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# ALMOST GLOBAL-IN-TIME SOLUTION OF THE PRANDTL EQUATIONS IN SOBOLEV SPACE

CHAO-JIANG XU AND XU ZHANG

ABSTRACT. In this paper, we study the long time well-posedness for the non-linear Prandtl boundary layer equation on the half plane. The initial data is the perturbation of a monotonic shear profile, we prove the existence, uniqueness and stability of solutions in weighted Sobolev space, the life span of the solution can be arbitrarily long while the initial perturbation of the monotonic shear profile is enough small. We use the energy method to prove the existence of solution by a parabolic regularizing approximation. The nonlinear cancellation properties of the Prandtl equations under the monotonic assumption is the main ingredients to establish a new energy estimate.

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## 1. INTRODUCTION

In this work, we study the initial-boundary value problem for the Prandtl boundary layer equation in two dimension, which reads

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = \partial_y^2 u, & t > 0, (x, y) \in \mathbb{R}_+^2, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = U(t, x), \\ u|_{t=0} = u_0(x, y), \end{cases}$$

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where  $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$ ,  $u(t, x, y)$  represents the tangential velocity,  $v(t, x, y)$  normal velocity.  $p(t, x)$  and  $U(t, x)$  are the values on the boundary of the Euler's pressure and Euler's tangential velocity and determined by the Bernoulli's law:  $\partial_t U(t, x) + U(t, x)\partial_x U(t, x) + \partial_x p = 0$ .

The Prandtl boundary layer equation was firstly derived formally by Prandtl in 1904 ([22]). From the mathematical point of view, the well-posedness and justification of the Prandtl boundary layer theory don't have satisfactory theory yet, and remain open for general cases. During the past century, lots of mathematicians have investigated this problems. The Russian school has contributed a lot to the boundary layer theory and their works were collected in [21]. Up to now, the local existence theory for the Prandtl boundary layer equation has been achieved when the initial data belong to some special functional spaces: 1) the analytic space or analytic with respect to the tangential variable [15, 19, 24, 25, 27]; 2) Sobolev spaces or Hölder spaces under monotonicity assumption [1, 17, 20, 21, 26]; 3) recently [7] in Gevrey class with non-degenerate critical point. See also [16] where the initial data is monotone on a number of intervals and analytic on the complement. In all above works, the life span of the solution are very small.

On the other hand, as pointed out by Grenier-Guo-Nguyen [9, 10, 11], the long time behavior of the Prandtl equations is important to make progress towards the inviscid limit of the Navier-Stokes equations, we must understand its behavior on a longer time interval than the one which causes the instability used to prove ill-posedness. To the best of our knowledge, the long-time existence of the Prandtl equations has only been considered in [21, 26] and recently in preprints [14, 18, 27].

Under the monotonic assumption, by using the Crocco transformation, Oleinik ([21]) obtained the global-in-time smooth solution in Hölder space for the Prandtl equation defined on the interval  $0 \leq x \leq L$  with  $L$  very small. Xin-Zhang ([26]) proved the global existence of weak solutions if the pressure gradient has a favorable sign, that is  $\partial_x p \leq 0$ . See [18] for a similar work in 3-D case. The global existence of smooth solutions in the monotonic case remains open.

In the analytical frame, Ignatova-Vicol ([14]) recently get an almost global-in-time solution which is analytic with respect to the tangential variable, see also [27] for a same attempt work by using a refined Littlewood-Paley analysis. On the other side, without the monotonicity assumption, E and Engquist in [5] constructed finite time blowup solutions to the Prandtl equation. After this work, there are many un-stability or strong ill-posedness results. In particular, Gérard-Varet and Dormy [6] showed that the linearized Prandtl equation around the shear flow with a non-degenerate critical point is ill-posed in the sense of Hadamard in Sobolev spaces. See also [4, 8, 12, 13, 23] for the relative works.

Motivated by above analysis, in this work, using the energy method, we will prove the existence of smooth solution of Prandtl equation in Sobolev space, the life span can be arbitrarily long while the initial data is a small perturbation of a monotonic shear profile. In what follows, we consider the uniform outflow  $U(t, x) = 1$  which implies  $p_x = 0$ . In other words the following problem for the Prandtl equation is

considered :

$$\begin{cases} \partial_t u + u \partial_x u + v \partial_y u = \partial_y^2 u, & t > 0, (x, y) \in \mathbb{R}_+^2, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, & \lim_{y \rightarrow +\infty} u = 1, \\ u|_{t=0} = u_0(x, y) = u_0^s(y) + \tilde{u}_0(x, y), \end{cases} \quad (1.1)$$

where  $u_0^s(y)$  is a shear profile and  $\tilde{u}_0(x, y)$  is a perturbation. We will use the weighted Sobolev spaces (similar to [20]) with the following norm, for  $\lambda \in \mathbb{R}, m \in \mathbb{N}$ ,

$$\|f\|_{H_\lambda^m(\mathbb{R}_+^2)}^2 = \sum_{|\alpha_1 + \alpha_2| \leq m} \int_{\mathbb{R}_+^2} \langle y \rangle^{2\lambda + 2\alpha_2} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} f|^2 dx dy.$$

We use also the notation  $\|f\|_{L_\lambda^2(\mathbb{R}_+^2)} = \|f\|_{H_\lambda^0(\mathbb{R}_+^2)}$  and  $H^m$  stands for the usual Sobolev space.

For the initial data of the shear profile, we suppose that  $u_0^s$  satisfies the following conditions: for some even integer  $m \geq 6$ , and  $k > 1$ ,

$$\begin{cases} u_0^s \in C^{m+4}([0, +\infty[), & \lim_{y \rightarrow +\infty} u_0^s(y) = 1; \\ (\partial_y^{2p} u_0^s)(0) = 0, & 0 \leq 2p \leq m+4; \\ c_1 \langle y \rangle^{-k} \leq (\partial_y u_0^s)(y) \leq c_2 \langle y \rangle^{-k}, & \forall y \geq 0, \\ |(\partial_y^p u_0^s)(y)| \leq c_2 \langle y \rangle^{-k-p+1}, & \forall y \geq 0, 1 \leq p \leq m+4, \end{cases} \quad (1.2)$$

for certain  $c_1, c_2 > 0$ . Let  $u^s(t, y)$  be the solution of the heat equation (2.1) with initial data  $u_0^s(y)$ , then  $(u^s, 0)$  is a shear flow of the system (1.1). We will construct the solution of the Prandtl equation (1.1) as a perturbation of this shear flow. We have the following existence of almost global-in-time solution.

**Theorem 1.1.** *Let  $m \geq 6$  be an even integer,  $k > 1, 0 < \ell < \frac{1}{2}, \ell' > \frac{1}{2}, \ell' - \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ . Assume that  $u_0^s$  satisfies (1.2), the initial data  $(u_0 - u_0^s) \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  and  $(u_0 - u_0^s)$  satisfies the compatibility condition up to order  $m+2$ . Then for any  $T > 0$ , there exists  $\delta_0 > 0$  small enough such that if*

$$\|u_0 - u_0^s\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)} \leq \delta_0, \quad (1.3)$$

the initial-boundary value problem (1.1) admits a unique solution  $(u, v)$  with

$$(u - u^s) \in L^\infty([0, T]; H_{k+\ell-1}^m(\mathbb{R}_+^2)), \quad \partial_y(u - u^s) \in L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$$

and

$$v \in L^\infty([0, T]; L^\infty(\mathbb{R}_{y,+}; H^{m-1}(\mathbb{R}_x))), \quad \partial_y v \in L^\infty([0, T]; H_{k+\ell-1}^{m-1}(\mathbb{R}_+^2)).$$

Moreover, we have the stability with respect to the initial data in the following sense: given any two initial data

$$u_0^1 = u_0^s + \tilde{u}_0^1, \quad u_0^2 = u_0^s + \tilde{u}_0^2,$$

if  $u_0^s$  satisfies (1.2) and  $\tilde{u}_0^1, \tilde{u}_0^2$  satisfies (1.3), then the solutions  $u^1$  and  $u^2$  of (1.1) with initial data  $u_0^1$  and  $u_0^2$  respectively satisfy,

$$\|u^1 - u^2\|_{L^\infty([0, T]; H_{k+\ell-1}^{m-3}(\mathbb{R}_+^2))} \leq C \|u_0^1 - u_0^2\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}.$$

This article is arranged as follows. In Section 2, we explain the main difficulties for the study of the Prandtl equation and present an outline of our approach. In Section 3, we study the approximate solutions to (1.1) by a parabolic regularization. In Section 4, we prepare some technical tools and the formal transformation for the Prandtl equations. Section 5 is dedicated to the uniform estimates of approximate solutions obtained in Section 3. We prove finally the main theorem in Section 6-7.

**Notations:** The letter  $C$  stands for various suitable constants, independent with functions and the special parameters, which may vary from line to line and step to step. When it depends on some crucial parameters in particular, we put a sub-index such as  $C_\epsilon$  etc, which may also vary from line to line.

## 2. PRELIMINARY

**Difficulties and our approach.** Now, we explain the main difficulties in proving Theorem 1.1, and present the strategies of our approach.

It is well-known that the major difficulty for the study of the Prandtl equation (1.1) is the term  $v \partial_y u$ , where the vertical velocity behaves like

$$v(t, x, y) = - \int_0^y \partial_x u(t, x, \tilde{y}) d\tilde{y},$$

by using the divergence free condition and boundary conditions. So it introduces a loss of  $x$ -derivative. The  $y$ -integration create also a loss of weights with respect to  $y$ -variable. Then the standard energy estimates do not work. This explains why there are few existence results in the literatures.

Recalling that in [1] (see also [20] for a similar transformation), under the monotonic assumption  $\partial_y u > 0$ , we divide the Prandtl equations by  $\partial_y u$  and then take derivative with respect to  $y$ , to obtain an equation of the new unknown function  $f = \left( \frac{u}{\partial_y u} \right)_y$ . In the new equation, the term  $v$  disappears by using the divergence free condition. Here a little different from [1], we use  $g_m = \left( \frac{\partial_x^m u}{\partial_y u} \right)_y$ , where  $m$  stands for the highest derivative with  $x$ . From [20], we can observe that we only need to worry about the highest derivative with  $x$ . This is why we only define  $g_m$ .

In order to prove the existence of solutions, follows the idea of Masmoudi-Wong ([20]), we will construct an approximate scheme and study the parabolic regularized Prandtl equation (3.1), which preserves the nonlinear structure of the original Prandtl equation (1.1), as well as the nonlinear cancellation properties. Then by an uniform energy estimate for the approximate solutions, the existence of solutions to the original Prandtl equation (1.1) follows. This energy estimate also implies the uniqueness and the stability. The uniform energy estimate for the approximate solutions is the main duty of this paper.

**Analysis of shear flow.** We write the solution  $(u, v)$  of system (1.1) as

$$u(t, x, y) = u^s(t, y) + \tilde{u}(t, x, y), \quad v(t, x, y) = \tilde{v}(t, x, y),$$

where  $u^s(t, y)$  is the solution of the following heat equation

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0, \\ u^s|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u^s(t, y) = 1, \\ u^s|_{t=0} = u_0^s(y). \end{cases} \quad (2.1)$$

Then (1.1) can be written as

$$\begin{cases} \partial_t \tilde{u} + (u^s + \tilde{u})\partial_x \tilde{u} + \tilde{v}(u_y^s + \partial_y \tilde{u}) = \partial_y^2 \tilde{u}, \\ \partial_x \tilde{u} + \partial_y \tilde{v} = 0, \\ \tilde{u}|_{y=0} = \tilde{v}|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} \tilde{u} = 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0(x, y). \end{cases} \quad (2.2)$$

We first study the shear flow,

**Lemma 2.1.** *Assume that the initial data  $u_0^s$  satisfy (1.2), then for any  $T > 0$ , there exist  $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$  such that the solution  $u^s(t, y)$  of the initial boundary value problem (2.1) satisfies*

$$\begin{cases} \tilde{c}_1 \langle y \rangle^{-k} \leq \partial_y u^s(t, y) \leq \tilde{c}_2 \langle y \rangle^{-k}, \quad \forall (t, y) \in [0, T] \times \bar{\mathbb{R}}_+, \\ |\partial_y^p u^s(t, y)| \leq \tilde{c}_3 \langle y \rangle^{-k-p+1}, \quad \forall (t, y) \in [0, T] \times \bar{\mathbb{R}}_+, \quad 1 \leq p \leq m+4, \end{cases} \quad (2.3)$$

where  $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$  depend on  $T$ .

*Proof.* Firstly, the solution of (2.1) can be written as

$$\begin{aligned} u^s(t, y) &= \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} \left( e^{-\frac{(y-\tilde{y})^2}{4t}} - e^{-\frac{(y+\tilde{y})^2}{4t}} \right) u_0^s(\tilde{y}) d\tilde{y} \\ &= \frac{1}{\sqrt{\pi}} \left( \int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} u_0^s(2\sqrt{t}\xi + y) d\xi - \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} u_0^s(2\sqrt{t}\xi - y) d\xi \right), \end{aligned}$$

which gives

$$\begin{aligned} \partial_t u^s(t, y) &= \frac{1}{\sqrt{\pi t}} \left( \int_{-\frac{y}{2\sqrt{t}}}^{+\infty} \xi e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi + y) d\xi \right. \\ &\quad \left. - \int_{\frac{y}{2\sqrt{t}}}^{+\infty} \xi e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi - y) d\xi \right). \end{aligned}$$

By using  $(\partial_y^{2j} u_0^s)(0) = 0$  for  $0 \leq 2j \leq m+4$ , it follows

$$\begin{aligned} \partial_y^p u^s(t, y) &= \frac{1}{\sqrt{\pi}} \left( \int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} (\partial_y^p u_0^s)(2\sqrt{t}\xi + y) d\xi \right. \\ &\quad \left. + (-1)^{p+1} \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} (\partial_y^p u_0^s)(2\sqrt{t}\xi - y) d\xi \right) \\ &= \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} \left( e^{-\frac{(y-\tilde{y})^2}{4t}} + (-1)^{p+1} e^{-\frac{(y+\tilde{y})^2}{4t}} \right) (\partial_y^p u_0^s)(\tilde{y}) d\tilde{y}, \end{aligned} \quad (2.4)$$

for all  $1 \leq p \leq m+4$ .

For  $p = 1$ , we have,

$$\begin{aligned} \partial_y u^s(t, y) &= \frac{1}{\sqrt{\pi}} \left( \int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi + y) d\xi \right. \\ &\quad \left. + \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi - y) d\xi \right) \\ &= \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} \left( e^{-\frac{(y-\tilde{y})^2}{4t}} + e^{-\frac{(y+\tilde{y})^2}{4t}} \right) (\partial_y u_0^s)(\tilde{y}) d\tilde{y}. \end{aligned}$$

Thanks to the monotonic assumption (1.2), we have that

$$\begin{aligned}\partial_y u^s(t, y) &\approx \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} \left( e^{-\frac{(y-\tilde{y})^2}{4t}} + e^{-\frac{(y+\tilde{y})^2}{4t}} \right) \langle \tilde{y} \rangle^{-k} d\tilde{y} \\ &\approx \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+\tilde{y})^2}{4t}} \langle \tilde{y} \rangle^{-k} d\tilde{y}.\end{aligned}$$

Recalling now Peetre's inequality, for any  $\lambda \in \mathbb{R}$

$$\tilde{c}_0 \langle y \rangle^\lambda \langle y + \tilde{y} \rangle^{-|\lambda|} \leq \langle \tilde{y} \rangle^\lambda \leq \tilde{c}_0^{-1} \langle y \rangle^\lambda \langle y + \tilde{y} \rangle^{|\lambda|},$$

then for  $\lambda = -k$ , we get the first estimate of (2.3) with

$$\tilde{c}_1 = c_1 \tilde{c}_0 (1+T)^{-\frac{k}{2}}, \quad \tilde{c}_2 = c_2 \tilde{c}_0^{-1} (1+T)^{\frac{k}{2}}. \quad (2.5)$$

For the second estimate of (2.3), (2.4) implies

$$\begin{aligned}|\partial_y^p u^s(t, y)| &\leq \frac{c_2}{2\sqrt{\pi t}} \int_0^{+\infty} \left( e^{-\frac{(y-\tilde{y})^2}{4t}} + e^{-\frac{(y+\tilde{y})^2}{4t}} \right) \langle \tilde{y} \rangle^{-k-p+1} d\tilde{y} \\ &\leq \frac{c_2}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+\tilde{y})^2}{4t}} \langle \tilde{y} \rangle^{-k-p+1} d\tilde{y}.\end{aligned}$$

Using now Peetre's inequality, with  $\lambda = -k - p + 1$ , we get

$$|\partial_y^p u^s(t, y)| \leq c_2 \tilde{c}_0^{-1} (1+T)^{\frac{k+p-1}{2}} \langle y \rangle^{-k-p+1},$$

for any  $(t, y) \in [0, T] \times \mathbb{R}_+$ .  $\square$

**Compatibility conditions and reduction of boundary data.** We give now the precise version of the compatibility condition for the nonlinear system (2.2) and the reduction properties of boundary data.

**Proposition 2.2.** *Let  $m \geq 6$  be an even integer, and assume that  $\tilde{u}$  is a smooth solution of the system (2.2), then the initial data  $\tilde{u}_0$  have to satisfy the following compatibility conditions up to order  $m + 2$ :*

$$\begin{cases} \tilde{u}_0(x, 0) = 0, & (\partial_y^2 \tilde{u}_0)(x, 0) = 0, \quad \forall x \in \mathbb{R}, \\ (\partial_y^4 \tilde{u}_0)(x, 0) = (\partial_y u_0^s(0) + (\partial_y \tilde{u}_0)(x, 0)) (\partial_y \partial_x \tilde{u}_0)(x, 0), \quad \forall x \in \mathbb{R}, \end{cases} \quad (2.6)$$

and for  $4 \leq 2p \leq m$ ,

$$(\partial_y^{2(p+1)} \tilde{u}_0)(x, 0) = \sum_{q=2}^p \sum_{(\alpha, \beta) \in \Lambda_q} C_{\alpha, \beta} \prod_{j=1}^q \partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u_0^s + \tilde{u}_0)|_{y=0}, \quad \forall x \in \mathbb{R}, \quad (2.7)$$

where

$$\begin{aligned} \Lambda_q &= \left\{ (\alpha, \beta) = (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_q) \in \mathbb{N}^q \times \mathbb{N}^q; \right. \\ &\quad \alpha_j + \beta_j \leq 2p - 1, \quad 1 \leq j \leq q; \quad \sum_{j=1}^q 3\alpha_j + \beta_j = 2p + 1; \\ &\quad \left. \sum_{j=1}^q \beta_j \leq 2p - 2, \quad 0 < \sum_{j=1}^q \alpha_j \leq p - 1 \right\}. \end{aligned} \quad (2.8)$$

Remark that for  $\alpha_j > 0$ , we have  $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s + \tilde{u}) = \partial_x^{\alpha_j} \partial_y^{\beta_j+1} \tilde{u}$ . So the condition  $0 < \sum_{j=1}^q \alpha_j$  implies that, for each terms of (2.7), there is at last one factor like  $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} \tilde{u}_0$ .

*Proof.* By the assumption of this Proposition,  $\tilde{u}$  is a smooth solution. If we need the existence of the trace of  $\partial_y^{m+2} \tilde{u}$  on  $y = 0$ , then we at least need to assume that  $\tilde{u} \in L^\infty([0, T]; H_{k+\ell-1}^{m+3}(\mathbb{R}_+^2))$ .

