



HAL
open science

The tail empirical process of regularly varying functions of geometrically ergodic Markov chains

Rafal Kulik, Philippe Soulier, Olivier Wintenberger

► **To cite this version:**

Rafal Kulik, Philippe Soulier, Olivier Wintenberger. The tail empirical process of regularly varying functions of geometrically ergodic Markov chains. *Stochastic Processes and their Applications*, 2019, 129 (11), pp.4209-4238. 10.1016/j.spa.2018.11.014 . hal-01228825v2

HAL Id: hal-01228825

<https://hal.science/hal-01228825v2>

Submitted on 21 Sep 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The tail empirical process of regularly varying functions of geometrically ergodic Markov chains

Rafał Kulik* Philippe Soulier† Olivier Wintenberger‡

September 21, 2018

Abstract

We consider a stationary regularly varying time series which can be expressed as a function of a geometrically ergodic Markov chain. We obtain practical conditions for the weak convergence of the tail array sums and feasible estimators of cluster statistics. These conditions include the so-called geometric drift or Foster-Lyapunov condition and can be easily checked for most usual time series models with a Markovian structure. We illustrate these conditions on several models and statistical applications. A counterexample is given to show a different limiting behavior when the geometric drift condition is not fulfilled.

1 Introduction

Let $\{X_j, j \in \mathbb{Z}\}$ be a stationary, regularly varying univariate time series with marginal distribution function F and tail index $\alpha > 0$. This means that for each integer $h \geq 0$, there exists a non zero Radon measure $\nu_{0,h}$ on $\overline{\mathbb{R}}^{h+1} \setminus \{\mathbf{0}\}$ such that $\nu_{0,h}(\overline{\mathbb{R}}^{h+1} \setminus \mathbb{R}^{h+1}) = 0$ and

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}((X_0, \dots, X_h) \in xA)}{\mathbb{P}(X_0 > x)} = \nu_{0,h}(A), \quad (1.1)$$

for all relatively compact sets $A \subset \overline{\mathbb{R}}^{h+1} \setminus \{\mathbf{0}\}$ satisfying $\nu_{0,h}(\partial A) = 0$. The measure $\nu_{0,h}$, called the exponent measure of (X_0, \dots, X_h) , is homogeneous with index $-\alpha$, i.e.

*Department of Mathematics and Statistics, University of Ottawa, K1N 6N5 Ottawa, Ontario Canada

†Université Paris Nanterre, Département de mathématique et informatique, Laboratoire MODAL'X, 200 avenue de la République, 92000 Nanterre, France

‡Sorbonne Universités, LSTA, 4 place Jussieu, BP 158 75005 Paris, FRANCE

$\nu_{0,h}(tA) = t^{-\alpha}\nu_{0,h}(A)$. The choice of the denominator $\mathbb{P}(X_0 > x)$ entails that $\nu_{0,h}((1, \infty) \times \mathbb{R}^h) = 1$ i.e. that the right tail of the stationary distribution is not trivial and that X_0 satisfies the so-called balanced tail condition.

Many statistical methods for extreme value characteristics of a time series are based on tail array sums of the form

$$M_n(\phi) = \frac{1}{n\bar{F}(u_n)} \sum_{j=1}^n \phi((X_j, \dots, X_{j+h})/u_n), \quad (1.2)$$

for a fixed non negative integer h , a non decreasing sequence $\{u_n\}$ such that $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} n\bar{F}(u_n) = \infty$ and a measurable function ϕ on \mathbb{R}^{h+1} such that $\mathbb{E}[|\phi((X_0, \dots, X_h)/u_n)|] < \infty$ for all n . An important example is the tail empirical distribution function, defined by

$$\tilde{T}_n(s) = \frac{1}{n\bar{F}(u_n)} \sum_{j=1}^n 1\{X_j > u_n s\}, \quad (1.3)$$

which is used for the estimation of univariate extremal characteristics such as the tail index. We are interested in the weak convergence of the centered and renormalized process

$$\mathbb{M}_n(\phi) = \sqrt{n\bar{F}(u_n)} \{M_n(\phi) - \mathbb{E}[M_n(\phi)]\}. \quad (1.4)$$

There is a huge literature on this problem. Essential references are [Rootzén et al. \(1998\)](#) which investigate weak convergence of tail array sums, [Drees \(1998, 2000, 2002, 2003\)](#) who developed techniques to study tail empirical and tail quantile processes for β -mixing time series, [Rootzén \(2009\)](#) which reviews the results for the functional convergence of the tail empirical process in the case of i.i.d. and weakly dependent (strong or β -mixing) univariate time series. Recently, [Drees and Rootzén \(2010\)](#) investigated the weak convergence of $\{\mathbb{M}_n(\phi), \phi \in \mathcal{G}\}$ as a sequence of random elements in the space $\ell_\infty(\mathcal{G})$, where \mathcal{G} is a class of functions. [Drees et al. \(2014\)](#) applied the latter reference to the estimation of the empirical distribution function of the spectral tail process (see definition in [Example 2.10](#)).

In all these references, results are proved under strong or β -mixing conditions and under additional assumptions guaranteeing the existence of the limiting variances and tightness in the case of functional convergence. These additional conditions are notably hard to check and have been verified for a handful of particular models such as solutions of stochastic recurrence equations (including some GARCH processes) and certain linear processes such as AR(1) processes. See e.g. [Drees \(2000, 2003\)](#) and [Davis and Mikosch \(2009\)](#).

The main purpose of this paper is to show that for time series which can be expressed as functions of an underlying Markov chain, these results can be proved under essentially a single, easily checked condition, namely the geometric or Foster-Lyapunov drift condition. See ([Meyn and Tweedie, 2009](#), Chapter 15). In order to use it in the context of extreme value theory, the drift function must have an additional homogeneity property.

This geometric drift condition was first used in the context of extreme value theory by [Roberts et al. \(2006\)](#) who used it to prove that the extremal index is positive. It was later used by [Mikosch and Wintenberger \(2013\)](#) to obtain large deviations and weak convergence to stable laws for heavy tailed functions of a Markov chain.

The main result of this paper is Theorem 2.3 in Section 2.1 which proves the joint asymptotic normality of tail array sums of the form (1.2) for functions of irreducible Markov chains which satisfy the geometric drift condition. In the first place, irreducibility and the drift condition imply that the time series is β -mixing with geometric decay of the β -mixing coefficients. Then, it is possible to use the regenerative properties of irreducible Markov chains to obtain the bounds and other ingredients needed to prove the central limit theorem, such as the existence of the limiting variance and asymptotic negligibility. The link between this condition and those used in the literature will be discussed at the end of Section 2.1. In Section 2.2, we will strengthen the finite dimensional convergence to functional convergence over classes of functions. Such a strengthening is needed for statistical applications.

The geometric drift condition is well known and has been established for many models in the Markov chain literature, but in order to be useful for extreme value problems, the drift function must have some homogeneity properties. In Section 2.3, we will provide a practical method to obtain a suitable drift function.

The geometric decay of the β -mixing coefficient is actually not an essential ingredient of the proof of our results. However, as illustrated in Section 2.4, there are examples of non geometrically ergodic Markov chains for which the centered and normalized tail empirical process has a non Gaussian limit, the normalization being different from the usual $\sqrt{nF(u_n)}$. Thus the geometric drift condition cannot be relaxed easily and it is the subject of further research to find practical sufficient conditions for non geometrically ergodic Markov chains.

The rest of the paper is organized as follows. Section 2 contains our main results on functions of Markov chains, examples and the aforementioned counterexample. Section 3 contains a central limit theorem for tail array sums and Section 4 contains the proof of the results of Section 2.

2 Main results for functions of Markov chains

Our context is a slight extension of the one in [Mikosch and Wintenberger \(2013\)](#). We now assume that $\{X_j, j \in \mathbb{N}\}$ is a function of a stationary Markov chain $\{\mathbb{Y}_j, j \in \mathbb{N}\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in a measurable space (E, \mathcal{E}) . That is, there exists a measurable real valued function $g : E \rightarrow \mathbb{R}$ such that $X_j = g(\mathbb{Y}_j)$.

Assumption 2.1. (i) *The Markov chain $\{\mathbb{Y}_j, j \in \mathbb{Z}\}$ is strictly stationary under \mathbb{P} .*

(ii) The sequence $\{X_j = g(\mathbb{Y}_j), j \in \mathbb{Z}\}$ is regularly varying with tail index $\alpha > 0$.

(iii) There exist a measurable function $V : E \rightarrow [1, \infty)$, $\gamma \in (0, 1)$ and $b > 0$ such that for all $y \in E$,

$$\mathbb{E}[V(\mathbb{Y}_1) \mid \mathbb{Y}_0 = y] \leq \gamma V(y) + b. \quad (2.1)$$

(iv) There exist an integer $m \geq 1$ and $x_0 \geq 1$ such that for all $x \geq x_0$, there exists a probability measure ν on (E, \mathcal{E}) and $\epsilon > 0$ such that, for all $y \in \{V \leq x\}$ and all measurable sets $B \in \mathcal{E}$,

$$\mathbb{P}(\mathbb{Y}_m \in B \mid \mathbb{Y}_0 = y) \geq \epsilon \nu(B). \quad (2.2)$$

(v) There exist $q_0 \in (0, \alpha)$ and a constant $c > 0$ such that

$$|g|^{q_0} \leq cV. \quad (2.3)$$

(vi) For every $s > 0$,

$$\limsup_{n \rightarrow \infty} \frac{1}{u_n^{q_0} \bar{F}(u_n)} \mathbb{E}[V(\mathbb{Y}_0) 1_{\{|g(\mathbb{Y}_0)| > u_n s\}}] < \infty. \quad (2.4)$$

Under Assumption 2.1, it is well known that the chain $\{\mathbb{Y}_j\}$ is irreducible and geometrically ergodic and $\mathbb{E}[V(\mathbb{Y}_0)] < \infty$. This implies that the chain $\{\mathbb{Y}_j\}$ and the sequence $\{X_j\}$ are β -mixing and there exists $c > 0$ such that $\beta_n = O(e^{-cn})$, where $\{\beta_n, n \geq 1\}$ is the β -mixing coefficients sequence; see (Bradley, 2005, Theorem 3.7). This is a very strong requirement. However, it is satisfied by many time series models. We will provide in Section 2.3 a fairly general methodology to check Assumption 2.1.

2.1 Convergence of tail arrays sums

Throughout the paper we will write $\mathbf{x}_{a,b}$ for (x_a, \dots, x_b) , $a \leq b \in \mathbb{Z}$, for any sequence $\mathbf{x} = (x_j)_{j \in \mathbb{Z}}$. For $q \geq 0$, let \mathcal{L}_q be the space of measurable functions ϕ defined on \mathbb{R}^{h+1} such that

- (i) there exists a constant $\epsilon > 0$ such that $|\phi(\mathbf{x})| \leq \epsilon^{-1}(|\mathbf{x}|^q \vee 1) 1_{\{|\mathbf{x}| > \epsilon\}}$ for $\mathbf{x} \in \mathbb{R}^{h+1}$;
- (ii) for all $j \geq 0$, the function $\mathbf{x}_{0,j+h} \mapsto \phi(\mathbf{x}_{j,j+h})$ is almost surely continuous with respect to $\nu_{0,j+h}$.

Note that $\mathcal{L}_q \subset \mathcal{L}_{q'}$ if $q \leq q'$. As an important example, the continuity condition is satisfied for functions of the form $\mathbf{x} \mapsto \psi(\mathbf{x}) 1_{(-\infty, \mathbf{u}]^c}(\mathbf{x})$, for $\mathbf{u} \in [\mathbf{0}, \infty)^{h+1} \setminus \{\mathbf{0}\}$ and a continuous function ψ , because of the homogeneity of the exponent measures.

Lemma 2.2. *Under Assumption 2.1, for $\phi \in \mathcal{L}_{q_0/2}$,*

$$\int_{\mathbb{R}^{h+1}} \phi^2(\mathbf{x}) \nu_{0,h}(\mathrm{d}\mathbf{x}) + \sum_{j=1}^{\infty} \int_{\mathbb{R}^{j+h+1}} |\phi(\mathbf{x}_{0,h}) \phi(\mathbf{x}_{j,j+h})| \nu_{0,j+h}(\mathrm{d}\mathbf{x}) < \infty . \quad (2.5)$$

This lemma will be proved in Section 3.1. Therefore, for $\phi, \phi' \in \mathcal{L}_{q_0/2}$, we can define

$$\sigma^2(\phi) = \int_{\mathbb{R}^{h+1}} \phi^2(\mathbf{x}) \nu_{0,h}(\mathrm{d}\mathbf{x}) + 2 \sum_{j=1}^{\infty} \int_{\mathbb{R}^{j+h+1}} \phi(\mathbf{x}_{0,h}) \phi(\mathbf{x}_{j,j+h}) \nu_{0,j+h}(\mathrm{d}\mathbf{x}) , \quad (2.6)$$

$$C(\phi, \phi') = \frac{1}{2} \{ \sigma^2(\phi + \phi') - \sigma^2(\phi) - \sigma^2(\phi') \} . \quad (2.7)$$

Let \mathbb{M} be a Gaussian process indexed by $\mathcal{L}_{q_0/2}$ with covariance function C . Our main result is the finite dimensional convergence of the sequence of the processes \mathbb{M}_n defined in (1.4) and indexed by \mathcal{L}_q for $q < q_0/2$. The proof is given in Section 4.2.

Theorem 2.3. *Let Assumption 2.1 hold and let $\{u_n\}$ be an increasing sequence such that*

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} n \bar{F}(u_n) = +\infty \quad (2.8)$$

and there exists $\eta > 0$ such that

$$\lim_{n \rightarrow \infty} \log^{1+\eta}(n) \bar{F}(u_n) = 0 . \quad (2.9)$$

Assume moreover that either $q = 0$ or there exists $\delta > 0$ such that $q(2 + \delta) \leq q_0$ and

$$\lim_{n \rightarrow \infty} \frac{\log^{1+\eta}(n)}{\{n \bar{F}(u_n)\}^{\delta/2}} = 0 . \quad (2.10)$$

Then $\mathbb{M}_n \xrightarrow{\text{fidi}} \mathbb{M}$ on \mathcal{L}_q .

Comments on the assumptions

Theorem 2.3 is obtained under Assumption 2.1 and the very mild restrictions (2.8), (2.9) and (2.10) on the choice of the sequence u_n . This simplicity is due to the Markovian assumption. In the literature on the fidi convergence of tail array sums for general mixing time series, assumptions are usually more involved. We briefly review some of them.

In the first place note that the geometric drift condition implies that the β -mixing coefficients decay geometrically fast. This allows to apply the blocking method for the proof with blocks of size r_n of order $\log^{1+\eta}(n)$. Even though β -mixing is restrictive, it is a commonly made assumption and conditions which ensure for β mixing are well known. Geometric ergodicity may also be considered restrictive since it excludes many Markov

chains, but as will be illustrated in Section 2.4, non geometrically ergodic Markov chains may have a non standard extremal behaviour and in particular may have a vanishing extremal index. See e.g. Roberts et al. (2006).

