



**HAL**  
open science

# Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach

Jean-Michel Coron, Hoai-Minh Nguyen

► **To cite this version:**

Jean-Michel Coron, Hoai-Minh Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. 2015. hal-01228895

**HAL Id: hal-01228895**

**<https://hal.science/hal-01228895>**

Preprint submitted on 14 Nov 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach

Jean-Michel Coron <sup>\*</sup>      Hoai-Minh Nguyen <sup>†</sup>

## Abstract

Using the backstepping approach we recover the null controllability for the heat equations with variable coefficients in space in one dimension and prove that these equations can be stabilized in finite time by means of periodic time-varying feedback laws. To this end, on one hand, we provide a new proof of the well-posedness and the “optimal” bound with respect to damping constants for the solutions of the kernel equations; this allows to deal with variable coefficients, even with a weak regularity of these coefficients. On another hand, we establish the well-posedness and estimates for the heat equations with a nonlocal boundary condition at one side.

**Keywords.** Null controllability, time-varying feedback laws, stabilization in finite time, backstepping, heat equation.

**AMS subject classification.** 93B05, 93B17, 93B52.

## 1 Introduction

The null controllability of the heat equations has been extensively investigated since several decades. This was pioneered in [12] by the moment method. Since then, there have been a few other methods to prove the null controllability of the heat equations. One is based on the construction of the fundamental solution as proposed [18, 24]. One is based on Carleman estimates, as initiated in [14, 23]; see also [11] and references therein for recent results. One, as proposed in [29], is via the transmutation method, which relates the null-controllability of the heat equation to the exact controllability of the wave equation. One, as proposed in [28], is via the flatness approach; in this approach,  $x$  is considered as the time-variable, see also [15].

In this paper, we give a new approach to obtain the null controllability for the heat equations with Dirichlet boundary control. This new approach is based on backstepping

---

<sup>\*</sup>Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, 4 place Jussieu, F-75252, Paris, France, coron@ann.jussieu.fr. JMC was supported by ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7).

<sup>†</sup>EPFL SB MATHAA CAMA, Station 8, CH-1015 Lausanne, Switzerland, hoai-minh.nguyen@epfl.ch. HMN was supported by ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7).

design in which the kernel also depends on time. The backstepping method has been used as a standard tool to stabilize finite dimensional control systems, see, e.g., [4, 20]. This method was initiated in [5, 26] to design feedback laws stabilizing control systems modeled by partial differential equations. Later on, Krstic and his collaborators introduced a key modification of the method: Using a Volterra transform of the second kind, coming from the application of the classical backstepping method applied to a spatial discretization of the partial differential equation, they map the original equation into an asymptotically stable one. In this context, the first continuous backstepping designs were proposed for the heat equation [25, 32]. The applications to wave equation appeared later in [19, 34, 31]. Since then, it has been applied to study the stability of parabolic equations in [32, 33], of hyperbolic systems in [21, 8, 9, 16, 17], of nonlinear parabolic equations in [35], of Korteweg-de Vries equations in [3] and of Kuramoto–Sivashinsky equations [27]. A concise introduction of this method applied to numerous partial differential equations can be found in [22]. In this paper, we implement the backstepping idea to obtain the null-controllability of 1-d heat equations. We also provide a new method to prove the existence of and to establish the “optimal” bound for a solution to the kernel equation. Finally, we show how this approach can be used to stabilize in finite time these equations by means of time-varying feedback laws. For this end, we establish the well-posedness and estimates for the heat equations with a nonlocal boundary condition at one side.

**Remark 1.** For some equations, one needs to use more general transformations than the one given by the backstepping approach (i.e. a Volterra transform of the second kind). See, in particular, [1] for wave equations, [2] for compensating the distributed effect of diffusion and counter-convection in Multi-Input and Multi-Output LTI systems, [6] for KdV equations, [7] for Kuramoto–Sivashinsky equations. In these last two papers, the existence of these more general transformations is shown to be equivalent to the controllability (of the linearized control system).

We consider the control system

$$\begin{cases} u_t(t, x) = (a(x)u_x(t, x))_x + c(x)u(t, x) & \text{in } (\tau_1, \tau_2) \times [0, 1], \\ u(t, 0) = 0, \quad u(t, 1) = U(t) & \text{for } t \in (\tau_1, \tau_2), \end{cases} \quad (1.1)$$

where, at  $t \in (\tau_1, \tau_2)$ , the state is  $u(t, \cdot) \in L^2(0, 1)$  and the control is  $U(t) \in \mathbb{R}$ . Throughout this paper, we assume that  $a \in H^2(0, 1)$ ,  $c \in H^1(0, 1)$ , and

$$a(x) > 0 \text{ for every } x \in [0, 1], \quad (1.2)$$

which implies the existence of  $\Lambda \geq 1$  such that  $0 < 1/\Lambda \leq a(x) \leq \Lambda$  in  $[0, 1]$  since  $a$  is continuous on  $[0, 1]$ .

The first goal of this paper is to provide a new way (via backstepping design) to obtain constructive controls  $U$  which steer the control system (1.1) from a given  $u_0 \in L^2(0, 1)$  to 0 during the interval of time  $[0, T]$ , with  $T > 0$  given, i.e.,  $u(t, \cdot) \rightarrow 0$  as  $t \rightarrow T_-$  where  $u$  is the solution of

$$\begin{cases} u_t(t, x) = (a(x)u_x(t, x))_x + c(x)u(t, x) & \text{in } (0, T) \times [0, 1], \\ u(t, 0) = 0, \quad u(t, 1) = U(t) & \text{for } t \in (0, T), \\ u(t = 0, \cdot) = u_0 & \text{for } x \in [0, 1]. \end{cases} \quad (1.3)$$

Let us denote by  $L^2(0,1)^*$  the set of continuous linear maps from  $L^2(0,1)$  into  $\mathbb{R}$ . We have the following theorem.

**Theorem 1.** *Let  $T > 0$ . There exists a piecewise constant functional  $\mathcal{K} : [0, T) \rightarrow L^2(0,1)^*$  such that, for every  $u_0 \in L^2(0,1)$ , if  $u \in C^0([0, T); L^2(0,1))$  is the solution of (1.3) with  $U(t)$  defined by*

$$U(t) := \mathcal{K}(t)u(t, \cdot), \quad (1.4)$$

then

$$u(t, \cdot) \rightarrow 0 \text{ in } L^2(0,1) \text{ as } t \rightarrow T_-, \quad (1.5)$$

$$U(t) \rightarrow 0 \text{ as } t \rightarrow T_-, \quad (1.6)$$

where  $u$  is the solution of (1.3).

Let us briefly describe here the idea of the proof of Theorem 1. The operator  $\mathcal{K}(t)$  in Theorem 1 is of the form

$$\mathcal{K}(t)v = \int_0^1 k_n(1, y)v(y) dy \text{ for every } v \in L^2(0,1), \quad (1.7)$$

for  $t_n \leq t < t_{n+1}$  for some sequence  $(t_n)_{n \in \mathbb{N}} \rightarrow T_-$  and for some sequence of functions (called kernels)  $(k_n)_{n \in \mathbb{N}}$  defined in  $D$  which is defined by

$$D := \{(x, y) \in [0, 1]^2; y \leq x\}.$$

The choice of  $(t_n)_{n \in \mathbb{N}}$  and  $(k_n)_{n \in \mathbb{N}}$  are derived from the backstepping approach as follows. Let  $(t_n)_{n \in \mathbb{N}}$  be a strictly increasing sequence of real numbers such that  $t_0 = 0$  and  $t_n \rightarrow T$  as  $n \rightarrow \infty$ . We construct the kernel  $k_n$  (used for the interval of time  $[t_n, t_{n+1})$ ) using backstepping design: define, for  $t_n \leq t < t_{n+1}$ ,

$$w(t, x) = u(t, x) - \int_0^x k_n(x, y)u(t, y) dy, \quad (1.8)$$

where  $k_n$  is chosen such that, for  $t_n \leq t < t_{n+1}$  and for  $u$  solution of (1.1),

$$w_t(t, x) - (a(x)w_x(t, x))_x + \lambda_n w(t, x) = 0 \text{ for } x \in [0, 1], \quad (1.9)$$

for some  $\lambda_n > 0$ , damping coefficients. By requiring that (1.9) holds for any  $u$  satisfying (1.1) with  $(\tau_1, \tau_2) = (t_n, t_{n+1})$ , one obtains the following system for  $k_n$ :

$$\begin{cases} 2a(x)\frac{d}{dx}k_n(x, x) + a_x(x)k_n(x, x) + [\lambda_n + c(x)] = 0 & \text{for } x \in [0, 1], \\ k_n(x, 0) = 0 & \text{for } x \in [0, 1], \\ (a(x)k_{n,x}(x, y))_x - (a(y)k_{n,y}(x, y))_y - [\lambda_n + c(y)]k_n(x, y) = 0 & \text{in } D. \end{cases} \quad (1.10)$$

Here and in what follows, we use the notation:

$$\frac{d}{dx}k(x, x) := k_x(x, x) + k_y(x, x),$$

where  $k_x$  and  $k_y$  denotes the partial derivative of  $k : D \rightarrow \mathbb{R}$  with respect to  $x$  and  $y$ . In fact, we can verify that such a  $k_n$  exists (Lemma 2) and that, indeed, (1.9) holds if  $u$  satisfies (1.1) with  $(\tau_1, \tau_2) = (t_n, t_{n+1})$  and if  $w$  is defined by (1.8) with  $k_n$  satisfying (1.10) (see Lemma 3). The control  $U$  in (1.4) is chosen as usual by requiring (1.7) in order to have

$$w(t, 1) = 0 \text{ for } t_n < t < t_{n+1}. \quad (1.11)$$

Let us point out that, from (1.1) and (1.8), we have

$$w(t, 0) = 0 \text{ for } t_n < t < t_{n+1}. \quad (1.12)$$

We derive from (1.9), (1.11), and (1.12) that

$$\|w(t, \cdot)\|_{L^2} \leq e^{-\lambda_n(t-t_n)} \|w(t_{n+1}, \cdot)\|_{L^2}. \quad (1.13)$$

From (1.13), we obtain the decay of  $w$  as  $t \rightarrow T_-$ . To compute  $u$  from  $w$ , one searches the kernel  $l_n$  such that

$$u(t, x) = w(t, x) + \int_0^x l_n(x, y) w(t, y) dy. \quad (1.14)$$

By requiring that, for  $t_n \leq t < t_{n+1}$ ,

$$u_t(t, x) = (a(x)u_x(t, x))_x + c(x)u(t, x) \text{ for } x \in [0, 1], \quad (1.15)$$

one gets

$$\begin{cases} 2a(x) \frac{d}{dx} l_n(x, x) + a_x(x) l_n(x, x) + \lambda_n + c(x) = 0 & \text{for } x \in [0, 1], \\ l_n(x, 0) = 0 & \text{for } x \in [0, 1], \\ (a(x)l_{n,x}(x, y))_x - (a(y)l_{n,y}(x, y))_y + [\lambda_n + c(x)]l_n(x, y) = 0 & \text{in } D. \end{cases} \quad (1.16)$$

In fact, we can prove that  $l_n$  exists (Lemma 2) and that, if  $l_n$  satisfies (1.16) and if  $w$  is defined by (1.8) where  $k_n$  satisfies (1.10), then (1.14) holds (Lemma 4). We establish the following crucial estimates for  $k_n$  and  $l_n$  (Lemma 2):

$$\|k_n\|_{H^1(D)} \leq e^{C\lambda_n^{1/2}} \quad \text{and} \quad \|l_n\|_{H^1(D)} \leq C\lambda_n, \quad (1.17)$$

for some positive constant  $C$  which depends only on  $a$  and  $c$ : it does not depend on  $n$ . Let us point out that related estimates already appear in [23, Proposition 1]. These related estimates are proved thanks to Carleman estimates. Our approach is completely different. From (1.13) and (1.17), we can derive that  $u(t, \cdot) \rightarrow 0$  in  $L^2(0, 1)$  and  $U(t) \rightarrow 0$  in  $\mathbb{R}$  as  $t \rightarrow T_-$  by appropriate choices of  $(t_n)_{n \in \mathbb{N}}$  and of  $(\lambda_n)_{n \in \mathbb{N}}$  (see also [23, p. 343-344]). More precisely, we prove the following proposition.

