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# WELL-POSEDNESS OF THE PRANDTL EQUATION IN SOBOLEV SPACE WITHOUT MONOTONICITY

CHAO-JIANG XU AND XU ZHANG

ABSTRACT. We study the well-posedness theory for the Prandtl boundary layer equation on the half plane with initial data in Sobolev spaces. We consider a class of initial data which admit the non-degenerate critical points, so it is not monotonic. For this kind of initial data, we prove the local-in-time existence, uniqueness and stability of solutions for the nonlinear Prandtl equation in weighted Sobolev space. We use the energy method to prove the existence of solution by a parabolic regularizing approximation. The nonlinear cancellation properties of the Prandtl equations and non-degeneracy of the critical points are the main ingredients to establish a new energy estimate. Our result improves the classical local well-posedness results for the initial data that are monotone, analytic or Gevrey class, and it will also help us to understand the ill-posedness and instability results for the Prandtl equation.

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## 1. INTRODUCTION

In this work, we study the initial-boundary value problem for the Prandtl boundary layer equation in two dimension, which reads

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = \partial_y^2 u, & t > 0, (x, y) \in \mathbb{R}_+^2, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = U(t, x), \\ u|_{t=0} = u_0(x, y), \end{cases}$$

where  $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$ ,  $u(t, x, y)$  represents the tangential velocity,  $v(t, x, y)$  normal velocity.  $p(t, x, y)$  and  $U(t, x)$  are the values on the boundary of the Euler's pressure and Euler's tangential velocity and determined by the Bernoulli's law:

$$\partial_t U(t, x) + U(t, x)\partial_x U(t, x) + \partial_x p = 0.$$

The Prandtl boundary layer equation was firstly derived formally by Ludwig Prandtl in 1904 ([22]). From the mathematical point of view, the well-posedness and justification of the Prandtl boundary layer theory don't have satisfactory theory yet, and remain open for general cases. During the past century, lots of mathematicians have investigated this problems. The Russian school has contributed a lot to the boundary layer theory and their works were collected in [21]. Up to now, the existence theory for the Prandtl boundary layer equation has been achieved only when the initial data belong to some special functional spaces: 1) the analytic space [17, 24, 25, 28]; 2) Sobolev spaces or Hölder spaces under monotonicity assumption [1, 16, 19, 21, 26]; 3) recently [7] in Gevrey class with non-degenerate critical point. See also [13] where the initial data is monotone on a number of intervals and analytic on the complement. On the other hand, without the monotonicity assumption, E and Engquist in [5] constructed finite time blowup solutions to the Prandtl equation. After this work, there are many un-stability or strong ill-posedness results. In particular, Gérard-Varet and Dormy [6] constructed a highly persuasive profile to reveal that the linearized Prandtl equation around the shear flow with a non-degenerate critical point is ill-posed in the sense of Hadamard in Sobolev spaces. In [8] the authors proved the strong ill-posedness for linearized Prandtl equation around a shear flow, meaning the non-existence of distribution solution. [11] extends the above ill-posedness results to nonlinear Prandtl equations. See also [4, 10, 14, 20, 23] for the relative works.

In this work, we consider the uniform outflow  $U(t, x) = 1$  which implies  $p_x = 0$ . In other words the following problem for the Prandtl equation is considered :

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = \partial_y^2 u, & t > 0, (x, y) \in \mathbb{R}_+^2, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = 1, \\ u|_{t=0} = u_0(x, y). \end{cases} \quad (1.1)$$

The aim of this work is to prove the local well-posedness in Sobolev spaces of the system (1.1) around a **non-monotonic shear profile**. More precisely, we assume that

$$u_0(x, y) = u_0^s(y) + \tilde{u}_0(x, y),$$

where the shear profile  $u_0^s$  satisfies the following conditions: for some even integer  $m \geq 6$ ,  $u_0^s \in C^{m+4}([0, +\infty[)$ ,  $\lim_{y \rightarrow +\infty} u_0^s(y) = 1$  with the compatibility condition  $(\partial_y^{2p} u_0^s)(0) = 0$ ,  $0 \leq 2p \leq m + 4$ , there exists  $k > 1$  and  $2 < a < +\infty$  such that

$$\begin{cases} (\partial_y u_0^s)(a) = 0, (\partial_y u_0^s)(y) \neq 0, \forall y \in \overline{\mathbb{R}_+} \setminus \{a\}; \\ |(\partial_y^p u_0^s)(y)| \leq B_p \langle y \rangle^{-k-p+1}, \forall y \geq 0, 1 \leq p \leq m+4, \end{cases} \quad (1.2)$$

moreover, there exists  $b_1, b_2, \tilde{B}_2 > 0$  and  $y_0 > a$  **big enough** such that

$$\begin{cases} b_1 \langle y \rangle^{-k} \leq |\partial_y u_0^s(y)|, y \in [a+1, +\infty[; \\ b_2 \langle y \rangle^{-k-1} \leq |\partial_y^2 u_0^s(y)|, y \in [a-1, y_0+2], \\ |\partial_y^2 u_0^s(y)| \leq \tilde{B}_2 \langle y \rangle^{-k-1}, y \in [y_0-2, +\infty[, \end{cases} \quad (1.3)$$

where  $\langle y \rangle = (1 + |y|^2)^{1/2}$ . See Appendix B for some example of shear profile.

Therefore,  $u_0^s$  has a **non-degenerate critical point** at  $a \in \mathbb{R}_+$  and, in particular, it is not monotonic on  $\mathbb{R}_+$ .

We will use the weighted Sobolev spaces (similar to [19]) with the following norm, for  $\lambda \in \mathbb{R}, m \in \mathbb{N}$ ,

$$\|f\|_{H_\lambda^m(\mathbb{R}_+^2)}^2 = \sum_{|\alpha_1 + \alpha_2| \leq m} \int_{\mathbb{R}_+^2} \langle y \rangle^{2\lambda + 2\alpha_2} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} f|^2 dx dy. \quad (1.4)$$

We use also the notation  $\|f\|_{L_\lambda^2(\mathbb{R}_+^2)} = \|f\|_{H_\lambda^0(\mathbb{R}_+^2)}$  and  $H^m$  stands for the usual Sobolev space.

Let  $u^s(t, y)$  be the solution of the heat equation (2.1) with initial data  $u_0^s(y)$ , then  $(u^s, 0)$  is a shear flow of the system (1.1). We will construct the solution of the Prandtl equation (1.1) as a perturbation of this shear flow. For the initial-boundary values problem (1.1), the existence of the smooth solution requires the high order compatibility conditions for the initial data  $u_0$ , see Proposition 2.2 below for the precise description.

The main result of this paper is stated as following :

**Theorem 1.1.** *Let  $m \geq 6$  be an even integer,  $k > 1$ ,  $0 < \ell < \frac{1}{2}$ ,  $k + 2\ell > 2$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ . Assume that  $u_0^s$  satisfies (1.2)-(1.3), the initial data  $(u_0 - u_0^s) \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  and  $(u_0 - u_0^s)$  satisfies the compatibility condition up to order  $m + 2$ . Then there exists  $T > 0$  such that if*

$$\|(u_0 - u_0^s)\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)} \leq \delta_0, \quad (1.5)$$

for some  $\delta_0 > 0$  small enough, the initial-boundary value problem (1.1) admits a unique solution  $(u, v)$  with

$$(u - u^s) \in L^\infty([0, T]; H_{k+\ell-1}^m(\mathbb{R}_+^2)), \quad \partial_y(u - u^s) \in L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2)), \quad (1.6)$$

and

$$v \in L^\infty([0, T]; L^\infty(\mathbb{R}_{y,+}; H^{m-1}(\mathbb{R}_x))), \quad \partial_y v \in L^\infty([0, T]; H_{k+\ell-1}^{m-1}(\mathbb{R}_+^2)).$$

Moreover, we have the stability with respect to the initial data in the following sense: given any two initial data

$$u_0^1 = u_0^s + \tilde{u}_0^1, \quad u_0^2 = u_0^s + \tilde{u}_0^2,$$

if  $u_0^s$  satisfies (1.2)-(1.3) and  $\tilde{u}_0^1, \tilde{u}_0^2$  satisfies (1.5), then the solutions  $u^1$  and  $u^2$  of (1.1) with initial data  $u_0^1$  and  $u_0^2$  respectively satisfy,

$$\|u^1 - u^2\|_{L^\infty([0, T]; H_{k+\ell-1}^{m-2}(\mathbb{R}_+^2))} \leq C \| (u_0^1 - u_0^2) \|_{H_{k+\ell-1}^m(\mathbb{R}_+^2)}.$$

**Remark 1.2.** .

- (1) It is not difficult to see from the proof of Theorem 1.1 that the above main results hold also for the problem defined on the torus  $\mathbb{T}$  for the  $x$ -variable.
- (2) The lifespan  $T$  of solution is not required to be very small. In fact, it is equal to the lifespan  $T_1$  of shear flow  $u^s(t, y)$  such that it preserves the monotonicity and convexity on  $[0, T_1]$  (see Lemma 2.1).
- (3) Comparing the order of regularity and the decay rate of initial data (1.5) with those of the solution (1.6), we observe that there are a loss of the regularity and also a loss of the decay rate. While the loss of the regularity seems natural for a degenerate parabolic type equation, the loss of decay rate of order  $\ell' - \ell > 0$  (which can be very small) is also a special phenomenon for the Prandtl equation.
- (4) Under the assumption of Theorem 1.1, there are many initial data  $u_0$  with a curve of non-degenerate critical points : for some smooth curve  $a(x) > 0$  with  $|a(x) - a| \ll 1, x \in \mathbb{R}$ ,

$$(\partial_y u_0)(x, a(x)) = 0, \text{ and } (\partial_y^2 u_0)(x, a(x)) > 0, \forall x \in \mathbb{R}.$$

See some precise examples in Appendix B.

- (5) It is well-known that to obtain the smooth solution of the initial-boundary value problem, the assumption on the high order compatibility condition is necessary. We will make it precise in Proposition 2.2 and Remark 3.3. See [2] and [3] for the Prandtl equation under the incompatibility condition.

Now, we give some comments on the differences and compatibility between our results and the ill-posedness works. In [6, 8] and [11], the authors showed that the linear and nonlinear Prandtl equations around the shear profile with non degenerate critical point are ill-posed in the sense of Hadamard in the Sobolev space. The results of our Theorem 1.1 observe contrary phenomenons. We can't say that there is contradictory between our Theorem 1.1 and the results of [6, 8, 11], even though in all of those works, the initial data (the shear profile for linear equation) has non-degenerate critical points. We list differences of those hypothesis and results. For the ill-posedness works, the decay rate of the shear profile is exponential (also possible with polynomial weights) and the initial perturbations are with same weights. But for the well-posedness results, we assume that the first and the second derivatives of the shear flow has positive lower and upper bounds of polynomial type. Moreover, the decay rate of the initial perturbations around shear flows are faster than that of the shear flow, which is crucial for us to close a new energy estimate. Despite with some loss of decay rate, our solutions  $(u - u^s)$  still decay faster than the shear flow. But for the ill-posedness work, the decay rate of the shear flow, the initial perturbations and the solutions are with the same weights. Together, we conclude that, with an initial perturbation faster than the shear flow,

we look for solutions in the function space with a decay rate more faster . In the other words, we work with a smaller set of initial perturbations, which excludes the possible growing eigenfunctions constructed by Gérard-Varet and Dormy [6]. So this yields no contradictory between our results and the ill-posedness works.

From the physics point of view, it is reasonable to assume that the solution of the Prandtl equations tend to Euler out flow with a polynomial decay rate. This was pointed out by Kukavica-Vicol in [12], this is also conformed by the results of [19] and our Theorem 1.1, so we believe that the suitable function space to study the nonlinear Prandtl equation is Sobolev space with an algebraic weights defined in (1.4). For the existence of the solution to the Cauchy problem, the loss of derivatives and loss of decay relative to the shear flow and to the initial data seem natural for the degenerate parabolic type equations. But on the other hand, under the monotonic assumption, the Prandtl equation is hypo-elliptic, and for  $t > 0$ , any Sobolev solution belongs to the Gevrey class of index  $3(1+k)$  (see [15]), where the decay rate  $k$  gives an effective reflection for the Gevrey index. See also [27] for the more general nonlinear hypo-elliptic equations which yields also the  $C^\infty$  smoothness of the solution of the Prandtl equation.

This article is arranged as follows. In Section 2, we explain the main difficulties for the study of the Prandtl equation and present an outline of our approach. In Section 3, we study the approximate solutions to (1.1) by a parabolic regularization. In Section 4, we prepare some technical tools and two formal transformations for the Prandtl equations. Sections 5-6 are dedicated to the uniform estimates of approximate solutions obtained in Section 3. We prove finally the main theorem in Section 7-8.

**Notations:** The letter  $C$  stands for various suitable constants, independent with functions and the special parameters, which may vary from line to line and step to step. When it depends on some crucial parameters in particular, we put a sub-index such as  $C_\epsilon$  etc, which may also vary from line to line.

## 2. OUTLINE OF OUR APPROACH AND PRELIMINARY

**2.1. Difficulties and our approach.** Now, we explain the main difficulties in proving Theorem 1.1, and present the strategies of our approach.

It is well-known that the major difficulty for the study of the Prandtl equation (1.1) is the term  $v \partial_y u$ , where the vertical velocity behaves like

$$v(t, x, y) = - \int_0^y \partial_x u(t, x, \tilde{y}) d\tilde{y},$$

by using the divergence free condition and boundary conditions. So it introduces a loss of  $x$ -derivative. The  $y$ -integration create also a loss of weights with respect to  $y$ -variable. Then the standard energy estimates do not work. This explains why there are few existence results in the literatures.

Recalling that in [1] (see also [19] for a similar transformation), under the monotone assumption  $\partial_y u > 0$ , we divide the Prandtl equations by  $\partial_y u$  and then take derivative with respect to  $y$ , to obtain an equation of the new unknown function  $g = \left( \frac{u}{\partial_y u} \right)_y$ . In the new equation, the term  $v$  disappears by using the divergence free condition. We call this property *the first type nonlinear cancellation*. By using this cancellation, under the monotone assumption, we can establish an energy estimate for the new equation in a suitable weighted Sobolev space to deduce the

existence of classical solutions. Now, under the assumption of Theorem 1.1, we use this properties together with a cut-off functions to treat the monotonic part, and establish a partial energy estimate.

On the other hand, near the non-degenerate critical points, saying  $\partial_y^2 u > 0$  on a small domain, we consider the vorticity equation ( $w = \partial_y u$ ) of (1.1),

$$\partial_t w + u \partial_x w + v \partial_y w = \partial_y^2 w.$$

Since  $\partial_y w = \partial_y^2 u > 0$ , we can try the same approach as the monotone case. But it doesn't work well because the divergence free condition can't be used here to make  $v$  disappear. In [9, 10], for the hydrostatic Euler equation, E. Grenier proposed another approach(see also [18]), after dividing the vorticity equation by  $\sqrt{\partial_y w}$ , we get then an equation of the following form

$$\partial_t h + u \partial_x h + v \sqrt{\partial_y w} = \partial_y^2 h + \text{commutators}, \quad \text{on a convex domain,}$$

with  $h = \frac{w}{\sqrt{\partial_y w}}$ . One can then take advantage of the following *second type nonlinear cancellation*

$$\begin{aligned} \int v \sqrt{\partial_y w} h dx dy &= \int v \partial_y u dx dy = - \int (\partial_y v) u dx dy \\ &= \int (\partial_x u) u dx dy = \frac{1}{2} \int \partial_x (u^2) dx dy = 0. \end{aligned}$$

This approach works well for the hydrostatic Euler equation, because, in that case, the integration is taken over  $\mathbb{T} \times ]0, 1[$  and  $\partial_y w > 0$  everywhere. But for the Prandtl equation, it is defined over  $\mathbb{T} \times \mathbb{R}_+$  or  $\mathbb{R} \times \mathbb{R}_+$ , so we can't use directly this approach. In the Prandtl case, we then need to introduce a cut-off function, and in this case, *a new problem appears!* The above calculation doesn't preserve the cancellation property. There are remaining terms from the commutators with cut-off functions. So it is standard that we can only close the energy estimate in Gevrey class(see [7]) with a suitably chosen cut-off functions in Gevrey class.

In this work, our approach is to combine above two type nonlinear cancellations with carefully chosen cut-off functions. We decompose  $\mathbb{R} \times \mathbb{R}_+$  into two parts, namely monotonic parts and convex part. We prove the energy estimates on each part. Of course, we can't close the energy estimate on each part separately. But it is a miracle that we can close a full energy estimate when combining those two partial energy estimates together. By using this full energy estimate, we prove then the existence of classical solutions.

In order to prove the existence of solutions, follows the idea of Masmoudi-Wong ([19]), we will construct an approximate scheme and study the parabolic regularized Prandtl equation (3.1), which preserves the nonlinear structure of the original Prandtl equation (1.1), as well as the nonlinear cancellation properties. Then by an uniform energy estimate for the approximate solutions, the existence of solutions to the original Prandtl equation (1.1) follows. This energy estimate also implies the uniqueness and the stability. The uniform energy estimate for the approximate solutions is the main duty of this paper.

**2.2. Analysis of shear flow.** We write the solution  $(u, v)$  of system (1.1) as

$$u(t, x, y) = u^s(t, y) + \tilde{u}(t, x, y), \quad v(t, x, y) = \tilde{v}(t, x, y),$$

where  $u^s(t, y)$  is the solution of the following heat equation

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0, \\ u^s|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u^s(t, y) = 1, \\ u^s|_{t=0} = u_0^s(y). \end{cases} \quad (2.1)$$

Then (1.1) can be written as

$$\begin{cases} \partial_t \tilde{u} + (u^s + \tilde{u}) \partial_x \tilde{u} + \tilde{v} (u_y^s + \partial_y \tilde{u}) = \partial_y^2 \tilde{u}, \\ \partial_x \tilde{u} + \partial_y \tilde{v} = 0, \\ \tilde{u}|_{y=0} = \tilde{v}|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} \tilde{u} = 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0(x, y). \end{cases} \quad (2.2)$$

We first study the shear flow, and give a precise version for the condition (1.2)-(1.3) on  $u_0^s$ : there exists  $k > 1, a > 0$  and  $a_0, c_1, c_2, c_3, B_p > 0$  with  $a_0 \leq \frac{a}{20}$ , such that

$$\begin{cases} u_0^s \in C^{m+4}([0, +\infty[), \quad \lim_{y \rightarrow +\infty} u_0^s(y) = 1; \\ (\partial_y^{2p} u_0^s)(0) = 0, \quad 0 \leq 2p \leq m+4; \\ \partial_y u_0^s(a) = 0; \quad \partial_y u_0^s(y) \neq 0, \quad \forall y \in \overline{\mathbb{R}}_+ \setminus \{a\}; \\ |\partial_y u_0^s(y)| \geq 2c_1, \quad 0 \leq y \leq a - a_0; \\ |\partial_y u_0^s(y)| \geq 2c_1 \langle y \rangle^{-k}, \quad y \geq a + a_0; \\ |\partial_y^p u_0^s(y)| \leq B_p \langle y \rangle^{-k-p+1}, \quad y \geq 0, \quad 1 \leq p \leq m+4. \end{cases} \quad (2.3)$$

and there exists also  $y_0 > 0$  **big enough** such that

$$\begin{cases} |\partial_y^2 u_0^s(y)| \geq 2c_2, \quad y \in [a - 4a_0, a + 4a_0] \\ |\partial_y^2 u_0^s(y)| \geq 2c_3 \langle y \rangle^{-k-1}, \quad y \in [a + a_0, y_0 + 2] \\ |\partial_y^2 u_0^s(y)| \leq \tilde{B}_2 \langle y \rangle^{-k-1}, \quad y \in [y_0 - 2, +\infty[ \end{cases} \quad (2.4)$$

where the positives constants  $c_1, c_2, c_3$  and  $\tilde{B}_2$  are independents of  $y_0$ .

Since

$$|\partial_y u_0^s(y)| > 0, \quad y \in ]a, +\infty[; \quad \partial_y u_0^s(a) = 0, \quad \lim_{y \rightarrow +\infty} |\partial_y u_0^s(y)| = 0,$$

then there exists  $\tilde{y}_0 > a$  such that  $M = \sup_{y \in \mathbb{R}_+} |\partial_y u_0^s(y)| = |\partial_y u_0^s(\tilde{y}_0)| > 0$ , and  $|\partial_y^2 u_0^s(\tilde{y}_0)| = 0$ , then the condition (2.4) means that we can take  $y_0 < \tilde{y}_0 \leq y_0 + 10$  for  $y_0$  big enough. We remark that for  $B_1$ , we need to take

$$B_1 = M \langle \tilde{y}_0 \rangle^k.$$

We will give some example and their sketch map in the Appendix B.

With this initial shear profile, we have first,

**Lemma 2.1.** *Assume that  $u_0^s$  satisfies (2.3). There exists a  $T_1 > 0$  such that the solution  $u^s(t, y)$  to the initial-boundary value problem (2.1) satisfies all the following properties on  $[0, T_1] \times \mathbb{R}^+$ ,*

- (1)  $u^s \in L^\infty([0, T_1]; C^{m+4}([0, +\infty[)) \cap C^1([0, T_1]; C^{m+2}([0, +\infty[));$   
 $(\partial_y^{2p} u^s)(t, 0) = 0, \quad 0 \leq 2p \leq m+2, \quad \lim_{y \rightarrow +\infty} u^s(t, y) = 1, \quad t \in [0, T_1];$
- (2) *there exists  $\alpha \in C^{\frac{m}{2}+1}([0, T_1])$ ,  $\alpha(0) = a$ , such that*  
 $a - \frac{a_0}{2} \leq \alpha(t) \leq a + \frac{a_0}{2}$  *and  $(\partial_y u^s)(t, \alpha(t)) = 0$ ;*
- (3)  $|(\partial_y u^s)(t, y)| \geq c_1, \quad 0 \leq y \leq a - 2a_0, \quad t \in [0, T_1];$



- (4)  $|(\partial_y u^s)(t, y)| \geq c_1 \langle y \rangle^{-k}$ ,  $y \geq a + 2a_0$ ,  $t \in [0, T_1]$ ,
- (5)  $|(\partial_y^2 u^s)(t, y)| \geq c_2$ ,  $y \in [a - 3a_0, a + 3a_0]$ ,  $t \in [0, T_1]$ ,
- (6)  $|(\partial_y^2 u^s)(t, y)| \geq c_3 \langle y \rangle^{-k-1}$ ,  $y \in [a + 2a_0, y_0 + 1]$ ,  $t \in [0, T_1]$ ,
- (7)  $|(\partial_y^2 u^s)(t, y)| \leq 2\tilde{B}_2 \langle y \rangle^{-k-1}$ ,  $y \in [y_0 - 1, +\infty[$ ,  $t \in [0, T_1]$ ,
- (8)  $|\partial_y^p u^s(t, y)| \leq 2B_p \langle y \rangle^{-k-p+1}$ ,  $y \geq 0$ ,  $t \in [0, T_1]$ ,  $1 \leq p \leq m + 4$ .

*Proof.* Firstly, the solution of (2.1) can be written as

$$\begin{aligned} u^s(t, y) &= \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} \left( e^{-\frac{(y-\tilde{y})^2}{4t}} - e^{-\frac{(y+\tilde{y})^2}{4t}} \right) u_0^s(\tilde{y}) d\tilde{y} \\ &= \frac{1}{\sqrt{\pi}} \left( \int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} u_0^s(2\sqrt{t}\xi + y) d\xi - \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} u_0^s(2\sqrt{t}\xi - y) d\xi \right), \end{aligned}$$

which gives

$$\begin{aligned} \partial_t u^s(t, y) &= \frac{1}{\sqrt{\pi t}} \left( \int_{-\frac{y}{2\sqrt{t}}}^{+\infty} \xi e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi + y) d\xi \right. \\ &\quad \left. - \int_{\frac{y}{2\sqrt{t}}}^{+\infty} \xi e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi - y) d\xi \right). \end{aligned}$$

By using  $(\partial_y^{2j} u_0^s)(0) = 0$  for  $0 \leq 2j \leq m + 4$ , it follows

$$\begin{aligned} \partial_y^p u^s(t, y) &= \frac{1}{\sqrt{\pi}} \left( \int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} (\partial_y^p u_0^s)(2\sqrt{t}\xi + y) d\xi \right. \\ &\quad \left. + (-1)^{p+1} \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} (\partial_y^p u_0^s)(2\sqrt{t}\xi - y) d\xi \right) \quad (2.5) \\ &= \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} \left( e^{-\frac{(y-\tilde{y})^2}{4t}} + (-1)^{p+1} e^{-\frac{(y+\tilde{y})^2}{4t}} \right) (\partial_y^p u_0^s)(\tilde{y}) d\tilde{y}, \end{aligned}$$

for all  $1 \leq p \leq m + 4$ .

This gives the conclusion (1). The conclusions (3) and (5) can be obtained by the continuity of  $\partial_y u^s(t, y)$  and  $\partial_y^2 u^s(t, y)$  on  $[0, T] \times [0, a + 6a_0]$ , where the smallness of  $T_1 > 0$  is required. For the conclusion (2), we use (5) and the implicit function theorem to the equation

$$(\partial_y u^s)(t, y(t)) = 0, \quad (t, y(t)) \in [0, T_1] \times [a - a_0/2, a + a_0/2], \quad \partial_y u^s(0, a) = 0,$$

where  $T_1 > 0$  is again required to be small. In fact, for (2), the curve  $\alpha(t)$  is the solution of the following ordinary differential equation

$$\begin{cases} \alpha'(t) = \frac{(\partial_y^3 u^s)(t, \alpha(t))}{(\partial_y^2 u^s)(t, \alpha(t))}, & t \in [0, T_1], \\ \alpha(0) = a, \end{cases}$$

where  $\frac{(\partial_y^3 u^s)(t, y)}{(\partial_y^2 u^s)(t, y)} \in C^{\frac{m}{2}}([0, T_1] \times [a - a_0/2, a + a_0/2])$ .

For the upper bound (8), (2.5) implies

$$\begin{aligned} |\partial_y^p u^s(t, y)| &\leq \frac{B_p}{2\sqrt{\pi t}} \int_0^{+\infty} \left( e^{-\frac{(y-\tilde{y})^2}{4t}} + e^{-\frac{(y+\tilde{y})^2}{4t}} \right) \langle \tilde{y} \rangle^{-k-p+1} d\tilde{y} \\ &\leq \frac{B_p}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+\tilde{y})^2}{4t}} \langle \tilde{y} \rangle^{-k-p+1} d\tilde{y}. \end{aligned}$$

Using now Peetre's inequality, for any  $\lambda \in \mathbb{R}$

$$\tilde{c}_0 \langle y \rangle^\lambda \langle y + \tilde{y} \rangle^{-|\lambda|} \leq \langle \tilde{y} \rangle^\lambda \leq \tilde{c}_0^{-1} \langle y \rangle^\lambda \langle y + \tilde{y} \rangle^{|\lambda|},$$

we get then, with  $\lambda = -k - p + 1$ ,

$$|\partial_y^p u^s(t, y)| \leq \tilde{c}_0^{-1} B_p (1+t)^{\frac{k+p-1}{2}} \langle y \rangle^{-k-p+1}.$$

For the upper bound (7), we just need to replace  $(\partial_y^2 u_0^s)(\tilde{y})$  by  $\chi(\tilde{y})(\partial_y^2 u_0^s)(\tilde{y})$  where  $\chi \in C^\infty(\mathbb{R}_+)$  with  $\chi(y) = 1, y \geq y_0 - 1$  and  $\chi(y) = 0, y \leq y_0 - 2$ .

For the lower bound (4) of  $|\partial_y u^s(t, y)|$  on  $[a + 2a_0, +\infty[$ , we use the maximal principal, similar to [19]. We suppose that  $\partial_y u_0^s(y) > 0$  for  $y \geq R_0 = a + 2a_0$ , denote  $H(t, y) = \langle y \rangle^k \partial_y u^s(t, y)$  (if  $\partial_y u_0^s(y) < 0$  for  $y \geq R_0$  and take  $H(t, y) = -\langle y \rangle^k \partial_y u^s(t, y)$ ). Then  $H(t, y)$  satisfies the following equation

$$\begin{cases} \partial_t H(t, y) - \partial_y^2 H(t, y) + h_1(y) H_y(t, y) + h_2(y) H(t, y) = 0, \\ H|_{y=R_0} = \langle R_0 \rangle^k \partial_y u^s(t, R_0), \\ H|_{t=0} = \langle y \rangle^k \partial_y u_0^s(y), \end{cases}$$

where

$$h_1(y) = 2 \frac{\partial_y \langle y \rangle^k}{\langle y \rangle^k}, \quad h_2(y) = -2 \left( \frac{\partial_y \langle y \rangle^k}{\langle y \rangle^k} \right)^2 + \frac{\partial_y^2 \langle y \rangle^k}{\langle y \rangle^k}.$$

Firstly noticing

$$|h_1(y)| \leq 2k, \quad |h_2(y)| \leq 8k^2,$$

then we have

$$\begin{aligned} \partial_t (H(t, y) e^{-8k^2 t}) - \partial_y^2 (H(t, y) e^{-8k^2 t}) + h_1(y) \partial_y (H(t, y) e^{-8k^2 t}) \\ + \left( h_2(y) + 8k^2 \right) (H(t, y) e^{-8k^2 t}) = 0, \end{aligned}$$

where the key point is that  $h_2(y) + 8k^2 \geq 0$ .

Denoting

$$b(T_1) = \min \left\{ \min_{0 \leq s \leq T_1} H(s, R_0), \min_{y \geq R_0} H(0, y) \right\} \geq \frac{3c_0}{4},$$

and, for  $\nu > 0$

$$E(t, y) = \left( H(t, y) - b(T_1) \right) e^{-8k^2 t} + (8k^2 b(T_1) + \nu 2k)t + \nu \ln(1 + y),$$

also  $E(t, y)$  satisfies

$$\partial_t E(t, y) - \partial_y^2 E(t, y) + h_1(y) E_y(t, y) + \left( h_2(y) + 8k^2 \right) E(t, y) \geq 0.$$

Using the upper bound of (8),  $|H(t, y)|_{L^\infty([0, T_1] \times \mathbb{R}^+)} \leq 2B_p$ , setting

$$R_\nu = \exp \left\{ \frac{(|H(t, y)|_{L^\infty([0, T_1] \times \mathbb{R}^+)} + b(T_1))}{\nu} \right\} - 1,$$

we have

$$\begin{aligned} E(t, y)|_{y=R_\nu} &= \left( H(t, R_\nu) - b(T_1) \right) e^{-8k^2 t} + (8k^2 b(T_1) + \nu 2k)t \\ &\quad + (|H(t, y)|_{L^\infty([0, T_1] \times \mathbb{R}^+)} + b(T_1)) \geq 0, \end{aligned}$$

and the choose of  $b(T_1)$  imply also

$$E(t, y)|_{y=R_0} \geq 0, \quad t \in [0, T_1], \quad \text{and } E(t, y)|_{t=0} \geq 0, \quad y \geq R_0.$$

Now thanks to the maximal principal of heat equation, we have

$$E(t, y) \geq 0, \quad (t, y) \in [0, T_1] \times [R_0, R_\nu].$$

Let  $\nu \rightarrow 0$ , we get

$$H(t, y) \geq \frac{3c_1}{4} - (8k^2)te^{8k^2t}.$$

Then we can choose  $T_1$  such that

$$(8k^2)T_1e^{8k^2T_1} \leq \frac{c_1}{4},$$

then

$$H(t, y) \geq \frac{c_1}{2}, \quad (t, y) \in [0, T_1] \times [R_0, \infty[.$$

The lower bound (6) of  $|\partial_y^2 u^s(t, y)|$  on the bounded interval  $[a + 2a_0, y_0 + 1]$  can be obtained by using the continuity of  $\tilde{H}(t, y) = \langle y \rangle^{k+1} \partial_y^2 u^s(t, y)$  with the initial data  $\tilde{H}(0, y) = \langle y \rangle^{k+1} \partial_y^2 u_0^s(y) \geq 2c_3 > 0$  on  $[a + a_0, y_0 + 2]$ .  $\square$

**2.3. Compatibility conditions and reduction of boundary data.** We give now the precise version of the compatibility condition for the nonlinear system (2.2) and the reduction properties of boundary data.

**Proposition 2.2.** *Let  $m \geq 6$  be an even integer, and assume that  $\tilde{u}$  is a smooth solution of the system (2.2), then the initial data  $\tilde{u}_0$  have to satisfy the following compatibility conditions up to order  $m + 2$ :*

$$\begin{cases} \tilde{u}_0(x, 0) = 0, & (\partial_y^2 \tilde{u}_0)(x, 0) = 0, \quad \forall x \in \mathbb{R}, \\ (\partial_y^4 \tilde{u}_0)(x, 0) = (\partial_y u_0^s(0) + (\partial_y \tilde{u}_0)(x, 0))(\partial_y \partial_x \tilde{u}_0)(x, 0), \quad \forall x \in \mathbb{R}, \end{cases} \quad (2.6)$$

and for  $4 \leq 2p \leq m$ ,

$$(\partial_y^{2(p+1)} \tilde{u}_0)(x, 0) = \sum_{q=2}^p \sum_{(\alpha, \beta) \in \Lambda_q} C_{\alpha, \beta} \prod_{j=1}^q \partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u_0^s + \tilde{u}_0)|_{y=0}, \quad \forall x \in \mathbb{R}, \quad (2.7)$$

where

$$\begin{aligned} \Lambda_q = & \left\{ (\alpha, \beta) = (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_q) \in \mathbb{N}^q \times \mathbb{N}^q; \right. \\ & \alpha_j + \beta_j \leq 2p - 1, \quad 1 \leq j \leq q; \quad \sum_{j=1}^q 3\alpha_j + \beta_j = 2p + 1; \\ & \left. \sum_{j=1}^q \beta_j \leq 2p - 2, \quad 0 < \sum_{j=1}^q \alpha_j \leq p - 1 \right\}. \end{aligned} \quad (2.8)$$

Remark that for  $\alpha_j > 0$ , we have  $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s + \tilde{u}) = \partial_x^{\alpha_j} \partial_y^{\beta_j+1} \tilde{u}$ . So the condition  $0 < \sum_{j=1}^q \alpha_j$  implies that, for each terms of (2.7), there is at last one factor like  $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} \tilde{u}_0$ .

