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# Swiss Cheese type operads and models for relative loop spaces

Alexandre Quesney

## Abstract

We construct a (coloured) operad  $\mathcal{RL}$  in the category of sets that may be thought of as a combinatorial model for the Swiss Cheese operad. By adapting Batanin-Berger's condensation process we obtain a topological (resp. chain) operad weakly equivalent to the topological (resp. chain) Swiss Cheese operad. As an application, we exhibit models for relative loop spaces acted on by Swiss Cheese type operads (in dimension 2).

## 1 Introduction

The Swiss Cheese operad  $\mathcal{SC}$  is a 2-coloured topological operad that mixes, in its  $m$ -dimensional part  $\mathcal{SC}_m$ , the  $m$ -dimensional and the  $(m-1)$ -dimensional parts of the little cubes operad  $\mathcal{C}$ . It was used by M. Kontsevich [Kon99] in deformation quantization as a natural way to define  $C_*(\mathcal{C}_m)$ -algebras acting on  $C_*(\mathcal{C}_{m-1})$ -algebras. In fact, it turns out that the Swiss Cheese operad  $\mathcal{SC}_m$  recognizes actions of  $m$ -fold loop spaces on  $m$ -fold relative loop spaces as announced in [HLS13]. Looking at  $m=1$  this means that the  $A_\infty$ -actions are recognized by  $\mathcal{SC}_1$ .

The purpose of this paper is two fold. We first provide a combinatorial model for the Swiss-Cheese operad  $\mathcal{SC}$ . That is, we construct a (coloured) operad  $\mathcal{RL}$  in the category of sets and, by mean of *condensation* and according to a choice of a cosimplicial object, we obtain an operad equivalent to the Swiss Cheese operad  $\mathcal{SC}$  as well as an operad equivalent to the chains of  $\mathcal{SC}$  with integers coefficients.

In a second part, we use our newly obtained operads to exhibit algebraic models for actions of 2-fold loop spaces on 2-fold relative loop spaces. We pay close attention to the couple (cobar construction, relative cobar construction).

In [BB09], Batanin and Berger introduce the notion of *condensation* of a coloured operad. By applying this condensation to the *lattice path operad*  $\mathcal{L}$  they obtain a model for the little cubes operad. More precisely, this means that, in the category of topological spaces **Top** (resp. the category of chain complexes **Ch**( $\mathbb{Z}$ )), the *condensation operad* of  $\mathcal{L}$  is weakly equivalent to the topological (resp. chain) little cubes operad.

We construct a coloured operad  $\mathcal{RL}$  in the category of sets that may be thought of as a combinatorial model for the Swiss Cheese operad.

Let us fix a closed monoidal symmetric category  $\mathbf{C}$  with a zero object and a cosimplicial object  $\delta : \Delta \rightarrow \mathbf{C}$ . By adapting Batanin-Berger's method, we obtain a functor

$$F : \mathcal{RL}\text{-algebra} \longrightarrow \text{Coend}_{\mathcal{RL}}(\delta)\text{-algebra}$$

that sends algebras over  $\mathcal{RL}$  into algebras over the *condensation (2-coloured) operad*  $\text{Coend}_{\mathcal{RL}}(\delta)$  in  $\mathbf{C}$ .

The operad  $\mathcal{RL}$  is filtered by suboperads  $\mathcal{RL}_m$ ,  $m \geq 1$  and we have the corresponding condensed operads  $\text{Coend}_{\mathcal{RL}_m}(\delta)$ ,  $m \geq 1$ .

We are interested by two choices for  $\delta$ :

$$\delta_{\text{Top}} : \Delta \xrightarrow{\delta_{\text{yon}}} \mathbf{Set}^{\Delta^{\text{op}}} \xrightarrow{|\cdot|} \mathbf{Top}$$

and

$$\delta_{\mathbb{Z}} : \Delta \xrightarrow{\delta_{\text{yon}}} \mathbf{Set}^{\Delta^{\text{op}}} \xrightarrow{C_*(-; \mathbb{Z})} \mathbf{Ch}(\mathbb{Z}),$$

where  $\delta_{\text{yon}}([n]) = \text{Hom}_{\Delta}(-, [n])$  is the Yoneda functor. In this manner, the condensation of  $\mathcal{R}\mathcal{L}_m$  leads to a topological operad  $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta_{\text{Top}})$  and a chain operad  $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta_{\mathbb{Z}})$ .

The Swiss Cheese operad that we consider is denoted by  $\mathcal{S}\mathcal{C}_m$ ,  $m \geq 1$ , and is the augmented (cubical) version of Voronov's Swiss Cheese operad  $\mathcal{S}\mathcal{C}_m^{\text{vor}}$  defined in [Vor98].

For each  $m \geq 1$ , we construct a cellular decomposition of the Swiss Cheese operad  $\mathcal{S}\mathcal{C}_m$ . This provides a recognition principle that we use afterwards to show the following.

**Theorem 1.1.** *Let  $m \geq 1$ . Then, the operad  $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta_{\text{Top}})$  is weakly equivalent to the topological Swiss-Cheese operad  $\mathcal{S}\mathcal{C}_m$  and, the operad  $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta_{\mathbb{Z}})$  is weakly equivalent to the chain Swiss-Cheese operad  $C_*(\mathcal{S}\mathcal{C}_m)$ .*

For each  $m \geq 1$ , the operad  $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta_{\mathbb{Z}})$  admits a weakly equivalent suboperad  $\mathcal{R}\mathcal{S}_m$ . This operad  $\mathcal{R}\mathcal{S}_m$  may be thought of as a Swiss Cheese version (or relative version) of the *surjection operad*  $\mathcal{S}_m$  studied by McClure-Smith [MS02, MS03] and Berger-Fresse [BF04].

Our cellular decomposition of the Swiss Cheese operad  $\mathcal{S}\mathcal{C}_m$  generalizes Berger's cell decomposition of the little  $m$ -cubes operad [Ber97] and, likewise, gives rise to a recognition principle.

The cells are indexed by a poset operad  $\mathcal{R}\mathcal{K}_m$  which is a Swiss Cheese (or relative) version of the *extended complete graph operad*. Such a decomposition provides a zig-zag of weak equivalences between the Swiss Cheese operad  $\mathcal{S}\mathcal{C}_m$  and the classifying operad of  $\mathcal{R}\mathcal{K}_m$ . The latter arise as a comparison object: for any topological  $\mathcal{R}\mathcal{K}_m$ -cellular operad  $\mathcal{O}$  (see Definition 3.7) there is a zig-zag of weak equivalences linking  $\mathcal{O}$  to the classifying operad of  $\mathcal{R}\mathcal{K}_m$ . Thus, we prove the following.

**Theorem 1.2.** *Let  $m \geq 1$ . Then any topological  $\mathcal{R}\mathcal{K}_m$ -cellular operad is weakly equivalent to the Swiss Cheese operad  $\mathcal{S}\mathcal{C}_m$ .*

The second objective of this paper is to provide models for relative loop spaces.

It is well-known that  $m$ -fold loop spaces are recognized by the little  $m$ -cubes operad, [May72]. A similar pattern is announced in [HLS13] for relative loop spaces: couples ( $m$ -fold loop space,  $m$ -fold relative loop space) are recognized by  $\mathcal{S}\mathcal{C}_m$ . Then a "good" model for such a couple should be acted on by a model of  $\mathcal{S}\mathcal{C}_m$ .

We focus on the second stage filtration operad  $\mathcal{R}\mathcal{L}_2$ . This operad encodes couples  $(\mathcal{M}, \mathcal{Z})$  together with a map  $\mathcal{M} \rightarrow \mathcal{Z}$  satisfying some properties. In particular,  $\mathcal{M}$  is a multiplicative operad and  $\mathcal{Z}$  is an infinitesimal bimodule over the associative operad  $\mathcal{A}$ s. These structures naturally endow  $\mathcal{M}$  and  $\mathcal{Z}$  with a cosimplicial structure. Moreover, the functor  $F$  sends such a couple  $(\mathcal{M}, \mathcal{Z})$  to the couple of totalizations  $(\text{Tot}_{\delta}\mathcal{M}, \text{Tot}_{\delta}\mathcal{Z})$ . Thus, the condensed operad  $\text{Coend}_{\mathcal{R}\mathcal{L}_2}(\delta)$  acts on the couple  $(\text{Tot}_{\delta}\mathcal{M}, \text{Tot}_{\delta}\mathcal{Z})$ .

Given a pair of topological spaces  $(X, Y)$  pointed at  $*$  and such that  $* \subset Y \subset X$ , there exists a cosimplicial space  $\omega(X, Y)$  such that its totalization  $\text{Tot}_{\delta_{\text{Top}}}\omega(X, Y)$  is homeomorphic to  $\Omega(X, Y)$ . In particular,  $\omega(X) = \omega(X, *)$  is a model for the loop space  $\Omega(X, *) = \Omega X$ . We have the following.

**Theorem 1.3.** *Let  $(M, N)$  be a pair of topological monoids pointed at the unit such that  $N$  is a submonoid of  $M$ . Let  $\delta$  be  $\delta_{\text{Top}}$  (resp.  $\delta_{\mathbb{Z}}$ ). Then there exists an operad  $\text{Coend}_{\mathcal{R}\mathcal{L}_2}(\delta)$  which is weakly equivalent to the topological operad  $\mathcal{S}\mathcal{C}_2$  (resp. to the chain operad  $C_*(\mathcal{S}\mathcal{C}_2)$ ) and which acts on the totalization  $(\text{Tot}_{\delta}\omega(M), \text{Tot}_{\delta}\omega(M, N))$ .*

A model for chains of loop spaces can be provided by Adams' cobar construction [Ada56]. Given a 1-connected topological space  $X$ , Adams' quasi-isomorphism of dg-algebras takes the form

$$\Phi : \underline{\Omega}C_*^1X \rightarrow C_*^{\square}(\Omega_M X).$$

Here,  $\Omega_M X$  denotes the associative Moore loop space of  $X$  and  $C_*^{\square}(-)$  stands for the cubical normalized chain functor. The presence of the cubical chain on the right side makes natural asking if the cobar construction  $\underline{\Omega}C_*^1X$  can be thought of as the chain complex of a cubical set with a monoidal structure. This is the point of view developed in [KS05]. It leads to an elegant proof that Adams' morphism  $\Phi$  is a morphism of dg-bialgebras. Such a compatibility plays an essential role for iterating the cobar construction, see [Bau81]. By interpreting the *relative cobar construction*  $\underline{\Omega}(-, -)$  as a cubical set, we obtain the following.

**Proposition 1.4.** *Let  $(X, Y)$  be a pair of two 1-connected spaces pointed at  $*$  such that  $* \subset Y \subset X$ . Then  $\underline{\Omega}(C_*^1X, C_*^1Y)$  is naturally a coalgebra and an  $\underline{\Omega}C_*^1X$ -module. Moreover, there exists a  $\Phi$ -equivariant morphism  $\Psi : \underline{\Omega}(C_*^1X, C_*^1Y) \rightarrow C_*^{\square}(\Omega(X, Y))$  that is a quasi-isomorphism of dg-coalgebras.*

---

With regard to the  $\Phi$ -equivariance of the above morphism  $\Psi$ , a similar statement was obtained in [FHT92]. However, both the coalgebra structure on  $\underline{\Omega}(C_*^1X, C_*^1Y)$  and the coalgebra compatibility of  $\Psi$  are a direct consequence of the adopted point of view. In particular, Proposition 1.4 allows us to iterate the relative cobar construction by taking the cobar construction of the relative cobar construction, *i.e.*

$$\underline{\Omega}\underline{\Omega}(C_*^1X, C_*^1Y),$$

provided that  $X$  and  $Y$  are 2-connected. Note that another way to proceed is by taking the relative cobar of the cobar constructions, *i.e.*

$$\underline{\Omega}(\underline{\Omega}C_*^1X, \underline{\Omega}C_*^1Y).$$

The second choice is well-adapted to the following consideration. We show (Theorem 8.11) that  $\mathcal{RS}_2$  acts on the couple  $(\underline{\Omega}B, \underline{\Omega}(B, C))$  whenever  $B$  is a 1-reduced bialgebra and  $C$  a  $B$ -comodule in the category of unital algebras. By generalizing this result to differential graded objects and taking  $B = C_*^1M$  and  $C = C_*^1N$  whenever  $(M, N)$  is a pair of monoids, one obtains algebraic counterpart to the fact that  $\Omega(M, N) \simeq \Omega(\Omega\mathcal{B}M, \Omega\mathcal{B}N)$  is a relative double loop space (of the classifying spaces  $\mathcal{B}M$  and  $\mathcal{B}N$ ).

**Outline of the paper.** We begin by explaining how we *condense* a particular type of coloured operads (*SC-split operads*) to obtain 2-coloured operads. This result will be used in Section 4.

In section 3 we consider the (cubical) Swiss Cheese operad  $\mathcal{SC}$ . For each non zero natural number  $m$ , we construct a cellular decomposition of  $\mathcal{SC}_m$  indexed by a Swiss Cheese version of the *extended complete graph operad*, say  $\mathcal{RK}_m$ .

With the third section 4, we describe our main operad  $\mathcal{RL}$  which is an SC-split operad. Using the condensation process developed in Section 2, one obtains the 2-coloured operad  $\text{Coend}_{\mathcal{RL}}(\delta)$ . We use results of Section 3 to prove the following. In the topological setting ( $\delta = \delta_{\text{Top}}$ ) and the chain setting ( $\delta = \delta_{\mathbb{Z}}$ ), we show a weak equivalence of  $\text{Coend}_{\mathcal{RL}_m}(\delta)$  with the topological (resp. chain) Swiss Cheese operad  $\mathcal{SC}_m$ .

In Section 5 we exhibit the sub-operad  $\mathcal{RS}_m$  and show that the inclusion  $\mathcal{RS}_m \hookrightarrow \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$  is a weak equivalence.

In Section 6 we focus on the operad  $\mathcal{RL}_2$  and its representations.

The remaining Section 7 and Section 8 are devoted to models for relative loop spaces.

In Section 7 we show that the pair of cosimplicial spaces  $(\omega(M), \omega(M, N))$  is a representation of  $\mathcal{RL}_2$  whenever  $(M, N)$  are monoids. This implies that  $(\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M), \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N))$  is an algebra over  $\text{Coend}_{\mathcal{RL}_2}(\delta_{\mathbb{Z}})$ , and so is a representation of  $\mathcal{RS}_2$  by restriction. We make explicit the latter action of the sub-operad  $\mathcal{RS}_2$ .

Finally, Section 8 is devoted to the relative cobar construction.

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## 2 Preliminaries

In [BB09] Batanin and Berger introduced the notion of condensation of a coloured operad. It consists of a realization followed by a totalization what "condenses" all the colours into a single one.

We consider particular coloured operads that we call *SC-split operads*. Roughly speaking, the set of colours of an SC-split operad can be split into two subsets that yield two sub-operads. We modify Batanin-Berger's condensation process for the SC-split operads. Our modification consists in condensing separately the colours of each of the two subsets of colours into one colour. This provides, in particular, 2-coloured operads.

### 2.1 SC functor-operads

Let  $\mathbf{C}$  be a closed symmetric monoidal category. Let  $\mathbf{A}$  and  $\mathbf{B}$  be two  $\mathbf{C}$ -categories (*i.e.* enriched over  $\mathbf{C}$ ). We denote by  $\mathbf{A} \otimes \mathbf{B}$  the category with the pairs  $(\mathbf{a}, \mathbf{b})$  for  $\mathbf{a} \in \mathbf{A}$  and  $\mathbf{b} \in \mathbf{B}$  as objects and

$$\text{Hom}_{\mathbf{A} \otimes \mathbf{B}}((\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')) := \text{Hom}_{\mathbf{A}}(\mathbf{a}, \mathbf{a}') \otimes \text{Hom}_{\mathbf{B}}(\mathbf{b}, \mathbf{b}')$$

as hom-objects, where the tensor on the right hand side is the tensor of  $\mathbf{C}$ .

**Definition 2.1.** For a family of  $\mathbf{C}$ -functors

$$\{\xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} : \mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_k \rightarrow \mathbf{A}_{k+1}\}_{\mathbf{A}_i \in \{\mathbf{A}, \mathbf{B}\}}$$

and a permutation  $\sigma \in \Sigma_k$ , we denote by  $\xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}^\sigma : \mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_k \rightarrow \mathbf{A}_{k+1}$  the functor

$$\xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}^\sigma(X_1, \dots, X_k) = \xi_{\mathbf{A}_{\sigma^{-1}(1)}, \dots, \mathbf{A}_{\sigma^{-1}(k)}; \mathbf{A}_{k+1}}^\sigma(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(k)}).$$

A family of  $\mathbf{C}$ -functors  $\{\xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} : \mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_k \rightarrow \mathbf{A}_{k+1}\}_{\mathbf{A}_i \in \{\mathbf{A}, \mathbf{B}\}}$  is called *twisting symmetric* if there exist  $\mathbf{C}$ -natural transformations  $\phi_{\sigma, \mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} : \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} \rightarrow \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}^\sigma$  for  $\sigma \in \Sigma_k$ , such that

$$\phi_{\sigma_1 \sigma_2, \mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} = (\phi_{\sigma_1, \mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}})^{\sigma_2} \phi_{\sigma_2, \mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}$$

and such that  $\phi_{\text{id}, \mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}$  is the identity transformation where  $\text{id}$  denotes the neutral element of  $\Sigma_k$ .

**Definition 2.2.** An *SC functor-operad*  $\xi = \{\xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}\}_{k \geq 0}$  over  $(\mathbf{A}, \mathbf{B})$  is the data, for each  $k \geq 0$ , of twisted symmetric families

$$\xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} : \mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_k \rightarrow \mathbf{A}_{k+1}$$

indexed by the  $(k+1)$ -uples  $(\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1})$  such that  $\mathbf{A}_{k+1} = \mathbf{B}$  whenever it exists an  $1 \leq i \leq k$  such that  $\mathbf{A}_i = \mathbf{B}$ , together with natural transformations

$$\begin{aligned} \mu_{[\mathbf{A}]_{1, i_1}, \dots, [\mathbf{A}]_{k, i_k}; \mathbf{A}_{k+1}} : \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} \circ (\xi_{\mathbf{A}_{1,1}, \dots, \mathbf{A}_{1, i_1}; \mathbf{A}_1} \otimes \dots \otimes \xi_{\mathbf{A}_{k,1}, \dots, \mathbf{A}_{k, i_k}; \mathbf{A}_k}) \\ \rightarrow \xi_{\mathbf{A}_{1,1}, \dots, \mathbf{A}_{k, i_k}; \mathbf{A}_{k+1}}, \quad \text{for } i_1, \dots, i_k \geq 0, \end{aligned}$$

where  $[\mathbf{A}]_{a,b} = (\mathbf{A}_{a,1}, \dots, \mathbf{A}_{a,b}; \mathbf{A}_a)$ . These natural transformations have to satisfy the following three conditions.

1. For  $\mathbf{A}_0 \in \{\mathbf{A}, \mathbf{B}\}$ ,  $\xi_{\mathbf{A}_0; \mathbf{A}_0}$  is the identity functor and  $\xi_{\mathbf{A}_{k+1}; \mathbf{A}_{k+1}} \circ \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} = \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} = \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} \circ (\xi_{\mathbf{A}_1; \mathbf{A}_1} \otimes \dots \otimes \xi_{\mathbf{A}_k; \mathbf{A}_k})$  where the equalities are obtained via  $\mu_{(\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}); \mathbf{A}_{k+1}}$  and  $\mu_{(\mathbf{A}_1; \mathbf{A}_1), \dots, (\mathbf{A}_k; \mathbf{A}_k); \mathbf{A}_{k+1}}$  respectively.
2. The natural transformations  $\mu_{[\mathbf{A}]_{1, i_1}, \dots, [\mathbf{A}]_{k, i_k}; \mathbf{A}_{k+1}}$  are associative.
3. All the diagrams of the following forms commute:

$$\begin{array}{ccc} \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} \circ (\xi_{\mathbf{A}_{1,1}, \dots, \mathbf{A}_{1, i_1}; \mathbf{A}_1} \otimes \dots \otimes \xi_{\mathbf{A}_{k,1}, \dots, \mathbf{A}_{k, i_k}; \mathbf{A}_k}) & \xrightarrow{\mu_{[\mathbf{A}]_{1, i_1}, \dots, [\mathbf{A}]_{k, i_k}; \mathbf{A}_{k+1}}} & \xi_{\mathbf{A}_{1,1}, \dots, \mathbf{A}_{k, i_k}; \mathbf{A}_{k+1}} \\ \downarrow \phi_\sigma \circ (\phi_{\sigma_1} \otimes \dots \otimes \phi_{\sigma_{k+j}}) & & \downarrow \phi_{\sigma(\sigma_1, \dots, \sigma_{k+j})} \\ \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}^\sigma \circ (\xi_{\mathbf{A}_{1,1}, \dots, \mathbf{A}_{1, i_1}; \mathbf{A}_1}^{\sigma_1} \otimes \dots \otimes \xi_{\mathbf{A}_{k,1}, \dots, \mathbf{A}_{k, i_k}; \mathbf{A}_k}^{\sigma_k}) & \xrightarrow{\mu_{[\mathbf{A}]_{1, i_1}, \dots, [\mathbf{A}]_{k, i_k}; \mathbf{A}_{k+1}}} & \xi_{\mathbf{A}_{1,1}, \dots, \mathbf{A}_{k, i_k}; \mathbf{A}_{k+1}}^{\sigma(\sigma_1, \dots, \sigma_{k+j})} \end{array}$$

**Definition 2.3.** Let  $\xi = \{\xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}\}_{k \geq 0}$  be an SC functor-operad over  $(\mathbf{A}, \mathbf{B})$ . A  $\xi$ -*algebra*  $X$  is a couple  $(X_{\mathbf{A}}, X_{\mathbf{B}}) \in \mathbf{A} \otimes \mathbf{B}$  equipped with morphisms in  $\mathbf{A}_{k+1}$

$$\alpha_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} : \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}(X_{\mathbf{A}_1}, \dots, X_{\mathbf{A}_k}) \rightarrow X_{\mathbf{A}_{k+1}}, \quad k \geq 0,$$

subject to the following conditions.

