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PAINLEVÉ MONODROMY MANIFOLDS, DECORATED CHARACTER VARIETIES AND CLUSTER ALGEBRAS.

LEONID CHEKHOV, MARTA MAZZOCCO, VLADIMIR RUBTSOV

Abstract. In this paper we introduce the concept of *decorated character variety* for the Riemann surfaces arising in the theory of the Painlevé differential equations. Since all Painlevé differential equations (apart from the sixth one) exhibit Stokes phenomenon, it is natural to consider Riemann spheres with holes and bordered cusps on such holes. The decorated character is defined as complexification of the bordered cusped Teichmüller space introduced in [8]. We show that the decorated character variety of a Riemann sphere with s holes and $n > 1$ cusps is a Poisson manifold of dimension $3s + 2n - 6$ and we explicitly compute the Poisson brackets which are naturally of cluster type. We also show how to obtain the confluence procedure of the Painlevé differential equations in geometric terms.

1. Introduction

It is well known that the sixth Painlevé monodromy manifold is the $SL_2(\mathbb{C})$ character variety of a 4 holed Riemann sphere. The real slice of this character variety is the decorated Teichmüller space of a 4 holed Riemann sphere, and can be combinatorially described by a fat-graph and shear coordinates. By complexifying the shear coordinates, flat coordinates for the character variety of a 4 holed Riemann sphere were found in [7].

For the other Painlevé equations, the interpretation of their monodromy manifolds as "character varieties" of a Riemann sphere with boundary is still an extremely difficult problem due to the fact that the linear problems associated to the other Painlevé equations exhibit Stokes phenomenon. This implies that some of the boundaries have *bordered cusps* on them [8]. Being on the boundary, these bordered cusps escape the usual notion of character variety leading to the necessity of introducing a decoration.

In this paper we present a decoration which truly encodes the geometry of each cusped boundary. On the real slice of our decorated character variety, this decoration corresponds to choosing some horocycles to associate a λ-length to each bordered cusp.[1] This geometric description allows us to introduce flat coordinates in the corresponding *bordered cusped Teichmüller space* (see [8] for the definition of this notion) and by complexification on the decorated character variety.

This leads us to define explicitly a set of coordinates on the decorated character variety of the Riemann spheres with bordered cusps which arise in the theory of the

[1]We use the term bordered cusp meaning a vertex of an ideal triangle in the Poincaré metric in order to distinguish it from standard cusps (without borders) associated to punctures on a Riemann surface.

Painlevé differential equations and to compute the Poisson brackets in these coordinates. Such Poisson brackets coincide with the cluster algebra Poisson structure as predicted in [8].

We note that another approach to this problem was developed in [2] where the definition of wild character variety was proposed following a construction by Gaiotto, Moore and Neitzke [16] which consisted in introducing spurious punctures at the points of intersection between the Stokes lines and some fixed circles around each irregular singularity. This description does not seem compatible with the confluence procedure of the Painlevé equations, which is one of our motivations to propose a new approach.

We show that, if we exclude PVI, we have nine possible Riemann surfaces with bordered cusps, for which we define the decorated character variety. We show that in each case there is a singled Poisson sub-algebra which is the coordinate ring of an affine variety (the monodromy manifold of one of the Painlevé differential equations) We shall explain it in details in the Section 5. Indeed, all the Painlevé differential equations arise as monodromy preserving deformations of an auxiliary linear system of two first order ODEs. The monodromy data of such auxiliary linear system are encoded in their monodromy manifolds which can all be described by affine cubic surfaces in $\mathbb{C}^3$ defined by the zero locus of the corresponding polynomials in $\mathbb{C}[x_1, x_2, x_3]$ given in Table 1, where $\omega_1, \ldots, \omega_4$ are some constants related to the parameters appearing in the corresponding Painlevé equation as described in Section 2.

P-eqs	Polynomials
PVI	$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
PV	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + 1 + \omega_3^2 - \frac{\omega_3(\omega_2+\omega_1\omega_3)(\omega_1+\omega_2\omega_3)}{(\omega_3^2-1)^2}$
PV_{deg}	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_1 - 1$
PIV	$x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2(x_2 + x_3) + \omega_2(1 + \omega_1 - \omega_2)$
$PIII$	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_1 - 1$
$PIII^{D_7}$	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 - x_2$
$PIII^{D_8}$	$x_1x_2x_3 + x_1^2 + x_2^2 - x_2$
PII^{JM}	$x_1x_2x_3 - x_1 + \omega_2 x_2 - x_3 - \omega_2 + 1$
PII^{FN}	$x_1x_2x_3 + x_1^2 + \omega_1 x_1 - x_2 - 1$
PI	$x_1x_2x_3 - x_1 - x_2 + 1$

Table 1.

Note that in Table 1, we distinguish ten different monodromy manifolds, the $PIII^{PIII}$, $PIII^{PIII^{D_7}}$ and $PIII^{PIII^{D_8}}$ correspond to the three different cases of the third Painlevé equation according to Sakai's classification [27], and the two monodromy manifolds PII^{FN} and PII^{JM} associated to the same second Painlevé equation correspond to the two different isomonodromy problems found by Flaschka–Newell [13] and Jimbo–Miwa [21] respectively.

Our methodology consists in reproducing the famous confluence scheme for the Painlevé equations:

$$\begin{tikzcd}
 & & P_{III}^{D_6} \arrow[r] \arrow[dr] & P_{III}^{D_7} \arrow[r] \arrow[dr] & P_{III}^{D_8} \\
P_{VI} \arrow[r] & P_V \arrow[ur] \arrow[r] \arrow[dr] & P_V^{deg} \arrow[ur] \arrow[dr] & P_{II}^{JM} \arrow[r] & P_I \\
 & & P_{IV} \arrow[r] \arrow[ur] & P_{II}^{FN} \arrow[ur] &
\end{tikzcd}$$

in terms of the following two basic operations on the underlying Riemann sphere:

- *Chewing-gum:* hook two holes together and stretch to infinity by keeping the area between them finite (see Fig. 1).
- *Cusps removal:* pull two cusps on the same hole away by tearing off an ideal triangle (see Fig. 2).

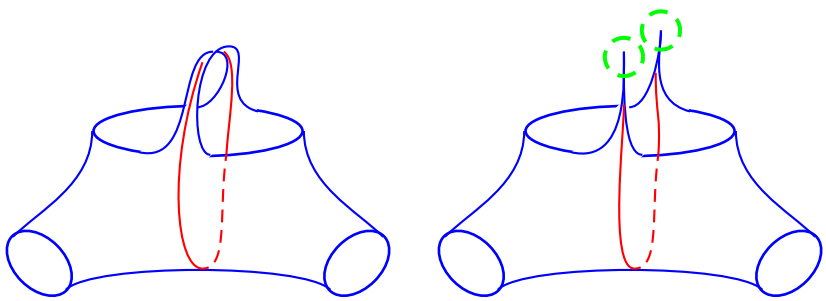

FIGURE 1. The process of confluence of two holes on the Riemann sphere with four holes: as a result we obtain a Riemann sphere with one less hole, but with two new cusps on the boundary of this hole. The red geodesic line which was initially closed becomes infinite, therefore two horocycles (the green dashed circles) must be introduced in order to measure its length.

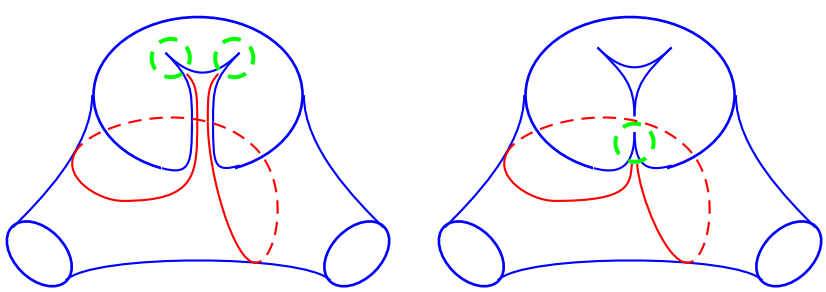

FIGURE 2. The process of breaking up a Riemann surface with boundary cusps: by grabbing together two cusps and pulling we tear apart an ideal triangle.

As shown by the first two authors in [8], these two operations correspond to certain asymptotics in the shear coordinates and perimeters. We will deal with such asymptotics in Section 5. The confluence process on the underlying Riemann spheres with cusped boundaries is described in Fig. 3.

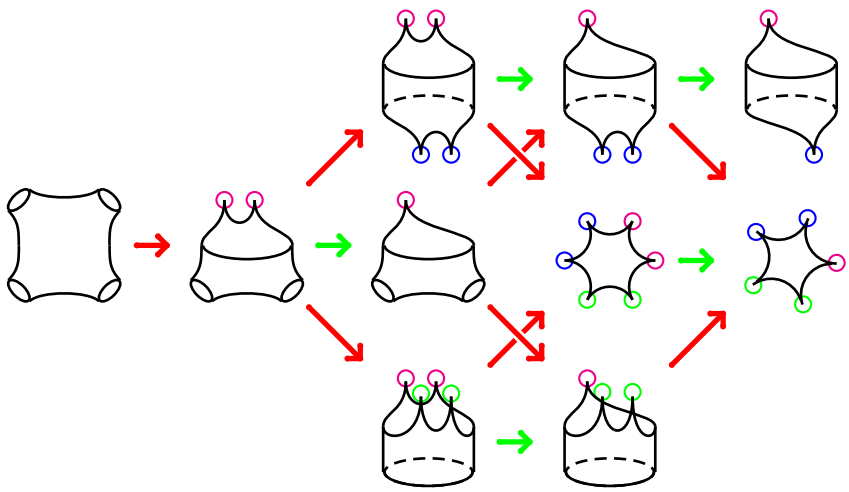

FIGURE 3. The table of confluences of Riemann surfaces from the Painlevé perspective. The red arrows correspond to chewing-gum moves, the green ones to cusp removal.

Note that these results agree with the work by T. Sutherland [28] who used the auxiliary linear problem to produce a quadratic differential on the same underlying Riemann spheres with cusped boundaries. In his work, Sutherland associated a quiver to each of the above Painlevé cusped Riemann spheres and explicitly exhibit the canonical connected component of the space of numerical stability conditions of the Painlevé quivers.

In our work, cluster algebras appear naturally when describing the bordered cusped Teichmüller space of each Riemann sphere with bordered cusps. Indeed, as shown in [8], when bordered cusps arise, it is possible to introduce a *generalised lamination* on the Riemann surface consisting only of geodesics which start and terminate at the cusps. The geodesic length functions (well defined by fixing horocycles at each cusp) in this lamination are the coordinates in the bordered cusped Teichmüller space, while the decoration itself is given by the choice of horocycles. In the Poisson structure given by the Goldman bracket, these coordinates satisfy the cluster algebra Poisson bracket. This is due to the fact that the geodesics in the lamination do not intersect in the interior of the Riemann sphere, but come together asymptotically in the bordered cusps. We also study the corresponding cluster mutations and show that in the case of a Riemann sphere with four holes they correspond to the procedure of analytic continuation for solutions to the sixth Painlevé equation, thus showing that this procedure of analytic continuation satisfies the Laurent phenomenon. For the other Painlevé equations, the cluster algebra

mutations correspond to the action of the Mapping Class Group on the cusped lamination.

We define the decorated character variety as the complexification of the bordered cusped Teichmüller space so that by complexifying the coordinates given by the generalised laminations we obtain coordinates on the decorated character varieties. We show that in the case of the Painlevé differential equations, the decorated character variety is a Poisson manifold of dimension $3s + 2n - 6$, where s is the number of holes and $n > 1$ is the number of cusps. We show that in each case the decorated character variety admits a special Poisson sub–manifold defined by the set of functions which Poisson commute with the frozen cluster variables. This sub–manifold is defined as a cubic surface $\mathcal{M}_\phi := \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3]/\langle \phi = 0\rangle)$, where ϕ is one of the polynomials in Table 1, with the natural Poisson bracket defined by:

$$\{x_1, x_2\} = \frac{\partial \phi}{\partial x_3}, \quad \{x_2, x_3\} = \frac{\partial \phi}{\partial x_1}, \quad \{x_3, x_1\} = \frac{\partial \phi}{\partial x_2}. \tag{1.1}$$

This paper is organised as follows: in Section 2, we recall the link between the parameters $\omega_1, \ldots, \omega_4$ and the Painlevé parameters α, β, γ and δ in each Painlevé equation and discuss the natural Poisson bracket (1.1) on each cubic. In Section 3, we remind some important notions on the combinatorial description on the bordered cusped Teichmüller space. In Section 4, we introduce the notion of decorated character variety. In Section 5, we present the flat coordinates for each cubic and describe the laminations and the corresponding cluster algebra structure. In Section 6, we explain the generalised cluster algebra structure appearing in the case of PVI, PV, $PIII$ and PIV. In the Appendix we discuss the singularity theory.