Recalling the boundary condition in (2.2):

$$\tilde{u}(t, x, 0) = 0, \quad \tilde{v}(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R},$$

then the following is obvious:

$$(\partial_t \partial_x^n \tilde{u})(t, x, 0) = 0, \quad (\partial_t \partial_x^n \tilde{v})(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad 0 \leq n \leq m.$$

Thus the first result of (2.6) is exactly the compatibility of the solution with the initial data at  $t = 0$ . For the second result of (2.6), using the equation of (2.2), we find that, for  $0 \leq n \leq m$

$$(\partial_y^2 \partial_x^n \tilde{u})(t, x, 0) = 0, \quad (\partial_t \partial_y^2 \partial_x^n \tilde{u})(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Derivating the equation of (2.2) with  $y$ ,

$$\partial_t \partial_y \tilde{u} + \partial_y((u^s + \tilde{u}) \partial_x \tilde{u}) + \partial_y(\tilde{v}(u_y^s + \partial_y \tilde{u})) = \partial_y^3 \tilde{u},$$

observing

$$\left( \partial_y((u^s + \tilde{u}) \partial_x \tilde{u}) + \partial_y(\tilde{v}(u_y^s + \partial_y \tilde{u})) \right) \Big|_{y=0} = 0,$$

then we get

$$(\partial_t \partial_y \tilde{u})|_{y=0} = (\partial_y^3 \tilde{u})|_{y=0}.$$

Derivating again the equation of (2.2) with  $y$ ,

$$\partial_t \partial_y^2 \tilde{u} + \partial_y^2((u^s + \tilde{u}) \partial_x \tilde{u}) + \partial_y^2(\tilde{v}(u_y^s + \partial_y \tilde{u})) = \partial_y^4 \tilde{u},$$

using Leibniz formula

$$\begin{aligned} & \partial_y^2((u^s + \tilde{u}) \partial_x \tilde{u}) + \partial_y^2(\tilde{v}(u_y^s + \partial_y \tilde{u})) \\ &= (\partial_y^2(u^s + \tilde{u})) \partial_x \tilde{u} + (\partial_y^2 \tilde{v})(u_y^s + \partial_y \tilde{u}) \\ & \quad + (u^s + \tilde{u}) \partial_y^2 \partial_x \tilde{u} + \tilde{v} \partial_y^2 (u_y^s + \partial_y \tilde{u}) \\ & \quad + 2(\partial_y(u^s + \tilde{u})) \partial_y \partial_x \tilde{u} + 2(\partial_y \tilde{v}) \partial_y (u_y^s + \partial_y \tilde{u}), \end{aligned}$$

thus,

$$(\partial_y^4 \tilde{u})(t, x, 0) = \left( u_y^s(t, 0) + (\partial_y \tilde{u})(t, x, 0) \right) (\partial_y \partial_x \tilde{u})(t, x, 0),$$

and

$$\begin{aligned} (\partial_t \partial_y^4 \tilde{u})(t, x, 0) &= \left( \partial_y u^s(t, 0) + (\partial_y \tilde{u})(t, x, 0) \right) \left( (\partial_y^3 \partial_x \tilde{u})(t, x, 0) \right) \\ & \quad + \left( \partial_y^3 u^s(t, 0) + (\partial_y^3 \tilde{u})(t, x, 0) \right) \left( (\partial_y \partial_x \tilde{u})(t, x, 0) \right). \end{aligned} \tag{2.9}$$



For  $p = 2$ , we have

$$\partial_t \partial_y^4 \tilde{u} + \partial_y^4 \left( (u^s + \tilde{u}) \partial_x \tilde{u} \right) + \partial_y^4 \left( \tilde{v} (u_y^s + \partial_y \tilde{u}) \right) = \partial_y^6 \tilde{u},$$

using Leibniz formula

$$\begin{aligned} & \partial_y^4 \left( (u^s + \tilde{u}) \partial_x \tilde{u} \right) + \partial_y^4 \left( \tilde{v} (u_y^s + \partial_y \tilde{u}) \right) \\ &= (\partial_y^4 (u^s + \tilde{u})) \partial_x \tilde{u} + (\partial_y^4 \tilde{v}) (u_y^s + \partial_y \tilde{u}) + (u^s + \tilde{u}) \partial_y^4 \partial_x \tilde{u} + \tilde{v} \partial_y^4 (u_y^s + \partial_y \tilde{u}) \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4 \left( (\partial_y^j (u^s + \tilde{u})) \partial_y^{4-j} \partial_x \tilde{u} + (\partial_y^j \tilde{v}) \partial_y^{4-j} (u_y^s + \partial_y \tilde{u}) \right), \end{aligned}$$

thus, by (2.9)

$$\begin{aligned} (\partial_y^6 \tilde{u})(t, x, 0) &= (\partial_t \partial_y^4 \tilde{u})(t, x, 0) - (\partial_y^3 \partial_x u)(u_y^s + \partial_y \tilde{u})(t, x, 0) \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4 \left( (\partial_y^j (u^s + \tilde{u})) \partial_y^{4-j} \partial_x \tilde{u} + (\partial_y^j \tilde{v}) \partial_y^{4-j} (u_y^s + \partial_y \tilde{u}) \right)(t, x, 0) \\ &= \left( \partial_y^3 u^s(t, 0) + (\partial_y^3 \tilde{u})(t, x, 0) \right) \left( (\partial_y \partial_x \tilde{u})(t, x, 0) \right) \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4 \left( (\partial_y^j (u^s + \tilde{u})) \partial_y^{4-j} \partial_x \tilde{u} - (\partial_y^{j-1} \partial_x \tilde{u}) \partial_y^{4-j} (u_y^s + \partial_y \tilde{u}) \right)(t, x, 0). \end{aligned} \tag{2.10}$$

Taking the values at  $t = 0$ , we have proven (2.7) for  $p = 2$ . The case of  $p \geq 3$  is then by induction.  $\square$

**Remark 2.3.** *By the similar methods, we can prove that if  $\tilde{u}$  is a smooth solution of the system (2.2), then we have*

$$\begin{cases} \tilde{u}(t, x, 0) = 0, \quad (\partial_y^2 \tilde{u})(t, x, 0) = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \\ (\partial_y^4 \tilde{u})(t, x, 0) = (u_y^s(t, 0) + (\partial_y \tilde{u})(t, x, 0)) (\partial_y \partial_x \tilde{u})(t, x, 0), \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \end{cases}$$

and for  $4 \leq 2p \leq m$ ,

$$(\partial_y^{2(p+1)} \tilde{u})(t, x, 0) = \sum_{q=2}^p \sum_{(\alpha, \beta) \in \Lambda_q} C_{\alpha, \beta} \prod_{j=1}^q \partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s(t, 0) + \tilde{u}(t, x, 0)), \tag{2.11}$$

for all  $(t, x) \in [0, T] \times \mathbb{R}$ , where  $\Lambda_q$  is defined in (2.8).

See Lemma 5.9 of [20] and Lemma 4 of [7] for the similar results.

Remark that the condition  $0 < \sum_{j=1}^q \alpha_j$  implies that, for each terms of (2.11), there is at last one factor like  $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} \tilde{u}(t, x, 0)$ .

### 3. THE APPROXIMATE SOLUTIONS

To prove the existence of solution of the Prandtl equation, we study a parabolic regularized equation for which we can get the existence by using the classical energy method.

**Nonlinear regularized Prandtl equation.** We study the following nonlinear regularized Prandtl equation, for  $0 < \epsilon \leq 1$ ,

$$\begin{cases} \partial_t \tilde{u}_\epsilon + (u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon + v_\epsilon (u_y^s + \partial_y \tilde{u}_\epsilon) = \partial_y^2 \tilde{u}_\epsilon + \epsilon \partial_x^2 \tilde{u}_\epsilon, \\ \partial_x \tilde{u}_\epsilon + \partial_y v_\epsilon = 0, \\ \tilde{u}_\epsilon|_{y=0} = v_\epsilon|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} \tilde{u}_\epsilon = 0, \\ \tilde{u}_\epsilon|_{t=0} = \tilde{u}_{0,\epsilon} = \tilde{u}_0 + \epsilon \mu_\epsilon, \end{cases} \quad (3.1)$$

where we choose the corrector  $\epsilon \mu_\epsilon$  such that  $\tilde{u}_0 + \epsilon \mu_\epsilon$  satisfies the compatibility condition up to order  $m+2$  for the regularized system (3.1).

We study now the boundary data of the solution for the regularized nonlinear system (3.1) which give also the precise version of the compatibility condition for the system (3.1), see [2, 3] for the Prandtl equation with non-compatible data.

**Proposition 3.1.** *Let  $m \geq 6$  be an even integer, and assume that  $\tilde{u}_0$  satisfies the compatibility conditions (2.6) and (2.7) for the system (2.2), and  $\mu_\epsilon \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  such that  $\tilde{u}_0 + \epsilon \mu_\epsilon$  satisfies the compatibility conditions up to order  $m+2$  for the regularized system (3.1). If  $\tilde{u}_\epsilon \in L^\infty([0, T]; H_{k+\ell}^{m+3}(\mathbb{R}_+^2)) \cap Lip([0, T]; H_{k+\ell}^{m+1}(\mathbb{R}_+^2))$  is a solution of the system (3.1), then we have*

$$\begin{cases} \tilde{u}_\epsilon(t, x, 0) = 0, \quad (\partial_y^2 \tilde{u}_\epsilon)(t, x, 0) = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \\ (\partial_y^4 \tilde{u}_\epsilon)(t, x, 0) = (u_y^s(t, 0) + (\partial_y \tilde{u}_\epsilon)(t, x, 0)) (\partial_y \partial_x \tilde{u}_\epsilon)(t, x, 0), \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \end{cases}$$

and for  $4 \leq 2p \leq m$ ,

$$\begin{aligned} (\partial_y^{2(p+1)} \tilde{u}_\epsilon)(t, x, 0) &= \sum_{q=2}^p \sum_{l=0}^{q-1} \epsilon^l \sum_{(\alpha^l, \beta^l) \in \Lambda_q^l} C_{\alpha^l, \beta^l} \\ &\quad \times \prod_{j=1}^q \partial_x^{\alpha_j^l} \partial_y^{\beta_j^l+1} (u^s(t, 0) + \tilde{u}_\epsilon(t, x, 0)), \end{aligned} \quad (3.2)$$

for all  $(t, x) \in [0, T] \times \mathbb{R}$ , where

$$\begin{aligned} \Lambda_q^l &= \left\{ (\alpha, \beta) = (\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_p) \in \mathbb{N}^q \times \mathbb{N}^q; \right. \\ &\quad \alpha_j + \beta_j \leq 2p - 1, \quad , \quad 1 \leq j \leq q; \quad \sum_{j=1}^q 3\alpha_j + \beta_j = 2p + 4l + 1; \\ &\quad \left. \sum_{j=1}^q \beta_j \leq 2p - 2l - 2, \quad 0 < \sum_{j=1}^q \alpha_j \leq p + 2l - 1 \right\}. \end{aligned}$$

Remark that the condition  $0 < \sum_{j=1}^q \alpha_j^l$  implies that, for each terms of (3.2), there are at last one factor like  $\partial_x^{\alpha_j^l} \partial_y^{\beta_j^l+1} \tilde{u}_\epsilon(t, x, 0)$ .

*Proof.* Firstly, for  $p \leq \frac{m}{2}$ , we have  $\partial_y^{2p+2} \tilde{u}_\epsilon \in L^\infty([0, T]; H_{k+l+2p+1}^1(\mathbb{R}_+^2))$ . So the trace of  $\partial_y^{2p+2} \tilde{u}_\epsilon$  exists on  $y = 0$ .

Using the boundary condition of (3.1), we have, for  $0 \leq n \leq m+2$ ,

$$\partial_x^n \tilde{u}_\epsilon(t, x, 0) = 0, \quad \partial_x^n v_\epsilon(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R},$$

and for  $0 \leq n \leq m$

$$(\partial_t \partial_x^n \tilde{u}_\epsilon)(t, x, 0) = 0, \quad (\partial_t \partial_x^n v_\epsilon)(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

From the equation of (3.1), we get also

$$(\partial_y^2 \partial_x^n \tilde{u}_\epsilon)(t, x, 0) = 0, \quad (\partial_t \partial_y^2 \partial_x^n \tilde{u}_\epsilon)(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}. \quad (3.3)$$

On the other hand,

$$\partial_t \partial_y \tilde{u}_\epsilon + \partial_y ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y (v_\epsilon (u_y^s + \partial_y \tilde{u}_\epsilon)) = \partial_y^3 \tilde{u}_\epsilon + \epsilon \partial_x^2 \partial_y \tilde{u}_\epsilon,$$

observing

$$\left[ \partial_y ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y (v_\epsilon (u_y^s + \partial_y \tilde{u}_\epsilon)) \right] \Big|_{y=0} = 0,$$

we get

$$(\partial_t \partial_y \tilde{u}_\epsilon) \Big|_{y=0} = (\partial_y^3 \tilde{u}_\epsilon) \Big|_{y=0} + \epsilon (\partial_x^2 \partial_y \tilde{u}_\epsilon) \Big|_{y=0}.$$

We have also

$$\partial_t \partial_y^2 \tilde{u}_\epsilon + \partial_y^2 ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y^2 (v_\epsilon (u_y^s + \partial_y \tilde{u}_\epsilon)) = \partial_y^4 \tilde{u}_\epsilon + \epsilon \partial_x^2 \partial_y^2 \tilde{u}_\epsilon,$$

using Leibniz formula

$$\begin{aligned} & \partial_y^2 ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y^2 (v_\epsilon (u_y^s + \partial_y \tilde{u}_\epsilon)) \\ &= (\partial_y^2 (u^s + \tilde{u}_\epsilon)) \partial_x \tilde{u}_\epsilon + (\partial_y^2 v_\epsilon) (u_y^s + \partial_y \tilde{u}_\epsilon) \\ & \quad + (u^s + \tilde{u}_\epsilon) \partial_y^2 \partial_x \tilde{u}_\epsilon + v_\epsilon \partial_y^2 (u_y^s + \partial_y \tilde{u}_\epsilon) \\ & \quad + 2(\partial_y (u^s + \tilde{u}_\epsilon)) \partial_y \partial_x \tilde{u}_\epsilon + 2(\partial_y v_\epsilon) \partial_y (u_y^s + \partial_y \tilde{u}_\epsilon), \end{aligned}$$

thus,

$$(\partial_y^4 \tilde{u}_\epsilon)(t, x, 0) = (u_y^s(t, 0) + (\partial_y \tilde{u}_\epsilon)(t, x, 0)) (\partial_y \partial_x \tilde{u}_\epsilon)(t, x, 0). \quad (3.4)$$

Applying  $\partial_t$  to (3.4), we have

$$\begin{aligned} (\partial_t \partial_y^4 \tilde{u}_\epsilon)(t, x, 0) &= (\partial_y^3 u^s(t, 0) + (\partial_y^3 \tilde{u}_\epsilon)(t, x, 0) + \epsilon (\partial_x^2 \partial_y \tilde{u}_\epsilon)(t, x, 0)) (\partial_y \partial_x \tilde{u}_\epsilon)(t, x, 0) \\ & \quad + (u_y^s(t, 0) + (\partial_y \tilde{u}_\epsilon)(t, x, 0)) ((\partial_y^3 \partial_x \tilde{u}_\epsilon)(t, x, 0) + \epsilon (\partial_x^3 \partial_y \tilde{u}_\epsilon)(t, x, 0)). \end{aligned}$$

On the other hand, we have

$$\partial_t \partial_y^4 \tilde{u}_\epsilon + \partial_y^4 ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y^4 (v_\epsilon (u_y^s + \partial_y \tilde{u}_\epsilon)) = \partial_y^6 \tilde{u}_\epsilon + \epsilon \partial_x^2 \partial_y^4 \tilde{u}_\epsilon,$$

using Leibniz formula

$$\begin{aligned} & \partial_y^4 ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y^4 (v_\epsilon (u_y^s + \partial_y \tilde{u}_\epsilon)) \\ &= (\partial_y^4 (u^s + \tilde{u}_\epsilon)) \partial_x \tilde{u}_\epsilon + (\partial_y^4 v_\epsilon) (u_y^s + \partial_y \tilde{u}_\epsilon) \\ & \quad + (u^s + \tilde{u}_\epsilon) \partial_y^4 \partial_x \tilde{u}_\epsilon + v_\epsilon \partial_y^4 (u_y^s + \partial_y \tilde{u}_\epsilon) \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4 ((\partial_y^j (u^s + \tilde{u}_\epsilon)) \partial_y^{4-j} \partial_x \tilde{u}_\epsilon + (\partial_y^j v_\epsilon) \partial_y^{4-j} (u_y^s + \partial_y \tilde{u}_\epsilon)), \end{aligned}$$

thus,

$$\begin{aligned} (\partial_y^6 \tilde{u}_\epsilon)(t, x, 0) &= (\partial_t \partial_y^4 \tilde{u}_\epsilon)(t, x, 0) - (\partial_y^3 \partial_x v_\epsilon) (u_y^s + \partial_y \tilde{u}_\epsilon)(t, x, 0) \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4 [(\partial_y^j (u^s + \tilde{u}_\epsilon)) \partial_y^{4-j} \partial_x \tilde{u}_\epsilon + (\partial_y^j v_\epsilon) \partial_y^{4-j} (u_y^s + \partial_y \tilde{u}_\epsilon)](t, x, 0) \\ & \quad - \underline{\epsilon \partial_x^2 \partial_y^4 \tilde{u}_\epsilon}(t, x, 0). \end{aligned}$$

Using (3.4), we get then

$$\begin{aligned} (\partial_y^6 \tilde{u}_\epsilon)(t, x, 0) &= (\partial_y^3 u^s(t, 0) + \partial_y^3 \tilde{u}_\epsilon(t, x, 0)) \partial_y \partial_x \tilde{u}_\epsilon(t, x, 0) \\ &\quad - \underline{2\epsilon \partial_x \partial_y \tilde{u}_\epsilon(t, x, 0) (\partial_y \partial_x^2 \tilde{u}_\epsilon)(t, x, 0)} \\ &+ \sum_{1 \leq j \leq 3} C_j^4 [(\partial_y^j (u^s + \tilde{u}_\epsilon)) \partial_y^{4-j} \partial_x \tilde{u}_\epsilon - \partial_y^{j-1} \partial_x \tilde{u}_\epsilon \partial_y^{4-j} (u_y^s + \partial_y \tilde{u}_\epsilon)](t, x, 0), \end{aligned} \quad (3.5)$$

Compared to (2.10), the underlined term is the new term.

This is the Proposition 3.1 for  $p = 2$ . We can complete the proof of Proposition 3.1 by induction.  $\square$

The proof of the above Proposition implies also the following result.

**Corollary 3.2.** *Let  $m \geq 6$  be an even integer, assume that  $\tilde{u}_0$  satisfies the compatibility conditions (2.6) - (2.7) for the system (2.2) and  $\partial_y \tilde{u}_0 \in H_{k+\ell'}^{m+2}(\mathbb{R}_+^2)$ , then there exists  $\epsilon_0 > 0$ , and for any  $0 < \epsilon \leq \epsilon_0$  there exists  $\mu_\epsilon \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  such that  $\tilde{u}_0 + \epsilon \mu_\epsilon$  satisfies the compatibility condition up to order  $m + 2$  for the regularized system (3.1). Moreover, for any  $m \leq \tilde{m} \leq m + 2$*

$$\|\partial_y \tilde{u}_{0,\epsilon}\|_{H_{k+\ell'}^{\tilde{m}}(\mathbb{R}_+^2)} \leq \frac{3}{2} \|\partial_y \tilde{u}_0\|_{H_{k+\ell'}^{\tilde{m}}(\mathbb{R}_+^2)},$$

and

$$\lim_{\epsilon \rightarrow 0} \|\partial_y \tilde{u}_{0,\epsilon} - \partial_y \tilde{u}_0\|_{H_{k+\ell'}^{\tilde{m}}(\mathbb{R}_+^2)} = 0.$$

*Proof.* We use the proof of the Proposition 3.1.