*Proof.* By the assumption of this Proposition,  $\tilde{u}$  is a smooth solution. If we need the existence of the trace of  $\partial_y^{m+2}\tilde{u}$  on  $y = 0$ , then we at least need to assume that  $\tilde{u} \in L^\infty([0, T]; H_{k+\ell-1}^{m+3}(\mathbb{R}_+^2))$ .

Recalling the boundary condition in (2.2):

$$\tilde{u}(t, x, 0) = 0, \quad \tilde{v}(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R},$$

then the following is obvious:

$$(\partial_t \partial_x^n \tilde{u})(t, x, 0) = 0, \quad (\partial_t \partial_x^n \tilde{v})(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad 0 \leq n \leq m.$$

Thus the first result of (2.6) is exactly the compatibility of the solution with the initial data at  $t = 0$ . For the second result of (2.6), using the equation of (2.2), we find that, for  $0 \leq n \leq m$

$$(\partial_y^2 \partial_x^n \tilde{u})(t, x, 0) = 0, \quad (\partial_t \partial_y^2 \partial_x^n \tilde{u})(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Derivating the equation of (2.2) with  $y$ ,

$$\partial_t \partial_y \tilde{u} + \partial_y((u^s + \tilde{u})\partial_x \tilde{u}) + \partial_y(\tilde{v}(u_y^s + \partial_y \tilde{u})) = \partial_y^3 \tilde{u},$$

observing

$$[\partial_y((u^s + \tilde{u})\partial_x \tilde{u}) + \partial_y(\tilde{v}(u_y^s + \partial_y \tilde{u}))]|_{y=0} = 0,$$

then we get

$$(\partial_t \partial_y \tilde{u})|_{y=0} = (\partial_y^3 \tilde{u})|_{y=0}.$$

Derivating again the equation of (2.2) with  $y$ ,

$$\partial_t \partial_y^2 \tilde{u} + \partial_y^2((u^s + \tilde{u})\partial_x \tilde{u}) + \partial_y^2(\tilde{v}(u_y^s + \partial_y \tilde{u})) = \partial_y^4 \tilde{u},$$

using Leibniz formula

$$\begin{aligned} & \partial_y^2((u^s + \tilde{u})\partial_x \tilde{u}) + \partial_y^2(\tilde{v}(u_y^s + \partial_y \tilde{u})) \\ &= (\partial_y^2(u^s + \tilde{u}))\partial_x \tilde{u} + (\partial_y^2 \tilde{v})(u_y^s + \partial_y \tilde{u}) \\ & \quad + (u^s + \tilde{u})\partial_y^2 \partial_x \tilde{u} + \tilde{v}\partial_y^2(u_y^s + \partial_y \tilde{u}) \\ & \quad + 2(\partial_y(u^s + \tilde{u}))\partial_y \partial_x \tilde{u} + 2(\partial_y \tilde{v})\partial_y(u_y^s + \partial_y \tilde{u}), \end{aligned}$$

thus,

$$(\partial_y^4 \tilde{u})(t, x, 0) = (u_y^s(t, 0) + (\partial_y \tilde{u})(t, x, 0))(\partial_y \partial_x \tilde{u})(t, x, 0),$$

and

$$\begin{aligned} (\partial_t \partial_y^4 \tilde{u})(t, x, 0) &= (\partial_y u^s(t, 0) + (\partial_y \tilde{u})(t, x, 0))((\partial_y^3 \partial_x \tilde{u})(t, x, 0)) \\ & \quad + (\partial_y^3 u^s(t, 0) + (\partial_y^3 \tilde{u})(t, x, 0))((\partial_y \partial_x \tilde{u})(t, x, 0)). \end{aligned} \tag{2.9}$$

For  $p = 2$ , we have

$$\partial_t \partial_y^4 \tilde{u} + \partial_y^4((u^s + \tilde{u})\partial_x \tilde{u}) + \partial_y^4(\tilde{v}(u_y^s + \partial_y \tilde{u})) = \partial_y^6 \tilde{u},$$

using Leibniz formula

$$\begin{aligned} & \partial_y^4((u^s + \tilde{u})\partial_x \tilde{u}) + \partial_y^4(\tilde{v}(u_y^s + \partial_y \tilde{u})) \\ &= (\partial_y^4(u^s + \tilde{u}))\partial_x \tilde{u} + (\partial_y^4 \tilde{v})(u_y^s + \partial_y \tilde{u}) \\ & \quad + (u^s + \tilde{u})\partial_y^4 \partial_x \tilde{u} + \tilde{v}\partial_y^4(u_y^s + \partial_y \tilde{u}) \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4((\partial_y^j(u^s + \tilde{u}))\partial_y^{4-j} \partial_x \tilde{u} + (\partial_y^j \tilde{v})\partial_y^{4-j}(u_y^s + \partial_y \tilde{u})), \end{aligned}$$

thus, by (2.9)

$$\begin{aligned}
(\partial_y^6 \tilde{u})(t, x, 0) &= (\partial_t \partial_y^4 \tilde{u})(t, x, 0) - (\partial_y^3 \partial_x u)(u_y^s + \partial_y \tilde{u})(t, x, 0) \\
&\quad + \sum_{1 \leq j \leq 3} C_j^4 [(\partial_y^j (u^s + \tilde{u})) \partial_y^{4-j} \partial_x \tilde{u} + (\partial_y^j \tilde{v}) \partial_y^{4-j} (u_y^s + \partial_y \tilde{u})](t, x, 0) \\
&= (\partial_y^3 u^s(t, 0) + (\partial_y^3 \tilde{u})(t, x, 0)) (\partial_y \partial_x \tilde{u})(t, x, 0) \\
&\quad + \sum_{1 \leq j \leq 3} C_j^4 [(\partial_y^j (u^s + \tilde{u})) \partial_y^{4-j} \partial_x \tilde{u} - (\partial_y^{j-1} \partial_x \tilde{u}) \partial_y^{4-j} (u_y^s + \partial_y \tilde{u})](t, x, 0).
\end{aligned} \tag{2.10}$$

Taking the values at  $t = 0$ , we have proven (2.7) for  $p = 2$ . The case of  $p \geq 3$  is then by induction.  $\square$

**Remark 2.3.** *By the similar methods, we can prove that if  $\tilde{u}$  is a smooth solution of the system (2.2), then we have*

$$\begin{cases} \tilde{u}(t, x, 0) = 0, & (\partial_y^2 \tilde{u})(t, x, 0) = 0, & \forall (t, x) \in [0, T] \times \mathbb{R}, \\ (\partial_y^4 \tilde{u})(t, x, 0) = (u_y^s(t, 0) + (\partial_y \tilde{u})(t, x, 0)) (\partial_y \partial_x \tilde{u})(t, x, 0), & \forall (t, x) \in [0, T] \times \mathbb{R}, \end{cases}$$

and for  $4 \leq 2p \leq m$ ,

$$(\partial_y^{2(p+1)} \tilde{u})(t, x, 0) = \sum_{q=2}^p \sum_{(\alpha, \beta) \in \Lambda_q} C_{\alpha, \beta} \prod_{j=1}^q \partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s(t, 0) + \tilde{u}(t, x, 0)), \tag{2.11}$$

for all  $(t, x) \in [0, T] \times \mathbb{R}$ , where  $\Lambda_q$  is defined in (2.8).

See Lemma 5.9 of [19] and Lemma 4 of [7] for the similar results.

Remark that the condition  $0 < \sum_{j=1}^q \alpha_j$  implies that, for each terms of (2.11), there is at last one factor like  $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} \tilde{u}(t, x, 0)$ .

### 3. THE APPROXIMATE SOLUTIONS

To prove the existence of solution of the Prandtl equation, we study a parabolic regularized equation for which we can get the existence by using the classical energy method.

**3.1. Nonlinear regularized Prandtl equation.** In this section, we study the following nonlinear regularized Prandtl equation, for  $0 < \epsilon \leq 1$ ,

$$\begin{cases} \partial_t \tilde{u}_\epsilon + (u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon + v_\epsilon (u_y^s + \partial_y \tilde{u}_\epsilon) = \partial_y^2 \tilde{u}_\epsilon + \epsilon \partial_x^2 \tilde{u}_\epsilon, \\ \partial_x \tilde{u}_\epsilon + \partial_y v_\epsilon = 0, \\ \tilde{u}_\epsilon|_{y=0} = v_\epsilon|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} \tilde{u}_\epsilon = 0, \\ \tilde{u}_\epsilon|_{t=0} = \tilde{u}_{0, \epsilon} = \tilde{u}_0 + \epsilon \mu_\epsilon, \end{cases} \tag{3.1}$$

where we choose the corrector  $\epsilon \mu_\epsilon$  such that  $\tilde{u}_0 + \epsilon \mu_\epsilon$  satisfies the compatibility condition up to order  $m + 2$  for the regularized system (3.1).

We study now the boundary data of the solution for the regularized nonlinear system (3.1) which give also the precise version of the compatibility condition for the system (3.1).

**Proposition 3.1.** *Let  $m \geq 6$  be an even integer, and assume that  $\tilde{u}_0$  satisfies the compatibility conditions (2.6) and (2.7) for the system (2.2), and  $\mu_\epsilon \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  such that  $\tilde{u}_0 + \epsilon \mu_\epsilon$  satisfies the compatibility conditions up to order  $m + 2$  for the*

regularized system (3.1). If  $\tilde{u}_\epsilon \in L^\infty([0, T]; H_{k+\ell}^{m+3}(\mathbb{R}_+^2)) \cap Lip([0, T]; H_{k+\ell}^{m+1}(\mathbb{R}_+^2))$  is a solution of the system (3.1), then we have

$$\begin{cases} \tilde{u}_\epsilon(t, x, 0) = 0, \quad (\partial_y^2 \tilde{u}_\epsilon)(t, x, 0) = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \\ (\partial_y^4 \tilde{u}_\epsilon)(t, x, 0) = (u_y^s(t, 0) + (\partial_y \tilde{u}_\epsilon)(t, x, 0))(\partial_y \partial_x \tilde{u}_\epsilon)(t, x, 0), \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \end{cases}$$

and for  $4 \leq 2p \leq m$ ,

$$\begin{aligned} (\partial_y^{2(p+1)} \tilde{u}_\epsilon)(t, x, 0) &= \sum_{q=2}^p \sum_{l=0}^{q-1} \epsilon^l \sum_{(\alpha^l, \beta^l) \in \Lambda_q^l} C_{\alpha^l, \beta^l} \\ &\quad \times \prod_{j=1}^q \partial_x^{\alpha_j^l} \partial_y^{\beta_j^l + 1} (u^s(t, 0) + \tilde{u}_\epsilon(t, x, 0)), \end{aligned} \quad (3.2)$$

for all  $(t, x) \in [0, T] \times \mathbb{R}$ , where

$$\begin{aligned} \Lambda_q^l &= \left\{ (\alpha, \beta) = (\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_p) \in \mathbb{N}^q \times \mathbb{N}^q; \right. \\ &\quad \alpha_j + \beta_j \leq 2p - 1, \quad 1 \leq j \leq q; \quad \sum_{j=1}^q 3\alpha_j + \beta_j = 2p + 4l + 1; \\ &\quad \left. \sum_{j=1}^q \beta_j \leq 2p - 2l - 2, \quad 0 < \sum_{j=1}^q \alpha_j \leq p + 2l - 1 \right\}. \end{aligned}$$

Remark that the condition  $0 < \sum_{j=1}^q \alpha_j^l$  implies that, for each terms of (3.2), there are at last one factor like  $\partial_x^{\alpha_j^l} \partial_y^{\beta_j^l + 1} \tilde{u}_\epsilon(t, x, 0)$ .

*Proof.* Firstly, for  $p \leq \frac{m}{2}$ , we have  $\partial_y^{2p+2} \tilde{u}_\epsilon \in L^\infty([0, T]; H_{k+\ell+2p+1}^1(\mathbb{R}_+^2))$ . So the trace of  $\partial_y^{2p+2} \tilde{u}_\epsilon$  exists on  $y = 0$ .

Using the boundary condition of (3.1), we have, for  $0 \leq n \leq m + 2$ ,

$$\partial_x^n \tilde{u}_\epsilon(t, x, 0) = 0, \quad \partial_x^n v_\epsilon(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R},$$

and for  $0 \leq n \leq m$

$$(\partial_t \partial_x^n \tilde{u}_\epsilon)(t, x, 0) = 0, \quad (\partial_t \partial_x^n v_\epsilon)(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

From the equation of (3.1), we get also

$$(\partial_y^2 \partial_x^n \tilde{u}_\epsilon)(t, x, 0) = 0, \quad (\partial_t \partial_y^2 \partial_x^n \tilde{u}_\epsilon)(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}. \quad (3.3)$$

On the other hand,

$$\partial_t \partial_y \tilde{u}_\epsilon + \partial_y((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y(v_\epsilon(u_y^s + \partial_y \tilde{u}_\epsilon)) = \partial_y^3 \tilde{u}_\epsilon + \epsilon \partial_x^2 \partial_y \tilde{u}_\epsilon,$$

observing

$$\left[ \partial_y((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y(v_\epsilon(u_y^s + \partial_y \tilde{u}_\epsilon)) \right] \Big|_{y=0} = 0,$$

we get

$$(\partial_t \partial_y \tilde{u}_\epsilon) \Big|_{y=0} = (\partial_y^3 \tilde{u}_\epsilon) \Big|_{y=0} + \epsilon (\partial_x^2 \partial_y \tilde{u}_\epsilon) \Big|_{y=0}.$$

We have also

$$\partial_t \partial_y^2 \tilde{u}_\epsilon + \partial_y^2((u^s + \tilde{u}_\epsilon) \partial_x \tilde{u}_\epsilon) + \partial_y^2(v_\epsilon(u_y^s + \partial_y \tilde{u}_\epsilon)) = \partial_y^4 \tilde{u}_\epsilon + \epsilon \partial_x^2 \partial_y^2 \tilde{u}_\epsilon,$$

using Leibniz formula

$$\begin{aligned} & \partial_y^2((u^s + \tilde{u}_\epsilon)\partial_x \tilde{u}_\epsilon) + \partial_y^2(v_\epsilon(u_y^s + \partial_y \tilde{u}_\epsilon)) \\ &= (\partial_y^2(u^s + \tilde{u}_\epsilon))\partial_x \tilde{u}_\epsilon + (\partial_y^2 v_\epsilon)(u_y^s + \partial_y \tilde{u}_\epsilon) \\ & \quad + (u^s + \tilde{u}_\epsilon)\partial_y^2 \partial_x \tilde{u}_\epsilon + v_\epsilon \partial_y^2(u_y^s + \partial_y \tilde{u}_\epsilon) \\ & \quad + 2(\partial_y(u^s + \tilde{u}_\epsilon))\partial_y \partial_x \tilde{u}_\epsilon + 2(\partial_y v_\epsilon)\partial_y(u_y^s + \partial_y \tilde{u}_\epsilon), \end{aligned}$$

thus,

$$(\partial_y^4 \tilde{u}_\epsilon)(t, x, 0) = (u_y^s(t, 0) + (\partial_y \tilde{u}_\epsilon)(t, x, 0)) (\partial_y \partial_x \tilde{u}_\epsilon)(t, x, 0). \quad (3.4)$$

Applying  $\partial_t$  to (3.4), we have

$$\begin{aligned} (\partial_t \partial_y^4 \tilde{u}_\epsilon)(t, x, 0) &= (\partial_y^3 u^s(t, 0) + (\partial_y^3 \tilde{u}_\epsilon)(t, x, 0) + \epsilon(\partial_x^2 \partial_y \tilde{u}_\epsilon)(t, x, 0)) (\partial_y \partial_x \tilde{u}_\epsilon)(t, x, 0) \\ & \quad + (u_y^s(t, 0) + (\partial_y \tilde{u}_\epsilon)(t, x, 0)) ((\partial_y^3 \partial_x \tilde{u}_\epsilon)(t, x, 0) + \epsilon(\partial_x^3 \partial_y \tilde{u}_\epsilon)(t, x, 0)). \end{aligned}$$

On the other hand, we have

$$\partial_t \partial_y^4 \tilde{u}_\epsilon + \partial_y^4((u^s + \tilde{u}_\epsilon)\partial_x \tilde{u}_\epsilon) + \partial_y^4(v_\epsilon(u_y^s + \partial_y \tilde{u}_\epsilon)) = \partial_y^6 \tilde{u}_\epsilon + \epsilon \partial_x^2 \partial_y^4 \tilde{u}_\epsilon,$$

using Leibniz formula

$$\begin{aligned} & \partial_y^4((u^s + \tilde{u}_\epsilon)\partial_x \tilde{u}_\epsilon) + \partial_y^4(v_\epsilon(u_y^s + \partial_y \tilde{u}_\epsilon)) \\ &= (\partial_y^4(u^s + \tilde{u}_\epsilon))\partial_x \tilde{u}_\epsilon + (\partial_y^4 v_\epsilon)(u_y^s + \partial_y \tilde{u}_\epsilon) \\ & \quad + (u^s + \tilde{u}_\epsilon)\partial_y^4 \partial_x \tilde{u}_\epsilon + v_\epsilon \partial_y^4(u_y^s + \partial_y \tilde{u}_\epsilon) \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4 ((\partial_y^j(u^s + \tilde{u}_\epsilon))\partial_y^{4-j} \partial_x \tilde{u}_\epsilon + (\partial_y^j v_\epsilon)\partial_y^{4-j}(u_y^s + \partial_y \tilde{u}_\epsilon)), \end{aligned}$$

thus,

$$\begin{aligned} (\partial_y^6 \tilde{u}_\epsilon)(t, x, 0) &= (\partial_t \partial_y^4 \tilde{u}_\epsilon)(t, x, 0) - (\partial_y^3 \partial_x u_\epsilon)(u_y^s + \partial_y \tilde{u}_\epsilon)(t, x, 0) \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4 [(\partial_y^j(u^s + \tilde{u}_\epsilon))\partial_y^{4-j} \partial_x \tilde{u}_\epsilon + (\partial_y^j v_\epsilon)\partial_y^{4-j}(u_y^s + \partial_y \tilde{u}_\epsilon)](t, x, 0) \\ & \quad - \underline{\epsilon \partial_x^2 \partial_y^4 \tilde{u}_\epsilon(t, x, 0)}. \end{aligned}$$

Using (3.4), we get then

$$\begin{aligned} (\partial_y^6 \tilde{u}_\epsilon)(t, x, 0) &= (\partial_y^3 u^s(t, 0) + \partial_y^3 \tilde{u}_\epsilon(t, x, 0)) \partial_y \partial_x \tilde{u}_\epsilon(t, x, 0) \\ & \quad - \underline{2\epsilon \partial_x \partial_y \tilde{u}_\epsilon(t, x, 0) (\partial_y \partial_x^2 \tilde{u}_\epsilon)(t, x, 0)} \\ & \quad + \sum_{1 \leq j \leq 3} C_j^4 [(\partial_y^j(u^s + \tilde{u}_\epsilon))\partial_y^{4-j} \partial_x \tilde{u}_\epsilon - \partial_y^{j-1} \partial_x \tilde{u}_\epsilon \partial_y^{4-j}(u_y^s + \partial_y \tilde{u}_\epsilon)](t, x, 0), \end{aligned} \quad (3.5)$$

Compared to (2.10), the underlined term is the new term.

This is the Proposition 3.1 for  $p = 2$ . We can complete the proof of Proposition 3.1 by induction.  $\square$

The proof of the above Proposition implies also the following result.

**Corollary 3.2.** *Let  $m \geq 6$  be an even integer, assume that  $\tilde{u}_0$  satisfies the compatibility conditions (2.6) - (2.7) for the system (2.2) and  $\partial_y \tilde{u}_0 \in H_{k+\ell'}^{m+2}(\mathbb{R}_+^2)$ , then there exists  $\epsilon_0 > 0$ , and for any  $0 < \epsilon \leq \epsilon_0$  there exists  $\mu_\epsilon \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  such that*

$\tilde{u}_0 + \epsilon\mu_\epsilon$  satisfies the compatibility condition up to order  $m + 2$  for the regularized system (3.1). Moreover, for any  $m \leq \tilde{m} \leq m + 2$

$$\|\partial_y \tilde{u}_{0,\epsilon}\|_{H_{k+\ell'}^{\tilde{m}}(\mathbb{R}_+^2)} \leq \frac{3}{2} \|\partial_y \tilde{u}_0\|_{H_{k+\ell'}^{\tilde{m}}(\mathbb{R}_+^2)},$$

and

$$\lim_{\epsilon \rightarrow 0} \|\partial_y \tilde{u}_{0,\epsilon} - \partial_y \tilde{u}_0\|_{H_{k+\ell'}^{\tilde{m}}(\mathbb{R}_+^2)} = 0.$$

*Proof.* We use the proof of the Proposition 3.1.

Taking the values at  $t = 0$  for (3.3), then (2.6) implies that the function  $\mu_\epsilon$  satisfies

$$(\partial_x^n \mu_\epsilon)(x, 0) = 0, \quad (\partial_y^2 \partial_x^n \mu_\epsilon)(x, 0) = 0, \quad x \in \mathbb{R}.$$

Taking  $t = 0$  for (3.4), we have

$$\begin{aligned} (\partial_y^4 \tilde{u}_0)(x, 0) + \epsilon(\partial_y^4 \mu_\epsilon)(x, 0) &= [\partial_y u_0^s(0) + (\partial_y \tilde{u}_0)(x, 0) + \epsilon(\partial_y \mu_\epsilon)(x, 0)] \\ &\quad \times [(\partial_y \partial_x \tilde{u}_0)(x, 0) + \epsilon(\partial_y \partial_x \mu_\epsilon)(x, 0)], \end{aligned}$$

using (2.6), we have that  $\mu_\epsilon$  satisfies

$$\begin{aligned} (\partial_y^4 \mu_\epsilon)(x, 0) &= (\partial_y u_0^s(0) + (\partial_y \tilde{u}_0)(x, 0))(\partial_y \partial_x \mu_\epsilon)(x, 0) \\ &\quad + (\partial_y \mu_\epsilon)(x, 0)(\partial_y \partial_x \tilde{u}_0)(x, 0) \\ &\quad + \epsilon(\partial_y \partial_x \mu_\epsilon)(x, 0)(\partial_y \partial_x \mu_\epsilon)(x, 0). \end{aligned}$$

We have also

$$\begin{aligned} (\partial_t \partial_y^4 \tilde{u}_\epsilon)(0, x, 0) &= (\partial_y^3 u_0^s(0) + (\partial_y^3 \tilde{u}_\epsilon)(0, x, 0) + \epsilon(\partial_x^2 \partial_y \tilde{u}_\epsilon)(0, x, 0)) \\ &\quad \times ((\partial_y^3 \partial_x \tilde{u}_\epsilon)(0, x, 0) + \epsilon(\partial_x^3 \partial_y \tilde{u}_\epsilon)(0, x, 0)). \end{aligned}$$

Taking the values at  $t = 0$  for (3.5), we obtain a restraint condition for  $(\partial_y^6 \mu_\epsilon)(x, 0)$ ,

$$\begin{aligned} \partial_y^6 \mu_\epsilon(x, 0) &= ((\partial_y^3 u_0^s + \partial_y^3 \tilde{u}_0) \partial_y \partial_x \mu_\epsilon)|_{y=0} + \partial_y^3 \mu_\epsilon \partial_y \partial_x \tilde{u}_0|_{y=0} + \epsilon \partial_y^3 \mu_\epsilon \partial_y \partial_x \mu_\epsilon|_{y=0} \\ &\quad - \underline{2\partial_x \partial_y \tilde{u}_0(x, 0) (\partial_y \partial_x^2 \tilde{u}_0)(x, 0)} - \underline{2\epsilon \partial_x \partial_y \tilde{u}_0(x, 0) (\partial_y \partial_x^2 \mu_\epsilon)(t, x, 0)} \\ &\quad - \underline{2\epsilon \partial_x \partial_y \mu_\epsilon(t, x, 0) (\partial_y \partial_x^2 \tilde{u}_0)(t, x, 0)} - \underline{2\epsilon^2 \partial_x \partial_y \mu_\epsilon(x, 0) (\partial_y \partial_x^2 \mu_\epsilon)(x, 0)} \\ &\quad + \sum_{1 \leq j \leq 3} C_j^4 [\partial_y^j (u_0^s + \tilde{u}_0) \partial_y^{4-j} \partial_x \mu_\epsilon + \partial_y^j \mu \partial_y^{4-j} \partial_x \tilde{u}_0 + \epsilon \partial_y^j \mu \partial_y^{4-j} \partial_x \mu_\epsilon]|_{y=0} \\ &\quad - \sum_{1 \leq j \leq 3} C_j^4 [\partial_y^{j-1} \partial_x \tilde{u}_0 \partial_y^{4-j} \mu_\epsilon + \epsilon \partial_y^{j-1} \partial_x \mu_\epsilon \partial_y^{4-j} \partial_y \mu_\epsilon]|_{y=0} \\ &\quad - \sum_{1 \leq j \leq 3} C_j^4 \partial_y^{j-1} \partial_x \mu_\epsilon \partial_y^{4-j} (\partial_y u_0^s + \partial_y \tilde{u}_0)|_{y=0}, \end{aligned}$$

thus

$$\begin{aligned} \partial_y^6 \mu_\epsilon(x, 0) &= - \frac{2\partial_x \partial_y \tilde{u}_0(x, 0) (\partial_y \partial_x^2 \tilde{u}_0)(x, 0)}{\phantom{+}} \\ &\quad + \sum_{\alpha_1, \beta_1; \alpha_2, \beta_2} C_{\alpha_1, \beta_1; \alpha_2, \beta_2} \partial_x^{\alpha_1} \partial_y^{\beta_1+1} (u_0^s + \tilde{u}_0) \partial_x^{\alpha_1} \partial_y^{\beta_1+1} \mu_\epsilon(x, 0) \\ &\quad + \sum_{\alpha_1, \beta_1; \alpha_2, \beta_2} C_{\alpha_1, \beta_1; \alpha_2, \beta_2} \partial_x^{\alpha_1} \partial_y^{\beta_1+1} \mu_\epsilon \partial_x^{\alpha_1} \partial_y^{\beta_1+1} \mu_\epsilon(x, 0), \end{aligned} \tag{3.6}$$

where the summation is for the index  $\alpha_2 + \beta_2 \leq 3$ ;  $\alpha_1 + \beta_1 + \alpha_2 + \beta_2 \leq 3$ . The underlined term in the above equality is deduced from the underlined term in (3.5). All these underlined terms are from the added regularizing term  $\epsilon \partial_x^2 \tilde{u}$  in the



equation (3.1). This means that the regularizing term  $\epsilon \partial_x^2 \tilde{u}$  has an affect on the boundary. This is why we add a corrector term.

More generally, for  $6 \leq 2p \leq m$ , we have that  $(\partial_y^{2(p+1)} \mu_\epsilon)(x, 0)$  is a linear combination of the terms of the form

$$\prod_{j=1}^{q_1} \left( \partial_x^{\alpha_j^1} \partial_y^{\beta_j^1+1} (u_0^s + \tilde{u}_0) \right) \Big|_{y=0}, \quad \prod_{i=1}^{q_2} \left( \partial_x^{\alpha_i^2} \partial_y^{\beta_i^2+1} \mu_\epsilon \right) \Big|_{y=0},$$

and

$$\prod_{j=1}^{q_1} \left( \partial_x^{\alpha_j^1} \partial_y^{\beta_j^1+1} (u_0^s + \tilde{u}_0) \right) \Big|_{y=0} \times \prod_{i=1}^{q_2} \left( \partial_x^{\alpha_i^2} \partial_y^{\beta_i^2+1} \mu_\epsilon \right) \Big|_{y=0},$$

where the coefficients of the combination can be depends on  $\epsilon$  but with a non-negative power. We have also  $\alpha_j^l + \beta_j^l + 1 \leq 2p, l = 1, 2$ , thus  $(\partial_y^{2(p+1)} \mu_\epsilon)(x, 0)$  is determined by the low order derivatives of  $\mu_\epsilon$  and these of  $\tilde{u}_0$ .

We now construct a polynomial function  $\tilde{\mu}_\epsilon$  on  $y$  by the following Taylor expansion,

$$\tilde{\mu}_\epsilon(x, y) = \sum_{p=3}^{\frac{m}{2}+1} \tilde{\mu}_\epsilon^{2p}(x) \frac{y^{2p}}{(2p)!},$$

where

$$\tilde{\mu}_\epsilon^6(x) = -2(\partial_x \partial_y \tilde{u}_0)(x, 0)(\partial_y \partial_x^2 \tilde{u}_0)(x, 0),$$

and  $\tilde{\mu}_\epsilon^{2p}(x)$  will give successively by  $(\partial_y^{2q} \mu_\epsilon)(x, 0)$  with  $(\partial_y^{2q+1} \mu_\epsilon)(x, 0) = 0, q = 0, \dots, m$ , and it is then determined by  $(\partial_x^\alpha \partial_y^\beta \tilde{u}_0)|_{y=0}$ . Finally we take  $\mu_\epsilon = \chi(y) \tilde{\mu}_\epsilon$  with  $\chi \in C^\infty([0, +\infty[); \chi(y) = 1, 0 \leq y \leq 1; \chi(y) = 0, y \geq 2$ . Thus we complete the proof of the Corollary.  $\square$

**Remark 3.3.** Suppose that  $\tilde{u}_0$  satisfies the compatibility conditions up to order  $m+2$  for the system (2.2) with  $m \geq 4$ , then for the regularized system (3.1), if we want to obtain the smooth solution  $\tilde{w}_\epsilon$ , we have to add a non-trivial corrector  $\mu_\epsilon$  to the initial data such that  $\tilde{u}_0 + \epsilon \mu_\epsilon$  satisfies the compatibility conditions up to order  $m+2$  for the system (3.1). In fact, if we take  $\mu_\epsilon$  with

$$(\partial_y^j \mu_\epsilon)(x, 0) = 0, \quad 0 \leq j \leq 5,$$

then (3.6) implies

$$(\partial_y^6 \mu_\epsilon)(x, 0) = -2(\partial_x \partial_y \tilde{u}_0)(x, 0)(\partial_y \partial_x^2 \tilde{u}_0)(x, 0),$$

which is not equal to 0. So added a corrector is necessary for the initial data of the regularized system.