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2. The Painlevé the monodromy manifolds and their Poisson structure

According to [26], the monodromy manifolds $\mathcal{M}^{(d)}$ have all the form

$$x_1x_2x_3 + \epsilon_1^{(d)}x_1^2 + \epsilon_2^{(d)}x_2^2 + \epsilon_3^{(d)}x_3^2 + \omega_1^{(d)}x_1 + \omega_2^{(d)}x_2 + \omega_3^{(d)}x_3 + \omega_4^{(d)} = 0, \tag{2.2}$$

where d is an index running on the list of the Painlevé cubics PVI, PV, PV_{deg}, PIV, $PIII^{D_6}, PIII^{D_7}, PIII^{D_8}, PII^{JM}, PII^{FN}, PI$ and the parameters $\epsilon_i^{(d)}$, $\omega_i^{(d)}$, $i = 1, 2, 3$ are given by:

$$\epsilon_1^{(d)} = \begin{cases} 1 & \text{for } d = PVI, PV, PIII, PV_{deg}, PIII^{D_7}, PIII^{D_8}, PIV, PII^{FN}, \\ 0 & \text{for } d = PII^{JM}, PI, \end{cases}$$

$$\epsilon_2^{(d)} = \begin{cases} 1 & \text{for } d = PVI, PV, PIII, PV_{deg}, PIII^{D_7}, PIII^{D_8} \\ 0 & \text{for } d = PIV, PII^{FN}, PII^{JM}, PI, \end{cases}$$

$$\epsilon_3^{(d)} = \begin{cases} 1 & \text{for } d = PVI, \\ 0 & \text{for } d = PV, PIII, PV_{deg}, PIII^{D_7}, PIII^{D_8}, PIV, PII^{FN}, PII^{JM}, PI. \end{cases}$$

while

$$
\begin{aligned}
&\omega_1^{(d)} = -G_1^{(d)}G_\infty^{(d)} - \epsilon_1^{(d)}G_2^{(d)}G_3^{(d)}, \quad \omega_2^{(d)} = -G_2^{(d)}G_\infty^{(d)} - \epsilon_2^{(d)}G_1^{(d)}G_3^{(d)},\\
&\omega_3^{(d)} = -G_3^{(d)}G_\infty^{(d)} - \epsilon_3^{(d)}G_1^{(d)}G_2^{(d)},\\
&\omega_4^{(d)} = \epsilon_2^{(d)}\epsilon_3^{(d)}\left(G_1^{(d)}\right)^2 + \epsilon_1^{(d)}\epsilon_3^{(d)}\left(G_2^{(d)}\right)^2 + \epsilon_1^{(d)}\epsilon_2^{(d)}\left(G_3^{(d)}\right)^2 + \left(G_\infty^{(d)}\right)^2 +\\
&\qquad +G_1^{(d)}G_2^{(d)}G_3^{(d)}G_\infty^{(d)} - 4\epsilon_1^{(d)}\epsilon_2^{(d)}\epsilon_3^{(d)},
\end{aligned}
\tag{2.3}
$$

where $G_1^{(d)}, G_2^{(d)}, G_3^{(d)}, G_\infty^{(d)}$ are some constants related to the parameters appearing in the Painlevé equations as follows:

$$
G_1^{(d)} = \begin{cases} 2\cos\pi\theta_0 & d = PVI, PV, PIII, PV_{deg}, PIV, PII^{FN}\\ 1 & d = PIII^{D_8}, PII^{JM}, PI\\ \infty & d = PIII^{D_7},\\ 0 & d = PIII^{D_8}, \end{cases}
$$

$$
G_2^{(d)} = \begin{cases} 2\cos\pi\theta_1 & d = PVI, PV,\\ 2\cos\pi\theta_\infty & d = PIII, PV_{deg}, PIV,\\ e^{i\pi\theta_0} & d = PII^{JM}\\ 1 & d = PIII^{D_8}, PII^{FN}\, PI\\ \infty & d = PIII^{D_7}, PIII^{D_8}, \end{cases}
$$

$$
G_3^{(d)} = \begin{cases} 2\cos\pi\theta_t & d = PVI,\\ 2\cos\pi\theta_\infty & d = PV, PIV\\ 1 & d = PII^{JM},\\ 0 & d = PIII, PV_{deg}, PIII^{D_7}, PIII^{D_8}, PII^{FN}, PI \end{cases}
\tag{2.4}
$$

$$
G_\infty^{(d)} = \begin{cases} 2\cos\pi\theta_\infty & d = PVI, PIV\\ 1 & d = PV, PV_{deg}, D_8, PII^{JM}, PII^{FN}, PI\\ e^{i\pi\theta_0} & d = PIII\\ 0 & d = PIII^{D_7}, PIII^{D_8}. \end{cases}
$$

where the parameters $\theta_0, \theta_1, \theta_t, \theta_\infty$ are related to the Painlevé equations parameters in the usual way [21]. Note that for $PIII^{D_7}$ the parameters G_1 and G_2 tend to infinity - we take this limiti in such a way that $\omega_1 = -G_1G_\infty$ and $\omega_2 = -G_2G_\infty$ are not zero, while $\omega_4 = 0$. Similarly for $PIII^{D_8}$.

Remark 2.1. Observe that in the original article [26] the cubic corresponding to the Flaschka–Newell isomonodromic problem [13] is in the form $x_1x_2x_3 + x_1 - x_2 + x_3 + 2\cos\pi\theta_0 = 0$. This can be obtained from our cubic $PII : E_7^*$ $x_1x_2x_3 + x_1^2 + \omega_1x_1 - x_2 + 1$ by the following diffeomorphism (away from $x_2x_3 = 0$):

$$
x_1 \to -sx_1, \qquad x_2 \to \frac{1}{s}x_2, \quad x_3 \to \frac{s^2x_1^2 - \frac{1+x_1x_2}{s}x_3}{x_1x_2},
$$

where $s = 2\cos\pi\theta_0$. The reason to choose the cubic in the form $PII : E_7^*$ will be clear in Section 5.

Remark 2.2. Note that the $PIII^{D_7}$, $PIII^{D_8}$ and PI cubics have different signs in [26], which can both be obtained by a trivial rescaling of the variables x_1, x_2, x_3.

2.1. **Natural Poisson bracket on the monodromy manifold.** We would like to address here some natural facts that arise when comparing the various descriptions of family of affine cubics surfaces with 3 lines at infinity (2.2).

First of all, the projective completion of the family of cubics 2.2 with $\epsilon_i^{(d)} \neq 0$ for all $i = 1, 2, 3$ has singular points only in the finite part of the surface and if any of $\epsilon_i^{(d)}, i = 1, 2, 3$ vanish, then $\mathcal{M}^{(d)}$ is singular at infinity with singular points in homogeneous coordinates $X_i = 1$ and $X_j = 0, j \neq i$ ([24]). Here $x_i = \frac{X_i}{X_0}$.

One can consider this family of cubics as a variety $\mathcal{S} = \{(\bar{x}, \bar{\omega}) \in \mathbb{C}^3 \times \Omega) : S(\bar{x}, \bar{\omega}) = 0\}$ where $\bar{x} = (x_1, x_2, x_3)$, $\bar{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and the "$\bar{x}-$forgetful" projection $\pi : \mathcal{S} \to \Omega : \pi(\bar{x}, \bar{\omega}) = \bar{\omega}$. This projection defines a family of affine cubics with generically non–singular fibres $\pi^{-1}(\bar{\omega})$ (we will discuss the nature of these singularities in Subsection 6.3).

The cubic surface $S_{\bar{\omega}}$ has a volume form $\vartheta_{\bar{\omega}}$ given by the Poincaré residue formulae:

$$\vartheta_{\bar{\omega}} = \frac{dx_1 \wedge dx_2}{(\partial S_{\bar{\omega}})/(\partial x_3)} = \frac{dx_2 \wedge dx_3}{(\partial S_{\bar{\omega}})/(\partial x_1)} = \frac{dx_3 \wedge dx_1}{(\partial S_{\bar{\omega}})/(\partial x_2)}. \tag{2.5}$$

The volume form is a holomorphic 2-form on the non-singular part of $S_{\bar{\omega}}$ and it has singularities along the singular locus. This form defines the Poisson brackets on the surface in the usual way as

$$\{x_1, x_2\}_{\bar{\omega}} = \frac{\partial S_{\bar{\omega}}}{\partial x_3} \tag{2.6}$$

and the other brackets are defined by circular transposition of x_1, x_2, x_3. It is a straightforward computation to show that for $(i, j, k) = (1, 2, 3)$:

$$\{x_i, x_j\}_{\bar{\omega}} = \frac{\partial S_{\bar{\omega}}}{\partial x_k} = x_i x_j + 2\epsilon_i^d x_k + \omega_i^d \tag{2.7}$$

and the volume form (2.5) reads as

$$\vartheta_{\bar{\omega}} = \frac{dx_i \wedge dx_j}{(\partial S_{\bar{\omega}})/(\partial x_k)} = \frac{dx_i \wedge dx_j}{(x_i x_j + 2\epsilon_i^d x_k + \omega_i^d)}. \tag{2.8}$$

In a special case of PVI , i.e. the $\widetilde{D}_4$ cubic with parameters $\omega_i = 0$ for $i = 1, 2, 3$ and $\omega_4 = -4$, there is an isomorphism $\pi : \mathbb{C}^* \times \mathbb{C}^*/\eta \to S_{\bar{\omega}}$: [4]

$$\pi(u, v) \to (x_1, x_2, x_3) = (-u - 1/u, -v - 1/v, -uv - 1/uv), \tag{2.9}$$

where η is the involution of $\mathbb{C}^* \times \mathbb{C}^*$ given by $u \to 1/u$, $v \to 1/v$. The log-canonical 2-form $\bar{\vartheta} = \frac{du \wedge dv}{uv}$ defines a symplectic structure on $\mathbb{C}^* \times \mathbb{C}^*$ which is invariant with respect the involution η and therefore defines a symplectic structure on the non-singular part of the cubic surface $S_{\bar{\omega}}$ for $\omega_i = 0$ for $i = 1, 2, 3$ and $\omega_4 = -4$.

The relation between the log-canonical 2-form $\bar{\vartheta} = \frac{du \wedge dv}{uv}$ and the Poisson brackets on the surface $S_{\bar{\omega}}$ can be extended to all values of the parameters $\bar{\omega}$ and for all the Painlevé cubics as we shall show in this paper. In fact the flat coordinates that we will introduce in Section 5 are such that their exponentials satisfy the log-canonical Poisson bracket.

Remark 2.3. The cubic $\mathcal{M}^{(PVI)}$ appears in many different contexts outside of the Painlevé theory. For example, it was studied in Oblomkov' s work (see [24]) in relation to Cherednik algebras and M. Gross, P. Hacking and S.Keel (see Example 5.12 of [18]) claim that the family 2.2 can be "uniformized" by some analogues of theta-functions related to toric mirror data on log-Calabi-Yau surfaces. More precisely, the projectivisation Y of 2.2 with the cubic divisor $\Delta : X_1 X_2 X_3 = 0$ is an

example of so called *Looeijnga pair* and $Y \setminus \Delta$ is a log-symplectic Calabi-Yau variety with the holomorphic 2-form 2.5. We shall quantize this log-Calabi-Yau variety in the subsequent paper [9].

3. Combinatorial description of the bordered cusped Teichmüller space

Let us start with the standard case, i.e. when no cusps are present. According to Fock [14] [15], the fat graph associated to a Riemann surface $\Sigma_{g,n}$ of genus g and with n holes is a connected three–valent graph drawn without self-intersections on $\Sigma_{g,n}$ with a prescribed cyclic ordering of labelled edges entering each vertex; it must be a maximal graph in the sense that its complement on the Riemann surface is a set of disjoint polygons (faces), each polygon containing exactly one hole (and becoming simply connected after gluing this hole). In the case of a Riemann sphere $\Sigma_{0,4}$ with 4 holes, the fat–graph is represented in Fig. 4.

The geodesic length functions, which are traces of hyperbolic elements in the Fuchsian group $\Delta_{g,s}$ such that

$$\Sigma_{g,s} \sim \mathbb{H}/\Delta_{g,s}$$

are obtained by decomposing each hyperbolic matrix $\gamma \in \Delta_{g,s}$ into a product of the so–called *right, left and edge matrices:*

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad L := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \qquad X_{s_i} := \begin{pmatrix} 0 & -\exp\left(\frac{s_i}{2}\right) \\ \exp\left(-\frac{s_i}{2}\right) & 0 \end{pmatrix},$$

where s_i is the shear coordinate associated to the i-th edge in the fat graph.

In [8] the notion of fat-graph was extended to allow cusps. Here we present this definition adapted to the special cases dealt in the current paper:

Definition 3.1. We call *cusped fat graph* (a graph with the prescribed cyclic ordering of edges entering each vertex) $\mathcal{G}_{g,s,n}$ a *spine of the Riemann surface* $\Sigma_{g,s,n}$ with g handles, s and $n > 0$ decorated bordered cusps if

(a) this graph can be embedded without self-intersections in $\Sigma_{g,s,n}$;
(b) all vertices of $\mathcal{G}_{g,s,n}$ are three-valent except exactly n one-valent vertices (endpoints of the open edges), which are placed at the corresponding bordered cusps;
(c) upon cutting along all *nonopen* edges of $\mathcal{G}_{g,s,n}$ the Riemann surface $\Sigma_{g,s,n}$ splits into s polygons each containing exactly one hole and being simply connected upon contracting this hole.

Definition 3.2. We call geometric *cusped geodesic lamination* (CGL) on a bordered cusped Riemann surface a set of nondirected curves up to a homotopic equivalence such that

(a) these curves are either closed curves (γ) or *arcs* ($\mathfrak{a}$) that start and terminate at bordered cusps (which can be the same cusp);
(b) these curves have no (self)intersections inside the Riemann surface (but can be incident to the same bordered cusp);
(c) these curves are not empty loops or empty loops starting and terminating at the same cusp.