Taking the values at  $t = 0$  for (3.3), then (2.6) implies that the function  $\mu_\epsilon$  satisfies

$$(\partial_x^n \mu_\epsilon)(x, 0) = 0, \quad (\partial_y^2 \partial_x^n \mu_\epsilon)(x, 0) = 0, \quad x \in \mathbb{R}.$$

Taking  $t = 0$  for (3.4), we have

$$\begin{aligned} (\partial_y^4 \tilde{u}_0)(x, 0) + \epsilon (\partial_y^4 \mu_\epsilon)(x, 0) &= [\partial_y u_0^s(0) + (\partial_y \tilde{u}_0)(x, 0) + \epsilon (\partial_y \mu_\epsilon)(x, 0)] \\ &\quad \times [(\partial_y \partial_x \tilde{u}_0)(x, 0) + \epsilon (\partial_y \partial_x \mu_\epsilon)(x, 0)], \end{aligned}$$

using (2.6), we have that  $\mu_\epsilon$  satisfies

$$\begin{aligned} (\partial_y^4 \mu_\epsilon)(x, 0) &= (\partial_y u_0^s(0) + (\partial_y \tilde{u}_0)(x, 0)) (\partial_y \partial_x \mu_\epsilon)(x, 0) \\ &\quad + (\partial_y \mu_\epsilon)(x, 0) (\partial_y \partial_x \tilde{u}_0)(x, 0) \\ &\quad + \epsilon (\partial_y \partial_x \mu_\epsilon)(x, 0) (\partial_y \partial_x \mu_\epsilon)(x, 0). \end{aligned}$$

We have also

$$\begin{aligned} (\partial_t \partial_y^4 \tilde{u}_\epsilon)(0, x, 0) &= (\partial_y^3 u_0^s(0) + (\partial_y^3 \tilde{u}_\epsilon)(0, x, 0) + \epsilon (\partial_x^2 \partial_y \tilde{u}_\epsilon)(0, x, 0)) \\ &\quad \times ((\partial_y^3 \partial_x \tilde{u}_\epsilon)(0, x, 0) + \epsilon (\partial_x^3 \partial_y \tilde{u}_\epsilon)(0, x, 0)). \end{aligned}$$

Taking the values at  $t = 0$  for (3.5), we obtain a restraint condition for  $(\partial_y^6 \mu_\epsilon)(x, 0)$ ,

$$\begin{aligned} \partial_y^6 \mu_\epsilon(x, 0) &= ((\partial_y^3 u_0^s + \partial_y^3 \tilde{u}_0) \partial_y \partial_x \mu_\epsilon)|_{y=0} + \partial_y^3 \mu_\epsilon \partial_y \partial_x \tilde{u}_0|_{y=0} + \epsilon \partial_y^3 \mu_\epsilon \partial_y \partial_x \mu_\epsilon|_{y=0} \\ &\quad - \underline{2 \partial_x \partial_y \tilde{u}_0(x, 0) (\partial_y \partial_x^2 \tilde{u}_0)(x, 0)} - 2\epsilon \partial_x \partial_y \tilde{u}_0(x, 0) (\partial_y \partial_x^2 \mu_\epsilon)(x, 0) \\ &\quad - 2\epsilon \partial_x \partial_y \mu_\epsilon(t, x, 0) (\partial_y \partial_x^2 \tilde{u}_0)(t, x, 0) - 2\epsilon^2 \partial_x \partial_y \mu_\epsilon(x, 0) (\partial_y \partial_x^2 \mu_\epsilon)(x, 0) \\ &\quad + \sum_{1 \leq j \leq 3} C_j^4 [\partial_y^j (u_0^s + \tilde{u}_0) \partial_y^{4-j} \partial_x \mu_\epsilon + \partial_y^j \mu_\epsilon \partial_y^{4-j} \partial_x \tilde{u}_0 + \epsilon \partial_y^j \mu_\epsilon \partial_y^{4-j} \partial_x \mu_\epsilon] |_{y=0} \end{aligned}$$

$$\begin{aligned}
& - \sum_{1 \leq j \leq 3} C_j^4 [\partial_y^{j-1} \partial_x \tilde{u}_0 \partial_y^{4-j} \mu_\epsilon + \epsilon \partial_y^{j-1} \partial_x \mu_\epsilon \partial_y^{4-j} \partial_y \mu_\epsilon] \Big|_{y=0} \\
& - \sum_{1 \leq j \leq 3} C_j^4 \partial_y^{j-1} \partial_x \mu_\epsilon \partial_y^{4-j} (\partial_y u_0^s + \partial_y \tilde{u}_0) \Big|_{y=0},
\end{aligned}$$

thus

$$\begin{aligned}
\partial_y^6 \mu_\epsilon(x, 0) &= - \frac{2 \partial_x \partial_y \tilde{u}_0(x, 0) (\partial_y \partial_x^2 \tilde{u}_0)(x, 0)}{\phantom{2 \partial_x \partial_y \tilde{u}_0(x, 0) (\partial_y \partial_x^2 \tilde{u}_0)(x, 0)}} \\
&+ \sum_{\alpha_1, \beta_1; \alpha_2, \beta_2} C_{\alpha_1, \beta_1; \alpha_2, \beta_2} \partial_x^{\alpha_1} \partial_y^{\beta_1+1} (u_0^s + \tilde{u}_0) \partial_x^{\alpha_1} \partial_y^{\beta_1+1} \mu_\epsilon(x, 0) \\
&+ \sum_{\alpha_1, \beta_1; \alpha_2, \beta_2} C_{\alpha_1, \beta_1; \alpha_2, \beta_2} \partial_x^{\alpha_1} \partial_y^{\beta_1+1} \mu_\epsilon \partial_x^{\alpha_1} \partial_y^{\beta_1+1} \mu_\epsilon(x, 0),
\end{aligned} \tag{3.6}$$

where the summation is for the index  $\alpha_2 + \beta_2 \leq 3$ ;  $\alpha_1 + \beta_1 + \alpha_2 + \beta_2 \leq 3$ . The underlined term in the above equality is deduced from the underlined term in (3.5). All these underlined terms are from the added regularizing term  $\epsilon \partial_x^2 \tilde{u}$  in the equation (3.1). This means that the regularizing term  $\epsilon \partial_x^2 \tilde{u}$  has an affect on the boundary. This is why we add a corrector term.

More generally, for  $6 \leq 2p \leq m$ , we have that  $(\partial_y^{2(p+1)} \mu_\epsilon)(x, 0)$  is a linear combination of the terms of the form

$$\prod_{j=1}^{q_1} \left( \partial_x^{\alpha_j^1} \partial_y^{\beta_j^1+1} (u_0^s + \tilde{u}_0) \right) \Big|_{y=0}, \quad \prod_{i=1}^{q_2} \left( \partial_x^{\alpha_i^2} \partial_y^{\beta_i^2+1} \mu_\epsilon \right) \Big|_{y=0},$$

and

$$\prod_{j=1}^{q_1} \left( \partial_x^{\alpha_j^1} \partial_y^{\beta_j^1+1} (u_0^s + \tilde{u}_0) \right) \Big|_{y=0} \times \prod_{i=1}^{q_2} \left( \partial_x^{\alpha_i^2} \partial_y^{\beta_i^2+1} \mu_\epsilon \right) \Big|_{y=0},$$

where the coefficients of the combination can be depends on  $\epsilon$  but with a non-negative power. We have also  $\alpha_j^l + \beta_j^l + 1 \leq 2p, l = 1, 2$ , thus  $(\partial_y^{2(p+1)} \mu_\epsilon)(x, 0)$  is determined by the low order derivatives of  $\mu_\epsilon$  and these of  $\tilde{u}_0$ .

We now construct a polynomial function  $\tilde{\mu}_\epsilon$  on  $y$  by the following Taylor expansion,

$$\tilde{\mu}_\epsilon(x, y) = \sum_{p=3}^{\frac{m}{2}+1} \tilde{\mu}_\epsilon^{2p}(x) \frac{y^{2p}}{(2p)!},$$

where

$$\tilde{\mu}_\epsilon^6(x) = -2(\partial_x \partial_y \tilde{u}_0)(x, 0) (\partial_y \partial_x^2 \tilde{u}_0)(x, 0),$$

and  $\tilde{\mu}_\epsilon^{2p}(x)$  will give successively by  $(\partial_y^{2q} \mu_\epsilon)(x, 0)$  with  $(\partial_y^{2q+1} \mu_\epsilon)(x, 0) = 0, q = 0, \dots, m$ , and it is then determined by  $(\partial_x^\alpha \partial_y^\beta \tilde{u}_0)|_{y=0}$ . Finally we take  $\mu_\epsilon = \chi(y) \tilde{\mu}_\epsilon$  with  $\chi \in C^\infty([0, +\infty[); \chi(y) = 1, 0 \leq y \leq 1; \chi(y) = 0, y \geq 2$ . Thus we complete the proof of the Corollary.  $\square$

**Remark 3.3.** Suppose that  $\tilde{u}_0$  satisfies the compatibility conditions up to order  $m+2$  for the system (2.2) with  $m \geq 4$ , then for the regularized system (3.1), if we want to obtain the smooth solution  $\tilde{w}_\epsilon$ , we have to add a non-trivial corrector  $\mu_\epsilon$  to the initial data such that  $\tilde{u}_0 + \epsilon \mu_\epsilon$  satisfies the compatibility conditions up to order  $m+2$  for the system (3.1). In fact, if we take  $\mu_\epsilon$  with

$$(\partial_y^j \mu_\epsilon)(x, 0) = 0, \quad 0 \leq j \leq 5,$$

then (3.6) implies

$$(\partial_y^6 \mu_\epsilon)(x, 0) = -2(\partial_x \partial_y \tilde{u}_0)(x, 0)(\partial_y \partial_x^2 \tilde{u}_0)(x, 0),$$

which is not equal to 0. So added a corrector is necessary for the initial data of the regularized system.

We will prove the the existence of the approximate solutions of the system (3.1) by using the following equation of vorticity  $\tilde{w}_\epsilon = \partial_y \tilde{u}_\epsilon$ , it reads

$$\begin{cases} \partial_t \tilde{w}_\epsilon + (u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon + v_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon) = \partial_y^2 \tilde{w}_\epsilon + \epsilon \partial_x^2 \tilde{w}_\epsilon, \\ \partial_y \tilde{w}_\epsilon|_{y=0} = 0, \\ \tilde{w}_\epsilon|_{t=0} = \tilde{w}_{0,\epsilon} = \tilde{w}_0 + \epsilon \partial_y \mu_\epsilon, \end{cases} \quad (3.7)$$

where

$$\tilde{u}_\epsilon(t, x, y) = - \int_y^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y}, \quad \tilde{v}_\epsilon(t, x, y) = - \int_0^y \partial_x \tilde{u}_\epsilon(t, x, \tilde{y}) d\tilde{y}. \quad (3.8)$$

We have the following theorem for the existence of approximate solutions

**Theorem 3.4.** *Let  $\partial_y \tilde{u}_0 \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$ , and  $m \geq 6$  be an even integer,  $k > 1, 0 \leq \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ , assume that  $\tilde{u}_0$  satisfies the compatibility conditions of order  $m+2$  for the system (2.2). Suppose that the shear flow satisfies*

$$|\partial_y^{p+1} u^s(t, y)| \leq C \langle y \rangle^{-k-p}, \quad (t, y) \in [0, T_1] \times \mathbb{R}_+, \quad 0 \leq p \leq m+2.$$

Then, for any  $0 < \epsilon \leq \epsilon_0$  and  $0 < \bar{\zeta}$ , there exists  $T_\epsilon > 0$  which depends on  $\epsilon$  and  $\bar{\zeta}$ , such that if

$$\|\tilde{w}_0\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \bar{\zeta},$$

then the system (3.7)-(3.8) admits a unique solution

$$\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2)),$$

which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \frac{4}{3} \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq 2 \|\tilde{w}_0\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}. \quad (3.9)$$

**Remark 3.5.** .

- (1) Remark that  $T_\epsilon$  depends on  $\epsilon$  and  $\bar{\zeta}$ , and  $T_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ . So this is not a bounded estimate for the approximate solution sequences  $\{u^s + \tilde{u}_\epsilon; 0 < \epsilon \leq \epsilon_0\}$  where  $\epsilon_0 > 0$  is given in Corollary 3.2. When the initial data  $\tilde{u}_0$  is small enough, we observe that  $u^s + \tilde{u}_\epsilon$  preserves the monotonicity and convexity of the shear flow on  $[0, T_\epsilon]$ .
- (2) In this theorem, for the regularized Prandtl equation, there are not constrain conditions on the initial date, meaning that we don't need the monotonicity or convexity of shear flow  $u^s$ , and  $\bar{\zeta}$  is also arbitrary.

If  $\tilde{w}_\epsilon$  is a solution of the system (3.7)-(3.8), then (A.1) with  $\lim_{y \rightarrow +\infty} \tilde{u}_\epsilon = 0$  imply

$$\tilde{u}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell-1}^{m+2}(\mathbb{R}_+^2)),$$

and

$$\tilde{v}_\epsilon \in L^\infty([0, T_\epsilon]; L^\infty(\mathbb{R}_{y,+}; H^{m+1}(\mathbb{R}_x))).$$

Integrating the equation of (3.7) over  $[y, +\infty[$  imply that  $(\tilde{u}_\epsilon, \tilde{v}_\epsilon)$  is a solution of the system (3.1), except the boundary condition to check:

$$\tilde{u}_\epsilon(t, x, 0) = - \int_0^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y} = 0, \quad (t, x) \in [0, T_\epsilon] \times \mathbb{R}. \quad (3.10)$$

In fact, noting  $f(t, x) = - \int_0^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y} = \tilde{u}_\epsilon(t, x, 0)$ , a direct calculate give

$$\begin{cases} \partial_t f + f \partial_x f = \epsilon \partial_x^2 f, & (t, x) \in ]0, T_\epsilon] \times \mathbb{R}; \\ f|_{t=0} = 0, \end{cases} \quad (3.11)$$

here we use

$$\begin{aligned} \int_0^{+\infty} v_\epsilon(u_{yy}^s + \partial_y \tilde{w}_\epsilon) dy &= [v_\epsilon(u_y^s + \tilde{w}_\epsilon)]_0^{+\infty} - \int_0^{+\infty} (\partial_y v_\epsilon)(u_y^s + \tilde{w}_\epsilon) dy \\ &= \int_0^{+\infty} (\partial_x u_\epsilon) \partial_y (u^s + \tilde{u}_\epsilon) dy \\ &= [(\partial_x u_\epsilon)(u^s + \tilde{u}_\epsilon)]_0^{+\infty} - \int_0^{+\infty} (\partial_x w_\epsilon)(u^s + \tilde{u}_\epsilon) dy \\ &= -f \partial_x f - \int_0^{+\infty} (\partial_x w_\epsilon)(u^s + \tilde{u}_\epsilon) dy. \end{aligned}$$

Since  $f \in L^\infty([0, T_\epsilon], H^{m+2}(\mathbb{R}))$ , the uniqueness of solution for equation (3.11) imply that  $f = 0$  on  $[0, T_\epsilon] \times \mathbb{R}$ . (3.10) imply also

$$\tilde{u}_\epsilon(t, x, y) = - \int_y^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y} = \int_0^y \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y}, \quad (t, x, y) \in [0, T_\epsilon] \times \mathbb{R}_+^2.$$

We will prove Theorem 3.4 by the following three Propositions, where the first one is devoted to the local existence of approximate solution  $\tilde{w}_\epsilon$  of (3.7).

**Proposition 3.6.** *Let  $\tilde{w}_{0,\epsilon} \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$ ,  $m \geq 6$  be an even integer,  $k > 1, 0 \leq \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ , and satisfy the compatibility conditions up to order  $m + 2$  for (3.7). Suppose that the shear flow satisfies*

$$|\partial_y^{p+1} u^s(t, y)| \leq C \langle y \rangle^{-k-p}, \quad (t, y) \in [0, T_1] \times \mathbb{R}_+, \quad 0 \leq p \leq m + 2.$$

Then, for any  $0 < \epsilon \leq 1$  and  $\bar{\zeta} > 0$ , there exists  $T_\epsilon > 0$  such that if

$$\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \bar{\zeta},$$

then the system (3.7) admits a unique solution

$$\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2)).$$

**Remark 3.7.** *If  $\tilde{w}_0 \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$  is the initial data in Theorem 3.4, using Corollary 3.2, there exists  $\epsilon_0 > 0$ , and for any  $0 < \epsilon \leq \epsilon_0$ , there exists  $\mu_\epsilon \in H_{k+\ell}^{m+3}(\mathbb{R}_+^2)$  such that  $\tilde{w}_{0,\epsilon} = \tilde{w}_0 + \epsilon \partial_y \mu_\epsilon$  satisfies the compatibility conditions up to order  $m + 2$  for the system (3.7), and*

$$\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \frac{3}{2} \|\tilde{w}_0\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}.$$

Then, using Proposition 3.6, we obtain also the existence of the approximate solution under the assumption of Theorem 3.4.

The proof of this Proposition is standard since the equation in (3.7) is a parabolic type equation. Firstly, we establish the *à priori* estimate and then prove the existence of solution by the standard iteration and weak convergence methods. Because we work in the weighted Sobolev space and the computation is not so trivial, we give a detailed proof in the Appendix B, to make the paper self-contained. So the rest of this section is devoted to proving the estimate (3.9).

**Uniform estimate with loss of  $x$ -derivative** In the proof of the Proposition 3.6 (see Lemma B.2), we already get the *à priori* estimate for  $\tilde{w}_\epsilon$ . Now we try to prove the estimate (3.9) in a new way, and our object is to establish an uniform estimate with respect to  $\epsilon > 0$ . We first treat the easy part in this subsection.

We define the non-isotropic Sobolev norm,

$$\|f\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 = \sum_{|\alpha_1+\alpha_2|\leq m, \alpha_1\leq m-1} \|\langle y \rangle^{k+\ell+\alpha_2} \partial_x^{\alpha_1} \partial_y^{\alpha_2} f\|_{L^2(\mathbb{R}_+^2)}^2, \quad (3.12)$$

where we don't have the  $m$ -order derivative with respect to  $x$ -variable. Then

$$\|f\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 = \|f\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + \|\partial_x^m f\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2.$$

**Proposition 3.8.** *Let  $m \geq 6$  be an even integer,  $k > 1, 0 \leq \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ , and assume that  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$  is a solution to (3.7), then we have*

$$\begin{aligned} & \frac{d}{dt} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \\ & + \epsilon \|\partial_x \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \leq C_1 \left( \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^m \right), \end{aligned} \quad (3.13)$$

where  $C_1 > 0$  is independent of  $\epsilon$ .

**Remark.** The above estimate is uniform with respect to  $\epsilon > 0$ , but on the left hand of (3.13), we missing the terms  $\|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2$ . This is because that we can't control the term

$$\partial_x^m \tilde{v}_\epsilon(t, x, y) = - \int_0^y \partial_x^{m+1} \tilde{u}_\epsilon(t, x, \tilde{y}) d\tilde{y},$$

which is the major difficulty in the study of the Prandtl equation. We will study this term in the next Proposition with a non-uniform estimate firstly, and then focus on proving the uniform estimate in the rest part of this paper.

