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Stability of distributed delay systems via a robust approach*

Frédéric Gouaisbaut^{a,b}, Yassine Ariba^{a,c}, Alexandre Seuret^{a,d}

Abstract—This paper is dedicated to the stability analysis of a class of distributed delay systems with a non constant kernel. By the use of appropriate orthogonal polynomials, this kernel is expressed as the sum of a polynomial and an additive bounded function. The resulting system is then modeled by an interconnected system between a nominal finite dimensional linear system and a infinite dimensional system. This last system is considered as a structured uncertainty and embedded into some well defined structured uncertainties. Appropriate robust control tools, i.e. quadratic separation, are then used to address the stability issue. Finally, numerical examples show the effectiveness of the proposed method.

I. INTRODUCTION

Since a few decades, the emergence of well established theory and potential applications led to a huge number of papers dedicated to the stability analysis of time delay systems [26]. Among them, few of them focused on the class of distributed time delay systems, for which the delay operator, via an integral action, has a cumulative effect over the past values of the dynamics. For specific integral kernels, including constant, rationals, γ distribution function, stability of such systems can be tackled by a direct method based on pole location [3], [15], [19], [25]. For kernels which can be interpreted as impulse responses of a linear system, the system can be modeled as a linear pointwise delay system and hence classical techniques on pole location can be used (see [19]). Nevertheless, these techniques are often restricted to the case of nominal systems without uncertainties [26] and for a constant delay.

Another widely used technique for stability analysis relies on Lyapunov Theory. The construction of Lyapunov Krasovskii functionals is modified in order to take into account the distributed nature of the delay. It results some Lyapunov structure which depends on double and triple integral terms [4], [5], [11], [13], [14], [28]. Different bounding techniques are then employed to derive numerically efficient stability tests. In general, the kernel is then restricted to be a constant function over the delay interval. In order to avoid this strong constraint, another possibility, described by Gu et al [10], is to use a very general class of Lyapunov Krasovskii functional (see for instance [21]) and a discretization scheme to obtain numerically tractable stability conditions. This technique designed for piecewise constant delay kernel leads to a

reduction of conservatism at the expense of the numerical burden. The case of general kernels have been studied successfully by [27] which proposed the use of new integral inequalities. These new bounding techniques which were especially designed for distributed delay systems led to new Lyapunov Krasovskii functional structures and efficient results on simulations. Another approach widely used in the literature stems from the robust analysis which use the same driving forces namely the comparisons systems and integral inequalities [18]. In [16], a combined full block S procedure and Lyapunov analysis is performed to prove the stability of a distributed delay system with a rational kernel. More recently, using quadratic separation approach, [7] has provided some LMI criteria for polynomial kernels. In this paper, we aim at extending the procedure proposed recently in [8] to the case of distributed delay systems and an arbitrary continuous and known kernel. The basic idea is to project this function upon a polynomial set whose a basis is orthogonal like Legendre polynomials. Choosing adequate extra states related to the projection of the kernel upon Legendre polynomials, the original system can be modeled as a classical time delay state submitted to an exogenous bounded perturbation. Using classical procedure in robust analysis, this last system can also be viewed as a linear system connected to a set of operators composed of the operators related to the distributed delayed dynamics. This last part is then embedded into some relevant structured uncertainties whose size is controlled by new inequalities from Bessel inequalities [8].

The paper is organized as follows. Next section is devoted to some preliminaries on the distributed delay system and quadratic separation. Then section 3 formulates a way to describe the distributed delay system as an interconnected uncertain system. Section 4 gives the central results of the paper. Section 5 presents some numerical simulations.

Notations: Throughout the paper, the following notations are used. $\mathbf{1}_n$ and $\mathbf{0}_{m \times n}$ denote respectively the identity matrix of size n and null matrix of size $m \times n$. For two symmetric matrices, A and B , $A > (\geq) B$ means that $A - B$ is (semi-) positive definite. A^T denotes the transpose of A .