In the case of arcs, the geodesic length functions are now replaced by the signed geodesic lengths of the parts of arcs contained between two horocycles decorating the corresponding bordered cusps; the sign is negative when these horocycles intersect.

Combinatorially speaking this corresponds to calculating the lengths of such arcs by associating to each arc a matrix in $SL_2(\mathbb{R})$ in the same way as before, i.e. by taking products of left, right and edge matrices, but then by taking the trace of such product of matrices *multiplied by the cusp matrix:* $K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ *at the right hand side of the whole expression.* For example the arc b in Fig. 6 has length l_b such that

$$\exp\left(\frac{l_b}{2}\right) = \mathrm{Tr}\left(X(k_1)RX(s_3)RX(s_2)RX(p_2)RX(s_2)LX(s_3)LX(k_1)K\right).$$

Note that in all fat graphs in this paper we distinguish the shear coordinates s_1, s_2, s_3 which correspond to the edges in the central T shaped part of the graph and the shear coordinates $k_1, \ldots, k_6$ which arise when breaking holes.

In [8] it is proved that for every cusped fat-graph with the additional property that the polygons containing holes with no cusps are monogons, there exists a complete cusped geodesic lamination which consists only of arcs and simple loops around the un-cusped holes. Loosely speaking, this means that all lengths of any closed geodesic or of any arc in the Riemann surface is a Laurent polynomial of the lengths of the elements in the lamination.

The Poisson brackets between lengths of arcs and closed geodesics can be computed by using the Weil–Petersson bracket , which is shear coordinates becomes [5, 6]

$$\{f(\mathbf{Z}), g(\mathbf{Z})\} = \sum_{\substack{\text{3-valent} \\ \text{vertices } \alpha = 1}}^{4g+2s+|\delta|-4} \sum_{i=1}^{3 \bmod 3} \left(\frac{\partial f}{\partial Z_{\alpha_i}}\frac{\partial g}{\partial Z_{\alpha_{i+1}}} - \frac{\partial g}{\partial Z_{\alpha_i}}\frac{\partial f}{\partial Z_{\alpha_{i+1}}}\right), \tag{3.10}$$

where Z_α are the shear coordinates on each edge and the sum ranges all three-valent vertices of a graph and α_i are the labels of the cyclically (clockwise) ordered ($\alpha_4 \equiv \alpha_1$) edges incident to the vertex with the label α.

This bracket gives rise to the *Goldman bracket* on the space of geodesic length functions [17] and in [8] it is proved that on the lengths of the elements of a complete cusped geodesic lamination which consists only of arcs and simple loops this Poisson bracket gives rise to the cluster algebra Poisson structure.

In order to describe this Poisson structure more explicitly, notice that for cusped fat-graph with the additional property that the polygons containing holes with no cusps are monogons, every hole with no associated bordered cusps is contained inside a closed loop, which is an edge starting and terminating at the same three-valent vertex. Vice versa, every such closed loop corresponds to a hole with no associated bordered cusps. Therefore every open edge corresponding to a bordered cusp "protrudes" towards the interior of some face of the graph, and we have exactly one hole contained inside this face.

As a consequence of these facts, we can fix an orientation of the fat graph and of each open edge which allows us to determine a natural partition of the set of bordered cusps into nonintersecting (maybe empty) subsets δ_k, $k = 1, \ldots, s$ of cusps incident to the corresponding holes, and to set a cyclic ordering in every

such subset. This means that all arcs in the lamination are uniquely determined by 4 indices, telling us in which cusp they originate and terminate and in what order they enter or exit the cusp. For example, if we orient the fat graph in Fig. 6 counterclockwise so that the arc d originates in cusp 2 and is the first arc in that cusp, then it terminates in cusp 1 and it is the eight arc in that cusp (we count arcs starting from the side of open edge that goes into the fat-graph), we can denote $d = g_{2_1,1_8}$. Analogously $b = g_{1_3,1_4}$ and so on. The formula for the Poisson brackets is then completely combinatorial:

$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \frac{\epsilon_{i-r}\delta_{s,p} + \epsilon_{j-r}\delta_{t,p} + \epsilon_{i-l}\delta_{s,q} + \epsilon_{j-l}\delta_{t,q}}{4}, \tag{3.11}$$

where $\epsilon_k := \mathrm{sign}(k)$.

In [8] it is proved that the abstract bracket defined by (3.11) is indeed a Poisson bracket.

4. Decorated character variety

The classical character varieties are moduli spaces of monodromy data of regular or singular connections, which can be considered like representation spaces of the fundamental group of a Riemann surface. N. Hitchin proved that they are endowed with a holomorphic symplectic structure [19].

It is well-known that so-called Stokes data should be added to the classical monodromy in the case of non-fuchsian irregular singularities. That is why we want to generalise the previous representation space description to define an appropriate generalisation of the classical (or "tame") character variety.

Various descriptions of generalised character varieties as spaces of representations of a "wild fundamental groupoid" [25], "Stokes groupoid" [11] or as "fissions" varieties of Stokes representations associated with a complex reductive linear algebraic group G [2].

In this paper we propose a different notion of decorated character variety which is based on the combinatorial description of Teichmüller space explained in the previous section. Our construction is based on the fact that topologically speaking a Riemann surface $\Sigma_{g,s,n}$ with n holes, with s bordered cusps is equivalent to a Riemann surface $\tilde{\Sigma}_{g,s,n}$ of genus g, with s holes and n marked points $m_1, \dots, m_n$ on the boundaries.

We introduce *the fundamental groupoid of arcs* $\mathfrak{A}$ as the set of all directed paths $\gamma_{ij} : [0,1] \to \tilde{\Sigma}_{g,s,n}$ such that $\gamma_{ij}(0) = m_i$ and $\gamma_{ij}(1) = m_j$ modulo homotopy. The groupoid structure is dictated by the usual path–composition rules.

In this groupoid we have n subgroups $\Pi_j = \{\gamma_{jj} | \gamma_{jj} : [0,1] \to \tilde{\Sigma}_{g,s,n}, \gamma_{jj}(0) = m_j, \gamma_{jj}(1) = m_j\}$. Each of these subgroups is isomorphic to the usual fundamental group and $\Pi_j = \gamma_{ij}^{-1}\Pi_i\gamma_{ij}$ for any arc $\gamma_{ij} \in \mathfrak{A}$.

Now, we use the geometry: using the decoration at each cusp, we associate to each arc γ_{ij} a matrix $M_{ij} \in SL_2(\mathbb{R})$ as explained in the previous section, for example $M_{ij} = X(k_j)LX(z_n)R \cdots LX(z_1)RX(k_i)$. In order to associate a matrix in $SL_2(\mathbb{C})$, we complexify the coordinates $Z_i \in \mathbb{C}$ in all formulas. We define two different characters:

$$\begin{aligned} \mathrm{Tr}_K : \quad & SL_2(\mathbb{C}) \to \mathbb{C} \\ & M \mapsto \mathrm{Tr}(MK) \end{aligned}$$

and the usual character (i.e. trace) which is only defined for the images of Π_i, $i = 1, \dots n$.

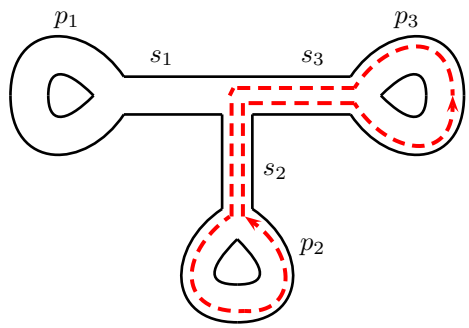


FIGURE 4. The fat graph of the 4 holed Riemann sphere. The red dashed geodesic is x_1.

Finally, we define the decorated character variety as

$$\mathrm{Hom}\,(\mathfrak{A}, \{\,\mathrm{Tr}\,_K(M_{ij})\} \cup \{\,\mathrm{Tr}\,(M_{ii})\})$$

To equip the decorated character variety with a Poisson bracket, we extend Poisson brackets (3.10) to the complexified shear coordinates. For $Z_i, Z_j \in \mathbb{C}$ we postulate $\{Z_i, Z_j\} = \{\overline{Z}_i, \overline{Z}_j\} := \{Z_i, Z_j\}_{\mathbb{R}}$ and $\{Z_i, \overline{Z}_j\} \equiv 0$ or, explicitly, $\{\Re Z_i, \Re Z_j\} = -\{\Im Z_i, \Im Z_j\} = \frac{1}{2}\{Z_i, Z_j\}_{\mathbb{R}}$, $\{\Re Z_i, \Im Z_j\} \equiv 0$ where we let $\{Z_i, Z_j\}_{\mathbb{R}}$ denote the (constant) Poisson bracket (3.10). All formulas for Poisson brackets between characters then remain valid irrespectively on whether we consider real or complexified generalised shear coordinates Z_i.

5. Decorated character varieties and Painlevé monodromy manifolds

In the case of a Riemann sphere with 4 holes and no cusps, the fat graph is given in Fig. 4 and the three geodesics lengths x_1, x_2, x_3 of the thee geodesics which go around two holes without self–intersections are enough to close the Poisson algebra.

By following the rules explained in Section 3, the following parameterization of x_1, x_2, x_3 in shear coordinates on the fat-graph of a 4–holed sphere was found in [7]:

$$\begin{aligned}
x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - e^{-s_2-s_3-\frac{p_2}{2}-\frac{p_3}{2}} - e^{s_2-s_3+\frac{p_2}{2}-\frac{p_3}{2}} - G_2 e^{-s_3-\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}},\\
x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - e^{-s_3-s_1-\frac{p_3}{2}-\frac{p_1}{2}} - e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_3 e^{-s_1-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}},\\
x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} - G_1 e^{-s_2-\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}},
\end{aligned} \tag{5.12}$$

where

$$G_i = e^{\frac{p_i}{2}} + e^{-\frac{p_i}{2}}, \qquad i = 1, 2, 3,$$

and

$$G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}} + e^{-s_1-s_2-s_3-\frac{p_1}{2}-\frac{p_3}{2}-\frac{p_3}{2}}.$$

Note that by complexifying $s_1, s_2, s_3, p_1, p_2, p_3$, we obtain a parameterisation of the PVI cubic, i.e. of the character variety of $SL_2(\mathbb{C})$ character variety of a Riemann sphere with 4 holes.

We are now going to produce a similar coordinate description of each of the other Painlevé cubics. We will provide a geometric description of the corresponding Riemann surface and its fat-graph and discuss the corresponding decorated character variety.

5.1. **Shear coordinates for PV.** The confluence from the cubic associated to PVI to the one associated to PV is realised by

$$p_3 \to p_3 - 2\log[\epsilon],$$

in the limit $\epsilon \to 0$. We obtain the following shear coordinate description for the PV cubic:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}}, \\ x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_3 e^{-s_1-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} - G_1 e^{-s_2-\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}}, \end{aligned} \tag{5.13}$$

where

$$G_i = e^{\frac{p_i}{2}} + e^{-\frac{p_i}{2}}, \quad i = 1, 2, \quad G_3 = e^{\frac{p_3}{2}}, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$\begin{aligned} &x_1x_2x_3 + x_1^2 + x_2^2 - (G_1G_\infty + G_2G_3)x_1 - (G_2G_\infty + G_1G_3)x_2 - \\ &-G_3G_\infty x_3 + G_\infty^2 + G_3^2 + G_1G_2G_3G_\infty = 0. \end{aligned} \tag{5.14}$$

Note that the parameter p_3 is now redundant, we can eliminate it by rescaling. To obtain the correct PV cubic, we need to pick $p_3 = -p_1 - p_2 - 2s_1 - 2s_2 - 2s_3$ so that $G_\infty = 1$.

Geometrically speaking, sending the perimeter p_3 to infinity means that we are performing a chewing-gum move: two holes, one of perimeter p_3 and the other of perimeter $s_1 + s_2 + s_3 + \frac{p_1}{2} + \frac{p_2}{2} + \frac{p_3}{2}$, become infinite, but the area between them remains finite, thus leading to a Riemann sphere with three holes and two cusps on one of them. In terms of the fat-graph, this is represented by Fig. 5.

The geodesic x_3 corresponds to the closed loop obtained going around p_1 and p_2 (green and red loops), while x_1 and x_2 are arcs starting at one cusp, going around p_1 and p_2 respectively, and coming back to the other cusp.

As explained in Section 3, according to [8], the Poisson algebra related to the character variety of a Riemann sphere with three holes and two cusps on one boundary is 7-dimensional. The fat-graph admits a complete cusped lamination as displayed in Fig. 6 so that a full set of coordinates on the character variety is given by the complexification of the five elements in the lamination and of the two parameters G_1 and G_2 which determine the perimeter of the two non-cusped holes.

Notice that there are two shear coordinates associated to the two cusps, they are denoted by k_1 and k_2, their sum corresponds to what we call p_3 in (5.13). These shear coordinates do not commute with the other ones, they in fact satisfy the following relations:

$$\{s_3, k_1\} = \{k_1, k_2\} = \{k_2, s_3\} = 1.$$

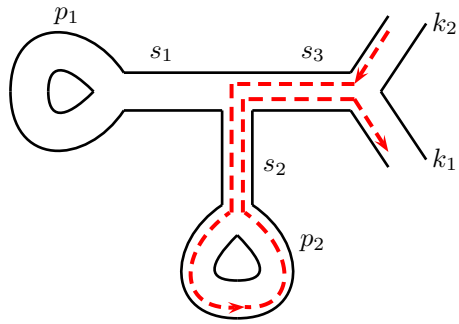


FIGURE 5. The fat graph corresponding to PV. The geodesic x_3 remains closed, while x_1 (see red arc) and x_2 become arcs.