*Proof.* For  $|\alpha| = \alpha_1 + \alpha_2 \leq m, \alpha_1 \leq m - 1$ , we have

$$\begin{aligned} & \partial_t \partial^\alpha \tilde{w}_\epsilon - \epsilon \partial_x^2 \partial^\alpha \tilde{w}_\epsilon - \partial_y^2 \partial^\alpha \tilde{w}_\epsilon \\ & = -\partial^\alpha ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon) - \partial^\alpha (\tilde{v}_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon)). \end{aligned} \quad (3.14)$$

Multiplying the (3.14) with  $\langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \tilde{w}_\epsilon$ , and integrating over  $\mathbb{R}_+^2$ ,

$$\begin{aligned} & \int_{\mathbb{R}_+^2} (\partial_t \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy - \epsilon \int_{\mathbb{R}_+^2} (\partial_x^2 \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy \\ & - \int_{\mathbb{R}_+^2} (\partial_y^2 \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy \\ & = - \int_{\mathbb{R}_+^2} \partial^\alpha ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon - \tilde{v}_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon)) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy. \end{aligned}$$



Remark that for  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ , all above integrations are in the classical sense. We deal with each term on the left hand respectively. After integration by part, we have

$$\begin{aligned} & \int_{\mathbb{R}_+^2} (\partial_t \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy = \frac{1}{2} \frac{d}{dt} \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)}^2, \\ & - \epsilon \int_{\mathbb{R}_+^2} (\partial_x^2 \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy = \epsilon \|\partial_x \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)}^2, \end{aligned}$$

and

$$\begin{aligned} & - \int_{\mathbb{R}_+^2} \partial_y^2 \partial^\alpha \tilde{w}_\epsilon \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy \\ & = \|\partial_y \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \partial^\alpha \partial_y \tilde{w}_\epsilon (\langle y \rangle^{2(k+\ell)+2\alpha_2})' \partial^\alpha \tilde{w}_\epsilon dx dy \\ & \quad + \int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) \Big|_{y=0} dx. \end{aligned}$$

Cauchy-Schwarz inequality implies

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \partial^\alpha \partial_y \tilde{w}_\epsilon (\langle y \rangle^{2(k+\ell)+2\alpha_2})' \partial^\alpha \tilde{w}_\epsilon dx dy \right| \\ & \leq \frac{1}{16} \|\partial_y \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)}^2 + C \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2-1}^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

We study now the term

$$\int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) \Big|_{y=0} dx.$$

**Case :**  $|\alpha| \leq m-1$ , using the trace Lemma A.2, we have

$$\begin{aligned} \left| \int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) \Big|_{y=0} dx \right| & \leq \|(\partial^\alpha \partial_y \tilde{w}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \|(\partial^\alpha \tilde{w}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \\ & \leq C \|\partial^\alpha \partial_y^2 \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \|\partial^\alpha \partial_y \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \\ & \leq C \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \\ & \leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2. \end{aligned}$$

**Case :**  $\alpha_1 = m-1, \alpha_2 = 1$ , using (3.3), we have

$$(\partial^\alpha \tilde{w}_\epsilon) \Big|_{y=0} = (\partial_x^{\alpha_1} \partial_y^2 \tilde{u}_\epsilon) \Big|_{y=0} = 0,$$

thus

$$\int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) \Big|_{y=0} dx = 0.$$

**Case :**  $\alpha_1 = 0, \alpha_2 = m$ . Only in this case, we need to suppose that  $m$  is even. Using again the trace Lemma A.2, we have

$$\begin{aligned} \left| \int_{\mathbb{R}} (\partial_y^{m+1} \tilde{w}_\epsilon \partial_y^m \tilde{w}_\epsilon) \Big|_{y=0} dx \right| & \leq \|(\partial_y^{m+2} \tilde{u}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \|(\partial_y^m \tilde{w}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \\ & \leq C \|(\partial_y^{m+2} \tilde{u}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \|\partial_y^{m+1} \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \\ & \leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|(\partial_y^{m+2} \tilde{u}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Using Proposition 3.1 and the trace Lemma A.2, we can estimate the above last term  $\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}^2$  by a finite summation of the following forms

$$\left\| \prod_{j=1}^p (\partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s + \tilde{u}_\epsilon)) \Big|_{y=0} \right\|_{L^2(\mathbb{R})}^2 \leq C \left\| \partial_y \prod_{j=1}^p (\partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s + \tilde{u}_\epsilon)) \right\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+^2)}^2$$

with  $2 \leq p \leq \frac{m}{2}$ ,  $\alpha_j + \beta_j \leq m - 1$  and  $\{j; \alpha_j > 0\} \neq \emptyset$ . Then using Sobolev inequality and  $m \geq 6$ , we get

$$\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})} \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^{m/2}.$$

**Case :**  $1 \leq \alpha_1 \leq m - 2$ ,  $\alpha_1 + \alpha_2 = m$ ,  $\alpha_2$  **even**, using the same argument to the precedent case, we have

$$\begin{aligned} \left| \int_{\mathbb{R}} (\partial_x^{\alpha_1} \partial_y \tilde{w}_\epsilon \partial_x^{\alpha_2} \tilde{w}_\epsilon) \Big|_{y=0} dx \right| &= \left| \int_{\mathbb{R}} (\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon \partial_x^{\alpha_1} \partial_y^{\alpha_2} \tilde{w}_\epsilon) \Big|_{y=0} dx \right| \\ &\leq \|(\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \|(\partial_x^{\alpha_1} \partial_y^{\alpha_2} \tilde{w}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|(\partial_x^{\alpha_1} \partial_y^{\alpha_2+2} \tilde{u}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^{\alpha_2}. \end{aligned}$$

**Case :**  $1 \leq \alpha_1 \leq m - 2$ ,  $\alpha_1 + \alpha_2 = m$ ,  $\alpha_2$  **odd**, integration by part with respect to  $x$  variable implies

$$\begin{aligned} \left| \int_{\mathbb{R}} (\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon \partial_x^{\alpha_1} \partial_y^{\alpha_2} \tilde{w}_\epsilon) \Big|_{y=0} dx \right| &= \left| \int_{\mathbb{R}} (\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon \partial_x^{\alpha_1+1} \partial_y^{\alpha_2} \tilde{w}_\epsilon) \Big|_{y=0} dx \right| \\ &\leq \|(\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \|(\partial_x^{\alpha_1+1} \partial_y^{\alpha_2} \tilde{w}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|(\partial_x^{\alpha_1+1} \partial_y^{\alpha_2+1} \tilde{u}_\epsilon) \Big|_{y=0}\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^{\alpha_2-1}. \end{aligned}$$

Finally, we have proven

$$\begin{aligned} &\int_{\mathbb{R}_+^2} (\partial_t \partial^\alpha \tilde{w}_\epsilon - \partial_y^2 \partial^\alpha \tilde{w}_\epsilon - \epsilon \partial_x^2 \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \tilde{w}_\epsilon dx dy \\ &\geq \frac{1}{2} \frac{d}{dt} \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2 + \epsilon \|\partial_x \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2 + \|\partial_y \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2 \\ &\quad - \frac{1}{4} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 - C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^m. \end{aligned}$$

We estimate now the right hand of (3.14). For the first item, we need to split it into two parts

$$-\partial^\alpha ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon) = -(u^s + \tilde{u}_\epsilon) \partial_x \partial^\alpha \tilde{w}_\epsilon + [(u^s + \tilde{u}_\epsilon), \partial^\alpha] \partial_x \tilde{w}_\epsilon.$$

Firstly, we have

$$\int_{\mathbb{R}_+^2} ((u^s + \tilde{u}_\epsilon) \partial_x \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \tilde{w}_\epsilon dx dy \leq \|\partial_x \tilde{u}_\epsilon\|_{L^\infty} \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2,$$

then using (A.2), we get

$$\left| \int_{\mathbb{R}_+^2} ((u^s + \tilde{u}_\epsilon) \partial_x \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(\ell+\alpha_2)} \partial^\alpha \tilde{w}_\epsilon dx dy \right| \leq \|\tilde{w}_\epsilon\|_{H_1^3} \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}.$$

For the commutator operator, in fact, it can be written as

$$[(u^s + \tilde{u}_\epsilon), \partial^\alpha] \partial_x \tilde{w}_\epsilon = \sum_{\beta \leq \alpha, 1 \leq |\beta|} C_\alpha^\beta \partial^\beta (u^s + \tilde{u}_\epsilon) \partial^{\alpha-\beta} \partial_x \tilde{w}_\epsilon.$$

Then for  $|\alpha| \leq m, m \geq 4$ , using the Sobolev inequality again and Lemma A.1,

$$\|[(u^s + \tilde{u}), \partial^\alpha] \partial_x \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2} \leq C(\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m} + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2).$$

Thus

$$\left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2(k+\ell+\alpha_2)} ([(u^s + \tilde{u}_\epsilon), \partial^\alpha] \partial_x \tilde{w}_\epsilon) \cdot \partial^\alpha \tilde{w}_\epsilon dx dy \right| \leq C(\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3),$$

and

$$\left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2(k+\ell+\alpha_2)} (\partial^\alpha ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon)) \partial^\alpha \tilde{w}_\epsilon dx dy \right| \leq C(\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3),$$

where  $C$  is independent of  $\epsilon$ .

For the next one, similar to the first term in (3.14), we have

$$\partial^\alpha (\tilde{v}_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon)) = \tilde{v}_\epsilon \partial_y \partial^\alpha \tilde{w}_\epsilon - [\tilde{v}_\epsilon, \partial^\alpha] \partial_y \tilde{w}_\epsilon + \partial^\alpha (\tilde{v}_\epsilon u_{yy}^s).$$

Then

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} \tilde{v}_\epsilon \langle y \rangle^{2(k+\ell+\alpha_2)} (\partial_y \partial^\alpha \tilde{w}_\epsilon) \cdot \partial^\alpha \tilde{w}_\epsilon dx dy \right| &\leq \|\tilde{v}_\epsilon\|_{L^\infty(\mathbb{R}_+^2)} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^m} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m} \\ &\leq \frac{1}{4} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 + C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^4 \end{aligned}$$

where we have used

$$\begin{aligned} \|\tilde{v}_\epsilon\|_{L^\infty(\mathbb{R}_+^2)} &\leq C \|\partial_x \tilde{u}_\epsilon\|_{L^\infty(\mathbb{R}_x; L_{\frac{1}{2}+\delta}^2(\mathbb{R}_{y,+}))} \\ &\leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{1+2\delta} (|\partial_x \tilde{u}_\epsilon|^2 + |\partial_x^2 \tilde{u}_\epsilon|^2) dx dy \\ &\leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{3+2\delta} (|\partial_x \tilde{w}_\epsilon|^2 + |\partial_x^2 \tilde{w}_\epsilon|^2) dx dy \leq C \|\tilde{w}_\epsilon\|_{H_{\frac{3}{2}+\delta}^2}, \end{aligned}$$

where  $\delta > 0$  is small.

Noticing that

$$[\tilde{v}_\epsilon, \partial^\alpha] \partial_y \tilde{w}_\epsilon = \sum_{\beta \leq \alpha, 1 \leq |\beta|} C_\alpha^\beta \partial^\beta \tilde{v}_\epsilon \partial^{\alpha-\beta} \partial_y \tilde{w}_\epsilon.$$

Since  $H_\ell^m$  is an algebra for  $m \geq 6$ , we only need to pay attention to the order of derivative in the above formula. Firstly for  $|\beta| \geq 1$ , we have for  $|\alpha - \beta| + 1 \leq m$ ,

$$-\partial^\beta \tilde{v}_\epsilon = \partial_x^{\beta_1} \partial_y^{\beta_2} \int_0^y \tilde{u}_{\epsilon,x} d\tilde{y} = \begin{cases} \partial_x^{\beta_1+1} \partial_y^{\beta_2-1} \tilde{u}_\epsilon, & \beta_2 \geq 1, \\ \int_0^y \partial_x^{\beta_1+1} \tilde{u}_\epsilon d\tilde{y}, & \beta_2 = 0. \end{cases}$$

Now using the hypothesis  $\beta \leq \alpha, 1 \leq |\beta|$  and  $\beta_1 \leq \alpha_1 \leq m - 1$ , using Lemma A.1, we get

$$\|[\tilde{v}_\epsilon, \partial^\alpha] \partial_y \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2} \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2.$$

On the other hand, if  $\alpha_2 = 0$ , using  $-1 + \ell < -\frac{1}{2}$ , we can get

$$\|\partial_x^{m-1}(\tilde{v}_\epsilon u_{yy}^s)\|_{L_{k+\ell}^2} \leq C \|\partial_x^m \tilde{u}_\epsilon\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} \|u_{yy}^s\|_{L_{k+\ell}^2(\mathbb{R}_+)} \leq C \|\tilde{w}_\epsilon\|_{H_{\frac{3}{2}+\delta}^m}.$$

Similar computation for other cases, we can get, for  $\alpha_2 > 0, \alpha_1 + \alpha_2 \leq m$ ,

$$\|\partial^\alpha(\tilde{v}_\epsilon u_{yy}^s)\|_{L_{k+\ell+\alpha_2}^2} \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}.$$

Combining the above estimates, we have finished the proof of the Proposition 3.8.  $\square$

**Smallness of approximate solutions.** To close the energy estimate, we still need to estimate the term  $\partial_x^m \tilde{w}_\epsilon$ .

**Proposition 3.9.** *Under the hypothesis of Theorem 3.4, and with the same notations as in Proposition 3.8, we have*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \frac{3\epsilon}{4} \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \frac{3}{4} \|\partial_y \partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 \\ & \leq C (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3) + \frac{32}{\epsilon} (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^4 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2). \end{aligned} \quad (3.15)$$

*Proof.* We have

$$\partial_t \partial_x^m \tilde{w}_\epsilon - \partial_y^2 \partial_x^m \tilde{w}_\epsilon - \epsilon \partial_x^m \partial_x^2 \tilde{w}_\epsilon = -\partial_x^m ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon) - \partial_x^m (\tilde{v}_\epsilon (\partial_y \tilde{w}_\epsilon + u_{yy}^s)),$$

then the same computations as in Proposition 3.8 give

$$\begin{aligned} & \frac{d}{2dt} \|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \epsilon \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \frac{3}{4} \|\partial_y \partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 \\ & \leq C (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3) \\ & + \left| \int_{\mathbb{R}_+^2} \partial_x^m (\tilde{v}_\epsilon (\partial_y \tilde{w}_\epsilon + u_{yy}^s)) \langle y \rangle^{2(k+\ell)} \partial_x^m \tilde{w}_\epsilon dx dy \right|, \end{aligned} \quad (3.16)$$

where the boundary terms is more easy to control, since

$$(\partial_y \partial_x^m \tilde{w}_\epsilon)(t, x, 0) = (\partial_y^2 \partial_x^m \tilde{u}_\epsilon)(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

The estimate of the last term on right hand is the main obstacle for the study of the Prandtl equations.

$$\begin{aligned} \partial_x^m (\tilde{v}_\epsilon (\partial_y \tilde{w}_\epsilon + u_{yy}^s)) &= \tilde{v}_\epsilon \partial_x^m \partial_y \tilde{w}_\epsilon + (\partial_x^m \tilde{v}_\epsilon) (\partial_y \tilde{w}_\epsilon + u_{yy}^s) \\ &+ \sum_{1 \leq j \leq m-1} C_m^j \partial_x^j \tilde{v}_\epsilon \partial_x^{m-j} \partial_y \tilde{w}_\epsilon. \end{aligned}$$

For the first term

$$\begin{aligned} \int_{\mathbb{R}_+^2} \tilde{v}_\epsilon (\partial_x^m \partial_y \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)} (\partial_x^m \tilde{w}_\epsilon) dx dy &= \frac{1}{2} \int \tilde{v}_\epsilon \langle y \rangle^{2(k+\ell)} \partial_y (\partial_x^m \tilde{w}_\epsilon)^2 dx dy \\ &= \frac{1}{2} \int \tilde{u}_{\epsilon,x} \langle y \rangle^{2(k+\ell)} (\partial_x^m \tilde{w}_\epsilon)^2 dx dy \\ &- \ell \int \tilde{v}_\epsilon \langle y \rangle^{2(k+\ell)-1} (\partial_x^m \tilde{w}_\epsilon)^2 dx dy \\ &\leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3, \end{aligned}$$

where we have used  $\tilde{v}_\epsilon|_{y=0} = 0$ , and

$$\left| \int_{\mathbb{R}_+^2} \left( \sum_{1 \leq j \leq m-1} C_m^j \partial_x^j \tilde{v}_\epsilon \partial_x^{m-j} \partial_y \tilde{w}_\epsilon \langle y \rangle^{2(k+\ell)} (\partial_x^m \tilde{w}_\epsilon) \right) dx dy \right| \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3.$$

Finally for the worst term, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} (\partial_x^m \tilde{v}_\epsilon) (\partial_y \tilde{w}_\epsilon + u_{yy}^s) \langle y \rangle^{2(k+\ell)} (\partial_x^m \tilde{w}_\epsilon) dx dy \right| \\ & \leq C \|\partial_x^m \tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} \|\partial_y \tilde{w}_\epsilon\|_{L^\infty(\mathbb{R}_x; L_{k+\ell}^2(\mathbb{R}_+))} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m} \\ & \quad + \|\partial_x^m \tilde{v}_\epsilon u_{yy}^s\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}. \end{aligned}$$

On the other hand, observing

$$\partial_x^m \tilde{v}_\epsilon(t, x, y) = - \int_0^y \partial_x^{m+1} \tilde{u}_\epsilon(t, x, \tilde{y}) d\tilde{y},$$

then using Sobolev inequality and Lemma A.1, for  $\delta > 0$  small,

$$\|\partial_x^m \tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} \leq C \|\partial_x^{m+1} \tilde{u}_\epsilon\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} \leq C \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{\frac{3}{2}+\delta}^2(\mathbb{R}_+^2)},$$

we get

$$\|\partial_x^m \tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} \leq C \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{\frac{3}{2}+\delta}^2(\mathbb{R}_+^2)}.$$

Using the hypothesis for the shear flow  $u^s$  and  $\ell - 1 < -\frac{1}{2}$ ,

$$\begin{aligned} \|\partial_x^m (\tilde{v}_\epsilon u_{yy}^s)\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} & \leq \|\partial_x^m \tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} \|u_{yy}^s\|_{L_{k+\ell}^2(\mathbb{R}_+)} \\ & \leq C \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{\frac{3}{2}+\delta}^2(\mathbb{R}_+^2)}, \end{aligned}$$

and for  $k + \ell \geq \frac{3}{2} + \delta$ ,

$$\|\partial_y \tilde{w}_\epsilon\|_{L^\infty(\mathbb{R}_x; L_{k+\ell}^2(\mathbb{R}_+))} \leq C \|\partial_y \tilde{w}_\epsilon\|_{H^1(\mathbb{R}_x; L_{k+\ell}^2(\mathbb{R}_+))} \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}.$$

Thus, we have

$$\begin{aligned} & \int (\partial_x^m (\tilde{v}_\epsilon (\partial_y \tilde{w}_\epsilon + u_{yy}^s))) \langle y \rangle^{2(k+\ell)} \partial_x^m \tilde{w}_\epsilon dx dy \\ & \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3 + \frac{32}{\epsilon} (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^4 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2) + \frac{\epsilon}{4} \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{\frac{3}{2}+\delta}^2}^2. \end{aligned} \quad (3.17)$$

From (3.16) and (3.17), we have, if  $k + \ell > \frac{3}{2}$ ,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \frac{3\epsilon}{4} \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \frac{3}{4} \|\partial_y \partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 \\ & \leq C (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3) + \frac{32}{\epsilon} (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^4 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2). \end{aligned}$$

□

**End of proof of Theorem 3.4.** Combining (3.13) and (3.15), for  $m \geq 6$ ,  $k > 1$ ,  $\frac{3}{2} - k < \ell < \frac{1}{2}$  and  $0 < \epsilon \leq 1$ , we get

$$\frac{d}{dt} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \leq \frac{C}{\epsilon} (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^m), \quad (3.18)$$

with  $C > 0$  independent of  $\epsilon$ .