We will prove the the existence of the approximate solutions of the system (3.1) by using the following equation of vorticity  $\tilde{w}_\epsilon = \partial_y \tilde{u}_\epsilon$ , it reads

$$\begin{cases} \partial_t \tilde{w}_\epsilon + (u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon + v_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon) = \partial_y^2 \tilde{w}_\epsilon + \epsilon \partial_x^2 \tilde{w}_\epsilon, \\ \partial_y \tilde{w}_\epsilon|_{y=0} = 0, \\ \tilde{w}_\epsilon|_{t=0} = \tilde{w}_{0,\epsilon} = \tilde{w}_0 + \epsilon \partial_y \mu_\epsilon, \end{cases} \quad (3.7)$$

where

$$\tilde{u}_\epsilon(t, x, y) = - \int_y^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y}, \quad \tilde{v}_\epsilon(t, x, y) = - \int_0^y \partial_x \tilde{u}_\epsilon(t, x, \tilde{y}) d\tilde{y}. \quad (3.8)$$

We have the following theorem for the existence of approximate solutions

**Theorem 3.4.** *Let  $\partial_y \tilde{u}_0 \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$ , and  $m \geq 6$  be an even integer,  $k > 1, 0 \leq \ell < \frac{1}{2}, k+\ell > \frac{3}{2}$ , assume that  $\tilde{u}_0$  satisfies the compatibility conditions of order  $m+2$  for the system (2.2). Suppose that the shear flow satisfies*

$$|\partial_y^{p+1} u^s(t, y)| \leq C \langle y \rangle^{-k-p}, \quad (t, y) \in [0, T_1] \times \mathbb{R}_+, \quad 0 \leq p \leq m+2.$$

*Then, for any  $0 < \epsilon \leq \epsilon_0$  and  $0 < \bar{\zeta}$ , there exists  $T_\epsilon > 0$  which depends on  $\epsilon$  and  $\bar{\zeta}$ , such that if*

$$\|\tilde{w}_0\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \bar{\zeta},$$

*then the system (3.7)-(3.8) admits a unique solution*

$$\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2)),$$

*which satisfies*

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \frac{4}{3} \|\tilde{w}_{0, \epsilon}\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq 2 \|\tilde{w}_0\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}. \quad (3.9)$$

**Remark 3.5.** .

- (1) *Remark that  $T_\epsilon$  depends on  $\epsilon$  and  $\bar{\zeta}$ , and  $T_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ . So this is not a bounded estimate for the approximate solution sequences  $\{u^s + \tilde{u}_\epsilon; 0 < \epsilon \leq \epsilon_0\}$  where  $\epsilon_0 > 0$  is given in Corollary 3.2. When the initial data  $\tilde{u}_0$  is small enough, we observe that  $u^s + \tilde{u}_\epsilon$  preserves the monotonicity and convexity of the shear flow on  $[0, T_\epsilon]$ .*
- (2) *In this theorem, for the regularized Prandtl equation, there are not constrain conditions on the initial date, meaning that we don't need the monotonicity or convexity of shear flow  $u^s$ , and  $\bar{\zeta}$  is also arbitrary.*

If  $\tilde{w}_\epsilon$  is a solution of the system (3.7)-(3.8), then (A.1) with  $\lim_{y \rightarrow +\infty} \tilde{u}_\epsilon = 0$  imply

$$\tilde{u}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell-1}^{m+2}(\mathbb{R}_+^2)),$$

and

$$\tilde{v}_\epsilon \in L^\infty([0, T_\epsilon]; L^\infty(\mathbb{R}_{y,+}; H^{m+1}(\mathbb{R}_x))).$$

Integrating the equation of (3.7) over  $[y, +\infty[$  imply that  $(\tilde{u}_\epsilon, \tilde{v}_\epsilon)$  is a solution of the system (3.1), except the boundary condition to check:

$$\tilde{u}_\epsilon(t, x, 0) = - \int_0^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y} = 0, \quad (t, x) \in [0, T_\epsilon] \times \mathbb{R}. \quad (3.10)$$

In fact, noting  $f(t, x) = - \int_0^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y} = \tilde{u}_\epsilon(t, x, 0)$ , a direct calculate give

$$\begin{cases} \partial_t f + f \partial_x f = \epsilon \partial_x^2 f, & (t, x) \in ]0, T_\epsilon] \times \mathbb{R}; \\ f|_{t=0} = 0, \end{cases} \quad (3.11)$$

here we use

$$\begin{aligned} \int_0^{+\infty} v_\epsilon(u_{yy}^s + \partial_y \tilde{w}_\epsilon) dy &= [v_\epsilon(u_y^s + \tilde{w}_\epsilon)]_0^{+\infty} - \int_0^{+\infty} (\partial_y v_\epsilon)(u_y^s + \tilde{w}_\epsilon) dy \\ &= \int_0^{+\infty} (\partial_x u_\epsilon) \partial_y (u^s + \tilde{u}_\epsilon) dy \\ &= [(\partial_x u_\epsilon)(u^s + \tilde{u}_\epsilon)]_0^{+\infty} - \int_0^{+\infty} (\partial_x w_\epsilon)(u^s + \tilde{u}_\epsilon) dy \\ &= -f \partial_x f - \int_0^{+\infty} (\partial_x w_\epsilon)(u^s + \tilde{u}_\epsilon) dy. \end{aligned}$$

Since  $f \in L^\infty([0, T_\epsilon], H^{m+2}(\mathbb{R}))$ , the uniqueness of solution for equation (3.11) imply that  $f = 0$  on  $[0, T_\epsilon] \times \mathbb{R}$ . (3.10) imply also

$$\tilde{u}_\epsilon(t, x, y) = - \int_y^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y} = \int_0^y \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y}, \quad (t, x, y) \in [0, T_\epsilon] \times \mathbb{R}_+^2.$$

We will prove Theorem 3.4 by the following three Propositions, where the first one is devoted to the local existence of approximate solution  $\tilde{w}_\epsilon$  of (3.7).

**Proposition 3.6.** *Let  $\tilde{w}_{0,\epsilon} \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$ ,  $m \geq 6$  be an even integer,  $k > 1, 0 \leq \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ , and satisfy the compatibility conditions up to order  $m + 2$  for (3.7). Suppose that the shear flow satisfies*

$$|\partial_y^{p+1} u^s(t, y)| \leq C \langle y \rangle^{-k-p}, \quad (t, y) \in [0, T_1] \times \mathbb{R}_+, \quad 0 \leq p \leq m + 2.$$

Then, for any  $0 < \epsilon \leq 1$  and  $\bar{\zeta} > 0$ , there exists  $T_\epsilon > 0$  such that if

$$\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \bar{\zeta},$$

then the system (3.7) admits a unique solution

$$\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2)).$$

**Remark 3.7.** *If  $\tilde{w}_0 \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$  is the initial data in Theorem 3.4, using Corollary 3.2, there exists  $\epsilon_0 > 0$ , and for any  $0 < \epsilon \leq \epsilon_0$ , there exists  $\mu_\epsilon \in H_{k+\ell}^{m+3}(\mathbb{R}_+^2)$  such that  $\tilde{w}_{0,\epsilon} = \tilde{w}_0 + \epsilon \partial_y \mu_\epsilon$  satisfies the compatibility conditions up to order  $m + 2$  for the system (3.7), and*

$$\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \frac{3}{2} \|\tilde{w}_0\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}.$$

Then, using Proposition 3.6, we obtain also the existence of the approximate solution under the assumption of Theorem 3.4.

The proof of this Proposition is standard since the equation in (3.7) is a parabolic type equation. Firstly, we establish the *à priori* estimate and then prove the existence of solution by the standard iteration and weak convergence methods. Because we work in the weighted Sobolev space and the computation is not so trivial, we give a detailed proof in the Appendix, to make the paper self-contained. So the rest of this section is devoted to proving the estimate (3.9).

**3.2. Uniform estimate with loss of  $x$ -derivative.** In the proof of the Proposition 3.6 (see Lemma C.2), we already get the *à priori* estimate for  $\tilde{w}_\epsilon$ . Now we try to prove the estimate (3.9) in a new way, and our object is to establish an uniform estimate with respect to  $\epsilon > 0$ . We first treat the easy part in this subsection.

We define the non-isotropic Sobolev norm,

$$\|f\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 = \sum_{|\alpha_1 + \alpha_2| \leq m, \alpha_1 \leq m-1} \|\langle y \rangle^{k+\ell+\alpha_2} \partial_x^{\alpha_1} \partial_y^{\alpha_2} f\|_{L^2(\mathbb{R}_+^2)}^2, \quad (3.12)$$

where we don't have the  $m$ -order derivative with respect to  $x$ -variable. Then

$$\|f\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 = \|f\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + \|\partial_x^m f\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2.$$

**Proposition 3.8.** *Let  $m \geq 6$  be an even integer,  $k > 1, 0 \leq \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ , and assume that  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$  is a solution to (3.7), then we have*

$$\begin{aligned} & \frac{d}{dt} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \\ & + \epsilon \|\partial_x \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \leq C_1 \left( \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^m \right), \end{aligned} \quad (3.13)$$

where  $C_1 > 0$  is independent of  $\epsilon$ .

**Remark.** The above estimate is uniform with respect to  $\epsilon > 0$ , but on the left hand of (3.13), we missing the terms  $\|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2$ . This is because that we can't control the term

$$\partial_x^m \tilde{v}_\epsilon(t, x, y) = - \int_0^y \partial_x^{m+1} \tilde{u}_\epsilon(t, x, \tilde{y}) d\tilde{y},$$

which is the major difficulty in the study of the Prandtl equation. We will study this term in the next Proposition with a non-uniform estimate firstly, and then focus on proving the uniform estimate in the rest part of this paper.

*Proof.* For  $|\alpha| = \alpha_1 + \alpha_2 \leq m, \alpha_1 \leq m - 1$ , we have

$$\begin{aligned} & \partial_t \partial^\alpha \tilde{w}_\epsilon - \epsilon \partial_x^2 \partial^\alpha \tilde{w}_\epsilon - \partial_y^2 \partial^\alpha \tilde{w}_\epsilon \\ & = -\partial^\alpha ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon) - \partial^\alpha (\tilde{v}_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon)). \end{aligned} \quad (3.14)$$

Multiplying the (3.14) with  $\langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \tilde{w}_\epsilon$ , and integrating over  $\mathbb{R}_+^2$ ,

$$\begin{aligned} & \int_{\mathbb{R}_+^2} (\partial_t \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy - \epsilon \int_{\mathbb{R}_+^2} (\partial_x^2 \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy \\ & - \int_{\mathbb{R}_+^2} (\partial_y^2 \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy \\ & = - \int_{\mathbb{R}_+^2} \partial^\alpha ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon - \tilde{v}_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon)) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy. \end{aligned}$$

Remark that for  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ , all above integrations are in the classical sense. We deal with each term on the left hand respectively. After integration by part, we have

$$\begin{aligned} & \int_{\mathbb{R}_+^2} (\partial_t \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy = \frac{1}{2} \frac{d}{dt} \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)}^2, \\ & - \epsilon \int_{\mathbb{R}_+^2} (\partial_x^2 \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy = \epsilon \|\partial_x \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)}^2, \end{aligned}$$

and

$$\begin{aligned} & - \int_{\mathbb{R}_+^2} \partial_y^2 \partial^\alpha \tilde{w}_\epsilon \langle y \rangle^{2(k+\ell)+2\alpha_2} \partial^\alpha \tilde{w}_\epsilon dx dy \\ & = \|\partial_y \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \partial^\alpha \partial_y \tilde{w}_\epsilon (\langle y \rangle^{2(k+\ell)+2\alpha_2})' \partial^\alpha \tilde{w}_\epsilon dx dy \\ & + \int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) |_{y=0} dx. \end{aligned}$$

Cauchy-Schwarz inequality implies

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \partial^\alpha \partial_y \tilde{w}_\epsilon (\langle y \rangle^{2(k+\ell)+2\alpha_2})' \partial^\alpha \tilde{w}_\epsilon dx dy \right| \\ & \leq \frac{1}{16} \|\partial_y \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)}^2 + C \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2-1}^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

We study now the term

$$\int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) |_{y=0} dx.$$

**Case :**  $|\alpha| \leq m-1$ , using the trace Lemma A.2, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) |_{y=0} dx \right| \\ & \leq \|(\partial^\alpha \partial_y \tilde{w}_\epsilon) |_{y=0}\|_{L^2(\mathbb{R})} \|(\partial^\alpha \tilde{w}_\epsilon) |_{y=0}\|_{L^2(\mathbb{R})} \\ & \leq C \|\partial^\alpha \partial_y^2 \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \|\partial^\alpha \partial_y \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \\ & \leq C \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \\ & \leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2. \end{aligned}$$

**Case :**  $\alpha_1 = m-1, \alpha_2 = 1$ , using (3.3), we have

$$(\partial^\alpha \tilde{w}_\epsilon) |_{y=0} = (\partial_x^{\alpha_1} \partial_y^2 \tilde{u}_\epsilon) |_{y=0} = 0,$$

thus

$$\int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) |_{y=0} dx = 0.$$

**Case :**  $\alpha_1 = 0, \alpha_2 = m$ . Only in this case, we need to suppose that  $m$  is even.

Using again the trace Lemma A.2, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}} (\partial_y^{m+1} \tilde{w}_\epsilon \partial_y^m \tilde{w}_\epsilon) |_{y=0} dx \right| \\ & \leq \|(\partial_y^{m+2} \tilde{u}_\epsilon) |_{y=0}\|_{L^2(\mathbb{R})} \|(\partial_y^m \tilde{w}_\epsilon) |_{y=0}\|_{L^2(\mathbb{R})} \\ & \leq C \|(\partial_y^{m+2} \tilde{u}_\epsilon) |_{y=0}\|_{L^2(\mathbb{R})} \|\partial_y^{m+1} \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \\ & \leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|(\partial_y^{m+2} \tilde{u}_\epsilon) |_{y=0}\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Using Proposition 3.1 and the trace Lemma A.2, we can estimate the above last term  $\|(\partial_y^{m+2} \tilde{u}_\epsilon) |_{y=0}\|_{L^2(\mathbb{R})}^2$  by a finite summation of the following forms

$$\left\| \prod_{j=1}^p (\partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s + \tilde{u}_\epsilon)) |_{y=0} \right\|_{L^2(\mathbb{R})}^2 \leq C \|\partial_y \prod_{j=1}^p (\partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s + \tilde{u}_\epsilon))\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)}^2$$

with  $2 \leq p \leq \frac{m}{2}$ ,  $\alpha_j + \beta_j \leq m-1$  and  $\{j; \alpha_j > 0\} \neq \emptyset$ . Then using Sobolev inequality and  $m \geq 6$ , we get

$$\|(\partial_y^{m+2} \tilde{u}_\epsilon) |_{y=0}\|_{L^2(\mathbb{R})} \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m/2}(\mathbb{R}_+^2)}^{m/2}.$$

**Case :**  $1 \leq \alpha_1 \leq m-2, \alpha_1 + \alpha_2 = m, \alpha_2$  **even**, using the same argument to the precedent case, we have

$$\left| \int_{\mathbb{R}} (\partial^\alpha \partial_y \tilde{w}_\epsilon \partial^\alpha \tilde{w}_\epsilon) |_{y=0} dx \right| = \left| \int_{\mathbb{R}} (\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon \partial_x^{\alpha_1} \partial_y^{\alpha_2} \tilde{w}_\epsilon) |_{y=0} dx \right|$$

$$\begin{aligned}
&\leq \|(\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})} \|(\partial_x^{\alpha_1} \partial_y^{\alpha_2} \tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})} \\
&\leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|(\partial_x^{\alpha_1} \partial_y^{\alpha_2+2} \tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}^2 \\
&\leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^{\alpha_2}.
\end{aligned}$$

**Case :**  $1 \leq \alpha_1 \leq m-2, \alpha_1 + \alpha_2 = m, \alpha_2$  **odd**, integration by part with respect to  $x$  variable implies

$$\begin{aligned}
&\left| \int_{\mathbb{R}} (\partial_x^{\alpha_1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon \partial_x^{\alpha_1} \partial_y^{\alpha_2} \tilde{w}_\epsilon)|_{y=0} dx \right| = \left| \int_{\mathbb{R}} (\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon \partial_x^{\alpha_1+1} \partial_y^{\alpha_2} \tilde{w}_\epsilon)|_{y=0} dx \right| \\
&\leq \|(\partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})} \|(\partial_x^{\alpha_1+1} \partial_y^{\alpha_2} \tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})} \\
&\leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|(\partial_x^{\alpha_1+1} \partial_y^{\alpha_2+1} \tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}^2 \\
&\leq \frac{1}{16} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^{\alpha_2-1}.
\end{aligned}$$

Finally, we have proven

$$\begin{aligned}
&\int_{\mathbb{R}_+^2} (\partial_t \partial^\alpha \tilde{w}_\epsilon - \partial_y^2 \partial^\alpha \tilde{w}_\epsilon - \epsilon \partial_x^2 \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \tilde{w}_\epsilon dx dy \\
&\geq \frac{1}{2} \frac{d}{dt} \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2 + \epsilon \|\partial_x \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2 + \|\partial_y \partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2 \\
&\quad - \frac{1}{4} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 - C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^m.
\end{aligned}$$

We estimate now the right hand of (3.14). For the first item, we need to split it into two parts

$$-\partial^\alpha ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon) = -(u^s + \tilde{u}_\epsilon) \partial_x \partial^\alpha \tilde{w}_\epsilon + [(u^s + \tilde{u}_\epsilon), \partial^\alpha] \partial_x \tilde{w}_\epsilon.$$

Firstly, we have

$$\int_{\mathbb{R}_+^2} ((u^s + \tilde{u}_\epsilon) \partial_x \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \tilde{w}_\epsilon dx dy \leq \|\partial_x \tilde{u}_\epsilon\|_{L^\infty} \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2,$$

then using (A.2), we get

$$\left| \int_{\mathbb{R}_+^2} ((u^s + \tilde{u}_\epsilon) \partial_x \partial^\alpha \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \tilde{w}_\epsilon dx dy \right| \leq \|\tilde{w}_\epsilon\|_{H_1^3} \|\partial^\alpha \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2}^2.$$

For the commutator operator, in fact, it can be written as

$$[(u^s + \tilde{u}_\epsilon), \partial^\alpha] \partial_x \tilde{w}_\epsilon = \sum_{\beta \leq \alpha, 1 \leq |\beta|} C_\alpha^\beta \partial^\beta (u^s + \tilde{u}_\epsilon) \partial^{\alpha-\beta} \partial_x \tilde{w}_\epsilon.$$

Then for  $|\alpha| \leq m, m \geq 4$ , using the Sobolev inequality again and Lemma A.1,

$$\|[(u^s + \tilde{u}), \partial^\alpha] \partial_x \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2} \leq C (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m} + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2).$$

Thus

$$\left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2(k+\ell+\alpha_2)} ([ (u^s + \tilde{u}_\epsilon), \partial^\alpha ] \partial_x \tilde{w}_\epsilon) \cdot \partial^\alpha \tilde{w}_\epsilon dx dy \right| \leq C (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3),$$

and

$$\left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2(k+\ell+\alpha_2)} (\partial^\alpha ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon)) \partial^\alpha \tilde{w}_\epsilon dx dy \right| \leq C (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3),$$

where  $C$  is independent of  $\epsilon$ .

For the next one, similar to the first term in (3.14), we have

$$\partial^\alpha (\tilde{v}_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon)) = \tilde{v}_\epsilon \partial_y \partial^\alpha \tilde{w}_\epsilon - [\tilde{v}_\epsilon, \partial^\alpha] \partial_y \tilde{w}_\epsilon + \partial^\alpha (\tilde{v}_\epsilon u_{yy}^s).$$

Then

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \tilde{v}_\epsilon \langle y \rangle^{2(k+\ell+\alpha_2)} (\partial_y \partial^\alpha \tilde{w}_\epsilon) \cdot \partial^\alpha \tilde{w}_\epsilon dx dy \right| \\ & \leq \|\tilde{v}_\epsilon\|_{L^\infty(\mathbb{R}_+^2)} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^m} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m} \\ & \leq \frac{1}{4} \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 + C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^4 \end{aligned}$$

where we have used

$$\begin{aligned} \|\tilde{v}_\epsilon\|_{L^\infty(\mathbb{R}_+^2)} & \leq C \|\partial_x \tilde{u}_\epsilon\|_{L^\infty(\mathbb{R}_x; L_{\frac{1}{2}+\delta}^2(\mathbb{R}_{y,+}))} \\ & \leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{1+2\delta} (|\partial_x \tilde{u}_\epsilon|^2 + |\partial_x^2 \tilde{u}_\epsilon|^2) dx dy \\ & \leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{3+2\delta} (|\partial_x \tilde{w}_\epsilon|^2 + |\partial_x^2 \tilde{w}_\epsilon|^2) dx dy \leq C \|\tilde{w}_\epsilon\|_{H_{\frac{3}{2}+\delta}^2}, \end{aligned}$$

where  $\delta > 0$  is small.

Noticing that

$$[\tilde{v}_\epsilon, \partial^\alpha] \partial_y \tilde{w}_\epsilon = \sum_{\beta \leq \alpha, 1 \leq |\beta|} C_\alpha^\beta \partial^\beta \tilde{v}_\epsilon \partial^{\alpha-\beta} \partial_y \tilde{w}_\epsilon.$$

Since  $H_\ell^m$  is an algebra for  $m \geq 6$ , we only need to pay attention to the order of derivative in the above formula. Firstly for  $|\beta| \geq 1$ , we have for  $|\alpha - \beta| + 1 \leq m$ ,

$$-\partial^\beta \tilde{v}_\epsilon = \partial_x^{\beta_1} \partial_y^{\beta_2} \int_0^y \tilde{u}_{\epsilon,x} d\tilde{y} = \begin{cases} \partial_x^{\beta_1+1} \partial_y^{\beta_2-1} \tilde{u}_\epsilon, & \beta_2 \geq 1, \\ \int_0^y \partial_x^{\beta_1+1} \tilde{u}_\epsilon d\tilde{y}, & \beta_2 = 0. \end{cases}$$

Now using the hypothesis  $\beta \leq \alpha, 1 \leq |\beta|$  and  $\beta_1 \leq \alpha_1 \leq m - 1$ , using Lemma A.1, we get

$$\|[\tilde{v}_\epsilon, \partial^\alpha] \partial_y \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2} \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2.$$

On the other hand, if  $\alpha_2 = 0$ , using  $-1 + \ell < -\frac{1}{2}$ , we can get

$$\|\partial_x^{m-1} (\tilde{v}_\epsilon u_{yy}^s)\|_{L_{k+\ell}^2} \leq C \|\partial_x^m \tilde{u}_\epsilon\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} \|u_{yy}^s\|_{L_{k+\ell}^2(\mathbb{R}_+)} \leq C \|\tilde{w}_\epsilon\|_{H_{\frac{3}{2}+\delta}^m}.$$

Similar computation for other cases, we can get, for  $\alpha_2 > 0, \alpha_1 + \alpha_2 \leq m$ ,

$$\|\partial^\alpha (\tilde{v}_\epsilon u_{yy}^s)\|_{L_{k+\ell+\alpha_2}^2} \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}.$$

Combining the above estimates, we have finished the proof of the Proposition 3.8.  $\square$

**3.3. Smallness of approximate solutions.** To close the energy estimate, we still need to estimate the term  $\partial_x^m \tilde{w}_\epsilon$ .

**Proposition 3.9.** *Under the hypothesis of Theorem 3.4, and with the same notations as in Proposition 3.8, we have*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \frac{3\epsilon}{4} \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \frac{3}{4} \|\partial_y \partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 \\ & \leq C (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3) + \frac{32}{\epsilon} (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^4 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2). \end{aligned} \quad (3.15)$$

*Proof.* We have

$$\partial_t \partial_x^m \tilde{w}_\epsilon - \partial_y^2 \partial_x^m \tilde{w}_\epsilon - \epsilon \partial_x^m \partial_x^2 \tilde{w}_\epsilon = -\partial_x^m ((u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon) - \partial_x^m (\tilde{v}_\epsilon (\partial_y \tilde{w}_\epsilon + u_{yy}^s)),$$

then the same computations as in Proposition 3.8 give

$$\begin{aligned} & \frac{d}{2dt} \|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \epsilon \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 + \frac{3}{4} \|\partial_y \partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2}^2 \\ & \leq C (\|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3) \\ & \quad + \left| \int_{\mathbb{R}_+^2} \partial_x^m (\tilde{v}_\epsilon (\partial_y \tilde{w}_\epsilon + u_{yy}^s)) \langle y \rangle^{2(k+\ell)} \partial_x^m \tilde{w}_\epsilon dx dy \right|, \end{aligned} \quad (3.16)$$

where the boundary terms is more easy to control, since

$$(\partial_y \partial_x^m \tilde{w}_\epsilon)(t, x, 0) = (\partial_y^2 \partial_x^m \tilde{w}_\epsilon)(t, x, 0) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

The estimate of the last term on right hand is the main obstacle for the study of the Prandtl equations.

$$\begin{aligned} \partial_x^m (\tilde{v}_\epsilon (\partial_y \tilde{w}_\epsilon + u_{yy}^s)) &= \tilde{v}_\epsilon \partial_x^m \partial_y \tilde{w}_\epsilon + (\partial_x^m \tilde{v}_\epsilon) (\partial_y \tilde{w}_\epsilon + u_{yy}^s) \\ & \quad + \sum_{1 \leq j \leq m-1} C_m^j \partial_x^j \tilde{v}_\epsilon \partial_x^{m-j} \partial_y \tilde{w}_\epsilon. \end{aligned}$$

For the first term

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \tilde{v}_\epsilon (\partial_x^m \partial_y \tilde{w}_\epsilon) \langle y \rangle^{2(k+\ell)} (\partial_x^m \tilde{w}_\epsilon) dx dy \\ &= \frac{1}{2} \int \tilde{v}_\epsilon \langle y \rangle^{2(k+\ell)} \partial_y (\partial_x^m \tilde{w}_\epsilon)^2 dx dy \\ &= \frac{1}{2} \int \tilde{u}_{\epsilon,x} \langle y \rangle^{2(k+\ell)} (\partial_x^m \tilde{w}_\epsilon)^2 dx dy \\ & \quad - \ell \int \tilde{v}_\epsilon \langle y \rangle^{2(k+\ell)-1} (\partial_x^m \tilde{w}_\epsilon)^2 dx dy \\ & \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3, \end{aligned}$$

where we have used  $\tilde{v}_\epsilon|_{y=0} = 0$ , and

$$\left| \int_{\mathbb{R}_+^2} \left( \sum_{1 \leq j \leq m-1} C_m^j \partial_x^j \tilde{v}_\epsilon \partial_x^{m-j} \partial_y \tilde{w}_\epsilon \right) \langle y \rangle^{2(k+\ell)} (\partial_x^m \tilde{w}_\epsilon) dx dy \right| \leq C \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m}^3.$$

Finally for the worst term, we have

$$\left| \int_{\mathbb{R}_+^2} (\partial_x^m \tilde{v}_\epsilon) (\partial_y \tilde{w}_\epsilon + u_{yy}^s) \langle y \rangle^{2(k+\ell)} (\partial_x^m \tilde{w}_\epsilon) dx dy \right|$$



$$\begin{aligned} &\leq C \|\partial_x^m \tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} \|\partial_y \tilde{w}_\epsilon\|_{L^\infty(\mathbb{R}_x; L^2_{k+\ell}(\mathbb{R}_+))} \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}} \\ &\quad + \|\partial_x^m \tilde{v}_\epsilon u_{yy}^s\|_{L^2_{k+\ell}(\mathbb{R}_+^2)} \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}. \end{aligned}$$

On the other hand, observing

$$\partial_x^m \tilde{v}_\epsilon(t, x, y) = - \int_0^y \partial_x^{m+1} \tilde{u}_\epsilon(t, x, \tilde{y}) d\tilde{y},$$

then using Sobolev inequality and Lemma A.1, for  $\delta > 0$  small,

$$\|\partial_x^m \tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} \leq C \|\partial_x^{m+1} \tilde{u}_\epsilon\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+^2)} \leq C \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L^2_{\frac{3}{2}+\delta}(\mathbb{R}_+^2)},$$

we get

$$\|\partial_x^m \tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} \leq C \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L^2_{\frac{3}{2}+\delta}(\mathbb{R}_+^2)}.$$

Using the hypothesis for the shear flow  $u^s$  and  $-1 + \ell < -\frac{1}{2}$ ,

$$\begin{aligned} \|\partial_x^m (\tilde{v}_\epsilon u_{yy}^s)\|_{L^2_{k+\ell}(\mathbb{R}_+^2)} &\leq \|\partial_x^m \tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} \|u_{yy}^s\|_{L^2_{k+\ell}(\mathbb{R}_+)} \\ &\leq C \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L^2_{\frac{3}{2}+\delta}(\mathbb{R}_+^2)}, \end{aligned}$$

and for  $k + \ell \geq \frac{3}{2} + \delta$ ,

$$\|\partial_y \tilde{w}_\epsilon\|_{L^\infty(\mathbb{R}_x; L^2_{k+\ell}(\mathbb{R}_+))} \leq C \|\partial_y \tilde{w}_\epsilon\|_{H^1(\mathbb{R}_x; L^2_{k+\ell}(\mathbb{R}_+))} \leq C \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}_+^2)}.$$

Thus, we have

$$\begin{aligned} &\int (\partial_x^m (\tilde{v}_\epsilon (\partial_y \tilde{w}_\epsilon + u_{yy}^s))) \langle y \rangle^{2(k+\ell)} \partial_x^m \tilde{w}_\epsilon dx dy \\ &\leq C \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}^3 + \frac{32}{\epsilon} (\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}^4 + \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}^2) + \frac{\epsilon}{4} \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L^2_{\frac{3}{2}+\delta}}^2. \end{aligned} \quad (3.17)$$

From (3.16) and (3.17), we have, if  $k + \ell > \frac{3}{2}$ ,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_x^m \tilde{w}_\epsilon\|_{L^2_{k+\ell}}^2 + \frac{3\epsilon}{4} \|\partial_x^{m+1} \tilde{w}_\epsilon\|_{L^2_{k+\ell}}^2 + \frac{3}{4} \|\partial_y \partial_x^m \tilde{w}_\epsilon\|_{L^2_{k+\ell}}^2 \\ &\leq C (\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}^2 + \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}^3) + \frac{32}{\epsilon} (\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}^4 + \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}^2). \end{aligned}$$

□

**End of proof of Theorem 3.4.** Combining (3.13) and (3.15), for  $m \geq 6, k > 1, \frac{3}{2} - k < \ell < \frac{1}{2}$  and  $0 < \epsilon \leq 1$ , we get

$$\frac{d}{dt} \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}_+^2)}^2 \leq \frac{C}{\epsilon} (\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}_+^2)}^m), \quad (3.18)$$

with  $C > 0$  independent of  $\epsilon$ .

From (3.18), by the nonlinear Gronwall's inequality, we have

$$\|\tilde{w}_\epsilon(t)\|_{H^m_{k+\ell}(\mathbb{R}_+^2)}^{m-2} \leq \frac{\|\tilde{w}_\epsilon(0)\|_{H^m_{k+\ell}}^{m-2}}{e^{-\frac{C}{\epsilon}t(\frac{m}{2}-1)} - (\frac{m}{2}-1)\frac{C}{\epsilon}t\|\tilde{w}_\epsilon(0)\|_{H^m_{k+\ell}}^{m-2}}, \quad 0 < t \leq T_\epsilon,$$

where we choose  $T_\epsilon > 0$  such that

$$\left( e^{-\frac{C}{\epsilon}T_\epsilon(\frac{m}{2}-1)} - (\frac{m}{2}-1)\frac{C}{\epsilon}T_\epsilon\|\tilde{w}_\epsilon(0)\|_{H^m_{k+\ell}}^{m-2} \right)^{-1} = \left( \frac{4}{3} \right)^{m-2}. \quad (3.19)$$

Finally, we get for any  $\|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m} \leq \bar{\zeta}$ , and  $0 < \epsilon \leq \epsilon_0$ ,

$$\|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \frac{4}{3} \|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq 2 \|\tilde{w}_0\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}, \quad 0 < t \leq T_\epsilon.$$

□

The rest of this paper is dedicated to improve the results of Proposition 3.9, and try to get an uniform estimate with respect to  $\epsilon$ . Of course, we have to recall the assumption on the shear flow in the main Theorem 1.1.

#### 4. FORMAL TRANSFORMATIONS

Since the estimate (3.13) is independent of  $\epsilon$ , we only need to treat (3.15) in a new way to get an estimate which is also independent of  $\epsilon$ . To simplify the notations, from now on, we drop the notation tilde and sub-index  $\epsilon$ , that is, with no confusion, we take

$$u = \tilde{u}_\epsilon, \quad v = \tilde{v}_\epsilon, \quad w = \tilde{w}_\epsilon.$$

**4.1. The cut-off functions and *à priori* assumptions.** We will decompose the approximate solution into the monotone part and convex part, so we need to introduce the cut-off functions. Choose  $\phi_1, \phi_2, \psi \in C^\infty(\mathbb{R}_+)$ ,  $0 \leq \phi_1, \phi_2, \psi \leq 1$  with

$$\begin{aligned} \phi_1(y) &= 1 \text{ for } 0 \leq y \leq a - 3a_0; & \phi_1(y) &= 0 \text{ for } y \in [a - 2a_0, +\infty[; \\ \phi_2(y) &= 0 \text{ for } 0 \leq y \leq a + 2a_0; & \phi_2(y) &= 1 \text{ for } y \in [a + 3a_0, +\infty[; \\ \psi(y) &= 0 \text{ for } 0 \leq y \leq a - 5a_0; & \psi(y) &= 1 \text{ for } y \in [a - 4a_0, y_0 - 1]; \\ \psi(y) &= 0 \text{ for } y \in [y_0 + 1, +\infty[. \end{aligned}$$

Also the support of those cut-off functions are as follows

$$\begin{aligned} I_{\phi_1} &= \{y; 0 \leq y \leq a - 2a_0\}; & I_{\phi_1'} &= \{y; a - 3a_0 \leq y \leq a - 2a_0\}; \\ I_{\phi_2} &= \{y; y \geq a + 2a_0\}; & I_{\phi_2'} &= \{y; a + 2a_0 \leq y \leq a + 3a_0\}; \\ I_\psi &= \{y; a - 5a_0 \leq y \leq y_0 + 1\}; \\ I_{\psi'} &= \{y; a - 5a_0 \leq y \leq a - 4a_0\} \cup \{y; y_0 - 1 \leq y \leq y_0 + 1\}. \end{aligned}$$

And we have

$$I_{\phi_1'} \cup I_{\phi_2'} \subset \{y; \psi(y) = 1\}, \quad I_{\psi'} \subset \{y; \phi_1(y) = 1\} \cup \{y; \phi_2(y) = 1\}.$$

Let  $w \in L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$  be a classical solution of (3.7) which satisfies the following *à priori* condition

$$\|w\|_{L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta. \quad (4.1)$$

Then (A.2) gives, for any  $\delta > 0$  very small and  $m \geq 4$ ,

$$\begin{aligned} & \|\langle y \rangle^{k+\ell} w\|_{L^\infty([0, T] \times \mathbb{R}_+^2)} + \|\langle y \rangle^{k+\ell} \partial_y w\|_{L^\infty([0, T] \times \mathbb{R}_+^2)} \\ & \leq C(\|\langle y \rangle^{\frac{1}{2}+\delta} (\langle y \rangle^{k+\ell} w)_y\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \|\langle y \rangle^{\frac{1}{2}+\delta} (\langle y \rangle^{k+\ell} w)_{xy}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))}) \\ & + C(\|\langle y \rangle^{\frac{1}{2}+\delta} (\langle y \rangle^{k+\ell} \partial_y w)_y\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))} + \|\langle y \rangle^{\frac{1}{2}+\delta} (\langle y \rangle^{k+\ell} \partial_y w)_{xy}\|_{L^\infty([0, T]; L^2(\mathbb{R}_+^2))}) \\ & \leq C_m \|w\|_{L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))}, \end{aligned}$$

where  $C_m$  denote the Sobolev embedding constant. Then we have, for  $\ell > 0$ ,

$$\begin{aligned} |\partial_y u(t, x, y)| &= |w(t, x, y)| \leq C_m \zeta \langle y \rangle^{-k-\ell}, \quad (t, x, y) \in [0, T] \times \mathbb{R}_+^2, \\ |\partial_y^2 u(t, x, y)| &= |\partial_y w(t, x, y)| \leq C_m \zeta \langle y \rangle^{-k-1-\ell}, \quad (t, x, y) \in [0, T] \times \mathbb{R}_+^2. \end{aligned} \quad (4.2)$$

We assume that  $\zeta$  is small enough such that

$$C_m \zeta \leq \frac{\min\{c_1, c_2, c_3\}}{2}. \quad (4.3)$$

Then we get

$$\frac{c_1}{4} \langle y \rangle^{-k} \leq |u_y^s + w| \leq 3B_1 \langle y \rangle^{-k}, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times (I_{\phi_1} \cup I_{\phi_2}); \quad (4.4)$$

$$\frac{c_3}{4} \langle y \rangle^{-k-1} \leq |u_{yy}^s + \partial_y w| \quad (t, x, y) \in [0, T] \times \mathbb{R} \times I_\psi, \quad (4.5)$$

and

$$\sup_{(t, x, y) \in [0, T] \times \mathbb{R}_+^2} |u_y^s + w| \leq 2M \text{ with } M = \sup_{(t, y) \in [0, T] \times \mathbb{R}_+} |u_y^s(t, y)|. \quad (4.6)$$

**4.2. The formal transformation of equations.** Under the conditions (4.4) and (4.5), in this subsection, we will introduce two transformations for the system (3.1), one to be used for the monotone domain, and the other for the convex domain.