**Proposition 1.** *Let  $T > 0$ ,  $(\lambda_n)_{n \geq 0}$  be an increasing sequence of positive numbers converging to infinity, and let  $(t_n)_{n \geq 0}$  be an increasing sequence which converges to  $T$  with  $t_0 = 0$ . Define, for  $t_n \leq t < t_{n+1}$ ,*

$$U(t) := \int_0^1 k_n(1, y) u(t, y) dy, \quad (1.18)$$

where  $k_n$  is given in (1.10). Set  $s_0 := 0$  and  $s_n := \sum_{k=0}^{n-1} \lambda_k(t_{k+1} - t_k)$  for  $n \geq 1$ . There exists a positive constant  $\gamma$ , depending only on  $a$  and  $c$ , such that if, for large  $n$ ,

$$(t_{n+1} - t_n)\lambda_n \geq \gamma\sqrt{\lambda_{n+1}}, \quad (1.19)$$

then, for  $t_n \leq t \leq t_{n+1}$ ,

$$\|u(t, \cdot)\|_{L^2} \leq Ce^{-s_{n-1}/4+C(n-1)}\|u_0\|_{L^2}, \quad (1.20)$$

$$|U(t)| \leq Ce^{-s_{n-1}/4+C(n-1)+C\sqrt{\lambda_n}}\|u_0\|_{L^2}. \quad (1.21)$$

for some positive constant  $C$  independent of  $n$  and  $u_0$ . In particular, if in addition that

$$\lim_{n \rightarrow +\infty} \frac{s_n}{n + \sqrt{\lambda_{n+1}}} = +\infty, \quad (1.22)$$

then

$$\lim_{t \rightarrow T_-} \|u(t, \cdot)\|_{L^2} = 0, \quad (1.23)$$

$$\lim_{t \rightarrow T_-} U(t) = 0. \quad (1.24)$$

There are sequences  $(t_n)_{n \in \mathbb{N}}$  and  $(\lambda_n)_{n \in \mathbb{N}}$  which satisfy (1.19) and (1.22): for example the sequences such that  $t_n = T - 1/n^2$  and  $\lambda_n = n^8$  for large  $n$ . Hence, Theorem 1 is a corollary of Proposition 1.

In this paper, we therefore provide a new constructive control to reach the null controllability for the heat equation with variable coefficients via backstepping approach. Our idea is to use a sequence of kernels  $(k_n)_{n \in \mathbb{N}}$  in an appropriate choice of time interval  $[t_n, t_{n+1})$  corresponding for a sequence of positive numbers  $(\lambda_n)_{n \in \mathbb{N}}$  used for damping effect converging to infinity to stabilize the equations more and more as  $t$  goes to  $T$ . To implement this idea, we need to prove the existence of  $k_n$  and  $l_n$ , and establish estimates given in (1.17). Our analysis is variational and different from the standard one and hence requires some new ideas. First, to handle the existence of  $k_n$  and  $l_n$  in the variable coefficients case, we show, in Lemma 1, a connection of (1.10) and (1.16) with the wave equation defined in  $[0, 1]^2$ . To this end, we establish a property of the finite speed of propagation type (see (2.6) in Lemma 1). Known methods to prove the existence of  $k_n$  and  $l_n$  (in the case  $a$  is constant) are based on special functions or by fixed point arguments (see, in particular, [22, Chapter 4] and [10]). Second, the proof of (1.17) is only known for constant  $a$  and  $c$  using the information of special functions (see, e.g., [22, Chapter 4]). In the case,  $a$  is constant and  $c$  is not, it is known that (1.17) holds provided that the exponent  $\lambda_n^{1/2}$  is replaced by  $\lambda_n$  (see, e.g., [22, Chapter 4]). Nevertheless, exponent  $\lambda_n$  is not sufficient to get the decay of  $u_n$  to 0 due to (1.13). Estimates in (1.17) are given in (2.5) of Lemma 1. These follow from an energy type estimate for the wave equation which is somehow nonstandard in the sense that the energy not only contains the gradient of the solutions but also the solutions, see (2.14); the standard energy estimate only gives the exponent  $\lambda_n$ .

The second goal of this article is to show that the control system (1.1) can be semi-globally stabilized in arbitrary time by means of time-varying feedback laws  $(t, v) \in \mathbb{R} \times L^2(0, 1) \mapsto F(t, v) \in \mathbb{R}$ . We look for feedback laws  $F$  satisfying the following three properties.

( $\mathcal{P}_1$ ). The feedback law  $F$  is  $T$ -periodic with respect to time:

$$F(t, v) = F(t + T, v) \text{ for every } (t, v) \in \mathbb{R} \times L^2(0, 1). \quad (1.25)$$

( $\mathcal{P}_2$ ). There exists a strictly increasing sequence  $(t_n)_{n \in \mathbb{N}}$  of real numbers such that

$$t_0 = 0, \quad (1.26)$$

$$\lim_{n \rightarrow +\infty} t_n = T, \quad (1.27)$$

$$F \text{ is of class } C^1 \text{ in } [t_n, t_{n+1}) \times L^2(0, 1) \text{ for every } n \in \mathbb{N}. \quad (1.28)$$

( $\mathcal{P}_3$ ). The map  $F$  vanishes on  $\mathbb{R} \times \{0\}$  and there exists a continuous function  $M : [0, T) \rightarrow [0, +\infty)$  such that

$$|F(t, v_2) - F(t, v_1)| \leq M(t) \|v_2 - v_1\|_{L^2} \quad \forall (t, v_1, v_2) \in [0, T) \times L^2(0, 1) \times L^2(0, 1). \quad (1.29)$$

Before stating our second result, let us comment on the Cauchy problem

$$\begin{cases} u_t(t, x) = (a(x)u_x(t, x))_x + c(x)u(t, x) & \text{for } (t, x) \in (s, \tau) \times [0, 1], \\ u(t, 0) = 0, \quad u(t, 1) = F(t, u(t, \cdot)) & \text{for } t \in (s, \tau), \\ u(s, \cdot) = u_0 & \text{for } x \in [0, 1], \end{cases} \quad (1.30)$$

where  $-\infty < s < \tau < +\infty$  and  $u_0 \in L^2(0, 1)$  are given. We use the following definition:  $u : [s, \tau) \times (0, 1) \rightarrow \mathbb{R}$  is a solution of (1.30) if  $u$  is in  $C^0([s, \tau); L^2(0, 1))$  is such that, for every  $\xi \in C^2([s, \tau) \times [0, 1])$  with compact support in  $[s, \tau) \times [0, 1]$  and which vanishes on  $[s, \tau) \times \{0, 1\}$ , one has

$$\begin{aligned} & - \int_0^1 u_0(x) \xi(s, x) dx - \int_s^\tau \int_0^1 u(t, x) \xi_t(t, x) dx, dt + \int_s^\tau a(1) F(t, u(t, \cdot)) \xi_x(t, 1) dt \\ & - \int_s^\tau \int_0^1 u(t, x) ((a(x) \xi_x(t, x))_x + c(x) \xi(t, x)) dx = 0. \end{aligned} \quad (1.31)$$

Using, properties ( $\mathcal{P}_1$ ) and ( $\mathcal{P}_3$ ), one gets the uniqueness of the solution to the Cauchy problem (1.30): two solutions  $u_1 : [s, \tau_1) \times (0, 1) \rightarrow \mathbb{R}$  and  $u_2 : [s, \tau_2) \times (0, 1) \rightarrow \mathbb{R}$  to (1.30) are equal on  $[s, \min\{\tau_1, \tau_2\}) \times (0, 1)$  (see the proof of Lemma 6). A solution  $u_1 : [s, \tau_1) \times (0, 1) \rightarrow \mathbb{R}$  to the Cauchy problem (1.30) is said to be maximal if there is no solution  $u_2 : [s, \tau_2) \times (0, 1) \rightarrow \mathbb{R}$  to the Cauchy problem (1.30) with  $\tau_2 > \tau_1$  and  $u_1 = u_2$  on  $[s, \tau_1) \times (0, 1)$ . From now on, all the solutions to the Cauchy problem (1.30) considered are maximal. As just mentioned this solution is unique. We denote by  $t \in [s, \mathfrak{t}(s, u_0)) \rightarrow \Phi(t, s, u_0)$  this solution. Let us point out that, as proved in Lemma 6, Properties ( $\mathcal{P}_1$ ), ( $\mathcal{P}_2$ ), and ( $\mathcal{P}_3$ ) imply that  $\mathfrak{t}(s, u_0) > s$  for every  $(s, u_0) \in \mathbb{R} \times L^2(0, 1)$ .

Our second result states that the control system (1.1) can be semi-globally stabilized in arbitrary time by means of time-varying feedback laws  $(t, v) \in \mathbb{R} \times L^2(0, 1) \mapsto F(t, v) \in \mathbb{R}$  satisfying Properties ( $\mathcal{P}_1$ ), ( $\mathcal{P}_2$ ), and ( $\mathcal{P}_3$ ).