In the literature, the convergence of the variance of the sum within one block is often assumed: e.g. (Rootzén et al., 1998, Theorem 4.1) or Rootzén (2009). For the tail empirical process (1.3) and other bounded functions which vanish in a neighborhood of zero, the following condition, introduced by Smith (1992), has been used:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{j=m}^{r_n} \mathbb{P}(X_j > u_n s \mid X_0 > u_n s) = 0. \quad (2.11)$$

See for instance Drees et al. (2014) whose Condition C is equivalent to (2.11). For sums of unbounded functions (which vanish in a neighborhood of zero), ad hoc conditions are usually assumed to ensure convergence of the block variance and the Lindeberg asymptotic negligibility condition for the central limit theorem; see for instance (Leadbetter et al., 1988, Condition 4.2) and (Drees and Rootzén, 2010, Condition 3.15). In this paper, we consider an extension of (2.11) to unbounded functions. We will prove in Lemma 4.3 that if $\phi \in \mathcal{L}_q$ for $q \leq q_0/2$, then Assumption 2.1 yields

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{j=m}^{r_n} \frac{\mathbb{E}[|\phi((X_0, \dots, X_h)/u_n)| |\phi((X_j, \dots, X_{j+h})/u_n)|]}{\bar{F}(u_n)} = 0. \quad (2.12)$$

This property in turn will allow to prove convergence of the variance and also some technical conditions related to tightness in the functional central limit theorem. Again, this is a consequence of geometric ergodicity which may seem to be a high price to pay, but on the other hand the condition (2.11) does not hold for the example of Section 2.4.

2.2 Statistical applications

For statistical purposes, we will consider the process \mathbb{M}_n indexed by a class \mathcal{G} of functions and convergence of \mathbb{M}_n to \mathbb{M} must be strengthened to weak convergence in $\ell_\infty(\mathcal{G})$, in particular in order to replace the deterministic threshold u_n by an appropriate sequence of order statistics. The general theory of weak convergence in $\ell_\infty(\mathcal{G})$ is developed in van der Vaart and Wellner (1996) and Giné and Nickl (2016) and was adapted in full generality in the context of cluster statistics in Drees and Rootzén (2010). We give here an illustration adapted from (van der Vaart and Wellner, 1996, Theorem 2.11.1). We state it under the simplifying assumption of linear ordering since it is sufficient for the forthcoming examples. More sophisticated examples can be treated as in Drees and Rootzén (2010) using entropy conditions but are beyond the scope of this paper.

We say that a class \mathcal{G} of functions is pointwise separable (cf. (van der Vaart and Wellner, 1996, Section 2.3.3)) if there exists a countable subclass $\mathcal{G}_0 \subset \mathcal{G}$ such that every $g \in \mathcal{G}$ is the pointwise limit of a sequence in \mathcal{G}_0 .

Let ρ_h be the pseudometric defined on \mathcal{L}_q by

$$\rho_h^2(\phi, \psi) = \nu_{0,h}((\phi - \psi)^2).$$

Note that ρ_h is well defined under the assumptions of Theorem 2.3 which imply $q < q_0/2$.

Theorem 2.4. *Let the assumptions of Theorem 2.3 hold and let $\mathcal{G} \subset \mathcal{L}_q$. Assume moreover that*

- (i) \mathcal{G} is pointwise separable and linearly ordered;
- (ii) the envelope function $\Phi_{\mathcal{G}} = \sup_{\phi \in \mathcal{G}} |\phi|$ is in \mathcal{L}_q ;
- (iii) (\mathcal{G}, ρ_h) is totally bounded;
- (iv) for every sequence $\{\delta_n\}$ which decreases to zero,

$$\limsup_{n \rightarrow \infty} \sup_{\substack{\phi, \psi \in \mathcal{G} \\ \rho_h(\phi, \psi) \leq \delta_n}} \frac{\mathbb{E}[\{\phi(u_n^{-1} \mathbf{X}_{0,h}) - \psi(u_n^{-1} \mathbf{X}_{0,h})\}^2]}{\bar{F}(u_n)} = 0. \quad (2.13)$$

Then $\mathbb{M}_n \Rightarrow \mathbb{M}$ in $\ell_\infty(\mathcal{G})$.

The proof is in Section 4.3.

In order to obtain convenient expressions for the limiting variances, we consider the tail process $\{Y_j, j \in \mathbb{Z}\}$, introduced in Basrak and Segers (2009) and defined as the weak limit (in the sense of finite dimensional distributions) of the sequence $\{X_j/x, j \in \mathbb{Z}\}$ given that $|X_0| > x$, as $x \rightarrow \infty$: for $i \leq j \in \mathbb{Z}$,

$$\mathbb{P}((Y_i, \dots, Y_j) \in \cdot) = \lim_{x \rightarrow \infty} \mathbb{P}(x^{-1}(X_i, \dots, X_j) \in \cdot \mid |X_0| > x). \quad (2.14)$$

Then $|Y_0|$ is a Pareto random variable with tail index α . The spectral tail process $\{\Theta_j, j \in \mathbb{Z}\}$ is then defined as $\Theta_j = |Y_0|^{-1}Y_j$.

As a first corollary of Theorem 2.4, we obtain functional convergence of the univariate tail empirical process. Note that contrary to most of the related literature, we do not have additional conditions to ensure existence of the limiting variance or tightness. Geometric ergodicity of the underlying Markov chain is sufficient.

Corollary 2.5. *Let the assumptions of Theorem 2.3 hold. Let $0 < s_0 < 1 < t_0$ and assume moreover that*

$$\lim_{n \rightarrow \infty} \sqrt{n \bar{F}(u_n)} \sup_{s_0 \leq s \leq t_0} \left| \frac{\bar{F}(u_n s)}{\bar{F}(u_n)} - s^{-\alpha} \right| = 0. \quad (2.15)$$

Then,

$$\sqrt{n\bar{F}(u_n)} \left\{ \frac{1}{n\bar{F}(u_n)} \sum_{j=1}^n 1\{X_j > u_n s\} - s^{-\alpha} \right\} \Rightarrow \mathbb{W}, \quad (2.16)$$

in $\ell_\infty([s_0, t_0])$, where \mathbb{W} is a Gaussian process with covariance function

$$\text{cov}(\mathbb{W}(s), \mathbb{W}(t)) = (s \vee t)^{-\alpha} + \sum_{j=1}^{\infty} \mathbb{E}[\{(\Theta_j/t) \wedge (1/s)\}_+^\alpha + \{(\Theta_j/s) \wedge (1/t)\}_+^\alpha \mid \Theta_0 = 1].$$

Assume that F is continuous and let $k = k(n)$ be an intermediate sequence of integers, that is $\lim_{n \rightarrow \infty} k = \lim_{n \rightarrow \infty} n/k = \infty$. Assume that the sequence u_n is such that $k = n\bar{F}(u_n)$. Let $X_{n:1} \leq \dots \leq X_{n:n}$ be the increasing order statistics. Consequently, by Vervaat's lemma (Resnick (2007, Proposition 3.3)) we obtain that $X_{n:n-k}/u_n \xrightarrow{P} 1$. For statistical applications, the deterministic threshold u_n will be replaced by $X_{n:n-k}$. Define the processes $\widehat{\mathbb{M}}_n$ and $\widehat{\mathbb{M}}$ on \mathcal{L}_q by

$$\begin{aligned} \widehat{\mathbb{M}}_n(\phi) &= \sqrt{k} \left\{ \frac{1}{k} \sum_{i=1}^n \phi(X_{n:n-k}^{-1} \mathbf{X}_{j,j+h}) - \boldsymbol{\nu}_{0,h}(\phi) \right\}, \\ \widehat{\mathbb{M}}(\phi) &= \mathbb{M}(\phi) - \boldsymbol{\nu}_{0,h}(\phi) \mathbb{M}(1_{(1,\infty) \times \mathbb{R}^h}). \end{aligned}$$

Corollary 2.6. *Let the assumptions of Theorem 2.3 and (2.15) hold. Let $0 < s_0 < 1 < t_0$ and let $\mathcal{G}_0 \subset \mathcal{L}_q$. Define $\mathcal{G} = \{\phi_s, \phi \in \mathcal{G}_0, s \in [s_0, t_0]\}$ with $\phi_s(\mathbf{x}) = \phi(\mathbf{x}/s)$. If \mathcal{G} satisfies the assumptions of Theorem 2.4 and*

$$\lim_{n \rightarrow \infty} \sqrt{k} \sup_{s_0 \leq s \leq t_0} \sup_{\phi \in \mathcal{G}} \left| \frac{\mathbb{E}[\phi(\mathbf{X}_{0,h}/(u_n s))]}{\bar{F}(u_n)} - s^{-\alpha} \boldsymbol{\nu}_{0,h}(\phi) \right| = 0, \quad (2.17)$$

then $\widehat{\mathbb{M}}_n \Rightarrow \widehat{\mathbb{M}}$ on $\ell_\infty(\mathcal{G}_0)$.

Remark 2.7. The term $\boldsymbol{\nu}_{0,h}(\phi) \mathbb{M}(1_{(1,\infty) \times \mathbb{R}^h})$ is an effect of the random threshold. Conditions (2.15) and (2.17) allow to get rid of the bias terms. These conditions are certainly fulfilled for some choices of k but they might be in conflict with (2.10) if the convergence in the definition of regular variation is very slow. However, conditions (2.15) and (2.17) are generally obtained by means of so-called second order conditions which yield polynomial rates of convergence. We do not pursue in this direction, nor in the very important practical issue of a data-driven choice of k , both problems being largely beyond the scope of this paper.

The proof of Corollaries 2.5 and 2.6 is in Section 4.4. We now give several examples.

Example 2.8 (Estimation of the (cluster) large deviation index). Assume for simplicity that the random variables X_j are nonnegative. For $A_h = \{x_0 + \dots + x_h > 1\}$ we consider the

following quantity which exists by regular variation:

$$b_{+,h} = \frac{1}{h+1} \lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_0 + \cdots + X_h > x)}{\bar{F}(x)} = \frac{1}{h+1} \nu_{0,h}(A_h) = \frac{1}{h+1} \int_{A_h} \nu_{0,h}(\mathrm{d}x) .$$

If $\{X_j\}$ is a sequence of i.i.d. regularly varying nonnegative random variables, then $b_{+,h} = 1$ for all h . In general, regular variation implies that $(h+1)^{-1} \leq b_{+,h} \leq (h+1)^\alpha$. It is shown in Mikosch and Wintenberger (2014) that the drift condition (2.1) implies that

$$b_+ = \lim_{h \rightarrow \infty} b_{+,h} \in [0, \infty) .$$

The quantity b_+ thus defined is called the cluster index and is related to the large deviation behavior of the partial sums $\sum_{j=1}^n X_j$; see Mikosch and Wintenberger (2014). No estimators of b_+ have been provided yet in the literature. Here we consider an estimator of $b_{+,h}$. Define

$$\widehat{b}_{n,+,h} = \frac{1}{k(h+1)} \sum_{j=1}^n 1\{X_j + \cdots + X_{j+h} > X_{n:n-k}\} .$$

With the notation of Corollary 2.6, $b_{+,h} = \nu_{0,h}(\phi)$ with $\phi(\mathbf{x}) = (h+1)^{-1} 1\{x_0 + \cdots + x_h > 1\}$. The class \mathcal{G}_0 consists of the single function ϕ and $\mathcal{G} = \{\phi_s, s \in [s_0, t_0]\}$. The class \mathcal{G} is pointwise separable, linearly ordered and its envelope function is ϕ_{s_0} which belongs to \mathcal{L}_0 . Conditions (iii) and (iv) of Theorem 2.4 are checked in Section 4.4.1, while (2.17) becomes

$$\lim_{n \rightarrow \infty} \sqrt{k} \sup_{s \geq s_0} \left| \frac{\mathbb{P}(X_0 + \cdots + X_h > u_n s)}{\bar{F}(u_n)} - s^{-\alpha} \nu_{0,h}(A_h) \right| = 0 . \quad (2.18)$$

Therefore, under the assumptions of Theorem 2.3 with $q = 0$ and (2.18), we can apply Corollary 2.6 and we obtain

$$\sqrt{k}(\widehat{b}_{n,+,h} - b_{+,h}) \xrightarrow{\mathrm{d}} N(0, \sigma_{+,h}^2) ,$$

with

$$\begin{aligned} \sigma_{+,h}^2 &= b_{+,h} \{b_{+,h} - 1/(h+1)\} + 2 \sum_{j=1}^{\infty} \left\{ \sum_{i=0}^h (h+1)^{-\alpha-2} \times \right. \\ &\quad \mathbb{P}(Y_{-i} \leq 1, \dots, Y_{-1} \leq 1, Y_{-i} + \cdots + Y_{h-i} > h+1, Y_{j-i} + \cdots + Y_{j+h-i} > h+1) \\ &\quad \left. - \frac{b_{+,h}}{h+1} \{ \mathbb{P}(Y_j + \cdots + Y_{j+h} > 1) + \mathbb{P}(Y_{-j} + \cdots + Y_{-j+h} > 1) \} + (b_{+,h})^2 \mathbb{P}(Y_j > 1) \right\} . \end{aligned}$$

Example 2.9 (Conditional tail expectation). We assume for simplicity that the time series $\{X_j\}$ is non negative and has tail index $\alpha > 1$. Then the following limit exists:

$$\lim_{x \rightarrow \infty} \frac{1}{x} \mathbb{E}[X_h \mid X_0 > x] = \int_{x_0=1}^{\infty} \int_{\mathbb{R}_+^h} x_h \nu_{0,h}(\mathrm{d}\mathbf{x}) = \mathbb{E}[Y_h] = \frac{\alpha \mathbb{E}[\Theta_h]}{\alpha - 1} = \text{CTE}_h . \quad (2.19)$$

Indeed, regular variation implies that the distribution of $x^{-1}(X_0, \dots, X_h)$ conditionally on $X_0 > x$ converges weakly to the probability measure equal to $\nu_{0,h}$ restricted to $[1, \infty) \times \mathbb{R}^h$ and if $\alpha > 1$, Potter's bounds ensure that $x^{-1}X_h$ is uniformly integrable conditionally on $X_0 > x$. Define

$$\widehat{C}_{n,h} = \frac{1}{k} \sum_{j=1}^n \frac{X_{j+h}}{X_{n:n-k}} 1\{X_j > X_{n:n-k}\}.$$

The bias condition (2.17) becomes

$$\lim_{n \rightarrow \infty} \sqrt{k} \sup_{s_0 \leq s \leq t_0} \left| \frac{\mathbb{E}[X_h 1\{X_0 > u_n s\}]}{u_n \bar{F}(u_n)} - \text{CTE}_h s^{1-\alpha} \right| = 0. \quad (2.20)$$

In order to apply Corollary 2.6, we assume that $\alpha > 2$. We set $\phi(\mathbf{x}) = x_h 1\{x_0 > 1\}$, $\mathcal{G}_0 = \{\phi\}$ and $\mathcal{G} = \{\phi_s, s \in [s_0, t_0]\}$ with $0 < s_0 < 1 < t_0$. The class \mathcal{G} is pointwise separable, linearly ordered and its envelop function is $s_0^{-1}x_h 1\{x_0 > s_0\}$ which belongs to \mathcal{L}_2 ; it is totally bounded for the metric ρ_h and Condition (2.13) holds (see Section 4.4.2 for a proof of the latter two points).