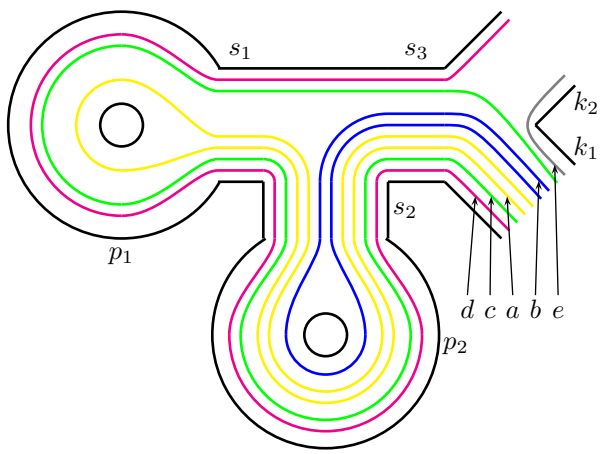


FIGURE 6. The system of arcs for PV.

As a consequence, the elements G_3 and G_∞ are not Casimirs in this Poisson algebra, despite being frozen variables in the cluster algebra setting (see Section 6)

In terms of shear coordinates, the elements in the lamination correspond to two loops (whose hyperbolic cosin length is denoted by G_1 and G_2 respectively) and five arcs whose lengths are expressed as follows:

$$\begin{aligned} &a = e^{k_1+s_1+2s_2+s_3+\frac{p_1}{2}+p_2}, \qquad b = e^{k_1+s_2+s_3+\frac{p_2}{2}}, \qquad e = e^{\frac{k_1}{2}+\frac{k_2}{2}}, \\ &c = e^{k_1+s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}, \qquad d = e^{\frac{k_1}{2}+\frac{k_2}{2}+s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}. \end{aligned} \tag{5.15}$$

They satisfy the following Poisson relations, which can be deduced by formula (3.11):

$$\{a,b\} = ab, \quad \{a,c\} = 0, \quad \{a,d\} = -\frac{1}{2}ad, \quad \{a,e\} = \frac{1}{2}ae, \tag{5.16}$$

$$\{b,c\}=0,\quad \{b,d\}=-\frac{1}{2}bd,\quad \{b,e\}=\frac{1}{2}be, \tag{5.17}$$

$$\{c,d\}=-\frac{1}{2}cd,\quad \{c,e\}=\frac{1}{2}ce,\quad \{d,e\}=0,\quad \{G_1,\cdot\}=\{G_2,\cdot\}=0, \tag{5.18}$$

so that the elements G_1, G_2 and $G_3G_\infty = de$ are central.

The generic family of symplectic leaves are determined by the common level set of the three Casimirs G_1, G_2 and $G_3G_\infty = de$ and are 4-dimensional (rather than 2-dimensional like in the PVI case).

On each symplectic leaf, the PV monodromy manifold (5.14) is the subspace defined by those functions of a, b, c, d (and of the Casimir values G_1, G_2, $G_3G_\infty = de$) which commute with the frozen variables, i.e. with $G_3 = e$ (and therefore with d as well, since de is a Casimir). To see this, we can use relations (5.15) to determine the exponentiated shear coordinates in terms of a, b, c, d, and then deduce the expressions of x_1, x_2, x_3 in terms of the lamination. We obtain the following expressions:

$$x_1=-e\frac{a}{c}-d\frac{b}{c},\qquad x_2=-e\frac{b}{c}-G_1d\frac{b}{a}-d\frac{b^2}{ac}-d\frac{c}{a}, \tag{5.19}$$

$$x_3=-G_2\frac{c}{b}-G_1\frac{c}{a}-\frac{b}{a}-\frac{c^2}{ab}-\frac{a}{b}, \tag{5.20}$$

which automatically satisfy (5.14).

Due to the Poisson relations (5.16) the functions that commute with e are exactly the functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}, d$. Such functions may depend on the Casimir values G_1, G_2 and G_3G_∞ and e itself, so that $d = G_\infty$ becomes a parameter now. With this in mind, it is easy to prove that x_1, x_2, x_3 are algebraically independent functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}, d$ so that x_1, x_2, x_3 form a basis in the space of functions which commute with e.

5.2. **Shear coordinates for PV_{deg}.** The confluence from PV to PV_{deg} is realised by the substitution

$$s_3 \to s_3 - \log[\epsilon],$$

in formulae (5.13). In the limit $\epsilon \to 0$ we obtain:

$$\begin{aligned}
x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}},\\
x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}}-e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}}-G_1e^{s_3+\frac{p_3}{2}},\\
x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}}-e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}}-e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}}-G_1e^{-s_2-\frac{p_2}{2}}-G_2e^{s_1+\frac{p_1}{2}},
\end{aligned} \tag{5.21}$$

where

$$G_i=e^{\frac{p_i}{2}}+e^{-\frac{p_i}{2}},\quad i=1,2,\quad G_\infty=e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1x_2x_3+x_1^2+x_2^2-G_1G_\infty x_1-G_2G_\infty x_2+G_\infty^2=0. \tag{5.22}$$

Note that the parameter p_3 is now redundant, we can eliminate it by rescaling. To obtain the correct PV_{deg} cubic, we need to pick $p_3 = -p_1 - p_2 - 2s_1 - 2s_2 - 2s_3$.

Geometrically speaking, sending the shear coordinate s_3 to infinity means that we are performing a cusp-removing move. In terms of the fat-graph, this is represented by Fig. 7.

The character variety of a Riemann sphere with three holes and one cusp on one boundary is 5-dimensional. The fat-graph admits a complete cusped lamination so

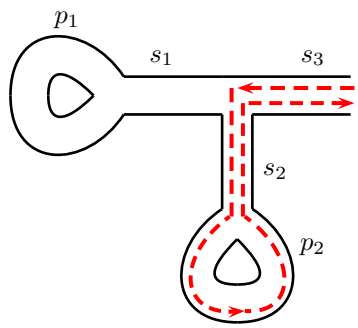


FIGURE 7. The fat graph corresponding to PV_{deg} . The red arc corresponds to x_1.

that a full set of coordinates on the character variety is given by the complexification of the geodesic length functions of the elements in the lamination. Now we have only one shear coordinate associated to the cusp, denoted by s_3, which does not commute with the other shear coordinates.

We omit the picture of the PV_{deg} lamination as it is very similar to Fig. 6, in which the edges labelled by k_1 and k_2 are removed and the geodesics d and e are lost.

In terms of shear coordinates, the elements in the lamination are two loops corresponding to the parameters G_1 and G_2 and three arcs for which the lengths are expressed as follows:

$$a = e^{s_1+2s_2+s_3+\frac{p_1}{2}+p_2}, \qquad b = e^{s_2+s_3+\frac{p_2}{2}}, \qquad c = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}, \tag{5.23}$$

They satisfy the following Poisson relations, which can be deduced by formula (3.11):

$$\{a,b\} = ab, \quad \{a,c\} = 0, \quad \{b,c\} = 0, \tag{5.24}$$

so that the element c is a Casimir as well as the parameters G_1, G_2. Each symplectic leaf is two-dimensional and corresponds to the PV_{deg} monodromy manifold (5.22). Indeed, we can use relations (5.23) to determine the exponentiated shear coordinates in terms of a, b, c, and then deduce he expressions of x_1, x_2, x_3 in terms of the lamination. We obtain the following expressions:

$$x_1 = -b, \qquad x_2 = -G_1\frac{bc}{a} - \frac{b^2}{a} - \frac{c^2}{a}, \tag{5.25}$$

$$x_3 = -G_2\frac{c}{b} - G_1\frac{c}{a} - \frac{b}{a} - \frac{c^2}{ab} - \frac{a}{b}, \tag{5.26}$$

which automatically satisfy (5.22).

In terms of lamination, the confluence from PV to PV_{deg} is given by the following rules:

$$a \to a, \quad b \to b, \quad c \to c, \quad d \to c, \quad e \to 0.$$

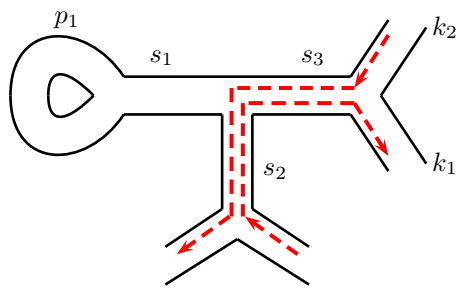


FIGURE 8. The fat graph corresponding to PIV. The variable x_1 now corresponds to the product of the two red arcs.

5.3. **Shear coordinates for** PIV**.** The confluence from the generic PV cubic (5.14) to the PIV one is realised by the substitution

$$p_2 \to p_2 - 2\log[\epsilon],$$

in formulae (5.13). In the limit $\epsilon \to 0$ we obtain:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}}, \\ x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_3 e^{-s_1-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}}, \end{aligned} \tag{5.27}$$

where

$$G_1 = e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}}, \quad G_2 = e^{+\frac{p_2}{2}}, \quad G_3 = e^{+\frac{p_3}{2}}, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$\begin{aligned} &x_1x_2x_3 + x_1^2 - (G_1G_\infty + G_2G_3)x_1 - G_2G_\infty x_2 - \\ &-G_3G_\infty x_3 + G_\infty^2 + G_1G_2G_3G_\infty = 0. \end{aligned} \tag{5.28}$$

Note that the parameters p_3, p_2 are now redundant, we can eliminate it by rescaling. To obtain the correct PIV cubic, we need to pick $p_2 = p_3 = -p_1 - 2s_1 - 2s_2 - 2s_3$ so that $G_2 = G_3 = G_\infty$.

Similarly to the previous case, this means that we send the perimeter p_2 to infinity, which is a chewing-gum move leading to a Riemann sphere with two holes, one of which has 4 cusps on it. The corresponding fat-graph is given in Fig. 8, where we see 4 new shear coordinates, one for each cusp, so that in formulae (5.27) $p_2 = k_3 + k_4$ and $p_3 = k_1 + k_2$.

The character variety is now 8 dimensional and the complete cusped lamination is given in Fig. 9.

In terms of shear coordinates, the elements in the lamination are expressed as follows:

$$a = e^{s_1+s_2+k_3+\frac{p_1}{2}}, \quad b = e^{\frac{s_2}{2}+\frac{s_3}{2}+\frac{k_1}{2}+\frac{k_4}{2}}, \quad c = e^{s_1+\frac{s_2}{2}+\frac{s_3}{2}+\frac{p_1}{2}+\frac{k_1}{2}+\frac{k_3}{2}}, \tag{5.29}$$

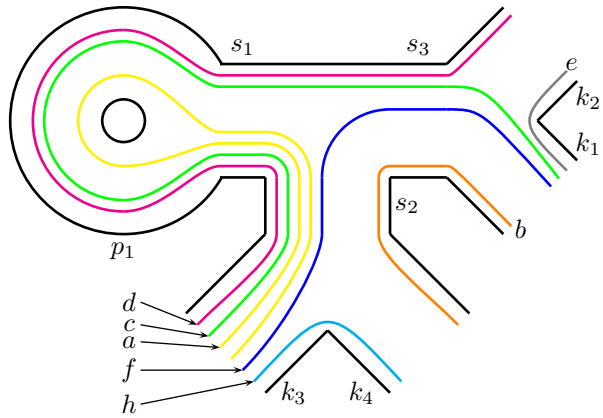


FIGURE 9. The system of arcs for PIV.

$$d = e^{s_1+\frac{s_2}{2}+\frac{s_3}{2}+\frac{p_1}{2}+\frac{k_2}{2}+\frac{k_3}{2}}, \quad e = e^{\frac{k_1}{2}+\frac{k_2}{2}}, \quad f = e^{\frac{s_2}{2}+\frac{s_3}{2}+\frac{k_1}{2}+\frac{k_3}{2}}, \quad g = e^{\frac{k_3}{2}+\frac{k_4}{2}}.$$

We omit the formulae for the Poisson brackets as these can be easily extracted from (3.11). Let us notice that the element $bdeh$ is a Casimir as well as the perimeter G_1. Each symplectic leaf is six-dimensional and the PIV monodromy manifold (5.28) is the subspace of those functions of $a, b, \ldots, g$ which commute with e and g. The proof of this statement is quite similar to the previously considered analogous assertion for the PV-case and we omit it.

5.4. **Shear coordinates for $PIII$.** The confluence from PV to $PIII$ is obtained by the following substitution:

$$s_1 \to s_1 + 2\log[\epsilon], \quad p_2 \to p_2 - 2\log[\epsilon], \quad p_1 \to p_1 - 2\log[\epsilon].$$

In the limit as $\epsilon \to 0$ we obtain:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}}, \\ x_2 &= -e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_3 e^{-s_1-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - G_1 e^{-s_2-\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}}, \end{aligned} \tag{5.30}$$

where

$$\widetilde{G}_i = e^{\frac{p_i}{2}}, \quad i = 1,2,3 \quad \widetilde{G}_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1x_2x_3+x_1^2+x_2^2-(\widetilde{G}_1\widetilde{G}_\infty+\widetilde{G}_2\widetilde{G}_3)x_1-(\widetilde{G}_2\widetilde{G}_\infty+\widetilde{G}_1\widetilde{G}_3)x_2+\widetilde{G}_1\widetilde{G}_2\widetilde{G}_3\widetilde{G}_\infty = 0. \tag{5.31}$$

We can pick $p_2 = p_3 = 0$ in order to obtain the correct $PIII$ cubic. Note that there is a slight discrepancy between the $\widetilde{G}_i$s in the cubic (5.31) and the G_is dictated by our formulae (2.4). This is easily solved by a simple transformation

$$G_\infty = \sqrt{\widetilde{G}_1\widetilde{G}_\infty}, \quad G_1 = \sqrt{\widetilde{G}_1\widetilde{G}_\infty} + \frac{1}{\sqrt{\widetilde{G}_1\widetilde{G}_\infty}}, \quad G_2 = \sqrt{\frac{\widetilde{G}_\infty}{\widetilde{G}_1}} + \sqrt{\frac{\widetilde{G}_1}{\widetilde{G}_\infty}}.$$

To understand the geometry of this confluence, we first need to flip the PV fat-graph to the equivalent graph given in Fig. 10.

