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A NEW STABILITY RESULT FOR THE WULFF SHAPE

JULIEN ROTH

ABSTRACT. We prove that a closed convex hypersurface of a Euclidean space with almost constant anisotropic first and second mean curvatures is close (up to rescaling) to the Wulff shape.

1. INTRODUCTION

Geodesic hyperspheres in Euclidean spaces can be characterized among closed hypersurfaces by various properties. In particular, it is well known that geodesic hyperspheres are the only totally geodesic closed connected hypersurfaces in Euclidean spaces. The question of the stability of this characterization has been intensively studied in the last years by many authors (see [5, 6, 7, 1] for instance). In the anisotropic setting, the so-called Wulff shape plays the role of geodesic spheres and can be characterized by similar results (see [3, 4] for instance). Analogously to spheres, the Wulff shape is, up to homotheties, the only closed convex hypersurfaces with vanishing traceless anisotropic second fundamental form. Very recently, De Rosa and Giofrè [2] studied the stability of this characterization. Namely, he proved that if the traceless part of the anisotropic second fundamental form is sufficiently small, then the hypersurface is closed to the Wulff shape. The aim of the present note is to obtain a new stability result concerning the Wulff shape. Namely, we prove the following

Theorem 1.1. *Let $n \geq 2$ an integer, $F : \mathbb{S}^n \rightarrow \mathbb{R}$ be an elliptic integrand, $h > 0$, $p > 1$ and $R > 0$. Assume that $V(M) = V(\mathcal{W}_F)$ and that the extrinsic radius of M is smaller than R . Then, there exists $\varepsilon_0(n, p, h, R, F) > 0$ such that if for $\varepsilon \leq \varepsilon_0$, we have*

- $|H^F - h| < \varepsilon h$ and
- $|H_2^F - h_2| < \varepsilon h^2$ for a constant h_2 ,

then $|h^2 - h_2| \leq A\varepsilon$, for a positive constant A depending on n, h, R and F , and M is closed to the Wulff shape in the following sense. There exists a smooth parametrisation $\psi : \mathcal{W}_F \rightarrow M$, a vector $c_0 \in \mathbb{R}^{n+1}$ and a constant K depending on n, p, h, R and F so that

$$\|\psi - \text{Id} - c_0\|_{W^{2,p}(\mathcal{W})} \leq K\varepsilon^{\frac{p}{2}}.$$

Remark 1.2.

- Here $V(M)$ is the volume of M for the induced metric g .
- An elliptic integrand is a function F satisfying (1) stated in the preliminaries section.
- We recall that the extrinsic radius of M is the radius of the smallest closed ball in \mathbb{R}^{n+1} containing M .

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- Note that the right-hand sides in both pinching conditions of the theorem are respectively $h\varepsilon$ and $h^2\varepsilon$ for some homogeneity reasons, since for the Wulff shape, we have $H_2^F = (H^F)^2$. Equivalently, they can be replaced by $\frac{\varepsilon}{R}$ and $\frac{\varepsilon}{\sqrt{R}}$.

2. PRELIMINARIES

Let $F : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ be a smooth function satisfying the following convexity assumption

$$(1) \quad A_F = (\nabla dF + F \text{Id}_{|T_x \mathbb{S}^n})_x > 0,$$

for all $x \in \mathbb{S}^n$, where ∇dF is the Hessian of F . Here, > 0 mean positive definite in the sense of quadratic forms. Now, we consider the following map

$$\begin{aligned} \phi : \mathbb{S}^n &\rightarrow \mathbb{R}^{n+1} \\ x &\mapsto F(x)x + (\text{grad}_{|\mathbb{S}^n} F)_x \end{aligned}$$

The image $\mathcal{W}_F = \phi(\mathbb{S}^n)$ is called the Wulff shape of F and is a smooth hypersurface of \mathbb{R}^{n+1} . Moreover, by the condition (1), \mathcal{W}_F is convex. Note that if $F = 1$, the the Wulff shape is the sphere \mathbb{S}^n .

Now, let (M^n, g) be a n -dimensional compact, connected, oriented manifold without boundary, isometrically immersed into by X into \mathbb{R}^{n+1} . We denote by ν a normal unit vector field globally defined on M , that is, we have $\nu : M \rightarrow \mathbb{S}^n$. We set $S_F = A_F \circ d\nu$, where A_F is defined in (1). S_F is called the F -Weingarten operator or anisotropic shape operator, and its eigenvalues are the anisotropic principal curvatures that we will denote $\kappa_1, \kappa_2, \dots, \kappa_n$. Finally, for $r \in \{1, \dots, n\}$, the r -th anisotropic mean curvature is defined by

$$H_r^F = \frac{1}{\binom{n}{r}} \sum_{i_1 < \dots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}.$$

We also set $H_0^F = 1$ for convenience. Note that if $F = 1$ then this is the definition of the classical r -th mean curvatures. Finally, we recall these integral formulas proved by He and Li in [3] and which generalize the classical Hsiung-Minkowski formulas in the anisotropic setting.

$$(2) \quad \int_M (F(\nu)H_{r-1}^F + H_r^F \langle X, \nu \rangle) dv_g = 0,$$

Now, we have all the ingredients to prove Theorem 1.1.