Thus we obtain, under the assumptions of Theorem 2.3 with $q = 2$,

$$\sqrt{k} \left\{ \widehat{C}_{n,h} - \text{CTE}_h \right\} \xrightarrow{d} N(0, \sigma_h^2),$$

with

$$\sigma_h^2 = \mathbb{E}[(Y_h - \text{CTE}_h)^2] + 2 \sum_{j=1}^{\infty} \mathbb{E}[(Y_h - \text{CTE}_h)(Y_{j+h} - \text{CTE}_h) 1\{Y_j > 1\}].$$

Example 2.10 (Estimation of the distribution of the spectral tail process). For $h > 0$, let L_h be the distribution function of the spectral tail process Θ_h at lag h ,

$$L_h(y) = \lim_{x \rightarrow \infty} \frac{\mathbb{P}(|X_0| > x, X_h \leq |X_0|y)}{\mathbb{P}(|X_0| > x)}, \quad y \in \mathbb{R}.$$

An estimator $\widehat{L}_{n,h}$ of L_h is defined by

$$\widehat{L}_{n,h}(y) = \frac{1}{k} \sum_{i=1}^n 1\{|X_i| > X_{n:n-k}\} 1\{X_{i+h} \leq |X_i|y\}, \quad (2.21)$$

where k is a non decreasing sequence. For $y \in \mathbb{R}$, define the function J_y on \mathbb{R}^{h+1} by

$$J_y(\mathbf{x}) = 1\{|x_0| > 1\} 1\{x_h \leq |x_0|y\},$$

and set $\mathbb{L}_h(y) = \mathbb{M}(J_y) - L_h(y)\mathbb{M}(1_{(1,\infty] \times \mathbb{R}^h})$.

Theorem 2.11. *Let the assumptions of Theorem 2.3 with $q = 0$ and (2.15) hold. Assume that the distribution function L_h of Θ_h is continuous on $[a, b]$ with $a < b \in \mathbb{R}$ and for each $y \in [a, b]$,*

$$\lim_{n \rightarrow \infty} \sqrt{k} \sup_{s_0 \leq s \leq t_0} \left| \frac{\mathbb{P}(|X_0| > u_n s, X_h \leq |X_0| y)}{\bar{F}(u_n)} - s^{-\alpha} L_h(y) \right| = 0, \quad (2.22)$$

where u_n is such that $k = n\bar{F}(u_n)$. Then $\sqrt{k}(\widehat{L}_{n,h} - L_h) \xrightarrow{\text{fidi}} \mathbb{L}_h$ on $[a, b]$.

The proof is in Section 4.4.3. We only consider fidi convergence since a proof of tightness for a two-parameter process would be much more technical since the linearly ordered property of the class of functions would fail to hold. Under additional assumptions on the distribution of the spectral tail process and if the bias condition (2.22) holds uniformly on $[a, b]$, tightness with respect to y could be proved by the same techniques as in (Drees and Rootzén, 2010, Example 4.4) for a plausibly slower rate of convergence.

Consider a univariate non negative Markov chain which in addition to Assumption 2.1 satisfies $\Theta_{j+1} = A_{j+1}\Theta_j$, $j \geq 0$ where $\{A_j, j \geq 1\}$ is a sequence of i.i.d. random variables with the same distribution as Θ_1 . This is the case for most usual Markovian time series, see Janssen and Segers (2014). For $h = 1$, we can calculate

$$\text{var}(\mathbb{L}_1(y)) = \mathbb{P}(\Theta_1 \leq y)\mathbb{P}(\Theta_1 > y).$$

This is the same as the variance found in (Drees et al., 2014, Corollary 5.2) for their forward estimator which instead of using a random threshold replaces the scaling k by the random number of exceedances of the Markov chain $\{X_j\}$ above the deterministic threshold u_n .

2.3 Checking Assumption 2.1

We now show how to check Assumption 2.1. Irreducibility and the drift condition (2.1) are well known for most Markovian time series models, but the link with the conditions of extreme value theory has not been yet fully investigated. Janssen and Segers (2014) used the functional autoregressive representation of most Markov chains to obtain conditions for the whole time series to be regularly varying when the stationary distribution is regularly varying. We build on this approach to check Assumption 2.1. Assume that $\{\mathbb{Y}_j, j \in \mathbb{N}\}$ is a \mathbb{R}^d -valued Markov chain which admits a functional autoregressive representation

$$\mathbb{Y}_{j+1} = \Phi(\mathbb{Y}_j, Z_{j+1}), \quad j \geq 0, \quad (2.23)$$

where $\{Z, Z_j, j \geq 1\}$ is a sequence of i.i.d. random variables with values in a measurable space \mathbf{E} and $\Phi : \mathbb{R}^d \times \mathbf{E} \rightarrow \mathbb{R}^d$ is a measurable map. Fix a norm $\|\cdot\|$ and denote the unit sphere by \mathbb{S}^{d-1} . We assume that there exist $q_0 < \alpha$, $\zeta_1, \zeta_2 > 0$ and a map $V : \mathbb{R}^d \rightarrow [1, \infty)$

such that

$$\zeta_1(\|\mathbf{x}\| \vee 1)^{q_0} \leq V(\mathbf{x}) \leq \zeta_2(\|\mathbf{x}\| \vee 1)^{q_0}, \quad (2.24a)$$

$$\sup_{\mathbf{x} \in \mathbb{R}^d} \|\mathbf{x}\|^{-q_0} \mathbb{E}[\|\Phi(\mathbf{x}, Z)\|^{q_0}] < \infty, \quad (2.24b)$$

$$\limsup_{\|\mathbf{x}\| \rightarrow \infty} \frac{\mathbb{E}[V(\Phi(\mathbf{x}, Z))]}{V(\mathbf{x})} < 1. \quad (2.24c)$$

Under these conditions, (2.1) holds. Indeed, under (2.24c), we can choose $\gamma \in (0, 1)$ such that for r sufficiently large

$$\sup_{\|\mathbf{x}\| > r} \frac{\mathbb{E}[V(\Phi(\mathbf{x}, Z))]}{V(\mathbf{x})} < \gamma.$$

The upper bounds in (2.24a) and (2.24b) ensure that $\mathbb{E}[V(\Phi(\mathbf{x}, Z))]$ is bounded on compact sets, and the lower bound in (2.24a) ensures that V is unbounded outside compact sets; thus (2.1) holds.

In most examples, $\{\mathbb{Y}_j\}$ is itself a regularly varying time series and g is a homogeneous function, so that $\{X_j\}$ is regularly varying and (2.4) holds by the conditions $|g|^{q_0} \leq cV$ and (2.24a). We now consider two examples.

AR(p) with regularly varying innovations

Convergence of the tail empirical processes of exceedances for infinite order moving averages has been obtained in the case of finite variance innovation; for infinite variance innovations it was proved only in the case of an AR(1) process in Drees (2003). We next show that Assumption 2.1 holds for general causal invertible AR(p) models.

Corollary 2.12. *Assume that $\{X_j\}$ is an AR(p) time series*

$$X_j = \varphi_1 X_{j-1} + \cdots + \varphi_p X_{j-p} + \varepsilon_j, \quad j \geq 1,$$

that satisfies the following conditions:

- $\{\varepsilon_j\}$ is a sequence of i.i.d. random variables, regularly varying with index α whose common density possesses an absolutely continuous component;
- the spectral radius of the matrix

$$A = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \cdots & \varphi_p \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

is smaller than 1.

- if $\alpha \leq 2$, then $\sum_{i=1}^p |\varphi_i|^r < 1$ for $r = \min\{1, \alpha\}$.

Let $\{\mathbb{Y}_j, j \geq 1\}$ be the \mathbb{R}^p -valued vector-autoregressive Markov chain

$$\mathbb{Y}_j = A\mathbb{Y}_{j-1} + Z_j \quad (2.25)$$

with

$$\mathbb{Y}_j = (X_j, \dots, X_{j-p+1})^T, \quad Z_j = (\varepsilon_j, 0, \dots, 0)^T.$$

Then there exists a norm $\|\cdot\|$ on \mathbb{R}^p such that Assumption 2.1 holds with $V(\mathbf{x}) = 1 + \|\mathbf{x}\|^{q_0}$ for any $q_0 < \alpha$.

Proof. Under the stated assumptions, the Markov chain $\{\mathbb{Y}_j, j \geq 1\}$ is positive Harris recurrent on \mathbb{R}^p , all compact sets are small sets, see (Alsmeyer, 2003, Example 2.6 (d)); the stationary distribution is given by $\mathbb{Y}_j = \sum_{k=0}^{\infty} A^k Z_{j-k}$ and is regularly varying, see Hult and Samorodnitsky (2008). The $AR(p)$ process admits the representation (2.23) with Φ given by

$$\Phi(\mathbf{x}, \mathbf{z}) = A\mathbf{x} + \mathbf{z}.$$

Therefore (2.24a) and (2.24b) hold for $V = 1 + \|\cdot\|^{q_0}$ for any norm $\|\cdot\|$ on \mathbb{R}^p with $\varphi(\mathbf{x}, \mathbf{z}) = A\mathbf{x}$ and for any q_0 such that $\mathbb{E}[|\varepsilon_0|^{q_0}] < \infty$. We must show that there exists a norm such that condition (2.24c) is fulfilled. Let λ be the spectral radius of the matrix A . Fix ϵ such that $\gamma = \lambda + \epsilon < 1$. Then there exists a norm (depending on A) such that

$$\sup_{\substack{\mathbf{x} \in \mathbb{R}^p \\ \|\mathbf{x}\|_p = 1}} \|A\mathbf{x}\| \leq \gamma;$$

see e.g. (Douc et al., 2014, Proposition 4.24 and Example 6.35). This yields (2.24c) with $V(\mathbf{x}) = 1 + \|\mathbf{x}\|^{q_0}$. \square

Threshold ARCH

We consider the Threshold-ARCH model. It was proved to have a regularly varying stationary distribution by Cline (2007). We show here that it satisfies Assumption 2.1.

Corollary 2.13. *Let $\xi \in \mathbb{R}$. Assume that $\{X_j\}$ follows a Threshold-ARCH model,*

$$X_j = (b_{10} + b_{11}X_{j-1}^2)^{1/2} Z_j 1\{X_{j-1} < \xi\} + (b_{20} + b_{21}X_{j-1}^2)^{1/2} Z_j 1\{X_{j-1} \geq \xi\}, \quad (2.26)$$

that satisfies the following conditions:

- $b_{10}, b_{11}, b_{20}, b_{21} > 0$;

- $\{Z_j, j \in \mathbb{Z}\}$ is a sequence of i.i.d. random variables such that $\mathbb{E}[|Z_1|^\beta] < \infty$ for all $\beta > 0$;
- the distribution of Z_1 has a bounded density with respect to Lebesgue's measure not vanishing in a neighbourhood of zero;
- $\mathbb{E}[\log\{|Z_1|(\sqrt{b_{11}}1\{Z_1 < 0\} + \sqrt{b_{21}}1\{Z_1 \geq 0\})\}] < 0$.

Then the Markov chain $\{X_j\}$ is an irreducible and aperiodic chain; its stationary distribution is regularly varying with index α obtained by solving

$$b_{11}^{\alpha/2}\mathbb{E}[|Z_1|^\alpha 1\{Z_1 < 0\}] + b_{21}^{\alpha/2}\mathbb{E}[|Z_1|^\alpha 1\{Z_1 \geq 0\}] = 1. \quad (2.27)$$

Assumption 2.1 holds with $V(x) = 1 + |x|^{q_0}(b_{11}^{q_0/2}1\{x < 0\} + b_{21}^{q_0/2}1\{x \geq 0\})$ for any $q_0 < \alpha$.

Proof. The statements about ergodicity and the tail of the marginal distribution are proved in Cline (2007). The chain has the representation (2.23) with

$$\Phi(x, z) = (b_{10} + b_{11}x^2)^{1/2}z1\{x < \xi\} + (b_{20} + b_{21}x^2)^{1/2}z1\{x \geq \xi\}.$$

Thus (2.24a) and (2.24b) hold for all $q_0 > 0$. For $q_0 < \alpha$, set $\lambda_{q_0} = b_{11}^{q_0/2}\mathbb{E}[|Z_1|^{q_0}1\{Z_1 < 0\}] + b_{21}^{q_0/2}\mathbb{E}[|Z_1|^{q_0}1\{Z_1 \geq 0\}]$. Then (2.27) guarantees that $\lambda_{q_0} < 1$ and it is readily checked that

$$\lim_{|x| \rightarrow \infty} \frac{\mathbb{E}[V(\Phi(x, Z_1))]}{V(x)} = \lambda_{q_0} < 1.$$

This proves (2.24c). □

2.4 A counterexample

Considering the sum of an i.i.d. regularly varying sequence and a non geometrically ergodic lighter tailed Markov chain, one can easily see that the geometric drift condition is not a necessary condition for the results of Sections 2.1 and 2.2 to hold. However, when the geometric drift condition does not hold, it is easy to build counterexamples of non geometrically ergodic Markov chains which exhibit a highly non standard behaviour of their tail empirical process. In particular, their extremal index is 0. We now provide a toy example of such a non standard behaviour.

Let $\{Z_j, j \in \mathbb{Z}\}$ be a sequence of i.i.d. positive integer valued random variables with regularly varying right tail with index $\beta > 1$. Define the Markov chain $\{X_j, j \geq 0\}$ by the following recursion:

$$X_j = \begin{cases} X_{j-1} - 1 & \text{if } X_{j-1} > 1, \\ Z_j & \text{if } X_{j-1} = 1. \end{cases}$$

Since $\beta > 1$, the chain admits a stationary distribution π on \mathbb{N} given by

$$\pi(n) = \frac{\mathbb{P}(Z_0 \geq n)}{\mathbb{E}[Z_0]}, \quad n \geq 1.$$

To avoid confusion, we will denote the distributions functions of Z_0 and X_0 (when the initial distribution is π) by F_Z and F_X , respectively. The tail \bar{F}_X of the stationary distribution is then regularly varying with index $\alpha = \beta - 1$, since it is given by

$$\bar{F}_X(x) = \frac{\mathbb{E}[(Z_0 - [x])_+]}{\mathbb{E}[Z_0]} \sim \frac{x\bar{F}_Z(x)}{\beta\mathbb{E}[Z_0]}, \quad x \rightarrow \infty. \quad (2.28)$$

We assume for simplicity that $\mathbb{P}(Z_0 = n) > 0$ for all $n \geq 1$; this implies that the chain is irreducible and aperiodic and the state $\{1\}$ is a recurrent atom. The distribution of the return time τ_1 to the atom $\{1\}$, when the chains start from $\{1\}$, is the distribution of Z_0 . Hence the chain is not geometrically ergodic since under the assumption on Z_0 , $\mathbb{E}_1[\kappa^{\tau_1}] = \mathbb{E}[\kappa^{Z_0}] = \infty$ for all $\kappa > 1$. Moreover, the extremal index of the chain is 0, by an application of (Rootzén, 1988, Theorem 3.2 and Eq. (4.2)).

Let $\{u_n\}$ be a scaling sequence. Consider the tail empirical distribution function defined in (1.3) and $T_n(s) = \mathbb{E}[\tilde{T}_n(s)] = \bar{F}_X(u_n s) / \bar{F}_X(u_n)$. Let $\{a_n\}$ be a scaling sequence such that $\lim_{n \rightarrow \infty} n\mathbb{P}(Z_0 > a_n) = 1$.