From (3.18), by the nonlinear Gronwall's inequality, we have

$$\|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)}^{m-2} \leq \frac{\|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^{m-2}}^{m-2}}{e^{-\frac{C}{\epsilon}t(\frac{m}{2}-1)} - (\frac{m}{2}-1)\frac{C}{\epsilon}t\|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m}^{m-2}}, \quad 0 < t \leq T_\epsilon,$$

where we choose  $T_\epsilon > 0$  such that

$$\left( e^{-\frac{C}{\epsilon}T_\epsilon(\frac{m}{2}-1)} - (\frac{m}{2}-1)\frac{C}{\epsilon}T_\epsilon\bar{\zeta}^{m-2} \right)^{-1} = \left( \frac{4}{3} \right)^{m-2}. \quad (3.19)$$

Finally, we get for any  $\|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m} \leq \bar{\zeta}$ , and  $0 < \epsilon \leq \epsilon_0$ ,

$$\|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \frac{4}{3}\|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq 2\|\tilde{w}_0\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}, \quad 0 < t \leq T_\epsilon.$$

□

The rest of this paper is dedicated to improve the results of Proposition 3.9, and try to get an uniform estimate with respect to  $\epsilon$ . Of course, we have to recall the assumption on the shear flow in the main Theorem 1.1.

#### 4. FORMAL TRANSFORMATIONS

Since the estimate (3.13) is independent of  $\epsilon$ , we only need to treat (3.15) in a new way to get an estimate which is also independent of  $\epsilon$ . To simplify the notations, from now on, we drop the notation tilde and sub-index  $\epsilon$ , that is, with no confusion, we take

$$u = \tilde{u}_\epsilon, \quad v = \tilde{v}_\epsilon, \quad w = \tilde{w}_\epsilon.$$

Let  $w \in L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$ ,  $m \geq 6$ ,  $k > 1$ ,  $0 < \ell < \frac{1}{2}$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$  be a classical solution of (3.7) which satisfies the following *à priori* condition

$$\|w\|_{L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta. \quad (4.1)$$

Then (A.2) gives

$$\begin{aligned} \|\langle y \rangle^{k+\ell} w\|_{L^\infty([0, T] \times \mathbb{R}_+^2)} &\leq C(\|\langle y \rangle^{\frac{1}{2}+\delta}(\langle y \rangle^{k+\ell} w)_y\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} \\ &\quad + \|\langle y \rangle^{\frac{1}{2}+\delta}(\langle y \rangle^{k+\ell} w)_{xy}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))}) \\ &\leq C_m \|w\|_{L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))}, \end{aligned}$$

which implies

$$|\partial_y u(t, x, y)| = |w(t, x, y)| \leq C_m \zeta \langle y \rangle^{-k-\ell}, \quad (t, x, y) \in [0, T] \times \mathbb{R}_+^2.$$

We assume that  $\zeta$  is **small enough** such that

$$C_m \zeta \leq \frac{\tilde{c}_1}{4}, \quad (4.2)$$

where  $C_m$  is the above Sobolev embedding constant. Then we have for  $\ell \geq 0$ ,

$$\frac{\tilde{c}_1}{4} \langle y \rangle^{-k} \leq |u_y^s + u_y| \leq 4\tilde{c}_2 \langle y \rangle^{-k}, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+. \quad (4.3)$$

**The formal transformation of equations.** Under the conditions (4.2) and (4.3), in this subsection, we will introduce the following formal transformations of system (3.1). Set, for  $0 \leq n \leq m$

$$g_n = \left( \frac{\partial_x^n u}{u_y^s + u_y} \right)_y, \quad \eta_1 = \frac{u_{xy}}{u_y^s + u_y}, \quad \eta_2 = \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y}, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}_+^2.$$

Formally, we will use the following notations

$$\partial_y^{-1} g_n(t, x, y) = \frac{\partial_x^n u}{u_y^s + u_y}(t, x, y), \quad \partial_y \partial_y^{-1} g_n = g_n, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}_+^2$$

Applying  $\partial_x^n$  to (3.1), we have

$$\begin{aligned} \partial_t \partial_x^n u + (u^s + u) \partial_x \partial_x^n u + (\partial_x^n v)(u_y^s + \partial_y u) \\ = \partial_y^2 \partial_x^n u + \epsilon \partial_x^2 \partial_x^n u + A_n^1 + A_n^2, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} A_n^1 &= -[\partial_x^n, (u^s + u)] \partial_x u = - \sum_{i=1}^n C_n^i \partial_x^i u \partial_x^{n+1-i} u, \\ A_n^2 &= -[\partial_x^n, (u_y^s + \partial_y u)] v = - \sum_{i=1}^n C_n^i \partial_x^i v \partial_x^{n-i} v. \end{aligned}$$

Dividing (4.4) with  $(u_y^s + u_y)$  and performing  $\partial_y$  on the resulting equation, observing

$$\partial_x \partial_x^n u + \partial_y \partial_x^n v = \partial_x^n (\partial_x u + \partial_y v) = 0,$$

we have for  $j = 1, 2$ ,

$$\begin{aligned} \partial_y \left( \frac{\partial_t \partial_x^n u}{u_y^s + u_y} \right) + (u^s + u) \partial_y \left( \frac{\partial_x \partial_x^n u}{u_y^s + u_y} \right) \\ = \partial_y \left( \frac{\partial_y^2 \partial_x^n u + \epsilon \partial_x^2 \partial_x^n u}{u_y^s + u_y} \right) + \partial_y \left( \frac{A_n^1 + A_n^2}{u_y^s + u_y} \right). \end{aligned}$$

We compute each term on the support of ,

$$\begin{aligned} \partial_y \left( \frac{\partial_t \partial_x^n u}{u_y^s + u_y} \right) &= \partial_y \left( \partial_t \frac{\partial_x^n u}{u_y^s + u_y} + \partial_y^{-1} g_n \frac{\partial_t u_y + \partial_t u_y^s}{u_y^s + u_y} \right) \\ &= \partial_t g_n + \partial_y \left( \partial_y^{-1} g_n \frac{\partial_t u_y^s + \partial_t u_y}{u_y^s + \tilde{u}_y} \right), \end{aligned}$$

$$\begin{aligned} (u^s + u) \partial_y \left( \frac{\partial_x \partial_x^n u}{u_y^s + u_y} \right) &= (u^s + u) \left\{ \partial_x \partial_y \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) + \partial_y \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) \frac{u_{xy}}{u_y^s + u_y} \right. \\ &\quad \left. + \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) \partial_y \left( \frac{u_{xy}}{u_y^s + u_y} \right) \right\} \\ &= (u^s + u) (\partial_x g_n + g_n \eta_1 + \partial_y^{-1} g_n \partial_y \eta_1), \end{aligned}$$

$$\frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} = \partial_y^2 \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) + 2 \left( \frac{\partial_y u}{u_y^s + u_y} \right) \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} - \partial_x^n u \partial_y^2 \left( \frac{1}{u_y^s + u_y} \right),$$

$$\partial_y^2 \left( \frac{1}{u_y^s + u_y} \right) = -\partial_y \left( \frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)^2} \right) = -\frac{u_{yyy}^s + u_{yyy}}{(u_y^s + u_y)^2} + 2 \left( \frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)} \right)^2 \frac{1}{u_y^s + u_y},$$

$$\frac{\partial_y \partial_x^n u}{u_y^s + u_y} \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} = \left( \frac{\partial_x^n u}{u_y^s + u_y} \right)_y \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} - \frac{\partial_x^n u}{u_y^s + u_y} \left( \frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)} \right)^2.$$

So

$$\begin{aligned} \frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} &= \partial_y g_n + 2(g_n \eta_2 - 2\partial_y^{-1} g_n \eta_2^2) + \partial_y^{-1} g_n \left( \frac{u_{yyy}^s + u_{yyy}}{(u_y^s + \tilde{u}_y)} \right), \\ \partial_y \left( \frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} \right) &= \partial_y^2 g_n + 2(\partial_y g_n) \eta_2 + 2g_n \partial_y \eta_2 - 4g_n \eta_2^2 \\ &\quad - 8\partial_y^{-1} g_n \eta_2 \partial_y \eta_2 + \partial_y \left( \partial_y^{-1} g_n \frac{u_{yyy}^s + u_{yyy}}{u_y^s + u_y} \right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{\partial_x^2 \partial_x^n u}{u_y^s + u_y} &= \partial_x^2 \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) + 2 \left( \frac{\partial_x^n u}{u_y^s + u_y} \right)_x \frac{u_{xy}}{u_y^s + u_y} \\ &\quad - 2 \frac{\partial_x^n u}{u_y^s + u_y} \left( \frac{u_{xy}}{(u_y^s + u_y)} \right)^2 + \frac{\partial_x^n u}{u_y^s + u_y} \frac{u_{xxy}}{(u_y^s + u_y)}, \\ \partial_y \left( \frac{\partial_x^2 \partial_x^n u}{u_y^s + u_y} \right) &= \partial_x^2 g_n + 2\partial_x g_n \eta_1 + 2\partial_x \partial_y^{-1} g_n \partial_y \eta_1 \\ &\quad - 2g_n \eta_1^2 - 4\partial_y^{-1} g_n \eta_1 \partial_y \eta_1 + \partial_y \left( \partial_y^{-1} g_n \frac{u_{xxy}}{u_y^s + u_y} \right). \end{aligned}$$

For the boundary condition, we only need to pay attention to  $j = 1$ . From (4.4) and the boundary condition for  $(u, v)$  in (3.1), we observe

$$\partial_x^n u|_{y=0} = 0, \quad \partial_y^2 \partial_x^n u|_{y=0} = 0, \quad (u_y^s + u_y)|_{y=0} \neq 0.$$

At the same time,

$$\begin{aligned} 0 = \frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} \Big|_{y=0} &= \partial_y g_n|_{y=0} + 2(g_n \eta_2 - 2(\partial_y^{-1} g_n) \eta_2^2)|_{y=0} \\ &\quad + \partial_y^{-1} g_n \left( \frac{u_{yyy}^s + u_{yyy}}{(u_y^s + \tilde{u}_y)} \right) \Big|_{y=0}, \end{aligned}$$

and

$$\eta_2|_{y=0} = \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} \Big|_{y=0} = 0, \quad \partial_y^{-1} g_n(t, x, y)|_{y=0} = \frac{\partial_x^n u}{u_y^s + u_y}(t, x, y) \Big|_{y=0} = 0,$$

we get then

$$(\partial_y g_n)|_{y=0} = 0, \quad 0 \leq n \leq m.$$

Finally, we have, for  $j = 1, 2$ ,

$$\begin{cases} \partial_t g_n + (u^s + u) \partial_x g_n - \partial_y^2 g_n - \epsilon \partial_x^2 g_n \\ \quad - \epsilon 2 (\partial_x \partial_y^{-1} g_n) \partial_y \eta_1 = M_n, \\ (\partial_y g_n)|_{y=0} = 0, \\ g_n|_{t=0} = g_{n,0}, \end{cases} \quad (4.5)$$

with  $M_n = \sum_{j=1}^6 M_j^n$ ,

$$M_1^n = -(u^s + u)(g_n \eta_1 + (\partial_y^{-1} g_n) \partial_y \eta_1),$$



$$\begin{aligned}
M_2^n &= 2(\partial_y g_n) \eta_2 + 2g_n(\partial_y \eta_2 - 2\eta_2^2) - 8(\partial_y^{-1} g_n) \eta_2 \partial_y \eta_2, \\
M_3^n &= \epsilon(2(\partial_x g_n) \eta_1 - 2g_n \eta_1^2 - 4(\partial_y^{-1} g_n) \eta_1 \partial_y \eta_1), \\
M_4^n &= \partial_y \left( \partial_y^{-1} g_n \frac{(u^s + u)w_x + v(w_y + u_{yy}^s)}{u_y^s + u_y} \right), \\
M_5^n &= -\partial_y \left( \frac{\sum_{i=1}^n C_n^i \partial_x^i u \cdot \partial_x^{n+1-i} u}{u_y^s + u_y} \right), \\
M_6^n &= -\partial_y \left( \frac{\sum_{i=1}^n C_n^i \partial_x^i w \cdot \partial_x^{n-i} v}{u_y^s + u_y} \right),
\end{aligned}$$

where we have used the relation,

$$\partial_t u_y^s + \partial_t u_y - (u_{yyy}^s + u_{yyy}) - \epsilon u_{xxy} = -(u^s + u)w_x + v(u_{yy}^s + w_y).$$

## 5. UNIFORM ESTIMATE

In the future application (see Lemma 6.3), we need that the weight of  $g_m$  big then  $\frac{1}{2}$ , but from the definition,  $w \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$  imply only  $g_m \in H_\ell^2(\mathbb{R}_+^2)$  with  $0 < \ell < \frac{1}{2}$ . So the first step is to improve this weights if the weight of the initial data is more big. We first have

**Lemma 5.1.** *If  $\tilde{w}_0 \in H_{k+\ell'}^{m+2}(\mathbb{R}_+^2)$ ,  $m \geq 6$ ,  $k > 1$ ,  $0 < \ell < \frac{1}{2}$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$  which satisfies (4.1)-(4.2) with  $0 < \zeta \leq 1$ , then  $(g_m)(0) \in H_{k+\ell}^2(\mathbb{R}_+^2)$ , and we have*

$$\|(g_m)(0)\|_{H_\ell^2(\mathbb{R}_+^2)} \leq C \|\tilde{w}_0\|_{H_{k+\ell'}^{m+2}(\mathbb{R}_+^2)}.$$

**Remark.** In fact, observing

$$g_m(0) = \left( \frac{\partial_x^m \tilde{w}_0}{u_{0,y}^s + \tilde{w}_{0,y}} \right)_y = \frac{\partial_y \partial_x^m \tilde{w}_0}{u_{0,y}^s + \tilde{w}_{0,y}} - \frac{\partial_x^m \tilde{w}_0}{u_{0,y}^s + \tilde{w}_{0,y}} \eta_2(0),$$

then (4.3) implies

$$\langle y \rangle^{k+\ell} |g_m(0)| \leq C \langle y \rangle^{k+\ell'} |\partial_x^m \tilde{w}_0| + C \langle y \rangle^{k+\ell'-1} |\partial_x^m \tilde{w}_0|,$$

which finishes the proof of this Lemma.

**Proposition 5.2.** *Let  $w \in L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ ,  $m \geq 6$ ,  $k > 1$ ,  $0 \leq \ell < \frac{1}{2}$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$ , satisfy (4.1)-(4.2) with  $0 < \zeta \leq 1$ . Assume that the shear flow  $u^s$  verifies the conclusion of Lemma 2.1, and  $g_n$  satisfies the equation (4.5) for  $1 \leq n \leq m$ , then we have the following estimates, for  $t \in [0, T]$*

$$\begin{aligned}
\frac{d}{dt} \sum_{n=1}^m \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\partial_y g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \epsilon \sum_{n=1}^m \|\partial_x g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
\leq C_2 \left( \sum_{n=1}^m \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \right),
\end{aligned} \tag{5.1}$$

where  $C_2$  is independent of  $\epsilon$ .

**Approach of the proof for the Proposition 5.2:** We can't prove (5.1) directly, since the approximate solution  $w_\epsilon$  obtained in Theorem 3.4 is belongs to  $L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ , which implies only  $g_n \in L^\infty([0, T_\epsilon]; H_\ell^2(\mathbb{R}_+^2))$ . Then we can't use  $\langle y \rangle^{2\ell'} g_n \in L^\infty([0, T_\epsilon]; H_{\ell-2\ell'}^2(\mathbb{R}_+^2))$  as the test function to the equation

(4.5). To overcome this difficulty, we consider that (4.5) as a linear system for  $g_n, n = 1, \dots, m$  with the coefficients and the source terms depends on  $w$  and their derivatives up to order  $m$ , we will clarify this confirmation in the following proof of the Proposition 5.2. We prove now the estimate (5.1) by the following approach: For the linear system (4.5), we prove firstly (5.1) as *à priori* estimate. Lemma 5.1 imply that  $g_n(0) \in H_{\ell'}^2(\mathbb{R}_+^2), n = 1, \dots, m$ , then by using Hahn-Banach theorem, this *à priori* estimate imply the existence of solutions

$$g_n \in L^\infty([0, T]; H_{\ell'}^2(\mathbb{R}_+^2)), \quad n = 1, \dots, m.$$

Finally, by uniqueness, we can prove the estimate (5.1) by proving it as *à priori* estimate. So that the proof of the Proposition 5.2 is reduced to the proof of the *à priori* estimate (5.1).

**Proof of the *à priori* estimate (5.1).** Multiplying the linear system (4.5) by  $\langle y \rangle^{2\ell'} g_n \in L^\infty([0, T]; H_{-\ell'}^2(\mathbb{R}_+^2))$  and integrating over  $\mathbb{R} \times \mathbb{R}^+$ . We start to deal with the left hand of (4.5) first, we have

$$\int_{\mathbb{R}_+^2} \partial_t g_n \langle y \rangle^{2\ell'} g_n dx dy = \frac{1}{2} \frac{d}{dt} \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2,$$

and

$$\begin{aligned} \int_{\mathbb{R}_+^2} (u^s + u) \partial_x g_n \langle y \rangle^{2\ell'} g_n dx dy &= \frac{1}{2} \int_{\mathbb{R}_+^2} (u^s + u) \cdot \partial_x (\langle y \rangle^{2\ell'} g_n^2) dx dy \\ &\leq \frac{1}{2} \|u_x\|_{L^\infty(\mathbb{R}_+^2)} \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ &\leq C \|w\|_{H_1^2(\mathbb{R}_+^2)} \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}. \end{aligned}$$

Integrating by part, where the boundary value is vanish,

$$\begin{aligned} - \int_{\mathbb{R}_+^2} \partial_y^2 g_n \langle y \rangle^{2\ell'} g_n dx dy &= \|\partial_y g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \partial_y g_n (\langle y \rangle^{2\ell'})' g_n dx dy \\ &\geq \frac{3}{4} \|\partial_y g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 - 4 \|g_n\|_{L^2(\mathbb{R}_+^2)}^2, \end{aligned}$$

and

$$-\epsilon \int_{\mathbb{R}_+^2} \partial_x^2 g_n \langle y \rangle^{2\ell'} g_n dx dy = \epsilon \|\partial_x g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2.$$

We have also

$$\begin{aligned} &-\epsilon \int_{\mathbb{R}_+^2} (\partial_x \partial_y^{-1} g_n) \partial_y \eta_1 \langle y \rangle^{2\ell'} g_n dx dy \\ &= \epsilon \int_{\mathbb{R}_+^2} \partial_y^{-1} g_n \partial_y \eta_1 \langle y \rangle^{2\ell'} \partial_x g_n dx dy \\ &\quad + \epsilon \int_{\mathbb{R}_+^2} \partial_y^{-1} g_n (\partial_y \partial_x \eta_1) \langle y \rangle^{2\ell'} g_n dx dy \\ &\leq \epsilon \|\partial_y^{-1} g_n \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \frac{\epsilon}{8} \|\partial_x g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ &\quad + \epsilon \|\partial_y^{-1} g_n \partial_y \partial_x \eta_1\|_{L^2(\mathbb{R}_+^2)}^2 + \epsilon \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

So by (4.5) and  $0 < \epsilon \leq 1$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\partial_y g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \epsilon \|\partial_x g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ & \leq C \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|(\partial_y^{-1} g_n) \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ & \quad + \|(\partial_y^{-1} g_n) \partial_y \partial_x \eta_1\|_{L^2(\mathbb{R}_+^2)}^2 + 2 \sum_{j=1}^6 \left| \int_{\mathbb{R}_+^2} M_j \langle y \rangle^{2\ell'} g_n dx dy \right|. \end{aligned}$$