**Theorem 2.** *Let  $T > 0$  and  $\Gamma > 0$ . There exists a time-varying feedback laws  $(t, v) \in \mathbb{R} \times L^2(0, 1) \mapsto F(t, v) \in \mathbb{R}$  satisfying Properties  $(\mathcal{P}_1)$ ,  $(\mathcal{P}_2)$ , and  $(\mathcal{P}_3)$  such that*

$$\exists(C, \bar{T}) \in (0, +\infty) \times (0, T) \text{ such that } |F(t, v)| \leq C \|v\|_{L^2}^{1/2} \quad \forall (t, v) \in [\bar{T}, T) \times L^2(0, 1), \quad (1.32)$$

$$\mathfrak{t}(s, u_0) = +\infty \text{ for every } (s, u_0) \in \mathbb{R} \times L^2(0, 1), \quad (1.33)$$

$$\Phi(t + 2T, t, u_0) = 0 \text{ for every } (t, u_0) \in \mathbb{R} \times L^2(0, 1) \text{ such that } \|u_0\|_{L^2} \leq \Gamma, \quad (1.34)$$

and such that the following uniform stability condition

$$\begin{cases} \forall \varepsilon > 0, \exists \eta > 0 \text{ such that, } \forall t' \in \mathbb{R}, \forall t \in [t', +\infty), \text{ and } \forall u_0 \in L^2(0, 1), \\ (\|u_0\|_{L^2} \leq \eta) \Rightarrow (\|\Phi(t, t', u_0)\| \leq \varepsilon) \end{cases} \quad (1.35)$$

holds

The paper is organized as follows. In Section 2, we establish several lemmas which are used in the proof of Proposition 1 and Theorem 2. In particular, we prove the existence and uniqueness of  $k_n$  and  $l_n$ , together with estimates on these functions. The proof of Proposition 1 is given in Section 3. (Let us recall that, as already observed above, Proposition 1 implies Theorem 1.) In Section 4, we establish the well-posedness of (1.30) and various estimates for the flow  $\Phi$ . Finally, in Section 5, we give the proof of Theorem 2.

## 2 Preliminaries

In this section, we establish several lemmas used in the proof of Proposition 1 and Theorem 2. The first one is on the stability and a property of the wave equation, which plays an important role in our analysis and is interesting in itself.

**Lemma 1.** *Let  $\lambda \in \mathbb{R}$ ,  $f \in L^2((0, 1)^2)$ , and let  $a_1, a_2, b_1, b_2$  and  $c$  be bounded measurable functions defined in  $[0, 1]^2$  such that  $a_1$  and  $a_2$  are Lipschitz and*

$$1/\Lambda \leq a_1(x, y), a_2(x, y) \leq \Lambda \quad \text{and} \quad |b(x, y)|, |c(x, y)| \leq \Lambda \text{ for } (x, y) \in [0, 1]^2, \quad (2.1)$$

for some  $\Lambda \geq 1$ , where  $b = (b_1, b_2)$ . There exists a unique solution

$$K \in L^2((0, 1); H_0^1(0, 1)) \cap H^1((0, 1)^2) \quad (2.2)$$

to the equation

$$\begin{aligned} & (a_1(x, y)K_x(x, y))_x - (a_2(x, y)K_y(x, y))_y \\ & + b(x, y) \cdot \nabla K(x, y) - [\lambda + c(x, y)]K(x, y) = f(x, y) \text{ in } [0, 1]^2, \end{aligned} \quad (2.3)$$

such that

$$K(x, 0) = K(x, 1) = 0 \text{ for } x \in [0, 1] \quad \text{and} \quad K(0, y) = K_x(0, y) = 0 \text{ for } y \in [0, 1]. \quad (2.4)$$



Moreover,

$$\int_0^1 |\nabla K(x, y)|^2 dy \leq C \max\{e^{C \operatorname{sign}(\lambda) \sqrt{|\lambda|}}, 1\} \int_0^1 \int_0^1 |f(x, y)|^2 dy dx \quad \text{for } x \in [0, 1]. \quad (2.5)$$

Assume in addition that  $a_1(x, x) \geq a_2(x, x)$  for  $x \in [0, 1]$  and  $\operatorname{supp} f \subset D$ . We have

$$K(x, y) = 0 \text{ in } [0, 1]^2 \setminus D. \quad (2.6)$$

Here  $C$  denotes a positive constant depending only on  $\Lambda$  and the Lipschitz constant of  $(a_1, a_2)$ .

We consider (2.3) and (2.4) as a wave system in which  $x$  is seen as time variable and  $y$  is seen as space variable and use the following definition for solutions: A function  $K : (0, 1)^2 \rightarrow \mathbb{R}$  is said to be a solution of (2.3) and (2.4) if (2.2) is satisfied, (2.3) holds in the distributional sense, and

$$K(0, \cdot) = 0, \quad (2.7)$$

$$K_x(0, \cdot) = 0. \quad (2.8)$$

(Equality (2.7) is an equality in  $L^2(0, 1)$ , while (2.8) is an equality in  $H^{-1}(0, 1)$ : note that by (2.2) and (2.3),  $K \in H^2((0, 1); H^{-1}(0, 1))$ ).

**Proof.** The existence and uniqueness of  $K$  are standard and left to the reader. We next prove (2.5) and (2.6). We only give the proof in the case  $a_1, a_2, b, c$ , and  $f$  are smooth and satisfy suitable compatibility conditions (which are automatically satisfied if the support of  $f$  is included in a compact subset of  $(0, 1] \times [0, 1]$ ). The proof in the general case follows by a standard regularizing argument. Since  $a_1, a_2, b, c$ , and  $f$  are smooth, from the standard regularity theory of the wave equation, it follows that  $K$  is smooth. We begin with the proof of (2.5). We only consider the case  $\lambda > 0$  and large enough. The proof in the other case is similar and even simpler. Multiplying the equation of  $K$  by  $K_x(x, y)$ , integrating with respect to  $y$  from 0 to 1, and using an integration by parts, we have

$$\begin{aligned} & \int_0^1 \frac{1}{2} \left[ \frac{d}{dx} (a_1(x, y) K_x^2(x, y)) + a_{1,x}(x, y) K_x^2(x, y) + \frac{d}{dx} (a_2(x, y) K_y^2(x, y)) - a_{2,x}(x, y) K_y^2(x, y) \right. \\ & \quad \left. + 2b(x, y) \cdot \nabla K(x, y) K_x(x, y) - [\lambda + c(x, y)] \frac{d}{dx} K^2(x, y) \right] dy = \int_0^1 f(x, y) K_x(x, y) dy. \end{aligned}$$

This implies

$$\begin{aligned} & \frac{d}{dx} \int_0^1 \left[ a_1(x, y) K_x^2(x, y) + a_2(x, y) K_y^2(x, y) - \lambda K^2(x, y) \right] dy \\ & = 2 \int_0^1 f(x, y) K_x(x, y) dy - \int_0^1 \left[ a_{1,x}(x) K_x^2(x, y) - a_{2,x}(x, y) K_y^2(x, y) \right. \\ & \quad \left. + 2b(x, y) \cdot \nabla K(x, y) K_x(x, y) - 2c(x, y) K_x(x, y) K(x, y) \right] dy. \quad (2.9) \end{aligned}$$

Integrating (2.9) from 0 to  $x$ , using the ellipticity and the Lipschitz property of  $a_1$  and  $a_2$  and the boundedness of  $b$  and  $c$ , we obtain, for  $x \in [0, 1]$ ,

$$\begin{aligned} & \int_0^1 [K_x^2(x, y) + K_y^2(x, y)] dy \\ & \leq C \int_0^1 \lambda K^2(x, y) dy + C \int_0^x \int_0^1 [K_x^2(s, y) + K_y^2(s, y)] dy ds + \|f\|_{L^2(0,1)^2}^2 \end{aligned} \quad (2.10)$$

Set

$$\hat{K}(x, y) = K(\lambda^{-1/2}x, y) \text{ for } (x, y) \in [0, \lambda^{1/2}] \times [0, 1].$$

We derive from (2.10) that, for  $x \in [0, \lambda^{1/2}]$ ,

$$\begin{aligned} & \int_0^1 [\hat{K}_x^2(x, y) + \lambda^{-1} \hat{K}_y^2(x, y)] dy \\ & \leq C \int_0^1 \hat{K}^2(x, y) dy + C \int_0^x \int_0^1 [\hat{K}_x^2(s, y) + \lambda^{-1} \hat{K}_y^2(s, y)] dy ds + \|f\|_{L^2}^2. \end{aligned} \quad (2.11)$$

Define

$$V_1(x) = \int_0^1 [\hat{K}_x^2(x, y) + \lambda^{-1} \hat{K}_y^2(x, y)] dy \quad \text{and} \quad V_2(x) = \int_0^1 \hat{K}^2(x, y) dy.$$

We have

$$V_2'(x) = 2 \int_0^1 \hat{K}_x(x, y) \hat{K}(x, y) dy \leq 2V_1^{1/2}(x)V_2^{1/2}(x), \quad (2.12)$$

and from (2.11) we obtain

$$V_1(x) \leq C \left( V_2(x) + \int_0^x V_1(s) ds + \|f\|_{L^2}^2 \right). \quad (2.13)$$

A combination of (2.12) and (2.13) yields

$$V_1(x) + V_2'(x) \leq C \left( V_2(x) + \int_0^x V_1(s) ds + \|f\|_{L^2}^2 \right). \quad (2.14)$$

We derive that

$$\int_0^x V_1(s) ds + V_2(x) \leq C \|f\|_{L^2}^2 e^{Cx};$$

which, together with (2.11), implies that

$$\int_0^1 [\hat{K}_x^2(x, y) + \lambda^{-1} \hat{K}_y^2(x, y)] dy \leq C \|f\|_{L^2}^2 e^{Cx}.$$

Estimate (2.5) now follows by a change of variables and the definition of  $\hat{K}$ .

We next establish that  $K(x, y) = 0$  in  $[0, 1]^2 \setminus D$ . Define

$$E(x) = \frac{1}{2} \int_x^1 \left( a_1(x, y) K_x^2(x, y) + a_2(x, y) K_y^2(x, y) \right) dy.$$

We have

$$\begin{aligned} E'(x) &= \int_x^1 \left( a_1(x, y) K_{xx}(x, y) K_x(x, y) + \frac{1}{2} a_{1,x}(x, y) K_x^2(x, y) \right) dy \\ &\quad + \int_x^1 \left( a_2(x, y) K_y(x, y) K_{xy}(x, y) + \frac{1}{2} a_{2,x}(x, y) K_y^2(x, y) \right) dy \\ &\quad - \frac{1}{2} [a_1(x, x) K_x^2(x, x) + a_2(x, x) K_y^2(x, x)]. \end{aligned}$$

An integration by parts yields

$$\begin{aligned} E'(x) &= \int_x^1 \left[ (a_1(x, y) K_x(x, y))_x - (a_2(x, y) K_y(x, y))_y \right] K_x(x, y) dy \\ &\quad + \int_x^1 \left( -\frac{1}{2} a_{1,x}(x, y) K_x^2(x, y) + \frac{1}{2} a_{2,x}(x, y) K_y^2(x, y) \right) dy \\ &\quad - \frac{1}{2} [a_1(x, x) K_x^2(x, x) + a_2(x, x) K_y^2(x, x) - 2a_2(x, x) K_x(x, x) K_y(x, x)]. \end{aligned}$$

Since  $a_1(x, x) \geq a_2(x, x)$ , we derive that

$$E'(x) \leq C(\lambda) E(x).$$

Since  $E(0) = 0$ , it follows that  $E = 0$ . The proof is complete.  $\square$

**Remark 2.** The Lipschitz assumption on  $a_1$  and  $a_2$  can be weakened; however the uniqueness does not hold if one only assumes (2.1): see [30, Theorem 9].

Using Lemma 1, we can establish the following lemma.