Proposition 2.14. • *If $\lim_{n \rightarrow \infty} n\bar{F}_Z(u_n) = 0$, then $\lim_{n \rightarrow \infty} \mathbb{P}(\tilde{T}_n(s) \neq 0) = 0$.*

- *If $\beta \in (1, 2)$ and $\lim_{n \rightarrow \infty} n\bar{F}_Z(u_n) = \infty$, then there exists a β -stable random variable Λ such that for every $s > 0$, $a_n^{-1}n\bar{F}_X(u_n)\{\tilde{T}_n(s) - T_n(s)\} \xrightarrow{d} \Lambda$.*
- *If $\beta > 2$, $\lim_{n \rightarrow \infty} n\bar{F}_Z(u_n) = \infty$ and $s_0 > 0$, then the process $s \rightarrow \sqrt{n\bar{F}_Z(u_n)}\{\tilde{T}_n(s) - T_n(s)\}$ converges weakly in $\mathbb{D}([s_0, \infty))$ equipped with the Skorokhod J_1 -topology to a centered Gaussian process $\tilde{\mathbb{G}}$ with covariance function*

$$C(s, t) = \frac{(\beta + 1)t^{1-\beta}}{\beta(\beta - 1)} - \frac{st^{-\beta}}{\beta}, \quad s < t.$$

Remark 2.15. In the standard situation (for example, under the geometric drift condition), a non degenerate limit is expected if $n\bar{F}_X(u_n) \rightarrow \infty$. Since $\bar{F}_X(u_n) \sim u_n\bar{F}_Z(u_n)$, it may happen simultaneously that $n\bar{F}_X(u_n) \rightarrow \infty$ and $n\bar{F}_Z(u_n) \rightarrow 0$. The appropriate threshold is determined by the distribution of Z_0 and not by the stationary distribution of the chain.

3 Central limit theorem for tail array sums

In this section we prove the central limit theorem for tail array sums in the general framework of a strictly stationary regularly varying β -mixing sequence with values in \mathbb{R}^d .

3.1 Finite cluster condition

The main tool to prove our results is a modified form of the condition (2.11). Precisely, let $\{\xi_j, j \in \mathbb{Z}\}$ be a regularly varying stationary \mathbb{R}^d -valued sequence. Let $|\cdot|$ be an arbitrary norm on \mathbb{R}^d and let H be the distribution function of $|\xi_0|$.

Assumption 3.1 (Condition $\mathcal{S}(u_n, r_n, \psi)$). *Let $\{u_n\}$ and $\{r_n\}$ be sequences which tend to infinity, $\{r_n\}$ being integer valued, and $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function which vanishes in a neighborhood of zero (i.e. there exists $\epsilon > 0$ such that $\psi(\mathbf{x}) = 0$ if $|\mathbf{x}| \leq \epsilon$). For all $s, t > 0$,*

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{H(u_n)} \sum_{j=L+1}^{r_n} \mathbb{E} [|\psi(s\xi_0/u_n)| |\psi(t\xi_j/u_n)|] = 0. \quad (\mathcal{S}(u_n, r_n, \psi))$$

Note that by stationarity,

$$\sum_{L < -j \leq r_n} \mathbb{E} [|\psi(s\xi_0/u_n)| |\psi(t\xi_j/u_n)|] = \sum_{L < j \leq r_n} \mathbb{E} [|\psi(s\xi_j/u_n)| |\psi(t\xi_0/u_n)|].$$

Therefore Condition $\mathcal{S}(u_n, r_n, \psi)$ can be equivalently written as a one-sided or a two sided sum. Note also that for bounded ψ Condition $\mathcal{S}(u_n, r_n, \psi)$ is implied by (2.11) applied to $\{\xi_j\}$. An equivalent formulation of (2.11) is condition (C) of Drees et al. (2014).

By assumption, for $j \geq 0$ the vector (ξ_0, \dots, ξ_j) is regularly varying so we can define the exponent measure μ_{0j} of (ξ_0, \dots, ξ_j) , that is the Radon measure on $\mathbb{R}^{d(j+1)} \setminus \{\mathbf{0}\}$ such that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}((\xi_0, \dots, \xi_j) \in xA)}{\mathbb{P}(|\xi_0| > x)} = \mu_{0j}(A)$$

for relatively compact sets A in $\mathbb{R}^{d(j+1)} \setminus \{\mathbf{0}\}$ such that $\mu_{0j}(\partial A) = 0$. For $j = 0$ we simply write μ_0 for μ_{00} . Note that if $d = h + 1$ and $\xi_0 = (X_0, \dots, X_h)$, then μ_0 is proportional to the measure $\nu_{0,h}$ appearing in (1.1). For measurable functions ϕ, ϕ' , define formally

$$\begin{aligned} \Gamma(\phi, \phi') &= \int_{\mathbb{R}^d} \phi(\mathbf{x}_0) \phi'(\mathbf{x}_0) \mu_0(d\mathbf{x}_0) \\ &\quad + \sum_{j=1}^{\infty} \int_{\mathbb{R}^{d(j+1)}} \{\phi(\mathbf{x}_0) \phi'(\mathbf{x}_j) + \phi(\mathbf{x}_0) \phi'(\mathbf{x}_j)\} \mu_{0j}(d\mathbf{x}), \end{aligned} \quad (3.1)$$

with $\mathbf{x} = (\mathbf{x}_0, \dots, \mathbf{x}_j) \in \mathbb{R}^{d(j+1)}$. In order to provide conditions for the series in (3.1) to be summable, we define the following set of functions.

Definition 3.2. *Let ψ be a non negative function defined on \mathbb{R}^d . The space \mathcal{M}_ψ is the set of measurable functions ϕ defined on \mathbb{R}^d having the following properties:*

- $|\phi| \leq \text{cst} \cdot \psi$, where cst depends on ϕ ;
- for all $j \geq 0$, the function defined on $\mathbb{R}^{d(j+1)}$ by $(\mathbf{x}_0, \dots, \mathbf{x}_j) \rightarrow \phi(\mathbf{x}_j)$ is almost surely continuous with respect to μ_{0j} .

Obviously, \mathcal{M}_ψ is a linear space.

Lemma 3.3. *Let $\{\xi_j, j \in \mathbb{Z}\}$ be a strictly stationary regularly varying sequence and let H be the distribution function of $|\xi_0|$. Let $\{u_n\}$ and $\{r_n\}$ be non decreasing sequences tending to infinity, $\{r_n\}$ being integer valued. Let ψ be a non negative measurable function which vanishes in a neighborhood of zero, satisfying Condition $\mathcal{S}(u_n, r_n, \psi)$ and for which there exists $\delta > 0$ such that*

$$\sup_{n \geq 1} \frac{\mathbb{E} [\psi^{2+\delta}(\xi_0/u_n)]}{\bar{H}(u_n)} < \infty . \quad (3.2)$$

Then for all $\phi, \phi' \in \mathcal{M}_\psi$, the series defining $\Gamma(\phi, \phi')$ in (3.1) is absolutely summable and

$$\Gamma(\phi, \phi') = \lim_{n \rightarrow \infty} \frac{1}{r_n \bar{H}(u_n)} \mathbb{E} \left[\left(\sum_{j=1}^{r_n} \phi(\xi_j/u_n) \right) \left(\sum_{j=1}^{r_n} \phi'(\xi_j/u_n) \right) \right] \quad (3.3)$$

$$= \lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{j=-L}^L \frac{\mathbb{E}[\phi(\xi_0/u_n) \phi'(\xi_j/u_n)]}{\bar{H}(u_n)} . \quad (3.4)$$

If moreover

$$\lim_{n \rightarrow \infty} r_n \bar{H}(u_n) = 0 , \quad (3.5)$$

then

$$\Gamma(\phi, \phi') = \lim_{n \rightarrow \infty} \frac{1}{r_n \bar{H}(u_n)} \text{cov} \left(\sum_{j=1}^{r_n} \phi(\xi_j/u_n), \sum_{j=1}^{r_n} \phi'(\xi_j/u_n) \right) . \quad (3.6)$$

Throughout this section, we will use the following notation. Set

$$\phi_{n,j} = \phi(\xi_j/u_n) , \quad c_j(\phi) = \int_{\mathbb{R}^{d(j+1)}} \phi(\mathbf{x}_0) \phi(\mathbf{x}_j) \mu_{0j}(d\mathbf{x}) , \quad j \geq 0 .$$

Proof. Since \mathcal{M}_ψ is a linear space and by the identity $2xy = (x+y)^2 - x^2 - y^2$, it suffices to prove these identities for $\phi = \phi'$. We restrict ourselves to non negative functions ϕ , the extension for general $\phi \in \mathcal{M}_\psi$ being straightforward. We must first prove that condition (3.2) ensures that the coefficients $c_j(\phi)$ are well defined. Since the support of

ϕ is bounded away from zero and vague convergence coincides with weak convergence on such sets, if ϕ is bounded, then the continuous mapping theorem yields

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\phi_{n,0}\phi_{n,j}]}{\bar{H}(u_n)} = c_j(\phi). \quad (3.7)$$

If ϕ is unbounded and condition (3.2) holds, then for all $A > 0$, applying Markov and Hölder inequalities, we obtain

$$\lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{E}[\phi_{n,0}\phi_{n,j}1\{|\phi_{n,0}\phi_{n,j}| > A\}]}{\bar{H}(u_n)} \leq \lim_{A \rightarrow \infty} \text{cst} \cdot A^{-\delta/2} \sup_{n \geq 1} \frac{\mathbb{E}[\psi^{2+\delta}(\xi_0/u_n)]}{\bar{H}(u_n)} = 0.$$

This allows to use a truncation argument and prove that (3.7) holds.

We now prove that the series $\Gamma(\phi, \phi)$ is summable and without loss of generality we assume that ϕ is nonnegative. Fix $\eta > 0$. Applying $\mathcal{S}(u_n, r_n, \psi)$ and the fact that ϕ is bounded by a multiple of ψ , we can choose L such that, for every $R \geq L$

$$\lim_{n \rightarrow \infty} \sum_{j=L}^R \frac{\mathbb{E}[\phi_{n,0}\phi_{n,j}]}{\bar{H}(u_n)} \leq \eta.$$

This yields that for every $\eta > 0$, large enough L and all $R \geq L$, $\sum_{j=L}^R c_j(\phi) \leq \eta$ and this means that the series $\sum_{j=1}^{\infty} c_j(\phi)$ is summable and that (3.4) holds. To prove (3.3), write

$$\frac{\mathbb{E} \left[\left(\sum_{j=1}^{r_n} \phi_{n,j} \right)^2 \right]}{r_n \bar{H}(u_n)} = \frac{\mathbb{E}[\phi_{n,0}^2]}{\bar{H}(u_n)} + 2 \sum_{j=1}^L (1 - j/r_n) \frac{\mathbb{E}[\phi_{n,0}\phi_{n,j}]}{\bar{H}(u_n)} + 2 \sum_{j=L+1}^{r_n} (1 - j/r_n) \frac{\mathbb{E}[\phi_{n,0}\phi_{n,j}]}{\bar{H}(u_n)}.$$

By Condition $\mathcal{S}(u_n, r_n, \psi)$, for every $\eta > 0$, we can choose L in such a way that the last term above is less than η . This yields

$$\limsup_{n \rightarrow \infty} \left| \sum_{1 \leq j, j' \leq r_n} \frac{\mathbb{E}[\phi_{n,j}\phi_{n,j'}]}{r_n \bar{H}(u_n)} - c_0(\phi) - 2 \sum_{j=1}^L c_j(\phi) \right| \leq \eta.$$

Since the series $\sum_{j=1}^L c_j(\phi)$ is convergent we can also choose L in such a way that $\sum_{j=L+1}^{\infty} c_j(\phi) \leq \eta$. This yields

$$\limsup_{n \rightarrow \infty} \left| \sum_{1 \leq j, j' \leq r_n} \frac{\mathbb{E}[\phi_{n,j}\phi_{n,j'}]}{r_n \bar{H}(u_n)} - \Gamma(\phi, \phi) \right| \leq 3\eta.$$

Since η is arbitrary, this proves (3.3). Finally, with

$$S_n(\phi) = \sum_{j=1}^{r_n} \phi_{n,j},$$

we have

$$\frac{\text{var}(S_n(\phi))}{r_n \bar{H}(u_n)} = \frac{\mathbb{E}[S_n^2(\phi)]}{r_n \bar{H}(u_n)} - \frac{r_n^2 (\mathbb{E}[\phi_{n,0}])^2}{r_n \bar{H}(u_n)} = \frac{\mathbb{E}[S_n^2(\phi)]}{r_n \bar{H}(u_n)} + O(r_n \bar{H}(u_n)).$$

Under condition (3.5), the last term is $o(1)$. This proves (3.6). \square

3.2 Fidi convergence of tail array sums

In this section, we prove a theorem on convergence of tail array sums which complements the results of [Rootzén et al. \(1998\)](#) and ([Drees and Rootzén, 2010](#), Theorem 2.3). We show that under β -mixing, condition $\mathcal{S}(u_n, r_n, \psi)$ is the main ingredient of the proof of the central limit theorem.

Define the process \mathbb{W}_n on \mathcal{M}_ψ by

$$\mathbb{W}_n(\phi) = \frac{1}{\sqrt{n\bar{H}(u_n)}} \sum_{j=1}^n \{\phi(\xi_j/u_n) - \mathbb{E}[\phi(\xi_0/u_n)]\}, \quad \phi \in \mathcal{M}_\psi.$$

Theorem 3.4. *Let $\{\xi_j, j \in \mathbb{Z}\}$ be a strictly stationary regularly varying sequence and let H be the distribution function of $|\xi_0|$. Let $\{u_n\}$ and $\{r_n\}$ be non decreasing sequences which tend to infinity, $\{r_n\}$ being integer valued, such that*

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} n\bar{H}(u_n) = \infty, \quad \lim_{n \rightarrow \infty} r_n\bar{H}(u_n) = 0. \quad (3.8)$$

Let ψ be a function which vanishes in a neighborhood of zero and such that $\mathcal{S}(u_n, r_n, \psi)$ holds. Assume that either ψ is bounded or there exists $\delta \in (0, 1]$ such that (3.2) holds and

$$\lim_{n \rightarrow \infty} \frac{r_n}{(n\bar{H}(u_n))^{\delta/2}} = 0. \quad (3.9)$$

Assume that the sequence $\{\xi_j, j \in \mathbb{Z}\}$ is β -mixing with coefficients $\{\beta_n, n \geq 1\}$ and there exists a sequence $\{\ell_n\}$ such that

$$\ell_n \rightarrow \infty, \quad \ell_n/r_n \rightarrow 0, \quad \lim_{n \rightarrow \infty} n\beta_{\ell_n}/r_n = 0. \quad (3.10)$$

Let \mathbb{W} be a Gaussian process indexed by \mathcal{M}_ψ with covariance function Γ defined in (3.1). Then $\mathbb{W}_n \xrightarrow{\text{fidi}} \mathbb{W}$ on \mathcal{M}_ψ .

Remark 3.5. It is possible to find sequences ℓ_n and r_n that satisfy (3.10) if the β -mixing coefficients β_n satisfy $\beta_n = O(n^{-a})$ for some $a > 0$. A suitable choice is then $r_n = n^\zeta$ and $\ell_n = n^\eta$ with $0 < \eta < \zeta < 1$ and $\zeta + a\eta > 1$.