1.  $\alpha_{\mathbf{A}_1} = 1_{X_{\mathbf{A}_1}}$ ;
2.  $\alpha_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} \circ \phi_\sigma = \alpha_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}$ , for all  $\sigma \in \Sigma_k$ ;
3. all the diagrams of the following form commute

$$\begin{array}{ccc} \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}} \circ (\xi_{[\mathbf{A}]_{1, i_1}}(X_{1, i_1}) \otimes \dots \otimes \xi_{[\mathbf{A}]_{k, i_k}}(X_{k, i_k})) & \xrightarrow{\mu_{[\mathbf{A}]_{1, i_1}, \dots, [\mathbf{A}]_{k, i_k}; \mathbf{A}_{k+1}}} & \xi_{\mathbf{A}_{1,1}, \dots, \mathbf{A}_{k, i_k}; \mathbf{A}_{k+1}}(X_{\mathbf{A}_{1,1}}, \dots, X_{\mathbf{A}_{k, i_k}}) \\ \downarrow \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}(\alpha_{[\mathbf{A}]_{1, i_1}} \otimes \dots \otimes \alpha_{[\mathbf{A}]_{k, i_k}}) & & \downarrow \alpha_{\mathbf{A}_{1,1}, \dots, \mathbf{A}_{k, i_k}; \mathbf{A}_{k+1}} \\ \xi_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}(X_{\mathbf{A}_1}, \dots, X_{\mathbf{A}_{k+1}}) & \xrightarrow{\alpha_{\mathbf{A}_1, \dots, \mathbf{A}_k; \mathbf{A}_{k+1}}} & X_{\mathbf{A}_{k+1}} \end{array}$$

where  $X_{a,b}$  denotes  $X_{\mathbf{A}_{a,1}, \dots, \mathbf{A}_{a,b}}$ .

## 2.2 SC-split operads

Let us fix an arbitrary coloured operad  $\mathcal{O}$  in  $\mathbf{C}$ . We denote  $\text{Col}$  its set of colours, so that  $\mathcal{O}$  consists of objects

$$\mathcal{O}(n_1, \dots, n_k; n) \in \mathbf{C},$$

for  $n_1, \dots, n_k \in \text{Col}$  and  $k \geq 0$  together with the unit  $I_{\mathbf{C}} \rightarrow \mathcal{O}(n; n)$  and substitution maps

$$\mathcal{O}(n_1, \dots, n_k; n) \otimes \mathcal{O}(m_1, \dots, m_l; n_i) \rightarrow \mathcal{O}(n_1, \dots, n_{i-1}, m_1, \dots, m_l, n_{i+1}, \dots, n_k; n)$$

satisfying the natural unit, associativity and equivariance axioms.

The underlying category of  $\mathcal{O}$  is the category  $\mathcal{O}_{\mathbf{u}}$  with the colours  $n \in \text{Col}$  as objects and the unary operations  $\mathcal{O}_{\mathbf{u}}(m, n) = \mathcal{O}(m; n)$  as morphisms. This way, we have functors

$$\mathcal{O}(\underbrace{\phantom{(-, \dots, -)}}_k; -) : (\mathcal{O}_{\mathbf{u}}^{\text{op}})^{\otimes k} \otimes \mathcal{O}_{\mathbf{u}} \rightarrow \mathbf{C}, \quad k \geq 0.$$

From now, we suppose  $\mathbf{C}$  endowed with a zero object *i.e.* an element  $0 \in \mathbf{C}$  such that  $0 \otimes X = 0$  for all  $X \in \mathbf{C}$ . Let us suppose now that  $\mathcal{O}$  satisfies the following hypothesis.

- H1.  $\text{Col} = \text{Col}_{\mathbf{c}} \sqcup \text{Col}_{\mathbf{o}}$ .
- H2. The collection of the  $\mathcal{O}(n_1, \dots, n_k; n)$  for  $n_i, n \in \text{Col}_{\mathbf{c}}$ ,  $k \geq 0$  forms a sub-operad of  $\mathcal{O}$ .
- H3. The collection of the  $\mathcal{O}(n_1, \dots, n_j; n)$  for  $n_i, n \in \text{Col}_{\mathbf{o}}$ ,  $j \geq 0$  forms a sub-operad of  $\mathcal{O}$ .
- H4. The  $\mathcal{O}(n_1, \dots, n_j; n)$  are the zero object for any  $n \in \text{Col}_{\mathbf{c}}$  if there exists an  $1 \leq i \leq j$ , such that  $n_i \in \text{Col}_{\mathbf{o}}$ ,  $j \geq 1$ .

We call such an operad an *SC-split operad*. The sub-operad in H2 is called the *closed part of  $\mathcal{O}$* ; the sub-operad in H3 is called the *open part of  $\mathcal{O}$* .

The underlying category  $\mathcal{O}_{\mathbf{u}}$  contains two particular categories:

- $\mathcal{O}_{\mathbf{u}}^{\mathbf{c}}$  the sub-category of  $\mathcal{O}_{\mathbf{u}}$  with objects the colours in  $\text{Col}_{\mathbf{c}}$  and morphisms the  $\mathcal{O}_{\mathbf{u}}(n, m)$  for  $n, m \in \text{Col}_{\mathbf{c}}$ ;
- $\mathcal{O}_{\mathbf{u}}^{\mathbf{o}}$  the sub-category of  $\mathcal{O}_{\mathbf{u}}$  with objects the colours in  $\text{Col}_{\mathbf{o}}$  and morphisms the  $\mathcal{O}_{\mathbf{u}}(n, m)$  for  $n, m \in \text{Col}_{\mathbf{o}}$ .

By H2 and H3 both  $\mathcal{O}_{\mathbf{u}}^{\mathbf{c}}$  and  $\mathcal{O}_{\mathbf{u}}^{\mathbf{o}}$  are  $\mathbf{C}$ -categories. The category  $\mathbf{C}^{\mathcal{O}_{\mathbf{u}}^{\mathbf{c}}}$  (resp.  $\mathbf{C}^{\mathcal{O}_{\mathbf{u}}^{\mathbf{o}}}$ ) of  $\mathbf{C}$ -functors from  $\mathcal{O}_{\mathbf{u}}^{\mathbf{c}}$  (resp. from  $\mathcal{O}_{\mathbf{u}}^{\mathbf{o}}$ ) to  $\mathbf{C}$  is a  $\mathbf{C}$ -category.

For  $k \geq 0$ , let  $(c_1, \dots, c_k; c_{k+1})$  be a tuple of elements in  $\{\mathbf{c}; \mathbf{o}\}$  satisfying

$$(2.1) \quad c_{k+1} = \mathbf{o} \text{ if there is an } 1 \leq i \leq k \text{ such that } c_i = \mathbf{o}.$$

We set  $\mathbf{A}_i := \mathbf{C}^{\mathcal{O}_{\mathbf{u}}^{\mathbf{c}_i}}$  and we define the  $\mathbf{C}$ -functor

$$\xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}} : \mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_k \rightarrow \mathbf{A}_{k+1}$$

as the coend

$$\xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(X_{c_1}, \dots, X_{c_k})(n) = \mathcal{O}(\underbrace{\phantom{(-, \dots, -)}}_k; n) \otimes_{\mathcal{O}_{\mathbf{u}}^{\mathbf{c}_1} \otimes \dots \otimes \mathcal{O}_{\mathbf{u}}^{\mathbf{c}_k}} X_{c_1}(-) \otimes \dots \otimes X_{c_k}(-).$$

An  $\mathcal{O}$ -algebra  $X$  is a family  $\{X(n)\}_{n \in \text{Col}}$  of objects  $X(n) \in \mathbf{C}$  equipped with morphisms

$$(2.2) \quad \mathcal{O}(n_1, \dots, n_k; n) \otimes X(n_1) \otimes \dots \otimes X(n_k) \rightarrow X(n), \quad n_1, \dots, n_k, n \in \text{Col}$$

subject to the natural unit, associative and equivariance axioms. In particular, from the hypothesis on  $\mathcal{O}$ ,  $X$  can be seen as a pair  $(X_{\mathbf{c}}, X_{\mathbf{o}})$  where  $X_{\mathbf{c}}$  is the sub-family  $\{X_{\mathbf{c}}(n)\}_{n \in \text{Col}_{\mathbf{c}}}$  and  $X_{\mathbf{o}}$  is the sub-family  $\{X_{\mathbf{o}}(n)\}_{n \in \text{Col}_{\mathbf{o}}}$ . We have an SC analogue to [BB09, Proposition 1.8] or [DS03]:

**Proposition 2.4.** *The functors  $\xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}$  extend to an SC functor-operad  $\xi(\mathcal{O})$ , such that the category of  $\mathcal{O}$ -algebras and the category of  $\xi(\mathcal{O})$ -algebras are isomorphic.*

*Proof.* Let us first show that the family of the  $\xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}$  form an SC functor-operad. We set  $[c]_{a,b} = c_{a,1}, \dots, c_{a,b}; c_{a+1}$  and we denote by  $\mathbf{n}_{a,b}$  the list  $n_{a,1}, \dots, n_{a,b}$  of objects  $n_{r,s} \in \mathcal{O}_{\mathbf{u}}^{c_r, s}$ . The natural transformation  $\mu_{[c]_{1,i_1}, \dots, [c]_{k,i_k}; c_{k+1}}$  is given by

$$\begin{aligned} & \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}} \circ (\xi(\mathcal{O})_{[c]_{1,i_1}}(X_{1,1}, \dots, X_{1,i_1}), \dots, \xi(\mathcal{O})_{[c]_{k,i_k}}(X_{k,1}, \dots, X_{k,i_k})) \\ &= \int^{\mathbf{n}_1, \dots, \mathbf{n}_k} \mathcal{O}(\mathbf{n}_1, \dots, \mathbf{n}_k; -) \otimes \int^{\mathbf{n}_{1,i_1}} \mathcal{O}(\mathbf{n}_{1,i_1}; \mathbf{n}_1) \otimes X_{1,1}(\mathbf{n}_{1,1}) \otimes \dots \otimes X_{1,i_1}(\mathbf{n}_{1,i_1}) \\ & \quad \otimes \dots \otimes \int^{\mathbf{n}_{k,i_k}} \mathcal{O}(\mathbf{n}_{k,i_k}; \mathbf{n}_k) \otimes X_{k,1}(\mathbf{n}_{k,1}) \otimes \dots \otimes X_{k,i_k}(\mathbf{n}_{k,i_k}) \\ &\cong \int^{\mathbf{n}_1, \dots, \mathbf{n}_k, \mathbf{n}_{1,i_1}, \dots, \mathbf{n}_{k,i_k}} \mathcal{O}(\mathbf{n}_1, \dots, \mathbf{n}_k; -) \otimes \mathcal{O}(\mathbf{n}_{1,i_1}; \mathbf{n}_1) \otimes \dots \otimes \mathcal{O}(\mathbf{n}_{k,i_k}; \mathbf{n}_k) \otimes X_{1,1}(\mathbf{n}_{1,1}) \otimes \dots \otimes X_{k,i_k}(\mathbf{n}_{k,i_k}) \\ & \quad \rightarrow \int^{\mathbf{n}_{1,i_1}, \dots, \mathbf{n}_{k,i_k}} \mathcal{O}(\mathbf{n}_{1,i_1}, \dots, \mathbf{n}_{k,i_k}; -) \otimes X_{1,1}(\mathbf{n}_{1,1}) \otimes \dots \otimes X_{k,i_k}(\mathbf{n}_{k,i_k}) \\ & \quad \quad \quad =: \xi(\mathcal{O})_{c_{1,1}, \dots, c_{k,i_k}; c_{k+1}}(X_{1,1}, \dots, X_{k,i_k}), \end{aligned}$$

where the last map is induced by the composition map of  $\mathcal{O}$ . The associativity property of the latter implies that  $\xi$  satisfies the associativity axiom. The unit axioms 1 of Definition 2.2 is due to the Yoneda lemma. The twisted symmetric condition is obtained from the equivariance of the operad.

Via the hypothesis H2 and H3,  $X_c$  and  $X_o$  can be seen as functors  $X_c : \mathcal{O}_{\mathbf{u}}^c \rightarrow \mathbf{C}$  and  $X_o : \mathcal{O}_{\mathbf{u}}^o \rightarrow \mathbf{C}$  respectively. The maps (2.2) give:

$$(2.3) \quad \mathcal{O}(\mathbf{n}_1, \dots, \mathbf{n}_k; \mathbf{n}_{k+1}) \otimes X_{c_1}(\mathbf{n}_1) \otimes \dots \otimes X_{c_k}(\mathbf{n}_k) \rightarrow X_c(\mathbf{n}_{k+1}),$$

for any  $\mathbf{n}_i \in \text{Col}_{c_i}$  and any  $(c_1, \dots, c_k; c_{k+1})$  satisfying (2.1). Since these maps satisfy the associativity and unit axioms, they induce a map

$$\xi_{c_1, \dots, c_k; c_{k+1}}(\mathcal{O})(X_{c_1}, \dots, X_{c_k}) = \int^{\mathbf{n}_1, \dots, \mathbf{n}_k} \mathcal{O}(\mathbf{n}_1, \dots, \mathbf{n}_k; \mathbf{n}_{k+1}) \otimes X_{c_1}(\mathbf{n}_1) \otimes \dots \otimes X_{c_k}(\mathbf{n}_k) \rightarrow X_{c_{k+1}}(\mathbf{n}_{k+1}).$$

This way, we obtain

$$\alpha_{c_1, \dots, c_k; c_{k+1}} : \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(X_{c_1}, \dots, X_{c_k}) \rightarrow X_{c_{k+1}}, \quad k \geq 0.$$

We conclude that  $X$  is a  $\xi(\mathcal{O})$ -algebra because of the unit, associativity and equivariance properties of maps (2.3). Conversely, H4 says that the  $\xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}$ 's recover all maps in (2.2).  $\square$

## 2.3 Condensation

Let  $\delta^c : \mathcal{O}_{\mathbf{u}}^c \rightarrow \mathbf{C}$  and  $\delta^o : \mathcal{O}_{\mathbf{u}}^o \rightarrow \mathbf{C}$  be two functors. We set  $\delta = (\delta^c, \delta^o)$ . We define the coendomorphism operad  $\text{Coend}_{\xi(\mathcal{O})}(\delta)$  as the operad in  $\mathbf{C}$  with objects:

$$\text{Coend}_{\xi(\mathcal{O})}(\delta)(c_1, \dots, c_k; c_{k+1}) = \text{Hom}_{\mathcal{O}_{\mathbf{u}}^{c_{k+1}}}(\delta^{c_{k+1}}, \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(\delta^{c_1}, \dots, \delta^{c_k})),$$

for  $c_1, \dots, c_k; c_{k+1}$  satisfying (2.1).

The composition maps

$$\begin{aligned} & \text{Coend}_{\xi(\mathcal{O})}(\delta)(c_1, \dots, c_k; c_{k+1}) \otimes \text{Coend}_{\xi(\mathcal{O})}(\delta)(c_{1,1}, \dots, c_{1,i_1}; c_1) \otimes \dots \otimes \text{Coend}_{\xi(\mathcal{O})}(\delta)(c_{k,1}, \dots, c_{k,i_k}; c_k) \\ & \quad \rightarrow \text{Coend}_{\xi(\mathcal{O})}(\delta)(c_{1,1}, \dots, c_{k,i_k}; c_{k+1}) \end{aligned}$$

are given by sending maps  $f \otimes g_1 \otimes \dots \otimes g_k$  to the composite

$$\begin{array}{ccc} \delta^{c_{k+1}} & & \xi(\mathcal{O})_{c_{1,1}, \dots, c_{k,i_k}; c_{k+1}}(\delta^{c_{1,1}}, \dots, \delta^{c_{k,i_k}}) \\ \downarrow f & & \uparrow \alpha_{c_1, \dots, c_k; c_{k+1}} \\ \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(\delta^{c_1}, \dots, \delta^{c_k}) & \longrightarrow & \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(\xi(\mathcal{O})_{[c]_{1,i_1}}(\delta), \dots, \xi(\mathcal{O})_{[c]_{k,i_k}}(\delta)). \\ & & \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(g_1, \dots, g_k) \end{array}$$

Given an  $\mathcal{O}$ -algebra  $X = (X_c, X_o)$ , we denote by

$$\begin{aligned} \text{Tot}_{\delta^c} X_c &:= \text{Hom}_{\mathcal{C}^{\mathcal{O}}_{\xi}}(\delta^c, X_c); \\ \text{Tot}_{\delta^o} X_o &:= \text{Hom}_{\mathcal{C}^{\mathcal{O}}_{\xi}}(\delta^o, X_o). \end{aligned}$$

In virtue of Proposition 2.4, the couple  $(\text{Tot}_{\delta^c} X_c, \text{Tot}_{\delta^o} X_o)$  is a  $\text{Coend}_{\xi(\mathcal{O})}(\delta)$ -algebra. The action maps

$$\text{Coend}_{\xi(\mathcal{O})}(\delta)(c_1, \dots, c_k; c_{k+1}) \otimes \text{Tot}_{\delta^{c_1}} X_{c_1} \otimes \dots \otimes \text{Tot}_{\delta^{c_k}} X_{c_k} \rightarrow \text{Tot}_{\delta^{c_{k+1}}} X_{c_{k+1}}$$

are given by sending maps  $f \otimes g_1 \otimes \dots \otimes g_k$  to the composite

$$\begin{array}{ccc} \delta^{c_{k+1}} & & X_{c_{k+1}} \\ \downarrow f & & \uparrow \alpha_{c_1, \dots, c_k; c_{k+1}} \\ \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(\delta^{c_1}, \dots, \delta^{c_k}) & \longrightarrow & \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(X_{c_1}, \dots, X_{c_k}). \\ \xi(\mathcal{O})_{c_1, \dots, c_k; c_{k+1}}(g_1, \dots, g_k) & & \end{array}$$

Unit, associative and equivariance axioms are deduced from the SC functor-operad properties of  $\xi(\mathcal{O})$ .

### 3 A cellular decomposition of the Swiss Cheese operad

The little cubes operad  $\mathcal{C}$  has a cellular decomposition indexed by the extended complete graph operad  $\mathcal{K}$ , see [Ber97] and [BFV07, 4.1]. We extend this result to the Swiss Cheese operads  $\mathcal{SC}_m$ ,  $m \geq 1$  what provides a recognition principle for Swiss Cheese type operads. In particular, we construct a poset operad  $\mathcal{RK}_m$  that indexes the cells  $(\mathcal{SC}_m)^{(\alpha)}$  of  $\mathcal{SC}_m$ . This leads to a zig-zag of weak equivalences of operads

$$\mathcal{SC}_m \longleftarrow \widetilde{\text{hocolim}}_{\alpha \in \mathcal{RK}_m} (\mathcal{SC}_m)^{(\alpha)} \longrightarrow \widetilde{\mathcal{BRK}_m},$$

between the Swiss Cheese operad  $\mathcal{SC}_m$  and the classifying operad of  $\mathcal{RK}_m$ .

#### 3.1 The Swiss Cheese operad

The Swiss Cheese operad that we use is the cubical version of the one defined in [Kon99].

Let  $m \geq 1$ . Let  $\text{Sym} : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be the reflection  $\text{Sym}(x_1, \dots, x_m) = (x_1, \dots, -x_m)$ , and let  $\text{Half}_+$  be the upper half space

$$\text{Half}_+ = \{(x_1, \dots, x_m) \in \mathbb{R}^m | x_m > 0\}.$$

The standard cube  $C_0$  in  $\mathbb{R}^m$  is  $C_0 = [-1, 1]^{\times m}$ . A cube  $C$  in the standard cube is of the form  $C = [x_1, y_1] \times [x_2, y_2] \times \dots \times [x_m, y_m]$  with  $-1 < x_j < y_j < 1$  for  $1 \leq j \leq m$ .

**Definition 3.1.** For  $n \geq 0$  and  $c_i, c \in \{c, o\}$  we define a topological  $\Sigma_n$ -space  $\mathcal{SC}_m(c_1, \dots, c_n; c)$  as the empty-set if  $c = c$  and there exists  $1 \leq i \leq n$  such that  $c_i = o$ ; for the other cases, it is

- the space of the little  $m$ -cubes operad  $\mathcal{C}^{(m)}(n)$  defined in [May72] for  $c = c$ ;
- the empty set if  $n = 0$ ;
- the one-point space if  $n = 1$ ;
- in the case  $s + t = n \geq 2$  with  $s, t \geq 0$  such that  $s$  colours  $c_i$  are  $c$  and  $t$  colours  $c_j$  are  $o$ , the space of configuration of  $2s + t$  disjoint cubes  $(C_1, \dots, C_{2s+t})$  in the standard cube  $C_0 \in \mathbb{R}^m$  such that  $\text{Sym}(C_i) = C_{i+s}$  for  $1 \leq i \leq s$  and  $\text{Sym}(C_i) = C_i$  for  $2s + 1 \leq i \leq 2s + t$  and such that all the cubes  $(C_1, \dots, C_s)$  are in the upper half space.

*Remark 3.2.* Because of the symmetry conditions imposed by  $\text{Sym}$ , we may thought of  $\mathcal{SC}_m(c_1, \dots, c_n; o)$  as the configuration space of cubes  $(C_1, \dots, C_s)$  and semi-cubes  $(C_{s+1}, \dots, C_{s+t})$  lying into the standard semi-cube  $\text{Half}_+ \cap C_0$ .

Similarly to the little  $m$ -cubes operad  $\mathcal{C}^{(m)}$  the composition maps

$$\circ_i : \mathcal{SC}_m(c_1, \dots, c_n; c) \times \mathcal{SC}_m(d_1, \dots, d_r; c_i) \rightarrow \mathcal{SC}_m(c_1, \dots, c_{i-1}, d_1, \dots, d_r, c_{i+1}, \dots, c_n; c)$$

are defined as substitutions of cubes. We denote the resulting 2-coloured operad  $\mathcal{SC}_m$ .

### 3.2 The SC extended complete graph operad

We define the *SC (or relative) extended complete graph operad*  $\mathcal{RK}$ . It is a 2-coloured poset operad with filtration  $\{\mathcal{RK}_m\}_{m \geq 1}$ . Its closed part is  $\mathcal{K}_m$ , its open part is  $\mathcal{K}_{m-1}$ , where  $\{\mathcal{K}_m\}_{m \geq 1}$  denotes the extended complete graph operad defined in [BFV07, Section 4.1].