Then we can finish the proof of the *a priori* estimate (5.1) by the following four Lemmas.  $\square$

**Lemma 5.3.** *Under the assumption of Proposition 5.2, we have*

$$\|\partial_y^{-1} g_n \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\partial_y^{-1} g_n \partial_y \partial_x \eta_1\|_{L^2(\mathbb{R}_+^2)}^2 \leq C \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2,$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* Notice that (4.1) and (4.2) imply

$$\begin{aligned} |\eta_1| & \leq C \langle y \rangle^{-\ell}, \quad |\partial_x \eta_1| \leq C \langle y \rangle^{-\ell}, \\ |\partial_y \eta_1| & \leq C \langle y \rangle^{-\ell-1}, \quad |\partial_y \partial_x \eta_1| \leq C \langle y \rangle^{-\ell-1}. \end{aligned}$$

Then  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ , imply

$$\begin{aligned} \|\partial_y^{-1} g_n (\partial_y \partial_x \eta_1)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 & \leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{2(\ell' - \ell - 1)} \left( \int_0^y g_n(t, x, \tilde{y}) d\tilde{y} \right)^2 dx dy \\ & \leq C \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

Similarly, we also obtain

$$\|\partial_y^{-1} g_n \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \leq C \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2.$$

$\square$

**Lemma 5.4.** *Under the assumption of Proposition 5.2, we have*

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} \sum_{j=0}^4 \tilde{M}_j^n \langle y \rangle^{2\ell'} g_n dx dy \right| & \leq \frac{1}{8} \|\partial_y g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \frac{\epsilon}{8} \|\partial_x g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ & \quad + \tilde{C} (\|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2), \end{aligned}$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* Recalling  $M_1^n = -(u^s + u)(g_n \eta_1 + (\partial_y^{-1} g_n) \partial_y \eta_1)$ , by Lemma 5.3,

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} (u^s + u) g_n \eta_1 \langle y \rangle^{2\ell'} g_n dx dy \right| & \leq C \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2, \\ \int_{\mathbb{R}_+^2} |(u^s + u) (\partial_y^{-1} g_n) \partial_y \eta_1 \langle y \rangle^{2\ell'} g_n| dy dx & \leq C \|w\|_{H_{k+\ell}^m}^2 + C \|g_n\|_{L_{\ell'}^2}^2. \end{aligned}$$

Besides, we have

$$\left| \int_{\mathbb{R}_+^2} M_1^n \langle y \rangle^{2\ell'} g_n dx dy \right| \leq C (\|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{\ell'}^n(\mathbb{R}_+^2)}^2).$$

The estimates of  $M_2^n$  and  $M_3^n$  needs the following decay rate of  $\eta_2$ :

$$\begin{aligned} |\eta_2| &\leq C\langle y \rangle^{-1}, \quad |\partial_x \eta_2| \leq C\langle y \rangle^{-\ell-1}, \\ |\partial_y \eta_2| &\leq C\langle y \rangle^{-2}, \quad |\partial_y \partial_x \eta_2| \leq C\langle y \rangle^{-\ell-2}. \end{aligned}$$

Recall  $M_2^n = 2\partial_y g_n \eta_2 + 2g_n(\partial_y \eta_2 - 2\eta_2^2) - 8\partial_y^{-1} g_n \eta_2 \partial_y \eta_2$ . We have

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} g_n(\partial_y \eta_2 - \eta_2^2) \langle y \rangle^{2\ell'} g_n dx dy \right| &\leq C \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2, \\ \left| \int_{\mathbb{R}_+^2} (\partial_y g_n) \eta_2 \langle y \rangle^{2\ell'} g_n dx dy \right| &\leq C \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \frac{1}{8} \|\partial_y g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2, \\ \left| 2 \int_{\mathbb{R}_+^2} \partial_y^{-1} g_n \eta_2 \partial_y \eta_2 \langle y \rangle^{2\ell'} g_n dx dy \right| \\ &\leq C \|\langle y \rangle^{\ell'-3} \partial_y^{-1} g_n\|_{L^2}^2 + \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \leq C \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

All together, we conclude

$$\left| \int_{\mathbb{R}_+^2} M_2^n \langle y \rangle^{2\ell'} g_n dx dy \right| \leq C (\|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{\ell'}^n(\mathbb{R}_+^2)}^2) + \frac{1}{8} \|\partial_y g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2,$$

and exactly same computation gives also

$$\left| \int_{\mathbb{R}_+^2} M_3^n \langle y \rangle^{2\ell'} g_n dx dy \right| \leq C (\|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2) + \frac{\epsilon}{8} \|\partial_x g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2.$$

Now using (4.1)-(4.2) and  $m \geq 6$ , with the same computation as above, we can get

$$\left| \int_{\mathbb{R}_+^2} M_4^n \langle y \rangle^{2\ell'} g_n dx dy \right| \leq C \left( \|g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2 \right).$$

which finishes the proof of Lemma 5.4.  $\square$

**Lemma 5.5.** *Under the assumption of Proposition 5.2, we have*

$$\left| \int_{\mathbb{R}_+^2} M_5^n \langle y \rangle^{2\ell'} g_n dx dy \right| \leq \tilde{C} \left( \sum_{p=1}^n \|\tilde{g}_p\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2 \right),$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* Recall,

$$\begin{aligned} \tilde{M}_5^n &= \sum_{i \geq 4} C_n^i g_i \partial_x^{n+1-i} u + \sum_{1 \leq i \leq 3} C_n^i \partial_x^i u g_{n+1-i} \\ &\quad + \sum_{i \geq 4} C_n^i \partial_y^{-1} g_n \partial_x^{n+1-i} w + \sum_{1 \leq i \leq 3} C_n^i \partial_x^i w \partial_y^{-1} g_{n+1-i}, \end{aligned}$$

here if  $n \leq 3$ , we have only the last term. Then, for  $\|w\|_{H_{k+\ell}^n} \leq \zeta \leq 1, m \geq 6$ ,

$$\begin{aligned} &\sum_{i \geq 4} C_n^i \|g_i \partial_x^{n+1-i} u\|_{L_{\ell'}^2(\mathbb{R}_+^2)} + \sum_{1 \leq i \leq 3} \|\partial_x^i u g_{n+1-i}\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \\ &\leq \sum_{i \geq 4} C_n^i \|g_i\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \|\partial_x^{n+1-i} u\|_{L^\infty(\mathbb{R}_+^2)} \end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq i \leq 3} C_n^i \|\partial_x^i u\|_{L^\infty(\mathbb{R}_+^2)} \|\tilde{g}_{n+1-i}\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \\
& \leq C \sum_{i \geq 4} C_n^i \|g_i\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \|w\|_{H_1^{n+3-i}} \\
& \quad + C \sum_{1 \leq i \leq 3} C_n^i \|w\|_{H_1^{i+3}} \|g_{n+1-i}\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \\
& \leq C \sum_{i=1}^n \|g_i\|_{L_{\ell'}^2}.
\end{aligned}$$

Similarly, for the second line in  $M_5$ , by Lemma 5.3, we have

$$\begin{aligned}
\sum_{i \geq 4} C_n^i \|(\partial_y^{-1} g_i) \partial_x^{n+1-i} w\|_{L_{\ell'}^2(\mathbb{R}_+^2)} & \leq \sum_{i \geq 4} C_n^i \|\langle y \rangle^{\ell' - \ell - 1} (\partial_y^{-1} g_i)\|_{L^2(\mathbb{R}_+^2)} \|\langle y \rangle^{\ell+1} \partial_x^{n+1-i} w\|_{L^\infty} \\
& \leq C \sum_{i=1}^n \|g_i\|_{L_{\ell'}^2(\mathbb{R}_+^2)}.
\end{aligned}$$

We have proven Lemma 5.5.  $\square$

**Lemma 5.6.** *Under the assumption of Proposition 5.2, we have*

$$\begin{aligned}
& \left| \int_{\mathbb{R}_+^2} M_6^n \langle y \rangle^{2\ell'} g_n dx dy \right| \\
& \leq \frac{1}{8m} \sum_{p=1}^n \|\partial_y g_p\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \tilde{C} \left( \sum_{p=1}^n \|g_p\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2 \right),
\end{aligned}$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* Recall

$$\begin{aligned}
M_6 & = \sum_{i=1}^n C_n^i g_i \eta_2 \partial_x^{n-i} v + \sum_{i=1}^n C_n^i g_i \partial_x^{n+1-i} u + \sum_{i=1}^n C_n^i \partial_y g_i \partial_x^{n-i} v \\
& \quad + \sum_{i=1}^n \partial_y^{-1} g_i (C_n^i \partial_x^{n-i} v \partial_y \eta_2 + C_n^i \partial_x^{n+1-i} u \eta_2).
\end{aligned}$$

In  $M_6^n$ , we just study the term  $\partial_y g_1 \partial_x^{n-1} v$  as an example, the others terms are similar,

$$\begin{aligned}
\int_{\mathbb{R}_+^2} \partial_y g_1 \partial_x^{n-1} v \langle y \rangle^{2\ell'} g_n & = - \int_{\mathbb{R}_+^2} g_1 \partial_x^{n-1} v \langle y \rangle^{2\ell'} \partial_y g_n \\
& \quad + \int_{\mathbb{R}_+^2} g_1 \partial_x^n u \langle y \rangle^{2\ell'} g_n dx dy, \\
\int_{\mathbb{R}_+^2} g_1 \partial_x^{n-1} v \langle y \rangle^{2\ell'} \partial_y g_n dx dy & \leq \frac{1}{8m} \|\partial_y g_n\|_{L_{\ell'}^2}^2 + C \|g_1 \partial_x^{n-1} v\|_{L_{\ell'}^2}^2, \\
\|g_1 \partial_x^{n-1} v\|_{L_{\ell'}^2}^2 & \leq \sup_{x \in \mathbb{R}} \int_0^{+\infty} \langle y \rangle^{2\ell'} g_1^2 dy \sup_{y \in \mathbb{R}_+} \int_{-\infty}^{+\infty} \left| \int_0^y \partial_x^n u dz \right|^2 dy
\end{aligned}$$

$$\begin{aligned}
&\leq \left( \|g_1\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + \|\partial_x g_1\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \right) \int_{-\infty}^{+\infty} \left| \int_0^{+\infty} |\partial_x^n u| dz \right|^2 dy \\
&\leq C \left( \|g_1\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + \|\partial_x g_1\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \right) \\
&\quad \times \int_{-\infty}^{+\infty} \left| \int_0^{+\infty} \langle y \rangle^{-k-\ell+1} \langle y \rangle^{k+\ell-1} |\partial_x^n u| dz \right|^2 dy \\
&\leq C \left( \|g_1\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + \|g_2\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + \|w\|_{H^m_{k+\ell}}^2 \right) \\
&\quad \times \int_{-\infty}^{+\infty} \left| \int_0^{+\infty} \langle y \rangle^{-k-\ell+1} \langle y \rangle^{k+\ell-1} |\partial_x^n u| dz \right|^2 dy \\
&\leq C \sum_{i=1}^2 \|g_i\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + C \|w\|_{H^m_{k+\ell}}^2.
\end{aligned}$$

Here we have used Lemma 5.3 and

$$k + \ell - 1 > \frac{1}{2}, \quad \|w\|_{H^m_{k+\ell}} \leq 1,$$

and

$$\partial_x g_j = g_{j+1} - g_j \eta_1 - \partial_y^{-1} g_n \cdot \partial_y \eta_1.$$

By the similar trick, we have completed the proof of this lemma.  $\square$

## 6. EXISTENCE OF THE SOLUTION

Now, we can conclude the following energy estimate for the sequence of approximate solutions.

**Theorem 6.1.** *Assume  $u^s$  satisfies Lemma 2.1. Let  $m \geq 6$  be an even integer,  $k + \ell > \frac{3}{2}$ ,  $0 < \ell < \frac{1}{2}$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ , , and  $\tilde{u}_0 \in H^{m+3}_{k+\ell'-1}(\mathbb{R}_+^2)$  which satisfies the compatibility conditions (2.6)-(2.7). Suppose that  $\tilde{w}_\epsilon \in L^\infty([0, T]; H^{m+2}_{k+\ell}(\mathbb{R}_+^2))$  is a solution to (3.7) such that*

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T]; H^m_{k+\ell}(\mathbb{R}_+^2))} \leq \zeta$$

with

$$0 < \zeta \leq 1, \quad C_m \zeta \leq \frac{\tilde{c}_1}{2},$$

where  $0 < T \leq T_1$  and  $T_1$  is the lifespan of shear flow  $u^s$  in the Lemma 2.1,  $C_m$  is the Sobolev embedding constant in (4.2). Then there exists  $C_T > 0$ ,  $\tilde{C}_T > 0$  such that,

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T]; H^{m+2}_{k+\ell}(\mathbb{R}_+^2))} \leq C_T \|\tilde{u}_0\|_{H^{m+1}_{k+\ell'-1}(\mathbb{R}_+^2)}, \quad (6.1)$$

where  $C_T > 0$  is increasing with respect to  $0 < T \leq T_1$  and independent of  $0 < \epsilon \leq 1$ .

Firstly, we collect some results to be used from Section 3 - 5. We come back to the notations with tilde and the sub-index  $\epsilon$ . Then  $g_m^\epsilon, h_m^\epsilon$  are the the functions defined by  $\tilde{u}_\epsilon$ . Under the hypothesis of Theorem 6.1, we have proven the estimates (3.13) and (5.1)

$$\begin{aligned}
&\frac{d}{dt} \|\tilde{w}_\epsilon\|_{H^{m, m-1}_{k+\ell}(\mathbb{R}_+^2)}^2 + \|\partial_y \tilde{w}_\epsilon\|_{H^{m, m-1}_{k+\ell}(\mathbb{R}_+^2)}^2 \\
&\quad + \epsilon \|\partial_x \tilde{w}_\epsilon\|_{H^{m, m-1}_{k+\ell}(\mathbb{R}_+^2)}^2 \leq C_1 \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}_+^2)}^2,
\end{aligned} \quad (6.2)$$

$$\begin{aligned}
& \frac{d}{dt} \sum_{n=1}^m \|g_n^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\partial_y g_n^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \epsilon \sum_{n=1}^m \|\partial_x g_n^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
& \leq C_2 \left( \sum_{n=1}^m \|g_n^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \right),
\end{aligned} \tag{6.3}$$

**Lemma 6.2.** *For the initial data, we have*

$$\begin{aligned}
T_m^\epsilon(g, w)(0) &= \sum_{n=1}^m \|g_n^\epsilon(0)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \\
&\leq C \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2,
\end{aligned}$$

where  $C$  is independent of  $\epsilon$ .

*Proof.* Notice for any  $1 \leq n \leq m$ ,

$$g_n^\epsilon = \left( \frac{\partial_x^n \tilde{u}_\epsilon}{u_y^s + \tilde{w}_\epsilon} \right)_y = \frac{\partial_x^n \partial_y \tilde{u}_\epsilon}{u_y^s + \tilde{w}_\epsilon} - \frac{\partial_x^n \tilde{u}_\epsilon}{u_y^s + \tilde{w}_\epsilon} \eta_2,$$

and  $\tilde{u}_\epsilon(0) = \tilde{u}_0$ , then we deduce, for any  $1 \leq n \leq m$ ,

$$\begin{aligned}
\|g_n^\epsilon(0)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 &\leq 2 \left\| \frac{\partial_x^n \partial_y \tilde{u}_0}{u_{0,y}^s + \tilde{w}_0} \right\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + 2 \left\| \frac{\partial_x^n \tilde{u}_0}{u_{0,y}^s + \tilde{w}_0} \eta_2(0) \right\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
&\leq C (\|\partial_x^n \partial_y \tilde{u}_0\|_{L_{k+\ell'}^2(\mathbb{R}_+^2)}^2 + \|\partial_x^n \tilde{u}_0\|_{L_{k+\ell'-1}^2(\mathbb{R}_+^2)}^2) \leq C \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2.
\end{aligned}$$

□

From (6.2) and (6.3), we have

$$\begin{aligned}
& \|g_m^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \\
& \leq C_8 e^{C_2 t} \int_0^t e^{-C_2 \tau} \|\tilde{w}_\epsilon(\tau)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 d\tau + C_9 e^{C_2 t} \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2.
\end{aligned} \tag{6.4}$$

**Lemma 6.3.** *We have also the following estimate :*

$$\|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2 \leq \tilde{C} \|g_m^\epsilon\|_{L_{\ell'}^2}^2.$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* By the definition,

$$\partial_x^m \tilde{u}_\epsilon(t, x, y) = (u_y^s + \tilde{w}_\epsilon) \int_0^y g_m^\epsilon(t, x, \tilde{y}) d\tilde{y}, \quad y \in \mathbb{R}_+,$$

Therefore,

$$\partial_x^m \tilde{w} = (u_{yy}^s + (\tilde{w}_\epsilon)_y) \int_0^y g_m^\epsilon(t, x, \tilde{y}) d\tilde{y} - (u_y^s + \tilde{w}_\epsilon) g_m^\epsilon(t, x, y), \quad y \geq 0,$$

and

$$\begin{aligned}
\|\partial_x^m \tilde{w}\|_{L_{k+\ell}^2}^2 &\leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{2\ell-2} \left( \int_0^y g_m^\epsilon(t, x, z) dz \right)^2 dx dy + \|g_m^\epsilon(t)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
&\leq C \|g_m^\epsilon(t)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2,
\end{aligned}$$

where we have used  $\ell - 1 < -\frac{1}{2}$  and  $\frac{1}{2} < \ell'$ .  $\square$

**End of proof of Theorem 6.1.** Combining (6.4), Lemma 6.2 and Lemma 6.3, we get, for any  $t \in ]0, T]$ ,

$$\begin{aligned} \|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 &\leq \tilde{C}_8 e^{C_2 t} \int_0^t e^{-C_2 \tau} \|\tilde{w}_\epsilon(\tau)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 d\tau \\ &\quad + \tilde{C}_9 e^{C_2 t} \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2, \end{aligned}$$

with  $\tilde{C}_8, \tilde{C}_9$  independent of  $0 < \epsilon \leq 1$ . We have by Gronwell's inequality that, for any  $t \in ]0, T]$ ,

$$\|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \leq \tilde{C}_9 e^{(C_2 + \tilde{C}_8)t} \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2.$$

So it is enough to take

$$C_T^2 = \tilde{C}_9 e^{(C_2 + \tilde{C}_8)T}$$

which gives (6.1), and  $C_T$  is increasing with respect to  $T$ . We finish the proof of Theorem 6.1.  $\square$

**Theorem 6.4.** *Assume  $u^s$  satisfies Lemma 2.1, and let  $\tilde{u}_0 \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$ ,  $m \geq 6$  be an even integer,  $k > 1, 0 < \ell < \frac{1}{2}$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$ , and*

$$0 < \zeta \leq 1 \text{ with } C_m \zeta \leq \frac{\tilde{c}_1}{2},$$

where  $C_m$  is the Sobolev embedding constant. If there exists  $0 < \zeta_0$  small enough such that,

$$\|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)} \leq \zeta_0,$$

then, there exists  $\epsilon_0 > 0$  and for any  $0 < \epsilon \leq \epsilon_0$ , the system (3.7) admits a unique solution  $\tilde{w}_\epsilon$  such that

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_1]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta,$$

where  $T_1$  is the lifespan of shear flow  $u^s$  in the Lemma 2.1.