**Lemma 2.** *Let  $\lambda \in \mathbb{R}$ ,  $g \in H^1(0, 1)$  be such that  $x^{3/2}[g(x)/x]'' \in L^2(0, 1)$ . There exists a unique solution  $k \in H^1(D)$  to the system*

$$\begin{cases} k(x, x) = g(x) & \text{for } x \in [0, 1], \\ k(x, 0) = 0 & \text{for } x \in [0, 1], \\ (a(x)k_x(x, y))_x - (a(y)k_y(x, y))_y - [\lambda + c(y)]k(x, y) = 0 & \text{in } D. \end{cases} \quad (2.15)$$

Moreover, for every  $x \in [0, 1]$ ,

$$\int_0^x |\nabla k(x, y)|^2 dy \leq C \max\{e^{C \operatorname{sign}(\lambda) \sqrt{|\lambda|}}, 1\} \left( \|g\|_{L^2}^2 + \|s^{1/2} g'(s)\|_{L^2}^2 + \|s^{3/2} [g(s)/s]''\|_{L^2}^2 \right) \quad (2.16)$$

for some positive constant  $C$  independent of  $g$ ,  $x \in [0, 1]$ , and  $\lambda$ .

**Remark 3.** The existence, uniqueness, and estimate of  $k$  are known in the case where  $a$  and  $c$  are constant, see [22, Chapter 4].

**Proof.** We first establish the uniqueness of  $k$ . Assume that  $g = 0$ . We prove that  $k = 0$ . Extend  $k = 0$  in  $[0, 1]^2 \setminus D$  and still denote the extension by  $k$ . Then

$$(a(x)k_x(x, y))_x - (a(y)k_y(x, y))_y - [\lambda + c(y)]k(x, y) = 0 \text{ in } [0, 1]^2,$$

$k(x, 0) = k(x, 1) = 0$  for  $x \in [0, 1]$ , and  $k(0, y) = k_x(0, y) = 0$  for  $y \in [0, 1]$ . Hence  $k = 0$  by Lemma 1. To prove the existence of  $k$ , we proceed as follows. Define

$$\varphi(x, y) = g(x)y/x \text{ in } D$$

and set  $k_1 = k - \varphi$  in  $D$ . Then

$$\begin{cases} k_1(x, x) = 0 & \text{for } x \in [0, 1], \\ k_1(x, 0) = 0 & \text{for } x \in [0, 1], \\ (a(x)k_{1,x}(x, y))_x - (a(y)k_{1,y}(x, y))_y - [\lambda + c(y)]k_1(x, y) = f_1(x, y) & \text{in } D, \end{cases} \quad (2.17)$$

where

$$f_1(x, y) = -(a(x)\varphi_x(x, y))_x + (a(y)\varphi_y(x, y))_y + [\lambda + c(y)]\varphi(x, y) \text{ in } D.$$

One has

$$\|f_1\|_{L^2(D)} \leq C(\|g\|_{L^2} + \|s^{1/2}g'(s)\|_{L^2} + \|s^{3/2}[g(s)/s]''\|_{L^2}).$$

By Lemma 1, there exists  $k_2$  the unique solution to the system

$$(a(x)k_{2,x}(x, y))_x - (a(y)k_{2,y}(x, y))_y - [\lambda + c(y)]k_2(x, y) = f_2(x, y) \text{ in } [0, 1]^2, \quad (2.18)$$

$k_2(x, 0) = k_2(x, 1) = 0$  for  $x \in [0, 1]$ , and  $k_2(0, y) = k_{2,x}(0, y) = 0$  for  $y \in [0, 1]$ . Here  $f_2 = 1_D f_1$  where  $1_D$  denotes the characteristic function of  $D$ . Applying Lemma 1, we also have

$$k_2(x, y) = 0 \text{ in } [0, 1]^2 \setminus D.$$

Hence  $k_1$  can be chosen as the restriction of  $k_2$  in  $D$  and  $k = k_1 + \varphi$  in  $D$ . Estimate (2.16) follows immediately from the one of  $k_2$  obtained from Lemma 1.  $\square$

From Lemma 2, we get the following corollary, which gives the existence and uniqueness of  $k_n$  together with the first inequality of (1.17).

**Corollary 1.** *Let  $\lambda_0 > 0$ . For every  $\lambda \geq \lambda_0$ , there exists a unique solution  $k \in H^1(D)$  of the system*

$$\begin{cases} 2a(x)\frac{d}{dx}k(x, x) + a_x(x)k(x, x) + \lambda + c(x) = 0 & \text{for } x \in [0, 1], \\ k(x, 0) = 0 & \text{for } x \in [0, 1], \\ (a(x)k_x(x, y))_x - (a(y)k_y(x, y))_y - [\lambda + c(y)]k(x, y) = 0 & \text{in } D. \end{cases} \quad (2.19)$$

Moreover,

$$\|k\|_{H^1(D)} \leq e^{C\lambda^{1/2}}, \quad (2.20)$$

for some positive constant  $C$  independent of  $\lambda \in [\lambda_0, +\infty)$ .

**Proof.** Since  $a \in H^2((0, 1); (0, +\infty))$  and  $c \in H^1(0, 1)$ , there exists (a unique)  $g \in H^2(0, 1)$  such that

$$2a(x)g'(x) + a_x(x)g(x) + \lambda + c(x) = 0 \text{ in } [0, 1] \quad \text{and} \quad g(0) = 0.$$

One easily checks that  $x^{3/2}[g(x)/x]'' \in L^2(0, 1)$ . Then (2.19) is equivalent to

$$\begin{cases} k(x, x) = g(x) & \text{for } x \in [0, 1], \\ k(x, 0) = 0 & \text{for } x \in [0, 1], \\ (a(x)k_x(x, y))_x - (a(y)k_y(x, y))_y - [\lambda + c(y)]k(x, y) = 0 & \text{in } D. \end{cases}$$

The existence and uniqueness of  $k$  now follow from Lemma 2 and estimate (2.20) is a consequence of (2.16).  $\square$

Similar to Corollary 1, we have the following consequence of Lemma 2, which gives the existence and uniqueness of  $l_n$  together with the second inequality of (1.17) .

**Corollary 2.** *Let  $\lambda_0 > 0$ . For every  $\lambda \geq \lambda_0$ , there exists a unique solution  $l \in H^1(D)$  of the system*

$$\begin{cases} 2a(x)\frac{d}{dx}l(x, x) + a_x(x)l(x, x) + \lambda + c(x) = 0 & \text{for } x \in [0, 1], \\ l(x, 0) = 0 & \text{for } x \in [0, 1], \\ (a(x)l_x(x, y))_x - (a(y)l_y(x, y))_y + [\lambda + c(y)]l(x, y) = 0 & \text{in } D. \end{cases} \quad (2.21)$$

Moreover,

$$\|l\|_{H^1(D)} \leq C\lambda, \quad (2.22)$$

for some positive constant  $C$  independent of  $\lambda \in [\lambda_0, +\infty)$ .

The third lemma, whose proof is quite standard and which is known in the case where  $a$  is constant, is on the link between (1.1) and (1.9).

**Lemma 3.** *Let  $\lambda > 0$  and  $\tau_2 > \tau_1 \geq 0$ . Assume that  $u \in L^2((\tau_1, \tau_2); H^1(0, 1))$  is a solution to the system*

$$u_t(t, x) - (a(x)u_x(t, x))_x - c(x)u(t, x) = 0 \text{ in } (\tau_1, \tau_2) \times [0, 1], \quad (2.23)$$

$$u(t, 0) = 0 \text{ for } t \in (\tau_1, \tau_2). \quad (2.24)$$

Define, for  $\tau_1 < t < \tau_2$ ,

$$w(t, x) = u(t, x) - \int_0^x k(x, y)u(t, y) dy, \quad (2.25)$$

where  $k \in H^1(D)$  is the unique solution of the system (2.19). Then  $w \in L^2((\tau_1, \tau_2); H^1(0, 1))$  and satisfies the equation

$$w_t(t, x) - (a(x)w_x(t, x))_x + \lambda w(t, x) = 0 \text{ in } (\tau_1, \tau_2) \times [0, 1]. \quad (2.26)$$

**Proof.** We assume that  $a$  and  $c$  are smooth and establish (2.26); the general case follows by a regularizing argument. The smoothness of  $a$  and  $c$  imply that  $k$  and  $u$  and hence  $w$  are smooth in  $(\tau_1, \tau_2) \times [0, 1]$ . We have, from (2.25),

$$w_t(t, x) = u_t(t, x) - \int_0^x k(x, y) u_t(t, y) dy. \quad (2.27)$$

Using the fact that  $u_t(t, x) = (a(x)u_x(t, x))_x + c(x)u(t, x)$  in  $(\tau_1, \tau_2) \times [0, 1]$  and integrating by parts, we derive from (2.27) that

$$\begin{aligned} w_t(t, x) = u_t(t, x) - \int_0^x \left[ (a(y)k_y(x, y))_y + c(y)k(x, y) \right] u(t, y) dy \\ - k(x, y)a(y)u_y(t, y) \Big|_0^x + a(y)k_y(x, y)u(t, y) \Big|_0^x. \end{aligned}$$

This implies, using also (2.24),

$$\begin{aligned} w_t(t, x) = u_t(t, x) - \int_0^x \left[ (a(y)k_y(x, y))_y + c(y)k(x, y) \right] u(t, y) dy \\ - k(x, x)a(x)u_x(t, x) + a(x)k_y(x, x)u(t, x) + k(x, 0)a(0)u_x(t, 0). \end{aligned} \quad (2.28)$$

From (2.25), we also have

$$\begin{aligned} (a(x)w_x(t, x))_x = (a(x)u_x(t, x))_x - \int_0^x (a(x)k_x(x, y)u(t, y))_x dy - \frac{d}{dx}(a(x)k(x, x))u(t, x) \\ - a(x)k(x, x)u_x(t, x) - a(x)k_x(x, x)u(t, x). \end{aligned} \quad (2.29)$$

Using the fact that  $u_t(t, x) - (a(x)u_x(t, x))_x - c(x)u(t, x) = 0$  in  $(\tau_1, \tau_2) \times [0, 1]$ , we derive from (2.25), (2.28), and (2.29) that

$$\begin{aligned} w_t(t, x) - (a(x)w_x(t, x))_x + \lambda w(t, x) \\ = \left[ 2a(x)\frac{d}{dx}k(x, x) + a_x(x)k(x, x) + \lambda + c(x) \right] u(t, x) + k(x, 0)a(0)u_x(t, 0) \\ + \int_0^x \left[ (a(x)k_x(x, y))_x - (a(y)k_y(x, y))_y - [\lambda + c(y)]k(x, y) \right] u(t, y). \end{aligned}$$

With the choice  $k$  in (2.19), we have

$$w_t(t, x) - (a(x)w_x(t, x))_x + \lambda w(t, x) = 0 \text{ in } (\tau_1, \tau_2) \times [0, 1].$$

The proof is complete.  $\square$

The fourth lemma deals with the inverse transform of the map  $u \mapsto w$ , where  $w$  is defined by (1.8).