Proof of Theorem 3.4. Since \mathcal{M}_ψ is a linear space and Γ is a quadratic form, it suffices to prove the central limit theorem for an arbitrary $\phi \in \mathcal{M}_\psi$. For $i = 1, \dots, [n/r_n]$, define

$$S_{n,i}(\phi) = \sum_{j=(i-1)r_n+1}^{ir_n} \phi(\xi_j/u_n), \quad \bar{S}_{n,i}(\phi) = S_{n,i}(\phi) - \mathbb{E}[S_{n,i}(\phi)]. \quad (3.11)$$

Arguing as in ([Drees and Rootzén, 2010](#), Lemma 5.1), Condition (3.10) implies that it suffices to prove the central limit theorem for the sum with independent blocks of length r_n

having the same marginal distribution as the original blocks $(\xi_{(i-1)r_n+1}, \dots, \xi_{ir_n})$ and that we can remove a smaller block $(\xi_{ir_n-\ell_n+1}, \dots, \xi_{ir_n})$ of size ℓ_n at the end of each large block. To this end, we must prove the convergence of the variance and the Lindeberg asymptotic negligibility condition on the sum of independent big blocks. By Lemma 3.3, we already know that

$$\lim_{n \rightarrow \infty} \frac{\text{var}(S_{n,1}(\phi))}{r_n \bar{H}(u_n)} = \lim_{n \rightarrow \infty} \frac{\text{var}\left(\sum_{j=1}^{r_n} \phi(\xi_j/u_n)\right)}{r_n \bar{H}(u_n)} = \Gamma(\phi, \phi).$$

Since $\ell_n \leq r_n$, $\mathcal{S}(u_n, r_n, \psi)$ implies $\mathcal{S}(u_n, \ell_n, \psi)$ and hence the limit above holds with r_n replaced with ℓ_n . By $\ell_n/r_n \rightarrow 0$, this also entails that

$$\lim_{n \rightarrow \infty} \frac{1}{r_n \bar{H}(u_n)} \text{var}\left(\sum_{j=1}^{\ell_n} \phi(\xi_j/u_n)\right) = 0.$$

This means that the small blocks do not contribute to the limit. Therefore, we only need to prove the asymptotic negligibility condition. This is done in Lemma 3.6 in the bounded case and Lemma 3.7 in the unbounded case. \square

Lemma 3.6. *Let ψ be a bounded non negative function which vanishes in a neighborhood of zero and such that $\mathcal{S}(u_n, r_n, \psi)$ holds. If (3.8) holds, then for all $\eta > 0$ and all $\phi \in \mathcal{M}_\psi$,*

$$\begin{aligned} \lim_{n \rightarrow 0} \frac{1}{r_n \bar{H}(u_n)} \mathbb{E} \left[S_{n,1}^2(\phi) 1\{|S_{n,1}(\phi)| > \eta \sqrt{n \bar{H}(u_n)}\} \right] \\ = \lim_{n \rightarrow 0} \frac{1}{r_n \bar{H}(u_n)} \mathbb{E} \left[\bar{S}_{n,1}^2(\phi) 1\{|\bar{S}_{n,1}(\phi)| > \eta \sqrt{n \bar{H}(u_n)}\} \right] = 0. \end{aligned}$$

Proof. Write for brevity $v_n = \sqrt{n \bar{H}(u_n)}$, $S_n(\phi)$ for $S_{n,1}(\phi)$ and $\bar{S}_n(\phi)$ for $\bar{S}_{n,1}(\phi)$. At the first step we note that the centering can be omitted. By the assumptions on ψ and regular variation, $\mathbb{E}[|\phi_{n,0}|^q] = O(\bar{H}(u_n))$ for all $\phi \in \mathcal{M}_\psi$ and $q > 0$ which implies $\mathbb{E}[S_n(\phi)] = O(r_n \bar{H}(u_n))$. Since $r_n = o(n)$, we have, for large enough n ,

$$1\{|\bar{S}_{n,1}(\phi)| > \eta \sqrt{n \bar{H}(u_n)}\} \leq 1\{|S_{n,1}(\phi)| > \eta \sqrt{n \bar{H}(u_n)}/2\}.$$

Since η is arbitrary, we can remove the centering from the indicator. Furthermore,

$$\begin{aligned} \frac{1}{r_n \bar{H}(u_n)} \mathbb{E} \left[\bar{S}_n^2(\phi) 1\{|S_n(\phi)| > \eta v_n\} \right] \\ = \frac{1}{r_n \bar{H}(u_n)} \mathbb{E} \left[S_n^2(\phi) 1\{|S_n(\phi)| > \eta v_n\} \right] + \frac{O(\{\mathbb{E}[S_n(\phi)]\}^2)}{r_n \bar{H}(u_n)} \\ = \frac{1}{r_n \bar{H}(u_n)} \mathbb{E} \left[S_n^2(\phi) 1\{|S_n(\phi)| > \eta v_n\} \right] + O(r_n \bar{H}(u_n)). \end{aligned} \quad (3.12)$$

Hence, under condition (3.8), it suffices to study the main term on the right hand side of (3.12), which is developed as $I_1 + 2I_2$ with

$$I_1 = \frac{1}{r_n \bar{H}(u_n)} \sum_{j=1}^{r_n} \mathbb{E}[\phi_{n,j}^2 1\{|S_n(\phi)| > \eta v_n\}], \quad I_2 = \frac{1}{r_n \bar{H}(u_n)} \sum_{i=1}^{r_n} \sum_{j=i+1}^{r_n} \mathbb{E}[\phi_{n,i} \phi_{n,j} 1\{|S_n(\phi)| > \eta v_n\}].$$

By (3.6) and the Hölder inequality, we have $\mathbb{E}[|S_n(\phi)|] = O(\sqrt{r_n \bar{H}(u_n)})$. Applying Markov's inequality and the boundedness of ϕ , we obtain

$$\begin{aligned} I_1 &\leq \frac{1}{\eta v_n r_n \bar{H}(u_n)} \sum_{j=1}^{r_n} \mathbb{E}[\phi_{n,j}^2 |S_n(\phi)|] \\ &= O\left(\frac{1}{v_n}\right) \frac{1}{r_n \bar{H}(u_n)} \mathbb{E}\left[\left(\sum_{j=1}^{r_n} |\phi_{n,j}|\right)^2\right] = O\left(\frac{1}{v_n}\right) = o(1). \end{aligned}$$

Fix a positive integer L . Since ϕ is bounded, we have

$$I_2 \leq \frac{1}{r_n \bar{H}(u_n)} \sum_{i=1}^L \sum_{j=i+1}^{r_n} \mathbb{E}[|\phi_{n,i} \phi_{n,j}| 1\{|S_n(\phi)| > \eta v_n\}] \quad (3.13)$$

$$+ \frac{1}{r_n \bar{H}(u_n)} \sum_{i=L+1}^{r_n} \sum_{j=i+1}^{i+L} \mathbb{E}[|\phi_{n,i} \phi_{n,j}| 1\{|S_n(\phi)| > \eta v_n\}] \quad (3.14)$$

$$+ \frac{1}{r_n \bar{H}(u_n)} \sum_{i=L+1}^{r_n} \sum_{j=i+L+1}^{r_n} \mathbb{E}[|\phi_{n,i} \phi_{n,j}|]. \quad (3.15)$$

Since ϕ is bounded, the terms in (3.13) and (3.14) are each bounded by

$$\frac{L \|\phi\|_\infty}{r_n \bar{H}(u_n)} \sum_{j=1}^{r_n} \mathbb{E}[|\phi_{n,j}| 1\{|S_n(\phi)| > \eta v_n\}] = o(1),$$

by the same argument as for I_1 . Thus,

$$I_2 = o(1) + \frac{2}{\bar{H}(u_n)} \sum_{i=L+1}^{r_n} \mathbb{E}[|\phi_{n,0} \phi_{n,i}|]. \quad (3.16)$$

By Condition $\mathcal{S}(u_n, r_n, \psi)$, the last expression in (3.16) can be made arbitrarily small by choosing L large enough. \square

We extend Lemma 3.6 to the case of unbounded functions, at the cost of the extra restriction (3.9) on the sequence $\{r_n\}$.

Lemma 3.7. *Let ψ be a non negative measurable function which vanishes in a neighborhood of zero, such that Conditions $\mathcal{S}(u_n, r_n, \psi)$, (3.2), (3.8) and (3.9) hold for the same $\delta \in (0, 1]$. Then for all $\eta > 0$ and all $\phi \in \mathcal{M}_\psi$,*

$$\begin{aligned} & \lim_{n \rightarrow 0} \frac{1}{r_n \bar{H}(u_n)} \mathbb{E} \left[S_{n,1}^2(\phi) 1\{|S_{n,1}(\phi)| > \eta \sqrt{n \bar{H}(u_n)}\} \right] \\ &= \lim_{n \rightarrow 0} \frac{1}{r_n \bar{H}(u_n)} \mathbb{E} \left[\bar{S}_{n,1}^2(\phi) 1\{|\bar{S}_{n,1}(\phi)| > \eta \sqrt{n \bar{H}(u_n)}\} \right] = 0 . \end{aligned}$$

Proof. We follow closely the proof of Lemma 3.6 with appropriate modifications. Recall that we have set $v_n = \sqrt{n \bar{H}(u_n)}$. Since $\phi \in \mathcal{M}_\psi$, Lemma 3.3 implies that $\mathbb{E}[S_n(\phi)] = O(\sqrt{r_n \bar{H}(u_n)})$ and thus the centering can be removed inside the indicator. The calculations leading to (3.12) are still valid in the unbounded case. Since $\delta \in (0, 1]$, we have by Markov inequality,

$$\begin{aligned} I_1 &= O\left(\frac{1}{v_n^\delta}\right) \frac{1}{r_n \bar{H}(u_n)} \sum_{j=1}^{r_n} \mathbb{E}[\phi_{n,j}^2 | S_n(\phi)|^\delta] \\ &= O\left(\frac{1}{v_n^\delta}\right) \frac{1}{r_n \bar{H}(u_n)} \sum_{i=1}^{r_n} \sum_{j=1}^{r_n} \mathbb{E}[\phi_{n,i}^2 |\phi_{n,j}|^\delta] = O\left(\frac{r_n}{v_n^\delta}\right) = o(1) , \end{aligned}$$

by (3.2) and (3.9). As for I_2 , the second term in (3.16) is handled again by Condition $\mathcal{S}(u_n, r_n, \psi)$. Applying Markov inequality, the term in (3.14) is bounded by

$$\frac{1}{\eta v_n^\delta r_n \bar{H}(u_n)} \sum_{i=\ell+1}^{r_n} \sum_{j=i+1}^{i+\ell} \sum_{k=1}^{r_n} \mathbb{E}[|\phi_{n,i}| |\phi_{n,j}| |\phi_{n,k}|^\delta] \leq \frac{\ell r_n \mathbb{E}[|\phi_{n,0}|^{2+\delta}]}{\eta v_n^\delta \bar{H}(u_n)} ,$$

on account of the extended Hölder inequality with $p = q = 2 + \delta$ and $r = (2 + \delta)/\delta$. This again is $o(1)$ by (3.2) and (3.9). The term in (3.13) is treated analogously. \square

4 Proof of the results for functions of Markov chains

Let $\{X_j, j \in \mathbb{Z}\}$ be as in Section 2. We will apply the results of Section 3 to the sequence $\{\xi_j, j \in \mathbb{Z}\}$ defined as $\xi_j = \mathbf{X}_{j,j+h} = (X_j, \dots, X_{j+h})$ which is also regularly varying. Since the distribution of X_0 satisfies the balanced tail condition and the right tail of X_0 is not trivial, X_0 and $|\xi_0|$ are tail equivalent.

We first recall some consequences of the geometric drift condition. Under condition (2.2), the chain $\{\mathbb{Y}_j\}$ can be embedded into an extended Markov chain $\{(\mathbb{Y}_j, B_j)\}$ such that the latter chain possesses an atom A , that is $\bar{P}(s, \cdot) = \bar{P}(t, \cdot)$ for every $s, t \in A$, where \bar{P} is the transition kernel of the extended chain. This existence is due to the Nummelin splitting technique (see (Meyn and Tweedie, 2009, Chapter 5)). Denote by \mathbb{E}_A the

expectation conditionally to $(\mathbb{Y}_0, B_0) \in A$ and let τ_A be the first return time to A of the chain $\{(\mathbb{Y}_j, B_j), j \geq 0\}$. Note that τ_A is a stopping time with respect to the extended chain, but not with respect to the chain $\{\mathbb{Y}_j\}$. We assume that the extended chain is defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that the extended chain is stationary under \mathbb{P} . Then, by (Meyn and Tweedie, 2009, Theorem 15.4.1) for q_0 as in Assumption 2.1, there exist $\kappa > 1$ and a constant cst such that for all $y \in E$,

$$\mathbb{E} \left[\sum_{j=1}^{\tau_A} \kappa^j |X_j|^{q_0} \mid \mathbb{Y}_0 = y \right] \leq c \mathbb{E} \left[\sum_{j=1}^{\tau_A} \kappa^j V(\mathbb{Y}_j) \mid \mathbb{Y}_0 = y \right] \leq \text{cst} V(y). \quad (4.1)$$

By Jensen's inequality, this implies that for all $q_1 \leq q_0$, there exists $\kappa_1 \in (1, \kappa)$ such that

$$\mathbb{E} \left[\sum_{j=1}^{\tau_A} \kappa_1^j |X_j|^{q_1} \mid \mathbb{Y}_0 = y \right] \leq \text{cst} V^{q_1/q_0}(y). \quad (4.2)$$

Moreover, Kac's formula (Meyn and Tweedie, 2009, Theorem 10.0.1) gives an expression of the stationary distribution in terms of the return time to A . For every bounded measurable function f , it holds that

$$\mathbb{E}[f(\mathbb{Y}_0)] = \frac{1}{\mathbb{E}_A[\tau_A]} \mathbb{E}_A \left[\sum_{j=0}^{\tau_A-1} f(\mathbb{Y}_j) \right]. \quad (4.3)$$

Since $V \geq 1$, the inequality (4.1) integrated with respect to the stationary distribution implies that $\mathbb{E}[\kappa^{\tau_A}] < \infty$.

4.1 Checking the finite cluster condition

In the present context, the anticlustering condition $\mathcal{S}(u_n, r_n, \psi)$ can be re-written as

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\bar{F}(u_n)} \sum_{j=L+1}^{r_n} \mathbb{E} [|\psi(s\mathbf{X}_{0,h}/u_n)| |\psi(t\mathbf{X}_{j,j+h}/u_n)|] = 0, \quad (4.4)$$

where F is the distribution function of X_0 .