II. PRELIMINARIES

A. Definitions

We define by \mathcal{H} the vector space of complex valued square integrable functions on $[-h, 0]$. For any functions in \mathcal{H} , we define the hermitian inner product

$$\langle f, g \rangle = \int_{-h}^0 f(\theta)g^*(\theta)d\theta, \quad (1)$$

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where f and g belong to \mathcal{H} . Basically, $x_i, x_j \in \mathcal{H}, i \neq j$ are said orthogonal if $\langle x_i, x_j \rangle = 0$. A sequence of \mathcal{H} , $\{e_0, e_1, e_2, \dots, e_n\}$, is an orthonormal sequence if $\langle e_i, e_j \rangle = \kappa_{ij}$, where κ_{ij} is the kronecker notation. Based on these definitions, we recall the Bessel inequality that will be employed later in the paper.

Lemma 1 (Bessel inequality): Let $\{e_0, e_1, e_2, \dots, e_n\}$ be an orthonormal sequence in \mathcal{H} . Then, for any f in \mathcal{H} the following inequality holds:

$$\langle f, f \rangle \geq \sum_{i=0}^n |\langle f, e_i \rangle|^2$$

An example of polynomials for the inner product (1) is the well known Legendre polynomials whose definition and properties are stated below.

Definition 1 (Legendre polynomials): The Legendre polynomials considered over the interval $[-h, 0]$ are defined by

$$\forall k \in \mathbb{N}, \quad L_k(u) = (-1)^k \sum_{l=0}^k p_l^k \left(\frac{u+h}{h} \right)^l.$$

with $p_l^k = (-1)^l \binom{k}{l} \binom{k+1}{l}$ where $\binom{k}{l} = \frac{k!}{l!(k-l)!}$. Some properties are summarized below and will be useful in the sequel.

P1 Orthogonality: $\forall (k, l) \in \mathbb{N}^2$,

$$\int_{-h}^0 L_k(u) L_l(u) du = \begin{cases} 0, & k \neq l \\ \frac{h}{2k+1}, & k = l \end{cases} \quad (2)$$

P2 Boundary conditions:

$$\forall k \in \mathbb{N}, \quad L_k(0) = 1, \quad L_k(-h) = (-1)^k.$$

P3 Recurrence on the derivative:

$$\frac{d}{du} [L_{k+1}(u) - L_{k-1}(u)] = \frac{2(2k+1)}{h} L_k(u)$$

with

$$\frac{d}{du} L_0(u) = 0, \quad \frac{d}{du} L_1(u) = \frac{2}{h} L_0(u).$$

Proofs of these properties are standard results and can be found in [6].

B. Problem statement

Consider the distributed delay system given by:

$$\dot{x}(t) = Ax(t) + A_d \int_{-h}^0 f(\theta) x(t+\theta) d\theta, \quad (3)$$

where $x \in \mathbb{R}^n$ is the instantaneous state vector and $A, A_d \in \mathbb{R}^{n \times n}$ are constant matrices. The function f is a scalar continuous function, defined over the interval $[-h, 0]$, and represents the kernel of the distributed delay. We aim at assessing stability of the system (3) using robust analysis, more especially using the quadratic separation approach which has been already used for polynomial kernel in [8] or pointwise delay system in [20].

Remark 1: We have considered that the kernel f is a scalar function but this assumption can be easily extended to the case of matrix kernel by considering several distributed terms with adequate matrices A_d . This will not be presented in this paper because of space limitation.

C. Stability analysis via quadratic separation

As a part of robust control theory, the quadratic separation provides a fruitful framework to address stability of non-linear and uncertain systems [20], [12]. Recent studies [1] have shown that such a framework allows to propose an original approach to deal with the stability analysis of time-delay systems. The key idea is to reformulate the delay system (3) as an uncertain feedback system as in Figure 1. In our case, the uncertainty stems from some operators related to the delay dynamic. It has been shown that a suitable interconnection modeling, based on relevant choices of operators, may reduce the conservatism of the analysis [1]. The stability conditions presented in this paper are founded on the theorem introduced below.

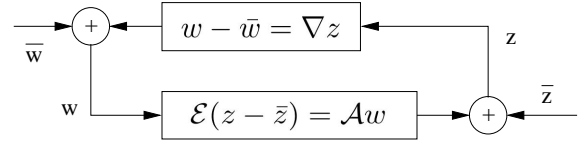


Fig. 1: Feedback system.