**Lemma 4.** Let  $\lambda > 0$ , let  $k \in H^1(D)$  be the unique solution of (2.19), and let  $l \in H^1(D)$  be the unique solution of (2.21). Let  $u \in L^2(0, 1)$  and define

$$w(x) := u(x) - \int_0^x k(x, y)u(y) dy \text{ for } x \in [0, 1].$$

We have

$$u(x) = w(x) + \int_0^x l(x, y)w(y) dy \text{ in } [0, 1]. \quad (2.30)$$

**Proof.** In what follows, we assume that  $a$  and  $c$  are smooth, the general case follows by a regularizing argument. Since  $a$  and  $c$  are smooth, it follows that  $k$  and  $l$  are smooth. We claim that

$$l(x, y) = k(x, y) + \int_y^x k(x, \xi)l(\xi, y) d\xi. \quad (2.31)$$

Admitting this claim, we prove (2.30). We have, by Fubini's theorem,

$$\begin{aligned} u(x) - \int_0^x k(x, y)u(y) + \int_0^x l(x, y) \left[ u(y) - \int_0^y k(y, \xi)u(\xi) d\xi \right] dy \\ = u(x) + \int_0^x \left[ l(x, y) - k(x, y) - \int_y^x l(x, \xi)k(\xi, y) d\xi \right] u(y) dy = u(x) : \end{aligned}$$

(2.30) is proved.

It remains to establish (2.31). Define

$$\hat{l}(x, y) = k(x, y) + \int_y^x k(x, \xi)l(\xi, y) d\xi. \quad (2.32)$$

We prove that  $\hat{l} = l$ . It is clear that  $\hat{l}(x, x) = k(x, x) = l(x, x)$  and  $\hat{l}(x, 0) = k(x, 0) = l(x, 0) = 0$  for  $x \in [0, 1]$ . We have, from (2.32),

$$\hat{l}_x(x, y) = k_x(x, y) + \int_y^x k_x(x, \xi)l(\xi, y) d\xi + k(x, x)l(x, y).$$

It follows that

$$\begin{aligned} (a(x)\hat{l}_x(x, y))_x &= (a(x)k_x(x, y))_x + \int_y^x (a(x)k_x(x, \xi))_x l(\xi, y) d\xi + a(x)k_x(x, x)l(x, y) \\ &+ \frac{d}{dx} [a(x)k(x, x)]l(x, y) + a(x)k(x, x)l_x(x, y). \end{aligned}$$

Using the fact that  $(a(x)k_x(x, y))_x = (a(y)k_y(x, y))_y + [\lambda + c(y)]k(x, y)$  in  $[0, 1]^2$ , we have

$$\begin{aligned}
(a(x)\hat{l}_x(x, y))_x &= (a(x)k_x(x, y))_x + \int_y^x \left[ (a(\xi)k_y(x, \xi))_y l(\xi, y) + [\lambda + c(\xi)]k(x, \xi)l(\xi, y) \right] d\xi \\
&\quad + a(x)k_x(x, x)l(x, y) + \frac{d}{dx} [a(x)k(x, x)]l(x, y) + a(x)k(x, x)l_x(x, y) \\
&= (a(x)k_x(x, y))_x + \int_y^x \left[ k(x, \xi)(a(\xi)l_x(\xi, y))_x + [\lambda + c(\xi)]k(x, \xi)l(\xi, y) \right] d\xi \\
&\quad + a(\xi)k_y(x, \xi)l(\xi, y) \Big|_y^x - k(x, \xi)a(\xi)l_x(\xi, y) \Big|_y^x + a(x)k_x(x, x)l(x, y) \\
&\quad + \frac{d}{dx} [a(x)k(x, x)]l(x, y) + a(x)k(x, x)l_x(x, y).
\end{aligned}$$

This implies

$$\begin{aligned}
(a(x)\hat{l}_x(x, y))_x &= (a(x)k_x(x, y))_x + \int_y^x \left[ (a(\xi)l_x(\xi, y))_x + [\lambda + c(\xi)]l(\xi, y) \right] k(x, \xi) d\xi \\
&\quad + a(x)k_y(x, x)l(x, y) - a(y)k_y(x, y)l(y, y) + k(x, y)a(y)l_x(y, y) \\
&\quad + a(x)k_x(x, x)l(x, y) + \frac{d}{dx} [a(x)k(x, x)]l(x, y).
\end{aligned}$$

Then, using the first equality of (2.21), it follows that

$$\begin{aligned}
(a(x)\hat{l}_x(x, y))_x &= (a(x)k_x(x, y))_x + \int_y^x \left[ (a(\xi)l_x(\xi, y))_x + [\lambda + c(\xi)]l(\xi, y) \right] k(x, \xi) d\xi \\
&\quad - [\lambda + c(x)]l(x, y) - a(y)k_y(x, y)l(y, y) + k(x, y)a(y)l_x(y, y). \quad (2.33)
\end{aligned}$$

Similarly, from (2.32), we obtain

$$\hat{l}_y(x, y) = k_y(x, y) + \int_y^x k(x, \xi)l_y(\xi, y) d\xi - k(x, y)l(y, y).$$

It follows that

$$\begin{aligned}
(a(y)\hat{l}_y(x, y))_y &= (a(y)k_y(x, y))_y + \int_y^x k(x, \xi)(a(y)l_y(\xi, y))_y dy - a(y)k(x, y)l_y(y, y) \\
&\quad - k(x, y)(a(y)l(y, y))_y - k_y(x, y)a(y)l(y, y). \quad (2.34)
\end{aligned}$$

Using the last equation of (2.19) and (2.21), we derive from (2.33) and (2.34) that

$$(a(x)\hat{l}_x(x, y))_x - (a(y)\hat{l}_y(x, y))_y = -[\lambda + c(x)]l(x, y).$$

On the other hand,

$$(a(x)l_x(x, y))_x - (a(y)l_y(x, y))_y = -[\lambda + c(x)]l(x, y).$$

Hence  $\hat{l} = l$  by the uniqueness of the solution to (2.21). The proof is complete.  $\square$



### 3 Proof of Proposition 1

This section is devoted to the proof of Proposition 1. We implement the ideas presented in the introduction. For  $t_n \leq t < t_{n+1}$ , Define  $w$  by (1.8) where  $k_n$  is given in (1.10). Applying Corollary 1, we have

$$\|w(t, \cdot)\|_{L^2}^2 \leq C e^{C\sqrt{\lambda_n}} \|u(t, \cdot)\|_{L^2}^2 \quad \text{for } t_n < t < t_{n+1}. \quad (3.1)$$

By Lemma 4, (1.14) holds where  $l_n$  is given by (1.16). We have, by applying Corollary 2,

$$\|u(t, \cdot)\|_{L^2}^2 \leq C \lambda_n^2 \|w(t, \cdot)\|_{L^2}^2 \quad \text{for } t_n \leq t < t_{n+1}. \quad (3.2)$$

and, by using Lemma 3,

$$w_t(t, x) - (a(x)w_x(t, x))_x + \lambda_n w(t, x) = 0 \quad \text{for } x \in [0, 1].$$

From the choice of the control (1.18), we obtain

$$w(t, 0) = w(t, 1) = 0 \quad \text{for } t_n < t < t_{n+1}.$$

It follows that

$$\|w(\xi_2, \cdot)\|_{L^2}^2 \leq \|w(\xi_1, \cdot)\|_{L^2}^2 e^{-2\lambda_n(\xi_2 - \xi_1)} \quad \text{for } t_n \leq \xi_1 < \xi_2 < t_{n+1}. \quad (3.3)$$

A combination of (3.1), (3.2), and (3.3) yields

$$\|u(t_{n+1}, \cdot)\|_{L^2}^2 \leq C \lambda_n^2 e^{-2\lambda_n(t_{n+1} - t_n) + C\sqrt{\lambda_n}} \|u(t_n, \cdot)\|_{L^2}^2. \quad (3.4)$$

We derive from (1.19) and (3.4) that, if  $\gamma$  is large enough, which will be always assumed,

$$\|u(t_{n+1}, \cdot)\|_{L^2}^2 \leq C e^{-\lambda_n(t_{n+1} - t_n)} \|u(t_n, \cdot)\|_{L^2}^2. \quad (3.5)$$

This, together with the definition of  $s_n$ , implies

$$\|u(t_{n+1}, \cdot)\|_{L^2}^2 \leq e^{-s_{n+1} + Cn} \|u(0, \cdot)\|_{L^2}^2. \quad (3.6)$$

We have, for  $t_n \leq t < t_{n+1}$ ,

$$\begin{aligned} \|u(t, \cdot)\|_{L^2}^2 &\leq C \lambda_n^2 \|w(t, \cdot)\|_{L^2}^2 \leq C \lambda_n^2 e^{-2\lambda_n(t - t_n)} \|w(t_n, \cdot)\|_{L^2}^2 \\ &\leq C \lambda_n^2 e^{-2\lambda_n(t - t_n) + C\sqrt{\lambda_n}} \|u(t_n, \cdot)\|_{L^2}^2 \\ &\leq C e^{-\lambda_{n-1}(t_n - t_{n-1})/2} \|u(t_{n-1}, \cdot)\|_{L^2}^2 \\ &\leq e^{-s_{n-1}/2 + C(n-2)} \|u(0, \cdot)\|_{L^2}^2, \end{aligned} \quad (3.7)$$

which gives (1.20). Here we use (3.2) in the first inequality of (3.7), (3.3) in the second inequality, (3.1) in the third inequality, (1.19) and (3.5) in the fourth inequality, and (3.6) (for  $t_{n-1}$  instead of  $t_{n+1}$ ) in the last inequality. Finally (1.21) follows from (1.17), (1.18), and (1.20).  $\square$

## 4 Some properties of the flow $\Phi$

In this section, we are interested in the flow  $\Phi$  introduced in the Introduction (Section 1). We start by mentioning a maximum principle for the Cauchy problem

$$\begin{cases} u_t(t, x) = (a(x)u_x(t, x))_x + c(x)u(t, x) + f(t, x) & \text{in } (\tau_1, \tau_2) \times (0, 1), \\ u(t, 0) = \alpha(t), \quad u(t, 1) = \beta(t) & \text{for } t \in (\tau_1, \tau_2), \\ u(0, x) = u_0(x), & \text{for } x \in (0, 1), \end{cases} \quad (4.1)$$

which we will use many times in this section. Let  $\tau_1$  and  $\tau_2$  be two real numbers such that  $\tau_1 < \tau_2$ , let  $\alpha$  and  $\beta \in L^2(\tau_1, \tau_2)$ , let  $f \in L^2((\tau_1, \tau_2) \times (0, 1))$ , and let  $u_0 \in H^{-1}(0, 1)$ . Let us recall that, with this regularity on  $\alpha$ ,  $\beta$  and  $f$ , the Cauchy problem (4.1) is well posed in  $C^0([\tau_1, \tau_2]; H^{-1}(0, 1))$ : it has one and only one solution in this set and there exists a constant  $C > 0$  independent of  $\alpha$  and  $\beta \in L^2(\tau_1, \tau_2)$ ,  $f \in L^2((\tau_1, \tau_2) \times (0, 1))$ , and  $u_0 \in H^{-1}(0, 1)$  such that

$$\|u\|_{C^0([\tau_1, \tau_2], H^{-1}(0, 1))} \leq C(\|\alpha\|_{L^2(\tau_1, \tau_2)} + \|\beta\|_{L^2(\tau_1, \tau_2)} + \|f\|_{L^2((\tau_1, \tau_2) \times (0, 1))} + \|u_0\|_{H^{-1}(0, 1)}). \quad (4.2)$$