Let $|\cdot|$ denote an arbitrary norm on \mathbb{R}^{h+1} . In this section, we prove that for all $\epsilon > 0$ Assumption 2.1 implies the condition (4.4) for the function ψ_ϵ defined by

$$\psi_\epsilon(\mathbf{x}) = |\mathbf{x}|^{q_0/2} 1\{|\mathbf{x}| > \epsilon\}.$$

This will be done in Lemma 4.3 below. First we introduce some notation and prove two preliminary results. For $0 < s < \infty$ define

$$Q_n(s) = \frac{1}{u_n^{q_0} \bar{F}(u_n)} \mathbb{E} [V(\mathbb{Y}_0) 1\{|X_0| > u_n s\}].$$

Lemma 4.1. *Let Assumption 2.1 holds. For every $s_0 > 0$, there exists a constant $C_0 > 0$ and $\kappa_0 > 1$ such that for $q_1 + q_2 \leq q_0$ and $s \geq s_0$,*

$$\frac{1}{\bar{F}(u_n)} \mathbb{E} \left[\sum_{j=L}^{\tau_A} 1\{|X_0| > su_n\} |X_0/u_n|^{q_1} |X_j/u_n|^{q_2} \right] \leq C_0 \kappa_0^{-L} Q_n^{(q_1+q_2)/q_0}(s). \quad (4.5)$$

Proof. Let κ be as in (4.1). Let the left hand side of (4.5) be denoted by $S_n(s)$. Conditioning on \mathbb{Y}_0 and applying (4.2), we obtain that there exists $\kappa_0 \in (1, \kappa)$ such that

$$\begin{aligned} S_n(s) &\leq \text{cst} \cdot \kappa_0^{-L} \frac{1}{u_n^{q_2} \bar{F}(u_n)} \mathbb{E} [V^{q_2/q_0}(\mathbb{Y}_0) |X_0/u_n|^{q_1} 1\{|X_0| > su_n\}] \\ &\leq \text{cst} \cdot \kappa_0^{-L} \frac{\bar{F}(u_n s)}{u_n^{q_2} \bar{F}(u_n)} \mathbb{E} [V^{q_2/q_0}(\mathbb{Y}_0) |X_0/u_n|^{q_1} \mid |X_0| > su_n] \\ &\leq \text{cst} \cdot \kappa_0^{-L} \frac{\bar{F}(u_n s)}{u_n^{q_1+q_2} \bar{F}(u_n)} \mathbb{E} [V^{(q_1+q_2)/q_0}(\mathbb{Y}_0) \mid |X_0| > su_n]. \end{aligned}$$

Applying Jensen's inequality to the conditional distribution given $|X_0| > u_n s$, we obtain

$$\begin{aligned} S_n(s) &\leq \text{cst} \cdot \kappa_0^{-L} \frac{\bar{F}(u_n s)}{u_n^{q_1+q_2} \bar{F}(u_n)} (\mathbb{E} [V(\mathbb{Y}_0) \mid |X_0| > su_n])^{(q_1+q_2)/q_0} \\ &= \text{cst} \cdot \kappa_0^{-L} \frac{\bar{F}(u_n s)}{u_n^{q_1+q_2} \bar{F}(u_n)} (\mathbb{E} [V(\mathbb{Y}_0) 1\{|X_0| > su_n\}] / \bar{F}(u_n))^{(q_1+q_2)/q_0} \left(\frac{\bar{F}(u_n)}{\bar{F}(u_n s)} \right)^{(q_1+q_2)/q_0} \\ &= \text{cst} \cdot \kappa_0^{-L} \left(\frac{\bar{F}(u_n s)}{\bar{F}(u_n)} \right)^{1-(q_1+q_2)/q_0} Q_n^{(q_1+q_2)/q_0}(s). \end{aligned} \quad (4.6)$$

This yields (4.5) since $\bar{F}(u_n s)/\bar{F}(u_n)$ is uniformly bounded on $[s_0, \infty)$ and $1 - (q_1 + q_2)/q_0 > 0$. \square

Lemma 4.2. *If Assumption 2.1 holds, $r_n \bar{F}(u_n) = o(1)$, and $q_1 + q_2 \leq q_0$, then*

$$\frac{1}{\bar{F}(u_n)} \mathbb{E} \left[\sum_{j=\tau_A+1}^{r_n+h} 1\{|X_0| > su_n\} 1\{|X_j| > su_n\} |X_0/u_n|^{q_1} |X_j/u_n|^{q_2} \right] = o(1), \quad (4.7)$$

uniformly with respect to $s \geq s_0$.

Proof. Let the left hand side of (4.7) be denoted by $R_n(s)$. Then, by the strong Markov

property,

$$\begin{aligned}
R_n(s) &\leq \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[1\{|X_0| > u_n s\} |X_0/u_n|^{q_1} \mathbb{E} \left[\sum_{j=\tau_A+1}^{r_n+h} 1\{|X_j| > u_n s_0\} |X_j/u_n|^{q_2} \mid \mathbb{Y}_{\tau_A} \right] \right] \\
&\leq \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[1\{|X_0| > u_n s_0\} |X_0/u_n|^{q_1} \mathbb{E} \left[\sum_{j=\tau_A+1}^{r_n+\tau_A+h} 1\{|X_j| > u_n s_0\} |X_j/u_n|^{q_2} \mid \mathbb{Y}_{\tau_A} \right] \right] \\
&\leq \frac{1}{\bar{F}(u_n)} \mathbb{E} [1\{|X_0| > u_n s_0\} |X_0/u_n|^{q_1}] \mathbb{E}_A \left[\sum_{j=1}^{r_n+h} 1\{|X_j| > u_n s_0\} |X_j/u_n|^{q_2} \right].
\end{aligned}$$

By classical regenerative arguments, Kac's formula (4.3) and regular variation, we obtain as $n \rightarrow \infty$ (and since h is fixed),

$$\begin{aligned}
\mathbb{E}_A \left[\sum_{j=1}^{r_n+h} 1\{|X_j| > u_n s_0\} |X_j/u_n|^{q_2} \right] &\sim \frac{r_n}{\mathbb{E}_A[\tau_A]} \mathbb{E}_A \left[\sum_{j=1}^{\tau_A} 1\{|X_j| > u_n s_0\} |X_j/u_n|^{q_2} \right] \\
&\leq r_n \bar{F}(u_n) \frac{\mathbb{E}[1\{|X_0| > u_n s_0\} |X_0/u_n|^{q_2}]}{\bar{F}(u_n)} = O(r_n \bar{F}(u_n)).
\end{aligned}$$

This yields, applying again the drift condition, Condition (2.4) and Jensen's inequality,

$$\begin{aligned}
\sup_{s \geq s_0} R_n(s) &= O(r_n) \mathbb{E} [1\{|X_0| > u_n s_0\} |X_0/u_n|^{q_1}] \\
&= O(r_n \bar{F}(u_n)) \frac{\mathbb{E} [V^{q_1/q_0}(\mathbb{Y}_0) \mid |X_0| > u_n s_0]}{u_n^{q_1}} \\
&= O(r_n \bar{F}(u_n)) \left(\frac{\mathbb{E} [V(\mathbb{Y}_0) \mid |X_0| > u_n s_0]}{u_n^{q_0}} \right)^{q_1/q_0} = o(1).
\end{aligned}$$

□

Lemma 4.3. *Let Assumption 2.1 hold. Then $\mathcal{S}(u_n, r_n, \psi_\epsilon)$ holds.*

Proof. For $j \geq h$, we have

$$\begin{aligned}
0 &\leq |\psi_\epsilon(\mathbf{X}_{0,h}/u_n)| |\psi_\epsilon(\mathbf{X}_{j,j+h}/u_n)| \\
&\leq \text{cst} u_n^{-q_0} \sum_{i_1, i_2, i_3, i_4=0}^h 1\{|X_{i_1}| > u_n \epsilon\} |X_{i_2}|^{q_0/2} 1\{|X_{j+i_3}| > u_n \epsilon\} |X_{j+i_4}|^{q_0/2}. \tag{4.8}
\end{aligned}$$

For all i, i' , we can write

$$\begin{aligned}
&1\{|X_i| > u_n \epsilon\} |X_{i'}/u_n|^{q_0/2} \\
&= 1\{|X_i| > u_n \epsilon\} 1\{|X_{i'}| \leq u_n \epsilon\} |X_{i'}/u_n|^{q_0/2} + 1\{|X_i| > u_n \epsilon\} 1\{|X_{i'}| > u_n \epsilon\} |X_{i'}/u_n|^{q_0/2} \\
&\leq 1\{|X_i| > u_n \epsilon\} \epsilon^{q_0/2} + 1\{|X_{i'}| > u_n \epsilon\} |X_{i'}/u_n|^{q_0/2} \\
&\leq 1\{|X_i| > u_n \epsilon\} |X_i/u_n|^{q_0/2} + 1\{|X_{i'}| > u_n \epsilon\} |X_{i'}/u_n|^{q_0/2}.
\end{aligned}$$

Thus, we can restrict the sum in (4.8) to the set of indices (i_1, i_2, i_3, i_4) such that $i_1 = i_2$ and $i_3 = i_4$. For $L > h$ and $i, i' \leq h$, we have by Lemmas 4.1 and 4.2 and by stationarity,

$$\begin{aligned}
& \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[\sum_{j=2L}^{r_n} 1\{|X_i| > u_n s\} 1\{|X_{j+i'}| > u_n s\} |X_i/u_n|^{q_0/2} |X_{j+i'}/u_n|^{q_0/2} \right] \\
& \leq \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[\sum_{j=2L}^{r_n} 1\{|X_0| > u_n s\} 1\{|X_{j+i'-i}| > u_n s\} |X_0/u_n|^{q_0/2} |X_{j+i'-i}/u_n|^{q_0/2} \right] \\
& \leq \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[\sum_{j=2L-h}^{r_n+h} 1\{|X_0| > u_n s\} 1\{|X_j| > u_n s\} |X_0/u_n|^{q_0/2} |X_j/u_n|^{q_0/2} \right] \\
& \leq \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[\sum_{j=L}^{\tau_A} 1\{|X_0| > u_n s\} 1\{|X_j| > u_n s\} |X_0/u_n|^{q_0/2} |X_j/u_n|^{q_0/2} \right] \\
& + \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[\sum_{j=\tau_A+1}^{r_n+h} 1\{|X_0| > u_n s\} 1\{|X_j| > u_n s\} |X_0/u_n|^{q_0/2} |X_j/u_n|^{q_0/2} \right] \\
& \leq \text{cst } \kappa_0^{-L} Q_n(s) + o(1).
\end{aligned}$$

where the $o(1)$ term is uniform with respect to $s \geq s_0$. The bound (2.4) in Assumption 2.1 implies that Q_n is asymptotically uniformly bounded on $[s_0, \infty)$. Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[\sum_{j=L}^{r_n} 1\{|X_0| > u_n s\} 1\{|X_j| > u_n s\} |X_0/u_n|^{q_0/2} |X_j/u_n|^{q_0/2} \right] = O(\kappa_0^{-L}).$$

Since $\kappa_0 > 1$, this proves (4.4). \square

4.2 Proof of Theorem 2.3

Fix $q \leq q_0/(2 + \delta)$. We apply Theorem 3.4 to the sequence $\xi_j = (X_j, \dots, X_{j+h})$, $j \in \mathbb{Z}$ in order to prove that $\mathbb{M}_n \xrightarrow{\text{f.d.i.}} \mathbb{M}$ on $\mathcal{M}_{h_{q,\epsilon}}$ for each $\epsilon > 0$, where $h_{q,\epsilon}$ is defined on \mathbb{R}^{h+1} by

$$h_{q,\epsilon}(\mathbf{x}) = |\mathbf{x}|^q 1\{|\mathbf{x}| > \epsilon\}. \quad (4.9)$$

Since $\mathcal{L}_q = \cup_{\epsilon > 0} \mathcal{M}_{h_{q,\epsilon}}$, this will prove Theorem 2.3.

(i) Assumption 2.1 implies $\mathcal{S}(u_n, r_n, h_{q,\epsilon})$ (cf. Lemma 4.3 and note that $h_{q_0/2,\epsilon} = \psi_\epsilon$) and β -mixing with geometric rate. Therefore we can choose $r_n = \log^{1+\eta}(n)$ and $\ell_n = c \log(n)$ for c large enough so that (3.10) holds.

(ii) Since $q(2 + \delta) \leq q_0$, assumptions (2.3) and (2.4) imply that Condition (3.2) holds for $\psi = h_{q,\epsilon}$.

(iii) With r_n as above, Conditions (2.9) and (2.10) are exactly (3.8) and (3.9).

4.3 Proof of Theorem 2.4

Let $\ell_0(\mathbb{R}^{h+1})$ be the set of \mathbb{R}^{h+1} -valued sequences $\mathbf{x} = (\mathbf{x}_j)_{j \in \mathbb{Z}}$ such that $\lim_{|j| \rightarrow \infty} |\mathbf{x}_j| = 0$. Let \mathcal{H}_q be the set of functions f defined on $\ell_0(\mathbb{R}^{h+1})$ for which there exists $\phi \in \mathcal{L}_q$ such that

$$f(\mathbf{x}) = \sum_{j \in \mathbb{Z}} \phi(\mathbf{x}_j), \quad \mathbf{x} \in \ell_0(\mathbb{R}^{h+1}).$$

Since functions in \mathcal{L}_q vanish in a neighborhood of zero, the series has finitely many non zero terms and the function ϕ is uniquely determined by f and will be denoted ϕ^f . We define a pseudometric ρ on \mathcal{H}_q (with $q < q_0/(2 + \delta)$) by

$$\rho^2(f, g) = \rho_h^2(\phi^f, \phi^g) = \nu_{0,h}(\{\phi^f - \phi^g\}^2). \quad (4.10)$$

Let $\{r_n\}$ be as in (i) of the proof of Theorem 2.3 and let $m_n = [n/r_n]$. Set $\mathbb{X}_{n,i} = u_n^{-1}(\mathbf{X}_{(i-1)r_n+1, (i-1)r_n+h+1}, \dots, \mathbf{X}_{ir_n, ir_n+h})$ and identify it with an element of $\ell_0(\mathbb{R}^{h+1})$ by adding zeros on both sides. Then

$$\begin{aligned} \mathbb{Z}_{n,i}(f) &= f(\mathbb{X}_{n,i}) = \sum_{j=(i-1)r_n+1}^{ir_n} \phi^f(u_n^{-1} \mathbf{X}_{j,j+h}), \quad \mathbb{Z}_n(f) = \sum_{i=1}^{m_n} \mathbb{Z}_{n,i}(f), \\ \bar{\mathbb{Z}}_n(f) &= \frac{1}{\sqrt{n\bar{F}(u_n)}} (\mathbb{Z}_n(f) - \mathbb{E}[\mathbb{Z}_n(f)]). \end{aligned}$$

Let \mathbb{Z} be the Gaussian process on \mathcal{H}_q defined by $\mathbb{Z}(f) = \mathbb{M}(\phi^f)$. Under the assumptions of Theorem 2.3 and with $r_n = \log^{1+\eta}(n)$, we have

$$\frac{1}{\sqrt{n\bar{F}(u_n)}} \sum_{i=r_n[n/r_n]+1}^n \{\phi^f(u_n^{-1} \mathbf{X}_{j,j+h}) - \mathbb{E}[\phi^f(u_n^{-1} \mathbf{X}_{0,h})]\} = o_P(1).$$

Moreover, since the envelope function belongs to \mathcal{L}_q , we can apply Lemma 3.3 (in view of Lemma 4.3) and we obtain

$$\begin{aligned} \sup_{f \in \hat{\mathcal{G}}} |\mathbb{M}_n(\phi^f) - \bar{\mathbb{Z}}_n(f)| &\leq \frac{1}{\sqrt{n\bar{F}(u_n)}} \sum_{j=m_n r_n+1}^n \Phi_{\mathcal{G}}(X_{n,j}/u_n) + \frac{r_n \bar{F}(u_n)}{\sqrt{n\bar{F}(u_n)}} \frac{\mathbb{E}[\Phi_{\mathcal{G}}(X_{n,0}/u_n)]}{\bar{F}(u_n)} \\ &= O_P\left(\sqrt{\frac{r_n}{n}}\right) + O\left(\frac{r_n \bar{F}(u_n)}{\sqrt{n\bar{F}(u_n)}}\right) = o_P(1). \end{aligned}$$

Thus $\bar{\mathbb{Z}}_n(f) = \mathbb{M}_n(\phi^f) + o_P(1)$ uniformly on \mathcal{H}_q and $\bar{\mathbb{Z}}_n \xrightarrow{\text{f.i.di.}} \mathbb{Z}$ on \mathcal{H}_q .