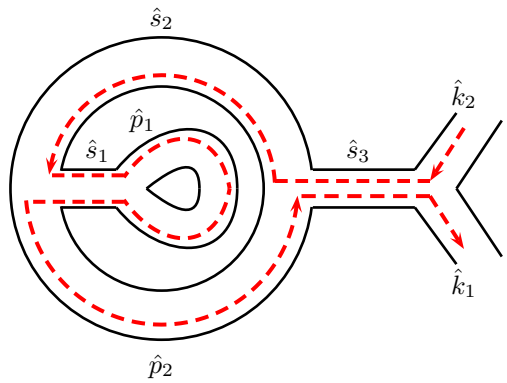


FIGURE 10. The flipped fat graph corresponding to PV. The red arc corresponds to x_1.

The new shear coordinates are expressed in terms of the old ones as follows:

$$\begin{aligned}
\hat{s}_1 &= -s_1 - p_1 - \log(1+e^{s_2}),\\
\hat{s}_2 &= -s_2 + \log\left((1+e^{p_1+s_1}+e^{p_1+s_1+s_2})(1+e^{s_1}+e^{s_1+s_2})\right),\\
\hat{s}_3 &= s_3 - \log(1+e^{-s_2}), \quad \hat{p}_1 = p_1,\\
\hat{p}_2 &= p_2 + s_2 + p_1 + 2s_1 + 2\log(1+e^{s_2}) +\\
&+ \log\left((1+e^{p_1+s_1}+e^{p_1+s_1+s_2})(1+e^{s_1}+e^{s_1+s_2})\right).
\end{aligned} \tag{5.32}$$

Remark 5.1. Note that this flip is obtained by composing two mapping class group transformations described in Figures 3 and 4 of [8]. This means that the fat-graph of PV is mapped to an intermediate fat-graph which does not satisfy the property that the polygons containing holes with no cusps are monogons. This is not a problem as in fact we can map the lamination in Fig. 6 to this new fat-graph and then to Fig.10.

In the new shear coordinates the substitution (5.30) becomes simply:

$$\hat{p}_1 \to \hat{p}_1 - 2\log[\epsilon],$$

which geometrically speaking corresponds to the fat-graph in Fig. 11, where we see 4 new shear coordinates, one for each cusp, so that $\hat{p}_2 = \hat{k}_3 + \hat{k}_4$ and $\hat{p}_1 = \hat{k}_5 + \hat{k}_6$. This is the fat-graph of a Riemann sphere with two holes each of them with two cusps.

Note that the coordinates in Fig. 11 are the true shear coordinates, namely they satisfy the Poisson brackets:

$$\begin{aligned}
&\{\hat{k}_2,\hat{s}_3\} = \{\hat{s}_3,\hat{k}_1\} = \{\hat{k}_1,\hat{k}_2\} = \{\hat{s}_3,\hat{s}_2\} = \{\hat{p}_2,\hat{s}_3\} = 1, \quad \{\hat{s}_2,\hat{p}_2\} = 2,\\
&\{\hat{s}_1,\hat{s}_2\} = \{\hat{p}_2,\hat{s}_1\} = \{\hat{s}_1,\hat{k}_5\} = \{\hat{k}_6,\hat{s}_1\} = \{\hat{k}_5,\hat{k}_6\} = 1,
\end{aligned} \tag{5.33}$$

If we use the limiting transformation of (5.32):

$$\begin{aligned}
\hat{s}_1 &= -s_1 - k_5 - k_6 - \log(1+e^{s_2}), \qquad \hat{s}_3 = s_3 - \log(1+e^{-s_2}),\\
\hat{s}_2 &= -s_2 + \log\left(1+e^{k_5+k_6+s_1}+e^{k_5+k_6+s_1+s_2}\right), \qquad \hat{p}_1 = p_1,
\end{aligned}$$

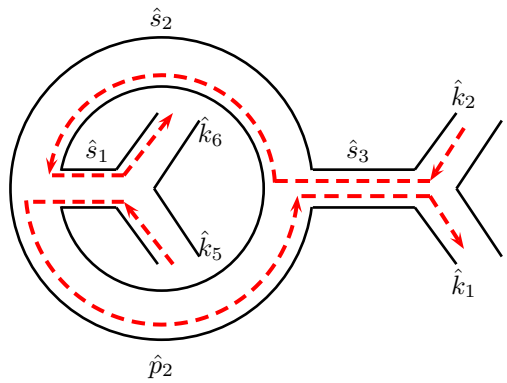


FIGURE 11. The fat graph corresponding to PIII.

$$
\begin{aligned}
\hat{k}_1 &= k_1, \qquad \hat{k}_2 = k_2, \qquad \hat{k}_5 = k_5, \qquad \hat{k}_6 = k_6, \\
\hat{p}_2 &= p_2 + s_2 + k_5 + k_6 + 2s_1 + 2\log(1+e^{s_2}) + \\
&+ \log\left((1+e^{k_5+k_6+s_1}+e^{k_5+k_6+s_1+s_2})(1+e^{s_1}+e^{s_1+s_2}\right).
\end{aligned} \tag{5.34}
$$

to go back to $s_1, s_2, s_3, p_2, k_1, k_2, k_5, k_6$, we see that k_5, k_6 have non standard Poisson brackets with s_1, s_2, s_3. This is due to the fact that this limiting transformation (5.34) destroys the geometry, as it essentially maps from a Riemann sphere with two holes each of them with two cusps to a Riemann sphere with two holes one of which has 4 cusps, and the other has no cusps (the PIV case). This implies that the correct coordinates to describe the character variety of a Riemann sphere with two holes each of them with two cusps are the complexified $\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{p}_2, \hat{k}_1, \hat{k}_2, \hat{k}_5, \hat{k}_6$.

This character variety is 8-dimensional. The fat-graph admits a complete cusped lamination as displayed in Fig. 12 so that a full set of coordinates on the character variety is given by the eight complexified elements in the lamination.

In terms of the shear coordinates $\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{p}_2, \hat{k}_1, \hat{k}_2, \hat{k}_5, \hat{k}_6$ the elements in the $PIII$ lamination are expressed as follows:

$$
\begin{aligned}
&a = e^{\frac{\hat{k}_1+\hat{k}_6-\hat{s}_1+\hat{s}_3+\hat{p}_2}{2}} + e^{\frac{\hat{k}_1+\hat{k}_6+\hat{s}_1+\hat{s}_3+\hat{p}_2}{2}}, \qquad e = e^{\frac{\hat{k}_1+\hat{k}_2}{2}}, \qquad g = e^{\frac{\hat{k}_5+\hat{k}_6}{2}}, \\
&b = e^{\frac{\hat{k}_1+\hat{k}_6-\hat{s}_1-\hat{s}_2+\hat{s}_3}{2}} + e^{\frac{\hat{k}_1+\hat{k}_6+\hat{s}_1-\hat{s}_2+\hat{s}_3}{2}} + e^{\frac{\hat{k}_1+\hat{k}_6+\hat{s}_1+\hat{s}_2+\hat{s}_3}{2}}, \\
&c = e^{\frac{2\hat{k}_1+\hat{s}_2+2\hat{s}_3+\hat{p}_2}{2}}, \qquad d = e^{\frac{\hat{k}_1+\hat{k}_2+\hat{s}_2+2\hat{s}_3+\hat{p}_2}{2}}, \\
&f = e^{\frac{\hat{k}_1+\hat{k}_5+\hat{s}_1+\hat{s}_3+\hat{p}_2}{2}}, \qquad h = e^{\frac{\hat{k}_5+\hat{k}_6+2\hat{s}_1+\hat{s}_2+\hat{p}_2}{2}}.
\end{aligned} \tag{5.35}
$$

The Poisson brackets admit two Casimirs, de and hg so that the symplectic leaves are 6-dimensional. Within each symplectic leaf, the $PIII$ monodromy manifold is given by the set of functions which commute with d, e, g, h. To see this, we proceed in the same way as in the case of PV. On the monodromy manifold we set $e = \tilde{G}_3$, $d = \tilde{G}_\infty$, $h = \tilde{G}_2$ and $g = \tilde{G}_1$.

Note that, unlike the cases of PV (Fig. 6), PIV (Fig. 9), and PII (Fig. 19 below), the expressions for the λ-lengths of arcs in (5.35) are not monomials in the exponentiated shear coordinates. This is because, unlike the other named cases, the fat

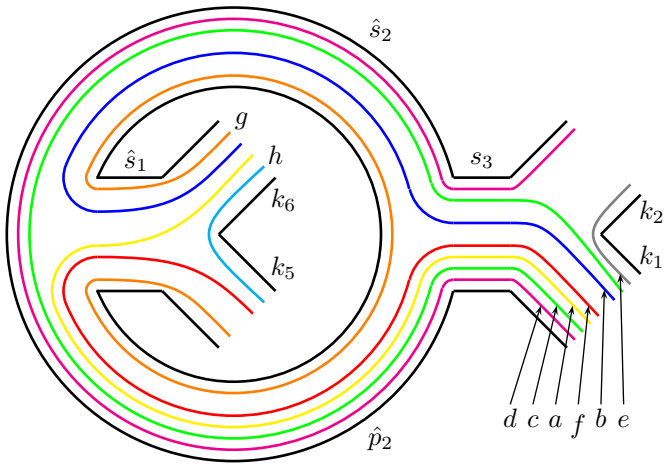


FIGURE 12. The character variety for the $\mathrm{P}_{III}^{D_6}$ system.

graph in Fig. 12 is not dual to the maximum cusped lamination $\{a, b, c, d, e, f, g, h\}$. We obtain the dual graph out of the one in Fig. 12 if we first flip the edge $\hat{s}_1$ subsequently flipping the edge $\hat{s}_2$. We depict the resulting fat graph and lamination in Fig. 13. In the transformed shear coordinates (indicated by tildes), the elements of the $PIII$ lamination read

$$
\begin{aligned}
&a = e^{\frac{\tilde{k}_1+\tilde{k}_6+\tilde{s}_1+\tilde{s}_3+2\tilde{s}_2+\tilde{p}_2}{2}}, \qquad e = e^{\frac{\tilde{k}_1+\tilde{k}_2}{2}}, \qquad g = e^{\frac{\tilde{k}_5+\tilde{k}_6+\tilde{p}_2}{2}},\\
&b = e^{\frac{\tilde{k}_1+\tilde{k}_6+\tilde{s}_2+\tilde{s}_3}{2}}, \qquad c = e^{\frac{2\tilde{k}_1+\tilde{s}_1+\tilde{s}_2+2\tilde{s}_3+\tilde{p}_2}{2}}, \qquad d = e^{\frac{\tilde{k}_1+\tilde{k}_2+\tilde{s}_1+\tilde{s}_2+2\tilde{s}_3+\tilde{p}_2}{2}},\\
&f = e^{\frac{\tilde{k}_1+\tilde{k}_5+\tilde{s}_2+\tilde{s}_3+\tilde{p}_2}{2}}, \qquad h = e^{\frac{\tilde{k}_5+\tilde{k}_6+\tilde{s}_1+\tilde{s}_2}{2}}.
\end{aligned}
\tag{5.36}
$$

In this form, the homogeneity of the Poisson relations on the set $\{a, b, c, d, e, f, g, h\}$ becomes obvious.

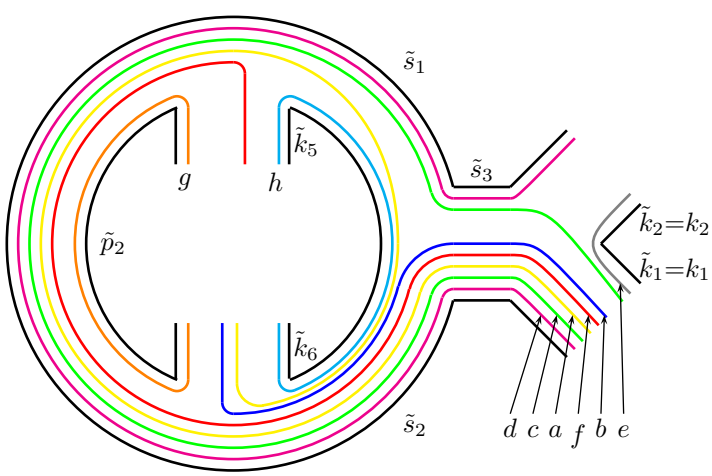


FIGURE 13. The character variety for the $\mathrm{P}_{III}^{D_6}$ system depicted on the fat graph dual to this variety.

It is interesting to construct the lamination and pattern in Fig. 12 as a limit of the corresponding system of PV. When we flip the fat-graph for PV, the lamination is flipped too as in Fig. 14.

