof. By [25, p. 58, Proposition 1.1 (ii)], each $\mathfrak{G}(V)$ is, in fact, an algebra containing all functions analytic at $x = x_0$. □

Remark A.2. *The sheaves $\mathfrak{G}_a^r$ and $\mathfrak{G}_\alpha^v$ considered in the beginning of §1.1 can be expressed as follows:*

$$\mathfrak{G}_a^r = \mathfrak{G}_{a,\tilde{a}}\,, \qquad \mathfrak{G}_\alpha^v = \mathfrak{G}_{\alpha,\frac{\pi}{2}}\,,$$

where $\tilde{a}$ is the same as the one in Remark A.1. □

A.2. **q-Borel-Laplace summation methods.** The Borel-Laplace summation matches each suitable power series with its *Borel-sum*. This process consists of two consecutive transforms: the Borel transform, applied to the given power series, and the Laplace transform, that gives the Borel-sum by means of an integral like as in (A.2). For example, one sees that the power series $\hat{E}(x) = \sum_{n\geq 0}(-1)^n n!\, x^{n+1}$, known as Euler series, is divergent for all $x \neq 0$. By definition, its *Borel-sum* is obtained as follows:

$$\sum_{n\geq 0}(-1)^n n!\, x^{n+1} \;\Rightarrow\; \sum_{n\geq 0}(-1)^n \xi^n = \frac{1}{1+\xi} \;\Rightarrow\; \int_0^\infty \frac{1}{1+\xi}\, e^{-\xi/x}\, d\xi\,. \tag{A.4}$$

In (A.4) in the above, the last integral is the Borel-sum of $\hat{E}(x)$, and it admits this series as the asymptotic expansion for $x \to 0$. Moreover, one can find that both $\hat{E}(x)$ and its Borel-sum satisfy the differential equation $x^2y' + y = x$. In this way, the Borel-Laplace summation, when it may be applied, gives an analytic solution in some sector whose asymptotic expansion is the given power series.

The usual q-analog $n!_q$ of $n!$ is defined by the relation $n!_q = \frac{(q;q)_n}{(1-q)^n}$. While replacing q with q^{-1}, one obtains the following q-analog of $n!$: $n!_q\, q^{-n(n-1)/2}$. This gives the divergent series $\sum_{n\geq 0}(-1)^n q^{-n(n-1)/2}\, n!_q\, x^{n+1}$, which satisfies the following q-difference equation:

$$x\,\frac{y(x) - y(\frac{x}{q})}{1 - \frac{1}{q}} + y(x) = x$$

or, equivalently,

$$(\frac{1}{q} - 1 - x)y(x) + xy(\frac{x}{q}) = (\frac{1}{q} - 1)x\,. \tag{A.5}$$

The point $x = 0$ is a non-Fuchsian singular point for (A.5). From the analytic classification viewpoint of q-difference equations (see [26]), this equation plays the same role as (2.2). It can be easily seen that the q-Euler series $\hat{E}(x; q)$ given in (0.6) is a formal solution of (2.2). In this way, the summation process in (A.4) is replaced with the following q-analog:

$$\sum_{n\geq 0}(-1)^n q^{-n(n-1)/2}\, x^n \;\Rightarrow\; \sum_{n\geq 0}(-1)^n \xi^n = \frac{1}{1+\xi} \;\Rightarrow\; L_q(\frac{1}{1+\xi})(x)\,, \tag{A.6}$$

where L_q denotes a q-Laplace transform such that

$$L_q(\xi^n)(x) = q^{-n(n-1)/2}\, x^n\,. \tag{A.7}$$

The condition in (A.7) can be really interpreted as the solution to the moment problem associated with the sequence $(q^{-n(n-1)/2})_n$. This problem being undetermined, one can have several q-analogs in (A.6). In [33] and [34], two q-Laplace transforms have been considered as follows.

First, assume $q \in (0,1)$, let

$$\omega(t;q) = \frac{1}{\sqrt{2\pi \ln(1/q)}}\, e^{\frac{\log^2(t/\sqrt{q})}{2\ln q}} \quad (t \in \tilde{\mathbb{C}}^*)\,, \tag{A.8}$$

and define

$$L_q(\varphi)(x) = \int_0^\infty \varphi(\xi)\, \omega(\frac{\xi}{x};q)\, \frac{d\xi}{\xi}\,. \tag{A.9}$$

Secondly, for $q \in \mathbb{C}$ with $0 < |q| < 1$, let $\theta(t;q) = \sum_{n\in\mathbb{Z}} q^{n(n-1)/2} x^n$ and define

$$L_q(\varphi)(x) = \sum_{\xi \in u q^{\mathbb{Z}}} \frac{\varphi(\xi)}{\theta(\frac{\xi}{x};q)}\,, \tag{A.10}$$

where u denotes a non-zero complex number which indicated the discrete integration-path.

Each of both q-summations constructed in the above, when it may be applied to a given power series, gives rise to a q-Borel sum for this series.

Proposition A.3. *Every q-Borel sum of a power series satisfying a linear q-difference equation is an analytic solution of the same functional equation in some suitable domain. Moreover, this sum-function admits the power series as asymptotic expansion as $x \to 0$.*

Proof. See [33] and [34]. □

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