**Monotone parts.** Set, for  $0 \leq n \leq m$

$$g_n = \left( \frac{\partial_x^n u}{u_y^s + u_y} \right)_y, \quad \eta_1 = \frac{u_{xy}}{u_y^s + u_y}, \quad \eta_2 = \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y}, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times (I_{\phi_1} \cup I_{\phi_2}).$$

Formally, we will use the following notation

$$\partial_y^{-1} g_n(t, x, y) = \frac{\partial_x^n u}{u_y^s + u_y}(t, x, y), \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times (I_{\phi_1} \cup I_{\phi_2}), \quad (4.7)$$

so that  $\partial_y(\partial_y^{-1} g_n) = g_n$

Applying  $\partial_x^n$  to (3.1), we have

$$\begin{aligned} \partial_t \partial_x^n u + (u^s + u) \partial_x \partial_x^n u + (\partial_x^n v)(u_y^s + \partial_y u) \\ = \partial_y^2 \partial_x^n u + \epsilon \partial_x^2 \partial_x^n u + A_n^1 + A_n^2, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} A_n^1 &= -[\partial_x^n, (u^s + u)] \partial_x u = - \sum_{i=1}^n C_n^i \partial_x^i u \partial_x^{n+1-i} u, \\ A_n^2 &= -[\partial_x^n, (u_y^s + \partial_y u)] v = - \sum_{i=1}^n C_n^i \partial_x^i w \partial_x^{n-i} v. \end{aligned}$$

Dividing (4.8) with  $(u_y^s + u_y)$  and performing  $\partial_y$  on the resulting equation, observing

$$\partial_x \partial_x^n u + \partial_y \partial_x^n v = \partial_x^n (\partial_x u + \partial_y v) = 0,$$

we have for  $j = 1, 2$ ,

$$\begin{aligned} \phi_j \partial_y \left( \frac{\partial_t \partial_x^n u}{u_y^s + u_y} \right) + \phi_j (u^s + u) \partial_y \left( \frac{\partial_x \partial_x^n u}{u_y^s + u_y} \right) \\ = \phi_j \partial_y \left( \frac{\partial_y^2 \partial_x^n u + \epsilon \partial_x^2 \partial_x^n u}{u_y^s + u_y} \right) + \phi_j \partial_y \left( \frac{A_n^1 + A_n^2}{u_y^s + u_y} \right). \end{aligned}$$

We compute each term on the support of  $\phi_j$ ,

$$\partial_y \left( \frac{\partial_t \partial_x^n u}{u_y^s + u_y} \right) = \partial_y \left( \partial_t \frac{\partial_x^n u}{u_y^s + u_y} + (\partial_y^{-1} g_n) \frac{\partial_t u_y + \partial_t u_y^s}{u_y^s + u_y} \right)$$

$$\begin{aligned}
&= \partial_t g_n + \partial_y \left( (\partial_y^{-1} g_n) \frac{\partial_t u_y^s + \partial_t u_y}{u_y^s + \tilde{u}_y} \right), \\
(u^s + u) \partial_y \left( \frac{\partial_x \partial_x^n u}{u_y^s + u_y} \right) &= (u^s + u) \left\{ \partial_x \partial_y \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) + \partial_y \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) \frac{u_{xy}}{u_y^s + u_y} \right. \\
&\quad \left. + \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) \partial_y \left( \frac{u_{xy}}{u_y^s + u_y} \right) \right\} \\
&= (u^s + u) (\partial_x g_n + g_n \eta_1 + (\partial_y^{-1} g_n) \partial_y \eta_1), \\
\frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} &= \partial_y^2 \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) + 2 \left( \frac{\partial_y u}{u_y^s + u_y} \right) \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} - \partial_x^n u \partial_y^2 \left( \frac{1}{u_y^s + u_y} \right), \\
\partial_y^2 \left( \frac{1}{u_y^s + u_y} \right) &= -\partial_y \left( \frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)^2} \right) = -\frac{u_{yyy}^s + u_{yyy}}{(u_y^s + u_y)^2} + 2 \left( \frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)} \right)^2 \frac{1}{u_y^s + u_y}, \\
\frac{\partial_y \partial_x^n u}{u_y^s + u_y} \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} &= \left( \frac{\partial_x^n u}{u_y^s + u_y} \right)_y \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} - \frac{\partial_x^n u}{u_y^s + u_y} \left( \frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)} \right)^2.
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} &= \partial_y g_n + 2(g_n \eta_2 - 2\partial_y^{-1} g_n \eta_2^2) + (\partial_y^{-1} g_n) \left( \frac{u_{yyy}^s + u_{yyy}}{(u_y^s + \tilde{u}_y)} \right), \\
\partial_y \left( \frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} \right) &= \partial_y^2 g_n + 2(\partial_y g_n) \eta_2 + 2g_n \partial_y \eta_2 - 4g_n \eta_2^2 \\
&\quad - 8\partial_y^{-1} g_n \eta_2 \partial_y \eta_2 + \partial_y \left( (\partial_y^{-1} g_n) \frac{u_{yyy}^s + u_{yyy}}{u_y^s + u_y} \right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{\partial_x^2 \partial_x^n u}{u_y^s + u_y} &= \partial_x^2 \left( \frac{\partial_x^n u}{u_y^s + u_y} \right) + 2 \left( \frac{\partial_x^n u}{u_y^s + u_y} \right)_x \frac{u_{xy}}{u_y^s + u_y} \\
&\quad - 2 \frac{\partial_x^n u}{u_y^s + u_y} \left( \frac{u_{xy}}{(u_y^s + u_y)} \right)^2 + \frac{\partial_x^n u}{u_y^s + u_y} \frac{u_{xxy}}{(u_y^s + u_y)}, \\
\partial_y \left( \frac{\partial_x^2 \partial_x^n u}{u_y^s + u_y} \right) &= \partial_x^2 g_n + 2\partial_x g_n \eta_1 + 2(\partial_x \partial_y^{-1} g_n) \partial_y \eta_1 \\
&\quad - 2g_n \eta_1^2 - 4\partial_y^{-1} g_n \eta_1 \partial_y \eta_1 + \partial_y \left( (\partial_y^{-1} g_n) \frac{u_{xxy}}{u_y^s + u_y} \right).
\end{aligned}$$

For the boundary condition, we only need to pay attention to  $j = 1$ . From (4.8) and the boundary condition for  $(u, v)$  in (3.1), we observe

$$\partial_x^n u|_{y=0} = 0, \quad \partial_y^2 \partial_x^n u|_{y=0} = 0, \quad (u_y^s + u_y)|_{y=0} \neq 0.$$

At the same time,

$$0 = \frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} \Big|_{y=0} = \partial_y g_n|_{y=0} + 2(g_n \eta_2 - 2(\partial_y^{-1} g_n) \eta_2^2)|_{y=0}$$

$$+ (\partial_y^{-1} g_n) \left( \frac{u_{yyy}^s + u_{yyy}}{(u_y^s + \tilde{u}_y)} \right) \Big|_{y=0},$$

and

$$\eta_2|_{y=0} = \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} \Big|_{y=0} = 0, \quad (\partial_y^{-1} g_n)(t, x, y)|_{y=0} = \frac{\partial_x^n u}{u_y^s + u_y}(t, x, y) \Big|_{y=0} = 0,$$

we get then

$$(\partial_y g_n)|_{y=0} = 0, \quad 0 \leq n \leq m.$$

Finally, we have, for  $j = 1, 2$ ,

$$\begin{cases} \partial_t \phi_j g_n + (u^s + u) \phi_j \partial_x g_n - \phi_j \partial_y^2 g_n - \epsilon \phi_j \partial_x^2 g_n \\ \quad - \epsilon 2 \phi_j (\partial_x \partial_y^{-1} g_n) \partial_y \eta_1 = \phi_j M_n, \\ \phi_j (\partial_y g_n)|_{y=0} = 0, \\ \phi_j g_n|_{t=0} = \phi_j g_{n,0}, \end{cases} \quad (4.9)$$

with  $M_n = \sum_{j=1}^6 M_j^n$ ,

$$\begin{aligned} M_1^n &= -(u^s + u)(g_n \eta_1 + (\partial_y^{-1} g_n) \partial_y \eta_1), \\ M_2^n &= 2(\partial_y g_n) \eta_2 + 2g_n (\partial_y \eta_2 - 2\eta_2^2) - 8(\partial_y^{-1} g_n) \eta_2 \partial_y \eta_2, \\ M_3^n &= \epsilon (2(\partial_x g_n) \eta_1 - 2g_n \eta_1^2 - 4(\partial_y^{-1} g_n) \eta_1 \partial_y \eta_1), \\ M_4^n &= \partial_y \left( (\partial_y^{-1} g_n) \frac{(u^s + u) w_x + v(w_y + u_{yy}^s)}{u_y^s + u_y} \right), \\ M_5^n &= -\partial_y \left( \frac{\sum_{i=1}^n C_n^i \partial_x^i u \cdot \partial_x^{n+1-i} u}{u_y^s + u_y} \right), \\ M_6^n &= -\partial_y \left( \frac{\sum_{i=1}^n C_n^i \partial_x^i w \cdot \partial_x^{n-i} v}{u_y^s + u_y} \right), \end{aligned}$$

where we have used the relation,

$$\partial_t u_y^s + \partial_t u_y - (u_{yyy}^s + u_{yyy}) - \epsilon u_{xxy} = -(u^s + u) w_x + v(u_{yy}^s + w_y).$$

**The convex part.** Taking the equation in (3.7) with derivative  $\partial_x^m$ ,

$$\begin{aligned} \partial_t \partial_x^m w + (u^s + u) \partial_x \partial_x^m w + (\partial_x^m v)(u_{yy}^s + \partial_y w) &= \partial_y^2 \partial_x^m w + \epsilon \partial_x^2 \partial_x^m w \\ &\quad - [\partial_x^m, (u^s + u)] \partial_x w - [\partial_x^m, (u_{yy}^s + \partial_y w)] v. \end{aligned} \quad (4.10)$$

On the support of  $\psi$ , without loss of the generality, we suppose that  $u_{yy}^s + w_y > 0$ , then set

$$h_m = \frac{\partial_x^m w}{\sqrt{u_{yy}^s + w_y}}, \quad \eta_3 = \frac{u_{yyy}^s + w_{yy}}{u_{yy}^s + w_y}, \quad \eta_4 = \frac{w_{xy}}{u_{yy}^s + w_y}.$$

Dividing (4.10) with  $\sqrt{u_{yy}^s + w_y}$ , we have, on the support of  $\psi$ ,

$$\begin{aligned} &\frac{\partial_t \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} + (u^s + u) \frac{\partial_x \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} + \sqrt{u_{yy}^s + w_y} \partial_x^m v \\ &= \frac{\partial_y^2 \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} + \epsilon \frac{\partial_x^2 \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} + \frac{-[\partial_x^m, (u^s + u)] \partial_x w - [\partial_x^m, (u_{yy}^s + \partial_y w)] v}{\sqrt{u_{yy}^s + w_y}}. \end{aligned}$$

We calculate each term

$$\frac{\partial_t \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} = \partial_t h_m + \frac{1}{2} h_m \frac{\partial_t (u_{yy}^s + w_y)}{(u_{yy}^s + w_y)},$$

$$\begin{aligned}
\frac{\partial_x \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} &= \partial_x h_m + \frac{1}{2} h_m \frac{w_{xy}}{(u_{yy}^s + w_y)}, \\
-\frac{\partial_y^2 \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} &= -\partial_y^2 \frac{\partial_x^m w}{\sqrt{u_{yy}^s + w_y}} + 2\partial_y \partial_x^m w \partial_y \frac{1}{\sqrt{u_{yy}^s + w_y}} + \partial_x^m w \partial_y^2 \frac{1}{\sqrt{u_{yy}^s + w_y}} \\
&= -\partial_y^2 h_m - \partial_y \partial_x^m w \frac{w_{yy} + u_{yyy}^s}{(\sqrt{u_{yy}^s + w_y})^3} - \partial_x^m w \partial_y \frac{w_{yy} + u_{yyy}^s}{2(\sqrt{u_{yy}^s + w_y})^3} \\
&= -\partial_y^2 h_m - (\partial_y h_m) \eta_3 + \frac{1}{2} h_m \eta_3^2 - \frac{1}{2} h_m \partial_y \eta_3 + \frac{1}{4} h_m \eta_3^2 \\
&= -\partial_y^2 h_m - (\partial_y h_m) \eta_3 + \frac{5}{4} h_m \eta_3^2 - \frac{1}{2} h_m \frac{w_{yyy} + u_{yyy}^s}{u_{yy}^s + w_y}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
-\frac{\partial_x^2 \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} &= -\partial_x^2 h_m + 2\partial_x \partial_x^m w \partial_x \left( \frac{1}{\sqrt{u_{yy}^s + w_y}} \right) + \partial_x^m w \partial_x^2 \left( \frac{1}{\sqrt{u_{yy}^s + w_y}} \right), \\
2\partial_x \partial_x^m w \partial_x \left( \frac{1}{\sqrt{u_{yy}^s + w_y}} \right) &= -\frac{\partial_x \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} \eta_4 = -\partial_x h_m \eta_4 - \frac{1}{2} h_m \eta_4^2,
\end{aligned}$$

and

$$\begin{aligned}
\partial_x^m w \partial_x^2 \left( \frac{1}{\sqrt{u_{yy}^s + w_y}} \right) &= -\partial_x^m w \partial_x \left( \frac{w_{xy}}{2(\sqrt{u_{yy}^s + w_y})^3} \right) \\
&= -\frac{1}{2} h_m \frac{w_{xxy}}{u_{yy}^s + w_y} + \frac{3}{4} h_m \eta_4^2.
\end{aligned}$$

Then

$$-\frac{\partial_x^2 \partial_x^m w}{\sqrt{u_{yy}^s + w_y}} = -\partial_x^2 h_m - \partial_x h_m \eta_4 + \frac{1}{4} h_m \eta_4^2 - \frac{1}{2} h_m \frac{w_{xxy}}{u_{yy}^s + w_y}.$$

Taking the equation (3.7) with derivative  $y$ , we get the relation

$$\begin{aligned}
\partial_t(w_y) + (u^s + u)w_{xy} - w_{yyy} - \epsilon w_{xxy} \\
= -v(w_{yy} + u_{yyy}^s) + u_x(u_{yy}^s + w_y) - (u_y^s + w)w_x,
\end{aligned}$$

and also

$$\partial_t(u_{yy}^s) - u_{yyy}^s = 0,$$

Finally we get

$$\begin{aligned}
\partial_t(\psi h_m) + (u^s + u)\partial_x(\psi h_m) + \sqrt{u_{yy}^s + w_y} \psi \partial_x^m v \\
= \psi \partial_y^2 h_m + \epsilon \psi \partial_x^2 h_m + \psi N_m,
\end{aligned} \tag{4.11}$$

where  $N_m = N_1^m + N_2^m + N_3^m + N_4^m$

$$\begin{aligned}
N_1^m &= -\frac{1}{4} h_m \eta_3^2 + \partial_y h_m \eta_3 - \frac{\epsilon}{4} h_m \eta_4^2 + \epsilon \partial_x h_m \eta_4, \\
N_2^m &= h_m \frac{v(u_{yyy}^s + w_{yy}) - u_x(u_{yy}^s + w_y) - (u_y^s + w)w_x}{2\sqrt{u_{yy}^s + w_y}}, \\
N_3^m &= -\frac{\sum_{p=0}^{m-1} C_m^p \partial_x^{m-p} u \partial_x^{p+1} w}{\sqrt{u_{yy}^s + w_y}},
\end{aligned}$$

$$N_4^m = -\frac{\sum_{p=0}^{m-1} C_m^p \partial_x^{m-p} \partial_y w \partial_x^p v}{\sqrt{u_{yy}^s + w_y}}.$$

### 5. UNIFORM ESTIMATE FOR THE MONOTONE PART

In the future application (see Lemma 7.3), we need that the weight of  $\phi_2 g_n$  big then  $\frac{1}{2}$ , but from the definition,  $w \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$  imply only  $\phi_2 g_n \in H_\ell^2(\mathbb{R}_+^2)$  with  $0 < \ell < \frac{1}{2}$ . So the first step is to improve this weights if the weight of the initial data is more big. We first have

**Lemma 5.1.** *If  $\tilde{w}_0 \in H_{k+\ell'}^{m+2}(\mathbb{R}_+^2)$ ,  $m \geq 6$ ,  $k > 1$ ,  $0 < \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$ ,  $\ell' > \frac{1}{2}$  which satisfies (4.1)-(4.3) with  $0 < \zeta \leq 1$ , then  $(\phi_2 g_m)(0) \in H_{\ell'}^2(\mathbb{R}_+^2)$ , and we have*

$$\|(\phi_2 g_m)(0)\|_{H_{\ell'}^2(\mathbb{R}_+^2)} \leq C \|\tilde{w}_0\|_{H_{k+\ell'}^{m+2}(\mathbb{R}_+^2)}.$$

**Remark.** With the same proof, we can also get

$$\|\phi_2 g_m\|_{H_{\ell'}^2(\mathbb{R}_+^2)} \leq C \|w\|_{H_{k+\ell'}^{m+2}(\mathbb{R}_+^2)}.$$

In fact, observing

$$\phi_2 g_m(0) = \phi_2 \left( \frac{\partial_x^m \tilde{u}_0}{u_{0,y}^s + \tilde{u}_{0,y}} \right)_y = \phi_2 \frac{\partial_y \partial_x^m \tilde{u}_0}{u_{0,y}^s + \tilde{u}_{0,y}} - \phi_2 \frac{\partial_x^m \tilde{u}_0}{u_{0,y}^s + \tilde{u}_{0,y}} \eta_2(0),$$

then (4.4) implies

$$\langle y \rangle^{\ell'} |\phi_2 g_m(0)| \leq C \langle y \rangle^{k+\ell'} |\phi_2 \partial_x^m \tilde{w}_0| + C \langle y \rangle^{k+\ell'-1} |\phi_2 \partial_x^m \tilde{u}_0|,$$

which finishes the proof of this Lemma.

**Proposition 5.2.** *Let  $w \in L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ ,  $m \geq 6$ ,  $k > 1$ ,  $\ell' > \frac{1}{2}$ ,  $0 < \ell < \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$  and  $k + \ell > \frac{3}{2}$ , satisfy (4.1)-(4.3) with  $0 < \zeta \leq 1$ . Assume that the shear flow  $u^s$  verifies the conclusion of Lemma 2.1, and  $g_n$  satisfies the equation (4.9) for  $1 \leq n \leq m$ , then we have the following estimates, for  $t \in [0, T]$*

$$\begin{aligned} \frac{d}{dt} \sum_{n=1}^m \|\phi_1 g_n\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\partial_y(\phi_1 g_n)\|_{L^2(\mathbb{R}_+^2)}^2 + \epsilon \sum_{n=1}^m \|\partial_x(\phi_1 g_n)\|_{L^2(\mathbb{R}_+^2)}^2 \\ \leq C_2 \left( \sum_{n=1}^m \|\phi_1 g_n\|_{L^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^m}^2 \right), \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} \frac{d}{dt} \sum_{n=1}^m \|\phi_2 g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\partial_y(\phi_2 g_n)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \epsilon \sum_{n=1}^m \|\partial_x(\phi_2 g_n)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ \leq C_2 \left( \sum_{n=1}^m \|\phi_2 g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \right), \end{aligned} \quad (5.2)$$

where  $C_2$  is independent of  $\epsilon$ .

**Remark.** This Proposition imply that if  $(\phi_2 g_m)(0) \in H_{\ell'}^2(\mathbb{R}_+^2)$ , then  $(\phi_2 g_m) \in L^\infty([0, T]; H_{\ell'}^2(\mathbb{R}_+^2))$  with  $\ell' > \frac{1}{2}$ .

**Approach of proof for the Proposition 5.2:** With the boundary condition  $(\partial_y g_n)|_{y=0} = 0$ , the estimate of  $\phi_1 g_n$  is simpler than  $\phi_2 g_n$ , moreover,  $\phi_1$  is compactly

supported, and then we can neglect the weight  $\langle y \rangle^{2\ell'}$  for  $\phi_1 g_n$  in the estimate (5.1). On the other hand, the proof of (5.2) is much more knotty, since the approximate solution  $w_\epsilon$  obtained in Theorem 3.4 is belongs to  $L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ , which implies only  $\phi_2 g_n \in L^\infty([0, T_\epsilon]; H_\ell^2(\mathbb{R}_+^2))$ . Then we can't use  $\langle y \rangle^{2\ell'} \phi_2 g_n$  as the test function to the equation (4.9) to establish the estimate (5.2), to overcome this difficulty, we study the following system which is a re-writing of the system (4.9), with  $\tilde{g}_n = \phi_2 g_n, n = 1, \dots, m$ ,

$$\begin{cases} \partial_t \tilde{g}_n + (u^s + u) \partial_x \tilde{g}_n - \partial_y^2 \tilde{g}_n - \epsilon \partial_x^2 \tilde{g}_n \\ \quad = 2\epsilon (\partial_x (\mathbf{Int}(\tilde{g}_n))) \partial_y \eta_1 + 2\epsilon (\partial_x \tilde{g}_n) \eta_1 + F_n(\tilde{g}_1, \dots, \tilde{g}_n, w), \\ (\partial_y \tilde{g}_n)|_{y=0} = 0, \\ \tilde{g}_n|_{t=0} = \phi_2 g_{n,0} \in H_{\ell'}^2(\mathbb{R}_+^2), \end{cases} \quad (5.3)$$

where  $\mathbf{Int}$  is the integrating operators defined as

$$\mathbf{Int}(f)(t, x, y) = \int_0^y f(t, x, \tilde{y}) d\tilde{y},$$

and  $F_n(\tilde{g}_1, \dots, \tilde{g}_n, w) = \sum_{j=0}^6 \tilde{M}_j$  with  $\tilde{M}_j$  defined as follows:

$$\begin{aligned} \tilde{M}_0^n &= (\partial_y^2 \phi_2) g_n - \partial_y (2\phi_2' g_n) - \epsilon \partial_x (\mathbf{Int}(2\phi_2' \partial_y^{-1} g_n)) \partial_y \eta_1, \\ \tilde{M}_1^n &= \phi_2 M_1^n = -(u^s + u) \left( \tilde{g}_n \eta_1 + \mathbf{Int}(\tilde{g}_n) \partial_y \eta_1 + \mathbf{Int}(\phi_2' \partial_y^{-1} g_n) \partial_y \eta_1 \right), \\ \tilde{M}_2^n &= \phi_2 M_2^n = 2(\partial_y \tilde{g}_n - \phi_2' g_n) \eta_2 + 2\tilde{g}_n (\partial_y \eta_2 - 2\eta_2^2) \\ &\quad - 8(\mathbf{Int}(\phi_2' \partial_y^{-1} g_n) + \mathbf{Int}(\tilde{g}_n)) \eta_2 \partial_y \eta_2, \\ \tilde{M}_3^n &= \phi_2 M_3^n = \epsilon \left( 2(\partial_x \tilde{g}_n) \eta_1 - 2\tilde{g}_n \eta_1^2 - 4(\mathbf{Int}(\phi_2' \partial_y^{-1} g_n) + \mathbf{Int}(\tilde{g}_n)) \eta_1 \partial_y \eta_1 \right), \\ \tilde{M}_4^n &= \phi_2 M_4 = \tilde{g}_n \frac{(u^s + u)w_x + v(w_y + u_{yy}^s)}{u_y^s + u_y} \\ &\quad + \left( \mathbf{Int}(\phi_2' \partial_y^{-1} g_n) + \mathbf{Int}(\tilde{g}_n) \right) \left( \frac{(u^s + u)w_x + v(w_y + u_{yy}^s)}{u_y^s + u_y} \right)_y, \\ \tilde{M}_5^n &= \phi_2 M_5^n = \sum_{i=4}^n C_n^i \tilde{g}_i \partial_x^{n+1-i} u + \sum_{1 \leq i \leq 3} C_n^i \partial_x^i u \tilde{g}_{n+1-i} \\ &\quad + \sum_{i=4}^n C_n^i \left( \mathbf{Int}(\phi_2' \partial_y^{-1} g_i) + \mathbf{Int}(\tilde{g}_n) \right) \partial_x^{n+1-i} w \\ &\quad + \sum_{1 \leq i \leq 3} C_n^i \partial_x^i w \left( \mathbf{Int}(\phi_2' \partial_y^{-1} g_{n+1-i}) + \mathbf{Int}(\tilde{g}_{n+1-i}) \right), \\ \tilde{M}_6^n &= \phi_2 M_6 = \sum_{i=1}^n C_n^i \tilde{g}_i \eta_2 \partial_x^{n-i} v + \sum_{i=1}^n C_n^i \tilde{g}_i \partial_x^{n+1-i} u \\ &\quad + \sum_{i=1}^n C_n^i (\partial_y \tilde{g}_i - \phi_2' g_i) \partial_x^{n-i} v \\ &\quad + \sum_{i=1}^n \left( \mathbf{Int}(\phi_2' \partial_y^{-1} g_i) + \mathbf{Int}(\tilde{g}_i) \right) \left( C_n^i \partial_x^{n-i} v \partial_y \eta_2 + C_n^i \partial_x^{n+1-i} u \eta_2 \right), \end{aligned}$$



where the notation  $\partial_y^{-1}g_n$  is defined in (4.7).

In fact, from (4.9), we just need to use the following commutator,

$$\phi_2 \partial_y g_n = \partial_y (\phi_2 g_n) - (\phi_2') g_n,$$

and

$$\begin{aligned} \phi_2 \partial_y^{-1} g_n(t, x, y) &= \int_0^y \phi_2 (\partial_y \partial_y^{-1} g_n)(t, x, \tilde{y}) d\tilde{y} + \int_0^y (\phi_2' \partial_y^{-1} g_n)(t, x, \tilde{y}) d\tilde{y} \\ &= \mathbf{Int}(\phi_2 g_n)(t, x, y) + \mathbf{Int}(\phi_2' \partial_y^{-1} g_n)(t, x, y), \end{aligned}$$

which is deduced by the integration by part with the boundary condition :

$$(\phi_2 \partial_y^{-1} g_n)|_{y=0} = 0.$$

With the above calculation, we find that (5.3) is a linear integro-differential system for  $\tilde{g}_n, n = 1, \dots, m$  with the coefficients and the source terms depends on  $w$  and their derivatives up to order  $m$ , we will clarify this confirmation in the proof of the Proposition 5.2. We prove now the estimate (5.2) by the following approach: For the linear integro-differential system (5.3), we prove firstly the *à priori* estimate, for  $\ell' > \frac{1}{2}, \ell' - \ell < \frac{1}{2}$ ,

$$\begin{aligned} \frac{d}{dt} \sum_{n=1}^m \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \epsilon \sum_{n=1}^m \|\partial_x \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ \leq C_2 \left( \sum_{n=1}^m \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \right). \end{aligned} \quad (5.4)$$

Lemma 5.1 confirms that  $\tilde{g}_n(0) = (\phi_2 g_n)(0) \in H_{\ell'}^2(\mathbb{R}_+^2), n = 1, \dots, m$ , then by using Hahn-Banach theorem, this *à priori* estimate implies the existence of solutions

$$\tilde{g}_n \in L^\infty([0, T]; H_{\ell'}^2(\mathbb{R}_+^2)), \quad n = 1, \dots, m.$$

Finally, the uniqueness of solution gives  $\tilde{g}_n = \phi_2 g_n$ , then we can prove the estimate (5.2) by proving (5.4). So that the proof of the Proposition 5.2 is reduced to the proof of the *à priori* estimate (5.4).

**Proof of the *à priori* estimate (5.4).** Multiplying the linear system (5.3) by  $\langle y \rangle^{2\ell'} \tilde{g}_n \in L^\infty([0, T]; H_{-\ell'}^2(\mathbb{R}_+^2))$  and integrating over  $\mathbb{R} \times \mathbb{R}^+$ . We start to deal with the left hand of (5.3) first, we have

$$\int_{\mathbb{R}_+^2} \partial_t \tilde{g}_n \langle y \rangle^{2\ell'} \tilde{g}_n dx dy = \frac{1}{2} \frac{d}{dt} \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2,$$

and integrating by part with respect to  $x$  variable,

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} (u^s + u) \partial_x \tilde{g}_n \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| &= \frac{1}{2} \left| \int_{\mathbb{R}_+^2} (u^s + u) \partial_x (\langle y \rangle^{2\ell'} \tilde{g}_n^2) dx dy \right| \\ &\leq \frac{1}{2} \|u_x\|_{L^\infty(\mathbb{R}_+^2)} \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ &\leq C \|w\|_{H_1^2(\mathbb{R}_+^2)} \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

Integrating by part with respect to  $y$  variable, where the boundary value is vanish,

$$- \int_{\mathbb{R}_+^2} \partial_y^2 \tilde{g}_n \langle y \rangle^{2\ell'} \tilde{g}_n dx dy$$

$$\begin{aligned}
&= \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \partial_y \tilde{g}_n \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \\
&\geq \frac{3}{4} \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 - 8 \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2,
\end{aligned}$$

and integrating by part with respect to  $x$  variable,

$$-\epsilon \int_{\mathbb{R}_+^2} \partial_x^2 \tilde{g}_n \langle y \rangle^{2\ell'} \tilde{g}_n dx dy = \epsilon \|\partial_x \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2.$$

We have also integrating by part with respect to  $x$  variable,

$$\begin{aligned}
&-\epsilon \int_{\mathbb{R}_+^2} (\partial_x \mathbf{Int}(\tilde{g}_n)) \partial_y \eta_1 \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \\
&= \epsilon \int_{\mathbb{R}_+^2} \mathbf{Int}(\tilde{g}_n) \partial_y \eta_1 \langle y \rangle^{2\ell'} \partial_x \tilde{g}_n dx dy \\
&\quad + \epsilon \int_{\mathbb{R}_+^2} \mathbf{Int}(\tilde{g}_n) (\partial_y \partial_x \eta_1) \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \\
&\leq \epsilon \|\mathbf{Int}(\tilde{g}_n) \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \frac{\epsilon}{8} \|\partial_x \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
&\quad + \epsilon \|\mathbf{Int}(\tilde{g}_n) \partial_y \partial_x \eta_1\|_{L^2(\mathbb{R}_+^2)}^2 + \epsilon \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2.
\end{aligned}$$

So by (5.3) and  $0 < \epsilon \leq 1$ , we obtain

$$\begin{aligned}
&\frac{d}{dt} \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \epsilon \|\partial_x \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
&\leq C \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|(\mathbf{Int}(\tilde{g}_n)) \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
&\quad + \|(\mathbf{Int}(\tilde{g}_n)) \partial_y \partial_x \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + 2 \sum_{j=0}^6 \left| \int_{\mathbb{R}_+^2} \tilde{M}_j \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right|.
\end{aligned}$$

Then we can finish the proof of the *à priori* estimate (5.4) by the following four Lemmas.  $\square$

**Lemma 5.3.** *Under the assumption of Proposition 5.2, we have*

$$\begin{aligned}
&\|\mathbf{Int}(\tilde{g}_n) \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\mathbf{Int}(\tilde{g}_n) \partial_y \partial_x \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \leq C \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
&\|\mathbf{Int}(\phi_2' \partial_y^{-1} g_n)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\phi_2' g_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \leq C \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2.
\end{aligned}$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* Notice that (4.1) and (4.3) imply

$$\begin{aligned}
|\mathbf{1}_{I_{\phi_2}} \eta_1| &\leq C \langle y \rangle^{-\ell}, & |\mathbf{1}_{I_{\phi_2}} \partial_x \eta_1| &\leq C \langle y \rangle^{-\ell}, \\
|\mathbf{1}_{I_{\phi_2}} \partial_y \eta_1| &\leq C \langle y \rangle^{-\ell-1}, & |\mathbf{1}_{I_{\phi_2}} \partial_y \partial_x \eta_1| &\leq C \langle y \rangle^{-\ell-1}.
\end{aligned}$$

Since  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < 1/2$ , we have

$$\begin{aligned} \|\mathbf{Int}(\tilde{g}_n)(\partial_y \partial_x \eta_1)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 &\leq C \|\langle y \rangle^{\ell' - \ell - 1} \mathbf{Int}(\tilde{g}_n)\|_{L^2(\mathbb{R}_+^2)}^2 \\ &\leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{2(\ell' - \ell - 1)} \left| \int_0^y \tilde{g}_n(t, x, \tilde{y}) d\tilde{y} \right|^2 dx dy \\ &\leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{2(\ell' - \ell - 1)} \|\tilde{g}_n(t, x, \cdot)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2 dx dy \\ &\leq C \|\tilde{g}_n(t)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}. \end{aligned}$$

Similarly, we also obtain

$$\|\mathbf{Int}(\tilde{g}_n) \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \leq C \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2.$$

For  $\mathbf{Int}(\phi'_2 \partial_y^{-1} g_n)$ , we have

$$\mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) = \int_0^y \phi'_2 \left( \frac{\partial_x^n u}{u_y^s + w} \right) dz,$$

since  $\text{Supp } \phi'_2 \subset [a + 2a_0, a + 3a_0]$ , we have

$$\begin{aligned} \|\mathbf{Int}(\phi'_2 \partial_y^{-1} g_n)\|_{L_{\ell'}^2}^2 &\leq C \int_{-\infty}^{+\infty} \int_0^{a+3a_0} \left( \int_0^y \left| \phi'_2 \left( \frac{\partial_x^n u}{u_y^s + w} \right) \right| dz \right)^2 dx dy \\ &\leq C \int_{-\infty}^{+\infty} \left( \int_{a+2a_0}^{a+3a_0} \left| \phi'_2 \left( \frac{\partial_x^n u}{u_y^s + w} \right) \right| dz \right)^2 dx \\ &\leq C \|w\|_{H_{k+\ell}^n}^2. \end{aligned}$$

We have also

$$\begin{aligned} \|\phi'_2 g_n\|_{L_{\ell'}^2}^2 &\leq C \int_{-\infty}^{+\infty} \int_{a+2a_0}^{a+3a_0} \left| \phi'_2 \left( \frac{\partial_x^n u}{u_y^s + w} \right) \right|_y^2 dx dy \\ &\leq C \int_{-\infty}^{+\infty} \int_{a+2a_0}^{a+3a_0} |\phi'_2|^2 (|\partial_x^n w|^2 + |\partial_x^n u|^2) dx dy \\ &\leq C \|w\|_{H_{k+\ell}^n}^2, \end{aligned}$$

which finishes the proof of Lemma 5.3.  $\square$

**Lemma 5.4.** *Under the assumption of Proposition 5.2, we have*

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} \sum_{j=0}^4 \tilde{M}_j^n \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| &\leq \frac{1}{8} \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \frac{\epsilon}{8} \|\partial_x \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ &\quad + \tilde{C} (\|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2), \end{aligned}$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* For  $\tilde{M}_0^n = (\partial_y^2 \phi_2) g_n - 2\partial_y(\phi'_2 g_n) - 2\epsilon \partial_x \mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) \partial_y \eta_1$ , the first term is compactly supported. The second and third term can be controlled like this after integrating by part

$$- \int_{\mathbb{R}_+^2} \left( \partial_y(\phi'_2 g_n) + \epsilon \partial_x \mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) \partial_y \eta_1 \right) \langle y \rangle^{2\ell'} \tilde{g}_n dx dy$$

$$\begin{aligned}
&= \int_{\mathbb{R}_+^2} (\phi'_2 g_n) \langle y \rangle^{2\ell'} \partial_y \tilde{g}_n dx dy + \int_{\mathbb{R}_+^2} \epsilon \mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) \partial_y \eta_1 \langle y \rangle^{2\ell'} \partial_x \tilde{g}_n dx dy \\
&+ \int_{\mathbb{R}_+^2} (\phi'_2 g_n) \partial_y (\langle y \rangle^{2\ell'}) \tilde{g}_n dx dy + \int_{\mathbb{R}_+^2} \epsilon \mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) \partial_x \partial_y \eta_1 \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \\
&\leq \frac{1}{8} \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \frac{\epsilon}{4} \|\partial_x \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \tilde{C} (\|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2),
\end{aligned}$$

where we have use Lemma 5.3, (4.1) and (4.3).