When we open the inside hole to obtain the fat-graph for PIII, the PV arcs a and b break giving rise to the new arcs a, b and f and G_1 breaks up giving rise to g so that

$$a \to af, \quad b \to bf, \quad c \to c, \quad d \to d, \quad e \to e, \quad G_2 \to h, \quad G_1 \to g.$$

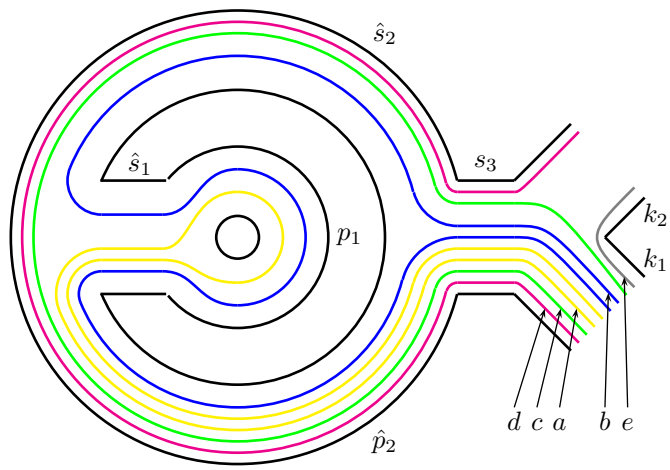


FIGURE 14. Transformation of the PV system of arcs from Fig. 6 under a sequence of two flips.

5.5. **Shear coordinates for $PIII^{D_7}$.** The confluence from the generic $PIII$ cubic (5.31) to the $PIII^{D_7}$ one is realised by the substitution

$$s_3 \to s_3 - \log[\epsilon],$$

in formulae (5.30). in the limit $\epsilon \to 0$ we obtain:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}}, \\ x_2 &= -e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - G_1 e^{-s_2-\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}}, \end{aligned} \tag{5.37}$$

where

$$G_i = e^{\frac{p_i}{2}}, \quad i = 1, 2, \quad G_3 = 0, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These same expressions can equivalently obtained by the substitution:

$$s_1 \to s_1 + 2\log(\epsilon), \quad p_1 \to p_1 - 2\log(\epsilon), \qquad p_2 \to p_2 - 2\log(\epsilon),$$

in formulae (5.21) and by taking the limit as $\epsilon \to 0$.

These coordinates satisfy the following cubic relation:

$$x_1 x_2 x_3 + x_1^2 + x_2^2 - G_1 G_\infty x_1 - G_2 G_\infty x_2 = 0. \tag{5.38}$$

We can pick $p_2 = p_3 = 0$ and $s_3 = -s_1 - s_2 - \frac{p_1}{2}$ in order to obtain the correct $PIII^{D_7}$ cubic.

In terms of fat graph, we need to work with the coordinates (5.32), for which the confluence gives

$$\hat{s}_3 \to \hat{s}_3 - \log[\epsilon],$$

which corresponds to the fat-graph in Fig. 15. This corresponds to a Riemann sphere with two holes, one with one cusp and one with two cusps.

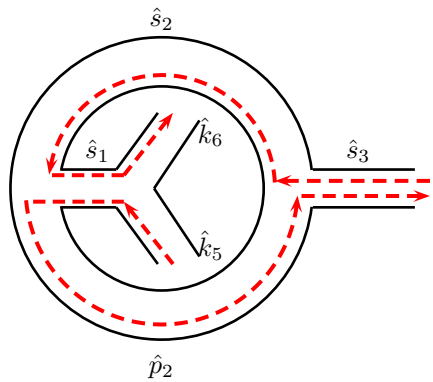


FIGURE 15. The fat graph corresponding to $PIII^{D_7}$.

The character variety is 6 dimensional and we omit the picture of the $PIII^{D_7}$ lamination as it is very similar to Fig. 12, in which the edges labelled by k_1 and k_2 are removed and the arcs d and e are lost. Again the Poisson brackets can be calculated using (3.11) and it is easy to check that there are two Casimirs so that the symplectic leaves are 4 dimensional. The $PIII^{D_7}$ monodromy manifold is given by those functions of a, b, c, d, f, h that commute with h.

5.6. **Shear coordinates for $PIII^{D_8}$.** The confluence from the generic $PIII^{D_7}$ cubic (5.38) to the $PIII^{D_8}$ one is realised by the substitution

$$s_1 \to s_1 + \log[\epsilon], \quad p_2 \to p_2 - 2\log[\epsilon]$$

in formulae (5.37). In the limit $\epsilon \to 0$ we obtain:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}}, \\ x_2 &= -e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}}, \end{aligned} \tag{5.39}$$

where

$$G_1 = 0, \quad G_2 = e^{\frac{p_2}{2}}, \quad i = 1, 2, \quad G_3 = 0, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1 x_2 x_3 + x_1^2 + x_2^2 - G_2 G_\infty x_2 = 0. \tag{5.40}$$

We can pick $p_2 = p_3 = 0$ and $s_3 = -s_1 - s_2 - \frac{p_1}{2}$ in order to obtain the correct $PIII^{D_8}$ cubic.

In terms of fat graph, we need to work with the coordinates (5.32), for which the confluence gives

$$\hat{s}_1 \to \hat{s}_1 - \log[\epsilon],$$

which corresponds to the fat-graph in Fig. 16. This corresponds to a Riemann sphere with two holes, each with one cusp on it.

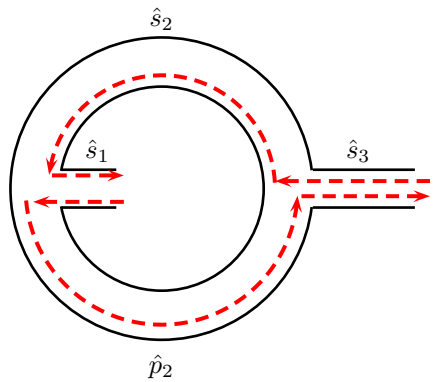


FIGURE 16. The fat graph corresponding to $PIII^{D_8}$.

The character variety is 4 dimensional and the Poisson brackets can be calculated using (3.11). It is easy to check that there are two Casimirs so that the symplectic leaves are 2 dimensional and coincide with the $PIII_{D_8}$ monodromy manifold.

5.7. **Shear coordinates for PII^{JM}.** The confluence from the generic PIV cubic (5.28) to the PII^{JM} cubic is realised by the substitution

$$p_1 \to p_1 - 2\log[\epsilon],$$

in formulae (5.27). In the limit $\epsilon \to 0$ we obtain:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}}, \\ x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}}, \end{aligned} \tag{5.41}$$

where

$$G_i = e^{\frac{p_i}{2}},\ i = 1,2,3, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$\begin{aligned} &x_1x_2x_3 - G_1G_\infty x_1 - G_2G_\infty x_2 - \\ &-G_3G_\infty x_3 + G_\infty^2 + G_1G_2G_3G_\infty = 0. \end{aligned} \tag{5.42}$$

Note that the parameters p_3, p_2, p_1 are now redundant, we can eliminate it by rescaling. To obtain the correct PII^{JM} cubic, we need to pick $p_2 = p_3 = 0$ and $p_1 = -2s_1 - 2s_2 - 2s_3$.

Again, this means that we send the perimeter p_1 to infinity, which is a chewing-gum move leading to a Riemann sphere with one hole with 6 cusps on it. The

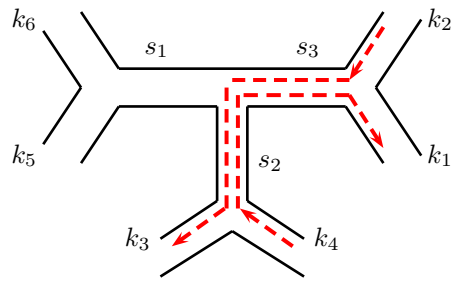


FIGURE 17. The fat graph corresponding to PII^{JM}.

corresponding fat-graph is given in Fig. 17 where we have 6 cusp shear coordinates $k_1, \dots, k_6$ such that in formulae (5.41) we must set $p_1 = k_5 + k_6$, $p_2 = k_3 + k_4$ and $p_3 = k_1 + k_2$. The character variety is now 9-dimensional, with only one Casimir and the PII^{JM} monodromy manifold is given by the set of functions that Poisson commute with $p_1 = k_5 + k_6$, $p_2 = k_3 + k_4$ and $p_3 = k_1 + k_2$. We omit all details in this case.

5.8. **Shear coordinates for PII^{FN}.** The confluence from the generic PIV cubic (5.28) to the PII^{FN} cubic is realised by the substitution

$$s_3 \to s_3 - \log[\epsilon],$$

in formulae (5.27). In the limit $\epsilon \to 0$ we obtain:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}}, \\ x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}}, \end{aligned}$$

where

$$G_1 = e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}}, \quad G_2 = e^{\frac{p_2}{2}}, \quad G_3 = 0, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}..$$

They satisfy the following cubic relation

$$x_1 x_2 x_3 + x_1^2 - G_1 G_\infty x_1 - G_2 G_\infty x_2 + G_\infty^2 = 0. \tag{5.43}$$

Observe that we can obtain exactly the same formulate by starting from the generic PV_{deg} cubic (5.22) by the substitution

$$p_2 \to p_2 - 2\log[\epsilon],$$

and by taking the limit as $\epsilon \to 0$.

To obtain the PII^{FN} cubic we pick $p_2 = p_3 = 0$ and $p_1 = -2s_1 - 2s_2 - 2s_3$.

Geometrically speaking, sending the shear coordinate s_3 to infinity means that we are performing a cusp-removing move. This gives a Riemann sphere with two holes, one of them having three cusps on its boundary. In terms of the fat-graph,

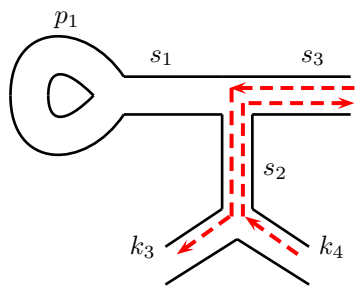


FIGURE 18. The fat graph corresponding to PII^{FN}.

this is represented by Fig. 18, where we wee three cusps of coordinates k_3, k_4 and s_3, so that in formulae (5.43) we must set $p_2 = k_3 + k_4$. .

The decorated character variety in this case is 6 dimensional. The lamination is given by the loop around the un-cusped-hole and the five arcs in Fig. 19.

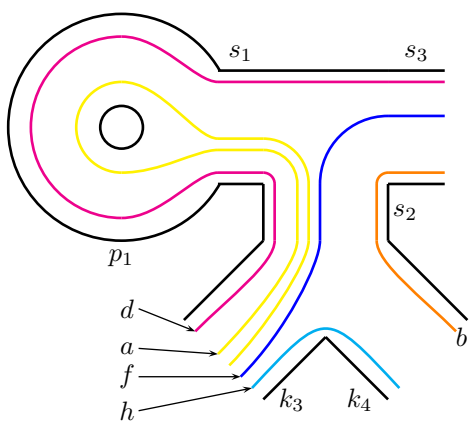


FIGURE 19. The system of arcs for $\mathrm{PII}^{\mathrm{FN}}$.

The lengths of the arcs are

$$a = e^{s_1+s_2+k_3+\frac{p_1}{2}}, \quad b = e^{\frac{s_2}{2}+\frac{s_3}{2}+\frac{k_4}{2}}, \tag{5.44}$$
$$d = e^{s_1+\frac{s_2}{2}+\frac{s_3}{2}+\frac{p_1}{2}+\frac{k_3}{2}}, \quad f = e^{\frac{s_2}{2}+\frac{s_3}{2}+\frac{k_3}{2}}, \quad g = e^{\frac{k_3}{2}+\frac{k_4}{2}}.$$

To show that our decorated character variety is not the same as the wild character variety (see [3] for the PII^{FN} case), we deal with this case in all details. The Poisson brackets among the complexified lamination arcs lengths are given by

$$\{a,b\} = \{d,f\} = 0, \quad \{a,d\} = -\frac{ad}{2}, \quad \{a,f\} = \frac{af}{2}, \quad \{a,h\} = \frac{af}{2},$$

$$\{b,d\}=\frac{bd}{4},\quad \{b,f\}=\frac{bf}{4},\quad \{b,h\}=-\frac{bh}{4},\quad \{d,h\}=\frac{dh}{4},\quad \{f,h\}=\frac{fh}{4}.$$

It is easy to check that there are two Casimirs, G_1 and bdh, so that the symplectic leaves are 4-dimensional and the PII^{FN} monodromy manifold (5.43) is the subspace of those functions of a,b,d,f,h which commute with h. To check that x_1,x_2 and x_3 commute with h it is enough to express them in terms of the lamination:

$$x_1=-bf,\quad x_2=-G_1\frac{df}{a}-\frac{d^2}{a}-\frac{f^2}{a},\quad x_3=-\frac{dh}{f}-\frac{ab}{f}.$$

Vice versa all functions commuting with h must have the form $a^\alpha b^\beta d^\delta f^\phi$ where $\alpha,\beta,\gamma,\delta,\phi$ are some numbers satisfying $2\alpha-\beta+\delta+\phi=0$. Using this fact, it is easy to prove that on each symplectic leaf the set of functions which commute with h is 2-dimensional.