**Remark 6.5.** *Under the uniform monotonic assumption (1.2), some results of above theorem holds for any fixed  $T > 0$ . But  $\zeta_0$  decreases as  $T$  increases, according to the (2.5).*

*Proof.* We fix  $0 < \epsilon \leq 1$ , then for any  $\tilde{w}_0 \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$  with  $m \geq 6, k > 1, 0 \leq \ell < \frac{1}{2}$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$ , Theorem 3.4 ensures that, there exists  $\epsilon_0 > 0$  and for any  $0 < \epsilon \leq \epsilon_0$ , there exists  $T_\epsilon > 0$  such that the system (3.7) admits a unique solution  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$  which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \frac{4}{3} \|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq 2 \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}.$$

Now choose  $\zeta_0$  such that

$$\max\{2, C_{T_1}\} \zeta_0 \leq \frac{\zeta}{2}.$$

On the other hand, taking  $\tilde{w}_\epsilon(T_\epsilon)$  as initial data for the system (3.7), Theorem 3.4 ensures that there exists  $T'_\epsilon > 0$ , which is defined by (3.19) with  $\tilde{\zeta} = \frac{\zeta}{2}$ , such that



the system (3.7) admits a unique solution  $\tilde{w}'_\epsilon \in L^\infty([T_\epsilon, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))$  which satisfies

$$\|\tilde{w}'_\epsilon\|_{L^\infty([T_\epsilon, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \frac{4}{3} \|\tilde{w}_\epsilon(T_\epsilon)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta.$$

Now, we extend  $\tilde{w}_\epsilon$  to  $[0, T_\epsilon + T'_\epsilon]$  by  $\tilde{w}'_\epsilon$ , then we get a solution  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))$  which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta.$$

So if  $T_\epsilon + T'_\epsilon < T_1$ , we can apply Theorem 6.1 to  $\tilde{w}_\epsilon$  with  $T = T_\epsilon + T'_\epsilon$ , and use (6.1), this gives

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq C_{T_1} \|\tilde{u}_0\|_{H_{k+\ell-1}^{m+1}(\mathbb{R}_+^2)} \leq \frac{\zeta}{2}.$$

Now taking  $\tilde{w}_\epsilon(T_\epsilon + T'_\epsilon)$  as initial data for the system (3.7), applying again Theorem 3.4, for the same  $T'_\epsilon > 0$ , the system (3.7) admits a unique solution  $\tilde{w}'_\epsilon \in L^\infty([T_\epsilon + T'_\epsilon, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))$  which satisfies

$$\|\tilde{w}'_\epsilon\|_{L^\infty([T_\epsilon + T'_\epsilon, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \frac{4}{3} \|\tilde{w}_\epsilon(T_\epsilon + T'_\epsilon)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta.$$

Now, we extend  $\tilde{w}_\epsilon$  to  $[0, T_\epsilon + 2T'_\epsilon]$  by  $\tilde{w}'_\epsilon$ , then we get a solution  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))$  which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta.$$

So if  $T_\epsilon + 2T'_\epsilon < T_1$ , we can apply Theorem 6.1 to  $\tilde{w}_\epsilon$  with  $T = T_\epsilon + 2T'_\epsilon$ , and use (6.1), this gives again

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq C_{T_1} \|\tilde{u}_0\|_{H_{k+\ell-1}^{m+1}(\mathbb{R}_+^2)} \leq \frac{\zeta}{2}.$$

Then by recurrence, we can extend the solution  $\tilde{w}_\epsilon$  to  $[0, T_1]$ , and then the lifespan of approximate solution is equal to that of shear flow if the initial data  $\tilde{u}_0$  is small enough.  $\square$

We have obtained the following estimate, for  $m \geq 6$  and  $0 < \epsilon \leq \epsilon_0$ ,

$$\|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta, \quad t \in [0, T_1].$$

By using the equation (3.7) and the Sobolev inequality, we get, for  $0 < \delta < 1$

$$\|\tilde{w}_\epsilon\|_{Lip([0, T_1]; C^{2, \delta}(\mathbb{R}_+^2))} \leq M < +\infty.$$

Then taking a subsequence, we have, for  $0 < \delta' < \delta$ ,

$$\tilde{w}_\epsilon \rightarrow \tilde{w} \ (\epsilon \rightarrow 0), \text{ locally strong in } C^0([0, T_1]; C^{2, \delta'}(\mathbb{R}_+^2)),$$

and

$$\partial_t \tilde{w} \in L^\infty([0, T_1]; H_{k+\ell}^{m-2}(\mathbb{R}_+^2)), \quad \tilde{w} \in L^\infty([0, T_1]; H_{k+\ell}^m(\mathbb{R}_+^2)),$$

with

$$\|\tilde{w}\|_{L^\infty([0, T_1]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta.$$

Then we have

$$\tilde{u} = \partial_y^{-1} w \in L^\infty([0, T_1]; H_{k+\ell-1}^m(\mathbb{R}_+^2)),$$

where we use the Hardy inequality (A.1), since

$$\lim_{y \rightarrow +\infty} \tilde{u}(t, x, y) = - \lim_{y \rightarrow +\infty} \int_y^{+\infty} \tilde{w}(t, x, \tilde{y}) d\tilde{y} = 0.$$

In fact, we also have

$$\lim_{y \rightarrow 0} \tilde{u}(t, x, y) = \lim_{y \rightarrow 0} \int_0^y \tilde{w}(t, x, \tilde{y}) d\tilde{y} = 0.$$

Using the condition  $k + \ell - 1 > \frac{1}{2}$ , we have also

$$\tilde{v} = - \int_0^y \tilde{u}_x d\tilde{y} \in L^\infty([0, T_1]; L^\infty(\mathbb{R}_{+, y}); H^{m-1}(\mathbb{R}_x)).$$

We have proven that,  $\tilde{w}$  is a classical solution to the following vorticity Prandtl equation

$$\begin{cases} \partial_t \tilde{w} + (u^s + \tilde{u}) \partial_x \tilde{w} + \tilde{v} \partial_y (u_y^s + \tilde{w}) = \partial_y^2 \tilde{w}, \\ \partial_y \tilde{w}|_{y=0} = 0, \\ \tilde{w}|_{t=0} = \tilde{w}_0, \end{cases}$$

and  $(\tilde{u}, \tilde{v})$  is a classical solution to (2.2). Finally,  $(u, v) = (u^s + \tilde{u}, \tilde{v})$  is a classical solution to (1.1), and satisfies (6.5). In conclusion, we have proved the following theorem

**Theorem 6.6.** *Let  $m \geq 6$  be an even integer,  $k > 1, 0 < \ell < \frac{1}{2}, \ell' > \frac{1}{2}, \ell' - \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ , assume that  $u_0^s$  satisfies (1.2), the initial date  $\tilde{u}_0 \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  and  $\tilde{u}_0$  satisfies the compatibility condition (2.6)-(2.7) up to order  $m + 2$ . Then there exists  $T > 0$  such that if*

$$\|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)} \leq \delta_0,$$

for some  $\delta_0 > 0$  small enough, then the initial-boundary value problem (2.2) admits a solution  $(\tilde{u}, \tilde{v})$  with

$$\tilde{u} \in L^\infty([0, T]; H_{k+\ell-1}^m(\mathbb{R}_+^2)), \quad \partial_y \tilde{u} \in L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2)).$$

Moreover, we have the following energy estimate,

$$\|\partial_y \tilde{u}\|_{L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq C \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2. \quad (6.5)$$

**Remark 6.7.** *We remark that  $\delta_0 \rightarrow 0$  when  $T \rightarrow +\infty$ .*

## 7. UNIQUENESS AND STABILITY

Now, we study the stability of solutions which implies immediately the uniqueness of solution.

Let  $\tilde{u}^1, \tilde{u}^2$  be two solutions obtained in Theorem 6.6 with respect to the initial date  $\tilde{u}_0^1, \tilde{u}_0^2$  respectively. Denote  $\bar{u} = \tilde{u}^1 - \tilde{u}^2$  and  $\bar{v} = \tilde{v}^1 - \tilde{v}^2$ , then

$$\begin{cases} \partial_t \bar{u} + (u^s + \tilde{u}_1) \partial_x \bar{u} + (u_y^s + \tilde{u}_{1,y}) \bar{v} = \partial_y^2 \bar{u} - \tilde{v}_2 \partial_y \bar{u} - (\partial_x \tilde{u}_2) \bar{u}, \\ \partial_x \bar{u} + \partial_y \bar{v} = 0, \\ \bar{u}|_{y=0} = \bar{v}|_{y=0} = 0, \\ \bar{u}|_{t=0} = \tilde{u}_0^1 - \tilde{u}_0^2. \end{cases}$$

So it is a linear equation for  $\bar{u}$ . We also have for the vorticity  $\bar{w} = \partial_y \bar{u}$ ,

$$\begin{cases} \partial_t \bar{w} + (u^s + \tilde{u}_1) \partial_x \bar{w} + (u_{yy}^s + \tilde{w}_{1,y}) \bar{v} = \partial_y^2 \bar{w} - \tilde{v}_2 \partial_y \bar{w} - (\partial_x \tilde{w}_2) \bar{u}, \\ \partial_y \bar{w}|_{y=0} = 0, \\ \bar{w}|_{t=0} = \tilde{w}_0^1 - \tilde{w}_0^2. \end{cases} \quad (7.1)$$

**Estimate with a loss of  $x$ -derivative.** Firstly, for the vorticity  $\bar{w} = \partial_y \bar{u}$ , we deduce an energy estimate with a loss of  $x$ -derivative with the anisotropic norm defined by (3.12).

**Proposition 7.1.** *Let  $\tilde{u}^1, \tilde{u}^2$  be two solutions obtained in Theorem 6.6 with respect to the initial data  $\tilde{u}_0^1, \tilde{u}_0^2$ , then we have*

$$\frac{d}{dt} \|\bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}^2 + \|\partial_y \bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}^2 \leq \bar{C}_1 \|\bar{w}\|_{H_{k+\ell}^{m-2}}^2, \quad (7.2)$$

where the constant  $\bar{C}_1$  depends on the norm of  $\tilde{w}^1, \tilde{w}^2$  in  $L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$ .

*Proof.* The proof of this Proposition is similar to the proof of the Proposition 3.8, and we need to use that  $m-2$  is even. We only give the calculation for the terms which need a different argument. Moreover we also explain why we only get the estimate on  $\|\bar{w}\|_{H_{k+\ell}^{m-2}}^2$  but require the norm of  $\tilde{w}^1, \tilde{w}^2$  in  $L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$ .

With out loss of the generality, we suppose that  $\|\bar{w}\|_{H_{k+\ell}^{m-2}} \leq 1, \|\tilde{w}^1\|_{H_{k+\ell}^m} \leq 1$  and  $\|\tilde{w}^2\|_{H_{k+\ell}^m} \leq 1$ .

Derivating the equation of (7.1) with  $\partial^\alpha = \partial_x^\alpha \partial_y^{\alpha_2}$ , for  $|\alpha| = \alpha_1 + \alpha_2 \leq m-2, \alpha_1 \leq m-3$ ,

$$\begin{aligned} \partial_t \partial^\alpha \bar{w} - \partial_y^2 \partial^\alpha \bar{w} &= -\partial^\alpha ((u^s + \tilde{u}_1) \partial_x \bar{w} + \tilde{v}_2 \partial_y \bar{w} \\ &\quad + (u_{yy}^s + \tilde{w}_{1,y}) \bar{v} + (\partial_x \tilde{w}_2) \bar{u}). \end{aligned} \quad (7.3)$$

Multiplying the above equation with  $\langle y \rangle^{k+\ell'+\alpha_2} \partial^\alpha \bar{w}$ , the same computation as in the proof of the Proposition 3.8, in particular, the reduction of the boundary-data are the same, gives

$$\begin{aligned} &\int_{\mathbb{R}_+^2} \left( \partial_t \partial^\alpha \bar{w} - \partial_y^2 \partial^\alpha \bar{w} \right) \langle y \rangle^{2(k+\ell'+\alpha_2)} \partial^\alpha \bar{w} dx dy \\ &\geq \frac{1}{2} \frac{d}{dt} \|\partial^\alpha \bar{w}\|_{L_{k+\ell'+\alpha_2}^2}^2 + \frac{3}{4} \|\partial_y \bar{w}\|_{H_{k+\ell}^{m-2, m-3}}^2 - C \|\bar{w}\|_{H_{k+\ell}^{m-2}}^2. \end{aligned}$$

As for the right hand of (7.3), for the first item, we split it into two parts

$$-\partial^\alpha \left( (u^s + \tilde{u}_1) \partial_x \bar{w} \right) = -(u^s + \tilde{u}_1) \partial_x \partial^\alpha \bar{w} + [(u^s + \tilde{u}_1), \partial^\alpha] \partial_x \bar{w}.$$

Firstly, we have

$$\left| \int_{\mathbb{R}_+^2} ((u^s + \tilde{u}_1) \partial_x \partial^\alpha \bar{w}) \langle y \rangle^{2(\ell'+\alpha_2)} \partial^\alpha \bar{w} dx dy \right| \leq \|\tilde{w}_1\|_{H^3} \|\partial^\alpha \bar{w}\|_{L_{k+\ell'+\alpha_2}^2}^2.$$

For the commutator operator, we have,

$$\|[(u^s + \tilde{u}_1), \partial^\alpha] \partial_x \tilde{w}_\epsilon\|_{L_{k+\ell'+\alpha_2}^2} \leq C \|\tilde{w}_1\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)} \|\bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}.$$

Notice that for this term, we don't have the loss of  $x$ -derivative.

With the similar method for the terms  $\tilde{v}_2 \partial_y \bar{w}$ , we get

$$\left| \int_{\mathbb{R}_+^2} \tilde{v}_2 \partial_y \bar{w} \langle y \rangle^{2(\ell+\alpha_2)} \partial^\alpha \bar{w} dx dy \right| \leq \|\tilde{w}_2\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)} \|\bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}^2.$$

For the next one, we have

$$\partial^\alpha \left( (u_{yy}^s + \partial_y \tilde{w}_1) \bar{v} \right) = \sum_{\beta \leq \alpha} C_\beta^\alpha \partial^\beta (u_{yy}^s + \partial_y \tilde{w}_1) \partial^{\alpha-\beta} \bar{v},$$

and thus

$$\begin{aligned} & \left\| \sum_{\beta \leq \alpha, 1 \leq |\beta| < |\alpha|} C_\beta^\alpha \partial^\beta (u_{yy}^s + \partial_y \tilde{w}_1) \partial^{\alpha-\beta} \bar{v} \right\|_{L_{k+\ell+\alpha_2}^2} \\ & \leq C \|\tilde{w}_1\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)} \|\bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}. \end{aligned}$$

On the other hand, using Lemma A.1 and  $\frac{3}{2} - k < \ell < \frac{1}{2}$ ,

$$\begin{aligned} & \left\| (\partial^\alpha (u_{yy}^s + \partial_y \tilde{w}_1)) \bar{v} \right\|_{L_{k+\ell+\alpha_2}^2} \leq \left\| (\partial^\alpha u_{yy}^s) \bar{v} \right\|_{L_{k+\ell+\alpha_2}^2} + \left\| (\partial^\alpha \partial_y \tilde{w}_1) \bar{v} \right\|_{L_{k+\ell+\alpha_2}^2} \\ & \leq C \|\bar{v}\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} + C \|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \|\bar{v}\|_{L^\infty(\mathbb{R}_+^2)} \\ & \leq C \|\bar{u}_x\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} + C \|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} (\|\bar{u}_x\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} + \|\bar{u}_{xx}\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)}) \\ & \leq C(1 + \|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}) \|\bar{w}\|_{H_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} \\ & \leq C(1 + \|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}) \|\bar{w}\|_{H_{k+\ell}^2(\mathbb{R}_+^2)}. \end{aligned}$$

So this term requires the norms  $\|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}$ .

Moreover, if  $\alpha_2 \neq 0$

$$\begin{aligned} & \left\| (u_{yy}^s + \partial_y \tilde{w}_1) \partial^\alpha \bar{v} \right\|_{L_{k+\ell+\alpha_2}^2} = \left\| (u_{yy}^s + \partial_y \tilde{w}_1) \partial_x^{\alpha_1} \partial^{\alpha_2-1} \bar{u}_x \right\|_{L_{k+\ell+\alpha_2}^2} \\ & \leq C(1 + \|\tilde{w}_1\|_{H_{k+\ell}^{m-1}(\mathbb{R}_+^2)}) \|\bar{w}\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)}, \end{aligned}$$

and also if  $\alpha_2 = 0$

$$\begin{aligned} & \left\| (u_{yy}^s + \partial_y \tilde{w}_1) \partial_x^{\alpha_1} \bar{v} \right\|_{L_{k+\ell}^2} = \left\| (u_{yy}^s + \partial_y \tilde{w}_1) \partial_y^{-1} \partial_x^{\alpha_1} \bar{u}_x \right\|_{L_{k+\ell}^2} \\ & \leq C(1 + \|\tilde{w}_1\|_{H_{k+\ell}^{m-1}(\mathbb{R}_+^2)}) \|\partial_x^{\alpha_1+1} \bar{w}\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)}. \end{aligned}$$

These two cases imply the loss of  $x$ -derivative.

Similar argument also gives

$$\left| \int_{\mathbb{R}_+^2} (\partial^\alpha (\partial_x \tilde{w}_2) \bar{u}) \langle y \rangle^{2(\ell+\alpha_2)} \partial^\alpha \bar{w} dx dy \right| \leq C \|\tilde{w}_2\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \|\bar{w}\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)},$$

which finishes the proof of the Proposition 7.1.  $\square$

**Estimate on the loss term.** To close the estimate (6.5), we need to study the terms  $\|\partial_x^{m-2} \bar{w}\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}$  which is missing in the left hand side of (7.2).

Similar to the argument in Section 6, we will recover this term by the estimate of functions

$$\bar{g}_n = \left( \frac{\partial_x^n \bar{u}}{u_y^s + \tilde{u}_{1,y}} \right)_y, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+.$$

**Proposition 7.2.** *Let  $\tilde{u}^1, \tilde{u}^2$  be two solutions obtained in Theorem 6.6 with respect to the initial data  $\tilde{u}_0^1, \tilde{u}_0^2$ , then we have*

$$\begin{aligned} \frac{d}{dt} \sum_{n=1}^{m-2} \|\bar{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^{m-2} \|\partial_y \bar{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ \leq C_2 \left( \sum_{n=1}^{m-2} \|\bar{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\bar{w}\|_{H_{k+\ell}^{m-2}}^2 \right), \end{aligned}$$

where the constant  $\bar{C}_2$  depends on the norm of  $\tilde{w}^1, \tilde{w}^2$  in  $L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$ .

These Propositions can be proven by using exactly the same calculation as in Section 5. The only difference is that when we use the Leibniz formula, for the term where the order of derivatives is  $|\alpha| = m - 2$ , it acts on the coefficient which depends on  $\tilde{u}^1, \tilde{u}^2$ . Therefore, we need their norm in the order of  $(m - 2) + 1$ . So we omit the proof of this Proposition here.

With the similar argument to the proof of Theorem 6.1, we get

$$\|\bar{w}\|_{L^\infty([0, T]; H_{k+\ell}^{m-2}(\mathbb{R}_+^2))} \leq C \|\bar{u}_0\|_{H_{k+\ell-1}^{m+1}(\mathbb{R}_+^2)},$$

which finishes the proof of Theorem 1.1.

#### APPENDIX A. SOME INEQUALITIES

We will use the following Hardy type inequalities.