We define the subclass $\hat{\mathcal{G}}$ of \mathcal{H}_q associated to the subclass \mathcal{G} of \mathcal{L}_q by $\hat{\mathcal{G}} = \{f : f(\mathbf{x}) = \sum_{j \in \mathbb{Z}} \phi(\mathbf{x}_j), \phi \in \mathcal{G}\}$. Arguing as in (Drees and Rootzén, 2010, Proof of Theorem 2.8), in order to prove weak convergence of $\bar{\mathbb{Z}}_n$, it suffices to prove the tightness of the process \mathbb{Z}_n^* summing independent copies of $\mathbb{Z}_{n,i}^*$ of $\mathbb{Z}_{n,i}$ indexed by the class $\hat{\mathcal{G}}$. For this purpose, we apply Theorem A.2.

- The pointwise separability of \mathcal{G} implies that $\widehat{\mathcal{G}}$ is also pointwise separable.
- The property (4.10) yields that $(\widehat{\mathcal{G}}, \rho)$ is totally bounded since (\mathcal{G}, ρ_h) is totally bounded by assumption.
- Since the envelope function $\Phi_{\mathcal{G}}$ is assumed to be in \mathcal{L}_q and $\mathcal{L}_q = \cup_{\epsilon>0} \mathcal{M}_{h_q, \epsilon}$, Lemma 3.7 implies that the Lindeberg condition (A.2) holds.
- We now check (A.3). For $f, g \in \widehat{\mathcal{G}}$, we have

$$\rho(f, g) = \lim_{n \rightarrow \infty} \frac{1}{\bar{F}(u_n)} \mathbb{E}[\{\phi^f(\mathbf{X}_{0,h}/u_n) - \phi^g(\mathbf{X}_{0,h}/u_n)\}^2], \quad (4.11)$$

Since the envelope function of \mathcal{G} is in \mathcal{L}_q , there exists $\epsilon > 0$ (which depends only on \mathcal{G}) such that $|\phi^f| \vee |\phi^g| \leq \epsilon^{-1} h_{q, \epsilon}$ (defined in (4.9)). Set $\phi_{n,j} = \phi^f(\mathbf{X}_{j,j+h}/u_n) - \phi^g(\mathbf{X}_{j,j+h}/u_n)$. For every integer $L > 0$, by stationarity, we have

$$\begin{aligned} & \frac{1}{r_n \bar{F}(u_n)} \mathbb{E}[\{\mathbb{Z}_{n,1}(f) - \mathbb{Z}_{n,1}(g)\}^2] \\ & \leq \frac{2}{\bar{F}(u_n)} \sum_{j=0}^L \mathbb{E}[|\phi_{n,0}| |\phi_{n,j}|] + \frac{2}{\bar{F}(u_n)} \sum_{j=L+1}^{r_n} \mathbb{E}[|\phi_{n,0}| |\phi_{n,j}|] \\ & \leq \frac{2(L+1) \mathbb{E}[\phi_{n,0}^2]}{\bar{F}(u_n)} + \frac{\text{cst}}{\bar{F}(u_n)} \sum_{j=L+1}^{r_n} \mathbb{E}[h_{q, \epsilon}(\mathbf{X}_{0,h}/u_n) h_{q, \epsilon}(\mathbf{X}_{j,j+h}/u_n)]. \end{aligned}$$

By Lemma 4.3, for every $\eta > 0$, we can choose L such that

$$\limsup_{n \rightarrow \infty} \frac{1}{\bar{F}(u_n)} \sum_{j=L+1}^{r_n} \mathbb{E}[h_{q, \epsilon}(\mathbf{X}_{0,h}/u_n) h_{q, \epsilon}(\mathbf{X}_{j,j+h}/u_n)] \leq \eta. \quad (4.12)$$

Applying this bound and the assumption (2.13) yields, for any sequence δ_n decreasing to zero,

$$\limsup_{n \rightarrow \infty} \sup_{\substack{f, g \in \widehat{\mathcal{G}} \\ \rho(f, g) \leq \delta_n}} \frac{1}{r_n \bar{F}(u_n)} \mathbb{E}[(\mathbb{Z}_{n,1}(f) - \mathbb{Z}_{n,1}(g))^2] \leq \eta. \quad (4.13)$$

Since η is arbitrary, this proves (A.3).

- Since \mathcal{G} is linearly ordered, so is $\widehat{\mathcal{G}}$ thus it is a VC subgraph class and the entropy condition (A.4) holds by (Giné and Nickl, 2016, Theorem 3.7.37).

We have checked all the assumptions of Theorem A.2. Therefore, \mathbb{Z}_n^* and hence $\bar{\mathbb{Z}}_n$ converges to \mathbb{Z} in $\ell_\infty(\widehat{\mathcal{G}})$ and since $\mathbb{M}(\phi^f) = \mathbb{Z}(f)$ by definition, this proves our result.

4.4 Proof of Corollaries 2.5 and 2.6

We first apply Theorem 2.4 with $\mathcal{G}_I = \{I_s = 1_{(s,\infty) \times \mathbb{R}^h}, s \in [s_0, t_0]\}$ for $0 < s_0 \leq 1 \leq t_0$. Then

$$\rho_h(I_s, I_t) = |s - t|^{-\alpha} \leq \alpha s_0^{-\alpha-1} |s - t| .$$

The class \mathcal{G}_I is pointwise separable, linearly ordered and totally bounded for the metric ρ_h . Condition (2.13) holds by regular variation and the uniform convergence theorem. The envelope of \mathcal{G}_I is I_{s_0} which belongs to \mathcal{L}_q . This proves Corollary 2.5.

We now prove Corollary 2.6. Set

$$B_{n,1}(s) = \sqrt{k}\{\mathbb{E}[M_n(I_s)] - s^{-\alpha}\} , \quad B_{n,2}(s) = \sqrt{k}\{\mathbb{E}[M_n(\phi_s)] - \nu_{0,h}(\phi_s)\} .$$

By conditions (2.15) and (2.17), we have $\lim_{n \rightarrow \infty} \sup_{s \in [s_0, t_0]} B_{n,i}(s) = 0$, $i = 1, 2$. Using this bound and $\nu_{0,h}(\phi_s) = s^{-\alpha} \nu_{0,h}(\phi)$, we obtain after some algebra,

$$\begin{aligned} \sqrt{k} \left(\frac{M_n(\phi_s)}{M_n(I_s)} - \nu_{0,h}(\phi) \right) &= \frac{\mathbb{M}_n(\phi_s) - \nu_{0,h}(\phi) \mathbb{M}_n(I_s) - \nu_{0,h}(\phi) B_{n,1}(s) + B_{n,2}(s)}{M_n(I_s)} \\ &= \frac{\mathbb{M}_n(\phi_s) - \nu_{0,h}(\phi) \mathbb{M}_n(I_s) + o(1)}{M_n(I_s)} , \end{aligned}$$

the term $o(1)$ begin uniform in $s \in [s_0, t_0]$ and $\phi \in \mathcal{G}_0$. Moreover,

$$M_n(I_s) = s^{-\alpha} + k^{-1/2} \mathbb{M}_n(I_s) + k^{-1/2} B_{n,1}(s) = s^{-\alpha} + o_P(1) ,$$

again uniformly in $s \in [s_0, t_0]$. Therefore,

$$\sqrt{k} \left(\frac{M_n(\phi_s)}{M_n(I_s)} - \nu_{0,h}(\phi) \right) = \frac{\mathbb{M}_n(\phi_s) - \nu_{0,h}(\phi) \mathbb{M}_n(I_s) + o(1)}{s^{-\alpha} + o_P(1)} , \quad (4.14)$$

the terms $o_P(1)$ begin uniform in $s \in [s_0, t_0]$ and $\phi \in \mathcal{G}_0$. Set $\zeta_n = X_{n:n-k}/u_n$. Since $M_n(I_{\zeta_n}) = 1$,

$$\widehat{\mathbb{M}}_n(\phi) = \sqrt{k} \left(\frac{M_n(\phi_{\zeta_n})}{M_n(I_{\zeta_n})} - \nu_{0,h}(\phi) \right) ,$$

and $\zeta_n \xrightarrow{P} 1$ (see comments after Corollary 2.5), we finally obtain that

$$\widehat{\mathbb{M}}_n(\phi) = \frac{\mathbb{M}_n(\phi_{\zeta_n}) - \nu_{0,h}(\phi) \mathbb{M}_n(I_{\zeta_n}) + o(1)}{\zeta_n^{-\alpha} + o_P(1)} \Rightarrow \mathbb{M}(\phi) - \nu_{0,h}(\phi) \mathbb{M}(1_{[1,\infty) \times \mathbb{R}^h}) ,$$

on $\ell_\infty(\mathcal{G}_0)$.

4.4.1 Proof for Example 2.8

By homogeneity of $\nu_{0,h}$, the semimetric ρ_h on the class \mathcal{G} for this example becomes, for $s_0 < s < t$,

$$\begin{aligned}\rho_h^2(\phi_s, \phi_t) &= \nu_{0,h}(\{1\{x_0 + \dots + x_h > s\} - 1\{x_0 + \dots + x_h > t\}\}^2) \\ &= \nu_{0,h}(A_h)(s^{-\alpha} - t^{-\alpha}) \leq \text{cst} \cdot (t - s).\end{aligned}$$

Thus (\mathcal{G}, ρ_h) is totally bounded. Moreover, by regular variation and the uniform convergence theorem, the convergence

$$\lim_{n \rightarrow \infty} \frac{1}{\bar{F}(u_n)} \mathbb{E} [\{1\{X_0 + \dots + X_h > su_n\} - 1\{X_0 + \dots + X_h > tu_n\}\}^2] = \rho_h^2(\phi_s, \phi_t)$$

is uniform on $[s_0, t_0]^2$. Thus (2.13) holds.

4.4.2 Proof for Example 2.9

Since $\alpha > 2$, the semimetric ρ_h on the class \mathcal{G} for to this example becomes, for $s_0 < s < t$,

$$\begin{aligned}\rho_h^2(\phi_s, \phi_t) &= \mathbb{E}[\{s^{-1}Y_h 1\{Y_0 > s\} - t^{-1}Y_h 1\{Y_0 > t\}\}^2] \\ &= \frac{\alpha \mathbb{E}[\Theta_h^2]}{\alpha - 2} \left\{ (s^{-\alpha} + t^{-\alpha} - 2 \frac{1}{st} (s \vee t)^{-\alpha+2}) \right\} \\ &\leq \frac{\alpha \mathbb{E}[\Theta_h^2]}{\alpha - 2} (s^{-\alpha} - t^{-\alpha}) \leq \text{cst} \cdot (t - s).\end{aligned}$$

Thus (\mathcal{G}, ρ_h) is totally bounded. Moreover, by regular variation and the uniform convergence theorem, the convergence

$$\lim_{n \rightarrow \infty} \frac{1}{u_n^2 \bar{F}(u_n)} \mathbb{E} [\{s^{-1}X_h 1\{X_0 > su_n\} - t^{-1}X_h 1\{X_0 > tu_n\}\}^2] = \frac{\alpha \mathbb{E}[\Theta_h^2]}{\alpha - 2} (s^{-\alpha} - t^{-\alpha})$$

is uniform on $[s_0, t_0]^2$. Thus (2.13) holds.

4.4.3 Proof of Theorem 2.11

Consider sets

$$C_{s,y}(\mathbf{x}) = \{\mathbf{x} \in \mathbb{R}^{h+1} : |x_0| > s, x_h \leq |x_0|y\}.$$

for $s_0 \leq s \leq t_0$ and $y \in [a, b]$. The fidi convergence on the class $\{1_{C_{s,y}}, s_0 \leq s \leq t_0, y \in [a, b]\}$ is a consequence of Theorem 2.3. We only need to prove tightness over $s \in [s_0, t_0]$ at one point y in order to conclude the tightness over a finite collection of points y_1, \dots, y_k ,

$k \geq 0$. Define the linearly ordered class $\mathcal{G} = \{1_{C_{s,y}}, s_0 \leq s \leq t_0\}$. The class \mathcal{G} is pointwise separable and its envelope function $1_{C_{s_0,y}}$ is in \mathcal{L}_0 . We now prove that (\mathcal{G}, ρ_h) is totally bounded. For $s_0 \leq s < t$, we have

$$\begin{aligned} \rho_h^2((1_{C_{s,y}} - 1_{C_{t,y}})^2) &= \{s^{-\alpha} + t^{-\alpha} - 2t^{-\alpha}\}L_h(y) \\ &\leq s^{-\alpha} + t^{-\alpha} - 2t^{-\alpha} \leq \alpha s_0^{-\alpha-1}|s - t|. \end{aligned} \quad (4.15)$$

Thus the class (\mathcal{G}, ρ_h) is totally bounded. Moreover, the convergence

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\bar{F}(u_n)} \mathbb{E} \left[(1\{|X_0| > su_n\}1\{X_h \leq |X_0|y\} - 1\{|X_0| > tu_n\}1\{X_h \leq |X_0|y'\})^2 \right] \\ = \{s^{-\alpha} + t^{-\alpha} - 2(s \vee t)^{-\alpha}\}L_h(y) \end{aligned}$$

is uniform on compact sets $[s_0, t_0]$ because of monotonicity. Therefore (2.13) holds.

5 Proof of Proposition 2.14

Let N_n be the number of returns to the state 1 before time n , that is

$$N_n = \sum_{j=0}^n 1\{X_j = 1\}.$$

Set also $\sigma_{-1} = -\infty$, $\sigma_0 = X_0 - 1$ and $\sigma_j = X_0 - 1 + \sum_{k=1}^j Z_{\sigma_{k-1}+1}$ for $j \geq 1$. Then, $\{N_n\}$ is the counting process associated to the delayed renewal process $\{\sigma_n\}$. That is, for $n, k \geq 0$,

$$N_n = k \Leftrightarrow \sigma_{k-1} \leq n < \sigma_k,$$

Since $\mathbb{E}[Z_0] < \infty$, setting $\lambda = 1/\mathbb{E}[Z_0]$, we have $N_n/n \rightarrow \lambda$ a.s. With this notation, we have, for every $s > 0$,

$$\sum_{j=0}^n 1\{X_j > u_n s\} = (X_0 - [u_n s])_+ + \sum_{j=1}^{N_n} (Z_{\sigma_{j-1}+1} - [u_n s])_+ + \varsigma_n, \quad (5.1)$$

where $\varsigma_n = (n - \sigma_{N_n}) \wedge (Z_{N_n} - [u_n s])_+$ is a correcting term accounting for the possibly incomplete last portion of the path. Since $\varsigma_n = O_P(1)$, it does not play any role in the asymptotics.