3. PROOF OF THEOREM 1.1

We begin the proof of Theorem 1.1 by the following technical lemma.

Lemma 3.1. *Let (M^n, g) be a closed and convex Riemannian manifold, isometrically immersed into \mathbb{R}^{n+1} and assume that the extrinsic radius of M is smaller than R . Let h and h_2 be two positive constants and $\varepsilon \in (0, 1)$. If the mean curvature and the scalar curvature satisfy*

- $|H^F - h| < \varepsilon h$ and
- $|H_2^F - h_2| < \varepsilon h^2$,

for some positive ε , then

$$|h^2 - h_2| \leq A\varepsilon,$$

where A is a positive constant depending on h , R and F .

Proof. The proof of this lemma is based on the Hsiung-Minkowski formulas (2) for $r = 1$ and $k = 2$. Indeed, the Hsiung-Minkowski formula for $r = 2$ is the following

$$(3) \quad \int_M \left(H_2^F \langle X, \nu \rangle + F(\nu) H^F \right) dv_g = 0.$$

Then, we get

$$\begin{aligned} 0 &= \int_M \left(H_2^F \langle X, \nu \rangle + F(\nu) H \right) dv_g \\ &= \int_M \left(h_2 \langle X, \nu \rangle + F(\nu) H \right) dv_g + \int_M (H_2^F - h_2) \langle X, \nu \rangle dv_g \\ &= \frac{h_2}{h} \int_M h \langle X, \nu \rangle + \int_M F(\nu) H^F dv_g + \int_M (H_2 - h_2) \langle X, \nu \rangle dv_g \\ &= \frac{h_2}{h} \int_M H^F \langle X, \nu \rangle dv_g + \frac{h_2}{h} \int_M (h - H^F) \langle X, \nu \rangle dv_g + \int_M F(\nu) h dv_g + \int_M F(\nu) (H^F - h) dv_g \\ &\quad + \int_M (H_2^F - h_2) \langle X, \nu \rangle dv_g \end{aligned}$$

Now, we use the Hsiung-Minkowski formula for $r = 1$, that is

$$(4) \quad \int_M \left(H^F \langle X, \nu \rangle + F(\nu) \right) dv_g = 0,$$

to get

$$\begin{aligned} 0 &= -\frac{h_2}{h} \int_M F(\nu) dv_g + \frac{h_2}{h} \int_M (h - H^F) \langle X, \nu \rangle dv_g + \int_M F(\nu) h dv_g + \int_M F(\nu) (H^F - h) dv_g \\ &\quad + \int_M (H_2^F - h_2) \langle X, \nu \rangle dv_g \\ &= \left(h - \frac{h_2}{h} \right) \int_M F(\nu) dv_g + \frac{h_2}{h} \int_M (h - H^F) \langle X, \nu \rangle dv_g + \int_M F(\nu) (H^F - h) dv_g \\ &\quad + \int_M (H_2^F - h_2) \langle X, \nu \rangle dv_g \end{aligned}$$

Then, using both conditions $|H^F - h| < \varepsilon h$ and $|H_2^F - h_2| < \varepsilon h^2$, we get

$$\left| h - \frac{h_2}{h} \right| \int_M F(\nu) dv_g \leq h_2 \varepsilon RV(M) + \varepsilon h \int_M F(\nu) dv_g + \varepsilon h^2 RV(M).$$

Using the fact that $|H_2^F| \leq (H^F)^2$, we deduce from the assumptions on h and h_2 that

$$\begin{aligned} |h_2| &\leq |H_2^F| + h^2 \varepsilon \\ &\leq (H^F)^2 + h^2 \varepsilon \\ &\leq (h + h\varepsilon)^2 + h^2 \varepsilon \\ &\leq 5h^2 \end{aligned}$$

for $\varepsilon < 1$. Thus, we have

$$\begin{aligned} \left| h^2 - h_2 \right| \int_M F(\nu) dv_g &\leq \varepsilon h^2 \int_M F(\nu) dv_g + (h^3 + hh_2) RV(M) \varepsilon \\ &\leq \varepsilon h^2 \int_M F(\nu) dv_g + 6h^3 RV(M) \varepsilon \end{aligned}$$

and we obtain

$$\begin{aligned} |h^2 - h_2| &\leq \left(h^2 + \frac{6h^3 R}{\inf(F)} \right) \varepsilon \\ &\leq A(h, R, F)\varepsilon, \end{aligned}$$

which gives the wanted assertion. \square

Now, using this lemma together with the result of De Rosa and Giofrè, we can prove Theorem 1.1. We recall the main result of [1].