Let consider the interconnection defined by Figure 1 where \mathcal{E} and \mathcal{A} are two, real valued, possibly non-square matrices and ∇ is a linear operator which represent the system uncertainty. This latter is assumed to belong to an uncertain set \mathbb{W} . In the present paper, we aim at embedding some delay-dependent operators into the matrix ∇ . For simplicity of notations, we assume that \mathcal{E} is full column rank. Then, we are interested in looking for conditions that assess the stability of the interconnection.

Theorem 1 ([20]): The uncertain feedback system of Figure 1 is well-posed and stable if and only if there exists a Hermitian matrix $\Theta = \Theta^*$ satisfying both conditions

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0, \quad (4)$$

$$\begin{bmatrix} 1 \\ \nabla \end{bmatrix}^* \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \leq 0, \quad \forall \nabla \in \mathbb{W}. \quad (5)$$

As a result, the stability is evaluated through two matrix inequalities (4) and (5) related respectively to the lower and the upper bloc. Basically, inequality (5) is built from definitions and informations on the system uncertainty. Then, the other one (4) provides the stability condition to be tested.

III. MODELING THE DELAY SYSTEM AS A FEEDBACK SYSTEM

A. Projection of the kernel

In light of the properties of Legendre polynomials, we consider the polynomial approximation of the kernel f . For a given integer r , it is trivial to write

$$\forall \theta \in [-h, 0], \quad f(\theta) = \sum_{i=0}^r a_i L_i(\theta) + g_r(\theta), \quad (6)$$

where

$$\begin{aligned} a_i &= \frac{2i+1}{h} \int_{-h}^0 f(u) L_k(u) du, \quad \forall i = 0, \dots, r \\ g_r(\theta) &= f(\theta) - \sum_{i=0}^r a_i L_i(\theta). \end{aligned} \quad (7)$$

Then, the distributed delay system (3) can be reformulated as

$$\dot{x}(t) = Ax(t) + A_d \sum_{i=0}^r a_i x_i(t) + A_d \int_{-h}^0 g_r(\theta) x(t+\theta) d\theta, \quad (8)$$

where we define a new state $x_i(t) = \int_{-h}^0 L_i(\theta) x(t+\theta) d\theta$.

B. An extended system

The initial system (3) has been reformulated introducing extra state variables x_i whose dynamics are:

$$\begin{aligned} \dot{x}_i(t) &= \int_{-h}^0 L_i(\theta) \dot{x}(t+\theta) d\theta, \\ &= L_i(0)x(t) - L_i(-h)x(t-h) \\ &\quad - \int_{-h}^0 \frac{d}{d\theta} L_i(\theta) x(t+\theta) d\theta, \end{aligned}$$

as $L_{i-1}(0) = L_{i+1}(0)$ and $L_{i-1}(-h) = L_{i+1}(-h)$, subtracting the $(i-1)$ -th to the $(i+1)$ -th equation, we have

$$\begin{aligned} \dot{x}_{i+1}(t) - \dot{x}_{i-1}(t) &= - \int_{-h}^0 [\dot{L}_{i+1}(\theta) - \dot{L}_{i-1}(\theta)] x(t+\theta) d\theta, \\ &= - \frac{2(2i+1)}{h} x_i(t). \end{aligned}$$

The last equality results from property P3 in Definition 1. So the additional dynamics are expressed as

$$\dot{x}_{i+1}(t) = \dot{x}_{i-1}(t) - \frac{2(2i+1)}{h} x_i(t), \quad (9)$$

with the initialization given by:

$$\begin{aligned} \dot{x}_0(t) &= \int_{-h}^0 \dot{x}(t+\theta) d\theta = x(t) - x(t-h) \\ \dot{x}_1(t) &= \int_{-h}^0 \left(\frac{2}{h}\theta + 1\right) \dot{x}(t+\theta) d\theta \\ &= x(t) + x(t-h) - \frac{2}{h} x_0(t). \end{aligned}$$

Let introduce the extra state $\zeta(t)$ and $x_g(t)$.

$$\zeta(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_r(t) \end{bmatrix}, \quad x_g(t) = \int_{-h}^0 g_r(\theta) x(t+\theta) d\theta$$