Given  $n$  colours  $c_i \in \{\mathfrak{c}, \mathfrak{o}\}$ , we denote by  $\{\tilde{c}_1, \dots, \tilde{c}_n\}$  the set with

$$\tilde{c}_i = \begin{cases} i & \text{if } c_i = \mathfrak{c}; \\ \underline{i} & \text{if } c_i = \mathfrak{o}. \end{cases}$$

A colouring and an orientation on a complete graph on  $\{\tilde{c}_1, \dots, \tilde{c}_n\}$  is, for each edge between  $\tilde{c}_i$  and  $\tilde{c}_j$ , an orientation  $\sigma_{i,j}$  (that is,  $\tilde{c}_i \rightarrow \tilde{c}_j$  or  $\tilde{c}_i \leftarrow \tilde{c}_j$ ) and a strict positive natural number  $\mu_{i,j} \in \mathbb{N}^{>0}$  as the colour. A *monochromatic acyclic orientation* of a complete graph is a colouring and orientation such that there exist no oriented cycles with the same colour, *i.e.* there are no configurations of the form  $\tilde{c}_{i_1} \rightarrow \tilde{c}_{i_2} \rightarrow \dots \rightarrow \tilde{c}_{i_k} \rightarrow \tilde{c}_{i_1}$  with  $\mu_{i_1, i_2} = \mu_{i_2, i_3} = \dots = \mu_{i_{k-1}, i_k} = \mu_{i_k, i_1}$ .

If there exists an  $i$  such that  $c_i = \mathfrak{o}$ , then we set  $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$  as the empty set. Else, the poset  $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$  is the set of pairs  $(\mu, \sigma)^{\mathfrak{c}}$  of monochromatic acyclic orientations of the complete graph on  $\{\tilde{c}_1, \dots, \tilde{c}_n\}$ . The colouring  $\mu$  is a collection of a colour  $\mu_{i,j}$  for each pair  $\{i, j\}$  and  $\sigma$  is a collection of an orientation  $\sigma_{i,j}$  for each pair  $\{i, j\}$ , with  $1 \leq i, j \leq n$ . This is equivalent to write  $(\mu, \sigma)$  as  $\{(\mu_{i,j}, \sigma_{i,j})\}_{1 \leq i < j \leq n}$  by setting  $\mu_{i,j} = \mu_{j,i}$  and  $\sigma_{i,j} = \tau_2 \sigma_{j,i}$  for  $1 \leq i < j \leq n$ , where  $\tau_2$  denotes the non-neutral element of  $\Sigma_2$ .

The poset structure is given by

$$(\mu, \sigma)^{\mathfrak{c}} \leq (\mu', \sigma')^{\mathfrak{c}} \Leftrightarrow \forall i < j, \text{ either } (\mu_{i,j}, \sigma_{i,j}) = (\mu'_{i,j}, \sigma'_{i,j}) \text{ or } \mu_{i,j} < \mu'_{i,j}.$$

The filtration  $(\mathcal{RK}_m)_{m \geq 1}$  is as follows.

For  $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$  with  $c_i = \mathfrak{c}$  for all  $i$ , we set

$$\mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{c}) = \{(\mu, \sigma)^{\mathfrak{c}} \in \mathcal{RK}(c_1, \dots, c_n; \mathfrak{c}) \mid \mu_{i,j} \leq m \forall i < j\}$$

For  $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{o})$ , we set

$$(3.1) \quad \mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{o}) = \{(\mu, \sigma)^{\mathfrak{o}} \in \mathcal{RK}(c_1, \dots, c_n; \mathfrak{o}) \mid \begin{array}{ll} \mu_{i,j} \leq m & \text{if } c_i = c_j = \mathfrak{c}, \\ \mu_{i,j} \leq m-1 & \text{if } c_i = c_j = \mathfrak{o}, \\ \mu_{i,j} \leq m & \text{if } \underline{i} \rightarrow j, \\ \mu_{i,j} \leq m-1 & \text{if } i \rightarrow \underline{j}. \end{array}\}$$

Given a permutation  $\sigma \in \Sigma_n$  and an element  $(\mu, \tau)^{\mathfrak{c}} \in \mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$ , the resulting element  $\sigma \cdot (\mu, \tau)^{\mathfrak{c}} \in \mathcal{RK}(c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)}; \mathfrak{c})$  is given by permuting the numbers  $i$  by  $\sigma$  without changing neither the underline nor the orientation nor the colouring. For example, the edges  $\underline{i} \rightarrow j$  of  $(\mu, \tau)^{\mathfrak{c}}$  with colours  $\mu_{i,j}$  become the edges  $\underline{\sigma(i)} \rightarrow \sigma(j)$  with the same colours  $\mu_{i,j}$ .

The compositions

$$\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c}) \times \mathcal{RK}(c_{1,1}, \dots, c_{1,k_1}; \mathfrak{c}_1) \times \dots \times \mathcal{RK}(c_{n,1}, \dots, c_{n,k_n}; \mathfrak{c}_n) \rightarrow \mathcal{RK}(c_{1,1}, \dots, c_{n,k_n}; \mathfrak{c})$$

send a tuple of  $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c}) \times \mathcal{RK}(c_{1,1}, \dots, c_{1,k_1}; \mathfrak{c}_1) \times \dots \times \mathcal{RK}(c_{n,1}, \dots, c_{n,k_n}; \mathfrak{c}_n)$  to an element in  $\mathcal{RK}(c_{1,1}, \dots, c_{n,k_n}; \mathfrak{c})$  obtained as follows. The sub complete graphs with vertices in the same block  $\{c_{i,1}, \dots, c_{i,k_i}\}$  is oriented and coloured as in  $\mathcal{RK}(c_{i,1}, \dots, c_{i,k_i}; \mathfrak{c}_i)$ ; the edges with vertices in two different blocks are oriented and coloured as the edges between the corresponding vertices in  $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$ .

*Remark 3.3.* For  $m = 1$  the conditions where  $\mu_{i,j} \leq m - 1$  cannot be satisfied. It follows that  $\mathcal{RK}_1(c_1, \dots, c_n; \mathfrak{o})$  is empty whenever the tuple  $(c_1, \dots, c_n)$  has more than one open colour.

### 3.3 Cellular decomposition of SC type operads

**Definition 3.4.** Let  $\mathbf{C}$  be a closed symmetric monoidal model category. A morphism of operads  $f : \mathcal{O} \rightarrow \mathcal{P}$  is a weak equivalence if each  $f_{c_1, \dots, c_k; \mathfrak{c}} : \mathcal{O}(c_1, \dots, c_k; \mathfrak{c}) \rightarrow \mathcal{P}(c_1, \dots, c_k; \mathfrak{c})$  is a weak equivalence in  $\mathbf{C}$ . Two operads  $\mathcal{O}$  and  $\mathcal{P}$  in  $\mathbf{C}$  are said *weakly equivalent* if there exists a zig-zag  $\mathcal{O} \leftarrow \dots \rightarrow \mathcal{P}$  of weak equivalences.

**Definition 3.5** ([Ber97]). Let  $X$  be a topological space and  $\mathcal{A}$  be a poset. We say that  $X$  admits an  $\mathcal{A}$ -cellulation if there is a functor  $\Theta : \mathcal{A} \rightarrow \mathbf{Top}$  such that:

1.  $\operatorname{colim}_{\alpha \in \mathcal{A}} \Theta(\alpha) \cong X$ ;
2.  $\beta \leq \alpha \in \mathcal{A} \Leftrightarrow \Theta(\beta) \subseteq \Theta(\alpha)$ ;
3. the inclusions  $\Theta(\beta) \subseteq \Theta(\alpha)$  are a closed cofibration;
4. for each  $\alpha \in \mathcal{A}$ , the ‘‘cell’’  $\Theta(\alpha)$  is contractible.

Given a cell  $\Theta(\alpha)$ , we denote its boundary  $\bigcup_{\beta < \alpha} \Theta(\beta)$  by  $\partial\Theta(\alpha)$ ; we denote its interior  $\Theta(\alpha) \setminus \partial\Theta(\alpha)$  by  $\overset{\circ}{\Theta}(\alpha)$ . A cell with a non empty interior is called *proper*. A cell with an empty interior is called *improper*.

**Lemma 3.6.** [Ber97, Lemma 1.7] *Let  $X$  be a topological space with an  $\mathcal{A}$ -cellulation. Then we have the weak equivalences*

$$X \cong \operatorname{colim}_{\alpha \in \mathcal{A}} \Theta(\alpha) \xleftarrow{\sim} \operatorname{hocolim}_{\alpha \in \mathcal{A}} \Theta(\alpha) \xrightarrow{\sim} \operatorname{hocolim}_{\alpha \in \mathcal{A}} (*) \cong \mathcal{BA},$$

where  $\mathcal{BA}$  denotes the realization of the nerve of the category  $\mathcal{A}$ .

*Proof.* Items 2 and 3 of Definition 3.5 give the left hand equivalence (see [BFSV03, Proposition 6.9] for details); the item 4 gives the right one.  $\square$

**Definition 3.7.** A topological 2-coloured operad  $\mathcal{O}$  with colours  $\{\mathfrak{c}, \mathfrak{o}\}$  is called an *SC type operad* if  $\mathcal{O}(c_1, \dots, c_n; \mathfrak{c})$  is empty whenever there is an  $i$  such that  $c_i = \mathfrak{o}$ . For such an operad  $\mathcal{O}$ , suppose we have given an  $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$ -cellulation of  $\mathcal{O}(c_1, \dots, c_n; \mathfrak{c})$

$$\Theta_{c_1, \dots, c_n; \mathfrak{c}} : \mathcal{RK}(c_1, \dots, c_n; \mathfrak{c}) \rightarrow \mathbf{Top},$$

for each  $c_1, \dots, c_n; \mathfrak{c}$ ,  $n \geq 0$ . This families of cellulations  $\Theta_{c_1, \dots, c_n; \mathfrak{c}}$  is said *compatible with the operad structure* of  $\mathcal{O}$  if

$$\begin{aligned} \gamma^{\mathcal{O}} \left( \Theta_{c_1, \dots, c_n; \mathfrak{c}}(\alpha) \times \Theta_{c_{1,1}, \dots, c_{1,k_1}; \mathfrak{c}_1}(\alpha_1) \times \cdots \times \Theta_{c_{n,1}, \dots, c_{n,k_n}; \mathfrak{c}_n}(\alpha_n) \right) \\ \subseteq \Theta_{c_{1,1}, \dots, c_{n,k_n}; \mathfrak{c}}(\gamma^{\mathcal{RK}}(\alpha; \alpha_1, \dots, \alpha_n)), \end{aligned}$$

for all variables  $c, c_i, c_{i,j}, \alpha, \alpha_i$ , where  $\gamma^{\mathcal{O}}$  and  $\gamma^{\mathcal{RK}}$  denote the composition map of  $\mathcal{O}$  and  $\mathcal{RK}$  respectively.

**Definition 3.8.** Let  $m \geq 1$ . A topological SC type operad  $\mathcal{O}$  is called an  $\mathcal{RK}_m$ -cellular operad if there are  $\mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{c})$ -cellulations of  $\mathcal{O}(c_1, \dots, c_n; \mathfrak{c})$

$$\Theta_{c_1, \dots, c_n; \mathfrak{c}} : \mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{c}) \rightarrow \mathbf{Top}$$

for each  $c_1, \dots, c_n; \mathfrak{c}$ ,  $n \geq 0$ , subject to the following two compatibilities.

1. The cellulations are compatible with the  $\Sigma_n$ -action:

$$\Theta_{c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)}; \mathfrak{c}}(\sigma \cdot \alpha) = \sigma \cdot \Theta_{c_1, \dots, c_n; \mathfrak{c}}(\alpha) \text{ for all } \sigma \in \Sigma_n.$$

2. The cellulations are compatible with the operadic structure of  $\mathcal{O}$ .

We have the ‘‘Swiss Cheese analogue’’ to Theorem 1.16 [Ber97]:

**Theorem 3.9.** *Let  $m \geq 1$ . Any two topological  $\mathcal{RK}_m$ -cellular operads are weakly equivalent. Moreover, the Swiss Cheese operad  $\mathcal{SC}_m$  has a structure of an  $\mathcal{RK}_m$ -cellular operad.*

*Proof.* Let  $\mathcal{O}$  be a cellular SC type operad. Analogue to [BFSV03, Lemma 6.11] is the fact that  $\{\operatorname{hocolim}_{\alpha \in \mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})} \Theta_{c_1, \dots, c_n; \mathfrak{c}}(\alpha)\}_{c_i, \mathfrak{c} \in \{\mathfrak{c}, \mathfrak{o}\}, n \geq 0}$  forms an operad. Moreover, the operad structures are compatible with the weak equivalences of Lemma 3.6.

We show that, for each  $m \geq 1$ , the operad  $\mathcal{SC}_m$  has a structure of a cellular SC type operad indexed by  $\mathcal{RK}_m$ . The ‘‘closed’’ part of  $\mathcal{SC}_m$ , that is the little  $m$ -cubes operad  $\mathcal{C}_m$ , is already shown to have a structure of a cellular operad (indexed by  $\mathcal{K}_m$ ), cf. [BFV07, Ber97].

We use the description of  $\mathcal{SC}_m$  via cubes and semi-cubes given in Remark 3.2. The number  $m \geq 1$  is fixed. For  $C_1$  either a cube or a semi-cube and  $C_2$  either a cube or a semi-cube, we note  $C_1 \square_\mu C_2$  if there are separated by a hyperplane  $H_i$  orthogonal to the  $i$ -th coordinate axis for some  $i \leq \mu$ , such that whenever there is no separating hyperplane  $H_i$  for  $i < \mu$ , the left element  $C_1$  lies in the negative side of  $H_\mu$  and  $C_2$  lies in the positive side of  $H_\mu$ .

Note that, whenever  $C_1$  is a semi-cube and  $C_2$  is a cube, if  $H_m$  exists, then  $C_1$  lies in the negative side of  $H_m$ .

For  $\alpha = (\mu, \sigma) \in \mathcal{RK}(c_1, \dots, c_k; \mathfrak{o})$ , we set  $\mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{o})^{(\alpha)}$  the cell

$$\{(C_1, \dots, C_k) \in \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{o}) \mid C_i \square_{\mu_{i,j}} C_j \text{ if } \tilde{c}_i \rightarrow \tilde{c}_j \text{ and } C_j \square_{\mu_{i,j}} C_i \text{ if } \tilde{c}_i \leftarrow \tilde{c}_j\}.$$

To see that  $\mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})$  is the colimit of its cells, the only delicate point is to show that if

$$x \in \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\alpha)} \cap \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\beta)},$$

with neither  $\alpha \leq \beta$  nor  $\alpha \geq \beta$  then

$$x \in \partial \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\alpha)} \cap \partial \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\beta)}.$$

Here  $\partial \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\alpha)}$  denotes the boundary  $\bigcup_{\gamma < \alpha} \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\gamma)}$ . For such an  $x$ , we construct  $\gamma \in \mathcal{RK}(c_1, \dots, c_k; \mathfrak{c})$  such that  $\gamma \leq \alpha$  and  $\gamma \leq \beta$  as follows. For each  $\tilde{c}_i$  and  $\tilde{c}_j$  we define a colouring and an orientation as the minimum among  $(\mu_{i,j}^\alpha, \sigma_{i,j}^\alpha)$  and  $(\mu_{i,j}^\beta, \sigma_{i,j}^\beta)$ . This minimum exists since  $\alpha$  and  $\beta$  represent the same configuration  $x$ . This defines an element  $\gamma \in \mathcal{RK}(c_1, \dots, c_k; \mathfrak{c})$ .

The compatibility with the operadic structure of  $\mathcal{SC}_m$  is clear.

Let us explain how works the contractibility of the cells. Recall that the interior of a cell  $\mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\alpha)}$  is  $\mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\alpha)} \setminus (\bigcup_{\beta < \alpha} \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\beta)})$ . Contractibility of proper cells (cells with a non empty interior) onto an interior point is obtained by coordinate-wise contractions starting from the last coordinate, see [Ber97, Theorem 1.16] for details. For improper cells we remark the following. If a cell  $\mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{c})^{(\alpha)}$  is improper then at least three cubes/semi-cubes are involved *i.e.*  $k \geq 3$ .

Two elements  $\tilde{c}_i$  and  $\tilde{c}_j$  of  $\alpha$  are said related by a *positive (resp. negative) monochromatic path of colour  $\nu$*  if there exist an  $l \geq 2$  and indices  $i =: i_0, i_1, \dots, i_{l-1}, i_l := j$  such that  $\tilde{c}_{i_r} \rightarrow \tilde{c}_{i_{r+1}}$  (resp.  $\tilde{c}_{i_r} \leftarrow \tilde{c}_{i_{r+1}}$ ) and  $\mu_{i_r, i_{r+1}} = \nu$  for all  $0 \leq r \leq l-1$ . Two elements  $\tilde{c}_i$  and  $\tilde{c}_j$  are called an *improper pair* if there are related by at least one monochromatic path of colour  $\nu$  such that  $\mu_{i,j} > \nu$ .

Then, an improper cell is exactly a cell indexed by an elements  $\alpha$  which contains at least one improper pair.

Let  $\alpha$  be such an element indexing an improper cell. To such an improper pair  $\tilde{c}_i$  and  $\tilde{c}_j$  of  $\alpha$  one assigns the colour  $\mu'_{i,j}$  defined as the minimal  $\nu$  among all the monochromatic paths of colour  $\nu$  satisfying  $\mu_{i,j} > \nu$ ; also, one assigns the following orientation  $\tilde{c}_i \rightarrow \tilde{c}_j$  if the<sup>1</sup> path of the minimal colour  $\nu = \mu'_{i,j}$  is positive, and  $\tilde{c}_i \leftarrow \tilde{c}_j$  if this path is negative.

Let  $\beta$  be the element obtained by applying such an assignment for each improper pair of  $\alpha$ . Then  $\beta$  is such that  $\beta \leq \alpha$  and it has no improper pairs, so that its corresponding cell is proper. Moreover, by construction it is unique and maximal for  $\alpha$ . Thus, any improper cell has a unique maximal proper cell and then is contractible.  $\square$

## 4 The operad $\mathcal{RL}$

### 4.1 Definition of the operad $\mathcal{RL}$

We describe an SC-split operad  $\mathcal{RL}$  in the category of sets, **Set**.

The operad  $\mathcal{RL}$  has a natural filtration by sub operads  $\mathcal{RL}_m$  for  $m \geq 1$ . For each  $m \geq 1$ ,  $\mathcal{RL}_m$  can be thought of as a mix between the sub operads  $\mathcal{L}_m$  and  $\mathcal{L}_{m-1}$  of the Lattice paths operad  $\mathcal{L}$  introduced in [BB09].

For  $m = 2$ , a description of  $\mathcal{RL}_2$  using planar trees is given in Section 6.

<sup>1</sup>More precisely, such a path is not necessarily unique. However, if  $\tilde{c}_i$  and  $\tilde{c}_j$  are related by two monochromatic paths with the same colour  $\nu$  then both have the same direction, either positive or negative.

**Definition of  $\mathcal{RL}$ .** Let  $\mathbf{Cat}_{*,*}$  be the category of bi-pointed small categories and functors preserving the two distinguished objects. An ordinal  $[i]$  defines a category freely generated by the linear graph  $\mathfrak{l}_i = \{0 \rightarrow 1 \rightarrow \dots \rightarrow i\}$ . For two ordinals  $[i]$  and  $[j]$ , the tensor product  $[i] \otimes [j]$  is the category freely generated by the graph  $\mathfrak{l}_i \otimes \mathfrak{l}_j$ . The category  $[i]$  is bi-pointed in  $(0, i)$ ; the tensor product  $[i] \otimes [j]$  is bi-pointed in  $((0, 0), (i, j))$ .

The set of colours of  $\mathcal{RL}$  is

$$\mathbf{Col} = \mathbf{Col}_c \sqcup \mathbf{Col}_o,$$

where  $\mathbf{Col}_c$  is the set  $\mathbb{N}$  of natural numbers and  $\mathbf{Col}_o$  is the set of natural numbers decorated with an underline. Hence,  $n \in \mathbf{Col}_c$  whereas  $\underline{n} \in \mathbf{Col}_o$ . In general a colour in  $\mathbf{Col}$  is denoted by  $\tilde{n}$ , so that it is either  $n$  or  $\underline{n}$ .

The set  $\mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$  is defined as:

$$\mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \emptyset$$

if  $\tilde{n} = n \in \mathbf{Col}_c$  and if there is an  $i$  such that  $\tilde{n}_i = \underline{n}_i \in \mathbf{Col}_o$ ;

$$\mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \mathbf{Cat}_{*,*}([\tilde{n} + 1], [\tilde{n}_1 + 1] \otimes [\tilde{n}_2 + 1] \otimes \dots \otimes [\tilde{n}_k + 1])$$

else.

The substitutions maps are given by tensor and composition in  $\mathbf{Cat}_{*,*}$ .

For instance, an element  $x \in \mathcal{RL}(n_1, \underline{n}_2; \underline{n})$  is a functor

$$(4.1) \quad x : [\underline{n} + 1] \rightarrow [n_1 + 1] \otimes [\underline{n}_2 + 1]$$

that sends  $(0, \underline{n} + 1)$  on  $((0, 0), (n_1 + 1, \underline{n}_2 + 1))$  and is determined by the image of the  $n$  remaining objects of  $[\underline{n} + 1]$  and the morphisms into the lattice  $[n_1 + 1] \otimes [\underline{n}_2 + 1]$ .

*Example 4.1.* The following lattice  $x$  belongs to  $\mathcal{RL}(3, \underline{2}; \underline{3})$ :

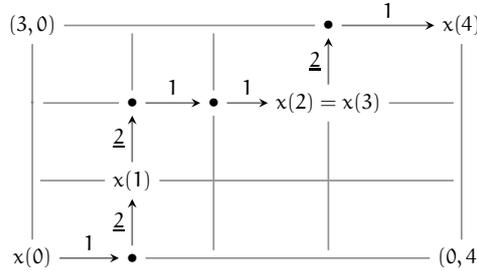


Figure 4.1: Lattice paths of  $(\underline{12|211|21})^\circ$ .