Theorem 3.2 (De Rosa-Giofrè [1]). *Let $n > 2$, $p \in (1, p)$ and $F : \mathbb{S}^n \rightarrow \mathbb{R}^+$ satisfying the convexity assumption (1). There exists a constant $\delta_0 = \delta_0(n, p, F) > 0$ such that if Σ is closed convex hypersurface into \mathbb{R}^{n+1} satisfying*

$$V(M) = V(\mathcal{W}_F) \quad \text{and} \quad \int_M \|S_F - H^F \text{Id}\|^p dv_g \leq \delta$$

with $\delta \leq \delta_0$ then there exists a smooth parametrisation $\psi : \mathcal{W}_F \rightarrow M$, a vector $c_0 \in \mathbb{R}^{n+1}$ and a constant C depending on n, p and F so that

$$\|\psi - \text{Id} - c_0\|_{W^{2,p}(\mathcal{W})} \leq C\delta.$$

Now, if $|H^F - h| < \varepsilon h$ and $|H_2^F - h_2| < \varepsilon h^2$, then from Lemma 3.1

$$|h^2 - h_2| \leq A\varepsilon,$$

with A a positive constant depending on h, R and F . Thus, we deduce that

$$\begin{aligned} (H^F)^2 - H_2^F &\leq |(H^F)^2 - h^2| + |h^2 - h_2| + |h_2 - H_2^F| \\ &\leq (|H^F| + h) \cdot |H^F - h| + A\varepsilon + h^2\varepsilon \\ &\leq (2h + h\varepsilon)h\varepsilon + A\varepsilon + h^2\varepsilon \\ &\leq A'\varepsilon, \end{aligned}$$

where A' is a positive constant depending only on h, R and F . On the other hand, we have

$$(H^F)^2 - H_2^F = \frac{1}{n^2(n-1)} \sum_{i,j=1}^n (\kappa_i - \kappa_j)^2,$$

so we get

$$\sum_{i,j=1}^n (\kappa_i - \kappa_j)^2 \leq A''\varepsilon.$$

where $A'' = n^2(n-1)A'$ is also a positive constant depending only on h, n, R and F . Hence, M has almost vanishing anisotropic second fundamental form. Indeed,

we have

$$\begin{aligned}
\|S_F - H^F \text{Id}\|_\infty^2 &= \sum_{i=1}^n (k_i - H^F)^2 \\
&= \sum_{i=1}^n \left(\kappa_i - \frac{1}{n} \sum_{j=1}^n \kappa_j \right)^2 \\
&= \frac{1}{n} \sum_{i,j=1}^n (\kappa_i - \kappa_j)^2 \\
&\leq \frac{1}{n} A'' \varepsilon.
\end{aligned}$$

Finally, we fix $p > 1$ and set $\varepsilon_0 = \inf \left\{ 1, \frac{n(\delta_0 V(\mathcal{W}_F))^{\frac{2}{p}}}{A''} \right\}$ where A'' is the constant defined above and δ_0 comes from Theorem 3.2. Note that ε_0 depends on n, p, h, R and F . Now, let $\varepsilon \leq \varepsilon_0$. We set $\delta = \frac{(A'' \varepsilon)^{\frac{p}{2}}}{n^{\frac{p}{2}} V(\mathcal{W}_F)}$. Since $\varepsilon \leq \varepsilon_0$ and from the definition of δ , we have $\delta \leq \delta_0$ and

$$\int_M \|S_F - H^F \text{Id}\|^p dv_g \leq \|S_F - H^F \text{Id}\|_\infty V(\mathcal{W}_F) \leq \delta.$$

Thus, since by assumption, we also have $V(M) = V(\mathcal{W}_F)$, we can apply Theorem 3.2 to obtain that there exists a smooth parametrisation $\psi : \mathcal{W}_F \rightarrow M$ a vector $c_0 \in \mathbb{R}^{n+1}$ and a constant C depending on n, p, h, R and F so that

$$\|\psi - \text{Id} - c_0\|_{W^{2,p}(\mathcal{W})} \leq C\delta = K\varepsilon^{\frac{p}{2}},$$

where $K = \frac{(nA'')^{\frac{p}{2}} C}{V(\mathcal{W}_F)}$ is a positive constant depending only on n, p, h, R and F since A'' depends on n, h, R and F , $V(\mathcal{W}_F)$ depends on n and F and C depends on n, p and F . This concludes the proof of Theorem 1.1.

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