The dynamics of the extended distributed delay system can be easily described as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \bar{A}_{dr} \zeta(t) + A_d x_g(t), \\ \begin{bmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n \times (r-1)n} \\ \hline & E \end{bmatrix} \dot{\zeta}(t) &= \begin{bmatrix} \mathbf{1}_n & \mathbf{0}_{n \times (r+1)n} \\ \hline \mathbf{0}_{(r-1)n \times n} & U \end{bmatrix} \zeta(t) \\ &\quad + \begin{bmatrix} -\mathbf{1}_n \\ \mathbf{1}_n \\ \mathbf{0}_{(r-1)n \times n} \end{bmatrix} x(t-h), \end{aligned}$$

with

$$\begin{aligned} \bar{A}_{dr} &= \begin{bmatrix} a_0 A_d & a_1 A_d & \dots & a_r A_d \\ \hline \alpha_0 \mathbf{1}_n & \dots & \mathbf{0}_n & \mathbf{0}_n \\ & \ddots & \vdots & \\ \mathbf{0}_n & & \alpha_{r-1} \mathbf{1}_n & \mathbf{0}_n \\ \hline -\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ & \ddots & \ddots & \ddots \\ \mathbf{0} & & -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \\ U &= \begin{bmatrix} \alpha_0 \mathbf{1}_n & \dots & \mathbf{0}_n & \mathbf{0}_n \\ & \ddots & \vdots & \\ \mathbf{0}_n & & \alpha_{r-1} \mathbf{1}_n & \mathbf{0}_n \\ \hline -\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ & \ddots & \ddots & \ddots \\ \mathbf{0} & & -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \\ E &= \begin{bmatrix} \alpha_0 \mathbf{1}_n & \dots & \mathbf{0}_n & \mathbf{0}_n \\ & \ddots & \vdots & \\ \mathbf{0}_n & & \alpha_{r-1} \mathbf{1}_n & \mathbf{0}_n \\ \hline -\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ & \ddots & \ddots & \ddots \\ \mathbf{0} & & -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \end{aligned}$$

and $\alpha_i = -2\frac{2i+1}{h}$. We obtain therefore a time delay system whose stability will be studied via the quadratic separation approach.

C. Uncertain feedback system

The methodology consists in expressing the delay system (3) into an interconnected system as depicted in Figure 1. To this end, we need to list all operators that are involved in the dynamic of the system as well as uncertainties. First of all, let define the following operators, for $i = \{0, 1, 2, \dots, r\}$

$$\begin{aligned} \delta_i &: \mathbb{R} \rightarrow \mathbb{R}, \\ x(t) &\rightarrow \int_{-h}^0 L_i(\theta) x(t+\theta) d\theta, \end{aligned} \quad (10)$$

$$\begin{aligned} \delta_g &: \mathbb{R} \rightarrow \mathbb{R}, \\ x(t) &\rightarrow \int_{-h}^0 g(\theta) x(t+\theta) d\theta. \end{aligned}$$

The first operator is the operator that defines state variables x_i , $x_i(t) = \delta_i[x](t)$. The second operator is associated to the remainder of the projection of the kernel f , namely $x_g(t) = \delta_g[x](t)$. Since h is constant, the operators above can be expressed in the Laplace domain as $\delta_i(s) = \int_{-h}^0 L_i(\theta) e^{s\theta} d\theta$ and $\delta_g(s) = \int_{-h}^0 g_r(\theta) e^{s\theta} d\theta$. Now, we gather all the operators that describe (3) according to our modeling:

$$\underbrace{\begin{bmatrix} x(t) \\ \zeta(t) \\ x(t-h) \\ \zeta(t) \\ x_g(t) \end{bmatrix}}_{w(t)} = \nabla \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{\zeta}(t) \\ x(t) \\ x(t) \\ x(t) \end{bmatrix}}_{z(t)}, \quad (11)$$

with $\bar{\delta} = [\delta_0, \delta_1, \dots, \delta_r]^T$ and

$$\nabla = \begin{bmatrix} s^{-1} \mathbf{1}_{n(r+2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & e^{-sh} \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\delta} \otimes \mathbf{1}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \delta_g \mathbf{1}_n \end{bmatrix}.$$