The elements of  $\mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$  correspond bijectively to a string of (decorated) natural numbers separated by vertical bars. Indeed, let us consider an  $x \in \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ . The relative lattice  $x$  is a path from  $x(0)$  to  $x(n + 1)$  made of edges in the grid  $\mathfrak{l}_1 \otimes \dots \otimes \mathfrak{l}_k$ . By running through  $x$  from  $x(0)$  to  $x(n + 1)$  we construct the integer-string with vertical bars as follows. To each parallel edge to the  $i$ -axis of the grid  $\mathfrak{l}_1 \otimes \dots \otimes \mathfrak{l}_k$  we assign  $i$  if  $\tilde{n}_i = n_i$  or  $\underline{i}$  if  $\tilde{n}_i = \underline{n}_i$ ; to each object  $x(s)$  for  $1 \leq s \leq n$  we assign a vertical bar. Additionally, we put an extra labelled according to the nature of the output colour.

*Example 4.2.*

$$(121)^\circ \in \mathcal{RL}(1, 0; \underline{0}) \text{ whereas } (121) \in \mathcal{RL}(1, 0; 0).$$

Let us expose the corresponding composition on integer-string representations via an example.

*Example 4.3.*

$$(\underline{12|14\underline{2}31|\underline{24})^\circ \circ_2 (1\underline{3|21\underline{3|31})^\circ = (124|1\underline{6}32451|\underline{426})^\circ$$

The composition is at  $\underline{2}$  and the second term has 3 outputs. Then one has renumbered the integer-string  $(\underline{12|14\underline{2}31|\underline{24})^\circ$  by increasing by  $2 = 3 - 1$  the numbers greater than  $\underline{2}$ ; one gets  $(\underline{12|1\underline{6}251|\underline{26})^\circ$ . One has increased the numbers of the second integer-string  $(1\underline{3|21\underline{3|31})^\circ$  by  $1 = 2 - 1$ :  $(24|324|42)^\circ$ . Finally, one has substituted the three occurrences of  $\underline{2}$  by the three sub-sequences  $\underline{24}$ ,  $\underline{324}$  and  $\underline{42}$ .

We use left action for the symmetric group: for  $\sigma \in \Sigma_k$  and  $x \in \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ , the string-integer representation of  $\sigma \cdot x \in \mathcal{RL}(\tilde{n}_{\sigma^{-1}(1)}, \dots, \tilde{n}_{\sigma^{-1}(k)}; \tilde{n})$  is obtained by permuting the number  $i$  (resp.  $\underline{i}$ ) of the string-integer representation of  $x$  by the number  $\sigma(i)$  (resp.  $\underline{\sigma(i)}$ ).

*Example 4.4.* For  $x = (1\underline{2}|3\underline{2}11||\underline{2}1)^o$  and  $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$  one has:

$$\sigma \cdot x = (2\underline{3}|1\underline{3}22||\underline{3}2)^o.$$

**The underlying category of  $\mathcal{RL}$ .** Let  $\mathbf{Cat}$  be the category of small categories. Via *Joyal-duality*:

$$\mathbf{Cat}_{*,*}([n+1], [m+1]) \cong \mathbf{Cat}([m], [n]).$$

The bijection is given by  $(\phi : [n+1] \rightarrow [m+1]) \leftrightarrow (\psi : [m] \rightarrow [n])$  given by  $\psi(i) + 1 = \min\{j | \phi(j) > i\}$  and  $\phi(j) - 1 = \max\{i | \psi(i) < j\}$ . One has immediately that:

$$\begin{aligned} \mathcal{RL}(n; m) &= \text{Hom}_{\Delta}([n], [m]); \\ \mathcal{RL}(\underline{n}; \underline{m}) &= \text{Hom}_{\Delta}([n], [m]); \\ \mathcal{RL}(n; \underline{m}) &= \text{Hom}_{\Delta}([n], [m]), \end{aligned}$$

where  $\Delta$  is the simplicial category. Thus, the underlying category  $(\mathcal{RL})_{\mathbf{u}}$  of  $\mathcal{RL}$  is the category with objects

$$\begin{aligned} [n] &= \{0 < 1 < \dots < n\} \text{ for } n \in \mathbb{N}; \text{ and} \\ \underline{[n]} &= \{\underline{0} < \underline{1} < \dots < \underline{n}\} \text{ for } n \in \mathbb{N}; \end{aligned}$$

and with hom-sets

$$\begin{aligned} \text{Hom}([n], [m]) &= \text{Hom}_{\Delta}([n], [m]); \\ \text{Hom}(\underline{[n]}, \underline{[m]}) &= \text{Hom}_{\Delta}([n], [m]); \\ \text{Hom}([n], \underline{[m]}) &= \text{Hom}_{\Delta}([n], [m]); \\ \text{Hom}(\underline{[n]}, [m]) &= \emptyset. \end{aligned}$$

The two sub-categories  $(\mathcal{RL})_{\mathbf{u}}^{\epsilon}$  and  $(\mathcal{RL})_{\mathbf{u}}^o$  are (canonically isomorphic to) the category  $\Delta$ .

This implies that, for each  $k \geq 0$ , the functor

$$(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) \mapsto \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$$

is a multisimplicial/cosimplicial set.

**Filtration by sub operads  $\mathcal{RL}_m$ .** Let us define two maps

$$c_{i,j}, c'_{i,j} : \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) \rightarrow \mathbb{N}.$$

For  $1 \leq i < j \leq k$ , we denote by

$$\phi_{ij} : \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) \rightarrow \mathcal{RL}(\tilde{n}_i, \tilde{n}_j; \tilde{n})$$

the projection induced by the canonical projection

$$p_{ij} : [\tilde{n}_1 + 1] \otimes \dots \otimes [\tilde{n}_k + 1] \rightarrow [\tilde{n}_i + 1] \otimes [\tilde{n}_j + 1].$$

For  $x \in \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$  and  $1 \leq i < j \leq k$ , we define  $c_{ij}(x)$  as the number of changes of directions in the lattice paths  $\phi_{ij}(x)$ .

The second number  $c'_{i,j}(x)$  is defined as follows. Since  $x \in \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ , its integer-string representation is in particular a sequence of numbers (underlined or not) between 1 and  $k$ . For  $1 \leq i \leq k$ , we set  $i^-$  (resp.  $\underline{i}^-$ ) the first occurrence of  $i$  (resp.  $\underline{i}$ ) in the integer-string representation. Equivalently,  $i^-$  (resp.  $\underline{i}^-$ ) is the first edge of the lattice  $x$  which is in the  $i$ -th direction. We write  $n^- < m^-$  if the element  $n^-$  precedes  $m^-$ .

For  $1 \leq i < j \leq k$ , we set:

$$(4.2) \quad c'_{i,j}(x) = \begin{cases} c_{i,j}(x) & \text{if } i^- < j^-; \\ c_{i,j}(x) + 1 & \text{if } i^- > j^-; \\ c_{i,j}(x) + 1 & \text{if } \underline{i}^- < j^-; \\ c_{i,j}(x) & \text{if } \underline{i}^- > j^-. \end{cases}$$

For  $m \geq 1$ , we define  $\mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$  as the set of elements  $x \in \mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$  satisfying the three conditions:

$$\begin{aligned} \max_{(i,j)} c_{i,j}(x) &\leq m; \\ \max_{(\underline{i},j)} c_{i,j}(x) &\leq m - 1; \text{ and,} \\ \max_{(\underline{i},j) \text{ or } (i,j)} c'_{i,j}(x) &\leq m. \end{aligned}$$

*Remark 4.5.* Changing the filtration defined in (4.2) by:

$$(4.3) \quad c''_{i,j}(x) = \begin{cases} c_{i,j}(x) + 1 & \text{if } i^- < j^-; \\ c_{i,j}(x) & \text{if } i^- > j^-; \\ c_{i,j}(x) & \text{if } \underline{i}^- < j^-; \\ c_{i,j}(x) + 1 & \text{if } \underline{i}^- > j^-. \end{cases}$$

we get another filtration of  $\mathcal{RL}$  by sub-operads  $\mathcal{RL}'_m$ . The operad  $\mathcal{RL}'_2$  seems to be more adapted for proving the Swiss Cheese version of Deligne's conjecture.

## 4.2 The operad $\text{Coend}_{\mathcal{RL}_m}(\delta)$ as a Swiss Cheese operad

We apply the method developed in [BB09, Sections 3.5-3.6]. More precisely, given a functor  $\delta : \Delta \rightarrow \mathbf{C}$  where  $\mathbf{C}$  is a *monoidal model category*, we construct a zig-zag of weak equivalences of operads

$$(4.4) \quad \text{Coend}_{\mathcal{RL}_m}(\delta) \xleftarrow{\sim} \text{Coend}_{\widehat{\mathcal{RL}}_m}(\delta) \xrightarrow{\sim} \mathcal{B}_\delta \mathcal{RK}_m,$$

whenever  $\delta$  satisfies some conditions. Here,  $\mathcal{B}_\delta \mathcal{A}$  denote the  $\delta$ -realization of the nerve of the category  $\mathcal{A}$ . The intermediate operad  $\widehat{\mathcal{RL}}_m$  is defined using homotopy colimits in  $\mathbf{C}$  applied on a decomposition of  $\mathcal{RL}_m$  indexed by  $\mathcal{RK}_m$ . Such a decomposition is a consequence of the following lemma.

**Lemma 4.6.** *There is a morphism  $q : \mathcal{RL} \rightarrow \mathcal{RK}$  of filtered operads.*

*Proof.* Let us recall that if  $x \in \mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$  then:

$$\begin{aligned} c_{i,j}(x) &\leq m && \text{if } \tilde{n}_i = n_i, \tilde{n}_j = n_j; \\ c_{i,j}(x) &\leq m - 1 && \text{if } \tilde{n}_i = \underline{n}_i, \tilde{n}_j = \underline{n}_j; \\ c_{i,j}(x) &\leq m && \text{if } i^- < j^-; \\ c_{i,j}(x) &\leq m - 1 && \text{if } i^- > j^-; \\ c_{i,j}(x) &\leq m - 1 && \text{if } \underline{i}^- < j^-; \\ c_{i,j}(x) &\leq m && \text{if } \underline{i}^- > j^-, \end{aligned}$$

for  $1 \leq i < j \leq k$ .

The element  $q(x) = (\mu, \sigma) \in \mathcal{RK}$  is defined, for  $1 \leq i < j \leq n$ , by:

$$\begin{aligned} \mu_{i,j} &= c_{i,j}(x) \\ \tilde{c}_i &\rightarrow \tilde{c}_j \text{ for } \tilde{i}^- > \tilde{j}^- \\ \tilde{c}_i &\leftarrow \tilde{c}_j \text{ for } \tilde{i}^- < \tilde{j}^-. \end{aligned}$$

The fact that  $q$  preserves the filtration is clear from its definition. Let us sketch the proof that  $q$  is a morphism of operad by using similar arguments to [BB09, Proposition 3.4]. It is straightforward to check

that the image of  $q$  is contained in a sub-operad  $\mathcal{R}\mathcal{K}_m^-$  of  $\mathcal{R}\mathcal{K}_m$ . This operad  $\mathcal{R}\mathcal{K}_m^-$  consists of colouring and orientation on complete graphs that are acyclic, that is, no (polychromatic) cyclic orientations are allowed. Such acyclic orientations on a complete graph on  $n$  elements correspond to the choice of a permutation in  $\Sigma_n$ . Then  $\mathcal{R}\mathcal{K}_m^-(c_1, \dots, c_n; \mathbf{c})$  is the set of pair  $(\mu, \sigma)$  of  $\{1, \dots, m\}^{\binom{n}{2}} \times \Sigma_n$  submitted to the same conditions than in (3.1) (i.e.  $\tilde{c}_i \rightarrow \tilde{c}_j$  corresponds to  $\sigma(i) < \sigma(j)$  and  $\tilde{c}_i \leftarrow \tilde{c}_j$  corresponds to  $\sigma(i) > \sigma(j)$ ). An explicit formula for operadic composition is given in [BB09, Proposition 3.2] for the non Swiss Cheese case; it is the same formula in our context. In particular, permutations are composed as in the (Swiss Cheese version of the) Symmetric operad  $\mathcal{R}\Sigma$ , and we have an operadic map  $\mathcal{R}\mathcal{K}_m^- \rightarrow \mathcal{R}\Sigma$ .

Then, it is sufficient to show that the composite  $\mathcal{R}\mathcal{L}_m \rightarrow \mathcal{R}\mathcal{K}_m^- \rightarrow \mathcal{R}\Sigma$  is a morphism of operads. This follows almost directly from the definition of  $q$ . Since  $q$  “reverses” the orientations we have to check that the morphisms  $\text{Rev}_n : \Sigma_n \rightarrow \Sigma_n$ , that sends  $\sigma$  to  $\text{Rev}_n \circ (\sigma)$  where  $\text{Rev}_n = \begin{pmatrix} 1 & 2 & \dots & n \\ n & \dots & 2 & 1 \end{pmatrix}$ , induces a morphism of operads  $\text{Rev} : \mathcal{R}\Sigma \rightarrow \mathcal{R}\Sigma$ . The Swiss Cheese symmetric operad  $\mathcal{R}\Sigma$  consists of sets  $\mathcal{R}\Sigma(c_1, \dots, c_n; \mathbf{c})$  to be either  $\Sigma_n$  or empty according to the usual “Swiss-Cheese” condition on the colours; the operadic compositions are defined as in the classical non colour case.  $\square$

*Remark 4.7.* Our morphism “reverses” the orientations. This is due to the choice of the cellulation of  $\mathcal{S}\mathcal{C}_m$  we have made. However, if  $\mathcal{R}\mathcal{L}'_m$  is the operad as in Remark 4.5, the similar morphism  $q' : \mathcal{R}\mathcal{L}'_m \rightarrow \mathcal{R}\mathcal{K}_m$  does not “reverse” the orientations.

Let us recall that for  $\delta = (\delta^c, \delta^o)$  with  $\delta^c, \delta^o : \Delta \rightarrow \mathbf{C}$ , the functor

$$\xi(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; \mathbf{c}}(\delta) : \Delta \rightarrow \mathbf{C}$$

denotes the realization of  $\mathcal{R}\mathcal{L}_m(c_1, \dots, c_k; \mathbf{c})$

$$\xi(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; \mathbf{c}}(\delta)(n) = \mathcal{R}\mathcal{L}_m(\underbrace{-, \dots, -}_k; n) \otimes_{\Delta^k} \delta^{c_1}(-) \otimes \dots \otimes \delta^{c_k}(-),$$

where we use implicitly the strong monoidal functor

$$\begin{aligned} \mathbf{Set} &\rightarrow \mathbf{C} \\ E &\mapsto \prod_{e \in E} 1_{\mathbf{C}}. \end{aligned}$$

We use the same functor  $\delta^c = \delta^o$  and we denote it by  $\delta$ . We fix two functors

$$\delta_{\mathbf{Top}} : \Delta \xrightarrow{\delta_{\text{yon}}} \mathbf{Set}^{\Delta^{\text{op}}} \xrightarrow{|\cdot|} \mathbf{Top}$$

and

$$\delta_{\mathbf{Z}} : \Delta \xrightarrow{\delta_{\text{yon}}} \mathbf{Set}^{\Delta^{\text{op}}} \xrightarrow{C_*(-; \mathbf{Z})} \mathbf{Ch}(\mathbf{Z})$$

where:

- $\delta_{\text{yon}}([n]) = \text{Hom}_{\Delta}(-, [n])$  is the Yoneda functor;
- $|\cdot| : \mathbf{Set}^{\Delta^{\text{op}}} \rightarrow \mathbf{Top}$  is the geometric realization; and,
- $C_*(-; \mathbf{Z}) : \mathbf{Set}^{\Delta^{\text{op}}} \rightarrow \mathbf{Ch}(\mathbf{Z})$  is the normalized chain complex.

For  $\alpha \in \mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; \mathbf{c})$  and  $\tilde{n}_i \in \text{Col}_{c_i}$ ,  $\tilde{n} \in \text{Col}_{\mathbf{c}}$ , we set

$$(\mathcal{R}\mathcal{L}_m)_{\alpha}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \{x \in \mathcal{R}\mathcal{L}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) \mid q(x) \leq \alpha\}.$$

Then we have:

$$\mathcal{R}\mathcal{L}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \text{colim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; \mathbf{c})} (\mathcal{R}\mathcal{L}_m)_{\alpha}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}),$$

for all  $(c_1, \dots, c_k; \mathbf{c})$  and  $\tilde{n}_i \in \text{Col}_{c_i}$ ,  $\tilde{n} \in \text{Col}_{\mathbf{c}}$ . Thus

$$\xi(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; \mathbf{c}}(\delta) = \text{colim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; \mathbf{c})} \xi(\mathcal{R}\mathcal{L}_m)_{\alpha}(\delta).$$

We suppose  $\mathbf{C}$  endowed with a *monoidal model structure*, cf. [Hov99]. The categories  $\mathbf{Set}^{\Delta^{\text{op}}}$  and  $\mathbf{Top}$  are considered with the Quillen model structure; the category  $\mathbf{Ch}(\mathbb{Z})$  is considered with its projective model structure.

In [BB09, Sections 3.5-3.6] it is proved that, given a *standard system of simplices*  $\delta$ , the operad  $\text{Coend}_{\mathcal{RL}_m}(\delta)$  is weakly equivalent to  $\mathcal{B}_\delta \mathcal{RK}$ , provided that operad  $\mathcal{RL}_m$  is *strongly  $\delta$ -reductive*.

A standard system of simplices  $\delta : \Delta \rightarrow \mathbf{C}$  (cf. [BM06, Definition A.6]) provides, in particular, a "monoidal symmetric" cosimplicial frame  $(-) \otimes_\Delta \delta$  so that homotopy colimits are compatible with the symmetric monoidal structure of  $\mathbf{C}$ . Moreover, for such a  $\delta$ , the realization functor  $(-) \otimes_\Delta \delta$  preserves and reflects weak equivalences.

It is proved in [BM06, A.13, A.16] that the functors  $\delta_{\text{yon}}$ ,  $\delta_{\text{Top}}$  and  $\delta_{\mathbb{Z}}$  are such standard system of simplices.

The *strong  $\delta$ -reductivity* allows us to show that the zig-zag of weak equivalence (4.4) follows essentially from the weak equivalences  $\xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^n \rightarrow \xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^0$  for all  $n, k \geq 0$ .

We define the *strong  $\delta$ -reductivity* condition for  $\text{Coend}_{\mathcal{RL}_m}(\delta)$  similarly to [BB09, Definition 3.7] by extending it to the functors

$$\xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^n \rightarrow \xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^0, \text{ for all } n \geq 0, k \geq 0,$$

see next definitions.

**Definition 4.8.** A weak equivalence in  $\mathbf{C}$  is called *universal* if any pullback of it is again a weak equivalence.

**Definition 4.9.** Let  $\delta$  be a standard system of simplices in  $\mathbf{C}$ . The operad  $\mathcal{RL}_m$  is called  *$\delta$ -reductive* if for any  $n \geq 0$  and  $k \geq 0$  and any colours  $c_i, c \in \{c, o\}$  satisfying (2.1), the map  $\xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^n \rightarrow \xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^0$  is a universal weak equivalence.

The operad  $\mathcal{RL}_m$  is called *strongly  $\delta$ -reductive* if in addition the induced maps  $\text{Coend}_{\mathcal{RL}_m}(\delta)(c_1, \dots, c_k; c) \rightarrow \xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^0$  are universal weak equivalence in  $\mathbf{C}$ .

The proof of [BB09, Theorem 3.8] can be applied *mutatis mutandis* to the functors  $\xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)$ . So, we have almost for free the analogue to [BB09, Theorem 3.8]:

**Theorem 4.10.** *Let  $\delta$  be a standard system of simplices in a model monoidal category  $\mathbf{C}$  with a zero object. If the operad  $\mathcal{RL}_m$  is strongly  $\delta$ -reductive, then the operad  $\text{Coend}_{\mathcal{RL}_m}(\delta)$  is weakly equivalent to  $\mathcal{B}_\delta \mathcal{RK}$ .*

*Proof.* We outline the proof.

We construct a zig-zag of weak equivalences of operads

$$(4.5) \quad \text{Coend}_{\mathcal{RL}_m}(\delta) \xleftarrow{\sim} \text{Coend}_{\widehat{\mathcal{RL}}_m}(\delta) \xrightarrow{\sim} \mathcal{B}_\delta \mathcal{RK}_m.$$

The intermediate operad  $\widehat{\mathcal{RL}}_m$  is defined as

$$\widehat{\mathcal{RL}}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \text{hocolim}_{\mathcal{RK}_m(c_1, \dots, c_k; c)} (\mathcal{RL}_m)_\alpha(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}),$$

for all  $(c_1, \dots, c_k; c)$  and  $\tilde{n}_i \in \text{Col}_{c_i}$ ,  $\tilde{n} \in \text{Col}_c$ . The properties of the standard system of simplices  $\delta$  imply that  $\text{Coend}_{\widehat{\mathcal{RL}}_m}(\delta)$  is an operad.

We denote by  $\widehat{\xi}(\mathcal{RL}_m)$  the corresponding SC functor-operad of  $\widehat{\mathcal{RL}}_m$ , so that

$$\widehat{\xi}(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta) = \text{hocolim}_{\mathcal{RK}_m(c_1, \dots, c_k; c)} \xi(\mathcal{RL}_m)_\alpha(\delta).$$

The left hand map in the zig-zag (4.5) is induced by the maps  $\widehat{\xi}(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta) \rightarrow \xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)$ . Because of the strong  $\delta$ -reductivity condition, it is sufficient to show that

$$\widehat{\xi}(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^0 \rightarrow \xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta)^0,$$

is a weak equivalence. This follows from general properties of standard system of simplices.