Recalling  $\tilde{M}_1^n = -(u^s + u)(\tilde{g}_n \eta_1 + \mathbf{Int}(\tilde{g}_n) \partial_y \eta_1 + \mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) \partial_y \eta_1)$ , by Lemma 5.3,

$$\begin{aligned}
&\left| \int_{\mathbb{R}_+^2} (u^s + u) \tilde{g}_n \eta_1 \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| \leq C \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2, \\
&\int_{\mathbb{R}_+^2} |(u^s + u) \mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) \partial_y \eta_1 \langle y \rangle^{2\ell'} \tilde{g}_n| dy dx \leq C \|w\|_{H_{k+\ell}^n}^2 + C \|\tilde{g}_n\|_{L_{\ell'}^2}^2. \\
&\left| \int_{\mathbb{R}_+^2} (u^s + u) \mathbf{Int}(\tilde{g}_n) \partial_y \eta_1 \cdot \langle y \rangle^{2(\ell')} \tilde{g}_n dx dy \right| \\
&\leq \|\mathbf{Int}(\tilde{g}_n) \partial_y \eta_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \leq C \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2.
\end{aligned}$$

Then, we have

$$\left| \int_{\mathbb{R}_+^2} \tilde{M}_1^n \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| \leq C (\|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2).$$

The estimates of  $\tilde{M}_2^n$  and  $\tilde{M}_3^n$  needs the following decay rate of  $\eta_2$ :

$$\begin{aligned}
|\mathbf{1}_{I_{\phi_2}} \eta_2| &\leq C \langle y \rangle^{-1}, \quad |\mathbf{1}_{I_{\phi_2}} \partial_x \eta_2| \leq C \langle y \rangle^{-\ell-1}, \\
|\mathbf{1}_{I_{\phi_2}} \partial_y \eta_2| &\leq C \langle y \rangle^{-2}, \quad |\mathbf{1}_{I_{\phi_2}} \partial_y \partial_x \eta_2| \leq C \langle y \rangle^{-\ell-2}.
\end{aligned}$$

Recall

$$\tilde{M}_2^n = 2(\partial_y \tilde{g}_n - \phi' g_n) \eta_2 + 2\tilde{g}_n (\partial_y \eta_2 - 2\eta_2^2) - 8(\mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) + \mathbf{Int}(\tilde{g}_n)) \eta_2 \partial_y \eta_2.$$

We have

$$\begin{aligned}
&\left| \int_{\mathbb{R}_+^2} \tilde{g}_n (\partial_y \eta_2 - \eta_2^2) \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| \leq C \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2, \\
&\left| \int_{\mathbb{R}_+^2} (\partial_y \tilde{g}_n) \eta_2 \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| \leq C \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \frac{1}{8} \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2, \\
&\left| 2 \int_{\mathbb{R}_+^2} \mathbf{Int}(\tilde{g}_n) \eta_2 \partial_y \eta_2 \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| \\
&\leq C \|\langle y \rangle^{\ell'-3} \mathbf{Int}(\tilde{g}_n)\|_{L^2}^2 + \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \leq C \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2, \\
&\left| \int_{\mathbb{R}_+^2} \left( -2\phi'_2 g_n \eta_2 - 8\mathbf{Int}(\phi'_2 \partial_y^{-1} g_n) \right) \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| \leq C \|\tilde{g}_n\|_{L_{\ell'}^2}^2 + C \|w\|_{H_{k+\ell}^n}^2.
\end{aligned}$$

All together, we conclude

$$\left| \int_{\mathbb{R}_+^2} \tilde{M}_2^n \langle y \rangle^{2(\ell')} \tilde{g}_n dx dy \right| \leq C (\|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{\ell'}^n(\mathbb{R}_+^2)}^2) + \frac{1}{8} \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2,$$

and exactly same computation gives also

$$\left| \int_{\mathbb{R}_+^2} \tilde{M}_3^n \langle y \rangle^{2(\ell')} \tilde{g}_n dx dy \right| \leq C (\|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2) + \frac{\epsilon}{8} \|\partial_x \tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2.$$

Now using (4.1)-(4.3) and  $m \geq 6$ , with the same computation as above, we can get

$$\left| \int_{\mathbb{R}_+^2} \tilde{M}_4^n \langle y \rangle^{2(\ell')} \tilde{g}_n dx dy \right| \leq C \left( \|\tilde{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2 \right).$$

which finishes the proof of Lemma 5.4.  $\square$

**Lemma 5.5.** *Under the assumption of Proposition 5.2, we have*

$$\left| \int_{\mathbb{R}_+^2} \tilde{M}_5^n \langle y \rangle^{2\ell'} \tilde{g}_n dx dy \right| \leq \tilde{C} \left( \sum_{p=1}^n \|\tilde{g}_p\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^n(\mathbb{R}_+^2)}^2 \right),$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* Recall,

$$\begin{aligned} \tilde{M}_5^n &= \sum_{i \geq 4} C_n^i \tilde{g}_i \partial_x^{n+1-i} u + \sum_{1 \leq i \leq 3} C_n^i \partial_x^i u \tilde{g}_{n+1-i} \\ &\quad + \sum_{i \geq 4} C_n^i (\mathbf{Int}(\phi_2' \partial_y^{-1} g_i) + \mathbf{Int}(\tilde{g}_n)) \partial_x^{n+1-i} w \\ &\quad + \sum_{1 \leq i \leq 3} C_n^i \partial_x^i w (\mathbf{Int}(\phi_2' \partial_y^{-1} g_{n+1-i}) + \mathbf{Int}(\tilde{g}_{n+1-i})), \end{aligned}$$

here if  $n \leq 3$ , we have only the last term. Then, under the hypothesis (4.1):  $\|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta \leq 1, m \geq 6$ , we have

$$\begin{aligned} &\sum_{i \geq 4} C_n^i \|\tilde{g}_i \partial_x^{n+1-i} u\|_{L_{\ell'}^2(\mathbb{R}_+^2)} + \sum_{1 \leq i \leq 3} \|\partial_x^i u \tilde{g}_{n+1-i}\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \\ &\leq \sum_{i \geq 4} C_n^i \|\tilde{g}_i\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \|\partial_x^{n+1-i} u\|_{L^\infty(\mathbb{R}_+^2)} \\ &\quad + \sum_{1 \leq i \leq 3} C_n^i \|\partial_x^i u\|_{L^\infty(\mathbb{R}_+^2)} \|\tilde{g}_{n+1-i}\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \\ &\leq C \sum_{i \geq 4} C_n^i \|\tilde{g}_i\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \|w\|_{H_1^{n+3-i}(\mathbb{R}_+^2)} \\ &\quad + C \sum_{1 \leq i \leq 3} C_n^i \|w\|_{H_1^{i+3}} \|\tilde{g}_{n+1-i}\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \\ &\leq C \sum_{i=1}^n \|\tilde{g}_i\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

Similarly, for the second line in  $\tilde{M}_5$ , by Lemma 5.3, we have also

$$\begin{aligned}
& \sum_{i \geq 4} C_n^i \|\mathbf{Int}(\tilde{g}_i) \partial_x^{n+1-i} w\|_{L_{\ell'}^2(\mathbb{R}_+^2)} + \sum_{i \geq 4} C_n^i \|\partial_x^{n+1-i} w \mathbf{Int}(\phi_2' \partial_y^{-1} g_i)\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \\
& \leq \sum_{i \geq 4} C_n^i \|\langle y \rangle^{\ell' - \ell - 1} \mathbf{Int}(\tilde{g}_i)\|_{L^2(\mathbb{R}_+^2)} \|\langle y \rangle^{1 + \ell - \ell'} \partial_x^{n+1-i} w\|_{L^\infty(\mathbb{R}_+^2)} \\
& \quad + \sum_{i \geq 4} C_n^i \|\mathbf{Int}(\phi_2' \partial_y^{-1} g_i)\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \|\partial_x^{n+1-i} w\|_{L^\infty(\mathbb{R}_+^2)} \\
& \leq C \sum_{i=1}^n \|\tilde{g}_i\|_{L_{\ell'}^2(\mathbb{R}_+^2)} + C \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}. \\
& \sum_{1 \leq i \leq 3} \|C_n^i \partial_x^i w (\mathbf{Int}(\phi_2' \partial_y^{-1} g_{n+1-i}) + \mathbf{Int}(\tilde{g}_{n+1-i}))\|_{L_{\ell'}^2(\mathbb{R}_+^2)} \\
& \leq C \sum_{i=1}^n \|\tilde{g}_i\|_{L_{\ell'}^2(\mathbb{R}_+^2)} + C \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}.
\end{aligned}$$

We have proven Lemma 5.5.  $\square$

**Lemma 5.6.** *Under the assumption of Proposition 5.2, we have*

$$\begin{aligned}
& \left| \int_{\mathbb{R}_+^2} \tilde{M}_6^n \langle y \rangle^{2\ell} \tilde{g}_n dx dy \right| \\
& \leq \frac{1}{8m} \sum_{p=1}^n \|\partial_y \tilde{g}_p\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \tilde{C} \left( \sum_{p=1}^n \|\tilde{g}_p\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \right),
\end{aligned}$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* Recall

$$\begin{aligned}
\tilde{M}_6 &= \sum_{i=1}^n C_n^i \tilde{g}_i \eta_2 \partial_x^{n-i} v + \sum_{i=1}^n C_n^i \tilde{g}_i \partial_x^{n+1-i} u \\
& \quad + \sum_{i=1}^n C_n^i (\partial_y \tilde{g}_i - \phi_2' g_i) \partial_x^{n-i} v \\
& \quad + \sum_{i=1}^n (\mathbf{Int}(\phi_2' \partial_y^{-1} g_i) + \mathbf{Int}(\tilde{g}_i)) (C_n^i \partial_x^{n-i} v \partial_y \eta_2 + C_n^i \partial_x^{n+1-i} u \eta_2).
\end{aligned}$$

In  $\tilde{M}_6^n$ , we just study the term  $(\partial_y \tilde{g}_1) \partial_x^{n-1} v$  as an example, the others terms are similar,

$$\begin{aligned}
& \int_{\mathbb{R}_+^2} (\partial_y \tilde{g}_1) \partial_x^{n-1} v \langle y \rangle^{2\ell'} \tilde{g}_n = - \int_{\mathbb{R}_+^2} \tilde{g}_1 \partial_x^{n-1} v \langle y \rangle^{2\ell'} \partial_y \tilde{g}_n \\
& \quad - \int_{\mathbb{R}_+^2} \tilde{g}_1 \partial_x^{n-1} v (\langle y \rangle^{2\ell'})' \tilde{g}_n + \int_{\mathbb{R}_+^2} \tilde{g}_1 \partial_x^n u \langle y \rangle^{2\ell'} \tilde{g}_n dx dy,
\end{aligned}$$

we study each terms

$$\int_{\mathbb{R}_+^2} \tilde{g}_1 \partial_x^{n-1} v \langle y \rangle^{2\ell'} \partial_y \tilde{g}_n dx dy \leq \frac{1}{8m} \|\partial_y \tilde{g}_n\|_{L_{\ell'}^2}^2 + C \|\tilde{g}_1 \partial_x^{n-1} v\|_{L_{\ell'}^2}^2,$$

and

$$\begin{aligned}
\|\tilde{g}_1 \partial_x^{n-1} v\|_{L_{\ell'}^2}^2 &\leq \left( \int_0^{+\infty} \sup_{x \in \mathbb{R}} \langle y \rangle^{2\ell'} |\tilde{g}_1|^2 dy \right) \int_{-\infty}^{+\infty} \sup_{y \in \mathbb{R}_+} \left| \int_0^y \partial_x^n u dz \right|^2 dx \\
&\leq \left( \|\tilde{g}_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\partial_x \tilde{g}_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \right) \int_{-\infty}^{+\infty} \left| \int_0^{+\infty} |\partial_x^n u| dz \right|^2 dx \\
&\leq C \left( \|\tilde{g}_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\partial_x \tilde{g}_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \right) \|\partial_x^n u\|_{L_{k+\ell-1}^2(\mathbb{R}_+^2)}^2 \\
&\leq C \left( \|\tilde{g}_1\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\tilde{g}_2\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|w\|_{H_{k+\ell}^m}^2 \right) \|\partial_x^n w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2 \\
&\leq C \sum_{i=1}^2 \|\tilde{g}_i\|_{L_{\ell'}^2}^2 + C \|w\|_{H_{k+\ell}^m}^2,
\end{aligned}$$

here we have used Lemma 5.3, (A.1) with  $\lim_{y \rightarrow +\infty} \partial_x^n u = 0$  and

$$k + \ell - 1 > \frac{1}{2}, \quad \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq 1,$$

and

$$\partial_x \tilde{g}_j = \tilde{g}_{j+1} - \tilde{g}_j \eta_1 - (\mathbf{Int}(\phi'_j \partial_y^{-1} g_n) + \mathbf{Int}(\tilde{g}_n)) \partial_y \eta_1.$$

By the similar trick, we have completed the proof of this lemma.  $\square$

## 6. UNIFORM ESTIMATE FOR THE CONVEX PART

In this section we consider the convex part. The goal is to obtain the  $L^2$  estimate of  $\partial_x^m w$  on the support of  $\psi$ .

**Proposition 6.1.** *Let  $w \in L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ ,  $m \geq 6$  be a solution of (3.7) such that it satisfies (4.1)-(4.3) with  $0 < \zeta \leq 1$ . Assume that the shear flow  $u^s$  verifies the conclusion of Lemma 2.1, then  $h_m$  satisfies the following estimate,*

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\psi h_m\|_{L^2(\mathbb{R}_+^2)}^2 + \frac{3}{4} \|\psi \partial_y h_m\|_{L^2(\mathbb{R}_+^2)}^2 + \frac{3\epsilon}{4} \|\psi \partial_x h_m\|_{L^2(\mathbb{R}_+^2)}^2 \\
&\leq C \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 - \int_{\mathbb{R}_+^2} \psi^2 \sqrt{u_{yy}^s + w_y} (\partial_x^m v) h_m dx dy,
\end{aligned}$$

where  $C$  is depends on  $c_1, c_2, c_3$ , but independent of  $0 < \epsilon \leq 1$ .

*Proof.* Multiplying (4.11) by  $\psi h_m$ , integrating over  $\mathbb{R}_+^2$ , then integrating by part, where the boundary value is vanish, we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\psi h_m\|_{L^2(\mathbb{R}_+^2)}^2 + \|\psi \partial_y h_m\|_{L^2(\mathbb{R}_+^2)}^2 + \epsilon \|\psi \partial_x h_m\|_{L^2(\mathbb{R}_+^2)}^2 \\
&= - \int_{\mathbb{R}_+^2} \psi^2 \sqrt{u_{yy}^s + w_y} (\partial_x^m v) h_m dx dy + \int_{\mathbb{R}_+^2} \psi^2 N_m h_m dx dy \\
&\quad + \int_{\mathbb{R}_+^2} (\partial_x u) |\psi h_m|^2 dx dy - 2 \int_{\mathbb{R}_+^2} (\partial_y \psi) \psi h_m \partial_y h_m dx dy.
\end{aligned}$$

Now, for  $m \geq 6$ ,  $\|w\|_{H_{k+\ell}^m} \leq \zeta$ ,  $k + \ell > \frac{3}{2}$  and (4.4)-(4.5), we have

$$\left| \frac{1}{\sqrt{u_{yy}^s + w_y}} \mathbf{1}_{I_\psi} \right| \leq c_3^{-1/2} \mathbf{1}_{I_\psi} \langle y \rangle^{\frac{k}{2} + \frac{1}{2}}, \quad |\eta_3 \mathbf{1}_{I_\psi}| + |\eta_4 \mathbf{1}_{I_\psi}| \leq c_3^{-1}, \quad (6.1)$$

and also

$$|h_m \mathbf{1}_{I_\psi}| = \left| \frac{\partial_x^m w}{\sqrt{u_{yy}^s + w_y}} \mathbf{1}_{I_\psi} \right| \leq c_3^{-1/2} \langle y \rangle^{\frac{k}{2} + \frac{1}{2}} \mathbf{1}_{I_\psi} |\partial_x^m w|, \quad (6.2)$$

then

$$\psi h_m \in L^2(\mathbb{R}_+^2),$$

and

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} (\partial_x u) |\psi h_m|^2 dx dy \right| + \left| 2 \int_{\mathbb{R}_+^2} (\partial_y \psi) \psi h_m \partial_y h_m dx dy \right| \\ & \leq C \|w\|_{H_{k+\ell}^m}^2 + \frac{1}{8} \|\psi \partial_y h_m\|_{L^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

For the terms  $N_m = N_1^m + N_2^m + N_3^m + N_4^m$  with

$$\begin{aligned} N_1^m &= -\frac{1}{4} h_m \eta_3^2 + \partial_y h_m \eta_3 - \frac{\epsilon}{4} h_m \eta_4^2 + \epsilon \partial_x h_m \eta_4, \\ N_2^m &= h_m \frac{v(u_{yyy}^s + w_{yy}) - u_x(u_{yy}^s + w_y) - (u_y^s + w)w_x}{2\sqrt{u_{yy}^s + w_y}}, \\ N_3^m &= -\frac{\sum_{p=0}^{m-1} C_m^p \partial_x^{m-p} u \partial_x^{p+1} w}{\sqrt{u_{yy}^s + w_y}}, \\ N_4^m &= -\frac{\sum_{p=0}^{m-1} C_m^p \partial_x^{m-p} \partial_y w \partial_x^p v}{\sqrt{u_{yy}^s + w_y}}. \end{aligned}$$

using (6.1) and (6.2), we can get directly

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \psi (N_1^m + N_2^m) \psi h_m dx dy \right| \\ & \leq \frac{1}{8} \|\psi \partial_y h_m\|_{L^2}^2 + \frac{\epsilon}{4} \|\psi \partial_x h_m\|_{L^2}^2 + C \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2. \end{aligned}$$

For the term

$$N_3^m = -\frac{\sum_{p=0}^{m-1} C_m^p \partial_x^{m-p} u \partial_x^{p+1} w}{\sqrt{u_{yy}^s + w_y}},$$

using (6.1) and (A.3), we can get

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \psi^2 N_3^m h_m dx dy \right| \leq \int_{\mathbb{R}_+^2} \psi^2 \frac{\sum_{p=0}^{m-1} C_m^p |\partial_x^{m-p} u \partial_x^{p+1} w \partial_x^m w|}{|u_{yy}^s + w_y|} dx dy \\ & \leq c_3^{-1} \sum_{p=0}^{m-1} \int_{\mathbb{R}_+^2} \langle y \rangle^{1-\ell} |\partial_x^{m-p} u \partial_x^{p+1} w| \langle y \rangle^{k+\ell} |\partial_x^m w| dx dy \\ & \leq C \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2. \end{aligned}$$

For the term

$$N_4^m = -\frac{\sum_{p=0}^{m-1} C_m^p (\partial_x^{m-p} \partial_y w) \partial_x^p v}{\sqrt{u_{yy}^s + w_y}},$$

observing that

$$\partial_x^p v = -\int_0^y \partial_x^{p+1} u d\tilde{y},$$



we can also get, by using (6.1) and (A.1), for  $\delta > 0$  small ( $\frac{3}{2} + \delta \leq k + \ell$ ),

$$\begin{aligned}
& \left| \int_{\mathbb{R}_+^2} \psi^2 \frac{\sum_{p=1}^{m-1} C_m^p (\partial_x^{m-p} \partial_y w) \partial_x^p v}{\sqrt{u_{yy}^s + w_y}} h_m dx dy \right| \\
& \leq c_3^{-1} \sum_{p=1}^{m-1} C_m^p \int_{\mathbb{R}_+^2} |(\partial_x^{m-p} \partial_y w) \partial_x^p v \langle y \rangle^{k+1} \partial_x^m w| dx dy \\
& \leq c_3^{-1} \sum_{p=1}^{m-1} C_m^p \left( \int_{\mathbb{R}_+^2} \left| (\partial_x^{m-p} \partial_y w) \left( \int_0^y \partial_x^{p+1} u d\tilde{y} \right) \langle y \rangle^{1-\ell} \right|^2 dx dy \right)^{\frac{1}{2}} \|\partial_x^m w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \\
& \leq C \sum_{p=1}^{m-1} C_m^p \left( \int_{\mathbb{R}_+^2} \left| (\partial_x^{m-p} \partial_y w) \|\partial_x^{p+1} u\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+)} \langle y \rangle^{1-\ell} \right|^2 dx dy \right)^{\frac{1}{2}} \|\partial_x^m w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \\
& \leq C \sum_{p=1}^{m-1} C_m^p \left( \int_{\mathbb{R}} \|(\partial_x^{m-p} \partial_y w)\|_{L_{1-\ell}^2(\mathbb{R}_+)} \|\partial_x^{p+1} w\|_{L_{k+\ell}^2(\mathbb{R}_+)}^2 dx \right)^{\frac{1}{2}} \|\partial_x^m w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)} \\
& \leq C \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^3.
\end{aligned}$$

We study finally the term

$$\begin{aligned}
\int_{\mathbb{R}_+^2} \psi^2 \frac{(\partial_x^m \partial_y w) v}{\sqrt{u_{yy}^s + w_y}} h_m dx dy &= \int_{\mathbb{R}_+^2} \psi^2 \frac{v (\partial_x^m \partial_y w) \partial_x^m w}{u_{yy}^s + w_y} dx dy \\
&= \frac{1}{2} \int_{\mathbb{R}_+^2} \psi^2 \frac{v \partial_y (\partial_x^m w)^2}{u_{yy}^s + w_y} dx dy \\
&= -\frac{1}{2} \int_{\mathbb{R}_+^2} \left( \frac{v \psi^2}{u_{yy}^s + w_y} \right)_y (\partial_x^m w)^2 dx dy,
\end{aligned}$$

where the boundary terms is

$$\left( \frac{v \psi^2}{u_{yy}^s + w_y} \right) (\partial_x^m w)^2 \Big|_0^{+\infty} = 0,$$

since  $\psi(y) = 0, y \leq a - 5a_0$ . On the other hand,

$$\left| \left( \frac{v \psi^2}{u_{yy}^s + w_y} \right)_y (\partial_x^m w)^2 \right| \leq C \langle y \rangle^{2(k+\ell)} |\partial_x^m w|^2.$$

We get then

$$\left| \int_{\mathbb{R}_+^2} \psi^2 \frac{(\partial_x^m \partial_y w) v}{\sqrt{u_{yy}^s + w_y}} h_m dx dy \right| \leq C \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2,$$

which conclude the Proposition 6.1.  $\square$

We prepare firstly the following Lemma,

**Lemma 6.2.** *Under the assumption of Proposition 6.1, let  $k + 2\ell > 2, \frac{1}{2} < \ell' < 1$ , we have*

$$\begin{aligned}
\left| \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m u)^2}{u_y^s + w} dx dy \right| &\leq C_5 \langle y_0 \rangle^{-2\ell'+1} (\|\phi_2 g_m\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\psi h_m\|_{L^2}^2) \\
&\quad + C_5 \|\phi_1 g_m\|_{L^2(\mathbb{R}_+^2)}^2,
\end{aligned} \tag{6.3}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} \frac{\psi\psi'(\partial_x^{m+1}u)^2}{u_y^s + w} dx dy \right| &\leq C_5 \langle y_0 \rangle^{-2\ell'+1} (\|\phi_2 \partial_x g_m\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\psi \partial_x h_m\|_{L^2}^2) \\ &\quad + C_5 \|\phi_1 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)}^2 + C_6 \langle y_0 \rangle^k \|\partial_x^m w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2, \end{aligned} \quad (6.4)$$

where  $C_5$  is independent of  $y_0$  and  $\epsilon$ , it depends on  $c_1, c_2, c_3, a, \tilde{B}_2$ , and  $C_6$  is independent of  $\epsilon$ .

*Proof.* First, for the boundary condition, we have, for  $k + 2\ell > 2$ ,

$$\left( \frac{\partial_x^m u}{u_y^s + w} \right) (t, x, 0) = 0, \quad \text{and} \quad \lim_{y \rightarrow +\infty} \frac{\partial_x^m u}{\sqrt{u_y^s + w}} (t, x, y) = 0,$$

because, for  $y \leq a + 2a_0$ ,

$$\begin{aligned} \left| \frac{\partial_x^m u}{\sqrt{u_y^s + w}} (t, x, y) \right| &\leq c_1^{-\frac{1}{2}} \langle y \rangle^{\frac{k}{2}} |\partial_x^m u(t, x, y)| \\ &\leq c_1^{-\frac{1}{2}} \langle y \rangle^{-\frac{k}{2} - \ell + 1} (\langle y \rangle^{k+\ell-1} |\partial_x^m u(t, x, y)|), \end{aligned}$$

and  $\langle y \rangle^{k+\ell-1} |\partial_x^m u(t, x, y)| \in L^2(\mathbb{R}_+^2)$ .

Recalling also

$$I_{\psi'} = \{y; a - 5a_0 \leq y \leq a - 4a_0\} \cup \{y; y_0 - 1 \leq y \leq y_0 + 1\} = J_1 \cup J_2,$$

and

$$|\psi| \leq 1, \quad |\mathbf{1}_{J_1}(y)\psi'| \leq C_0, \quad |\mathbf{1}_{J_2}(y)\psi'| \leq 2,$$

with  $C_0$  depends only on  $a_0$ . Then, (4.6)

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} \frac{\psi\psi'(\partial_x^m u)^2}{u_y^s + w} dx dy \right| &\leq \int_{\mathbb{R}_+^2} |\psi'| \left( \frac{\partial_x^m u}{\sqrt{u_y^s + w}} \right)^2 dx dy \\ &\leq 4C_0 M \int_{\mathbb{R}_+^2} \mathbf{1}_{J_1}(y) \left( \frac{\partial_x^m u}{u_y^s + w} \right)^2 dx dy + 4 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \frac{\partial_x^m u}{\sqrt{u_y^s + w}} \right)^2 dx dy \\ &\leq 4C_0 M \int_{\mathbb{R}_+^2} \left( \mathbf{1}_{J_1}(y) \int_0^{\min\{y, a-4a_0\}} \phi_1(\tilde{y}) g_m(t, x, \tilde{y}) d\tilde{y} \right)^2 dx dy \\ &\quad + 4 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \int_y^{+\infty} \left( \frac{\partial_x^m u}{u_y^s + w} \sqrt{u_{\tilde{y}}^s + w} \right)_y d\tilde{y} \right)^2 dx dy \\ &\leq 4C_0 M \|\phi_1 g_m\|_{L^2(\mathbb{R}_+^2)}^2 + 8 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \int_y^{+\infty} \left( \frac{\partial_x^m u}{u_y^s + w} \right)_y \sqrt{u_{\tilde{y}}^s + w} d\tilde{y} \right)^2 dx dy \\ &\quad + 2 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \int_y^{+\infty} \left( \frac{\partial_x^m u}{u_y^s + w} \right)_y \frac{u_{y\tilde{y}}^s + w_{\tilde{y}}}{\sqrt{u_{\tilde{y}}^s + w}} d\tilde{y} \right)^2 dx dy. \end{aligned}$$

We have for the second term

$$\begin{aligned} \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left( \frac{\partial_x^m u}{u_y^s + w} \right)_y \sqrt{u_{\tilde{y}}^s + w} d\tilde{y} \right|^2 &\leq \mathbf{1}_{J_2}(y) M \left| \int_y^{+\infty} \phi_2 |g_m| d\tilde{y} \right|^2 \\ &\leq \mathbf{1}_{J_2}(y) M \langle y \rangle^{-2\ell'+1} \|\phi_2 g_m(t, x, \cdot)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2. \end{aligned}$$

So that, using  $J_2 = \{y; y_0 - 1 \leq y \leq y_0 + 1\}$ ,

$$\begin{aligned} & \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left( \frac{\partial_x^m u}{u_y^s + w} \right)_y \sqrt{u_y^s + w} d\tilde{y} \right|^2 \\ & \leq 2M \langle y_0 \rangle^{-2\ell'+1} \mathbf{1}_{J_2}(y) \|\phi_2 g_m(t, x, \cdot)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2. \end{aligned}$$

On the other hand, we have also

$$\begin{aligned} & \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left( \frac{\partial_x^m u}{u_y^s + w} \right) \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\ & = \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left( \int_{a+2a_0}^{\tilde{y}} g_m(t, x, z) dz + \frac{\partial_x^m u}{u_y^s + w}(t, x, a + 2a_0) \right) \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\ & \leq 2 \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left| \int_{a+2a_0}^{\tilde{y}} \phi_2 |g_m(t, x, z)| dz \right| \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\ & \quad + 2 \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left| \frac{\partial_x^m u}{u_y^s + w}(t, x, a + 2a_0) \right| \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\ & \leq 4 \mathbf{1}_{J_2}(y) \langle a + 2a_0 \rangle^{-2\ell'+1} \|\phi_2 g_m(t, x, \cdot)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2 \left| \int_y^{+\infty} \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\ & \quad + 2 \mathbf{1}_{J_2}(y) \left| \frac{\partial_x^m u}{u_y^s + w}(t, x, a + 2a_0) \right|^2 \left| \int_y^{+\infty} \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2. \end{aligned}$$

We have also

$$\begin{aligned} \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 & \leq \tilde{B}_2^2 c_1^{-1} \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \langle \tilde{y} \rangle^{-\frac{k}{2}-1} d\tilde{y} \right|^2 \\ & \leq \tilde{B}_2^2 c_1^{-1} \mathbf{1}_{J_2}(y) \langle y_0 \rangle^{-k}. \end{aligned}$$

We have then obtained

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m u)^2}{u_y^s + w} dx dy \right| & \leq 4C_0 M \|\phi_1 g_m\|_{L^2(\mathbb{R}_+^2)}^2 \\ & \quad + \tilde{M} \langle y_0 \rangle^{-2\ell'+1} \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \|\phi_2 g_m(t, x, \cdot)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2 dx dy \\ & \quad + \tilde{B}_2^2 c_1^{-1} \langle y_0 \rangle^{-k} \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \|\phi_2 g_m(t, x, \cdot)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2 dx dy \\ & \quad + \tilde{B}_2^2 c_1^{-1} \langle y_0 \rangle^{-k} \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left| \frac{\partial_x^m u}{(u_y^s + w)}(t, x, a + 2a_0) \right|^2 dx dy \\ & \leq 4C_0 M \|\phi_1 g_m\|_{L^2(\mathbb{R}_+^2)}^2 \\ & \quad + (2M \langle y_0 \rangle^{-2\ell'+1} + \tilde{B}_2^2 c_1^{-1} \langle y_0 \rangle^{-k}) \|\phi_2 g_m\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ & \quad + \tilde{B}_2^2 c_1^{-1} \langle y_0 \rangle^{-k} \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left| \frac{\partial_x^m u}{(u_y^s + w)}(t, x, a + 2a_0) \right|^2 dx dy, \end{aligned}$$

where  $\tilde{B}_2, M$  are independent of  $y_0, \epsilon$ .