**Lemma A.1.** *Let  $f : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ . Then*

(i) *if  $\lambda > -\frac{1}{2}$  and  $\lim_{y \rightarrow \infty} f(x, y) = 0$ , then*

$$\|\langle y \rangle^\lambda f\|_{L^2(\mathbb{R}_+^2)} \leq C_\lambda \|\langle y \rangle^{\lambda+1} \partial_y f\|_{L^2(\mathbb{R}_+^2)}; \quad (\text{A.1})$$

(ii) *if  $-1 \leq \lambda < -\frac{1}{2}$  and  $f(x, 0) = 0$ , then*

$$\|\langle y \rangle^\lambda f\|_{L^2(\mathbb{R}_+^2)} \leq C_\lambda \|\langle y \rangle^{\lambda+1} \partial_y f\|_{L^2(\mathbb{R}_+^2)}.$$

Here  $C_\lambda \rightarrow +\infty$ , as  $\lambda \rightarrow -\frac{1}{2}$ .

We need the following trace theorem in the weighted Sobolev space.

**Lemma A.2.** *Let  $\lambda > \frac{1}{2}$ , then there exists  $C > 0$  such that for any function  $f$  defined on  $\mathbb{R}_+^2$ , if  $\partial_y f \in L_\lambda^2(\mathbb{R}_+^2)$ , it admits a trace on  $\mathbb{R}_x \times \{0\}$ , and satisfies*

$$\|\gamma_0(f)\|_{L^2(\mathbb{R}_x)} \leq C \|\partial_y f\|_{L_\lambda^2(\mathbb{R}_+^2)},$$

where  $\gamma_0(f)(x) = f(x, 0)$  is the trace operator.

The proof of the above two Lemmas is elementary, so we leave it to the reader.

We use also the following Sobolev inequality and algebraic properties of  $H_{k+\ell}^m(\mathbb{R}_+^2)$ ,

**Lemma A.3.** *For the suitable functions  $f, g$ , we have:*

1) *If the function  $f$  satisfies  $f(x, 0) = 0$  or  $\lim_{y \rightarrow +\infty} f(x, y) = 0$ , then for any small  $\delta > 0$ ,*

$$\|f\|_{L^\infty(\mathbb{R}_+^2)} \leq C (\|f_y\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} + \|f_{xy}\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)}). \quad (\text{A.2})$$

2) *For  $m \geq 6, k + \ell > \frac{3}{2}$ , and any  $\alpha, \beta \in \mathbb{N}^2$  with  $|\alpha| + |\beta| \leq m$ , we have*

$$\|(\partial^\alpha f)(\partial^\beta g)\|_{L_{k+\ell+\alpha_2+\beta_2}^2(\mathbb{R}_+^2)} \leq C \|f\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \|g\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}. \quad (\text{A.3})$$

3) For  $m \geq 6, k + \ell > \frac{3}{2}$ , and any  $\alpha \in \mathbb{N}^2, p \in \mathbb{N}$  with  $|\alpha| + p \leq m$ , we have,

$$\|(\partial^\alpha f)(\partial_x^p(\partial_y^{-1}g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}_+^2)} \leq C\|f\|_{H^m_{k+\ell}(\mathbb{R}_+^2)}\|g\|_{H^m_{\frac{1}{2}+\delta}(\mathbb{R}_+^2)},$$

where  $\partial_y^{-1}$  is the inverse of derivative  $\partial_y$ , meaning,  $\partial_y^{-1}g = \int_0^y g(x, \tilde{y}) d\tilde{y}$ .

*Proof.* For (1), using  $f(x, 0) = 0$ , we have

$$\begin{aligned} \|f\|_{L^\infty(\mathbb{R}_+^2)} &= \left\| \int_0^y (\partial_y f)(x, \tilde{y}) d\tilde{y} \right\|_{L^\infty(\mathbb{R}_+^2)} \leq C\|\partial_y f\|_{L^\infty(\mathbb{R}_x; L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+))} \\ &\leq C(\|\partial_y f\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+^2)} + \|\partial_x \partial_y f\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+^2)}). \end{aligned}$$

If  $\lim_{y \rightarrow +\infty} f(x, y) = 0$ , we use

$$f(x, y) = - \int_y^\infty (\partial_y f)(x, \tilde{y}) d\tilde{y}.$$

For (2), firstly,  $m \geq 6$  and  $|\alpha| + |\beta| \leq m$  imply  $|\alpha| \leq m - 2$  or  $|\beta| \leq m - 2$ , without loss of generality, we suppose that  $|\alpha| \leq m - 2$ . Then, using the conclusion of (1), we have

$$\begin{aligned} \|(\partial^\alpha f)(\partial^\beta g)\|_{L^2_{k+\ell+\alpha_2+\beta_2}(\mathbb{R}_+^2)} &\leq \|\langle y \rangle^{\alpha_2}(\partial^\alpha f)\|_{L^\infty(\mathbb{R}_+^2)}\|\partial^\beta g\|_{L^2_{k+\ell+\beta_2}(\mathbb{R}_+^2)} \\ &\leq C\|f\|_{H^{|\alpha|+2}_{\frac{1}{2}+\delta}(\mathbb{R}_+^2)}\|\partial^\beta g\|_{L^2_{k+\ell+\beta_2}(\mathbb{R}_+^2)}, \end{aligned}$$

which give (A.3).

For (3), if  $|\alpha| \leq m - 2$ , we have

$$\begin{aligned} \|(\partial^\alpha f)(\partial_x^p(\partial_y^{-1}g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}_+^2)} &\leq \|\langle y \rangle^{k+\ell+\alpha_2}(\partial^\alpha f)\|_{L^2(\mathbb{R}_{y,+}; L^\infty(\mathbb{R}_x))}\|\partial_x^p(\partial_y^{-1}g)\|_{L^\infty(\mathbb{R}_{y,+}; L^2(\mathbb{R}_x))} \\ &\leq C\|f\|_{H^{|\alpha|+2}_{k+\ell}(\mathbb{R}_+^2)}\|\partial_x^p g\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+^2)}. \end{aligned}$$

If  $p \leq m - 2$ , we have

$$\begin{aligned} \|(\partial^\alpha f)(\partial_x^p(\partial_y^{-1}g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}_+^2)} &\leq \|\langle y \rangle^{k+\ell+\alpha_2}(\partial^\alpha f)\|_{L^2(\mathbb{R}_+^2)}\|\partial_x^p(\partial_y^{-1}g)\|_{L^\infty(\mathbb{R}_+^2)} \\ &\leq C\|f\|_{H^{|\alpha|}_{k+\ell}(\mathbb{R}_+^2)}\|\partial_x^p g\|_{L^\infty(\mathbb{R}_x; L^2_{\frac{1}{2}+\delta}(\mathbb{R}_{y,+}))} \\ &\leq C\|f\|_{H^{|\alpha|}_{k+\ell}(\mathbb{R}_+^2)}\|g\|_{H^m_{\frac{1}{2}+\delta}(\mathbb{R}_+^2)}. \end{aligned}$$

We have completed the proof of the Lemma.  $\square$

## APPENDIX B. THE EXISTENCE OF APPROXIMATE SOLUTIONS

Now, we prove the Proposition 3.6, the existence of solution to the vorticity equation  $\tilde{w}_\epsilon = \partial_y \tilde{u}_\epsilon$  and suppose that  $m, k, \ell$  and  $u^s(t, y)$  satisfy the assumption of Proposition 3.6,

$$\begin{cases} \partial_t \tilde{w}_\epsilon + (u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon + v_\epsilon (u^s_{yy} + \partial_y \tilde{w}_\epsilon) = \partial_y^2 \tilde{w}_\epsilon + \epsilon \partial_x^2 \tilde{w}_\epsilon, \\ \partial_y \tilde{w}_\epsilon|_{y=0} = 0 \\ \tilde{w}_\epsilon|_{t=0} = \tilde{w}_{0,\epsilon}, \end{cases} \quad (\text{B.1})$$

where

$$\tilde{u}_\epsilon(t, x, y) = - \int_y^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y}, \quad \tilde{v}_\epsilon(t, x, y) = - \int_0^y \partial_x \tilde{u}_\epsilon(t, x, \tilde{y}) d\tilde{y}.$$

We will use the following iteration process to prove the existence of solution, where  $w^0 = \tilde{w}_{0,\epsilon}$ ,

$$\begin{cases} \partial_t w^n + (u^s + u^{n-1}) \partial_x w^n + (u_{yy}^s + \partial_y w^{n-1}) v^n = \partial_y^2 w^n + \epsilon \partial_x^2 w^n, \\ \partial_y w^n|_{y=0} = 0 \\ w^n|_{t=0} = \tilde{w}_{0,\epsilon}, \end{cases} \quad (\text{B.2})$$

with

$$u^{n-1}(t, x, y) = - \int_y^{+\infty} w^{n-1}(t, x, \tilde{y}) d\tilde{y},$$

and

$$\begin{aligned} v^n(t, x, y) &= - \int_0^y \partial_x u^n(t, x, \tilde{y}) d\tilde{y} \\ &= \int_0^y \int_{\tilde{y}}^{+\infty} \partial_x w^n(t, x, z) dz d\tilde{y}. \end{aligned}$$

Here for the boundary data, we have

$$\begin{aligned} \partial_y^3 w^n|_{y=0} &= ((u_y^s + w^{n-1}) \partial_x w^n)|_{y=0}, \\ &(\partial_y^5 w^n)(t, x, 0) \\ &= (\partial_y^3 u^s(t, 0) + \partial_y^2 w^{n-1}(t, x, 0) + \epsilon(\partial_x^2 w^{n-1})(t, x, 0)) (\partial_x w^n)(t, x, 0) \\ &\quad + (u_y^s(t, 0) + (w^{n-1})(t, x, 0)) ((\partial_y^2 \partial_x w^n)(t, x, 0) + \epsilon(\partial_x^3 w^n)(t, x, 0)) \\ &\quad - (\partial_y \partial_x w^n)(u_y^s + w^{n-1})(t, x, 0) \\ &\quad + \sum_{1 \leq j \leq 3} C_j^4 \left( (\partial_y^j (u^s + u^{n-1})) \partial_y^{4-j} \partial_x u^n - (\partial_y^{j-1} \partial_x \tilde{u}^n) \partial_y^{4-j} (u_y^s + w^{n-1}) \right) (t, x, 0) \\ &\quad - \epsilon \partial_x^2 \left( (u_y^s(t, 0) + (w^{n-1})(t, x, 0)) (\partial_x w^n)(t, x, 0) \right). \end{aligned}$$

and also for  $3 \leq p \leq \frac{m}{2} + 1$ ,  $\partial_y^{2p+1} w^n|_{y=0}$  is a linear combination of the terms of the form:

$$\prod_{j=1}^{q_1} \left( \partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s + u^n) \right) \Big|_{y=0} \times \prod_{l=1}^{q_2} \left( \partial_x^{\tilde{\alpha}_l} \partial_y^{\tilde{\beta}_l+1} (u^s + u^{n-i}) \right) \Big|_{y=0}, \quad (\text{B.3})$$

where  $2 \leq q_1 + q_2 \leq p$ ,  $1 \leq i \leq \min\{n, p\}$  and

$$\begin{aligned} \alpha_j + \beta_j &\leq 2p - 1, \quad 1 \leq j \leq q_1; \quad \tilde{\alpha}_l + \tilde{\beta}_l \leq 2p - 1, \quad 1 \leq l \leq q_2; \\ \sum_{j=1}^{q_1} (3\alpha_j + \beta_j) + \sum_{l=1}^{q_2} (3\tilde{\alpha}_l + \tilde{\beta}_l) &= 2p + 1; \\ \sum_{j=1}^{q_1} \beta_j + \sum_{l=1}^{q_2} \tilde{\beta}_l &\leq 2p - 2; \quad \sum_{j=1}^{q_1} \alpha_j + \sum_{l=1}^{q_2} \tilde{\alpha}_l \leq p - 1, \quad 0 < \sum_{j=1}^{q_1} \alpha_j. \end{aligned}$$

Remark that the condition  $0 < \sum_{j=1}^{q_1} \alpha_j$  implies that, in (B.3), there are at last one factor like  $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} u^n(t, x, 0)$ .

For given  $w^{n-1}$ , we have  $u^{n-1} = \partial_y^{-1} w^{n-1}$  and  $v^n = -\partial_y^{-1} u_x^n$ . We will prove the existence and boundness of the sequence  $\{w^n, n \in \mathbb{N}\}$  in  $L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$  to the linear equation (B.2) firstly, then the existence of solution to (B.1) is guaranteed by using the standard weak convergence methods.

**Lemma B.1.** *Assume that  $w^{n-i} \in L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ ,  $1 \leq i \leq \min\{n, \frac{m}{2} + 1\}$  and  $\tilde{w}_{0,\epsilon}$  satisfies the compatibility condition up to order  $m+2$  for the system (B.1), then the initial-boundary value problem (B.2) admit a unique solution  $w^n$  such that, for any  $t \in [0, T]$ ,*

$$\frac{d}{dt} \|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 \leq B_T^{n-1} \|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + D_T^{n-1} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2}, \quad (\text{B.4})$$

where

$$B_T^{n-1} = C \left( 1 + \sum_{i=1}^{\min\{n, m/2+1\}} \|w^{n-i}\|_{L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))} + \left(1 + \frac{1}{\epsilon}\right) \sum_{i=1}^{\min\{n, m/2+1\}} \|w^{n-i}\|_{L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))}^2 \right),$$

and

$$D_T^{n-1} = C \sum_{i=1}^{\min\{n, m/2+1\}} \|w^{n-i}\|_{L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))}^{m+2}.$$

*Proof.* Once we get *à priori* estimate for this linear problem, the existence of solution is guaranteed by the Hahn-Banach theorem. So we only prove the *à priori* estimate of the smooth solutions.

For any  $\alpha \in \mathbb{N}^2$ ,  $|\alpha| \leq m+2$ , taking the equation (B.2) with derivative  $\partial^\alpha$ , multiplying the resulting equation by  $\langle y \rangle^{2k+2\ell+2\alpha_2} \partial^\alpha w^n$  and integrating by part over  $\mathbb{R}_+^2$ , one obtains that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \|\partial_y w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \epsilon \|\partial_x w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 \\ &= \sum_{|\alpha| \leq m+2} \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} \partial^\alpha ((u^s + u^{n-1}) \partial_x w^n \\ & \quad - (\partial_y^{-1} u_x^n)(u_{yy}^s + \partial_y w^{n-1})) \partial^\alpha w^n dx dy \\ & + \sum_{|\alpha| \leq m+2} \int_{\mathbb{R}_+^2} (\langle y \rangle^{2k+2\ell+2\alpha_2})' \partial^\alpha \partial_y w^n \partial^\alpha \partial_y w^n dx dy \\ & + \sum_{|\alpha| \leq m+2} \int_{\mathbb{R}} (\partial^\alpha \partial_y w^n \partial^\alpha \partial_y w^n)|_{y=0} dx, \end{aligned} \quad (\text{B.5})$$



With similar analysis to Section 5, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} (u^s + u^{n-1}) \partial_x \partial^\alpha w^n \partial^\alpha w^n dx dy \right| \\ &= \left| -\frac{1}{2} \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} \partial_x (u^s + u^{n-1}) \partial^\alpha w^n \partial^\alpha w^n dx dy \right| \\ &\leq C \|u^{n-1}\|_{L^\infty(\mathbb{R}_+^2)} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2, \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} [\partial^\alpha, (u^s + u^{n-1})] \partial_x w^n \partial^\alpha w^n dx dy \right| \\ &\leq C(1 + \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}) \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2. \end{aligned}$$

For the second term on the right hand of (B.5), by using the Leibniz formula, we need to pay more attention to the following two terms

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} (\partial^\alpha \partial_y^{-1} u_x^n) (u_{yy}^s + \partial_y w^{n-1}) \partial^\alpha w^n dx dy \right| \\ &\leq C(1 + \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}) \|\partial_x w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \\ &\leq \frac{\epsilon}{2} \|\partial_x w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \frac{C}{\epsilon} (1 + \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)})^2 \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2, \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} v^n (\partial^\alpha \partial_y w^{n-1}) \partial^\alpha w^n dx dy \\ &= - \int_{\mathbb{R}_+^2} \partial_y (\langle y \rangle^{2k+2\ell+2\alpha_2} (\partial_y^{-1} u_x^n)) (\partial^\alpha w^{n-1}) \partial^\alpha w^n dx dy \\ &\quad - \int_{\mathbb{R}_+^2} (\langle y \rangle^{2k+2\ell+2\alpha_2} (\partial_y^{-1} u_x^n)) (\partial^\alpha w^{n-1}) \partial_y \partial^\alpha w^n dx dy, \end{aligned}$$

here we have used  $v^n|_{y=0} = 0$ , thus

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} v^n (\partial^\alpha \partial_y w^{n-1}) \partial^\alpha w^n dx dy \right| \\ &\leq C \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} (\|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \|\partial_y w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}). \end{aligned}$$

For the boundary term, similar to the proof of Proposition 3.8, we can get

$$\begin{aligned} & \sum_{|\alpha| \leq m+2} \left| \int_{\mathbb{R}} (\partial^\alpha \partial_y w^n \partial^\alpha \partial_y w^n)|_{y=0} dx \right| \\ &\leq \frac{1}{16} \|\partial_y w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + C \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2}. \end{aligned}$$

We get finally

$$\begin{aligned} & \frac{d}{dt} \|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \|\partial_y w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \epsilon \|\partial_x w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 \\ &\leq B_T^{n-1} \|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + D_T^{n-1} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2}. \end{aligned}$$

□

**Lemma B.2.** *Suppose that  $m, k, \ell$  and  $u^s(t, y)$  satisfy the assumption of Proposition 3.6,  $\bar{\zeta} > 0$ , then for any  $0 < \epsilon \leq 1$ , there exists  $T_\epsilon > 0$  such that for any  $\tilde{w}_{0,\epsilon} \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$  with*

$$\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \bar{\zeta},$$

*the iteration equations (B.2) admit a sequence of solution  $\{w^n, n \in \mathbb{N}\}$  such that, for any  $t \in [0, T_\epsilon]$ ,*

$$\|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \frac{4}{3} \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}, \quad \forall n \in \mathbb{N}.$$

**Remark.** Here  $\bar{\zeta}$  is arbitrary.

*Proof.* Integrating (B.4) over  $[0, t]$ , for  $0 < t \leq T$  and  $T > 0$  small,

$$\|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^m \leq \frac{\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^m}{e^{-\frac{m}{2}B_T^{n-1}t} - \frac{m}{2}D_T^{n-1}t\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^m}.$$

We prove the Lemma by induction. For  $n = 1$ , we have

$$\begin{aligned} B_T^0 &= C \left( 1 + \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} + \left(1 + \frac{1}{\epsilon}\right) \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 \right) \\ &\leq C \left( 1 + \bar{\zeta} + \left(1 + \frac{1}{\epsilon}\right) \bar{\zeta}^2 \right), \end{aligned}$$

and

$$D_T^{n-1} = C \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2} \leq C \bar{\zeta}^{m+2}.$$

Choose  $T_\epsilon > 0$  small such that

$$\left( e^{-\frac{m}{2}C(1+2\bar{\zeta}+4(1+\frac{1}{\epsilon})\bar{\zeta}^2)T_\epsilon} - \frac{m}{2}C(2\bar{\zeta})^{m+2}T_\epsilon(2\bar{\zeta})^m \right)^{-1} = \left( \frac{4}{3} \right)^m,$$

we get

$$\|w^1(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \frac{4}{3} \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}.$$

Now the induction hypothesis is: for  $0 \leq t \leq T_\epsilon$ ,

$$\|w^{n-1}(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \frac{4}{3} \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)},$$

thanks to the choose of  $T_\epsilon$ , we have also

$$\left( e^{-\frac{m}{2}B_{T_\epsilon}^{n-1}T_\epsilon} - \frac{m}{2}D_{T_\epsilon}^{n-1}T_\epsilon\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^m \right)^1 \leq \left( \frac{4}{3} \right)^m$$

for any  $t \in [0, T_\epsilon]$ , then we finish the proof of the Lemma B.2. □

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