To show the other weak equivalence

$$\text{Coend}_{\widehat{\mathcal{RL}}_m}(\delta) \rightarrow \mathcal{B}_\delta \mathcal{RK}_m,$$

we first remark that

$$\mathcal{B}_\delta \mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c) \cong \text{hocolim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c)}(cc\delta^0) \cong \text{Hom}(\delta, \widehat{\xi}(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; c}(cc\delta^0))$$

where  $cc\delta^0$  is the constant cosimplicial object at  $\delta^0 = 1_{\mathbf{C}}$ . Then it is sufficient to prove that

$$\text{Hom}_{\mathbf{C}^\Delta}(\delta, \widehat{\xi}(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; c}(cc\delta^0)) \rightarrow \text{Hom}_{\mathbf{C}^\Delta}(\delta, \widehat{\xi}(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; c}(\delta))$$

is a weak equivalence. Since  $\text{Hom}_{\mathbf{C}^\Delta}(-, -)$  preserves weak equivalences, this is satisfied if

$$\widehat{\xi}(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; c}(\delta)^n \rightarrow \widehat{\xi}(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; c}(cc\delta^0)^n$$

is a weak equivalence for all  $n \geq 0$ . Since we have

$$\widehat{\xi}(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; c}(\delta)^n \cong \text{hocolim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c)} \xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta),$$

it remains to show that

$$\xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta)^n \rightarrow \xi(\mathcal{R}\mathcal{L}_m)_\alpha(cc\delta^0)^n = 1_{\mathbf{C}}$$

is a weak equivalence for all  $\alpha$  and  $n \geq 0$ . By  $\delta$ -reductivity the left vertical arrow in

$$\begin{array}{ccc} \xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta)^n & \longrightarrow & \xi(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; c}(\delta)^n \\ \downarrow & & \downarrow \\ \xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta)^0 & \longrightarrow & \xi(\mathcal{R}\mathcal{L}_m)_{c_1, \dots, c_k; c}(\delta)^0 \end{array}$$

is a weak equivalence. Moreover, for each  $(c_1, \dots, c_k; c)$  and  $\alpha \in \mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c)$ , the object  $\xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta)^0$  is weakly contractible. The latter is due to the following fact. Properties of the standard system of simplices imply that: the realization functor  $(-)\otimes_{\Delta} \delta$  preserves and reflect weak equivalences; the two objects  $\xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta)^0$  and  $\xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta_{\text{yon}})^0 \otimes_{\Delta} \delta$  are weakly equivalent; so that, it is sufficient to prove that for  $\delta = \delta_{\text{Top}}$ , the space  $\xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta_{\text{Top}})^0$  is weakly contractible.  $\square$

In Theorem 4.10 we have used our version of [MS04, Lemma 14.8] (see also [BB09, Lemma 3.9]):

**Lemma 4.11.** *For each  $m \geq 1$ ,  $(c_1, \dots, c_k; c)$  and  $\alpha \in \mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c)$ , the space  $\xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta_{\text{Top}})^0$  is weakly contractible.*

*Proof.* The proof is quite similar to [MS04, Lemma 14.8]; it is sufficient to check that the arguments are compatible with our operads  $\mathcal{R}\mathcal{K}$  and  $\mathcal{R}\mathcal{L}$ . We outline the proof for convenience.

For each  $\alpha \in \mathcal{R}\mathcal{K}(c_1, \dots, c_k; c)$ , we construct a retraction

$$\iota: \xi(\mathcal{R}\mathcal{L}_m)_\alpha(\text{Cone}(\delta_{\text{Top}}), \dots, \text{Cone}(\delta_{\text{Top}}))^0 \hookrightarrow \xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta_{\text{Top}}, \dots, \delta_{\text{Top}})^0 : \rho$$

such that  $\rho \circ \iota = \text{id}$ . The functor  $\text{Cone}(\delta_{\text{Top}}) : \Delta \rightarrow \mathbf{Top}$  is defined as follows.

$$\text{Cone}(\delta_{\text{Top}})^n = \delta_{\text{Top}}^{n+1}$$

$$\text{Cone}(\delta_{\text{Top}})(f: [l] \rightarrow [n]) = \delta_{\text{Top}}(\tilde{f}: [l+1] \rightarrow [n+1]),$$

where  $\tilde{f}: [l+1] \rightarrow [n+1]$  is defined as  $\tilde{f}(0) = 0$  and  $\tilde{f}(k) = f(k-1) + 1$  for  $1 \leq k \leq l+1$ .

The map  $\rho$  sends the class of  $(x, u_1, \dots, u_k)$  to the class of  $(x, d^0 u_1, \dots, d^0 u_k)$  where  $d^0: \delta_{\text{Top}}^n \rightarrow \delta_{\text{Top}}^{n+1}$  is the *zeroth* cosimplicial face operator.

For  $x \in \mathcal{R}\mathcal{L}_\alpha(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ , with  $(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$  according to  $(c_1, \dots, c_k; c)$ , the map  $\iota$  assigns to the class of  $(x, u_1, \dots, u_k)$  the class of  $(\tilde{x}, u_1, \dots, u_k)$  where  $\tilde{x}$  is the lattice obtained from  $x$  by doubling the first occurrence of  $i$  (or  $\underline{i}$ ) for  $1 \leq i \leq k$  in its integer-string representation. Thus the lattice  $\tilde{x}$  belongs to  $\mathcal{R}\mathcal{L}_\alpha(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ .

Since  $\text{Cone}(\delta_{\text{Top}})$  is contractible, it provides a map

$$\xi(\mathcal{R}\mathcal{L}_m)_\alpha(\text{Cone}(\delta_{\text{Top}}), \dots, \text{Cone}(\delta_{\text{Top}}))^0 \rightarrow \xi(\mathcal{R}\mathcal{L}_m)_\alpha(*, \dots, *)^0 = *$$

which is a weak equivalence.  $\square$

Once again, the proof of the following is completely similar to [BB09, Examples 3.10].

**Proposition 4.12.** *For  $\delta$  being  $\delta_{\text{Top}}$  or  $\delta_{\mathbb{Z}}$ , the operad  $\mathcal{RL}_m$  is strongly  $\delta$ -reductive. Consequently, the operad  $\text{Coend}_{\mathcal{RL}_m}(\delta)$  is weakly equivalent to the topological (resp. chain) Swiss Cheese operad  $\mathcal{SC}_m$  (resp.  $\mathcal{C}_*\mathcal{SC}_m$ ) for  $\delta$  being  $\delta_{\text{Top}}$  (resp.  $\delta_{\mathbb{Z}}$ ).*

*Proof.* A) *The topological case  $\delta = \delta_{\text{Top}}$ .* Let us recall from [MS04, Proposition 13.4] that, for all  $k \geq 0$  and  $\mathbf{c}_i = \mathbf{c} = \mathbf{c}$ , we have an isomorphism of cosimplicial spaces

$$(4.6) \quad (\theta_{\text{Top}})_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}^* : \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\text{Top}})^* \rightarrow \delta_{\text{Top}}^* \times \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\text{Top}})^0.$$

The same argument<sup>2</sup> applies to show that, for all  $k \geq 0$  and all colours  $(\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{o})$ ,

$$(4.7) \quad (\theta_{\text{Top}})_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{o}}^* : \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{o}}(\delta_{\text{Top}})^* \rightarrow \delta_{\text{Top}}^* \times \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{o}}(\delta_{\text{Top}})^0$$

is an isomorphism of cosimplicial spaces. Moreover, both the cosimplicial isomorphisms (4.6) and (4.7) are compatible with the projection onto the second factor so that one obtains trivial fibrations  $(\theta_{\text{Top}})_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}^n$  for each  $n \geq 0$  and colours  $(\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c})$  as in (2.1). Moreover  $\theta$  induces a homeomorphism on  $\text{Coend}_{\mathcal{L}_m}(\delta_{\text{Top}})(\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}) \cong \text{Hom}_{\Delta}(\delta_{\text{Top}}, \delta_{\text{Top}}) \times \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\text{Top}})^0$ . Using the contractibility of  $\text{Hom}_{\Delta}(\delta_{\text{Top}}, \delta_{\text{Top}})$  one gets a weak equivalence

$$(4.8) \quad \text{Coend}_{\mathcal{RL}_m}(\delta_{\text{Top}})(\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}) \rightarrow \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\text{Top}})^0,$$

which is also universal.

B) *The chain complex case  $\delta = \delta_{\mathbb{Z}}$ .* We will show that the maps

$$(4.9) \quad (\theta_{\mathbb{Z}})_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}^* : \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^* \rightarrow \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^0$$

are trivial Reedy fibrations in  $\mathbf{Ch}(\mathbb{Z})^{\Delta}$ . Indeed, the trivial Reedy fibrations are objectwise trivial fibrations what implies the  $\delta_{\mathbb{Z}}$ -reductivity. Moreover, since  $\delta_{\mathbb{Z}}$  is a standard system of simplicies, the  $\delta_{\mathbb{Z}}$ -totalization functor  $\text{Hom}_{\mathbf{Ch}(\mathbb{Z})}(\delta_{\mathbb{Z}}, -)$  is a right Quillen functor. Then, the induced maps

$$(4.10) \quad \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})(\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}) \rightarrow \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^0$$

are trivial fibrations in  $\mathbf{Ch}(\mathbb{Z})$ .

Recall that the realization functors  $|-|_{\delta_{\text{Top}}}$  and  $|-|_{\delta_{\mathbb{Z}}}$  preserve and reflect the weak equivalences. Then, since  $(\theta_{\text{Top}})_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}^n$  is a weak equivalence for each  $n \geq 0$ , we deduce that for  $n \geq 0$ , the maps

$$(4.11) \quad (\theta_{\mathbb{Z}})_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}^n : \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^n \rightarrow \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^0$$

so are. It remains to show that the maps  $(\theta_{\mathbb{Z}})_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}^*$  are Reedy fibrations in  $\mathbf{Ch}(\mathbb{Z})^{\Delta}$ . Let us denote by  $M_n X$  the matching object of  $X$ . Since  $M_n \text{cc} \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^0 = \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^0$ , it is sufficient to show that the maps

$$(4.12) \quad j_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}^n : \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^n \rightarrow M_n \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^*$$

induced by the  $(\theta_{\mathbb{Z}})_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}^n$  are surjective for each  $n \geq 1$ . This can be shown by a careful investigation on  $M_n \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^*$ .  $\square$

## 5 The relative surjection operad

For  $m \geq 1$ , we define a  $\{\mathbf{c}; \mathbf{o}\}$ -coloured operad  $\mathcal{RS}_m$ , sub-operad of  $\text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ . We show that the inclusion  $\mathcal{RS}_m \hookrightarrow \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$  is a weak equivalence.

As complexes, we set  $\mathcal{RS}_m(\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}) := \xi(\mathcal{RL}_m)_{\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c}}(\delta_{\mathbb{Z}})^0$ .

For  $(\mathbf{c}_1, \dots, \mathbf{c}_k)$  where  $\mathbf{c}_i \in \{\mathbf{c}, \mathbf{o}\}$ , we set

$$\tilde{\mathbf{c}}_i = \begin{cases} \mathbf{i} & \text{if } \mathbf{c}_i = \mathbf{c}; \\ \underline{\mathbf{i}} & \text{if } \mathbf{c}_i = \mathbf{o}. \end{cases}$$

<sup>2</sup> In Proposition 12.7 and Proposition 13.4 from [MS04] it is sufficient to consider elements  $\bar{p} \sqcup \bar{q} \xleftarrow{f} \top \xrightarrow{h} s$  with  $\bar{p} \sqcup \bar{q}$  instead of  $\bar{k}$ , where  $p$  and  $q$  stand respectively for the number of closed colours and the number of open colours in  $(\mathbf{c}_1, \dots, \mathbf{c}_k; \mathbf{c})$ .

A surjection  $f : \{1, \dots, k+r\} \rightarrow \{\tilde{c}_1, \dots, \tilde{c}_k\}$  is called *degenerate* if there exists an  $i$  such that  $f(i) = f(i+1)$ .

Then, an element in  $\mathcal{RS}_m(c_1, \dots, c_k; c)$  is a sum of *non degenerate* surjections  $f : \{1, \dots, k+r\} \rightarrow \{\tilde{c}_1, \dots, \tilde{c}_k\}$ . Indeed, a generator of the complex

$$\xi(\mathcal{RL}_m)_{c_1, \dots, c_k; c}(\delta_{\mathbb{Z}})^0 = \mathcal{RL}_m(-, \dots, -; \tilde{0}) \otimes_{\Delta^k} \delta_{\mathbb{Z}} \otimes \dots \otimes \delta_{\mathbb{Z}}$$

is represented by an element in  $\mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_{k+j}; \tilde{0})$  without repetitions (that is without doubles  $\tilde{i}$  or  $\tilde{i}$  in the integer-string representation). The degree of  $f : \{1, \dots, k+r\} \rightarrow \{\tilde{c}_1, \dots, \tilde{c}_k\}$  is the total degree  $r = n_1 + \dots + n_k$  of the corresponding generator in

$$\mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{0}) \otimes (\delta_{\mathbb{Z}}^{\tilde{n}_1})_{n_1} \otimes \dots \otimes (\delta_{\mathbb{Z}}^{\tilde{n}_k})_{n_k}.$$

Following [BB09] we define maps

$$\vartheta_n : \xi_{c_1, \dots, c_k; c}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0 \otimes \delta_{\mathbb{Z}}^n \rightarrow \xi_{c_1, \dots, c_k; c}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^n, \quad n \geq 0,$$

as follows. Let  $T \in \mathcal{RS}_m(c_1, \dots, c_k; c) = \xi_{c_1, \dots, c_k; c}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0$  be a generator. We denote by  $T_{\#} \in \xi_{c_1, \dots, c_k; c}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^n$  the element represented by the relative lattice obtained by adding  $n$  vertical bars in its integer-string representation. Let us denote by  $e_n$  the generator of  $(\delta_{\mathbb{Z}}^n)_n = C_n(\text{Hom}_{\Delta}(-, [n]); \mathbb{Z})$ , so that  $(\delta_{\mathbb{Z}}^n)_n \cong \mathbb{Z}[e_n]$  for  $n \geq 0$ . Let  $T \in \xi_{c_1, \dots, c_k; c}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0$  be a generator so that  $T$  is the class of  $T \otimes e_{n_1} \otimes \dots \otimes e_{n_k}$ ; we denote  $[T \otimes e_{n_1} \otimes \dots \otimes e_{n_k}]$  such a class. We define  $\vartheta_n(T \otimes e_n)$  to be the signed sum of elements  $[T_{\#} \otimes e_{n_{\#}} \otimes \dots \otimes e_{n_{\#}}]$ ; the sum being over all possible relative lattices path of the form  $T_{\#}$ . We extend  $\vartheta_n$  on  $\xi_{c_1, \dots, c_k; c}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0 \otimes \delta_{\mathbb{Z}}^n$  as a cosimplicial map.

Let us described the partial compositions of  $\mathcal{RS}_m$ . For two surjections  $f \in \mathcal{RS}_m(c_1, \dots, c_k; c)$  and  $g \in \mathcal{RS}_m(d_1, \dots, d_j; c_i)$  we set

$$(5.1) \quad f \circ_{\tilde{c}_i}^{\mathcal{RS}} g = f \circ_i \vartheta_{n(i)}(g \otimes e_{n(i)}),$$

where  $n(i)$  denotes the number of occurrences of  $\tilde{c}_i$  in the surjection  $f$  (*i.e.*  $n(i)$  is the cardinal of  $f^{-1}(\tilde{c}_i)$ ) and  $\circ_i$  denotes the partial composition of  $\text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ . We extend the composition by linearity. Such partial compositions give an operadic structure on  $\mathcal{RS}_m$ .

*Example 5.1.*

$$(121)^{\circ} \circ_1^{\mathcal{RS}} (12) = (1312)^{\circ} + (1232)^{\circ}.$$

**Proposition 5.2.** *The inclusion  $\mathcal{RS}_m \hookrightarrow \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$  is a weak equivalence of operads.*

*Proof.* The maps  $\{\vartheta_n\}_n$  induce, by adjunction, a map  $\vartheta' : \mathcal{RS}_m \hookrightarrow \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ . Except for signs, the fact that  $\vartheta'$  is compatible with the operadic structures is straightforward from the definition. Such a compatibility implies (and then defines) the signs involved in the definition of  $\vartheta_n$ . In particular, signs can be determined by an induction process on the dimension of the generators of  $\mathcal{RS}_m$ .

Moreover, let us denote by

$$\pi : \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}}) \rightarrow \mathcal{RS}_m$$

the weak equivalence coming from (4.10); it satisfies  $\pi\vartheta' = \text{id}$ . Thus,  $\vartheta'$  is a weak equivalence.  $\square$

**Proposition 5.3.** *As an operad,  $\mathcal{RS}_2$  is generated by the following elements*

$$\begin{aligned} \mu_c &= (12) \\ T_k &= (1213 \cdots 1k1), \quad k \geq 2 \\ \mu_o &= (\underline{12})^{\circ} \\ T_j &= (\underline{121}\underline{3} \cdots 1j1)^{\circ}, \quad j \geq 2 \\ \text{inc} &= (1)^{\circ}, \end{aligned}$$

and the two unit elements  $\text{id}_c = (1)$  and  $\text{id}_o = (\underline{1})^{\circ}$ .

*Proof.* We suppose by induction on  $N$  that any homogeneous element of  $\mathcal{RS}_2$  with  $N \geq 1$  different integers in its integer-string representation is obtained by operadic compositions of elements cited in the statement. The cases  $N = 1$  and  $N = 2$  are trivially verified.

In what follows we abusively do not mark the distinction between integers with under bar and integers without under bar. Let  $x$  be a homogeneous element of  $\mathcal{RS}_2$  with  $N + 1$  different integers. Because

of the filtration condition (4.2),  $x$  can be written as a sequence  $(A_1 \cdots A_n)$  where the  $A_i$ 's are non empty sequences of integers such that, if  $j$  belongs to  $A_i$ , then  $j \notin A_s$  for  $s \neq i$ . Moreover, because of the symmetric group action, one can suppose that the integers of  $A_i$  are smaller than the integers of  $A_j$  whenever  $i < j$ . In this case, if  $n > 1$ , then  $x = ((12) \circ_1 (A_1 \cdots A_{n-1})) \circ_{\max A_{n-1} + 1} (A_n)$  where  $\tilde{A}_n$  is obtained from  $A_n$  by decreasing each number by  $\max A_{n-1}$ . Since the  $A_i$ 's are not empty,  $(A_1 \cdots A_{n-1})$  as well as  $A_n$  have at most  $N$  different integers and then satisfy the induction hypotheses. If  $n = 1$  then either  $x$  is  $T_k$  (or  $\underline{T}_k$ ) for some  $k$  or,  $x$  is such that  $A_1 = jB_1jB_2j \cdots jB_pj$  with  $1 \leq p < N$  and for some integer  $j$ . Thus there exists at least one  $B_{i_0}$  that contains  $2 \leq q \leq N - (p - 1)$  and  $x = (jB_1j \cdots B_{i_0-1}j \alpha j B_{i_0+1}j \cdots jB_pj) \circ_\alpha \tilde{B}_{i_0}$  for some  $\alpha$ , what concludes the proof.  $\square$

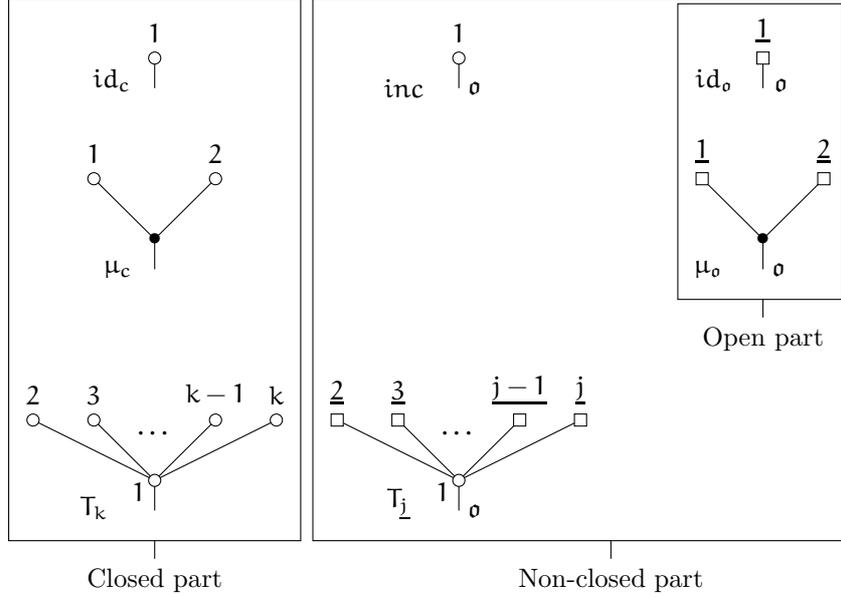


Figure 5.1: Tree representation of the generators of  $\mathcal{RS}_2$  (via the description of Proposition 6.3).

## 6 The operad $\mathcal{RL}_2$

### 6.1 The operad $\mathcal{RL}_2$ in term of trees

We describe an SC-split operad  $\mathcal{RL}_2$  in the category of sets, **Set**. Given any cocomplete, closed monoidal symmetric category  $\mathbf{C}$ , one has the strong monoidal functor

$$\begin{aligned} \mathbf{Set} &\rightarrow \mathbf{C} \\ E &\mapsto \coprod_{e \in E} 1_{\mathbf{C}} \end{aligned}$$

where  $1_{\mathbf{C}}$  denotes the unit for the monoidal structure  $\otimes$  of  $\mathbf{C}$ . This way we can consider algebras over the operad  $\mathcal{RL}_2$  that are (family of) objects in **Top** or **Ch**( $\mathbb{Z}$ ).

The operad  $\mathcal{RL}_2$  can be thought of as a mix between the Lattice paths operads  $\mathcal{L}_2$  and  $\mathcal{L}_1$  introduced in [BB09]. The closed part of  $\mathcal{RL}_2$  is the sub-operad  $\mathcal{L}_2$  of multiplicative operads. The open part of  $\mathcal{RL}_2$  is the sub-operad  $\mathcal{L}_1$  of cosimplicial  $\square$ -monoids (cf. [MS04, Definition 2.1]). In Section 6.2 we define the notion of *wide* and *stable wide* bimodules over an operad so that a cosimplicial  $\square$ -monoid is a wide bimodule over the non-symmetric operad of associative algebras  $\mathcal{A}s$ .

A *multiplicative operad*  $\mathcal{M}$  is a (non-symmetric) operad endowed with an inclusion  $\mathcal{A}s \hookrightarrow \mathcal{M}$  of non-symmetric operads.

The operad  $\mathcal{RL}_2$  encodes the couples  $(\mathcal{M}, \mathcal{Z})$  subject to the following conditions.