We study now the last term,

$$\begin{aligned}
& \left| \left( \frac{\partial_x^m u}{u_y^s + w} \right) (t, x, a + 2a_0) \right| \leq c_1^{-1} \langle a + 2a_0 \rangle^k |\partial_x^m u(t, x, a + 2a_0)| \\
& \leq c_1^{-1} \langle a + 2a_0 \rangle^k \left| \int_{a-3a_0}^{a+2a_0} \partial_x^m w(t, x, \tilde{y}) d\tilde{y} + \int_0^{a-3a_0} \partial_x^m w(t, x, \tilde{y}) d\tilde{y} \right| \\
& \leq c_1^{-1} \langle a + 2a_0 \rangle^k \left| \int_{a-3a_0}^{a+2a_0} \sqrt{u_{yy}^s + w_y} \frac{\partial_x^m w}{\sqrt{u_{yy}^s + w_y}}(t, x, \tilde{y}) d\tilde{y} \right| \\
& \quad + c_1^{-1} \langle a + 2a_0 \rangle^k \left| \int_0^{a-3a_0} \left( \frac{\partial_x^m u}{u_y^s + w}(u_y^s + w) \right)_y (t, x, \tilde{y}) d\tilde{y} \right| \\
& \leq c_1^{-1} \langle a + 2a_0 \rangle^k \|\sqrt{u_{yy}^s + w_y}(t, x, \cdot)\|_{L^2([a-3a_0, a+2a_0])} \|\psi h_m(t, x, \cdot)\|_{L^2(\mathbb{R}_+)} \\
& \quad + c_1^{-1} \langle a + 2a_0 \rangle^k \left| \int_0^{a-3a_0} \phi_1 g_m(t, x, \tilde{y})(u_y^s + w)(t, x, \tilde{y}) d\tilde{y} \right| \\
& \quad + c_1^{-1} \langle a + 2a_0 \rangle^k \left| \int_0^{a-3a_0} \left( \int_0^{\tilde{y}} \phi_1 g_m(t, x, z) dz \right) (u_{yy}^s + w_y)(t, x, \tilde{y}) d\tilde{y} \right| \\
& \leq c_1^{-1} \langle a + 2a_0 \rangle^k \|\sqrt{u_{yy}^s + w_y}(t, x, \cdot)\|_{L^2([a-3a_0, a+2a_0])} \|\psi h_m(t, x, \cdot)\|_{L^2(\mathbb{R}_+)} \\
& \quad + c_1^{-1} \langle a + 2a_0 \rangle^k \|\phi_1 g_m(t, x, \cdot)\|_{L^2(\mathbb{R}_+)} \\
& \quad \times \left( \|(u_y^s + w)(t, x, \cdot)\|_{L^2([0, a-3a_0])} + \int_0^{a-3a_0} |\tilde{y}|^{1/2} |(u_{yy}^s + w_y)(t, x, \tilde{y})| d\tilde{y} \right) \\
& \leq C(\|\psi h_m(t, x, \cdot)\|_{L^2(\mathbb{R}_+)} + \|\phi_1 g_m(t, x, \cdot)\|_{L^2(\mathbb{R}_+)}),
\end{aligned}$$

where  $C$  is depends on  $a$ , but independent of  $y_0, \epsilon$ .

Finally, we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m u)^2}{u_y^s + w} dx dy \right| \leq \bar{C}_0 \langle y_0 \rangle^{-2\ell'+1} (\|\phi_2 g_m\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\psi h_m\|_{L^2}^2) \\
& \quad + C_5 \|\phi_1 g_m\|_{L^2(\mathbb{R}_+^2)}^2,
\end{aligned}$$

where  $\bar{C}_0$  is independent of  $y_0, \epsilon$ , it is depends en  $c_1, c_2, c_3, \tilde{B}_2$ . So that we have proven (6.3).

Now, for (6.4), we have

$$\begin{aligned}
\phi_1 g_{m+1} &= \phi_1 \left( \frac{\partial_x^{m+1} u}{u_y^s + w} \right)_y = \partial_x \phi_1 \left( \frac{\partial_x^m u}{u_y^s + w} \right)_y + \phi_1 \left( \partial_x^m u \frac{w_x}{(u_y^s + w)^2} \right)_y \\
&= \phi_1 \partial_x g_m + \phi_j \partial_x^m w \frac{w_x}{(u_y^s + w)^2} + \phi_1 \partial_x^m u \left( \frac{w_x}{(u_y^s + w)^2} \right)_y,
\end{aligned}$$

then, using  $\partial_x^m u|_{y=0} = 0$ , we have

$$\|\phi_1 g_{m+1}\|_{L^2(\mathbb{R}_+^2)} \leq \|\phi_1 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)} + C \|\partial_x^m w\|_{L^2(\mathbb{R}_+^2)} \quad (6.5)$$

and

$$\psi h_{m+1} = \psi \frac{\partial_x^{m+1} w}{\sqrt{u_{yy}^s + w_y}} = \psi \partial_x \left( \frac{\partial_x^m w}{\sqrt{u_{yy}^s + w_y}} \right) + \psi \partial_x^m w \frac{w_{xy}}{2(u_{yy}^s + w_y)^{3/2}}$$

$$= \psi \partial_x h_m + \psi \partial_x^m w \frac{w_{xy}}{2(u_{yy}^s + w_y)^{3/2}},$$

we get also

$$\|\psi h_{m+1}\|_{L^2(\mathbb{R}_+^2)} \leq \|\psi \partial_x h_m\|_{L^2(\mathbb{R}_+^2)} + C \langle y_0 \rangle^{\frac{k}{2} + \frac{1}{2}} \|\partial_x^m w\|_{L^2(\mathbb{R}_+^2)}. \quad (6.6)$$

So that, we only need to pay attention for the term  $\phi_1 g_{m+1}$  in the above proof.

In fact,

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^{m+1} u)^2}{u_y^s + w} dx dy \right| \leq \int_{\mathbb{R}_+^2} |\psi'| \left( \frac{\partial_x^{m+1} u}{\sqrt{u_y^s + w}} \right)^2 dx dy \\ & \leq 4C_0 M \int_{\mathbb{R}_+^2} \mathbf{1}_{J_1}(y) \left( \frac{\partial_x^{m+1} u}{u_y^s + w} \right)^2 dx dy + 4 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \frac{\partial_x^{m+1} u}{\sqrt{u_y^s + w}} \right)^2 dx dy \\ & \leq 4C_0 M \|\phi_1 g_{m+1}\|_{L^2(\mathbb{R}_+^2)}^2 + 8 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \partial_x \left( \frac{\partial_x^m u}{u_y^s + w} \right) \sqrt{u_y^s + w} \right)^2 dx dy \\ & \quad + 8 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \frac{\partial_x^m u}{u_y^s + w} \frac{w_x}{\sqrt{u_y^s + w}} \right)^2 dx dy \\ & \leq 4C_0 M \|\phi_1 g_{m+1}\|_{L^2(\mathbb{R}_+^2)}^2 + 16 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \int_y^{+\infty} \partial_x \left( \frac{\partial_x^m u}{u_y^s + w} \right)_y \sqrt{u_y^s + w} d\tilde{y} \right)^2 dx dy \\ & \quad + 4 \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left( \int_y^{+\infty} \partial_x \left( \frac{\partial_x^m u}{u_y^s + w} \right) \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}} d\tilde{y} \right)^2 dx dy \\ & \quad + C \langle y_0 \rangle^k \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) |\partial_x^m u|^2 dx dy, \end{aligned}$$

where we use the condition

$$\lim_{y \rightarrow +\infty} \left| \partial_x \left( \frac{\partial_x^m u}{u_y^s + w} \right) \sqrt{u_y^s + w} \right| = 0.$$

The same proof as the case  $m$ , we have

$$\begin{aligned} & \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \partial_x \left( \frac{\partial_x^m u}{u_y^s + w} \right)_y \sqrt{u_y^s + w} d\tilde{y} \right|^2 \\ & \leq 2M \langle y_0 \rangle^{-2\ell' + 1} \mathbf{1}_{J_2}(y) \|\phi_2 \partial_x g_m(t, x, \cdot)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2. \end{aligned}$$

On the other hand, we have also

$$\begin{aligned} & \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \partial_x \left( \frac{\partial_x^m u}{u_y^s + w} \right) \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\ & = \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left( \int_{a+2a_0}^{\tilde{y}} \partial_x g_m(t, x, z) dz + \left( \frac{\partial_x^m u}{u_y^s + w} \right)_x(t, x, a + 2a_0) \right) \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\ & \leq 2 \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left| \int_{a+2a_0}^{\tilde{y}} \phi_2 |\partial_x g_m(t, x, z)| dz \right| \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \end{aligned}$$

$$\begin{aligned}
& + 2 \mathbf{1}_{J_2}(y) \left| \int_y^{+\infty} \left| \left( \frac{\partial_x^m u}{u_y^s + w} \right)_x (t, x, a + 2a_0) \right| \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\
& \leq 4 \mathbf{1}_{J_2}(y) \langle a + 2a_0 \rangle^{-2\ell'+1} \|\phi_2 \partial_x g_m(t, x, \cdot)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2 \left| \int_y^{+\infty} \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2 \\
& \quad + 2 \mathbf{1}_{J_2}(y) \left| \left( \frac{\partial_x^m u}{u_y^s + w} \right)_x (t, x, a + 2a_0) \right|^2 \left| \int_y^{+\infty} \frac{u_{yy}^s + w_y}{\sqrt{u_y^s + w}}(t, x, \tilde{y}) d\tilde{y} \right|^2.
\end{aligned}$$

Using (6.5), we have then obtained

$$\begin{aligned}
\left| \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^{m+1} u)^2}{u_y^s + w} dx dy \right| & \leq 4C_0 M \|\phi_1 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)}^2 + C \langle y_0 \rangle^k \|\partial_x^m w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2 \\
& \quad + (2M \langle y_0 \rangle^{-2\ell'+1} + \tilde{B}_2^2 c_1^{-1} \langle y_0 \rangle^{-k}) \|\phi_2 \partial_x g_m\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\
& \quad + \tilde{B}_2^2 c_1^{-1} \langle y_0 \rangle^{-k} \int_{\mathbb{R}_+^2} \mathbf{1}_{J_2}(y) \left| \left( \frac{\partial_x^m u}{u_y^s + w} \right)_x (t, x, a + 2a_0) \right|^2 dx dy.
\end{aligned}$$

We study now the last term,

$$\begin{aligned}
& \left| \left( \frac{\partial_x^m u}{u_y^s + w} \right)_x (t, x, a + 2a_0) \right| \leq c_1^{-1} \langle a + 2a_0 \rangle^k (|\partial_x^{m+1} u(t, x, a + 2a_0)| + |\partial_x^m u(t, x, a + 2a_0)|) \\
& \leq c_1^{-1} \langle a + 2a_0 \rangle^k \left| \int_{a-3a_0}^{a+2a_0} \partial_x^{m+1} w(t, x, \tilde{y}) d\tilde{y} + \int_0^{a-3a_0} \partial_x^{m+1} w(t, x, \tilde{y}) d\tilde{y} \right| \\
& \quad + c_1^{-1} \langle a + 2a_0 \rangle^k \left| \int_0^{a+2a_0} \partial_x^m w(t, x, \tilde{y}) d\tilde{y} \right| \\
& \leq C (\|\psi h_{m+1}(t, x, \cdot)\|_{L^2(\mathbb{R}_+)} + \|\phi_1 g_{m+1}(t, x, \cdot)\|_{L^2(\mathbb{R}_+)} + \|\partial_x^m w(t, x, \cdot)\|_{L^2(\mathbb{R}_+)}),
\end{aligned}$$

where  $C$  is depends on  $a$ , but independent of  $y_0, \epsilon$ .

Finally, using (6.5)-(6.6), we have

$$\begin{aligned}
\left| \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^{m+1} u)^2}{u_y^s + w} dx dy \right| & \leq \bar{C}_0 \langle y_0 \rangle^{-2\ell'+1} (\|\phi_2 \partial_x g_m\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\psi \partial_x h_m\|_{L^2}^2) \\
& \quad + C_5 (\|\phi_1 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)}^2 + \langle y_0 \rangle^k \|\partial_x^m w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2),
\end{aligned}$$

where  $\bar{C}_0$  is independent of  $y_0, \epsilon$ , it is depends en  $c_1, c_2, c_3, \tilde{B}_2$ .

The proof of the Lemma is complete.  $\square$

We now study the worst term

$$- \int_{\mathbb{R}_+^2} \psi^2 \sqrt{u_{yy}^s + w_y} (\partial_x^m v) h_m dx dy$$

which is the main difficulty for the study of the Prandtl equation. We have

**Proposition 6.3.** *Under the assumption of Proposition 6.1, we have*

$$\begin{aligned}
& - \int_{\mathbb{R}_+^2} \psi^2 \sqrt{u_{yy}^s + w_y} (\partial_x^m v) h_m dx dy \\
& \leq \epsilon C_5 \langle y_0 \rangle^{-2\ell'+1} (\|\phi_2 \partial_x g_m\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\psi \partial_x h_m\|_{L^2}^2) \\
& \quad + \epsilon C_5 \|\phi_1 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)}^2 + C_6 \langle y_0 \rangle^k \|\partial_x^m w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2 \\
& \quad - \frac{d}{dt} \int_{\mathbb{R}_+^2} (\partial_x^m u)^2 \frac{\psi \psi'}{u_y^s + u_y} dx dy
\end{aligned}$$

where  $C_5, C_6$  are the constants in Lemma 6.2.

*Proof.* By the definition of  $h_m, w$  and  $v$ , firstly, we have

$$\begin{aligned}
& - \int_{\mathbb{R}_+^2} \psi^2 \sqrt{u_{yy}^s + w_y} (\partial_x^m v) h_m dx dy \\
& = - \int_{\mathbb{R}_+^2} \psi^2 \sqrt{u_{yy}^s + w_y} (\partial_x^m v) \frac{\partial_x^m w}{\sqrt{u_{yy}^s + w_y}} dx dy \\
& = - \int_{\mathbb{R}_+^2} \psi^2 (\partial_x^m v) \partial_x^m w dx dy = - \int_{\mathbb{R}_+^2} \psi^2 (\partial_x^m v) \partial_y \partial_x^m u dx dy \\
& = - \int_{\mathbb{R}_+^2} \psi^2 \partial_x^{m+1} u \partial_x^m u dx dy + \int_{\mathbb{R}_+^2} \partial_y (\psi^2) (\partial_x^m v) \partial_x^m u dx dy \\
& = 2 \int_{\mathbb{R}_+^2} \psi \psi' (\partial_x^m v) \partial_x^m u dx dy,
\end{aligned}$$

where we use  $\partial_y \partial_x^m v = -\partial_x^{m+1} u$ , and the fact,

$$\int_{\mathbb{R}_+^2} \psi^2 \partial_x^{m+1} u \partial_x^m u dx dy = \frac{1}{2} \int_{\mathbb{R}_+^2} \psi^2 \partial_x (\partial_x^m u)^2 dx dy = 0.$$

Taking the first equation of (3.1) with derivative  $\partial_x^m$ , we have

$$\partial_t \partial_x^m u + (u^s + u) \partial_x^{m+1} u + (u_y^s + u_y) \partial_x^m v = \partial_x^m \partial_x^2 u + \epsilon \partial_x^{m+2} u + \mathbf{A}_1 + \mathbf{A}_2, \quad (6.7)$$

where  $\mathbf{A}_1 = [u^s + u, \partial_x^m] \partial_x u$ ,  $\mathbf{A}_2 = [u_y^s + u_y, \partial_x^m] v$ . Observing

$$\begin{aligned}
\text{Supp } \psi' &= [a - 5a_0, a - 4a_0] \cup [y_0 - 1, y_0 + 1] = J_1 \cup J_2 \\
&\subset \text{Supp } \psi \cap (\{y; \phi_1(y) = 1\} \cup \{y; \phi_2(y) = 1\}),
\end{aligned} \quad (6.8)$$

multiplying (6.7) with  $\frac{2\psi\psi'\partial_x^m u}{u_y^s + u_y}$ , we get

$$\begin{aligned}
2 \int_{\mathbb{R}_+^2} \psi\psi'(\partial_x^m v) \partial_x^m u dx dy &= - \int_{\mathbb{R}_+^2} \partial_t \partial_x^m u \frac{2\psi\psi'\partial_x^m u}{u_y^s + u_y} dx dy \\
&\quad - \int_{\mathbb{R}_+^2} (u^s + u) \partial_x^{m+1} u \frac{2\psi\psi'\partial_x^m u}{u_y^s + u_y} dx dy \\
&\quad + \int_{\mathbb{R}_+^2} \partial_y^2 \partial_x^m u \frac{2\psi\psi'\partial_x^m u}{u_y^s + u_y} dx dy \\
&\quad + \epsilon \int_{\mathbb{R}_+^2} \partial_x^{m+2} u \frac{2\psi\psi'\partial_x^m u}{u_y^s + u_y} dx dy \\
&\quad + \int_{\mathbb{R}_+^2} (\mathbf{A}_1 + \mathbf{A}_2) \frac{2\psi\psi'\partial_x^m u}{u_y^s + u_y} dx dy \\
&= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}_5.
\end{aligned} \tag{6.9}$$

Now, we study each term of (6.9).

$$\begin{aligned}
\mathbf{I}_1 &= - \int_{\mathbb{R}_+^2} \partial_t \partial_x^m u \frac{2\psi\psi'\partial_x^m u}{u_y^s + u_y} = - \int_{\mathbb{R}_+^2} \partial_t (\partial_x^m u)^2 \frac{\psi\psi'}{u_y^s + u_y} \\
&= - \frac{d}{dt} \int_{\mathbb{R}_+^2} (\partial_x^m u)^2 \frac{\psi\psi'}{u_y^s + u_y} dx dy + \int_{\mathbb{R}_+^2} (\partial_x^m u)^2 \partial_t \left( \frac{\psi\psi'}{u_y^s + u_y} \right) dx dy.
\end{aligned}$$

For the second term in the above equality, according to (3.7) and (6.8), we obtain, for  $m \geq 6$ ,

$$\begin{aligned}
\left\| \partial_t \left( \frac{\psi\psi'}{u_y^s + u_y} \right) \right\|_{L^\infty(\mathbb{R}_+^2)} &= \left\| \psi\psi' \left( \frac{\partial_t u_y^s + \partial_t u_y}{(u_y^s + u_y)^2} \right) \right\|_{L^\infty(\mathbb{R}_+^2)} \\
&\leq c_1^{-2} \|\psi\psi'(\partial_t u_y^s + \partial_t u_y)\|_{L^\infty(\mathbb{R}_+^2)} \\
&\leq C(1 + \|w\|_{H_1^m(\mathbb{R}_+^2)}^2).
\end{aligned} \tag{6.10}$$

Then under the assumption  $\|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta$ , we get

$$\mathbf{I}_1 \leq - \frac{d}{dt} \int_{\mathbb{R}_+^2} (\partial_x^m u)^2 \left( \frac{\psi\psi'}{u_y^s + u_y} \right) dx dy + C\|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2.$$

For the second term

$$\begin{aligned}
\mathbf{I}_2 &= \int_{\mathbb{R}_+^2} (u^s + u) \partial_x \partial_x^m u \frac{2\psi\psi'\partial_x^m u}{u_y^s + u_y} dx dy \\
&= \int_{\mathbb{R}_+^2} \frac{\psi\psi'(u^s + u)}{u_y^s + u_y} \partial_x (\partial_x^m u)^2 dx dy \\
&= - \int_{\mathbb{R}_+^2} \left( \frac{\psi\psi'(u^s + u)}{u_y^s + u_y} \right)_x (\partial_x^m u)^2 dx dy.
\end{aligned}$$

Similar to (6.10), we can get

$$\left\| \left( \frac{\psi\psi'(u^s + u)}{u_y^s + u_y} \right)_x \right\|_{L^\infty(\mathbb{R}_+^2)} \leq C(1 + \|w\|_{H_1^m(\mathbb{R}_+^2)}^2).$$



Again with the help of the assumption  $\|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta$ , we get

$$|\mathbf{I}_2| \leq C\|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2.$$

For the term  $\mathbf{I}_3$ , using  $w = \partial_y u$ , and integrating by part,

$$\begin{aligned} \mathbf{I}_3 &= \int_{\mathbb{R}_+^2} \partial_y^2 \partial_x^m u \frac{2\psi\psi' \partial_x^m u}{u_y^s + u_y} = \int_{\mathbb{R}_+^2} \partial_y \partial_x^m w \frac{2\psi\psi' \partial_x^m u}{u_y^s + u_y} \\ &= - \int_{\mathbb{R}_+^2} (\partial_x^m w)^2 \frac{2\psi\psi'}{u_y^s + u_y} dx dy - \int_{\mathbb{R}_+^2} \partial_x^m w \partial_x^m u \partial_y \left( \frac{2\psi\psi'}{u_y^s + u_y} \right) dx dy, \end{aligned}$$

we get

$$|\mathbf{I}_3| \leq C\|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2.$$

For the term  $\mathbf{I}_5$ , recalling

$$\mathbf{A}_1 = \sum_{i=1}^m C_m^i \partial_x^i u \partial_x^{m+1-i} u, \quad \mathbf{A}_2 = \sum_{i=1}^{m-1} C_m^i \partial_x^i w \partial_x^{m-i} v,$$

we don't need to worry about it, since the order of derivative is easy to be controlled. So we have

$$|\mathbf{I}_5| \leq C\|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2.$$

For the term  $\mathbf{I}_4$ , we have to be careful, since the order of derivative with respect to  $x$  is up to  $m+2$ . Integrating by parts with respect to  $x$ , we have

$$\begin{aligned} \mathbf{I}_4 &= \epsilon \int_{\mathbb{R}_+^2} \partial_x^{m+2} u \frac{2\psi\psi' \partial_x^m u}{u_y^s + u_y} dx dy \\ &= -\epsilon \int_{\mathbb{R}_+^2} \frac{2\psi\psi'}{u_y^s + u_y} (\partial_x^{m+1} u)^2 dx dy + \epsilon \int_{\mathbb{R}_+^2} \left( \frac{\psi\psi'}{u_y^s + u_y} \right)_{xx} (\partial_x^m u)^2 dx dy \\ &= \mathbf{I}_4^1 + \mathbf{I}_4^2. \end{aligned}$$

For the second term, observing

$$\left\| \left( \frac{\psi\psi'}{u_y^s + u_y} \right)_{xx} \right\|_{L^\infty(\mathbb{R}_+^2)} \leq C\|w\|_{H_1^4(\mathbb{R}_+^2)},$$

it implies

$$|\mathbf{I}_4^2| = \left| \epsilon \int_{\mathbb{R}_+^2} \left( \frac{\psi\psi'}{u_y^s + u_y} \right)_{xx} (\partial_x^m u)^2 dx dy \right| \leq C\|w\|_{H_1^m(\mathbb{R}_+^2)}^2.$$

For the first term, applying (6.4), we get then

$$\begin{aligned} |\mathbf{I}_4^1| &\leq \epsilon C_5 \langle y_0 \rangle^{-2\ell'+1} \left( \|\phi_2 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)}^2 + \|\psi \partial_x h_m\|_{L^2}^2 \right) \\ &\quad + \epsilon C_5 \|\phi_1 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)}^2 + C_6 \langle y_0 \rangle^k \|\partial_x^m w\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2. \end{aligned}$$

with  $C$  is independent of  $\epsilon$ , which finishes the proof of Proposition 6.3.  $\square$

Combining Proposition 6.1 and Proposition 6.3, for the convex part, we get the following results.

**Proposition 6.4.** *Under the assumption of Proposition 6.1, if  $y_0$  is enough big such that*

$$C_5 \langle y_0 \rangle^{-2\ell'+1} \leq \frac{1}{4}, \quad (6.11)$$

then we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\psi h_m\|_{L^2}^2 + \frac{3}{4} \|\psi \partial_y h_m\|_{L^2}^2 + \frac{\epsilon}{2} \|\psi \partial_x h_m\|_{L^2}^2 \\ & \leq C_3 \|w\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 + \epsilon C_4 (\|\phi_1 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_2 \partial_x g_m\|_{L^2(\mathbb{R}_+^2)}^2) \\ & \quad - \frac{d}{dt} \int_{\mathbb{R}_+^2} (\partial_x^m u)^2 \frac{\psi \psi'}{u_y^s + u_y} dx dy, \end{aligned} \quad (6.12)$$

where  $C_3, C_4$  are independent of  $\epsilon$ .

## 7. EXISTENCE OF THE SOLUTION

Now, we can conclude the following energy estimate for the sequence of approximate solutions.

**Theorem 7.1.** *Assume  $u^s$  verifies the conclusion of the Lemma 2.1. Let  $m \geq 6$  be an even integer,  $k > 1, 0 < \ell < \frac{1}{2}, k + \ell > \frac{3}{2}, \ell' > \frac{1}{2}, \ell' - \ell < \frac{1}{2}$ , and  $\tilde{u}_0 \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  which satisfies the compatibility conditions (2.6)-(2.7). Suppose that  $\tilde{w}_\epsilon \in L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$  is a solution to (3.7) such that*

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta$$

with

$$0 < \zeta \leq 1, \quad C_m \zeta \leq \frac{\min\{c_1, c_2, c_3\}}{2},$$

where  $0 < T \leq T_1$  and  $T_1$  is the lifespan of shear flow  $u^s$  in the Lemma 2.1,  $C_m$  is the Sobolev embedding constant in (4.3). Then there exists  $C_T > 0, \tilde{C}_T > 0$  such that,

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq C_T \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}. \quad (7.1)$$

Moreover,  $C_T > 0, \tilde{C}_T > 0$  is increasing with respect to  $0 < T \leq T_1$  and independent of  $0 < \epsilon \leq 1$ .

Firstly, we collect some results to be used from Section 3 - 6. We come back to the notations with tilde and the sub-index  $\epsilon$ . Then  $g_n^\epsilon, h_n^\epsilon$  are the functions defined by  $\tilde{w}_\epsilon$ . Under the hypothesis of Theorem 7.1, we have proven the estimates (3.13), (5.1), (5.2) and (6.12),

$$\begin{aligned} & \frac{d}{dt} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m, m-1}(\mathbb{R}_+^2)}^2 + \|\partial_y \tilde{w}_\epsilon\|_{H_{k+\ell}^{m, m-1}(\mathbb{R}_+^2)}^2 \\ & \quad + \epsilon \|\partial_x \tilde{w}_\epsilon\|_{H_{k+\ell}^{m, m-1}(\mathbb{R}_+^2)}^2 \leq C_1 \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2, \end{aligned} \quad (7.2)$$

$$\begin{aligned} & \frac{d}{dt} \sum_{n=1}^m \|\phi_1 g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_1 \partial_y g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \epsilon \sum_{n=1}^m \|\phi_1 \partial_x g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \\ & \leq C_2 \left( \sum_{n=1}^m \|\phi_1 g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \right), \end{aligned} \quad (7.3)$$

and

$$\begin{aligned} & \frac{d}{dt} \sum_{n=1}^m \|\phi_2 g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_2 \partial_y g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + \epsilon \sum_{n=1}^m \|\phi_2 \partial_x g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \\ & \leq C_2 \left( \sum_{n=1}^m \|\phi_2 g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \right), \end{aligned} \quad (7.4)$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \frac{3}{4} \|\psi \partial_y h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \frac{\epsilon}{2} \|\psi \partial_x h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \\ & \leq C_3 \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 + \epsilon C_4 (\|\phi_1 \partial_x g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_2 \partial_x g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2) \\ & \quad - \frac{d}{dt} \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m \tilde{u}_\epsilon)^2}{u_y^s + \tilde{w}_\epsilon} dx dy, \end{aligned} \quad (7.5)$$

where  $C_1, C_2, C_3, C_4$  are independent of  $0 < \epsilon \leq 1$ .

We have also directly, for  $\delta > 0$  small,

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m \tilde{u}_\epsilon)^2}{u_y^s + \tilde{w}_\epsilon} dx dy \right| & \leq C \int_{\mathbb{R}_+^2} |\psi'(y)| |\partial_x^m \tilde{u}_\epsilon|^2 dx dy \\ & \leq C \int_{\mathbb{R}_+^2} |\psi'| \left| \int_0^y \partial_x^m \tilde{w}_\epsilon d\tilde{y} \right|^2 dx dy \\ & \leq C_6 \|\tilde{w}_\epsilon\|_{H_{\frac{1}{2}+\delta}^m(\mathbb{R}_+^2)}^2, \end{aligned} \quad (7.6)$$

and

$$\|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \leq C_7 \|\partial_x^m \tilde{w}_\epsilon\|_{L^2_{\frac{1}{2}+\frac{1}{2}}(\mathbb{R}_+^2)}^2 \leq C_7 \|\partial_x^m \tilde{w}_\epsilon\|_{L^2_{k+\ell}(\mathbb{R}_+^2)}^2. \quad (7.7)$$

Denoting  $\tilde{C}_4 = \max\{C_4, 2C_5\}$ , and taking

$$\tilde{C}_4 \times \{(7.3) + (7.4)\} + (7.2) + (7.5),$$

we get

$$\begin{aligned} & \frac{d}{dt} \left\{ \tilde{C}_4 \left( \sum_{n=1}^m \|\phi_1 g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_2 g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \right) + \frac{1}{2} \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \right. \\ & \quad \left. + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m \tilde{u}_\epsilon)^2}{u_y^s + \tilde{w}_\epsilon} dx dy \right\} \\ & \leq \tilde{C}_4 C_2 \left( \sum_{n=1}^m \|\phi_1 g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_2 g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \right) \\ & \quad + (2\tilde{C}_4 C_2 + C_1 + C_3) \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2. \end{aligned}$$

Then we have

$$\begin{aligned}
& \frac{d}{dt} \left\{ e^{-C_2 t} \left[ \tilde{C}_4 \left( \sum_{n=1}^m \|\phi_1 g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_2 g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \right) + \frac{1}{2} \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \right. \right. \\
& \quad \left. \left. + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m \tilde{u}_\epsilon)^2}{u_y^s + \tilde{w}_\epsilon} dx dy \right] \right\} \\
& \leq (2\tilde{C}_4 C_2 + C_1 + C_3) e^{-C_2 t} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 - \frac{C_2}{2} e^{-C_2 t} \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \\
& \quad - C_2 e^{-C_2 t} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 - C_2 e^{-C_2 t} \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m \tilde{u}_\epsilon)^2}{u_y^s + \tilde{w}_\epsilon} dx dy.
\end{aligned}$$

According to (7.6) and (7.7), we obtain

$$\begin{aligned}
& \frac{d}{dt} \left\{ e^{-C_2 t} \left( \tilde{C}_4 \left( \sum_{n=1}^m \|\phi_1 g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_2 g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \right) + \frac{1}{2} \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \right. \right. \\
& \quad \left. \left. + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m \tilde{u}_\epsilon)^2}{u_y^s + \tilde{w}_\epsilon} dx dy \right) \right\} \\
& \leq C_8 e^{-C_2 t} \|\tilde{w}_\epsilon\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2.
\end{aligned}$$

Using (7.6) and (7.7) again, we have for any  $t \in ]0, T]$

$$\begin{aligned}
& \tilde{C}_4 \left( \sum_{n=1}^m \|\phi_1 g_n^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_2 g_n^\epsilon\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \right) + \frac{1}{2} \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \\
& + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \leq \left| \int_{\mathbb{R}_+^2} \frac{\psi \psi' (\partial_x^m \tilde{u}_\epsilon)^2}{u_y^s + \tilde{w}_\epsilon} dx dy \right| \\
& + C_8 e^{C_2 t} \int_0^t e^{-C_2 \tau} \|\tilde{w}_\epsilon(\tau)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 d\tau \\
& + e^{C_2 t} \tilde{C}_4 \left( \sum_{n=1}^m \|\phi_1 g_n^\epsilon(0)\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_2 g_n^\epsilon(0)\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 \right) \\
& + (1 + C_6 + C_7) e^{C_2 t} \|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2.
\end{aligned} \tag{7.8}$$

Firstly, we have the following upper bounded estimate.

**Lemma 7.2.** *For the initial date, we have*

$$\begin{aligned}
T_m^\epsilon(g, w)(0) &= \sum_{n=1}^m \|\phi_1 g_n^\epsilon(0)\|_{L^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^m \|\phi_2 g_n^\epsilon(0)\|_{L^2_{\ell'}(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \\
&\leq C \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2,
\end{aligned}$$

where  $C$  is independent of  $\epsilon$ .