5.9. **Shear coordinates for PI.** The confluence from the generic PII^{JM} cubic to the PI one is realised by

$$s_3\to s_3-\log[\epsilon],$$

in formulae (5.41). In the limit $\epsilon\to 0$ we obtain:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}},\\ x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}}-G_1e^{s_3+\frac{p_3}{2}},\\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}}-G_2e^{s_1+\frac{p_1}{2}}, \end{aligned}$$

(5.45)

where

$$G_i=e^{\frac{p_i}{2}},\ i=1,2,\quad G_3=0,\quad G_\infty=e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1x_2x_3-G_1G_\infty x_1-G_2G_\infty x_2+G_\infty^2=0. \tag{5.46}$$

Observe that we can obtain exactly the same formulate by starting from the generic PII^{FN} cubic (5.43) in Flashcka-Newell form by the substitution

$$p_1\to p_1-2\log[\epsilon],$$

and by taking the limit as $\epsilon\to 0$.

Note that the parameters p_3,p_2,p_1 and s_3 are now redundant, we can eliminate them by rescaling. We pick $p_1=p_2=p_3=0$ and $s_3=-s_1-s_2$, we obtain the correct PI by changing the sign of x_1 and x_2.

Geometrically speaking, sending the shear coordinate s_3 to infinity means that we are performing a cusp-removing move. In terms of the fat-graph, this is represented by Fig. 20.

The character variety is now 7-dimensional, with only one Casimir and the PI monodromy manifold is given by the set of functions that Poisson commute with $p_1=k_5+k_6$, $p_2=k_3+k_4$. We omit all details in this case.

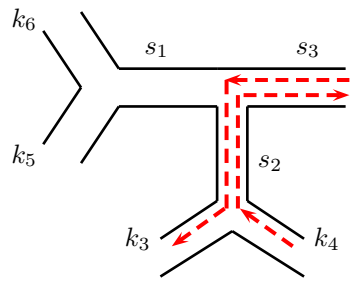


FIGURE 20. The fat graph corresponding to PI.

5.10. **Shear coordinates for the Weierstrass equation.** We now remove one further cusp from the generic PI cubic by replacing

$$s_1 \to s_1 - \log[\epsilon],$$

in formulae (5.45). In the limit $\epsilon \to 0$ we obtain:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}}, \\ x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - G_2 e^{s_1+\frac{p_1}{2}}, \end{aligned}$$

(5.47)

where

$$G_2 = e^{\frac{p_2}{2}}, \quad G_1 = G_3 = 0, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1 x_2 x_3 - G_2 G_\infty x_2 + G_\infty^2 = 0. \tag{5.48}$$

Note that the parameters p_3, p_2, p_1 and s_3 are now redundant, we can choose $p_1 = p_2 = p_3 = 0$ and $s_3 = -s_1 - s_2$ so that $G_2 = G_\infty = 1$. This gives us the following cubic

$$x_1 x_2 x_3 - x_2 + 1 = 0. \tag{5.49}$$

In order to relate this cubic to the Weiestrass elliptic curve, we need to projectivise it first:

$$x_1 x_2 x_3 - x_2 x_0^2 + x_0^3 = 0. \tag{5.50}$$

This cubic is now invariant under the following transformation

$$x_0 \to \alpha x_0, \quad x_1 \to \beta x_1, \quad x_2 \to \alpha x_2, \quad x_3 \to \frac{\alpha^2}{\beta} x_3,$$

so that we can rescale $x_2 \to 1$ and $x_3 \to x_1$, leading to the Weiestrass elliptic curve:

$$x_1^2 - x_0^2 + x_0^3 = 0.$$

6. Painlevé cluster algebras: braid-group and affine MCG actions

6.1. **Painlevé VI: analytic continuation and cluster mutations.** In [12, 23] it was proved that the procedure of analytic continuation of a local solution to the sixth Painlevé equation corresponds to the following action of the braid group on the monodromy manifold:

$$\beta_1 : \begin{array}{lcl} x_1 & \to & -x_1 - x_2x_3 + \omega_1, \\ x_2 & \to & x_3, \\ x_3 & \to & x_2, \end{array} \tag{6.51}$$

$$\beta_2 : \begin{array}{lcl} x_1 & \to & x_3, \\ x_2 & \to & -x_2 - x_1x_2 + \omega_2, \\ x_3 & \to & x_1, \end{array} \tag{6.52}$$

$$\beta_3 : \begin{array}{lcl} x_1 & \to & x_2, \\ x_2 & \to & x_1, \\ x_3 & \to & -x_3 - x_1x_2 + \omega_3. \end{array} \tag{6.53}$$

In [7] it was shown that flips on the shear coordinates correspond to the action of the braid group on the cubic. The flips f_1, f_2, f_3 of the shear coordinates which give rise to the braid transformations $\beta_1\beta_2$ and β_3 respectively have the following form

$$f_1 : \begin{array}{lcl} s_1 & \to & -p_1 - s_1, \qquad p_2 \to p_3, \qquad p_3 \to p_2, \\ s_2 & \to & s_3 + \log\left[1 + e^{s_1}\right] + \log\left[1 + e^{s_1+p_1}\right], \\ s_3 & \to & s_2 - \log\left[1 + e^{-s_1}\right] + \log\left[1 + e^{-s_1-p_1}\right], \end{array} \tag{6.54}$$

$$f_2 : \begin{array}{lcl} s_1 & \to & s_3 - \log\left[1 + e^{-s_2}\right] - \log\left[1 + e^{-s_2-p_2}\right], \\ s_2 & \to & -p_2 - s_2, \qquad p_1 \to p_3, \qquad p_3 \to p_1, \\ s_3 & \to & s_1 + \log\left[1 + e^{s_2}\right] + \log\left[1 + e^{s_2+p_2}\right], \end{array} \tag{6.55}$$

$$f_3 : \begin{array}{lcl} s_1 & \to & s_2 + \log\left[1 + e^{s_3}\right] + \log\left[1 + e^{s_3+p_3}\right], \\ s_2 & \to & s_1 - \log\left[1 + e^{-s_3}\right] - \log\left[1 + e^{-s_3-p_3}\right], \\ s_3 & \to & -p_3 - s_3 \qquad p_1 \to p_2, \qquad p_2 \to p_1. \end{array} \tag{6.56}$$

Remark 6.1. Observe that in [10] it was proved that shear coordinate flips (6.54), (6.55), (6.56) are indeed *dual* to the generalised cluster mutations (6.59) for the corresponding λ-lengths.

We are now going to show that when $G_\infty = 2$ (geometrically this means that we have a puncture at infinity), the action of the braid group coincides with a *generalised cluster algebra structure* [10].

In order to see this let us compose each braid with a Okamoto symmetry in order to obtain the following

$$\widetilde{\beta}_i : \begin{array}{lcl} x_i & \to & -x_i - x_jx_k + \omega_i, \quad j,k \neq i, \\ x_j & \to & x_j, \quad \text{for} \quad j \neq i \end{array} \tag{6.57}$$

By using (2.2) this transformation acquires a cluster flavour:

$$\widetilde{\beta}_i : x_ix_i' = x_j^2 + x_k^2 + \omega_jx_j + \omega_kx_k + \omega_4 \quad j,k \neq i. \tag{6.58}$$

Indeed let us introduce the shifted variables:

$$y_i := x_i - G_i, \quad i = 1, 2, 3,$$

they satisfy the *generalised cluster algebra relation:*

$$\mu_i : y_i y_i' = y_j^2 + y_k^2 + G_i y_j y_k \quad j,k \neq i. \tag{6.59}$$

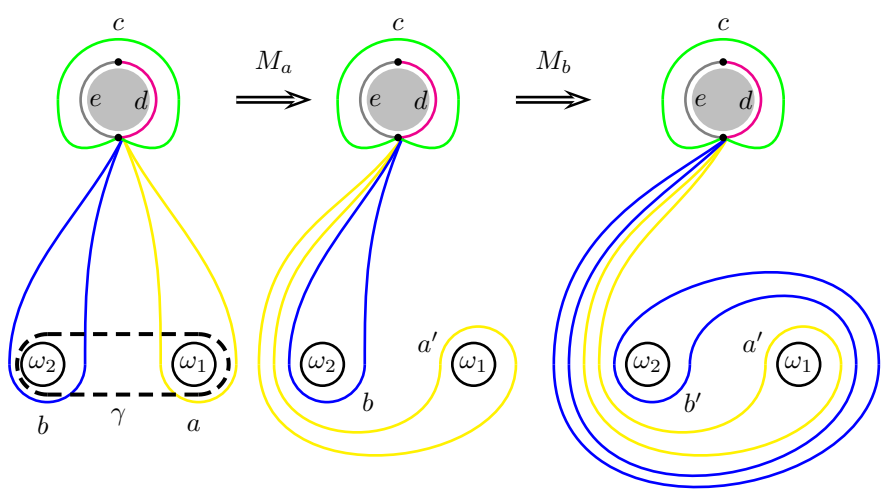
Note that generalised cluster algebras satisfy the Laurent phenomenon. In particular this result implies that procedure of analytic continuation of the solutions to the sixth Painlevé equation satisfies the Laurent phenomenon: if we start from a local solution corresponding to the point (y_1^0, y_2^0, y_3^0) on the shifted Painlevé cubic

$$y_1 y_2 y_3 + y_1^2 + y_2^2 + y_3^2 + G_1 y_2 y_3 + G_2 y_1 y_3 + G_3 y_1 y_2 = 0$$

any other branch of that solution will corresponds to points (y_1, y_2, y_3) on the same cubic such that each y_i is a Laurent polynomial of the initial (y_1^0, y_2^0, y_3^0).

6.2. **Generalised cluster algebra structure for PV and $PV_{\mathbf{deg}}$.** In this case, we have a Riemann surface $\Sigma_{0,3,2}$ with two bordered cusps on one hole. The only nontrivial Dehn twist is around the closed geodesic γ encircling these two holes (this geodesic is unique).

We now consider the effect of this MCG transformation on the system of arcs in Fig. 6.



Here the generalized mutations M_a and M_b are given by the formulas

$$a'a = b^2 + c^2 + \omega_1 bc; \qquad b'b = (a')^2 + c^2 + \omega_2 a'c,$$

or, explicitly,

$$\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} \dfrac{b^2 + c^2 + \omega_1 bc}{a} \\ \dfrac{(b^2 + c^2 + \omega_1 bc)^2}{a^2 b} + \omega_2 c \dfrac{b^2 + c^2 + \omega_1 bc}{ba} + \dfrac{c^2}{b} \end{bmatrix}. \tag{6.60}$$

The geodesic function of γ is

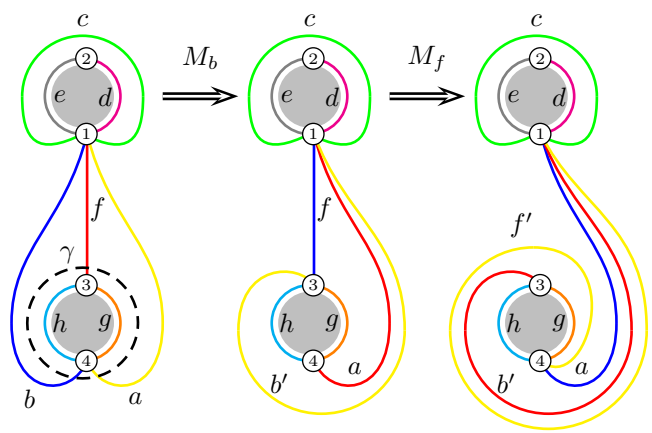
$$G_\gamma = \omega_2 \frac{c}{b} + \omega_1 \frac{c}{a} + \frac{a}{b} + \frac{b}{a} + \frac{c^2}{ab} \tag{6.61}$$

and this function is the so-called *Hamiltonian MCG invariant:* it generates the corresponding Dehn twist (see [22]), has nontrivial Poisson brackets with a and b, and is preserved by the MCG action (6.60).

In the case of $PV_{\rm deg}$, all the above formulas remain valid provided we replace c by the λ-length d of the boundary arc.

6.3. **Generalised cluster algebra structure for $PIII^{D_6}$, $PIII^{D_7}$, and $PIII^{D_8}$.** In all cases of $PIII$, we have a Riemann surface $\Sigma_{0,2,n}$ with $n_1>0$ and $n_2>0$, $n_1+n_2=n$, bordered cusps on the respective holes. For any n_1 and n_2, the only nontrivial Dehn twist is around the closed geodesic γ separating the holes (this geodesic is unique). Its geodesic function G_γ is the Hamiltonian MCG invariant. Besides this invariant, we have (non-Hamiltonian) invariants, which are λ-lengths of all arcs starting and terminating at the same boundary component.

We begin with the case of $PIII^{D_6}$ and consider the MCG action on the system of arcs in Fig. 12:



These transformations are governed by the standard mutation rules,

$$bb'=hc+fa,\qquad ff'=gc+ab';$$

in order to describe them in a more regular way, let us introduce the notation: we let $\lambda^{(i)}_{\alpha,\beta}$ denote the λ-length of the arc that goes between bordered cusps α and β (belonging to different boundary components) winding i times around the lower hole. For example,

$$f=\lambda^{(0)}_{1,3},\quad b=\lambda^{(0)}_{1,4},\quad a=\lambda^{(1)}_{1,4},\quad b'=\lambda^{(1)}_{1,3},\quad f'=\lambda^{(2)}_{1,4},\quad c=\lambda_{1,1}\text{ etc.}$$

Note that $\lambda_{\alpha,\beta}$ with the labels α and β pertaining to the same boundary component are unique and invariant under the MCG action.