The notion of solution to the Cauchy problem (4.1) has to be understood in the transposition sense (compare with (1.31)): a solution  $u$  of (4.1) is a function  $u$  in  $C^0([\tau_1, \tau_2]; H^{-1}(0, 1))$  such that, for every  $s \in [\tau_1, \tau_2]$  and for every  $\xi \in L^2((\tau_1, s); H^2(0, 1)) \cap H^1((\tau_1, s); H_0^1(0, 1))$  such that

$$-\xi_t = (a(x)\xi_x)_x + c(x)\xi \text{ in } L^2((\tau_1, s) \times (0, 1)), \quad (4.3)$$

one has

$$\begin{aligned} -\langle u_0, \xi(\tau_1, \cdot) \rangle_{H^{-1}, H_0^1} + \langle u(s, \cdot), \xi(s, \cdot) \rangle_{H^{-1}, H_0^1} + \int_{\tau_1}^s a(1)\beta(t)\xi_x(t, 1) dt \\ - \int_s^\tau a(0)\alpha(t)\xi_x(t, 0) dt - \int_{\tau_1}^s \int_0^1 f(t, x)\xi(t, x) dx dt = 0. \end{aligned} \quad (4.4)$$

See, e.g., [4, Definition 2.36 and Section 2.7.1] for this notion of solution, the well-posedness and (4.2). Then using a standard smoothing procedure and the classical maximum principle (see, e.g. [13, Chapter 2]) one gets the following proposition.

**Proposition 2** (Maximum principle). *Let  $\alpha$  and  $\beta \in L^2(\tau_1, \tau_2)$ , let  $f \in L^2((\tau_1, \tau_2) \times (0, 1))$ , and let  $u_0 \in H^{-1}(0, 1)$  be such that*

$$\alpha \geq 0, \beta \geq 0, f \geq 0, u_0 \geq 0. \quad (4.5)$$

then, for every  $t \in [\tau_1, \tau_2]$ ,

$$u(t, \cdot) \geq 0. \quad (4.6)$$

In the next lemma, we derive from this maximum principle an a priori estimate on the solutions of (1.30).

Let us denote by  $S(t) : L^2(0, 1) \rightarrow L^2(0, 1)$ ,  $t \geq 0$ , the semigroup generated by the operator  $v \mapsto (a(x)v_x)_x + c(x)v(t, x)$  with Dirichlet boundary condition: for  $t \geq 0$  and  $u_0 \in L^2(0, 1)$ ,  $(S(t)u_0)(x) = v(t, x)$  where  $v \in C^0([0, +\infty); L^2(0, 1))$  is the unique solution of

$$\begin{cases} v_t(t, x) = (a(x)v_x(t, x))_x + c(x)v(t, x) & \text{in } (0, +\infty) \times (0, 1), \\ v(t, 0) = v(t, 1) = 0 & \text{for } t \in (0, +\infty), \\ v(0, \cdot) = u_0. \end{cases} \quad (4.7)$$

**Lemma 5.** *Assume that  $F$  satisfies Properties  $(\mathcal{P}_1)$ ,  $(\mathcal{P}_2)$ , and  $(\mathcal{P}_3)$  and that  $|F(t, v)| \leq C\|v\|_{L^2}^{1/2}$  for  $(t, v) \in [T_1, T) \times L^2(0, 1)$  for some  $C > 0$  and for some  $0 < T_1 < T$ . Then, there exists  $C_1 > 0$  such that, for every  $u_0 \in L^2(0, 1)$ , for every  $T_1 \leq s < s' < T$  and for every solution  $u \in C^0([s, s']; L^2(0, 1))$  of*

$$\begin{cases} u_t(t, x) = (a(x)u_x(t, x))_x + c(x)u(t, x) & \text{for } (t, x) \in (s, s') \times [0, 1], \\ u(t, 0) = 0, \quad u(t, 1) = F(t, u(t, \cdot)) & \text{for } t \in (s, s'), \\ u(s, \cdot) = u_0 & \text{for } x \in [0, 1], \end{cases} \quad (4.8)$$

one has

$$\|u(t, \cdot) - S(t-s)u_0\|_{L^2} \leq C_1(t-s)^{1/4}(1 + \|u_0\|_{L^2})^{1/2} \forall T_1 \leq s \leq t \leq s'. \quad (4.9)$$

**Proof.** Let  $v(t, x)$  be the unique solution of the system

$$\begin{cases} v_t(t, x) = (a(x)v_x(t, x))_x + c(x)v(t, x) & \text{in } (s, s') \times (0, 1), \\ v(t, 0) = v(t, 1) = 0 & \text{for } t \in (s, s'), \\ v(s, \cdot) = u_0. \end{cases}$$

Set  $w(t, x) = u(t, x) - v(t, x)$ . Then

$$\begin{cases} w_t(t, x) = (a(x)w_x(t, x))_x + c(x)w(t, x) & \text{in } (s, s') \times (0, 1), \\ w(t, 0) = 0, \quad |w(t, 1)| \leq C\left(\|w(t, \cdot)\|_{L^2}^{1/2} + \|v(t, \cdot)\|_{L^2}^{1/2}\right) & \text{for } t \in (s, s'), \\ w(\tau, \cdot) = 0. \end{cases} \quad (4.10)$$

Let  $t \in (s, s']$ . Define

$$W(\tau, x) = \frac{1}{\sqrt{\tau - s + \tau_0}} e^{-\frac{(1-x+\tau_0^{1/2})^2}{\lambda(\tau-s+\tau_0)}} \text{ for } (\tau, x) \in [s, t] \times [0, 1], \quad (4.11)$$

where  $\tau_0 = t - s$ . By choosing  $\lambda > 0$  small enough, the smallness of  $\lambda$  depends only on  $a$  and  $c$ , we have

$$W_\tau(\tau, x) \leq (a(x)W_x(\tau, x))_x + c(x)W(\tau, x) \text{ in } (s, t) \times [0, 1].$$

Set

$$\mathcal{W}(\tau, x) = \hat{C}AW(\tau, x) \text{ for } (\tau, x) \in (s, t) \times (0, 1), \quad (4.12)$$

where  $\hat{C}$  is a positive constant defined later and

$$A = (t - s)^{1/2} + (t - s)^{1/2} \|u_0\|_{L^2}^{1/2}. \quad (4.13)$$

We have, for  $\tau \in (s, t)$ ,

$$\mathcal{W}(\tau, 1) = \hat{C}A(\tau - s + \tau_0)^{-1/2} e^{-\frac{\tau_0}{\lambda(\tau - s + \tau_0)}} \geq \hat{C}A[2(t - s)]^{-1/2} e^{-1/\lambda}, \quad (4.14)$$

and

$$\|\mathcal{W}(\tau, \cdot)\|_{L^2} \leq \lambda^{1/4} \hat{C}A(\tau - s + \tau_0)^{-1/4} e^{-\frac{\tau_0}{4\lambda(\tau - s + \tau_0)}} \leq \lambda^{1/4} \hat{C}A(t - s)^{-1/4} e^{-1/(4\lambda)}. \quad (4.15)$$

Here in the first inequality we used the fact that, for  $m > 0$ ,

$$\int_m^\infty e^{-y^2} dy \leq \frac{\sqrt{\pi}}{2} e^{-m^2/2}.$$

Fixing  $\lambda > 0$  small, taking  $\hat{C}$  large enough, the largeness of  $\hat{C}$  being independent of  $s, t$ , and  $u_0$ , we obtain, for  $t - s \leq e^{-7/\lambda}$ ,

$$\begin{aligned} \mathcal{W}(\tau, 1) &\geq 2C \left( \|\mathcal{W}(\tau, \cdot)\|_{L^2}^{1/2} + \|u_0\|_{L^2}^{1/2} \right) \\ &\geq 2C \left( \|\mathcal{W}(\tau, \cdot)\|_{L^2}^{1/2} + \|v(\tau, \cdot)\|_{L^2}^{1/2} \right) \text{ for } \tau \in (s, t). \end{aligned} \quad (4.16)$$

Let us check that, by the maximum principle (Proposition 2), this implies that

$$|w(t, x)| \leq \mathcal{W}(t, x) \text{ for almost every } x \in (0, 1). \quad (4.17)$$

Indeed, the maximum principle leads to (4.17) if

$$|w(\tau, 1)| \leq \mathcal{W}(\tau, 1) \forall \tau \in [s, t]. \quad (4.18)$$

Let us assume that (4.18) does not hold. Then, there exists  $t' \in (s, t)$  such that

$$|w(\tau, 1)| \leq \mathcal{W}(\tau, 1) \forall \tau \in [s, t'], \quad (4.19)$$

$$|w(t', 1)| = \mathcal{W}(t', 1). \quad (4.20)$$

From (4.19) and the maximum principle, we get that

$$|w(t', x)| \leq \mathcal{W}(t', x) \text{ for almost every } x \in (0, 1), \quad (4.21)$$

which, together with the second line of (4.10) leads to

$$|w(t', 1)| \leq C \left( \|\mathcal{W}(t', \cdot)\|_{L^2}^{1/2} + \|v(t', \cdot)\|_{L^2}^{1/2} \right). \quad (4.22)$$

From (4.16), (4.20) and (4.22), we get a contradiction, which shows that (4.18) and therefore also (4.17) hold. Finally (4.17) yields (4.9) for  $t - s \leq e^{-7/\lambda}$ . The conclusion in the general case follows immediately from this.  $\square$

Concerning the existence and uniqueness of the flow  $\Phi$ , let us prove the following lemma.