Combining expressions (8) and (9), we can build the linear equation (feedforward block in Figure 1) specifying all the relationships between vectors z and w :

$$\mathcal{E} \begin{bmatrix} \dot{x}(t) \\ \dot{\zeta}(t) \\ x(t) \\ x(t) \\ x(t) \end{bmatrix} = \mathcal{A} \begin{bmatrix} x(t) \\ \zeta(t) \\ x(t-h) \\ \zeta(t) \\ x_g(t) \end{bmatrix}, \quad (12)$$

with

$$\mathcal{E} = \begin{bmatrix} 1_{3n} & 0_{3n \times (r-1)n} & 0_{3n} \\ 0_{(r-1)n \times n} & E & 0_{(r-1)n \times 3n} \\ 0_{3n} & 0_{3n \times (r-1)n} & 1_{3n} \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A} =$$

$$\begin{bmatrix} A & \bar{A}_{dr} & 0_n & 0_{n \times (r+1)n} & A_d \\ 1_n & 0_{n \times (r+1)n} & -1_n & 0_{n \times (r+1)n} & 0_n \\ 1_n & U & 1_n & 0_{rn \times (r+1)n} & 0_{rn \times n} \\ 0_{(r-1)n \times n} & & 0_{(r-1)n \times n} & & \\ 1_n & 0_{3n \times (r+1)n} & 0_{3n \times n} & 0_{3n \times (r+1)n} & 0_{3n \times n} \\ 1_n & & & & \\ 1_n & & & & \\ 0_{n(r+1)} & 1_{n(r+1)} & 0_{n(r+1) \times n} & -1_{n(r+1)} & 0_{n(r+1) \times n} \end{bmatrix}.$$

IV. STABILITY CONDITION

A. Inequality constraints

The dynamic of our feedback system is described by a set of operators (11). We want to state now a set of inequalities that will be useful to build the constraint on ∇ (5).

Lemma 2 ([1]): A quadratic constraint for the operator s^{-1} is given by the following inequality for any positive definite matrix P ,

$$\begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix}^* \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix} \leq 0.$$

Lemma 3 ([1]): A quadratic constraint for the operator e^{-hs} is given by the following inequality for any positive definite matrix Q ,

$$\begin{bmatrix} 1_n \\ e^{-hs}1_n \end{bmatrix}^* \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 1_n \\ e^{-hs}1_n \end{bmatrix} \leq 0.$$

The proof of the two above lemmas can be found in [20], [1].

Lemma 4: [8] A quadratic constraint for the operator $\bar{\delta}$ is given by the following inequality for any positive definite matrix R_1 ,

$$\begin{bmatrix} 1_n \\ \delta_0 1_n \\ \delta_1 1_n \\ \vdots \\ \delta_r 1_n \end{bmatrix}^* \Xi \begin{bmatrix} 1_n \\ \delta_0 1_n \\ \delta_1 1_n \\ \vdots \\ \delta_r 1_n \end{bmatrix} \leq 0,$$

with

$$\Xi = \begin{bmatrix} -h^2 R_1 & 0 & 0 & \dots & 0 \\ 0 & R_1 & 0 & \dots & 0 \\ 0 & 0 & 3R_1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & (2r+1)R_1 \end{bmatrix}$$

Proof: Let apply the Bessel inequality (Lemma 1) on the delay function $e^{s\theta}$ with the normalized Legendre polynomials $\sqrt{\frac{2i+1}{h}} L_i$:

$$\sum_{i=0}^r |\langle e^{s\theta}, \sqrt{\frac{2i+1}{h}} L_i \rangle|^2 \leq \langle e^{s\theta}, e^{s\theta} \rangle$$

Upperbounding $\langle e^{s\theta}, e^{s\theta} \rangle$ by h , we obtain:

$$\sum_{i=0}^r (2i+1) \delta_i \delta_i^* \leq h \langle e^{s\theta}, e^{s\theta} \rangle \leq h^2,$$

The proof follows straightforwardly. \blacksquare

Lemma 5: A quadratic constraint for the operator δ_g is given by the following inequality for any positive definite matrix R_2 ,

$$\begin{bmatrix} 1_n \\ \delta_g 1_n \end{bmatrix}^* \begin{bmatrix} -\lambda R_2 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} 1_n \\ \delta_g 1_n \end{bmatrix} \leq 0,$$

with $\lambda = h \int_{-h}^0 |g_r(\theta)|^2 d\theta$.