The net result of the Dehn twist on the triple $\{\lambda^{(i-1)}_{1,4},\lambda^{(i-1)}_{1,3},\lambda^{(i)}_{1,4}\}$ reads:

$$\begin{bmatrix}\lambda^{(i-1)}_{1,4}\\ \lambda^{(i-1)}_{1,3}\\ \lambda^{(i)}_{1,4}\end{bmatrix}\to\begin{bmatrix}\lambda^{(i)}_{1,4}\\ \dfrac{\lambda^{(i-1)}_{1,3}\lambda^{(i)}_{1,4}+h\lambda_{1,1}}{\lambda^{(i-1)}_{1,4}}\\ \dfrac{(\lambda^{(i)}_{1,4})^2}{\lambda^{(i-1)}_{1,4}}+\dfrac{h\lambda_{1,1}\lambda^{(i)}_{1,4}}{\lambda^{(i-1)}_{1,4}\lambda^{(i-1)}_{1,3}}+\dfrac{g\lambda_{1,1}}{\lambda^{(i-1)}_{1,3}}\end{bmatrix}. \tag{6.62}$$

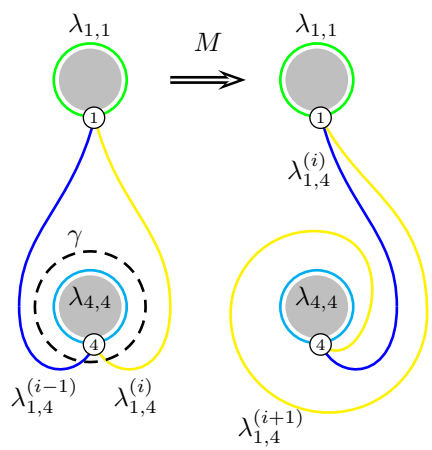
This action admits two invariants: G_γ and $\lambda_{4,4}$ (the latter is obtained by the mutation of the element f, or $\lambda_{1,3}^{(i)}$):

$$G_\gamma = \frac{\lambda_{1,4}^{(i)}}{\lambda_{1,4}^{(i-1)}} + \frac{\lambda_{1,4}^{(i-1)}}{\lambda_{1,4}^{(i)}} + \frac{h\lambda_{1,1}}{\lambda_{1,3}^{(i-1)}\lambda_{1,4}^{(i-1)}} + \frac{g\lambda_{1,1}}{\lambda_{1,3}^{(i-1)}\lambda_{1,4}^{(i)}} \tag{6.63}$$

$$\lambda_{4,4} = \frac{g\lambda_{1,4}^{(i-1)}}{\lambda_{1,3}^{(i-1)}} + \frac{h\lambda_{1,4}^{(i)}}{\lambda_{1,3}^{(i-1)}} \tag{6.64}$$

The case of $PIII^{D_7}$ coincides with that of $PIII^{D_6}$, the geodesic $\lambda_{1,1}$ now becomes the boundary geodesic after erasing the bordered cusp 2.

In the case of $PIII^{D_8}$ we erase bordered cusps 2 and 3; the only MCG transformation is



This MCG transformation reads:

$$\begin{bmatrix} \lambda_{1,4}^{(i-1)} \\ \lambda_{1,4}^{(i)} \end{bmatrix} \to \begin{bmatrix} \lambda_{1,4}^{(i)} \\ \dfrac{(\lambda_{1,4}^{(i)})^2 + \lambda_{1,1}\lambda_{4,4}}{\lambda_{1,4}^{(i-1)}} \end{bmatrix}. \tag{6.65}$$

and the (Hamiltonian) invariant is

$$G_\gamma = \frac{\lambda_{1,4}^{(i)}}{\lambda_{1,4}^{(i-1)}} + \frac{\lambda_{1,4}^{(i-1)}}{\lambda_{1,4}^{(i)}} + \frac{\lambda_{1,1}\lambda_{4,4}}{\lambda_{1,4}^{(i-1)}\lambda_{1,4}^{(i)}}. \tag{6.66}$$

The cases of PIV, PII, and PI correspond to finite cluster algebras admitting no nontrivial modular transformations.

Appendix A Singularity theory approach to the Painlevé cubics

As mentioned above, for special values of $\omega_1^{(d)}, \ldots, \omega_4^{(d)}$ the fibre may have a singularity. Such singularities were classified in [20] for PVI and in [26] for all other Painlevé equations. These results can be summarised in the following table:

The meaning of the table is the following: for each Painlevé equation of type specified by the first column in the table, there is at least one singular fibre with singularity of the type given in the second column of the table, and at least one

Dynkin	Painlevé equations	Surface singularity type
$\widetilde{D}_4$	P_{VI}	D_4
$\widetilde{D}_5$	P_V	A_3
$\widetilde{D}_6$	deg P_V=$P_{III}(\widetilde{D}_6)$	A_1
$\widetilde{D}_6$	$P_{III}(\widetilde{D}_6)$	A_1
$\widetilde{D}_7$	$P_{III}(\widetilde{D}_7)$	non-singular
$\widetilde{D}_8$	$P_{III}(\widetilde{D}_8)$	non-singular
$\widetilde{E}_6$	P_{IV}	A_2
$\widetilde{E}_7^*$	$P_{II}(FN)$	A_1
$\widetilde{E}_7^{**}$	$P_{II}(MJ)$	A_1
$\widetilde{E}_8$	P_I	non-singular

TABLE 2.

singular fibre with singularity of type specified by any Dynkin sub-diagram of the Dynkin diagram given in the second column of the table.

For example PIV is the equation corresponding to $\widetilde{E_6}$ and it has a two singular fibres with singularity of type A_2 and at three singular fibres with singularity of type A_1.

The scope of this section is to show that the non singular fibres of each family of affine cubics are locally diffeomorphic to the versal unfolding [1] of the singularity of the type given in the second column of the table.

A.3.1. $\widetilde{D}_4$. This case corresponds to the sixth Painlevé equation. The cubic in this case is (we drop the indices $(\widetilde{D}_4)$ for convenience):

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0. \tag{A.1}$$

To show that this is diffeomorphic to the versal unfolding of D_4 we need to map this cubic to Arnol'd form. To this aim we first shift all variables by $x_i \to x_i + 2$, $i = 1, 2, 3$ to obtain

$$x_1^2+x_2^2+x_3^2+2x_1x_2+2x_2x_3+2x_1x_3+x_1x_2x_3+\widetilde{\omega}_1x_1+\widetilde{\omega}_2x_2+\widetilde{\omega}_3x_3+\widetilde{\omega}_4 = 0, \tag{A.2}$$

where

$$\widetilde{\omega}_i = \omega_i + 8, \quad \text{for } i = 1, 2, 3, \qquad \widetilde{\omega}_4 = \omega_4 + 2(\omega_1 + \omega_2 + \omega_3) + 20.$$

As a second step we use the following diffeomorphism around the origin:

$$x \to x - \frac{1}{2}y, \quad y \to x + \frac{1}{2}x, \quad z \to z + \frac{y^2}{8} - 2x - \frac{x^2}{2} - \frac{\widetilde{\omega}_3}{2}$$

so that the new cubic (up to a Morse singularity that we throw away and after a shift $x \to x - \frac{\omega_3}{4}$) becomes indeed the versal unfolding of a D_4 singularity in Arnol'd form:

$$-2x_1^3 + \frac{x_1x_2^2}{2} + \widehat{\omega}_1 x_1 + \widehat{\omega}_2 x_2 + \widehat{\omega}_3 x_1^2 + \widehat{\omega}_4,$$

where

$$\widehat{\omega}_1 = \omega_1 + \omega_2 - 8 - 4\omega_3 - \frac{\omega_3^2}{8}, \quad \widehat{\omega}_2 = \frac{\omega_2 - \omega_1}{2},$$

$$\widehat{\omega}_3 = 8 + \omega_3, \quad \widehat{\omega}_4 = \omega_4 + 2\omega_3 - \frac{\omega_3(\omega_1 + \omega_2 - \omega_3)}{4} + 4.$$

The above formulae show that the versal unfolding parameters $\widehat{\omega}_1, \ldots, \widehat{\omega}_4$ are independent as long as $\omega_1, \ldots, \omega_4$ are.

A.4. $\widetilde{D}_5$**.** This case corresponds to the fifth Painlevé equation. The cubic in this case is (we drop the indices $(\widetilde{D}_5)$ for convenience):

$$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4 = 0, \tag{A.3}$$

where only three parameters are free:

$$\omega_1 = -G_1G_\infty - G_2, \quad \omega_2 = -G_2G_\infty - G_1, \quad \omega_3 = -G_\infty, \quad \omega_4 = 1 + G_\infty^2 + G_1G_2G_\infty.$$

Again we want to show that this is diffeomorphic to the versal unfolding of A_3. To this aim we impose the following change of variables:

(A.4)

$$x_1 \to x_1 - x_3 + \frac{G_\infty}{u(x_2)}, \quad x_2 \to u(x_2), \quad x_3 \to 2\frac{x_3}{u(x_2)} + \frac{G_2 + G_1G\infty}{u(x_2)} - \frac{2G_\infty}{u(x_2)^2},$$

where $u(x_2)$ is a function to be determined. This maps the PV cubic to:

$$x_1^2 - x_3^2 + 1 + G_1G_2G_\infty + G_\infty^2 + \frac{G_\infty^2}{u^2} - \frac{G_\infty(G_2 + G_1G_\infty)}{u} - (G_1 + G_2G_\infty)u + u^2.$$

It is easy to prove that any solution $u(x_2)$ of the equation

$$\frac{G_\infty^2}{u^2} - \frac{G_\infty(G_2 + G_1G_\infty)}{u} - (G_1 + G_2G_\infty)u + u^2 = x_2^4 + (G_2 + G_1G_\infty)x_2^2 + (G_1 + G_2G_\infty)x_2$$

will define a diffeomorphism by (A.4) mapping (A.3) to the versal unfolding of A_3.

A.5. $\widetilde{D}_6$**.** This case corresponds to the third Painlevé equation. The cubic in this case is (we drop the indices $(\widetilde{D}_3)$ for convenience):

$$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_4 = 0, \tag{A.5}$$

where only two parameters are free:

$$\omega_1 = -1 - G_\infty^2, \quad \omega_2 = -G_2G_\infty, \quad \omega_4 = G_\infty^2.$$

The most singular fibre is given by $G_\infty = 1$ and $G_2 = 2$ and has two singular points at $(1, 0, 2)$ and $(0, 1, 2)$ respectively. We can define two local diffeomorphisms, one around $(1, 0, 2)$, the other around $(0, 1, 2)$, which map our cubic to the versal unfolding of a A_1 singularity.

The first diffeomorphism is given by:

$$x_1 \to \frac{1 + G_\infty^2}{2} + x_1, \quad x_2 \to -x_2 + x_3, \quad x_3 \to \frac{2(G_2G_\infty - 2x_3)}{1 + G_\infty^2 + 2x_1}$$

The second diffeomorphism is:

$$x_1 \to -x_1 + x_3, \quad x_2 \to \frac{G_2G_\infty}{2} - x_2, \quad x_3 \to \frac{2(1 + G_\infty^2 - 2x_3)}{G_2G_\infty^2 - 2x_2}.$$

A.5.1. $\widetilde{E}_6$. This case corresponds to the fourth Painlevé equation. The cubic in this case is (we drop the indices $(\widetilde{E}_6)$ for convenience):

$$x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0, \tag{A.6}$$

where only two parameters are free:

$$\omega_1 = -G_1G_\infty - G_\infty^2, \quad \omega_2 = -G_\infty^2, \quad \omega_3 = -G_\infty^2, \quad \omega_4 = G_\infty^2 + G_1G_\infty^3.$$

Again we want to show that this is diffeomorphic to the versal unfolding of A_2. To this aim we impose the following change of variables:

$$x_1 \to x_1 - x_3 + \frac{G_\infty^2}{u}, \quad x_2 \to u, \quad x_3 \to \frac{2x_3}{u} + \frac{G_\infty}{u}(G_1 + G_\infty) - \frac{2G_\infty^2}{u^2} \tag{A.7}$$

where u is function of x_3 satisfying the following

$$\frac{G_\infty^4}{u^2} - \frac{G_\infty^3(G_\infty + G_1)}{u} - G_\infty^2 u = x_2^3 + G_\infty x_2.$$

It is easy to prove that this transformation is a local diffeomorphism mapping our cubic to

$$x_1^2 - x_3^2 + x_2^3 + G_\infty x_2 + G_\infty + G_1G_\infty^3,$$

the versal unfolding of the A_2 singularity.

A.5.2. $\widetilde{E}_7$. This case corresponds to the second Painlevé equation. Since the treatment of the two cubics $\widetilde{E}_7^*$ and $\widetilde{E}_7^{**}$ is completely equivalent, we choose to work with the former:

$$x_1x_2x_3 - x_1 - x_2 - x_3 + \omega_4 = 0, \tag{A.8}$$

where:

$$\omega_4 = G_\infty^2 + G_\infty^{-2}$$

The following change of variables:

$$x_1 \to x_1 - x_3 + \frac{1}{u}, \quad x_2 \to u, \quad x_3 \to \frac{x_1 + x_3 + 1}{u},$$

where u is a function of x_2 satisfying

$$-\frac{1}{u} - u = x_2^2,$$

is a local diffeomorphism mapping our cubic to the versal unfolding of the A_1 singularity:

$$x_1^2 - x_3^2 + x_2^2 + \omega_4.$$

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