*Proof.* Notice for any  $1 \leq n \leq m$ ,

$$\phi_2 g_n^\epsilon = \phi_2 \left( \frac{\partial_x^n \tilde{u}_\epsilon}{u_y^s + \tilde{w}_\epsilon} \right)_y = \phi_2 \frac{\partial_x^n \partial_y \tilde{u}_\epsilon}{u_y^s + \tilde{w}_\epsilon} - \phi_2 \frac{\partial_x^n \tilde{u}_\epsilon}{u_y^s + \tilde{w}_\epsilon} \eta_2,$$

and  $\tilde{u}_\epsilon(0) = \tilde{u}_0$ , then we deduce, for any  $1 \leq n \leq m$ ,

$$\begin{aligned} \|\phi_2 g_n^\epsilon(0)\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 &\leq 2 \left\| \phi_2 \frac{\partial_x^n \partial_y \tilde{u}_0}{u_{0,y}^s + \tilde{w}_0} \right\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + 2 \left\| \phi_2 \frac{\partial_x^n \tilde{u}_0}{u_{0,y}^s + \tilde{w}_0} \eta_2(0) \right\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ &\leq C (\|\partial_x^n \partial_y \tilde{u}_0\|_{L_{k+\ell'}^2(\mathbb{R}_+^2)}^2 + \|\partial_x^n \tilde{u}_0\|_{L_{k+\ell'-1}^2(\mathbb{R}_+^2)}^2) \leq C \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2. \end{aligned}$$

The estimate of  $\|\phi_1 g_m^\epsilon(0)\|_{L^2(\mathbb{R}_+^2)}^2$  is easier, since there is no weight. And

$$\|\tilde{w}_\epsilon(0)\|_{H_{k+\ell'}^m(\mathbb{R}_+^2)} = \|\partial_y \tilde{u}_0\|_{H_{k+\ell'}^m(\mathbb{R}_+^2)} \leq \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}.$$

This completes the proof of Lemma 7.2.  $\square$

Using (6.3), Lemma 7.2, we can deduce from (7.8) that for any  $t \in ]0, T[$

$$\begin{aligned} &\tilde{C}_4 \left( \|\phi_1 g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_2 g_m^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \right) + \frac{1}{2} \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \\ &\leq C_8 e^{C_2 t} \int_0^t e^{-C_2 \tau} \|\tilde{w}_\epsilon(\tau)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 d\tau + C_9 e^{C_2 t} \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2 \\ &\quad + C_5 \langle y_0 \rangle^{-2\ell'+1} (\|\phi_2 g_m^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\psi h_m^\epsilon\|_{L^2}^2) + C_5 \|\phi_1 g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2, \end{aligned}$$

then (6.11) and the choice of  $C_5 \leq \frac{\tilde{C}_4}{2}$  imply

$$\begin{aligned} &\frac{\tilde{C}_4}{2} \left( \|\phi_1 g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_2 g_m^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \right) + \frac{1}{4} \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|\tilde{w}_\epsilon\|_{H_{k+\ell}^{m,m-1}(\mathbb{R}_+^2)}^2 \\ &\leq C_8 e^{C_2 t} \int_0^t e^{-C_2 \tau} \|\tilde{w}_\epsilon(\tau)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 d\tau + C_9 e^{C_2 t} \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2. \end{aligned} \quad (7.9)$$

We recall the definition of the cut-off functions

$$\begin{aligned} \phi_1(y) &= 1, \quad 0 \leq y \leq a - 3a_0; \quad \phi_1(y) = 0, \quad y \geq a - 2a_0; \\ \phi_2(y) &= 0, \quad 0 \leq y \leq a + 2a_0; \quad \phi_2(y) = 1, \quad y \geq a + 3a_0; \\ \psi(y) &= 0, \quad y \in \mathbb{R}_+ \setminus [a - 5a_0, y_0 + 1]; \quad \psi(y) = 1, \quad y \in [a - 4a_0, y_0 - 1]. \end{aligned}$$

We have also the following low bounded estimate.

**Lemma 7.3.** *Let  $k > 1, 0 < \ell < \frac{1}{2}, \ell' > \frac{1}{2}, k + \ell > \frac{3}{2}$ , then we have also the following estimate :*

$$\|\partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2 \leq \tilde{C} \left( \|\phi_1 g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_2 g_m^\epsilon\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\psi h_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \right),$$

where  $\tilde{C}$  is independent of  $\epsilon$ .

*Proof.* First, for  $y \geq a + 3a_0$

$$\begin{aligned} \partial_x^m \tilde{u}_\epsilon(t, x, y) &= (u_y^s + \tilde{w}_\epsilon)(t, x, y) \int_{a+3a_0}^y g_m^\epsilon(t, x, \tilde{y}) dy \\ &\quad + \left( \frac{\partial_x^m \tilde{u}_\epsilon}{\partial_y u^s + \tilde{w}_\epsilon} \right) (t, x, a + 3a_0) (u_y^s + \tilde{w}_\epsilon)(t, x, y), \end{aligned}$$

Therefore,

$$\begin{aligned} \partial_x^m \tilde{w}_\epsilon(t, x, y) &= (u_{yy}^s + (\tilde{w}_\epsilon)_y)(t, x, y) \int_{a+3a_0}^y g_m^\epsilon(t, x, \tilde{y}) dy \\ &\quad + (u_y^s + \tilde{w}_\epsilon)(t, x, y) g_m^\epsilon(t, x, y) \end{aligned}$$

$$+ \left( \frac{\partial_x^m \tilde{w}_\epsilon}{\partial_y u^s + \tilde{w}_\epsilon} \right) (t, x, a + 3a_0) (\partial_y^2 u^s + \partial_y \tilde{w}_\epsilon)(t, x, y).$$

Recall

$$J_{\phi_2} = \{y \in \mathbb{R}_+; \phi_2(y) = 1\} = \{y \in \mathbb{R}_+; y \geq a + 3a_0\},$$

we have,

$$\begin{aligned} \|\mathbf{1}_{J_{\phi_2}} \partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2 &\leq \int_{\mathbb{R}_+^2} \langle y \rangle^{2(k+\ell)} (u_{yy}^s + (\tilde{w}_\epsilon)_y)^2 \\ &\quad \times \mathbf{1}_{J_{\phi_2}}(y) \left| \int_{a+3a_0}^y \phi_2(\tilde{y}) g_m^\epsilon(t, x, \tilde{y}) d\tilde{y} \right|^2 dx dy \\ &\quad + \int_{\mathbb{R}_+^2} \langle y \rangle^{2(k+\ell)} ((u_y^s + \tilde{w}_\epsilon) \phi_2 g_m^\epsilon(t, x, y))^2 dx dy \\ &\quad + \int_{\mathbb{R}_+^2} \left( \langle y \rangle^{k+\ell} \mathbf{1}_{J_{\phi_2}}(y) \frac{\partial_x^m \tilde{w}_\epsilon}{\partial_y u^s + \tilde{w}_\epsilon} (t, x, a + 3a_0) (\partial_y^2 u^s + \partial_y \tilde{w}_\epsilon) \right)^2 dx dy \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Recalling (4.2), we have

$$\langle y \rangle^{2(k+\ell)} (u_{yy}^s + (\tilde{w}_\epsilon)_y)^2 \leq C \langle y \rangle^{2\ell-2}.$$

Then,  $2\ell - 2 < -1, \ell' > \frac{1}{2}$  imply ,

$$\begin{aligned} I_1 &\leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{2\ell-2} \mathbf{1}_{J_{\phi_2}}(y) \left| \int_{a+3a_0}^y \phi_2(\tilde{y}) g_m^\epsilon(t, x, \tilde{y}) d\tilde{y} \right|^2 dx dy \\ &\leq C \int_{\mathbb{R}} \|\phi_2 g_m^\epsilon(t, x)\|_{L_{\ell'}^2(\mathbb{R}_+)}^2 dx \\ &\leq C \|\phi_2 g_m^\epsilon(t)\|_{L_{\ell'}^2}^2. \end{aligned}$$

Directly using  $\ell < \ell'$ , we have

$$\begin{aligned} I_2 &\leq \int_{\mathbb{R}_+^2} \langle y \rangle^{2\ell} \left( \phi_2 g_m^\epsilon(t, x, y) \right)^2 dx dy \\ &\leq C \|\phi_2 g_m^\epsilon(t)\|_{L_{\ell'}^2}^2. \end{aligned}$$

For the term  $I_3$ , we have

$$\begin{aligned} I_3 &\leq C \int_{\mathbb{R}_+^2} (\langle y \rangle^{k+\ell} \mathbf{1}_{J_{\phi_2}} \partial_x^m \tilde{w}_\epsilon(t, x, a + 3a_0) (\partial_y^2 u^s + \partial_y \tilde{w}_\epsilon))^2 dx dy \\ &\leq C \int_{\mathbb{R}_+^2} (\langle y \rangle^{\ell-1} \mathbf{1}_{J_{\phi_2}} \partial_x^m \tilde{w}_\epsilon(t, x, a + 3a_0))^2 dx dy \\ &\leq C \int_{\mathbb{R}_+^2} (\langle y \rangle^{\ell-1} \mathbf{1}_{J_{\phi_2}} \int_{a-4a_0}^{a+3a_0} \partial_x^m \tilde{w}_\epsilon(t, x, z))^2 dx dy \\ &\quad + C \int_{\mathbb{R}_+^2} (\langle y \rangle^{\ell-1} \mathbf{1}_{J_{\phi_2}} \partial_x^m \tilde{w}_\epsilon(t, x, a - 4a_0))^2 dx dy \\ &\leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{2(\ell-1)} \int_{a-4a_0}^{a+3a_0} (\partial_x^m \tilde{w}_\epsilon(t, x, z))^2 dz dx dy \end{aligned}$$

$$\begin{aligned}
& + C \int_{\mathbb{R}_+^2} \langle y \rangle^{2(\ell-1)} \left( (\partial_y u^s + \tilde{w}_\epsilon)(t, x, a - 4a_0) \int_0^{a-4a_0} g_m^\epsilon(t, x, z) dz \right)^2 dx dy \\
& \leq C \int_{\mathbb{R}_+^2} \langle y \rangle^{2(\ell-1)} \int_{a-4a_0}^{a+3a_0} (\psi h_\epsilon^m)^2 dz dx dy \\
& \quad + C \int_{\mathbb{R}_+^2} \langle y \rangle^{2(\ell-1)} \|\phi_1 g_m^\epsilon(t, x)\|_{L^2(\mathbb{R}_+)}^2 dx dy \\
& \leq C (\|\psi h_\epsilon^m\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_1 g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2),
\end{aligned}$$

here we have used  $\ell - 1 < -\frac{1}{2}$  and (4.4). We can conclude that

$$\|\mathbf{1}_{[a+3a_0, +\infty[} \partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2 \leq C (\|\psi h_\epsilon^m\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_1 g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_2 g_m^\epsilon\|_{L_{\ell'}^2}^2).$$

Moreover, we also have

$$\|\mathbf{1}_{[a-4a_0, a+4a_0]} \partial_x^m \tilde{w}_\epsilon\|_{L^2(\mathbb{R}_+^2)}^2 \leq C \|\psi h_\epsilon^m\|_{L^2(\mathbb{R}_+^2)}^2.$$

Similarly, for the other side,

$$\{y \in \mathbb{R}_+; \phi_1(y) = 1\} = \{y \in \mathbb{R}_+; 0 \leq y \leq a - 3a_0\},$$

we have,

$$\begin{aligned}
\mathbf{1}_{[0, a-3a_0]} \partial_x^m \tilde{w}_\epsilon &= \mathbf{1}_{[0, a-3a_0]} (u_{yy}^s + (\tilde{w}_\epsilon)_y) \int_0^y g_m^\epsilon(t, x, \tilde{y}) d\tilde{y} \\
& \quad + \mathbf{1}_{[0, a-3a_0]} (u_y^s + \tilde{w}_\epsilon) g_m^\epsilon(t, x, y) \\
&= \mathbf{1}_{[0, a-3a_0]} (u_{yy}^s + (\tilde{w}_\epsilon)_y) \int_0^{\min\{y, a-3a_0\}} \phi_1 g_m^\epsilon(t, x, \tilde{y}) d\tilde{y} \\
& \quad + \mathbf{1}_{[0, a-3a_0]} (u_y^s + \tilde{w}_\epsilon) \phi_1 g_m^\epsilon(t, x, y),
\end{aligned}$$

Thus

$$\|\mathbf{1}_{[0, a-3a_0]} \partial_x^m \tilde{w}_\epsilon\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}^2 \leq C \|\phi_1 g_m^\epsilon\|_{L^2(\mathbb{R}_+^2)}^2.$$

Then we can finish the proof of Lemma 7.3 by using

$$1 \leq \mathbf{1}_{[0, a-3a_0]}(y) + \mathbf{1}_{[a-4a_0, a+4a_0]}(y) + \mathbf{1}_{[a+3a_0, +\infty[}(y), \quad \forall y \in \mathbb{R}_+.$$

□

**End of proof of Theorem 7.1.** Combining (7.9), Lemma 7.2 and Lemma 7.3, we get, for any  $t \in ]0, T]$ ,

$$\begin{aligned}
\|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 &\leq \tilde{C}_8 e^{C_2 t} \int_0^t e^{-C_2 \tau} \|\tilde{w}_\epsilon(\tau)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 d\tau \\
& \quad + \tilde{C}_9 e^{C_2 t} \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2,
\end{aligned}$$

with  $\tilde{C}_8, \tilde{C}_9$  independent of  $0 < \epsilon \leq 1$ . We have by Gronwell's inequality that, for any  $t \in ]0, T]$ ,

$$\|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}^2 \leq \tilde{C}_9 e^{(C_2 + \tilde{C}_8)t} \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2.$$

So it is enough to take

$$C_T^2 = \tilde{C}_9 e^{(C_2 + \tilde{C}_8)T}$$

which gives (7.1), and  $C_T$  is increasing with respect to  $T$ .

We finish the proof of Theorem 7.1.  $\square$

**Theorem 7.4.** *Assume  $u^s$  verifies the conclusion of the Lemma 2.1, and let  $\tilde{u}_0 \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$ ,  $m \geq 6$  be an even integer,  $k > 1$ ,  $0 < \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ , and*

$$0 < \zeta \leq 1 \text{ with } C_m \zeta \leq \frac{\min\{c_1, c_2, c_3\}}{2},$$

where  $C_m$  is the Sobolev embedding constant. If there exists  $0 < \zeta_0$  small enough such that,

$$\|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)} \leq \zeta_0,$$

then, there exists  $\epsilon_0 > 0$  and for any  $0 < \epsilon \leq \epsilon_0$ , the system (3.7) admits a unique solution  $\tilde{w}_\epsilon$  such that

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_1]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta,$$

where  $T_1$  is the lifespan of shear flow  $u^s$  in the Lemma 2.1.

*Proof.* We fix  $0 < \epsilon \leq 1$ , then for any  $\tilde{w}_0 \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$  with  $m \geq 6$ ,  $k > 1$ ,  $0 < \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$ , Theorem 3.4 ensures that, there exists  $\epsilon_0 > 0$  and for any  $0 < \epsilon \leq \epsilon_0$ , there exists  $T_\epsilon > 0$  such that the system (3.7) admits a unique solution  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$  which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \frac{4}{3} \|\tilde{w}_\epsilon(0)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq 2 \|\tilde{u}_0\|_{H_{k+\ell-1}^{m+1}(\mathbb{R}_+^2)}.$$

Now choose  $\zeta_0$  such that

$$\max\{2, C_{T_1}\} \zeta_0 \leq \frac{\zeta}{2}.$$

On the other hand, taking  $\tilde{w}_\epsilon(T_\epsilon)$  as initial data for the system (3.7), Theorem 3.4 ensures that there exists  $T'_\epsilon > 0$ , which is defined by (3.19) with  $\tilde{\zeta} = \frac{\zeta}{2}$ , such that the system (3.7) admits a unique solution  $\tilde{w}'_\epsilon \in L^\infty([T_\epsilon, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))$  which satisfies

$$\|\tilde{w}'_\epsilon\|_{L^\infty([T_\epsilon, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \frac{4}{3} \|\tilde{w}_\epsilon(T_\epsilon)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta.$$

Now, we extend  $\tilde{w}_\epsilon$  to  $[0, T_\epsilon + T'_\epsilon]$  by  $\tilde{w}'_\epsilon$ , then we get a solution  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))$  which still satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta.$$

So if  $T_\epsilon + T'_\epsilon < T_1$ , we can apply Theorem 7.1 to  $\tilde{w}_\epsilon$  with  $T = T_\epsilon + T'_\epsilon$ , and use (7.1), this gives

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon + T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq C_{T_1} \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)} \leq \frac{\zeta}{2}.$$

Now taking  $\tilde{w}_\epsilon(T_\epsilon + T'_\epsilon)$  as initial data for the system (3.7), applying again Theorem 3.4, for the same  $T'_\epsilon > 0$ , the system (3.7) admits a unique solution  $\tilde{w}'_\epsilon \in L^\infty([T_\epsilon + T'_\epsilon, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))$  which satisfies

$$\|\tilde{w}'_\epsilon\|_{L^\infty([T_\epsilon + T'_\epsilon, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \frac{4}{3} \|\tilde{w}_\epsilon(T_\epsilon + T'_\epsilon)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta.$$

Now, we extend  $\tilde{w}_\epsilon$  to  $[0, T_\epsilon + 2T'_\epsilon]$  by  $\tilde{w}'_\epsilon$ , then we get a solution  $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))$  which satisfies again

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta.$$



So if  $T_\epsilon + 2T'_\epsilon < T_1$ , we can continue to apply Theorem 7.1 to  $\tilde{w}_\epsilon$  with  $T = T_\epsilon + 2T'_\epsilon$ , and use (7.1), this gives also

$$\|\tilde{w}_\epsilon\|_{L^\infty([0, T_\epsilon + 2T'_\epsilon]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq C_{T_1} \|\tilde{u}_0\|_{H_{k+\ell-1}^{m+1}(\mathbb{R}_+^2)} \leq \frac{\zeta}{2}.$$

Then by recurrence, for any fixed  $\epsilon \leq \epsilon_0$ , we can extend the approximate solution  $\tilde{w}_\epsilon$  over  $[0, T_1]$ , and the lifespan of approximate solution  $\tilde{w}_\epsilon$  is uniformly equal to that of shear flow if the initial data  $\tilde{u}_0$  is small enough.  $\square$

We have obtained the following estimate, for  $m \geq 6$  and  $0 < \epsilon \leq \epsilon_0$ ,

$$\|\tilde{w}_\epsilon(t)\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \leq \zeta, \quad t \in [0, T_1].$$

By using the equation (3.7) and the Sobolev inequality, we get, for  $0 < \delta < 1$

$$\|\tilde{w}_\epsilon\|_{Lip([0, T_1]; C^{2, \delta}(\mathbb{R}_+^2))} \leq M < +\infty.$$

Then taking a subsequence, we have, for  $0 < \delta' < \delta$ ,

$$\tilde{w}_\epsilon \rightarrow \tilde{w} \quad (\epsilon \rightarrow 0), \text{ locally strong in } C^0([0, T_1]; C^{2, \delta'}(\mathbb{R}_+^2)),$$

and

$$\partial_t \tilde{w} \in L^\infty([0, T_1]; H_{k+\ell}^{m-2}(\mathbb{R}_+^2)), \quad \tilde{w} \in L^\infty([0, T_1]; H_{k+\ell}^m(\mathbb{R}_+^2)),$$

with

$$\|\tilde{w}\|_{L^\infty([0, T_1]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq \zeta.$$

Then we have

$$\tilde{u} = - \int_y^{+\infty} \tilde{w}(t, x, \tilde{y}) d\tilde{y} \in L^\infty([0, T_1]; H_{k+\ell-1}^m(\mathbb{R}_+^2)),$$

where we use the Hardy inequality (A.1), since

$$\lim_{y \rightarrow +\infty} \tilde{u}(t, x, y) = - \lim_{y \rightarrow +\infty} \int_y^{+\infty} \tilde{w}(t, x, \tilde{y}) d\tilde{y} = 0.$$

In fact, we also have

$$\lim_{y \rightarrow 0} \tilde{u}(t, x, y) = \lim_{y \rightarrow 0} \int_0^y \tilde{w}(t, x, \tilde{y}) d\tilde{y} = 0.$$

Using the condition  $k + \ell - 1 > \frac{1}{2}$ , we have also

$$\tilde{v}(t, x, y) = - \int_0^y \tilde{u}_x(t, x, \tilde{y}) d\tilde{y} \in L^\infty([0, T_1]; L^\infty(\mathbb{R}_{+, y}; H^{m-1}(\mathbb{R}_x))).$$

We have proven that,  $\tilde{w}$  is a classical solution to the following vorticity Prandtl equation

$$\begin{cases} \partial_t \tilde{w} + (u^s + \tilde{u}) \partial_x \tilde{w} + \tilde{v} \partial_y (u_y^s + \tilde{w}) = \partial_y^2 \tilde{w}, \\ \partial_y \tilde{w}|_{y=0} = 0, \\ \tilde{w}|_{t=0} = \tilde{w}_0, \end{cases}$$

and  $(\tilde{u}, \tilde{v})$  is a classical solution to (2.2). Finally,  $(u, v) = (u^s + \tilde{u}, \tilde{v})$  is a classical solution to (1.1), and satisfies (7.10). In conclusion, we have proved the following theorem

**Theorem 7.5.** *Let  $m \geq 6$  be an even integer,  $k > 1$ ,  $0 < \ell < \frac{1}{2}$ ,  $k + \ell > \frac{3}{2}$ ,  $\ell' > \frac{1}{2}$ ,  $\ell' - \ell < \frac{1}{2}$ , assume that  $u_0^s$  satisfies (1.2), the initial date  $\tilde{u}_0 \in H_{k+\ell'-1}^{m+3}(\mathbb{R}_+^2)$  and  $\tilde{u}_0$  satisfies the compatibility condition (2.6)-(2.7) up to order  $m + 2$ . Then there exists  $T > 0$  such that if*

$$\|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)} \leq \delta_0,$$

for some  $\delta_0 > 0$  small enough, then the initial-boundary value problem (2.2) admits a solution  $(\tilde{u}, \tilde{v})$  with

$$\tilde{u} \in L^\infty([0, T]; H_{k+\ell-1}^m(\mathbb{R}_+^2)), \quad \partial_y \tilde{u} \in L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2)).$$

Moreover, we have the following energy estimate,

$$\|\partial_y \tilde{u}\|_{L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))} \leq C \|\tilde{u}_0\|_{H_{k+\ell'-1}^{m+1}(\mathbb{R}_+^2)}^2. \quad (7.10)$$

## 8. UNIQUENESS AND STABILITY

Now, we study the stability of solutions which implies immediately the uniqueness of solution.

Let  $\tilde{u}^1, \tilde{u}^2$  be two solutions obtained in Theorem 7.5 with respect to the initial date  $\tilde{u}_0^1, \tilde{u}_0^2$  respectively. Denote  $\bar{u} = \tilde{u}^1 - \tilde{u}^2$  and  $\bar{v} = \tilde{v}^1 - \tilde{v}^2$ , then

$$\begin{cases} \partial_t \bar{u} + (u^s + \tilde{u}_1) \partial_x \bar{u} + (u_y^s + \tilde{u}_{1,y}) \bar{v} = \partial_y^2 \bar{u} - \tilde{v}_2 \partial_y \bar{u} - (\partial_x \tilde{u}_2) \bar{u}, \\ \partial_x \bar{u} + \partial_y \bar{v} = 0, \\ \bar{u}|_{y=0} = \bar{v}|_{y=0} = 0, \\ \bar{u}|_{t=0} = \tilde{u}_0^1 - \tilde{u}_0^2. \end{cases}$$

So it is a linear equation for  $\bar{u}$ . We also have for the vorticity  $\bar{w} = \partial_y \bar{u}$ ,

$$\begin{cases} \partial_t \bar{w} + (u^s + \tilde{u}_1) \partial_x \bar{w} + (u_{yy}^s + \tilde{w}_{1,y}) \bar{v} = \partial_y^2 \bar{w} - \tilde{v}_2 \partial_y \bar{w} - (\partial_x \tilde{w}_2) \bar{u}, \\ \partial_y \bar{w}|_{y=0} = 0, \\ \bar{w}|_{t=0} = \tilde{w}_0^1 - \tilde{w}_0^2. \end{cases} \quad (8.1)$$

**8.1. Estimate with a loss of  $x$ -derivative.** Firstly, for the vorticity  $\bar{w} = \partial_y \bar{u}$ , we deduce an energy estimate with a loss of  $x$ -derivative with the anisotropic norm defined by (3.12).

**Proposition 8.1.** *Let  $\tilde{u}^1, \tilde{u}^2$  be two solutions obtained in Theorem 7.5 with respect to the initial date  $\tilde{u}_0^1, \tilde{u}_0^2$ , then we have*

$$\frac{d}{dt} \|\bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}^2 + \|\partial_y \bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}^2 \leq \bar{C}_1 \|\bar{w}\|_{H_{k+\ell}^{m-2}}^2, \quad (8.2)$$

where the constant  $\bar{C}_1$  depends on the norm of  $\tilde{w}^1, \tilde{w}^2$  in  $L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$ .

*Proof.* The proof of this Proposition is similar to the proof of the Proposition 3.8, and we need to use that  $m - 2$  is even. We only give the calculation for the terms which need a different argument. Moreover we also explain why we only get the estimate on  $\|\bar{w}\|_{H_{k+\ell}^{m-2}}^2$  but require the norm of  $\tilde{w}^1, \tilde{w}^2$  in  $L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$ .

Without loss of the generality, we suppose that  $\|\bar{w}\|_{H_{k+\ell}^{m-2}} \leq 1$ ,  $\|\tilde{w}^1\|_{H_{k+\ell}^m} \leq 1$  and  $\|\tilde{w}^2\|_{H_{k+\ell}^m} \leq 1$ .

Derivating the equation of (8.1) with  $\partial^\alpha = \partial_x^\alpha \partial_y^{\alpha_2}$ , for  $|\alpha| = \alpha_1 + \alpha_2 \leq m-2$ ,  $\alpha_1 \leq m-3$ ,

$$\begin{aligned} \partial_t \partial^\alpha \bar{w} - \partial_y^2 \partial^\alpha \bar{w} &= -\partial^\alpha ((u^s + \tilde{u}_1) \partial_x \bar{w} + \tilde{v}_2 \partial_y \bar{w}) \\ &\quad + (u_{yy}^s + \tilde{w}_{1,y}) \bar{v} + (\partial_x \tilde{w}_2) \bar{u}. \end{aligned} \quad (8.3)$$

Multiplying the above equation with  $\langle y \rangle^{k+\ell'+\alpha_2} \partial^\alpha \bar{w}$ , the same computation as in the proof of the Proposition 3.8, in particular, the reduction of the boundary-data are the same, gives

$$\begin{aligned} &\int_{\mathbb{R}_+^2} (\partial_t \partial^\alpha \bar{w} - \partial_y^2 \partial^\alpha \bar{w}) \langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \bar{w} dx dy \\ &\geq \frac{1}{2} \frac{d}{dt} \|\partial^\alpha \bar{w}\|_{L_{k+\ell+\alpha_2}^2}^2 + \frac{3}{4} \|\partial_y \bar{w}\|_{H_{k+\ell}^{m-2, m-3}}^2 - C \|\bar{w}\|_{H_{k+\ell}^{m-2}}^2. \end{aligned}$$

As for the right hand of (8.3), for the first item, we split it into two parts

$$-\partial^\alpha ((u^s + \tilde{u}_1) \partial_x \bar{w}) = -(u^s + \tilde{u}_1) \partial_x \partial^\alpha \bar{w} + [(u^s + \tilde{u}_1), \partial^\alpha] \partial_x \bar{w}.$$

Firstly, we have

$$\left| \int_{\mathbb{R}_+^2} ((u^s + \tilde{u}_1) \partial_x \partial^\alpha \bar{w}) \langle y \rangle^{2(\ell+\alpha_2)} \partial^\alpha \bar{w} dx dy \right| \leq \|\tilde{w}_1\|_{H_1^3} \|\partial^\alpha \bar{w}\|_{L_{k+\ell+\alpha_2}^2}^2.$$

For the commutator operator, we have,

$$\|[(u^s + \tilde{u}_1), \partial^\alpha] \partial_x \tilde{w}_\epsilon\|_{L_{k+\ell+\alpha_2}^2} \leq C \|\tilde{w}_1\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)} \|\bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}.$$

Notice that for this term, we don't have the loss of  $x$ -derivative.

With the similar method for the terms  $\tilde{v}_2 \partial_y \bar{w}$ , we get

$$\left| \int_{\mathbb{R}_+^2} \tilde{v}_2 \partial_y \bar{w} \langle y \rangle^{2(\ell+\alpha_2)} \partial^\alpha \bar{w} dx dy \right| \leq \|\tilde{w}_2\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)} \|\bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}^2.$$

For the next one, we have

$$\partial^\alpha ((u_{yy}^s + \partial_y \tilde{w}_1) \bar{v}) = \sum_{\beta \leq \alpha} C_\beta^\alpha \partial^\beta (u_{yy}^s + \partial_y \tilde{w}_1) \partial^{\alpha-\beta} \bar{v},$$

and thus

$$\begin{aligned} &\sum_{\beta \leq \alpha, 1 \leq |\beta| < |\alpha|} C_\beta^\alpha \|\partial^\beta (u_{yy}^s + \partial_y \tilde{w}_1) \partial^{\alpha-\beta} \bar{v}\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)} \\ &\leq C \|\tilde{w}_1\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)} \|\bar{w}\|_{H_{k+\ell}^{m-2, m-3}(\mathbb{R}_+^2)}. \end{aligned}$$

On the other hand, using Lemma A.1 and  $\frac{3}{2} - k < \ell < \frac{1}{2}$ ,

$$\begin{aligned} &\|(\partial^\alpha (u_{yy}^s + \partial_y \tilde{w}_1)) \bar{v}\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)} \leq \|(\partial^\alpha u_{yy}^s) \bar{v}\|_{L_{k+\ell+\alpha_2}^2} + \|(\partial^\alpha \partial_y \tilde{w}_1) \bar{v}\|_{L_{k+\ell+\alpha_2}^2} \\ &\leq C \|\bar{v}\|_{L^2(\mathbb{R}_x; L^\infty(\mathbb{R}_+))} + C \|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \|\bar{v}\|_{L^\infty(\mathbb{R}_+^2)} \\ &\leq C \|\bar{u}_x\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} + C \|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} (\|\bar{u}_x\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} + \|\bar{u}_{xx}\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)}) \\ &\leq C(1 + \|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}) \|\bar{w}\|_{H_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} \\ &\leq C(1 + \|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}) \|\bar{w}\|_{H_{k+\ell}^2(\mathbb{R}_+^2)}. \end{aligned}$$

So this term requires the norms  $\|\tilde{w}_1\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}$ .

Moreover, if  $\alpha_2 \neq 0$

$$\begin{aligned} \|(u_{yy}^s + \partial_y \tilde{w}_1) \partial_x^\alpha \bar{v}\|_{L_{k+\ell+\alpha_2}^2} &= \|(u_{yy}^s + \partial_y \tilde{w}_1) \partial_x^{\alpha_1} \partial_x^{\alpha_2-1} \bar{u}_x\|_{L_{k+\ell+\alpha_2}^2} \\ &\leq C(1 + \|\tilde{w}_1\|_{H_{k+\ell}^{m-1}(\mathbb{R}_+^2)}) \|\bar{w}\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)}, \end{aligned}$$

and also if  $\alpha_2 = 0$

$$\begin{aligned} \|(u_{yy}^s + \partial_y \tilde{w}_1) \partial_x^{\alpha_1} \bar{v}\|_{L_{k+\ell}^2} &= \|(u_{yy}^s + \partial_y \tilde{w}_1) \partial_y^{-1} \partial_x^{\alpha_1} \bar{u}_x\|_{L_{k+\ell}^2} \\ &\leq C(1 + \|\tilde{w}_1\|_{H_{k+\ell}^{m-1}(\mathbb{R}_+^2)}) \|\partial_x^{\alpha_1+1} \bar{w}\|_{L_{\frac{3}{2}+\delta}^2(\mathbb{R}_+^2)}. \end{aligned}$$

These two cases imply the loss of  $x$ -derivative.

Similar argument also gives

$$\left| \int_{\mathbb{R}_+^2} (\partial^\alpha (\partial_x \tilde{w}_2) \bar{u}) \langle y \rangle^{2(\ell+\alpha_2)} \partial^\alpha \bar{w} dx dy \right| \leq C \|\tilde{w}_2\|_{H_{k+\ell}^m(\mathbb{R}_+^2)} \|\bar{w}\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)},$$

which finishes the proof of the Proposition 8.1.  $\square$

**8.2. Estimate on the loss term.** To close the estimate (7.10), we need to study the terms  $\|\partial_x^{m-2} \bar{w}\|_{L_{k+\ell}^2(\mathbb{R}_+^2)}$  which is missing in the left hand side of (8.2).