- I.  $\mathcal{M}$  is a (non-symmetric) multiplicative operad.

- II.  $\mathcal{Z}$  is a stable wide **left**-module over  $\mathcal{M}$ ; in particular, using  $\mathcal{A}s \hookrightarrow \mathcal{M}$ , it is a wide **left**-module over  $\mathcal{A}s$ ; this is a part of the following structure III.
- III.  $\mathcal{Z}$  is a wide **bimodule** over  $\mathcal{A}s$ .

A stable wide bimodule is, in particular, an *infinitesimal bimodule* (called weak bimodule in [Tur10, Definition 4.1]).

**Lemma 6.1.** [Tur10, Lemma 4.2] *The structure of a cosimplicial vector space is equivalent to the structure of an infinitesimal bimodule over  $\mathcal{A}s$ .*

Our operad  $\mathcal{RL}_2$  is closed to the operad  $\mathcal{O}$  constructed in [DTT11, section 3.1]. The operad  $\mathcal{O}$  was constructed in order to obtain an action of a Swiss Cheese operad on the pair  $(\text{CC}^*(A, A), A)$  formed by an associative algebra  $A$  and its Hochschild cochain complex  $\text{CC}^*(A, A)$ . We enlarge the (non-closed part of the) operad  $\mathcal{O}$  in order to obtain an action of a Swiss Cheese operad on the couple  $(\text{Tot}_{\delta_{\mathcal{Z}}}\mathcal{M}, \text{Tot}_{\delta_{\mathcal{Z}}}\mathcal{Z})$  where  $(\mathcal{M}, \mathcal{Z})$  is a couple as above and  $\text{Tot}_{\delta_{\mathcal{Z}}}(-)$  denotes the  $\delta_{\mathcal{Z}}$ -totalization. In particular, our open part takes into account the cosimplicial structure of  $\mathcal{Z}$ .

Recall from Section 4.1 that the set of colours of  $\mathcal{RL}_2$  is

$$\text{Col} = \text{Col}_c \sqcup \text{Col}_o,$$

where  $\text{Col}_c$  is the set of natural numbers and  $\text{Col}_o$  is the set of natural numbers decorated with an underline.

**Definition 6.2.** Let  $T$  be a planar rooted tree. Let  $v$  be a vertex of  $T$ . We denote by  $T_v$  the maximal sub-tree of  $T$  such that  $v$  is the root vertex of  $T_v$ .

For  $n_i, \mathbf{n} \in \text{Col}_c$ , the set  $\mathcal{RT}(n_1, \dots, n_k; \mathbf{n})$  is the set of equivalence classes of planar rooted trees  $T$  satisfying:

- a subset of the set of vertices of the tree  $T$  is indexed by the set  $\{1, \dots, k\} \sqcup \{1, \dots, n\}$  in such a way that:
  - the vertices indexed by  $\{1, \dots, n\}$  are only terminal vertices,
- the ordered set of edges originating at the vertex indexed by  $s \in \{1, \dots, k\}$  is identified with  $[n_s - 1]$ . (We set  $[-1] = \emptyset$ ).

The subset of vertices identified with  $\{1, \dots, n\}$  benefits of an order given by  $\{1 < \dots < n\}$ . We require that this order coincides with the order which is given by turning around the tree in the clockwise direction starting from the root vertex. The equivalence class of equivalence is the same as in [DTT11, 3.2.1]. Explicitly, it is the finest one in which two trees are equivalent if one of them can be obtained from the other by either:

- the contraction of an edge with unmarked ends; or,
- removing an unmarked vertex with only one edge originating from it and joining the two edges adjacent to this vertex into one edge.

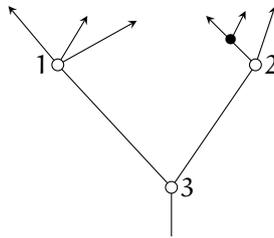


Figure 6.1: Element in  $\mathcal{RL}_2(3, 2, 2; 6)$ .

Let  $(\tilde{n}_1, \dots, \tilde{n}_{k+j})$  be a tuple of colour in  $\text{Col}$ . We set  $\{s_1, \dots, s_{k+j}\}$  the set such that

$$s_i = \begin{cases} i & \text{if } \tilde{n}_i = n_i; \\ \underline{i} & \text{if } \tilde{n}_i = \underline{n}_i. \end{cases}$$

The set  $\mathcal{RT}(\tilde{n}_1, \dots, \tilde{n}_{k+j}; \underline{\mathbf{n}})$  is the set of equivalence classes of planar rooted trees  $T$  satisfying:

- a subset of the set of vertices of a tree  $T$  is indexed by the set  $\{s_1, \dots, s_{k+j}\} \sqcup \{1, \dots, n\}$  in such a way that:
  - given a vertex  $v$  of  $T$  indexed by  $\underline{i} \in \{s_1, \dots, s_{k+j}\}$  then, in the tree  $T_v$  there is no vertex different of  $v$  indexed by an element  $s_j \in \{s_1, \dots, s_{k+j}\}$ ,
  - the vertices indexed by  $\{1, \dots, n\}$  are only terminal vertices,
- the ordered set of edges originating at the vertex indexed by  $s_i \in \{s_1, \dots, s_{k+j}\}$  is identified with  $[\tilde{n}_i - 1]$ ;
- the root is decorated by an  $\sigma$ .

The subset of vertices identified with  $\{1, \dots, n\}$  benefits of an order given by  $\{1 < \dots < n\}$ . We require that this order coincides with the order which is given by turning around the tree in the clockwise direction starting from the root vertex. The equivalence class is the same as the previous one.

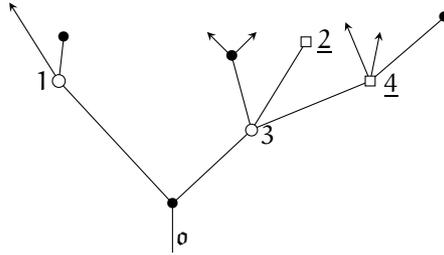


Figure 6.2: Element in  $\mathcal{RT}(2, \underline{0}, 3, \underline{3}, \underline{5})$ .

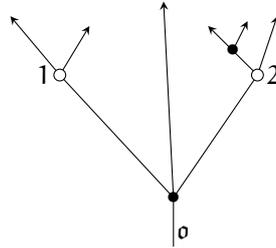


Figure 6.3: Element in  $\mathcal{RT}(2, 2, \underline{6})$ .

The composition maps in  $\mathcal{RT}$  are defined by substitution of trees into marked vertices.

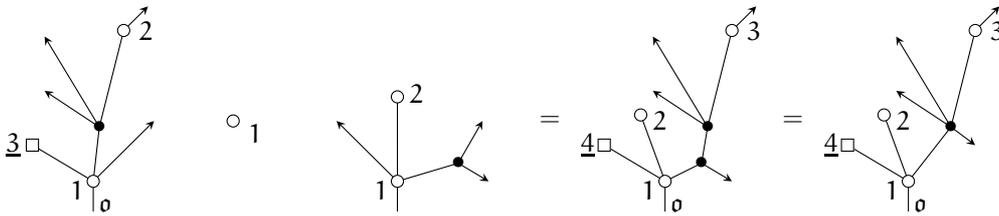


Figure 6.4: Example of composition.

**Proposition 6.3.** *The two operads  $\mathcal{RL}_2$  and  $\mathcal{RT}$  are isomorphic.*

*Proof.* Let us take a labelled tree  $T \in \mathcal{RT}$ . One runs through the tree  $T$  in clockwise direction starting from the root in such a way that one passes exactly two times on each edges (once per sense). One assigns the number an integer-string by writing down the corresponding label each time one meets a labelled vertex and by writing down a vertical bar each time one meets a terminal vertex. One add an extra label  $\sigma$  if the root is decorated by  $\sigma$ .

The inverse construction is given as follows. To an integer-string representation one assigns a tree with: one labelled vertex for each different integer and one terminal vertex for each vertical bar. The

labelled vertices have one output less than there are occurrences for the corresponding integer. The corresponding tree is constructed such that its order fits with the reading (from the left to the right) of the integer-string. Note that when two equal integers (or two vertical bars) are adjacent in the integer-string this forces the creation of an unlabelled vertex.

One checks that this two assignments provide an isomorphism of operads.

As an example, the tree from Figure 6.2 corresponds to the integer-string  $(1|113||3234|4|443)^\circ$ .  $\square$

## 6.2 The algebras over $\mathcal{RL}_2$

Let us fix a cocomplete, closed monoidal symmetric category  $(\mathbf{C}, \otimes, 1_{\mathbf{C}}, \tau_{\mathbf{C}})$ .

**Definition 6.4.** Let  $\mathcal{M}$  be a (1-coloured) non symmetric operad in  $\mathbf{C}$ . A family  $\mathcal{Z} = \{\mathcal{Z}(A)\}_{A \in \text{finite sets}}$  of objects in  $\mathbf{C}$  is a *wide left module over  $\mathcal{M}$*  if, for any three finite ordered sets  $A, A'$  and  $B$  such that  $A' \subset A$  and  $\alpha : B \rightarrow A \setminus A'$  a map of ordered sets, there is a map

$$\lambda'_\alpha : \mathcal{M}(A) \otimes \bigotimes_{a \in A \setminus A'} \mathcal{Z}(\alpha^{-1}(a)) \rightarrow \mathcal{Z}(B \sqcup A'),$$

such that all the diagrams of the following form commute

$$\begin{array}{ccc} \mathcal{M}(A) \otimes \mathcal{M}(B) \otimes \bigotimes_{e \in (A \cup_a B) \setminus (A' \sqcup B')} \mathcal{Z}((\alpha \sqcup \beta)^{-1}(e)) & \xrightarrow{\text{id} \otimes \lambda'_\beta} & \mathcal{M}(A) \otimes \bigotimes_{e \in A \setminus A'} \mathcal{Z}(\tilde{\alpha}^{-1}(e)) \\ \downarrow (- \circ_a -) \otimes \text{id} & & \downarrow \lambda_{\tilde{\alpha}} \\ \mathcal{M}(A \cup_a B) \otimes \bigotimes_{e \in (A \cup_a B) \setminus (A' \sqcup B')} \mathcal{Z}((\alpha \sqcup \beta)^{-1}(e)) & \xrightarrow{\lambda'_{\alpha \sqcup \beta}} & \mathcal{Z}(C_1 \sqcup A' \sqcup C_2 \sqcup B'), \end{array}$$

for  $A' \subset A \setminus \{a\}$ ,  $B' \subset B$ ,  $\alpha : C_1 \rightarrow (A \setminus \{a\}) \setminus A'$ ,  $\beta : C_2 \rightarrow B \setminus B'$ ,  $\alpha \sqcup \beta : C_1 \sqcup C_2 \rightarrow (A \cup_a B) \setminus (A' \sqcup B')$  and where  $\tilde{\alpha} : C_1 \sqcup (C_2 \sqcup B') \rightarrow A \setminus A'$  is defined as

$$\tilde{\alpha}(c) = \begin{cases} \alpha(c) & \text{if } c \in C_1; \\ a & \text{if } c \in C_2 \sqcup B'. \end{cases}$$

The maps  $\lambda'_\alpha$  are required to be natural in isomorphisms of ordered sets  $A, A', B$  and  $\alpha$ . Moreover, for all finite set  $B$  and  $\alpha : B \rightarrow \{1\}$ , the map

$$\lambda'_\alpha : \mathcal{Z}(B) = 1_{\mathcal{M}(1)} \otimes \mathcal{Z}(B) \rightarrow \mathcal{Z}(B),$$

is required to be the identity.

**Proposition 6.5.** A wide left module  $\mathcal{Z}$  over  $\mathcal{M}$  with actions maps

$$\lambda'_\alpha : \mathcal{M}(A) \otimes \bigotimes_{a \in A \setminus A'} \mathcal{Z}(\alpha^{-1}(a)) \rightarrow \mathcal{Z}(B \sqcup A'),$$

for  $A' \subset A$  and  $\alpha : B \rightarrow A \setminus A'$  is, in particular:

1. an infinitesimal left module over  $\mathcal{M}$  (see [Tur10, Definition 4.1]) for the action maps  $\lambda_\alpha := \lambda'_\alpha$  defined for  $A' = \emptyset$ ; and,
2. a left module over  $\mathcal{M}$  for the action maps  $\lambda_a := \lambda'_\alpha$  defined for  $A \setminus A' = \{a\}$ .

*Proof.* The diagram from Definition 6.4 gives the required properties for  $\lambda_\alpha$  and  $\lambda_a$ .  $\square$

**Definition 6.6.** A wide left module  $\mathcal{Z}$  over an operad  $\mathcal{M}$  is called *stable* if there are maps in  $\mathbf{C}$

$$\iota_A : \mathcal{M}(A) \rightarrow \mathcal{Z}(A)$$

for each finite set  $A$ , such that all the diagrams of the following form commute

$$\begin{array}{ccc}
\mathcal{M}(A) \otimes \bigotimes_{a \in A \setminus A'} \mathcal{M}(\alpha^{-1}(a)) & \xrightarrow{\text{id} \otimes \bigotimes_{b \in B} \iota_{\alpha^{-1}(b)}} & \mathcal{M}(A) \otimes \bigotimes_{a \in A \setminus A'} \mathcal{Z}(\alpha^{-1}(a)) \\
\downarrow - \circ_{\alpha} - & & \downarrow \lambda'_{\alpha} \\
\mathcal{M}(B \sqcup A') & \xrightarrow{\iota_{B \sqcup A'}} & \mathcal{Z}(B \sqcup A'),
\end{array}$$

where  $\alpha : B \rightarrow A \setminus A'$ .

**Definition 6.7.** A (resp. stable) wide bimodule over an operad  $\mathcal{M}$  is a (resp. stable) wide left-module over  $\mathcal{M}$  and a right-module over  $\mathcal{M}$  such that the underlying infinitesimal left-module structure together with the right-module structure form an infinitesimal bimodule structure over  $\mathcal{M}$ .

**Definition 6.8.** The non-symmetric operad  $\mathcal{A}s$  in  $\mathbf{C}$  is given as  $\mathcal{A}s(\mathfrak{n}) = 1_{\mathbf{C}}$  for  $\mathfrak{n} \geq 0$ .

Let  $\mathbf{E}$  be the category with objects the couples  $(\mathcal{M}, \mathcal{Z}) \in \mathbf{C} \otimes \mathbf{C}$  satisfying the three conditions I, II and III of section 6; and, with morphisms the pair  $(f, g) : (\mathcal{M}, \mathcal{Z}) \rightarrow (\mathcal{M}', \mathcal{Z}')$  where  $f : \mathcal{M} \rightarrow \mathcal{M}'$  is a morphism of multiplicative operads and  $g : \mathcal{Z} \rightarrow \mathcal{Z}'$  is a morphism of stable wide left modules over  $\mathcal{M}$  and wide bimodules over  $\mathcal{A}s$ .

**Lemma 6.9.** The category of cosimplicial  $\square$ -monoids in  $\mathbf{C}$  is isomorphic to the category of wide bimodules over  $\mathcal{A}s$  in  $\mathbf{C}$ .

**Proposition 6.10.** [BB09, Proposition 2.14] The category of  $\mathcal{L}_1$ -algebras (resp. of  $\mathcal{L}_2$ -algebras) in  $\mathbf{C}$  is isomorphic to the category of cosimplicial  $\square$ -monoids (resp. of multiplicative operads) in  $\mathbf{C}$ .

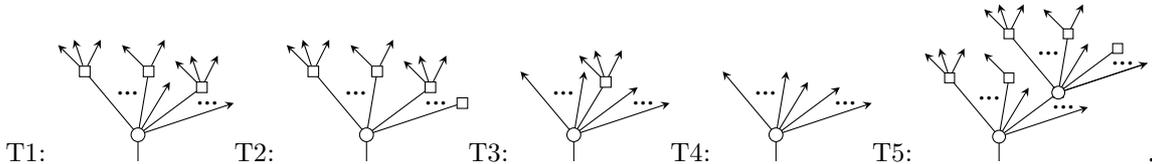
**Proposition 6.11.** Let  $\mathbf{C}$  be a cocomplete, closed monoidal symmetric category with a zero object. The category of  $\mathcal{R}\mathcal{L}_2$ -algebras in  $\mathbf{C}$  is isomorphic to the category  $\mathbf{E}$ .

*Proof.* We use the interpretation of  $\mathcal{R}\mathcal{L}_2$  in terms of planar trees, see Proposition 6.3. Given a tree in  $\mathcal{R}\mathcal{L}_2(\tilde{\mathfrak{n}}_1, \dots, \tilde{\mathfrak{n}}_k; \tilde{\mathfrak{n}})$  we recall that  $\{s_1, \dots, s_k\}$ , where

$$s_i = \begin{cases} \mathbf{i} & \text{if } \tilde{\mathfrak{n}}_i = \mathfrak{n}_i; \\ \underline{\mathbf{i}} & \text{if } \tilde{\mathfrak{n}}_i = \underline{\mathfrak{n}}_i, \end{cases}$$

denotes the set that labels "open" and "closed" vertices. The action of  $\mathcal{R}\mathcal{L}_2$  on an object  $(\mathcal{M}, \mathcal{Z})$  of  $\mathbf{E}$  is given as follows. One decorates the closed marked vertex indexed by  $\mathbf{i}$  with an element  $x_{\mathbf{i}} \in \mathcal{M}([\mathfrak{n}_{\mathbf{i}} - 1])$  and one decorates the open marked vertex indexed by  $\underline{\mathbf{j}}$  with an element  $y_{\underline{\mathbf{j}}} \in \mathcal{Z}([\underline{\mathfrak{n}}_{\underline{\mathbf{j}}}] - 1)$ ; the resulting element in  $\mathcal{Z}(\{1, \dots, \mathfrak{n}\})$  is obtained by composing the decorating elements along the tree, using the module structures of  $\mathcal{M}$  and  $\mathcal{Z}$ .

Conversely, the vertices of type T1, T2 and T3 give the maps  $\lambda'_{\alpha}$  and the vertices of type T4 give the map  $\iota : \mathcal{M} \rightarrow \mathcal{Z}$ . In particular, the vertices of type T2 give the maps  $\lambda_{\alpha}$  for the left module structure; the vertices of type T3 give the maps  $\lambda_{\alpha}$  for the left infinitesimal module structure. The elements of type T5 and other combinations give the diagram of Definition 6.4 using the operadic structure of  $\mathcal{R}\mathcal{L}_2$ . The diagram of Definition 6.6 is obtained similarly.



□

## 7 Cosimplicial relative loop space

In this section we define a cosimplicial model  $\omega(X, Y)$  for the relative loop spaces  $\Omega(X, Y)$ , that is, a cosimplicial space such that its  $\delta_{\text{Top}}$ -totalization  $\text{Tot}_{\delta_{\text{Top}}} \omega(X, Y)$  is homeomorphic to  $\Omega(X, Y)$ . For  $(M, N)$  a pair of monoids pointed at the unit  $*$  and such that  $N$  is a sub-monoid of  $M$ , we show

that  $\omega(M, N)$  is endowed with an additional structure. More precisely, we show that there exists an  $\mathcal{R}\mathcal{L}_2$ -algebra  $(\underline{\omega}(M), \underline{\omega}(M, N))$  associated to  $(\omega(M), \omega(M, N))$  such that, for  $\delta$  being  $\delta_{\text{Top}}$  or  $\delta_{\mathbb{Z}}$ , the totalization  $(\text{Tot}_{\delta}\omega(M), \text{Tot}_{\delta}\omega(M, N))$  is an algebra over  $\text{Coend}_{\mathcal{R}\mathcal{L}_2}(\delta)$ .

Let us start by a definition.

**Definition 7.1.** Let  $(X, Y)$  be two topological spaces pointed at  $*$  and such that  $* \subset Y \subset X$ . The relative loop space of  $(X, Y)$ ,  $\Omega(X, Y)$ , is the space of continuous maps  $\gamma : [0, 1] \rightarrow X$  satisfying  $\gamma(0) = *$  and  $\gamma(1) \in Y$ .

**Definition 7.2.** Let  $(X, Y)$  be a pair of topological spaces pointed at  $*$  such that  $* \subset Y \subset X$ . The *cosimplicial relative loop space*  $\omega(X, Y)$  is the cosimplicial space such that  $\omega(X, Y)^0 = Y$ , and  $\omega(X, Y)^k = X^{\times k} \times Y$  for  $k \geq 1$ , with

$$\begin{aligned} d^0(x_1, \dots, x_k, y) &= (*, x_1, \dots, x_k, y) \\ d^i(x_1, \dots, x_k, y) &= (x_1, \dots, x_i, x_i, \dots, x_k, y), 1 \leq i \leq k \\ d^{k+1}(x_1, \dots, x_k, y) &= (x_1, \dots, x_k, y, y) \\ s^i(x_1, \dots, x_k, y) &= (x_1, \dots, x_i, x_{i+2}, \dots, x_k, y), 0 \leq i \leq k. \end{aligned}$$

*Remark 7.3.* For  $Y = *$ , the cosimplicial space  $\omega(X, Y)$  is the cosimplicial space  $\omega X \times \{*\} \cong \omega X$  that is a model for the loop space  $\Omega X$  described in [Sal09].

**Proposition 7.4.** *The maps*

$$\begin{aligned} \Omega(X, Y) \times \Delta^k &\rightarrow X^k \times Y \\ (\gamma, (t_1, \dots, t_k)) &\mapsto (\gamma(t_1), \dots, \gamma(t_k), \gamma(1)), \quad k \in \mathbb{N}, \end{aligned}$$

*induce, by adjunction, a homeomorphism  $\Omega(X, Y) \cong \text{Tot}(\omega(X, Y))$ .*

## 7.1 Cosimplicial relative loop space of monoids

Given a topological monoid  $M$  and a sub-monoid  $N$ , the totalization of the cosimplicial relative loop space  $\omega(M, N)$  is homeomorphic to  $\Omega(M, N) \simeq \Omega(\Omega \mathcal{B}M, \Omega \mathcal{B}N) \cong \Omega^2(\mathcal{B}M, \mathcal{B}N)$  where  $\mathcal{B}G$  denotes the classifying space of the monoid  $G$ . From [Sal09] we know that  $\omega(M)$  can be seen as the cosimplicial space coming from a multiplicative operad  $\underline{\omega}(M)$ . This property implies that the totalization of  $\omega(M)$  is an  $E_2$ -algebra.