- Consider the case $\lim_{n \rightarrow \infty} n\bar{F}_Z(u_n) = 0$. Then, for an integer $m > \lambda$,

$$\begin{aligned} \mathbb{P} \left(\sum_{j=1}^{N_n} (Z_j - [u_n s])_+ \neq 0 \right) &\leq \mathbb{P}(N_n > mn) + \mathbb{P} \left(\sum_{j=1}^{mn} (Z_j - [u_n s])_+ \neq 0 \right) \\ &\leq \mathbb{P}(N_n > mn) + \mathbb{P}(\exists j \in \{1, \dots, mn\}, Z_j > [u_n s]) \\ &\leq \mathbb{P}(N_n > mn) + mn\bar{F}_Z([u_n s]) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

This proves our first claim.

We proceed with the case $\lim_{n \rightarrow \infty} n\bar{F}_Z(u_n) = \infty$. Using (2.28) and (5.1) we have

$$\begin{aligned} \sum_{j=0}^n \{1\{X_j > u_n s\} - \mathbb{P}(X_0 > u_n s)\} &= \sum_{j=1}^{N_n} \{(Z_{\sigma_{j-1}+1} - [u_n s])_+ - \mathbb{E}[(Z_0 - [u_n s])_+]\} \\ &\quad + (X_0 - [u_n s])_+ + \varsigma_n + \{N_n - \lambda n\} \mathbb{E}[(Z_0 - [u_n s])_+]. \end{aligned} \quad (5.2)$$

• Consider the case $n\bar{F}_Z(u_n) \rightarrow \infty$ and $\beta \in (1, 2)$. Since $\lim_{n \rightarrow \infty} \mathbb{E}[(Z_0 - [u_n s])_+] = 0$, we obtain, for every $s > 0$,

$$\begin{aligned} a_n^{-1} \sum_{j=0}^n \{1\{X_j > u_n s\} - \mathbb{P}(X_0 > u_n s)\} \\ = a_n^{-1} \sum_{j=1}^{N_n} \{Z_{\sigma_{j-1}+1} - \mathbb{E}[Z_0]\} - a_n^{-1} \sum_{j=1}^{N_n} \{Z_{\sigma_{j-1}+1} \wedge [u_n s] - \mathbb{E}[Z_0 \wedge [u_n s]]\} + o_P(1). \end{aligned}$$

By regular variation of \bar{F}_Z , we obtain

$$\text{var} \left(\sum_{j=1}^n \{Z_j \wedge [u_n s] - \mathbb{E}[(Z_0 \wedge [u_n s])]\} \right) = O(u_n^2 n \bar{F}_Z(u_n)).$$

The regular variation of \bar{F}_Z and the conditions $n\bar{F}_Z(u_n) \rightarrow \infty$ and $n\bar{F}_Z(a_n) \rightarrow 1$ imply that $u_n/a_n \rightarrow 0$. Define $h(x) = x\sqrt{\bar{F}_Z(x)}$. The function h is regularly varying at infinity with index $1 - \beta/2 > 0$ and thus

$$\lim_{n \rightarrow \infty} \frac{u_n \sqrt{n\bar{F}_Z(u_n)}}{a_n} = \lim_{n \rightarrow \infty} \frac{u_n \sqrt{\bar{F}_Z(u_n)}}{a_n \sqrt{\bar{F}_Z(a_n)}} = \lim_{n \rightarrow \infty} \frac{h(u_n)}{h(a_n)} = 0.$$

This yields

$$a_n^{-1} \sum_{j=0}^n \{1\{X_j > u_n s\} - \mathbb{P}(X_0 > u_n s)\} = a_n^{-1} \sum_{j=1}^{N_n} \{Z_{\sigma_{j-1}+1} - \mathbb{E}[Z_0]\} + o_P(1),$$

where the $o_P(1)$ term is locally uniform with respect to $s > 0$. Since the distribution of Z_0 is in the domain of attraction of the β -stable law and the sequence $\{Z_j\}$ is i.i.d., $\{a_n^{-1}(\sigma_{[ns]} - \lambda^{-1}ns), s > 0\} \Rightarrow \Lambda$, where Λ is a mean zero, totally skewed to the right β -stable Lévy process, and the convergence holds with respect to the J_1 topology on compact sets of $(0, \infty)$. Since $\lim_{n \rightarrow \infty} N_n/n = \lambda$ a.s., and since a Lévy process is stochastically continuous, this yields, by (Whitt, 2002, Proposition 13.2.1),

$$a_n^{-1} \sum_{j=1}^{N_n} \{Z_{\sigma_{j-1}+1} - \mathbb{E}[Z_0]\} \xrightarrow{d} \Lambda(\lambda), \quad n \rightarrow \infty.$$

This proves the second claim.

• Consider now the case $\beta > 2$. In that case, Vervaat's Lemma implies that $(N_n - \lambda n)/\sqrt{n}$ converges weakly to a gaussian distribution. Thus, (5.2) combined with $\mathbb{E}[(Z_0 - [u_n s])_+] = O(u_n \bar{F}_Z(u_n))$, yields

$$(N_n - \lambda n)\mathbb{E}[(Z_0 - [u_n s])_+] = O_P(u_n \bar{F}_Z(u_n)\sqrt{n}).$$

Next, we apply the Lindeberg central limit theorem for triangular arrays of independent random variables to prove that

$$\frac{1}{u_n \sqrt{n \bar{F}_Z(u_n)}} \sum_{j=1}^n \{(Z_{\sigma_{j-1}+1} - [u_n s])_+ - \mathbb{E}[(Z_0 - [u_n s])_+]\} \xrightarrow{d} N\left(0, \frac{2s^{1-\alpha}}{\alpha(\alpha-1)}\right).$$

By regular variation of \bar{F}_Z , we have, for all $\delta \in [2, \beta)$,

$$\mathbb{E}[(Z_0 - u_n s)_+]^\delta \sim C_\delta u_n^\delta \bar{F}_Z(u_n) s^{\delta-\beta},$$

with $C_\delta = \delta \int_1^\infty (z-1)^{\delta-1} z^{-\beta} dz$. Set

$$Y_{n,j}(s) = \frac{1}{u_n \sqrt{n \bar{F}_Z(u_n)}} \{(Z_j - [u_n s])_+ - \mathbb{E}[(Z_0 - [u_n s])_+]\}.$$

The previous computations yield, for $\delta \in (2, \beta)$ and $s > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} n \text{var}(Y_{n,1}(s)) &= \frac{2s^{2-\beta}}{(\beta-1)(\beta-2)}, \\ n \mathbb{E}[|Y_{n,1}|^\delta] &= O\left(\frac{nu_n^\delta \bar{F}_Z(u_n)}{u_n^\delta \{n \bar{F}_Z(u_n)\}^{\delta/2}}\right) = O(\{n \bar{F}_Z(u_n)\}^{1-\delta/2}) = o(1). \end{aligned}$$

We conclude that the Lindeberg central limit theorem holds. Convergence of the finite dimensional distribution is done along the same lines. Tightness with respect to the J_1 topology on $(0, \infty)$ is proved by applying (Billingsley, 1999, Theorem 13.5).

A Convergence in ℓ_∞

Theorem A.1 (Giné and Nickl (2016, Theorem 3.7.23)). *Let $\{\mathbb{Z}_n, n \in \mathbb{N}\}$, be a sequence of processes with values in $\ell_\infty(\mathcal{F})$. Then the following statements are equivalent.*

- (i) *The finite dimensional distributions of the processes \mathbb{Z}_n converge in law and there exists a pseudometric ρ on \mathcal{F} such that (\mathcal{F}, ρ) is totally bounded and for all $\epsilon > 0$,*

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}^* \left(\sup_{\rho(f,g) < \delta} |\mathbb{Z}_n(f) - \mathbb{Z}_n(g)| > \epsilon \right) = 0. \quad (\text{A.1})$$

(ii) There exists a process \mathbb{Z} whose law is a tight Borel probability measure on $\ell_\infty(\mathcal{F})$ and such that $\mathbb{Z}_n \Rightarrow \mathbb{Z}$ in $\ell_\infty(\mathcal{F})$.

Moreover, if (i) holds, then the process \mathbb{Z} in (ii) has a version with bounded uniformly continuous paths for ρ .

The following result provides a sufficient condition for (A.1) above. Let $\{X_{n,i}, 1 \leq i \leq m_n\}$, $n \geq 1$, be an array of row-wise i.i.d. random elements in a measurable space $(\mathbf{X}, \mathcal{X})$ and define $Z_{n,i}(f) = f(X_{n,i})$, $f \in \mathcal{F}$. Let a_n be a non decreasing sequence and \mathcal{F} be a set of measurable functions defined on \mathbf{X} . Define the random pseudometric d_n on \mathcal{F} by

$$d_n^2(f, g) = \frac{1}{a_n^2} \sum_{i=1}^{m_n} \{f(X_{n,i}) - g(X_{n,i})\}^2, \quad f, g \in \mathcal{F}.$$

Let $N(\epsilon, \mathcal{F}, d_n)$ be the minimum number of balls in the pseudometric d_n needed to cover \mathcal{F} . Let \mathbb{Z}_n be the empirical process defined by

$$\mathbb{Z}_n(f) = \frac{1}{a_n} \sum_{i=1}^{m_n} \{f(X_{n,i}) - \mathbb{E}[f(X_{n,i})]\}, \quad f \in \mathcal{F}.$$

Define finally the sup-norm $\|H\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |H(f)|$ for any functional H on \mathcal{F} . If \mathcal{F} is a pseudometric space and H is measurable on \mathcal{F} then the separability of \mathcal{F} implies that $\|H\|_{\mathcal{F}}$ is measurable.

Theorem A.2 (Adapted from [van der Vaart and Wellner \(1996, Theorem 2.11.1\)](#)). Assume that the pseudometric space \mathcal{F} is totally bounded and pointwise separable.

(i) For all $\eta > 0$,

$$\lim_{n \rightarrow \infty} a_n^{-2} m_n \mathbb{E}[\|\mathbb{Z}_{n,1}\|_{\mathcal{F}}^2 \mathbf{1}\{\|\mathbb{Z}_{n,1}\|_{\mathcal{F}} > \eta a_n\}] = 0. \quad (\text{A.2})$$

(ii) For every sequence $\{\delta_n\}$ which decreases to zero,

$$\lim_{n \rightarrow \infty} \sup_{\substack{f, g \in \mathcal{F} \\ \rho(f, g) \leq \delta_n}} \mathbb{E}[d_n^2(f, g)] = 0, \quad (\text{A.3})$$

$$\int_0^{\delta_n} \sqrt{\log N(\epsilon, \mathcal{F}, d_n)} d\epsilon \xrightarrow{\mathbb{P}} 0. \quad (\text{A.4})$$

Then \mathbb{Z}_n is asymptotically ρ -equicontinuous, i.e. (A.1) holds. .

Acknowledgements The research of Rafal Kulik was supported by the NSERC grant 210532-170699-2001. The research of Philippe Soulier and Olivier Wintenberger was partially supported by the ANR14-CE20-0006-01 project AMERISKA network. Philippe Soulier also acknowledges support from the LABEX MME-DII.

References

- Alsmeyer, G. (2003). On the harris recurrence of iterated random lipschitz functions and related convergence rate results. *Journal of Theoretical Probability*, 16(1):217–247.
- Basrak, B. and Segers, J. (2009). Regularly varying multivariate time series. *Stochastic Process. Appl.*, 119(4):1055–1080.
- Billingsley, P. (1999). *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition.
- Bradley, R. C. (2005). Basic properties of strong mixing conditions. A survey and some open questions. *Probability Surveys*, 2:107–144.
- Cline, D. B. H. (2007). Regular variation of order 1 nonlinear AR-ARCH models. *Stochastic Processes and their Applications*, 117(7):840—861.
- Davis, R. A. and Mikosch, T. (2009). The extremogram: A correlogram for extreme events. *Bernoulli*, 38A:977–1009. Probability, statistics and seismology.
- Douc, R., Moulines, E., and Stoffer, D. S. (2014). *Nonlinear time series*. Chapman & Hall/CRC Texts in Statistical Science Series. Chapman & Hall/CRC, Boca Raton, FL. Theory, methods, and applications with R examples.
- Drees, H. (1998). Optimal rates of convergence for estimates of the extreme value index. *The Annals of Statistics*, 26(1):434–448.
- Drees, H. (2000). Weighted approximations of tail processes for β -mixing random variables. *The Annals of Applied Probability*, 10(4):1274–1301.
- Drees, H. (2002). Tail empirical processes under mixing conditions. In *Empirical process techniques for dependent data*, pages 325–342. Birkhäuser Boston, Boston, MA.
- Drees, H. (2003). Extreme quantile estimation for dependent data, with applications to finance. *Bernoulli*, 9(4):617–657.
- Drees, H. and Rootzén, H. (2010). Limit theorems for empirical processes of cluster functionals. *Ann. Statist.*, 38(4):2145–2186.
- Drees, H., Segers, J., and Warchoł, M. (2014). Statistics for tail processes of markov chains. arxiv:1405.7721.
- Giné, E. and Nickl, R. (2016). *Mathematical Foundations of Infinite-Dimensional Statistical Models*. Cambridge University Press, New York.
- Hult, H. and Samorodnitsky, G. (2008). Tail probabilities for infinite series of regularly varying random vectors. *Bernoulli*, 14(3):838–864.

- Janssen, A. and Segers, J. (2014). Markov tail chains. *Journal of Applied Probability*, 51(4):1133–1153.
- Leadbetter, M. R., Rootzén, H., and de Haan, L. (1988). On the distribution of tail array sums for strongly mixing stationary sequences. *Ann. Appl. Probab.*, 8(3):868–885.
- Meyn, S. and Tweedie, R. L. (2009). *Markov chains and stochastic stability*. Cambridge University Press, Cambridge, second edition.
- Mikosch, T. and Wintenberger, O. (2013). Precise large deviations for dependent regularly varying sequences. *Probability Theory and Related Fields*, 156(3-4):851–887.
- Mikosch, T. and Wintenberger, O. (2014). The cluster index of regularly varying sequences with applications to limit theory for functions of multivariate Markov chains. *Probability Theory and Related Fields*, 159(1-2):157–196.
- Resnick, S. I. (2007). *Heavy-Tail Phenomena*. Springer Series in Operations Research and Financial Engineering. Springer, New York. Probabilistic and statistical modeling.
- Roberts, G. O., Rosenthal, J. S., Segers, J., and Sousa, B. (2006). Extremal indices, geometric ergodicity of markov chains, and mcmc. *Extremes*, 9(3-4):213–229.
- Rootzén, H. (1988). Maxima and exceedances of stationary Markov chains. *Adv. in Appl. Probab.*, 20(2):371–390.
- Rootzén, H. (2009). Weak convergence of the tail empirical process for dependent sequences. *Stoch. Proc. Appl.*, 119(2):468–490.
- Rootzén, H., Leadbetter, M. R., and de Haan, L. (1998). On the distribution of tail array sums for strongly mixing stationary sequences. *Ann. Appl. Probab.*, 8(3):868–885.
- Smith, R. L. (1992). The extremal index for a Markov chain. *Journal of Applied Probability*, 29(1):37–45.
- van der Vaart, A. W. and Wellner, J. A. (1996). *Weak convergence and empirical processes*. Springer, New York.
- Whitt, W. (2002). *Stochastic-process limits*. Springer-Verlag, New York.