**Lemma 6.** *Assume that  $F$  satisfies Properties  $(\mathcal{P}_1)$ ,  $(\mathcal{P}_2)$ , and  $(\mathcal{P}_3)$ . Let  $0 \leq s < T$ . There exists  $T_0 = T_0(s) > 0$  such that, for every  $u_0 \in L^2(0, 1)$ , there exists a unique solution  $u \in C^0([s, s + T_0]; L^2(0, 1))$  of (1.30). Moreover,*

$$\|u(t, \cdot)\|_{L^2} \leq C\|u_0\|_{L^2} \text{ for } t \in (s, s + T_0), \quad (4.23)$$

for some positive constant  $C = C(s)$  independent of  $u_0$  and the functions  $T_0 : [0, T] \rightarrow (0, +\infty)$  and  $C : [0, T] \rightarrow [0, +\infty)$  can be chosen such that, for every  $\delta \in (0, T]$ ,

$$\inf\{T_0(s); s \in [0, T - \delta]\} > 0 \text{ and } \sup\{C(s); s \in [0, T - \delta]\} < +\infty. \quad (4.24)$$

**Proof.** Let us first deal with the uniqueness. Let  $u_1$  and  $u_2$  be two solutions having the same initial data at time  $s$  and defined at least on  $[s, s'] \times (0, 1)$ , with  $s < s'$ . Let  $w : (s, s') \times (0, 1) \rightarrow \mathbb{R}$  be defined by  $w = u_2 - u_1$ . One has, for some  $C > 0$ ,

$$\begin{cases} w_t(t, x) = (a(x)w_x(t, x))_x + c(x)w(t, x) & \text{in } (s, s') \times (0, 1), \\ w(t, 0) = 0, \quad |w(t, 1)| \leq C\|w(t, \cdot)\|_{L^2} & \text{for } t \in (s, s'), \\ w(s, \cdot) = 0, \end{cases} \quad (4.25)$$

It suffices to prove that

$$w(t, \cdot) = 0 \text{ for } t > s \text{ close enough to } s. \quad (4.26)$$

We proceed as in the proof of Lemma 5. We define  $\mathcal{W}$  as in (4.11)-(4.12), with (4.13) now replaced by

$$A = 1. \quad (4.27)$$

From (4.14) and (4.15), fixing  $\lambda > 0$  small, we obtain, for  $t - s \leq e^{-4/\lambda}$  and for every  $\hat{C} > 0$ ,

$$\mathcal{W}(\tau, 1) \geq 2C\|\mathcal{W}(\tau, \cdot)\|_{L^2} \text{ for } \tau \in (s, t). \quad (4.28)$$

(Compare with (4.16).) Then, proceeding as in the proof of (4.17), we get, using the maximum principle,

$$|w(t, x)| \leq \mathcal{W}(t, x) \text{ for almost every } x \in (0, 1). \quad (4.29)$$

However, since  $\hat{C} > 0$  is arbitrary, it follows from (4.29) that  $w(t, \cdot) = 0$ . This concludes the proof of the uniqueness.

Let us now establish the existence of  $u$  using a fixed point argument. For notational ease, we assume that  $s = 0$ . Set  $U_0 = 0$  and, for  $n \geq 1$ , let  $U_n$  be the unique solution of

$$\begin{cases} U_{n,t}(t, x) = (a(x)U_{n,x}(t, x))_x + c(x)U_n(t, x) & \text{in } (0, T) \times (0, 1), \\ U_n(t, 0) = 0, \quad U_n(t, 1) = F(t, U_{n-1}(t, \cdot)) & \text{for } t \in (0, T), \\ U_n(t = 0, \cdot) = u_0 & \text{in } L^2(0, 1). \end{cases} \quad (4.30)$$

Define

$$W_n(t, x) = U_{n+1}(t, x) - U_n(t, x) \text{ in } (0, T) \times (0, 1).$$

Let  $0 < T_0 < T$ . Then, for some  $C \in (0, +\infty)$  independent of  $n$  and  $u_0$ ,

$$\begin{cases} W_{n,t}(t, x) = (a(x)W_{n,x}(t, x))_x + c(x)W_n(t, x) & \text{in } (0, T) \times [0, 1], \\ W_n(t, 0) = 0 & \text{for } t \in (0, T), \\ |W_n(t, 1)| \leq C\|W_{n-1}(t, \cdot)\|_{L^2} & \text{for } t \in (0, T_0), \\ W_n(t = 0, \cdot) = 0 & \text{in } L^2(0, 1). \end{cases} \quad (4.31)$$

By the maximum principle (Proposition 2) applied to the function

$$(t, x) \mapsto Re^{Rt} \sup_{\tau \in (0, T_0)} \|W_{n-1}(\tau, \cdot)\|_{L^2} \pm W_n(t, x)$$

with  $R > 0$  large enough, we have

$$|W_n(t, x)| \leq M \sup_{\tau \in (0, T_0)} \|W_{n-1}(\tau, \cdot)\|_{L^2} \text{ for } (t, x) \in (0, T_0) \times (0, 1), \quad (4.32)$$

for some  $M > 0$  independent of  $n$  and  $u_0$ . For  $0 < r < 1$ , let  $\varphi \in C^1(\mathbb{R})$  be such that  $\varphi(x) = 1$  for  $x \leq 1 - r$ ,  $\varphi(x) = 0$  for  $x \geq 1 - (r/2)$  and  $|\varphi'(x)| \leq 4/r$  for  $x \in [0, 1]$ . Multiplying the equation of  $W_n$  by  $W_n(t, x)\varphi^2(x)$  and integrating by parts, we obtain

$$\begin{aligned} \int_0^1 |W_n(t, x)\varphi(x)|^2 dx + \int_0^t \int_0^1 \left| (W_n(t, x)\varphi(x))_x \right|^2 \\ \leq C \int_0^t \int_0^1 |W_n(t, x)|^2 (|\varphi_x(x)|^2 + |\varphi(x)|^2) dx dt. \end{aligned}$$

It then follows from (4.32) that, for  $0 < r < 1$ ,

$$\begin{aligned} \int_0^{1-r} |W_n(t, x)|^2 dx + \int_0^t \int_0^{1-r} |W_{n,x}(t, x)|^2 \leq \\ Cr^{-2}T_0M^2 \sup_{\tau \in (0, T_0)} \|W_{n-1}(\tau, \cdot)\|_{L^2}^2 \text{ for } t \in (0, T_0). \end{aligned} \quad (4.33)$$

A combination of (4.32) and (4.33) yields the existence of  $\bar{C}$  independent of  $r \in (0, 1)$ , of  $n$ , and of  $u_0$ , such that

$$\sup_{\tau \in (0, T_0)} \|W_n(\tau, \cdot)\|_{L^2} \leq \bar{C}M(r^{-1}T_0^{1/2} + r^{1/2}) \sup_{\tau \in (0, T_0)} \|W_{n-1}(\tau, \cdot)\|_{L^2}.$$

Let us fix  $r \in (0, 1)$  small enough so that  $\bar{C}Mr^{1/2} \leq 1/4$ . Then, by choosing  $T_0$  small enough, we obtain

$$\sup_{\tau \in (0, T_0)} \|W_n(\tau, \cdot)\|_{L^2} \leq \frac{1}{2} \sup_{\tau \in (0, T_0)} \|W_{n-1}(\tau, \cdot)\|_{L^2}. \quad (4.34)$$

Hence  $(U_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $C^0([0, T_0]; L^2(0, 1))$ . Let  $u \in C^0([0, T_0]; L^2(0, 1))$  be its limit. One can easily verify that  $u$  is a solution of (1.30) on  $(0, T_0)$ . Moreover, by (4.34),

$$\int_0^1 |u(t, x)|^2 dx \leq C\|u_0\|_{L^2}^2 \text{ for } t \in (0, T_0). \quad (4.35)$$

Finally, it follows from our proof that (4.24) can also be imposed. □

As a consequence of Lemma 6 we have the following corollary.

**Corollary 3.** *Assume that  $F$  satisfies Properties  $(\mathcal{P}_1)$ ,  $(\mathcal{P}_2)$ , and  $(\mathcal{P}_3)$ . Let  $0 \leq s' < T$ . Then there exist  $C > 0$  such that, for every  $s \in [0, s']$  and for every  $u_0 \in L^2(0, 1)$ , there exists a unique solution  $u \in C^0([s, s']; L^2(0, 1))$  of (1.30) and this solution satisfies*

$$\|u(t, \cdot)\|_{L^2} \leq C \|u_0\|_{L^2} \text{ for every } t \in [s, s']. \quad (4.36)$$

We now assume that  $F$  satisfies Properties  $(\mathcal{P}_1)$ ,  $(\mathcal{P}_2)$ , and  $(\mathcal{P}_3)$  and that (1.32) holds. It follows from Corollary 3 that the flow  $\Phi$  is well defined on  $\{(t, s, u_0) \in [0, T] \times [0, T] \times L^2(0, 1); t \geq s\}$ . In order to prove that

$$\Phi \text{ is well defined on } \{(t, s, u_0) \in \mathbb{R} \times \mathbb{R} \times L^2(0, 1); t \geq s\}, \quad (4.37)$$

it only remains to check that, for every  $u_0 \in L^2(0, 1)$  and for every  $s \in [0, T]$ ,

$$\Phi(t, s, u_0) \text{ is converging in } L^2(0, 1) \text{ as } t \rightarrow T_-. \quad (4.38)$$

Let  $u_0 \in L^2(0, 1)$  and let  $s \in [0, T]$ . Define, for  $t \in [s, T]$ ,  $u(t) = \Phi(t, s, u_0)$ . Let  $\varepsilon > 0$ . It follows from Lemma 5 and Corollary 3 that  $t \in [s, T] \mapsto \|u(t)\|_{L^2}$  is bounded. It then follows from Lemma 5, that there exists  $\delta > 0$  such that

$$\|u(t) - S(t - T + \delta)u(T - \delta)\|_{L^2} \leq \varepsilon \quad \forall t \in [T - \delta, T]. \quad (4.39)$$

Let  $\delta' \in (0, \delta)$  be such that

$$\|S(t - T + \delta)u_0 - S(\delta)u_0\|_{L^2} \leq \varepsilon \quad \forall t \in [T - \delta', T]. \quad (4.40)$$

From (4.39) and (4.40), one gets

$$\|u(t) - u(t')\|_{L^2} \leq 4\varepsilon \quad \forall t \in [T - \delta', T], \forall t' \in [T - \delta', T]. \quad (4.41)$$

This shows that, for every sequence  $(t_n)_{n \in \mathbb{N}}$  of real numbers in  $(0, T)$  converging to  $T$  as  $n \rightarrow +\infty$ , the sequence  $(u(t_n))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(0, 1)$ . This concludes the proof of (4.38) and therefore of (4.37).

## 5 Proof of Theorem 2

In this section we prove Theorem 2. Let us first point out that the uniform stability property (1.35) follows from the other conditions. Indeed, we have, by Lemma 5,

$$\|\Phi(t, s, u_0)\|_{L^2} \leq C_1 \left( \|u_0\|_{L^2} + (t - s)^{1/4} (1 + \|u_0\|_{L^2})^{1/2} \right) \quad \forall \bar{T} \leq s \leq t < T, \forall u_0 \in L^2(0, 1). \quad (5.1)$$

and, by Corollary 3,

$$\|\Phi(t, s, u_0)\|_{L^2} \leq C_1 \|u_0\|_{L^2} \quad \forall 0 \leq s \leq t < (T + \bar{T})/2, \forall u_0 \in L^2(0, 1), \quad (5.2)$$

for some positive constant  $C_1$  independent of  $u_0$ . Let  $r \geq 0$ ,  $B(r)$  denote the closed ball of radius  $r$  in  $L^2(0, 1)$ . Let  $\varepsilon > 0$ . We derive from (5.1) and (5.2) that there exists  $\delta > 0$  such that

$$\|\Phi(t, t', u_0)\|_{L^2} \leq \varepsilon \quad \forall 0 \leq t' \leq t \leq T, \forall u_0 \in B(\delta). \quad (5.3)$$

Here we also use the fact

$$\Phi(t, t', u_0) = \Phi(t, \tau, \Phi(\tau, t', u_0)) \quad \forall t' \leq \tau \leq t, \forall u_0 \in L^2(0, 1) \quad (5.4)$$

and take  $\bar{T}$  close to  $T$ . Using (5.4) again, we derive from (5.3) the existence of  $\eta > 0$  such that

$$\|\Phi(t, t', u_0)\|_{L^2} \leq \varepsilon \quad \forall 0 \leq t' \leq t \leq 3T, \forall u_0 \in B(\eta), \quad (5.5)$$

which, together with (1.25) and (1.34), gives (1.35) provided that we decrease  $\eta > 0$  if necessary so that  $\eta \leq \Gamma$ . This implies the uniform stability (1.35).