Similar to the argument in Section 7, we will recover this term by the estimate of functions

$$\begin{aligned} \bar{g}_n &= \left( \frac{\partial_x^n \bar{u}}{u_y^s + \tilde{u}_{1,y}} \right)_y, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times (I_{\phi_1} \cup I_{\phi_2}), \\ \bar{h}_n &= \frac{\partial_x^n \bar{w}}{\sqrt{u_{yy}^s + \tilde{w}_{1,y}}}, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times (I_\psi). \end{aligned}$$

**Proposition 8.2.** *Let  $\tilde{u}^1, \tilde{u}^2$  be two solutions obtained in Theorem 7.5 with respect to the initial data  $\tilde{u}_0^1, \tilde{u}_0^2$ , then we have*

$$\begin{aligned} \frac{d}{dt} \|\phi_1 \bar{g}_{m-2}\|_{L^2(\mathbb{R}_+^2)}^2 + \|\phi_1 \partial_y \bar{g}_{m-2}\|_{L^2(\mathbb{R}_+^2)}^2 \\ \leq \bar{C}_2 (\|\phi_1 \bar{g}_{m-2}\|_{L^2(\mathbb{R}_+^2)}^2 + \|\bar{w}\|_{H_{k+\ell}^{m-2}}^2), \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \sum_{n=1}^{m-2} \|\phi_2 \bar{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \sum_{n=1}^{m-2} \|\phi_2 \partial_y \bar{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 \\ \leq C_2 \left( \sum_{n=1}^{m-2} \|\phi_2 \bar{g}_n\|_{L_{\ell'}^2(\mathbb{R}_+^2)}^2 + \|\bar{w}\|_{H_{k+\ell}^{m-2}}^2 \right), \end{aligned}$$

where the constant  $\bar{C}_2$  depends on the norm of  $\tilde{w}^1, \tilde{w}^2$  in  $L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$ .

**Proposition 8.3.** *With same assumption as in Proposition 8.2, we have*

$$\begin{aligned} \frac{d}{2dt} \|\psi \bar{h}_{m-2}\|_{L^2}^2 + \frac{3}{4} \|\psi \partial_y \bar{h}_{m-2}\|_{L^2}^2 \\ \leq \bar{C}_3 \|\bar{w}\|_{H_{k+\ell}^{m-2}(\mathbb{R}_+^2)}^2 - \frac{d}{dt} \int_{\mathbb{R}_+^2} (\partial_x^{m-2} \bar{u})^2 \frac{\psi \psi'}{u_y^s + u_y} dx dy, \end{aligned}$$

where  $\bar{C}_3$  depends on the norm of  $\tilde{w}^1, \tilde{w}^2$  in  $L^\infty([0, T]; H_{k+\ell}^m(\mathbb{R}_+^2))$ .

These two Propositions can be proven by using exactly the same calculation and also the same nonlinear cancellation as in Section 5 and Section 6. The only difference is that when we use the Leibniz formula, for the term where the order of derivatives is  $|\alpha| = m - 2$ , it acts on the coefficient which depends on  $\tilde{u}^1, \tilde{u}^2$ . Therefore, we need their norm in the order of  $(m - 2) + 1$ . So we omit the proof of the two Proposition here.

With the similar argument to the proof of Theorem 7.1, we get

$$\|\bar{w}\|_{L^\infty([0,T];H_{k+\ell}^{m-2}(\mathbb{R}_+^2))} \leq C\|\bar{u}_0\|_{H_{k+\ell-1}^m(\mathbb{R}_+^2)},$$

which finishes the proof of Theorem 1.1.

#### APPENDIX A. SOME INEQUALITIES

We will use the following Hardy type inequalities.

**Lemma A.1.** *Let  $f : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ . Then*

(i) *if  $\lambda > -\frac{1}{2}$  and  $\lim_{y \rightarrow \infty} f(x, y) = 0$ , then*

$$\|\langle y \rangle^\lambda f\|_{L^2(\mathbb{R}_+^2)} \leq C_\lambda \|\langle y \rangle^{\lambda+1} \partial_y f\|_{L^2(\mathbb{R}_+^2)}; \quad (\text{A.1})$$

(ii) *if  $-1 \leq \lambda < -\frac{1}{2}$  and  $f(x, 0) = 0$ , then*

$$\|\langle y \rangle^\lambda f\|_{L^2(\mathbb{R}_+^2)} \leq C_\lambda \|\langle y \rangle^{\lambda+1} \partial_y f\|_{L^2(\mathbb{R}_+^2)}.$$

Here  $C_\lambda \rightarrow +\infty$ , as  $\lambda \rightarrow -\frac{1}{2}$ .

We need the following trace theorem in the weighted Sobolev space.

**Lemma A.2.** *Let  $\lambda > \frac{1}{2}$ , then there exists  $C > 0$  such that for any function  $f$  defined on  $\mathbb{R}_+^2$ , if  $\partial_y f \in L_\lambda^2(\mathbb{R}_+^2)$ , it admits a trace on  $\mathbb{R}_x \times \{0\}$ , and satisfies*

$$\|\gamma_0(f)\|_{L^2(\mathbb{R}_x)} \leq C\|\partial_y f\|_{L_\lambda^2(\mathbb{R}_+^2)},$$

where  $\gamma_0(f)(x) = f(x, 0)$  is the trace operator.

The proof of the above two Lemmas is elementary, so we leave it to the reader.

We use also the following Sobolev inequality and algebraic properties of  $H_{k+\ell}^m(\mathbb{R}_+^2)$ ,

**Lemma A.3.** *For the suitable functions  $f, g$ , we have:*

1) *If the function  $f$  satisfies  $f(x, 0) = 0$  or  $\lim_{y \rightarrow +\infty} f(x, y) = 0$ , then for any small  $\delta > 0$ ,*

$$\|f\|_{L^\infty(\mathbb{R}_+^2)} \leq C(\|f_y\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} + \|f_{xy}\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)}). \quad (\text{A.2})$$

2) *For  $m \geq 6, k + 2\ell > 2$ , and any  $\alpha, \beta \in \mathbb{N}^2$  with  $|\alpha| + |\beta| \leq m$ , we have*

$$\|(\partial^\alpha f)(\partial^\beta g)\|_{L_{k+\ell+\alpha_2+\beta_2}^2(\mathbb{R}_+^2)} \leq C\|f\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}\|g\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}. \quad (\text{A.3})$$

3) *For  $m \geq 6, k + 2\ell > 2$ , and any  $\alpha \in \mathbb{N}^2, p \in \mathbb{N}$  with  $|\alpha| + p \leq m$ , we have, for  $\delta > 0$  small,*

$$\|(\partial^\alpha f)(\partial_x^p \mathbf{Int}(g))\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)} \leq C\|f\|_{H_{k+\ell}^m(\mathbb{R}_+^2)}\|g\|_{H_{\frac{1}{2}+\delta}^m(\mathbb{R}_+^2)}, \quad (\text{A.4})$$

where  $\mathbf{Int}(g) = \int_0^y g(x, \tilde{y}) d\tilde{y}$ .

*Proof.* For (1), using  $f(x, 0) = 0$ , we have

$$\begin{aligned} \|f\|_{L^\infty(\mathbb{R}_+^2)} &= \left\| \int_0^y (\partial_y f)(x, \tilde{y}) d\tilde{y} \right\|_{L^\infty(\mathbb{R}_+^2)} \leq C \|\partial_y f\|_{L^\infty(\mathbb{R}_x; L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+))} \\ &\leq C(\|\partial_y f\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)} + \|\partial_x \partial_y f\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)}). \end{aligned}$$

If  $\lim_{y \rightarrow +\infty} f(x, y) = 0$ , we use

$$f(x, y) = - \int_y^\infty (\partial_y f)(x, \tilde{y}) d\tilde{y}.$$

For (2), firstly,  $m \geq 6$  and  $|\alpha| + |\beta| \leq m$  imply  $|\alpha| \leq m - 2$  or  $|\beta| \leq m - 2$ , without loss of generality, we suppose that  $|\alpha| \leq m - 2$ . Then, using the conclusion of (1), we have

$$\begin{aligned} \|(\partial^\alpha f)(\partial^\beta g)\|_{L_{k+\ell+\alpha_2+\beta_2}^2(\mathbb{R}_+^2)} &\leq \|\langle y \rangle^{\alpha_2} (\partial^\alpha f)\|_{L^\infty(\mathbb{R}_+^2)} \|\partial^\beta g\|_{L_{k+\ell+\beta_2}^2(\mathbb{R}_+^2)} \\ &\leq C \|f\|_{H_{\frac{1}{2}+\delta}^{|\alpha|+2}(\mathbb{R}_+^2)} \|\partial^\beta g\|_{L_{k+\ell+\beta_2}^2(\mathbb{R}_+^2)}, \end{aligned}$$

which give (A.3).

For (3), if  $|\alpha| \leq m - 2$ , we have

$$\begin{aligned} \|(\partial^\alpha f)(\partial_x^p \mathbf{Int}(g))\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)} &\leq \|\langle y \rangle^{k+\ell+\alpha_2} (\partial^\alpha f)\|_{L^2(\mathbb{R}_{y,+}; L^\infty(\mathbb{R}_x))} \|\partial_x^p \mathbf{Int}(g)\|_{L^\infty(\mathbb{R}_{y,+}; L^2(\mathbb{R}_x))} \\ &\leq C \|f\|_{H_{k+\ell}^{|\alpha|+2}(\mathbb{R}_+^2)} \|\partial_x^p g\|_{L_{\frac{1}{2}+\delta}^2(\mathbb{R}_+^2)}. \end{aligned}$$

If  $p \leq m - 2$ , we have, for  $\delta > 0$  small,

$$\begin{aligned} \|(\partial^\alpha f)(\partial_x^p \mathbf{Int}(g))\|_{L_{k+\ell+\alpha_2}^2(\mathbb{R}_+^2)} &\leq \|\langle y \rangle^{k+\ell+\alpha_2} (\partial^\alpha f)\|_{L^2(\mathbb{R}_+^2)} \|\partial_x^p \mathbf{Int}(g)\|_{L^\infty(\mathbb{R}_+^2)} \\ &\leq C \|f\|_{H_{k+\ell}^{|\alpha|}(\mathbb{R}_+^2)} \|\partial_x^p g\|_{L^\infty(\mathbb{R}_x; L_{\frac{1}{2}+\delta}^2(\mathbb{R}_{y,+}))} \\ &\leq C \|f\|_{H_{k+\ell}^{|\alpha|}(\mathbb{R}_+^2)} \|g\|_{H_{\frac{1}{2}+\delta}^m(\mathbb{R}_+^2)}. \end{aligned}$$

We have completed the proof of the Lemma.  $\square$

## APPENDIX B. SOME EXAMPLE OF INITIAL DATA

We construct now some example of initial data  $u_0 = u_0^s + \tilde{u}_0$  with non degenerate critical points.

**Non-monotonic shear profile.** We construct now some example of shear profile  $u_0^s$  satisfying (2.3)- (2.4).

**Lemma B.1.** *There exists  $f \in C^{m+3}([0, +\infty[)$  satisfy : for  $k > 1, 2 < a < a + 10a_0 < y_0$  with  $y_0$  very big ,  $c_1, c_2, c_3, B_p > 0$  and  $M_0 < M_1 < 0 < M_2 < M_3 < M$ , such that*

$$\begin{cases} f(y) \leq -2c_1, y \leq a - a_0; f(y) < 0, y < a; f(a) = 0; \\ f(y) > 0, y > a; f(y) \geq 2c_1 y^{-k}, y \geq a + 3a_0, \\ f(0) = M_0, f(a - 3a_0) = M_1, f(a + 3a_0) = M_2; \\ f(y_0 + 2) = M_3, M = f(y_0 + 5) = \sup_{y \in \mathbb{R}_+} f(y), \\ \int_0^{+\infty} f(y) dy = 1; |\partial_y^p f(y)| \leq B_p \langle y \rangle^{-k-p}, y \geq 0, 0 \leq p \leq m + 3, \end{cases}$$

meaning that  $f$  is croissant on  $[0, y_0 + 5]$  from  $f(0) = M_0 < 0$  to the maximum  $M > 0$ . Moreover,  $\partial_y f(y)$  verifies the following conditions

$$\begin{cases} \partial_y f(y) \geq 0, & y \in [0, y_0 + 5[; & \partial_y f(y) < 0, & y \in ]y_0 + 5, +\infty[; \\ \partial_y f(0) = 0; & \partial_y f(y) \geq 2c_2 > 0, & y \in [a - 3a_0, a + 3a_0[; \\ \partial_y f(y) \geq 2c_3 y^{-k-1} > 0, & y \in [a + 3a_0, y_0 + 2[; \\ |\partial_y f(y)| \leq \tilde{B}_2 y^{-k-1}, & y \in [y_0 - 2, +\infty[, \end{cases}$$

where  $c_1, c_2, c_3, \tilde{B}_2, M_0, M_1, M_2, M_3$  and  $M$  are independents on the choose of  $y_0$ .

*Proof.* Set

$$g(y) = \begin{cases} 2\tilde{c}_2(y), & [0, a + 3a_0], \\ 2\tilde{c}_3(y)y^{-k-1}, & [a + 3a_0, y_0 + 2], \\ \alpha(y)y^{-k-1}(y_0 + 5 - y), & [y_0 + 2, y_0 + 7], \\ -2\tilde{c}_1(y)ky^{-k-1}, & [y_0 + 7, +\infty[, \end{cases}$$

where  $\tilde{c}_1(y), \tilde{c}_2(y), \tilde{c}_3(y), \alpha(y)$  are smooth positive functions with the proprieties to determine as following :

We choose  $\tilde{c}_2(y) \geq 0$  as

$$\tilde{c}_2(y) = \begin{cases} 0, & [0, a - 8a_0]; \\ c_2, & [a - 3a_0, a + 3a_0], \end{cases}$$

where

$$M_1 = \int_0^{a-3a_0} 2\tilde{c}_2(y)dy + M_0 < 0, \quad \int_0^a 2\tilde{c}_2(y)dy + M_0 = 0, \quad 0 < c_2 = \frac{M_2 - M_1}{12a_0}.$$

The function  $\tilde{c}_2(y)$  is defined on  $[0, a + 4a_0[$ , and it is smoothly superimposed over the intervals  $]a + 3a_0, a + 4a_0[$  with  $2\tilde{c}_3(y)y^{-k-1}$  which is defined on  $]a + 3a_0, y_0 + 3[$  and

$$\tilde{c}_3(y) = \bar{c}_3(y_0), \quad [a + 3a_0, y_0 + 2],$$

where we choose

$$2\bar{c}_3(y_0) \int_{a+3a_0}^{y_0+2} y^{-k-1} dy = M_3 - M_2,$$

thus

$$\begin{aligned} 2\bar{c}_3(y_0) &= \frac{(M_3 - M_2)(a + 3a_0)^k (y_0 + 2)^k}{(k + 1)((y_0 + 2)^k - (a + 3a_0)^k)} \\ &= \frac{(M_3 - M_2)(a + 3a_0)^k}{(k + 1)} + \frac{(M_3 - M_2)(a + 3a_0)^k}{(k + 1)((y_0 + 2)^k - (a + 3a_0)^k)}. \end{aligned}$$

For  $y_0$  very big (for example  $(y_0 + 2)^k - (a + 3a_0)^k \geq 2$ ), the above second terms is positive and small. So that we can choose

$$2c_3 = \frac{(M_3 - M_2)(a + a_0)^k}{(k + 1)},$$

with this choose, we have

$$g(y) = 2\tilde{c}_3(y_0)y^{-k-1} \geq 2c_3y^{-k-1}, \quad y \in [a + 3a_0, y_0 + 2].$$

The function  $\alpha \geq 0$  is monotonic on  $[y_0 + 1, y_0 + 8]$ , it is smoothly superimposed over the intervals  $]y_0 + 1, y_0 + 2[$  with  $\alpha(y)(y_0 + 5 - y)$ , and  $\tilde{c}_1(y)$  is smoothly superimposed with  $\alpha$  over  $]y_0 + 7, y_0 + 8$ , such that

$$\begin{cases} 2\tilde{c}_3(y) = \alpha(y)(y_0 + 5 - y), & [y_0 + 1, y_0 + 2]; \\ \alpha(y)(y_0 + 5 - y) = -2\tilde{c}_1(y)k, & [y_0 + 7, y_0 + 8]; \\ 2\tilde{c}_1(y) = 2c_1 = M_2 \langle (a + 3a_0) \rangle^k, & [y_0 + 8, +\infty[. \end{cases}$$

Then we have

$$|g(y)| \leq 8c_3 y^{-k-1}, \quad y \in [a + a_0, y_0 + 2],$$

and

$$|g(y)| \leq \tilde{B}_2 y^{-k-1}, \quad y \in [y_0 - 2, +\infty[,$$

with

$$\tilde{B}_2 = \max\{16c_3, 4kc_1\}.$$

We have also for  $p \geq 0$ ,

$$|\partial_y^p g(y)| \leq B_{p+1} y^{-k-p-1}, \quad y \geq 0,$$

with

$$B_{p+1} = \max \left\{ \sup_{0 \leq y \leq y_0 + 6} |\partial_y^p g(y)| \langle (y_0 + 6) \rangle^{k+p+1}, 2c_1 k \cdots (k + p + 1) \right\}.$$

Finally, we set

$$f(y) = \int_0^y g(z) dz + M_0,$$

then  $f$  satisfy the properties of Lemma except the condition

$$\int_0^{+\infty} f(y) dy = 1,$$

which can be obtained by the normalisation.  $\square$

Finally, set

$$u_0^s(y) = \int_0^y f(z) dz,$$

then  $u_0^s$  satisfy (2.3)-(2.4).

**The perturbation of initial data.** Let  $u_0^s$  be the initial data of shear profile satisfying (2.3), we suppose that  $u_0^s$  is concave, then we have

$$\partial_y u_0^s(y) < 0, \quad 0 \leq y < a; \quad \partial_y u_0^s(y) > 0, \quad a < y.$$

We construct now some perturbation  $\tilde{u}_0$  such that the initial data  $u_0 = u_0^s + \tilde{u}_0$  possesses a curves of the non degenerate critical points. For  $c_0 \ll 1$ , set

$$\tilde{u}_0 = \begin{cases} 0, & y \leq a - 6a_0 \\ (c_0)^j f_1(y) b(x), & a - 6a_0 \leq y \leq a - 5a_0 \\ (c_0)^j (y + 6a_0 - a)^2 b(x), & a - 5a_0 \leq y \leq a + 5a_0 \\ (c_0)^j f_2(y) b(x), & y \geq a + 5a_0. \end{cases}$$

where  $j$  is a big integer,  $f_1, f_2$  are two smooth joint functions with compact supports, and  $b \in H^{m+4}(\mathbb{R}_x)$ ,  $0 < b(x) \leq 1$  for all  $x \in \mathbb{R}$ .

One can check that if  $j$  is big enough, we have that

$$\|\partial_y \tilde{u}_0\|_{H_{k+\ell'}^m(\mathbb{R}_+^2)} + \|\partial_y^2 \tilde{u}_0\|_{H_{k+\ell'}^m(\mathbb{R}_+^2)} \ll 1,$$



which imply

$$\partial_y^2 u_0(x, y) = \partial_y^2 u_0^s(y) + \partial_y^2 \tilde{u}_0(x, y) \geq c_0 \langle y \rangle^{-k-1}, \quad (x, y) \in \mathbb{R} \times [a - 6a_0, +\infty[. \quad (\text{B.1})$$

If  $j \gg 1$  we have also

$$0 < \partial_y \tilde{u}_0 \leq 2(c_0)^j (y + 6a_0 - a)b(x) \leq 24a_0(c_0)^j b(x) \leq c_0^6, \quad a - 5a_0 \leq y \leq a + 5a_0.$$

For the shear profile, according to (2.3),

$$\partial_y u_0^s(y) \leq -2c_0, \quad 0 \leq y \leq a - a_0.$$

Then we have

$$\begin{aligned} \partial_y u_0^s(a - a_0) + \partial_y \tilde{u}_0(x, a - a_0) &\leq -2c_0 + c_0^6 < 0, \\ \partial_y u_0^s(a) + \partial_y \tilde{u}_0(x, a) &= \partial_y \tilde{u}_0(x, a) > 0. \end{aligned}$$

By the intermediate value theorem, there exist a point  $a(x)$  such that

$$\partial_y u_0(x, a(x)) = \partial_y u_0^s(a(x)) + \partial_y \tilde{u}_0(x, a(x)) = 0, \quad a(x) \in ]a - a_0, a[.$$

In detail, it is equal to

$$2c_0^j \left( a(x) + 6a_0 - a \right) b(x) = -\partial_y u_0^s(a(x)), \quad a(x) \in ]a - a_0, a[.$$

The smoothness of the curve  $a(x)$  can be deduced by implicit function theorem and (B.1).

### APPENDIX C. THE PROOF OF PROPOSITION 3.6

Now, we prove the existence of solution to the vorticity equation  $\tilde{w}_\epsilon = \partial_y \tilde{u}_\epsilon$  and suppose that  $m, k, \ell$  and  $u^s(t, y)$  satisfy the assumption of Proposition 3.6,

$$\begin{cases} \partial_t \tilde{w}_\epsilon + (u^s + \tilde{u}_\epsilon) \partial_x \tilde{w}_\epsilon + v_\epsilon (u_{yy}^s + \partial_y \tilde{w}_\epsilon) = \partial_y^2 \tilde{w}_\epsilon + \epsilon \partial_x^2 \tilde{w}_\epsilon, \\ \partial_y \tilde{w}_\epsilon|_{y=0} = 0 \\ \tilde{w}_\epsilon|_{t=0} = \tilde{w}_{0,\epsilon}, \end{cases} \quad (\text{C.1})$$

where

$$\tilde{u}_\epsilon(t, x, y) = - \int_y^{+\infty} \tilde{w}_\epsilon(t, x, \tilde{y}) d\tilde{y}, \quad \tilde{v}_\epsilon(t, x, y) = - \int_0^y \partial_x \tilde{u}_\epsilon(t, x, \tilde{y}) d\tilde{y}.$$

We will use the following iteration process to prove the existence of solution, where  $w^0 = \tilde{w}_{0,\epsilon}$ ,

$$\begin{cases} \partial_t w^n + (u^s + u^{n-1}) \partial_x w^n + (u_{yy}^s + \partial_y w^{n-1}) v^n = \partial_y^2 w^n + \epsilon \partial_x^2 w^n, \\ \partial_y w^n|_{y=0} = 0 \\ w^n|_{t=0} = \tilde{w}_{0,\epsilon}, \end{cases} \quad (\text{C.2})$$

with

$$u^{n-1}(t, x, y) = - \int_y^{+\infty} w^{n-1}(t, x, \tilde{y}) d\tilde{y},$$

and

$$\begin{aligned} v^n(t, x, y) &= - \int_0^y \partial_x u^n(t, x, \tilde{y}) d\tilde{y} \\ &= \int_0^y \int_{\tilde{y}}^{+\infty} \partial_x w^n(t, x, z) dz d\tilde{y}. \end{aligned}$$

Here for the boundary data, we have

$$\begin{aligned}
& \partial_y^3 w^n|_{y=0} = ((u_y^s + w^{n-1})\partial_x w^n)|_{y=0}, \\
& (\partial_y^5 w^n)(t, x, 0) \\
& = (\partial_y^3 u^s(t, 0) + \partial_y^2 w^{n-1}(t, x, 0) + \epsilon(\partial_x^2 w^{n-1})(t, x, 0)) (\partial_x w^n)(t, x, 0) \\
& \quad + (u_y^s(t, 0) + (w^{n-1})(t, x, 0)) ((\partial_y^2 \partial_x w^n)(t, x, 0) + \epsilon(\partial_x^3 w^n)(t, x, 0)) \\
& \quad - (\partial_y \partial_x w^n)(u_y^s + w^{n-1})(t, x, 0) \\
& \quad + \sum_{1 \leq j \leq 3} C_j^4 \left( (\partial_y^j (u^s + u^{n-1})) \partial_y^{4-j} \partial_x u^n - (\partial_y^{j-1} \partial_x \tilde{u}^n) \partial_y^{4-j} (u_y^s + w^{n-1}) \right) (t, x, 0) \\
& \quad - \epsilon \partial_x^2 \left( (u_y^s(t, 0) + (w^{n-1})(t, x, 0)) (\partial_x w^n)(t, x, 0) \right).
\end{aligned}$$

and also for  $3 \leq p \leq \frac{m}{2} + 1$ ,  $\partial_y^{2p+1} w^n|_{y=0}$  is a linear combination of the terms of the form:

$$\prod_{j=1}^{q_1} \left( \partial_x^{\alpha_j} \partial_y^{\beta_j+1} (u^s + u^n) \right) \Big|_{y=0} \times \prod_{l=1}^{q_2} \left( \partial_x^{\tilde{\alpha}_l} \partial_y^{\tilde{\beta}_l+1} (u^s + u^{n-i}) \right) \Big|_{y=0}, \quad (\text{C.3})$$

where  $2 \leq q_1 + q_2 \leq p$ ,  $1 \leq i \leq \min\{n, p\}$  and

$$\begin{aligned}
& \alpha_j + \beta_j \leq 2p - 1, \quad 1 \leq j \leq q_1; \quad \tilde{\alpha}_l + \tilde{\beta}_l \leq 2p - 1, \quad 1 \leq l \leq q_2; \\
& \sum_{j=1}^{q_1} (3\alpha_j + \beta_j) + \sum_{l=1}^{q_2} (3\tilde{\alpha}_l + \tilde{\beta}_l) = 2p + 1; \\
& \sum_{j=1}^{q_1} \beta_j + \sum_{l=1}^{q_2} \tilde{\beta}_l \leq 2p - 2; \quad \sum_{j=1}^{q_1} \alpha_j + \sum_{l=1}^{q_2} \tilde{\alpha}_l \leq p - 1, \quad 0 < \sum_{j=1}^{q_1} \alpha_j.
\end{aligned}$$

Remark that the condition  $0 < \sum_{j=1}^{q_1} \alpha_j$  implies that, in (C.3), there are at last one factor like  $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} u^n(t, x, 0)$ .

For given  $w^{n-1}$ , we have  $u^{n-1} = \partial_y^{-1} w^{n-1}$  and  $v^n = -\partial_y^{-1} u_x^n$ . We will prove the existence and boundness of the sequence  $\{w^n, n \in \mathbb{N}\}$  in  $L^\infty([0, T_\epsilon]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$  to the linear equation (C.2) firstly, then the existence of solution to (C.1) is guaranteed by using the standard weak convergence methods.

**Lemma C.1.** *Assume that  $w^{n-i} \in L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))$ ,  $1 \leq i \leq \min\{n, \frac{m}{2} + 1\}$  and  $\tilde{w}_{0,\epsilon}$  satisfies the compatibility condition up to order  $m+2$  for the system (C.1), then the initial-boundary value problem (C.2) admit a unique solution  $w^n$  such that, for any  $t \in [0, T]$ ,*

$$\frac{d}{dt} \|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 \leq B_T^{n-1} \|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + D_T^{n-1} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2}, \quad (\text{C.4})$$

where

$$\begin{aligned}
B_T^{n-1} = C & \left( 1 + \sum_{i=1}^{\min\{n, m/2+1\}} \|w^{n-i}\|_{L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))} \right. \\
& \left. + \left(1 + \frac{1}{\epsilon}\right) \sum_{i=1}^{\min\{n, m/2+1\}} \|w^{n-i}\|_{L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))}^2 \right),
\end{aligned}$$

and

$$D_T^{n-1} = C \sum_{i=1}^{\min\{n, m/2+1\}} \|w^{n-i}\|_{L^\infty([0, T]; H_{k+\ell}^{m+2}(\mathbb{R}_+^2))}^{m+2}.$$

*Proof.* Once we get *à priori* estimate for this linear problem, the existence of solution is guaranteed by the Hahn-Banach theorem. So we only prove the *à priori* estimate of the smooth solutions.

For any  $\alpha \in \mathbb{N}^2$ ,  $|\alpha| \leq m+2$ , taking the equation (C.2) with derivative  $\partial^\alpha$ , multiplying the resulting equation by  $\langle y \rangle^{2k+2\ell+2\alpha_2} \partial^\alpha w^n$  and integrating by part over  $\mathbb{R}_+^2$ , one obtains that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \|\partial_y w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \epsilon \|\partial_x w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 \\ &= \sum_{|\alpha| \leq m+2} \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} \partial^\alpha ((u^s + u^{n-1}) \partial_x w^n \\ & \quad - (\partial_y^{-1} u_x^n)(u_{yy}^s + \partial_y w^{n-1})) \partial^\alpha w^n dx dy \\ & + \sum_{|\alpha| \leq m+2} \int_{\mathbb{R}_+^2} (\langle y \rangle^{2k+2\ell+2\alpha_2})' \partial^\alpha \partial_y w^n \partial^\alpha \partial_y w^n dx dy \\ & \quad + \sum_{|\alpha| \leq m+2} \int_{\mathbb{R}} (\partial^\alpha \partial_y w^n \partial^\alpha \partial_y w^n)|_{y=0} dx, \end{aligned} \tag{C.5}$$

With similar analysis to Section 5, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} (u^s + u^{n-1}) \partial_x \partial^\alpha w^n \partial^\alpha w^n dx dy \right| \\ &= \left| -\frac{1}{2} \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} \partial_x (u^s + u^{n-1}) \partial^\alpha w^n \partial^\alpha w^n dx dy \right| \\ &\leq C \|u^{n-1}\|_{L^\infty(\mathbb{R}_+^2)} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2, \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} [\partial^\alpha, (u^s + u^{n-1})] \partial_x w^n \partial^\alpha w^n dx dy \right| \\ &\leq C(1 + \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}) \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2. \end{aligned}$$

For the second term on the right hand of (C.5), by using the Leibniz formula, we need to pay more attention to the following two terms

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} (\partial^\alpha \partial_y^{-1} u_x^n)(u_{yy}^s + \partial_y w^{n-1}) \partial^\alpha w^n dx dy \right| \\ &\leq C(1 + \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}) \|\partial_x w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \\ &\leq \frac{\epsilon}{2} \|\partial_x w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \frac{C}{\epsilon} (1 + \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)})^2 \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2, \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} v^n (\partial^\alpha \partial_y w^{n-1}) \partial^\alpha w^n dx dy \\ &= - \int_{\mathbb{R}_+^2} \partial_y (\langle y \rangle^{2k+2\ell+2\alpha_2} (\partial_y^{-1} u_x^n)) (\partial^\alpha w^{n-1}) \partial^\alpha w^n dx dy \\ & \quad - \int_{\mathbb{R}_+^2} (\langle y \rangle^{2k+2\ell+2\alpha_2} (\partial_y^{-1} u_x^n)) (\partial^\alpha w^{n-1}) \partial_y \partial^\alpha w^n dx dy, \end{aligned}$$

here we have used  $v^n|_{y=0} = 0$ , thus

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2\ell+2\alpha_2} v^n (\partial^\alpha \partial_y w^{n-1}) \partial^\alpha w^n dx dy \right| \\ & \leq C \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} (\|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \|\partial_y w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}). \end{aligned}$$

For the boundary term, similar to the proof of Proposition 3.8, we can get

$$\begin{aligned} & \sum_{|\alpha| \leq m+2} \left| \int_{\mathbb{R}} (\partial^\alpha \partial_y w^n \partial^\alpha \partial_y w^n)|_{y=0} dx \right| \\ & \leq \frac{1}{16} \|\partial_y w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + C \|w^{n-1}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2}. \end{aligned}$$

We get finally

$$\begin{aligned} & \frac{d}{dt} \|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \|\partial_y w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + \epsilon \|\partial_x w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 \\ & \leq B_T^{n-1} \|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 + D_T^{n-1} \|w^n\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2}. \end{aligned}$$

□

**Lemma C.2.** *Suppose that  $m, k, \ell$  and  $u^s(t, y)$  satisfy the assumption of Proposition 3.6,  $\bar{\zeta} > 0$ , then for any  $0 < \epsilon \leq 1$ , there exists  $T_\epsilon > 0$  such that for any  $\tilde{w}_{0,\epsilon} \in H_{k+\ell}^{m+2}(\mathbb{R}_+^2)$  with*

$$\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \bar{\zeta},$$

*the iteration equations (C.2) admit a sequence of solution  $\{w^n, n \in \mathbb{N}\}$  such that, for any  $t \in [0, T_\epsilon]$ ,*

$$\|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \frac{4}{3} \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}, \quad \forall n \in \mathbb{N}.$$

**Remark.** Here  $\bar{\zeta}$  is arbitrary.

*Proof.* Integrating (C.4) over  $[0, t]$ , for  $0 < t \leq T$  and  $T > 0$  small,

$$\|w^n(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^m \leq \frac{\|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^m}{e^{-\frac{m}{2} B_T^{n-1} t} - \frac{m}{2} D_T^{n-1} t \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^m}.$$

We prove the Lemma by induction. For  $n = 1$ , we have

$$\begin{aligned} B_T^0 &= C \left( 1 + \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} + \left(1 + \frac{1}{\epsilon}\right) \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^2 \right) \\ &\leq C \left( 1 + \bar{\zeta} + \left(1 + \frac{1}{\epsilon}\right) \bar{\zeta}^2 \right), \end{aligned}$$

and

$$D_T^{n-1} = C \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^{m+2} \leq C\bar{\zeta}^{m+2}.$$

Choose  $T_\epsilon > 0$  small such that

$$\left( e^{-\frac{m}{2}C(1+2\bar{\zeta}+4(1+\frac{1}{\epsilon})\bar{\zeta}^2)T_\epsilon} - \frac{m}{2}C(2\bar{\zeta})^{m+2}T_\epsilon(2\bar{\zeta})^m \right)^{-1} = \left( \frac{4}{3} \right)^m,$$

we get

$$\|w^1(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \frac{4}{3} \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}.$$

Now the induction hypothesis is: for  $0 \leq t \leq T_\epsilon$ ,

$$\|w^{n-1}(t)\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)} \leq \frac{4}{3} \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)},$$

thanks to the choose of  $T_\epsilon$ , we have also

$$\left( e^{-\frac{m}{2}B_{T_\epsilon}^{n-1}T_\epsilon} - \frac{m}{2}D_{T_\epsilon}^{n-1}T_\epsilon \|\tilde{w}_{0,\epsilon}\|_{H_{k+\ell}^{m+2}(\mathbb{R}_+^2)}^m \right)^1 \leq \left( \frac{4}{3} \right)^m$$

for any  $t \in [0, T_\epsilon]$ , then we finish the proof of the Lemma C.2.  $\square$

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