In the same spirit we show that  $\omega(M, N)$  comes from  $\underline{\omega}(M, N)$ , which is both a wide **bimodule** over  $\mathcal{A}s$  and a stable wide **left** module over the multiplicative operad  $\underline{\omega}(M)$ . Then, one obtains that the couple of totalizations  $(\text{Tot}_{\delta}\omega(M), \text{Tot}_{\delta}\omega(M, N))$  is a Swiss Cheese algebra for  $\delta = \delta_{\text{Top}}$  or  $\delta = \delta_{\mathbb{Z}}$ .

The structural map of the operad  $\underline{\omega}(M)$  is denoted by

$$\begin{aligned} \gamma : \underline{\omega}(M)(k) \times \underline{\omega}(M)(l_1) \times \dots \times \underline{\omega}(M)(l_k) &\rightarrow \underline{\omega}(M)(l_1 + \dots + l_k) \\ (f, g_1, \dots, g_k) &\mapsto \gamma(f; g_1, \dots, g_k). \end{aligned}$$

Since  $\underline{\omega}(M)$  has a unit,  $\gamma$  is equivalent to infinitesimal maps

$$\begin{aligned} \circ_i : \underline{\omega}(M)(k) \times \underline{\omega}(M)(l) &\rightarrow \underline{\omega}(M)(l_1 + \dots + l_k) \\ (f, g) &\mapsto f \circ_i g. \end{aligned}$$

Explicitly, if  $f = (x_1, \dots, x_k)$  and  $g = (y_1, \dots, y_l)$ , then

$$(7.1) \quad f \circ_i g := (x_1, \dots, x_{i-1}, x_i y_1, \dots, x_i y_l, x_{i+1}, \dots, x_k),$$

see [Sal09].

The family  $\underline{\omega}(M, N) = \{\underline{\omega}(M, N)(l)\}_{l \geq 0}$  has, as underlying spaces, the spaces  $\underline{\omega}(M, N)(l) := \omega(M, N)_l$ . A typical element in  $\underline{\omega}(M, N)(l)$  is denoted by a couple  $(g; n)$  so that  $g \in M^{\times l}$  and  $n \in N$ .

Let us define a stable wide left action. For  $k \geq 1$ ,  $1 \leq s \leq k$ ,  $l_i \geq 0$  and an injective order preserving map  $\beta : \{1 < \dots < s\} \rightarrow \{1 < \dots < k\}$ , we define

$$(7.2) \quad \sigma' : \underline{\omega}(M)(k) \times \underline{\omega}(M, N)(l_1) \times \dots \times \underline{\omega}(M, N)(l_s) \rightarrow \underline{\omega}(M, N)(l_1 + \dots + l_s + k - s)$$

by

$$\begin{aligned} \sigma'(f, (g_1; \mathbf{n}_1), \dots, (g_s; \mathbf{n}_s)) &= (\gamma(f; \mathbf{1}^{\beta(1)-1}, g_1, \mathbf{n}_1^{\beta(2)-\beta(1)-1}, g_2 \triangleright \mathbf{n}_1, \mathbf{n}_2 \mathbf{n}_1^{\beta(3)-\beta(2)-1}, \dots \\ &\quad \dots, g_s \triangleright \mathbf{n}_{s-1} \cdots \mathbf{n}_1, \mathbf{n}_s \cdots \mathbf{n}_1^{k-\beta(s)-1}); \mathbf{n}_s \cdots \mathbf{n}_1), \end{aligned}$$

where, for  $\mathbf{n}, \mathbf{m} \in \mathbb{N}$  and  $j \geq 0$ ,  $\mathbf{n}^j := \underbrace{\mathbf{n}, \dots, \mathbf{n}}_j$  and  $\mathbf{nm}^j := \underbrace{\mathbf{nm}, \dots, \mathbf{nm}}_j$  and where  $g \triangleright \mathbf{n}$  is the diagonal right action *i.e.* if  $g = (x_1, \dots, x_l)$  then  $g \triangleright \mathbf{n} = (x_1 \mathbf{n}, \dots, x_l \mathbf{n})$ .

For  $s = 1$  above, we get an infinitesimal left action

$$(7.3) \quad \begin{aligned} \sigma_i : \underline{\omega}(\mathcal{M})(k) \times \underline{\omega}(\mathcal{M}, \mathcal{N})(l) &\rightarrow \underline{\omega}(\mathcal{M}, \mathcal{N})(k+l-1) \\ (f, (g; \mathbf{n})) &\mapsto (\gamma(f; \underbrace{\mathbf{1}, \dots, \mathbf{1}}_{i-1}, g, \mathbf{n}, \dots, \mathbf{n}); \mathbf{n}), \end{aligned}$$

for  $1 \leq i \leq k$ .

For  $s = k$  above, we get a left action

$$(7.4) \quad \begin{aligned} \sigma : \underline{\omega}(\mathcal{M})(k) \times \underline{\omega}(\mathcal{M}, \mathcal{N})(l_1) \times \cdots \times \underline{\omega}(\mathcal{M}, \mathcal{N})(l_k) &\rightarrow \underline{\omega}(\mathcal{M}, \mathcal{N})(l_1 + \cdots + l_k) \\ (f, (g_1; \mathbf{n}_1), \dots, (g_k; \mathbf{n}_k)) &\mapsto (\gamma(f; g_1, g_2 \triangleright \mathbf{n}_1, g_3 \triangleright \mathbf{n}_2 \mathbf{n}_1, \dots, g_k \triangleright \mathbf{n}_{k-1} \cdots \mathbf{n}_1); \mathbf{n}_k \cdots \mathbf{n}_1), \end{aligned}$$

for all  $k \geq 1$  and  $l_i \geq 1$ .

For  $s = 0$  above, we set the inclusion

$$(7.5) \quad \begin{aligned} \iota : \underline{\omega}(\mathcal{M})(k) &\rightarrow \underline{\omega}(\mathcal{M}, \mathcal{N})(k) \\ f &\mapsto (f; \mathbf{1}), \end{aligned}$$

for all  $k \geq 1$ .

The right action is given by

$$(7.6) \quad \begin{aligned} \rho : \underline{\omega}(\mathcal{M}, \mathcal{N})(k) \times \underline{\omega}(\mathcal{M})(l_1) \times \cdots \times \underline{\omega}(\mathcal{M})(l_k) &\rightarrow \underline{\omega}(\mathcal{M}, \mathcal{N})(l_1 + \cdots + l_k) \\ ((f; \mathbf{n}), g_1, \dots, g_k) &\mapsto (\gamma(f; g_1, \dots, g_k); \mathbf{n}), \end{aligned}$$

where  $\gamma$  is the structural map of the operad  $\underline{\omega}(\mathcal{M})$ .

**Lemma 7.5.** *The maps  $\sigma'$ ,  $\iota$  and  $\rho$  endow  $\underline{\omega}(\mathcal{M}, \mathcal{N})$  with a stable wide bimodule structure over  $\underline{\omega}(\mathcal{M})$ . In particular, using the map  $\mathcal{A}s \hookrightarrow \underline{\omega}(\mathcal{M})$ , the module  $\underline{\omega}(\mathcal{M}, \mathcal{N})$  is a wide bimodule over  $\mathcal{A}s$ . With regard to the infinitesimal bimodule structure over  $\mathcal{A}s$ , the corresponding (via Lemma 6.1) cosimplicial space is  $\omega(\mathcal{M}, \mathcal{N})$ .*

*Proof.* The only non obvious property is that of wide left module structure. One has to prove that the following equation

$$\begin{aligned} \sigma'(f_1; \underbrace{\mathbf{1}, \dots, (g_1; \mathbf{n}_1), \dots, (g_s; \mathbf{n}_s), \dots, \mathbf{1}}_{i-1}, \sigma'(f_2; \mathbf{1}, \dots, (g_{s+1}; \mathbf{n}_{s+1}), \dots, (g_r; \mathbf{n}_r), \dots, \mathbf{1}), \dots, (g_t; \mathbf{n}_t), \dots, \mathbf{1}) \\ = \sigma'(f_1 \circ_i f_2; \mathbf{1}, \dots, (g_1; \mathbf{n}_1), \dots, (g_s; \mathbf{n}_s), \dots, (g_{s+1}; \mathbf{n}_{s+1}), \dots, (g_r; \mathbf{n}_r), \dots, (g_t; \mathbf{n}_t), \dots, \mathbf{1}) \end{aligned}$$

holds for all  $f_1 \in \underline{\omega}(\mathcal{M})(k)$ ,  $f_2 \in \underline{\omega}(\mathcal{M})(l)$  and  $(g_j; \mathbf{n}_j) \in \underline{\omega}(\mathcal{M}, \mathcal{N})$  and  $1 \leq i \leq k$ ,  $1 \leq t \leq k+l-1$ . Using the very definition of  $\sigma'$  in terms of  $\gamma$  and  $\triangleright$ , by the associativity of  $\gamma$ , proving the associativity for the left action  $\sigma'$  essentially amounts to prove that

$$(7.7) \quad \gamma(f; g_1, \dots, g_k) \triangleright \mathbf{n} = \gamma(f; g_1 \triangleright \mathbf{n}, \dots, g_k \triangleright \mathbf{n}),$$

for all  $f, g_1, \dots, g_k \in \underline{\omega}(\mathcal{M})$  and  $\mathbf{n} \in \mathbb{N}$ . This results from the following. Let us write  $x \triangleleft (y_1, \dots, y_l) := (xy_1, \dots, xy_l)$  for  $x, y_i \in \mathcal{M}$ . Let  $f = (x_1, \dots, x_k)$ . Then one has

$$\begin{aligned} \gamma(f; g_1, \dots, g_k) \triangleright \mathbf{n} &= (x_1 \triangleleft g_1, \dots, x_k \triangleleft g_k) \triangleright \mathbf{n} \\ &= ((x_1 \triangleleft g_1) \triangleright \mathbf{n}, \dots, (x_k \triangleleft g_k) \triangleright \mathbf{n}) \\ &= (x_1 \triangleleft (g_1 \triangleright \mathbf{n}), \dots, x_k \triangleleft (g_k \triangleright \mathbf{n})) \\ &= \gamma(f; g_1 \triangleright \mathbf{n}, \dots, g_k \triangleright \mathbf{n}). \end{aligned}$$

□

In virtue of the above Lemma 7.5, Proposition 6.11, Theorem 4.12 and Section 2.3, we have

**Theorem 7.6.** *Let  $(M, N)$  be a pair of topological monoids pointed at the unit  $*$  such that  $N$  is a submonoid of  $M$ . Let  $\delta$  be  $\delta_{\text{Top}}$  or  $\delta_{\mathbb{Z}}$ . Then the operad  $\text{Coend}_{\mathcal{RL}_2}(\delta)$ , which is weakly equivalent to the Swiss-Cheese operad  $\mathcal{SC}_2$  if  $\delta = \delta_{\text{Top}}$  and weakly equivalent to  $C_*(\mathcal{SC}_2)$  if  $\delta = \delta_{\mathbb{Z}}$ , acts on the couple of totalizations  $(\text{Tot}_{\delta}\omega(M), \text{Tot}_{\delta}\omega(M, N))$ .*

## 7.2 Action of $H_*(\mathcal{SC}_2)$

We write down a few operations on  $(\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M), \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N))$  when  $(M, N)$  is a pair of monoids pointed at the unit  $*$  with  $N$  submonoid of  $M$ .

The described operations are the sufficient to ensure the existence of an  $H_*(\mathcal{SC}_2)$ -algebra structure on  $(H_*\text{Tot}_{\delta_{\mathbb{Z}}}C_*(M), H_*\text{Tot}_{\delta_{\mathbb{Z}}}(C_*(M), C_*(N)))$ . They form a part of the  $\mathcal{RS}_2$ -algebra structure we make explicit in the next section.

Recall that an algebra over the homology operad  $H_*(\mathcal{SC}_2)$  is a triple  $(A, G, f)$  where  $A$  is an associative algebra;  $G$  is a Gerstenhaber algebra; and,  $f: G \rightarrow A$  is an algebra morphism such that  $f(G)$  belongs to the center of  $A$ , for example see [HL13, Proposition 3.2.1].

For a cosimplicial set  $K^*$ , the totalization  $\text{Tot}_{\delta_{\mathbb{Z}}}(K)$  is  $\prod_{n \geq 0} \mathbb{Z}[K^n]$  with the differential usually given as the sum of the cosimplicial face maps. We denote by  $K(n)$  the  $n$ -th component  $\mathbb{Z}[K^n]$ ; the degree of an element  $f \in K(n)$  is  $n$  and it is denoted by  $|f|$ .

Let us recall a consequence of McClure and Smith's work.

**Theorem 7.7** ([MS04]). *Let  $\mathcal{O}$  be a multiplicative operad. Then  $\text{Tot}_{\delta_{\mathbb{Z}}}\mathcal{O}$  is an  $E_2$ -algebra.*

In particular,  $\text{Tot}_{\delta_{\mathbb{Z}}}\mathcal{O}$  is endowed with a product,  $\cup$ , commutative up to a chain homotopy  $E_{1,1}$ .

The product  $\cup$  on  $\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M)$  is given by

$$(7.8) \quad f \cup g := (\mu \circ_2 g) \circ_1 f,$$

for  $f, g \in \omega(M)$ . We define an associative dg-product  $\sqcup$  on  $\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$  by

$$(7.9) \quad u \sqcup v := \sigma(\mu; u, v),$$

for  $u, v \in \omega(M, N)$  where  $\sigma$  is the map defined in (7.4).

In other words, for  $u = (f; m)$  and  $v = (g; n)$  one has

$$u \sqcup v = (f, g \triangleright m; nm).$$

The inclusion  $\text{inc}: \text{Tot}_{\delta_{\mathbb{Z}}}\omega M \hookrightarrow \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$  is given on its components by

$$(7.10) \quad \begin{aligned} \text{inc}: \omega(M)(k) &\rightarrow \omega(M, N)(k) \\ (a_1, \dots, a_k) &\mapsto (a_1, \dots, a_k; 1) \end{aligned}$$

and it is a chain map.

From this, we easily deduce that

$$\text{inc}(f \cup g - (-1)^{|f||g|}g \cup f) \sqcup u = \text{inc}(\partial E_{1,1}(f; g)) \sqcup u = \partial(\text{inc}(E_{1,1}(f; g)) \sqcup u),$$

for any homogeneous elements  $f, g \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M)$  and  $u \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$ . Finally, we define a chain homotopy  $H$  such that

$$\text{inc}(f) \sqcup u - (-1)^{|f||u|}u \sqcup \text{inc}(f) = \partial H(f, u)$$

for any two homogeneous elements  $f \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M)$  and  $u \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$ . We recall the infinitesimal left action  $\sigma_i$  defined in (7.3). The homotopy  $H$  is defined as

$$(7.11) \quad H(f, (g; n)) = \sum_{1 \leq i \leq k} (-1)^{i+|g|+|f||g|} \sigma_i(f; (g; n)),$$

for any  $f \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M)$  of degree  $k$  and  $(g; n) \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$ .

We have shown, in particular, the following.

**Proposition 7.8.** *Let  $(M, N)$  be a pair of 1-connected monoids pointed at the unit and such that  $N$  a sub-monoid of  $M$ . Then the above operations (7.8), (7.9) and (7.10) induce an  $H_*(\mathcal{SC}_2)$ -algebra structure on the pair  $(H_*\text{Tot}_{\delta_Z}\omega(M), H_*\text{Tot}_{\delta_Z}\omega(M, N))$ .*

*Remark 7.9.* For two paths  $\tau, \tau' : [0, 1] \rightarrow M$ , let us denote by  $\tau \cdot \tau'$  the path  $\tau \cdot \tau'(t) = \tau(t)\tau'(t)$  given by the product of  $M$ . For  $x \in M$ , one denotes by  $c_x$  the constant path at  $x$ .

Let us define:

- a (associative and commutative all two up to homotopy) product  $\mu_c$  for  $\Omega(M)$  as the concatenation  $\mu_c(\gamma, \gamma') = \gamma\gamma'$ ;
- a (associative up to homotopy) product  $\mu_o$  for  $\Omega(M, N)$  as  $\mu_o(\gamma, \gamma') = \gamma(\gamma' \cdot c_{\gamma(1)})$ ; and,
- an inclusion  $\Omega M \hookrightarrow \Omega(M, N)$  as the canonical inclusion of loops.

The above inclusion and multiplication  $\mu_o$  provide a left action  $l : \Omega M \times \Omega(M, N) \rightarrow \Omega(M, N)$  given by concatenation  $l(\gamma, \tau) = \gamma\tau$ . Since,  $M, N$  are monoids, it is easy to see that the right action  $r : \Omega(M, N) \times \Omega M \rightarrow \Omega(M, N)$  given by translation/concatenation  $r(\tau, \gamma) = \tau(\gamma \cdot c_{\tau(1)})$  is homotopic to the left one  $l$ .

One can show that these operations induce an  $H_*(\mathcal{SC}_2)$ -algebra structure on the pair  $(H_*\Omega(M), H_*\Omega(M, N))$ . It would be interesting to compare this structure with that one of Proposition 7.8.

### 7.3 Action of the whole operad $\mathcal{RS}_2$

The inclusion  $\mathcal{RS}_2 \hookrightarrow \text{Coend}_{\mathcal{RL}_2}(\delta_Z)$  implies an action of the operad  $\mathcal{RS}_2$  on  $(\text{Tot}_{\delta_Z}\omega(M), \text{Tot}_{\delta_Z}\omega(M, N))$ . We describe this action.

To do that it is sufficient to write down explicitly the operations corresponding to the generators of  $\mathcal{RS}_2$ .

The closed part of  $\mathcal{RS}_2$  acts as described in [Kad05]. We recall from Proposition 5.3 that  $\mathcal{RS}_2$  is generated by  $\mu_c, T_k, \mu_o, T_j$  and  $\text{inc}$ . By a slight abuse of notation, for an element  $T \in \mathcal{RS}_2(c_1, \dots, c_k; c)$  we denote also by  $T$  the corresponding operation

$$T : (\text{Tot}_{\delta_Z}\omega_{c_1})_* \otimes \dots \otimes (\text{Tot}_{\delta_Z}\omega_{c_k})_* \rightarrow (\text{Tot}_{\delta_Z}\omega_c)_{*+d},$$

where  $\omega_{c_i}$  is either  $\omega M$  if  $c_i = c$  or  $\omega(M, N)$  if  $c_i = o$ .

The multiplication  $\mu_c \in \mathcal{RS}_2(c, c; c)_0$  acts as the product  $\cup$  in (7.8).

The trees  $T_k \in \mathcal{RS}_2(\underbrace{c, \dots, c}_k; c)_{2k+1}$  act as follows. For any  $f \in \omega(M)(n)$  and  $g_1, \dots, g_{k-1} \in$

$\text{Tot}_{\delta_Z}\omega(M, N)$ , one has

$$T_k(f; g_1, \dots, g_{k-1}) = \sum_{\substack{0 \leq i_1 + \dots + i_{k-1} \leq n-k+1 \\ i_s \geq 0}} \pm \gamma(f; 1^{i_1}, g_1, 1^{i_2}, g_2, \dots, g_{k-1}, 1^{n-i_1+\dots+i_{k-1}}).$$

The generator  $\mu_o \in \mathcal{RS}_2(o, o; o)_0$  acts as the product  $\sqcup$  in (7.9).

The generator  $T_j \in \mathcal{RS}_2(c, \underbrace{o, \dots, o}_{j-1}; o)_{2j+1}$  acts as follows. For each homogeneous elements  $f \in$

$\text{Tot}_{\delta_Z}\omega(M)$  and  $(h_1; n_1), \dots, (h_{j-1}; n_{j-1}) \in \text{Tot}_{\delta_Z}\omega(M, N)$ , the element  $T_j(f; (h_1; n_1), \dots, (h_{j-1}; n_{j-1}))$  is given as

$$\sum_{\substack{0 \leq i_1 + i_2 + \dots + i_j \leq n-j \\ i_s \geq 0}} \pm \sigma(f; 1^{i_1}, h_1, 1^{i_2}, h_2, \dots, h_j, 1^{n-i_1+\dots+i_j}),$$

where  $\sigma$  is the map defined in (7.4). In other words,  $T_j(f; (h_1; n_1), \dots, (h_{j-1}; n_{j-1}))$  is the sum over  $0 \leq i_1 + i_2 + \dots + i_j \leq n-j$  of

$$(\gamma(f; 1^{i_1}, h_1, \underbrace{n_1, \dots, n_1}_{i_2}, h_2 \triangleright n_1, \dots, h_j \triangleright (n_{j-1} \cdots n_2 n_1), \underbrace{n_j \cdots n_2 n_1, \dots, n_j \cdots n_2 n_1}_{n-i_1+\dots+i_j}; n_j \cdots n_2 n_1).$$

The generator  $\text{inc} \in \mathcal{RS}_2(c; o)_0$  is the canonical inclusion of  $\text{Tot}_{\delta_Z}\omega(M)$  into  $\text{Tot}_{\delta_Z}\omega(M, N)$ .

## 8 Relative cobar construction

For  $C$  a 1-reduced (coaugmented, counital, coassociative) dg-coalgebra, we denote by  $\underline{\Omega}C$  its cobar construction as defined in [FHT92, p.538].

**Definition 8.1.** Let  $(C, N)$  be a pair of dg-module such that  $C$  is a 1-reduced dg-coalgebra (coaugmented, counital, coassociative) and  $N$  is a left  $C$ -comodule. The *relative cobar construction of  $(C, N)$*  is the free graded  $\underline{\Omega}C$ -module on  $N$ ,  $\underline{\Omega}C \otimes N$ , with the following differential  $D$ .