We next construct  $F$ . For  $n \in \mathbb{N}$ , let  $\lambda_n$  and  $t_n$  be defined by

$$\lambda_n = (n + 1)^8 \text{ for every } n \in \mathbb{N}, \quad (5.6)$$

$$t_0 = 0, \quad (5.7)$$

$$t_n = T \left( 1 - \frac{1}{2n^2} \right) \text{ for every } n \in \mathbb{N} \setminus \{0\}. \quad (5.8)$$

By Corollary 1, for some  $C_1$  large enough,

$$\|k_n\|_{L^2} \leq e^{C_1 \lambda_n^{1/2}} \text{ for every } n \in \mathbb{N}. \quad (5.9)$$

We also know from Proposition 1 that there exists some  $C_2 > 0$  such that, for  $t_n \leq t < t_{n+1}$ ,

$$\|u(t, \cdot)\|_{L^2} \leq C_2 e^{-s_{n-1}/4 + C_2(n-1)} \|u_0\|_{L^2}, \quad (5.10)$$

$$|U(t)| \leq C_2 e^{-s_{n-1}/4 + C_2(n-1) + C_2 \sqrt{\lambda_n}} \|u_0\|_{L^2}, \quad (5.11)$$

where  $s_n = \sum_{k=0}^{n-1} \lambda_k (t_{k+1} - t_k)$  for  $n \geq 1$  and  $s_0 = 1$ . It is easy to verify that

$$\lim_{n \rightarrow +\infty} \frac{(t_{n+1} - t_n) \lambda_n}{\sqrt{\lambda_{n+1}}} = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{s_n}{n + \sqrt{\lambda_{n+1}}} = +\infty.$$

Fix  $\alpha$  and  $\beta$  be two real numbers such that

$$4 < \alpha < \beta < 5. \quad (5.12)$$

Let  $(\mu_n)_{n \in \mathbb{N}}$  and  $(\nu_n)_{n \in \mathbb{N}}$  be defined by

$$\mu_n := e^{-n^\alpha} \text{ and } \nu_n := e^{-n^\beta} \quad \forall n \in \mathbb{N}. \quad (5.13)$$



For  $n \in \mathbb{N}$ , we choose a function  $\varphi_n \in C^1(\mathbb{R})$  such that  $0 \leq \varphi_n \leq 1$ ,  $\varphi_n(s) = 1$  for  $s \leq \mu_n$  and  $\varphi_n(s) = 0$  if  $s \geq 2\mu_n$ . Fix  $N$  a large (see below) positive integer. We define  $F$  in the following way for  $t_n \leq t < t_{n+1}$ ,

$$F(t, v) := \int_0^1 k_n(1, y)v(y) dy \quad \forall t \in [t_n, t_{n+1}) \text{ with } n \leq N-1, \quad (5.14)$$

$$F(t, v) := \varphi_n(\|v\|_{L^2}) \int_0^1 k_n(1, y)v(y) dy \quad \forall t \in [t_n, t_{n+1}) \text{ with } n \geq N. \quad (5.15)$$

We derive from (5.9) that, if  $N$  is large enough, which is always assumed from now on,

$$\|k_n\|_{L^2} \leq e^{C_1 \lambda_n^{1/2}} \leq (1/(2\mu_n))^{1/2} \quad \forall n \geq N. \quad (5.16)$$

It follows that, for  $t_n < t < t_{n+1}$  with  $n \geq N$ ,

$$|F(t, v)| \leq 2\|v\|_{L^2}^{1/2},$$

which gives (1.32). From (the proof of) Proposition 1 we get that

$$\Phi(T, t, u_0) = 0 \quad \forall u_0 \in B(\nu_n), \forall t \in [t_{N-1}, t_N], \quad (5.17)$$

$$\Phi(T, t, u_0) = 0 \quad \forall t \in [t_n, t_{n+1}) \text{ with } n \leq N-2, \forall u_0 \in B(1/\nu_n). \quad (5.18)$$

Let  $\Gamma$  be a positive real number. It follows from (5.18) that

$$\Phi(T, t, u_0) = 0 \quad \forall t \in [t_n, t_{n+1}) \text{ with } n \leq N-2, \forall u_0 \in B(\Gamma). \quad (5.19)$$

(Let us recall that we always assume  $N$  large enough and how large  $N$  is depends now on  $\Gamma$ .) Again, from the proof of Proposition 1, we have

$$\|\Phi(t_N, t, u_0)\|_{L^2} \leq \frac{1}{\mu_n} \forall u_0 \in B(\Gamma), \forall t \in [t_{N-1}, t_N]. \quad (5.20)$$

From (1.25), (5.1), and (5.18), we get that

$$\Phi(2T, t, u_0) = \Phi(2T, T, \Phi(T, t, u_0)) = 0 \quad \forall t \in [t_N, T], \forall u_0 \in B(1/\mu_n), \quad (5.21)$$

which, together with (5.19) and (5.20), concludes the proof of Theorem 2.  $\square$

**Conflict of Interest:** The authors declare that they have no conflict of interest.

## References

- [1] N. Bekiaris-Liberis and M. Krstic, *Compensating the distributed effect of a wave PDE in the actuation or sensing path of multi-input and MIMO LTI systems*, Systems Control Lett. **59** (2010), 713–719.

- [2] N. Bekiaris-Liberis and M. Krstic, *Compensating the distributed effect of diffusion and counter-convection in multi-input and multi-output LTI systems*, IEEE Trans. Automat. Control **56** (2011), 637–643.
- [3] E. Cerpa and J.-M. Coron, *Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition*, IEEE Trans. Automat. Control **58** (2013), 1688–1695.
- [4] J.-M. Coron, *Control and nonlinearity*, vol. 136 of Mathematical Surveys and Monographs, American Mathematical Soc., Providence, RI, 2007.
- [5] J.-M. Coron and B. d’Andréa-Novel, *Stabilization of a rotating body beam without damping*, IEEE Trans. Automat. Control **43** (1998), 608–618.
- [6] J.-M. Coron and Q. Lü, *Local rapid stabilization for a Korteweg–de Vries equation with a Neumann boundary control on the right*, J. Math. Pures Appl. **102** (2014), 1080–1120.
- [7] J.-M. Coron and Q. Lü, *Fredholm transform and local rapid stabilization for a Kuramoto–Sivashinsky equation*, J. Diff. Equations **259** (2015), 3683–3729.
- [8] J.-M. Coron, R. Vazquez, M. Krstic, and G. Bastin, *Local Exponential  $H^2$  Stabilization of a  $2 \times 2$  Quasilinear Hyperbolic System Using Backstepping*, SIAM J. Control Optim. **51** (2013), 2005–2035.
- [9] F. Di Meglio, R. Vazquez, and M. Krstic, *Stabilization of a system of  $n + 1$  coupled first-order hyperbolic linear PDEs with a single boundary input*, IEEE Trans. Automat. Control **58** (2013), 3097–3111.
- [10] A. Elharfi *Explicit construction of a boundary feedback law to stabilize a class of parabolic equations*, Differential and Integral Equations **21** (2008), 351–362.
- [11] L. Escauriaza and G. Alessandrini, *Null-controllability of one-dimensional parabolic equations*, ESAIM Control Optim. Calc. Var. **14** (2008), 284–293.
- [12] O. H. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. **43** (1971), 272–292.
- [13] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
- [14] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*, vol. 34, Seoul National University, Seoul, 1996.
- [15] Y.-J. L. Guo and W. Littman, *Null boundary controllability for semilinear heat equations*, Appl. Math. Optim. **32** (1995), 281–316.
- [16] L. Hu and F. Di Meglio, *Finite-time backstepping stabilization of  $3 \times 3$  hyperbolic systems*, IEEE European Control Conference, under review, 2015.
- [17] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic, *Boundary control design of homodirectional and general heterodirectional linear hyperbolic PDEs*, Preprint, 2015.

- [18] B. F. Jones, *A fundamental solution for the heat equation which is supported in a strip*, J. Math. Anal. Appl. **60** (1977), 314–324.
- [19] M. Krstic, B. Z. Guo, A. Balogh, and A. Smyshlyaev, *Output-feedback stabilization of an unstable wave equation*, Automatica **44** (2008), 63–74.
- [20] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and adaptive control design*, John Wiley & Sons, Inc., New York, 1995.
- [21] M. Krstic and A. Smyshlyaev, *Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays*, Systems Control Lett. **57** (2008), 750–758.
- [22] M. Krstic and A. Smyshlyaev, *Boundary control of PDEs: A course on backstepping designs*, vol. 16, SIAM, Philadelphia, 2008.
- [23] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), 335–356.
- [24] W. Littman, *Boundary control theory for hyperbolic and parabolic partial differential equations with constant coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **5** (1978), 567–580.
- [25] W.-J. Liu, *Boundary feedback stabilization of an unstable heat equation*, SIAM J. Control Optim., **42** (2003), 1033–1043.
- [26] W.-J. Liu and M. Krstic, *Backstepping boundary control of Burgers's equation with actuator dynamics*, Systems Control Lett. **41** (2000), 291–303.
- [27] W.-J. Liu and M. Krstic, *Stability enhancement by boundary control in the Kuramoto-Sivashinsky equation*, Nonlinear Anal. Ser. A: Theory Methods **43** (2001), 485–507.
- [28] P. Martin, L. Rosier, and P. Rouchon, *Null controllability of the heat equation using flatness*, Automatica **50** (2014), 3067–3076.
- [29] L. Miller, *The control transmutation method and the cost of fast controls*, SIAM J. Control Optim. **45** (2006), 762–772.
- [30] M. Reissig, *Hyperbolic equations with non-Lipschitz coefficients*, Rend. Sem. Mat. Univ. Politec. Torino **61** (2003), 135–181.
- [31] A. Smyshlyaev, E. Cerpa, and M. Krstic, *Boundary stabilization of a 1-D wave equation with in-domain antidamping*, SIAM J. Control Optim. **48** (2010), 4014–4031.
- [32] A. Smyshlyaev and M. Krstic, *Closed-form boundary state feedbacks for a class of 1 – D partial integro-differential equations*, IEEE Trans. Automat. Control **49** (2004), 2185–2202.
- [33] A. Smyshlyaev and M. Krstic, *On control design for PDEs with space-dependent diffusivity or time-dependent reactivity*, Automatica **41** (2005), 1601–1608.

- [34] A. Smyshlyaev and M. Krstic, *Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary*, Systems Control Lett. **58** (2009), 617–623.
- [35] R. Vazquez and M. Krstic, *Control of 1-D parabolic PDEs with Volterra nonlinearities, part I: design*, Automatica **44** (2008), 2778–2790.