Proof: Let apply the Cauchy-Schwartz inequality on the function δ_g :

$$\delta_g \delta_g^* = \left| \int_{-h}^0 g_r(\theta) e^{s\theta} d\theta \right|^2 \leq \lambda,$$

which concludes the proof. \blacksquare

B. Main result

We now propose a theorem that provides a sufficient stability condition for the distributed delay system system (3). This stability condition is expressed as a LMI condition.

Theorem 2: For a given positive scalar h and a positive integer r , if there exist positive definite matrices $P \in \mathbb{R}^{n(r+2) \times n(r+2)}$, Q , R_1 and $R_2 \in \mathbb{R}^{n \times n}$, then the system (3) is asymptotically stable if the following LMI is satisfied:

$$[\mathcal{E} \quad -\mathcal{A}]^{\perp *} \Theta [\mathcal{E} \quad -\mathcal{A}]^{\perp} > 0,$$

where matrices \mathcal{E} , \mathcal{A} and Θ are defined in (12) and (13).

Proof: In Section III-C, it has been shown that system (3) could be expressed as in Figure 1 with equations (11)-(12). Combining all the quadratic constraints stated in Section IV-A, we build the matrix (the separator)

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}, \quad (13)$$

with

$$\Theta_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -Q & 0 & 0 \\ 0 & 0 & -h^2 R_1 & 0 \\ 0 & 0 & 0 & -\lambda R_2 \end{bmatrix},$$

$$\Theta_{12} = \begin{bmatrix} -P & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$\Theta_{22} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & Q & 0 & \dots & 0 & 0 \\ 0 & 0 & R_1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & (2r+1)R_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & R_2 \end{bmatrix},$$

Theorems	h	nbr. of var.
[3] (“analytical”)	1.498	—
[27]	1.03	3
Th.2 ($r = 1$)	1.39	9
Th.2 ($r = 2$)	1.491	13
Th.2 ($r = 3$)	1.495	18

TABLE I: Maximal allowable delay h for system (14).

that fulfills condition (5). Hence, invoking Theorem 1, the aforementioned system is stable if the first condition (4) is also satisfied. ■

Remark 2: Following [9] or [18], we can show that the proposed result can also be interpreted as a Lyapunov-Krasovskii functional of the form:

$$\begin{aligned}
V &= \begin{bmatrix} x(t) \\ \zeta(t) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \zeta(t) \end{bmatrix} + \int_{-h}^0 x_t^T(\theta) Q x_t(\theta) d\theta \\
&+ \int_{-h}^0 \int_s^0 x_t^T(\theta) R_1 x_t(\theta) d\theta ds \\
&+ \int_{-h}^0 \int_s^0 x_g^T(t+\theta) R_2 x_g(t+\theta) d\theta ds
\end{aligned}$$

It appears that this functional is related to the functional used recently by [22] and [23], [24] to establish some stability criteria using either Wirtinger inequalities or Bessel inequalities.

V. NUMERICAL EXAMPLES

A. Example 1

Let consider a first scalar example:

$$\dot{x}(t) = -x(t) + \int_{-h}^0 e^{-\theta} \sin(\theta) x(t+\theta) d\theta \quad (14)$$

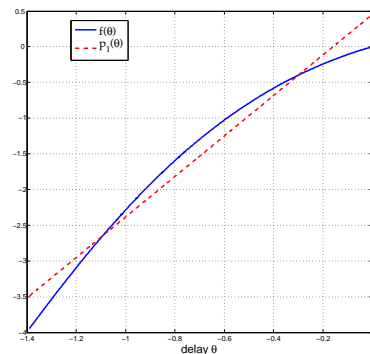
Using a numerical method [2], [3], system (14) is asymptotically stable for all delays less than 1.498. Table I shows results obtained with Theorem 2. As expected, better results are obtained as the degree of the polynomial r increases. Figure 2 shows the kernel function $f(\theta)$ and the projection polynomial $p_r(\theta) = \sum_0^r a_i L_i(\theta)$. The difference between the two curves represents $g_r(\theta)$ and it can be seen that for $r = 2$, the approximation is good enough to get a small λ .

B. Example 2

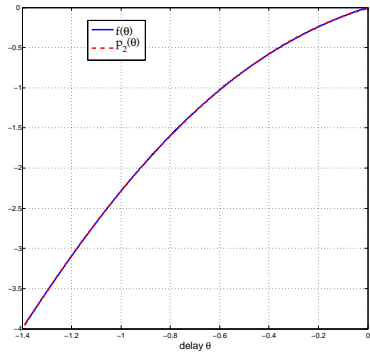
Let consider the following distributed delay system:

$$\dot{x}(t) = -2x(t) + \int_{-h}^0 (\theta + 3 \cos \theta) x(t+\theta) d\theta \quad (15)$$

This example is interesting because it has two stability pockets namely h in $[0, 0.964]$ and in $[1.372, 2.105]$ [3]. Table II shows results obtained with Theorem 2. Note that even if the theorem from [27] provides a very good result in estimating the first interval of stability, it is not able to detect the second one.



(a)



(b)

Fig. 2: The delay kernel f and the projection polynomial p_r (for $r = 1$ and $r = 2$).

Theorems	1 st interval	2 nd interval	nbr of var.
[3] (“analytical”)	[0 – 0.964]	[1.372 – 2.105]	—
[27]	[0 – 0.964]	—	3
Th.2 ($r = 1$)	[0 – 0.86]	—	9
Th.2 ($r = 2$)	[0 – 0.960]	[1.42 – 1.99]	13
Th.2 ($r = 3$)	[0 – 0.964]	[1.375 – 2.097]	18

TABLE II: Allowable delay h for system (15).

C. Example 3

The system we are considering now is interesting in that it is unstable if the delay h is zero.

$$\dot{x}(t) = 0.2x(t) - \int_{-h}^0 (1 - 0.3\theta) x(t+\theta) d\theta \quad (16)$$

In this case, although the dynamic matrix A (here a scalar) is positive, the delayed dynamic may stabilize the system. [27] shows that system (16) is stable for h in $[0.195 - 1.442]$. Invoking Theorem 2 with $r = 1$, a larger stability interval: $[0.195 - 1.658]$ is obtained. Hence, it is worthy to note that both results are able to assess the stability of the distributed delay system even when the delay-free system is unstable.

D. Example 4

This last example is extracted from [17]:

$$\dot{x}(t) = -2.1x(t) - \int_{-h}^0 \tilde{f}(\theta) x(t+\theta) d\theta, \quad (17)$$

with an uncertain kernel $\tilde{f}(\theta) = f(\theta) + \Delta_f(\theta)$ and $f(\theta) = \frac{-0.6\theta}{1+\theta^2}$. It is assumed that $\|\Delta_f(\theta)\| \leq 0.1$ for all $\theta \in [-h, 0]$. In order to handle this uncertainty and apply our result, a slight change on operator δ_g is required:

$$\delta_g(s) = \int_{-h}^0 (g(\theta) + \Delta_f(\theta)) e^{s\theta} d\theta,$$

and we have the inequality

$$\delta_g^* \delta_g \leq h \int_{-h}^0 |g(\theta)|^2 + |\Delta_f(\theta)|^2 d\theta.$$

Hence, Theorem 2 can be applied, adapting the scalar λ in (13). The stability condition in [17] states that (17) is stable for a delay interval h up to 5.2. The result in this paper is able to find out a larger delay interval (for which the system remains stable): $h = 7.8$ for $r = 2$ and $h = 8$ for $r = 4$.

VI. CONCLUSION

In this paper, we have provided a systematic procedure for the stability analysis of distributed delay systems with a general kernel using on a robust approach. The approach is based on the approximation of the kernel by orthogonal polynomials. The resulting system is then rewritten as a more classical time delay system submitted to a perturbation whose size can be estimated by integral inequalities issued from Bessel inequalities. The resulting criterion is expressed in terms of an LMI and give interesting results in simulations. Future works include the conservatism reduction analysis and the more tricky case of time-varying delay.

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