The differential is given by:  $D = d_0 + d_1$ , where  $d_0$  is the Koszul differential of the module  $\underline{\Omega}C \otimes N$  induces by  $d_C$  and  $d_N$ ; and  $d_1$  is the quadratic part induces by the quadratic part of the cobar construction  $\underline{\Omega}C$  and a twist with the reduced  $C$ -comodule structure of  $N$ . We write the reduced coproducts  $\overline{\nabla}_C(c) = \sum c^1 \otimes c^2 = \nabla_C(c) - c \otimes 1 - 1 \otimes c$  and  $\overline{\nabla}_N(n) = \sum z^1 \otimes n^2 = \nabla_N(n) - 1 \otimes n$ . One has:

$$\begin{aligned} d_1(s^{-1}c) &= \sum (-1)^{c^1} s^{-1}c^1 \otimes s^{-1}c^2, \quad c \in C_{>1} \\ d_1(n) &= \sum s^{-1}z^1 \otimes n^2, \quad n \in N, \end{aligned}$$

where  $s^{-1}$  denotes the desuspension *i.e.*  $(s^{-1}C)_i = C_{i+1}$ . The augmentation  $\epsilon : \underline{\Omega}C \otimes N \rightarrow k$  is given by  $\epsilon = \epsilon_{\underline{\Omega}C} \cdot \epsilon_N$ . The relative cobar construction  $(\underline{\Omega}C \otimes N, D, \epsilon)$  is denoted by  $\underline{\Omega}(C, N)$ .

The action of  $\underline{\Omega}C$  is given by concatenation

$$\begin{aligned} \alpha_{\underline{\Omega}(C, N)} : \underline{\Omega}C \otimes \underline{\Omega}(C, N) &\rightarrow \underline{\Omega}(C, N) \\ (s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_k) \otimes (s^{-1}x_{k+1} \otimes \cdots \otimes s^{-1}x_r \otimes n) &\mapsto s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_r \otimes n, \end{aligned}$$

and yields on  $\underline{\Omega}(C, N)$  a  $\underline{\Omega}C$ -module structure in the category of dg-modules.

*Data 8.2.* Let us fix a pair of module maps  $(f : C \rightarrow A, g : N \rightarrow M)$ , where:

- $C$  is a 1-connected coaugmented dg-coalgebra;
- $A$  is an augmented dg-algebra;
- $N$  is a left dg-comodule over  $C$  with coaction  $c_N : N \rightarrow C \otimes N$ ;
- $M$  is a left dg-module over  $A$  with action  $\alpha_M : A \otimes M \rightarrow M$ .

We denote by  $\overline{f} : \underline{\Omega}C \rightarrow A$  the induced algebra morphism of  $f$ . Let

$$(8.1) \quad \overline{(f, g)} : \underline{\Omega}C \otimes N \rightarrow M$$

be the map  $\overline{f} \cdot g$  whose the  $k$ -th component is

$$C^{\otimes k} \otimes N \xrightarrow{\overline{f} \otimes g} A \otimes M \xrightarrow{\alpha_M} M.$$

As an immediate consequence,  $\overline{(f, g)} \alpha_{\underline{\Omega}(C, N)} = \alpha_M(\overline{f} \otimes g)$ , so that  $\overline{(f, g)}$  is  $\overline{f}$ -equivariant.

Now, we define a relative version of twisting cochain. We recall first that the set of twisting cochains  $\text{Tw}(C, A)$  is the set of maps  $f : C \rightarrow A$  satisfying  $f \cup f = \partial f$ , where  $f_1 \cup f_2 = \mu_A(f_1 \otimes f_2) \nabla_C$  and  $\partial$  denotes the usual differential in  $\text{Hom}(C, A)$ .

**Definition 8.3.** Let  $(f : C \rightarrow A, g : N \rightarrow M)$  a pair as in *Data 8.2*. The pair  $(f, g)$  is called *relative twisting* if  $f$  is a twisting cochain and if

$$(8.2) \quad \partial g = \alpha_M(f \otimes g) c_N.$$

**Proposition 8.4.** *Let  $(f, g)$  be a pair as in Data 8.2. Suppose  $f$  is a twisting cochain. Then the following propositions are equivalent.*

- The pair  $(f, g)$  is a relative twisting pair.
- The  $\overline{f}$ -equivariant map  $\overline{(f, g)} : \underline{\Omega}C \otimes N \rightarrow M$  is a morphism of dg-modules.

### 8.1 Cubical model for relative loop spaces

In this section we define a model for Moore relative loop spaces. Given a pair  $(X, Y)$  of topological spaces, pointed at  $*$  and such that  $* \subset Y \subset X$ , the Moore relative loop space  $\Omega_M(X, Y)$  is defined as

$$\Omega_M(X, Y) = \{(\gamma, r) \in X^{[0, +\infty)} \times [0, +\infty) \mid \gamma(0) = * \text{ and } \gamma(t) = \gamma(r) \in Y \forall t \geq r\}.$$

We denote simply  $\Omega_M(X, *)$  by  $\Omega_M X$  which is the loop space of  $X$ . The Moore path space  $P_M X$  is defined as

$$P_M X = \{(\gamma, r) \in X^{[0, +\infty)} \times [0, +\infty) \mid \gamma(0) = * \text{ and } \gamma(t) = \gamma(r) \forall t \geq r\}.$$

Let us start with a few definitions about cubical sets. Those are extracted from [KS05, Section 2.4].

**Definition 8.5.** A cubical set  $Q$  is a graded set  $Q = \{Q_n\}_{n \geq 0}$  with face operators  $d_i^\epsilon : Q_n \rightarrow Q_{n-1}$ ,  $n \geq 1$ ,  $1 \leq i \leq n$ ,  $\epsilon = 0, 1$ , and degeneracy operators  $\eta_i : Q_n \rightarrow Q_{n+1}$ ,  $n \geq 0$ ,  $1 \leq i \leq n+1$ , satisfying:

$$\begin{aligned} d_j^\epsilon d_i^{\epsilon'} &= d_i^{\epsilon'} d_{j+1}^\epsilon & i \leq j \\ d_i^\epsilon \eta_j &= \begin{cases} \eta_{j-1} d_i^\epsilon & i < j \\ 1 & i = j \\ \eta_j d_{i-1}^\epsilon & i > j \end{cases} \\ \eta_i \eta_j &= \eta_{j+1} \eta_i & i \leq j. \end{aligned}$$

The product of two cubical sets  $Q$  and  $Q'$ , is

$$(8.3) \quad Q \times Q' = \{(Q \times Q')_n = \bigcup_{p+q=n} Q_p \times Q'_q\} / \sim$$

where  $(\eta_{p+1}(x), y) \sim (x, \eta_1(y))$  for  $(x, y) \in Q_p \times Q'_q$ . The face and degeneracy operators are induced by those of  $Q$  and  $Q'$  in the obvious way.

For  $n \geq 0$ , let  $I^n$  be the cube of dimension  $n$  given as the cartesian product of the interval  $I = [0, 1]$ ; then the cube  $I^0$  is a point. Let  $X$  be a pointed connected topological space. The cubical set  $\text{Sing}^\square X = \{\text{Sing}_n^\square X\}_{n \geq 0}$  is formed by the continuous maps  $I^n \rightarrow X$ .

Let  $Q$  be a cubical set. We denote by  $C_*^\square Q$  the normalized chain complex of  $Q$ . For a topological space  $X$ ,  $C_*^\square X$  denotes the normalized chain complex of  $\text{Sing}^\square X$ .

**Definition 8.6.** A *monoidal cubical set* is a cubical set  $Q$  with an associative cubical multiplication  $\mu : Q \times Q \rightarrow Q$  for which a distinguished element  $e \in Q_0$  is the unit.

The chain complex  $C_*^\square Q$  of a monoidal cubical set is a dg-bialgebra.

**Definition 8.7.** Let  $Q$  be a monoidal cubical set. A cubical set  $P$  is a  $Q$ -*module* if there is an associative cubical map  $Q \times P \rightarrow P$  with the unit of  $Q$  acting as identity.

The chain complex  $C_*^\square P$  of a  $Q$ -monoidal cubical set  $P$  is a  $C_*^\square Q$ -module.

**Definition 8.8.** For  $n \geq 0$ , a simplicial set  $K$  is  $n$ -*reduced* if  $K_0 = \dots = K_n = \{*\}$ .

Let  $X$  be a 1-connected space pointed at  $*$ . Let  $\text{Sing}^1 X = \{\text{Sing}_n^1 X\}_{n \geq 0}$  be the simplicial set formed by the singular simplexes  $\sigma_n : \Delta^n \rightarrow X$  such that  $\sigma_n$  sends the 1-skeleton of  $\Delta^n$  to the base point  $* \in X$ . For any topological space  $Z$ , let us denote by  $C_*^1 Z$  the normalized chain complex generated by singular simplexes whose the 1-skeleton is sent to the base point of  $Z$ ; that is  $C_*^1 Z = C_*(\text{Sing}^1 Z)$ .

In [KS05] the cubical cobar construction  $\Omega^\square(\text{Sing}^1 X)$  on the simplicial set  $\text{Sing}^1 X$  is constructed. It is a monoidal cubical set; it is shown to be homotopically equivalent to the monoidal cubical set  $\text{Sing}^\square \Omega_M X$ , cf. [KS05, Theorem 5.1]. In particular, taking the cubical chain complexes, one obtains Adams' morphism [Ada56]

$$\Phi : C_*^\square \Omega^\square(\text{Sing}^1 X) = \underline{\Omega} C_*^1 X \rightarrow C_*^\square \Omega_M X,$$

that is a quasi-isomorphism of dg-algebras. In fact, since  $\Phi$  comes from a map of cubical sets

$$(8.4) \quad \phi : \Omega^\square(\text{Sing}^1 X) \rightarrow \text{Sing}^\square \Omega_M X,$$

it is immediate that  $\Phi$  is a morphism of dg-coalgebras. Indeed, given a cubical set  $Q$ , its cubical chain complex  $C^\square(Q)$  is a dg-coalgebra; the coproduct we consider is the Serre diagonal defined using the face operator of  $Q$  (see [KS05, 2.4 equation (3)]). The compatibility of  $\phi$  with the monoidal structures implies that Adams' morphism is a quasi-isomorphism of dg-bialgebras.

The construction of  $\phi$  is realized by an induction process involving another map  $p : P^\square \text{Sing}^1 X \rightarrow \text{Sing}^1 PX$  between the *cubical path construction* and the cubical set of the path space  $PX$ . For a simplicial set  $K$ , the cubical set  $P^\square K$  is defined as the *twisted cartesian product* ([KS05, Definition 4.2])

$$(8.5) \quad P^\square(K) := \Omega^\square K \times_{\tau_u} K,$$

where  $\tau_u : K \rightarrow \Omega^\square K$  is the *universal truncating twisting function* [KS05, Section 4]. The maps  $p$  and  $\phi$  satisfy  $p(\sigma', \sigma) = \phi(\sigma') \cdot p(e, \sigma)$  where  $\cdot$  stands for the left action  $\Omega_M X \times P_M X \rightarrow P_M X$  and where  $e \in \Omega^\square X$  is the unit of the monoidal cubical set, see [KS05, Proof of Theorem 5.1].

For a pair  $(K, L)$  of 1-reduced simplicial sets such that  $L$  is a sub simplicial set of  $K$ , we define the *relative cubical cobar construction*  $\Omega^\square(K, L)$  as the twisted cartesian product

$$(8.6) \quad \Omega^\square(K, L) := \Omega^\square K \times_\tau L,$$

where  $\tau : L \rightarrow \Omega^\square K$  is the inclusion  $L \subset K$  composed with the universal truncating twisting function  $\tau_u : K \rightarrow \Omega^\square K$ .

It is easy to show that the concatenation  $\Omega^\square K \times \Omega^\square(K, L) \rightarrow \Omega^\square(K, L)$  makes  $\Omega^\square(K, L)$  into a cubical  $\Omega^\square K$ -module.

Let  $(X, Y)$  be a pair of 1-connected pointed spaces such that  $Y \subset X$ . We define a map of cubical sets

$$(8.7) \quad \psi : \Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) \rightarrow C_*^\square \Omega_M(X, Y)$$

as follows. The inclusion  $Y \subset X$  gives rise to an inclusion of cubical sets  $\Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) \subset P^\square(\text{Sing}^1 Y)$ . The map  $\psi$  is defined as the restriction  $p|_{\Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y)}$  and satisfies  $\psi(\sigma', \sigma) = \phi(\sigma') \cdot \psi(e, \sigma)$  where  $\cdot$  stands for the left action induced by  $\Omega_M X \times \Omega_M(X, Y) \rightarrow \Omega_M(X, Y)$ . The map  $\psi$  is then a map of cubical modules. The fact that  $\psi$  is a homotopy equivalence follows from the long exact sequences in homotopy induced by

$$\begin{aligned} \Omega_M X \rightarrow \Omega_M(X, Y) \rightarrow Y, \text{ and} \\ |\Omega^\square \text{Sing}^1 X| \rightarrow |\Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y)| \rightarrow |\text{Sing}^1 Y|, \end{aligned}$$

using that  $\phi$  is a homotopy equivalence.

By construction (see [KS05, (4) iii]), we have

$$C_* \Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) = \underline{\Omega}(C_*^1 X, C_*^1 Y).$$

Consequently, we obtain

**Proposition 8.9.** *Let  $(X, Y)$  be a pair of 1-connected topological spaces pointed at  $*$  such that  $*$   $\subset$   $Y \subset X$ . The map  $\psi : \Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) \rightarrow \text{Sing}^1 \Omega_M(X, Y)$  induces a  $\Phi$ -equivariant morphism  $\Psi : C_* \Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) = \underline{\Omega}(C_*^1 X, C_*^1 Y) \rightarrow C_*^\square \Omega_M(X, Y)$  that is a quasi-isomorphism of dg-coalgebras.*

## 8.2 Unreduced (relative) cobar construction

With regard to its structure, this section is analogous to Section 7.1: we exhibit a couple  $(\mathcal{M}_B, \mathcal{Z}_{B,C})$  giving rise to the both *unreduced* cobar and relative cobar constructions over a couple of a coalgebra/comodule  $(B, C)$  in the category of algebras; the couple  $(\mathcal{M}_B, \mathcal{Z}_{B,C})$  is shown to be an algebra over  $\mathcal{RL}_2$ .

Let  $(B, \nabla_B)$  be an (ungraded) unital/counital bialgebra with counit  $\epsilon$  (*i.e.*  $B$  is a counital coalgebra in the category of unital algebras) and let  $(C, \nabla_C)$  be a left  $B$ -comodule in the category of unital algebras.

Let us consider the *unreduced cobar construction*  $\underline{\Omega}_u B$  given by

$$\underline{\Omega}_u B = (TB, D)$$

where  $D$  is the differential

$$D(\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_n) = 1 \otimes \mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_n + \sum_{i=1}^n (-1)^i \mathbf{b}_1 \otimes \dots \otimes \nabla_{\mathbf{B}}(\mathbf{b}_i) \otimes \dots \otimes \mathbf{b}_n \\ + (-1)^{n+1} \mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_n \otimes 1,$$

for  $\mathbf{b}_i \in \mathbf{B}$ . In the same fashion, we define  $\underline{\mathcal{Q}}_{\mathbf{u}}(\mathbf{B}, \mathbf{C})$  to be the *unreduced relative cobar construction*

$$\underline{\mathcal{Q}}_{\mathbf{u}}(\mathbf{B}, \mathbf{C}) = (\mathbf{TB} \otimes \mathbf{C}, D')$$

with differential

$$D'(\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_n \otimes \mathbf{c}) = 1 \otimes \mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_n \otimes \mathbf{c} + \\ + \sum_{i=1}^n (-1)^i \mathbf{b}_1 \otimes \dots \otimes \nabla_{\mathbf{B}}(\mathbf{b}_i) \otimes \dots \otimes \mathbf{b}_n \otimes \mathbf{c} + (-1)^{n+1} \mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_n \otimes \nabla_{\mathbf{C}}(\mathbf{c}),$$

for  $\mathbf{b}_i \in \mathbf{B}$ ,  $\mathbf{c} \in \mathbf{C}$ .

Let us fix some notations. For  $k \geq 0$ ,  $\mathbf{B}^{\otimes k}$  is an algebra for the product  $(\mathbf{a}_1 \otimes \dots \otimes \mathbf{a}_k) \cdot (\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_k) = \mathbf{a}_1 \mathbf{b}_1 \otimes \dots \otimes \mathbf{a}_k \mathbf{b}_k$ . We denote by  $\nabla_{\mathbf{B}}^{(0)} = \epsilon$ ,  $\nabla_{\mathbf{B}}^{(1)} = \nabla_{\mathbf{B}}$ , and  $\nabla_{\mathbf{B}}^{(k)} = (\nabla_{\mathbf{B}} \otimes \text{id}) \nabla_{\mathbf{B}}^{(k-1)}$  for  $k \geq 2$ . Since  $\mathbf{B}$  is a bialgebra, the tensor product  $\mathbf{B}^{\otimes k}$ ,  $k \geq 0$ , is: a left  $\mathbf{B}$ -module with  $\mathbf{b} \triangleleft (\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_k) = \nabla_{\mathbf{B}}^{(k-1)}(\mathbf{b}) \cdot (\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_k)$  where the dot  $\cdot$  stands for the multiplication in  $\mathbf{B}^{\otimes k}$ ; and, a right  $\mathbf{B}$ -module for  $(\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_k) \triangleright \mathbf{b} = (\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_k) \cdot \nabla_{\mathbf{B}}^{(k-1)}(\mathbf{b})$ . For  $\mathbf{c} \in \mathbf{C}$ , let us denote  $\nabla_{\mathbf{C}}(\mathbf{c}) = \sum_{(c)} z_c^{(1)} \otimes \mathbf{c}^{(2)}$ , and more generally,  $(\nabla_{\mathbf{B}}^{(k-1)} \otimes \text{id}) \nabla_{\mathbf{C}}(\mathbf{c}) = \sum_{(c)} z_c^{(1)} \otimes \dots \otimes z_c^{(k)} \otimes \mathbf{c}^{(k+1)}$ .

Let us define  $\mathcal{M}_{\mathbf{B}}$  to be the following multiplicative operad (for instance see [Men04]). For  $k \geq 0$ , we set  $\mathcal{M}_{\mathbf{B}}(k) = \mathbf{B}^{\otimes k}$ .

The partial composition maps

$$\circ_i^{\mathbf{B}} : \mathcal{M}_{\mathbf{B}}(k) \otimes \mathcal{M}_{\mathbf{B}}(l) \rightarrow \mathcal{M}_{\mathbf{B}}(k+l-1),$$

for  $1 \leq i \leq k$ , are given by

$$(\mathbf{a}_1 \otimes \dots \otimes \mathbf{a}_k) \circ_i^{\mathbf{B}} (\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_l) := (\mathbf{a}_1 \otimes \dots \otimes \mathbf{a}_{i-1} \otimes \mathbf{a}_i \triangleleft (\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_l) \otimes \mathbf{a}_{i+1} \otimes \dots \otimes \mathbf{a}_k).$$

We denote by  $\gamma^{\mathbf{B}}$  the corresponding structural map of this operad. The multiplication  $\mu \in \mathcal{M}_{\mathbf{B}}(2)$  is  $\mu = 1 \otimes 1$ . The  $\delta_{\mathbb{Z}}$ -totalization of  $\mathcal{M}_{\mathbf{B}}$  is the unreduced cobar construction  $\underline{\mathcal{Q}}_{\mathbf{u}} \mathbf{B}$ .

We define  $\mathcal{Z}_{\mathbf{B}, \mathbf{C}}$  as  $\mathcal{Z}_{\mathbf{B}, \mathbf{C}}(k) = \mathbf{B}^{\otimes k} \otimes \mathbf{C}$  for  $k \geq 0$ .

Let us define a wide left action of  $\mathcal{M}_{\mathbf{B}}$  on  $\mathcal{Z}_{\mathbf{B}, \mathbf{C}}$ .

For  $k \geq 1$ ,  $1 \leq s \leq k$ ,  $l_i \geq 0$  and an injective order preserving map  $\beta : \{1 < \dots < s\} \rightarrow \{1 < \dots < k\}$ , we define

$$(8.8) \quad \lambda'_{\beta}{}^{\mathbf{B}} : \mathcal{M}_{\mathbf{B}}(k) \otimes \mathcal{Z}_{\mathbf{B}, \mathbf{C}}(l_1) \otimes \dots \otimes \mathcal{Z}_{\mathbf{B}, \mathbf{C}}(l_s) \rightarrow \mathcal{Z}_{\mathbf{B}, \mathbf{C}}(l_1 + \dots + l_s + k - s)$$

by

$$\lambda'_{\beta}{}^{\mathbf{B}}(f, (\mathbf{g}_1; \mathbf{c}_1), \dots, (\mathbf{g}_s; \mathbf{c}_s)) = \\ (\gamma(f; \mathbf{1}^{\beta(1)-1}, \mathbf{g}_1, z_{c_1}^{(1)}, \dots, z_{c_1}^{(\beta(2)-\beta(1)-1)}, \mathbf{g}_2 \triangleright z_{c_1}^{(\beta(2)-\beta(1))}, z_{c_2}^{(1)} z_{c_1}^{(\beta(2)-\beta(1)+1)}, \dots \\ \dots, \mathbf{g}_s \triangleright z_{c_{s-1}}^{(1)} \dots z_{c_1}^{(\beta(s)-1)}, \dots, z_{c_s}^{(k-\beta(s))} \dots z_{c_1}^{(k-\beta(1))}); \\ \mathbf{c}_s^{(k-\beta(s)+1)} \mathbf{c}_{s-1}^{(k-\beta(s)+2)} \dots \mathbf{c}_1^{(k-\beta(1)+1)}),$$

where  $\mathbf{1}^{\beta(1)-1} := \underbrace{1, \dots, 1}_{\beta(1)-1}$ .

For  $s = 1$  above, we get an infinitesimal left action

$$(8.9) \quad \lambda_i^{\mathbf{B}} : \mathcal{M}_{\mathbf{B}}(k) \otimes \mathcal{Z}_{\mathbf{B}, \mathbf{C}}(l) \rightarrow \mathcal{Z}_{\mathbf{B}, \mathbf{C}}(k+l-1)$$

given by

$$\lambda_i^B \left( (\mathbf{a}_1 \otimes \dots \otimes \mathbf{a}_k) \otimes (\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_l \otimes \mathbf{c}) \right) = \sum_{(c)} (\mathbf{a}_1 \otimes \dots \otimes \mathbf{a}_{i-1} \otimes \mathbf{a}_i \triangleleft (\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_l) \otimes (\mathbf{a}_{i+1} \otimes \dots \otimes \mathbf{a}_k) \triangleright z_c^{(1)} \otimes \mathbf{c}^{(2)}),$$

for  $1 \leq i \leq k$ .

For  $s = k$  above, we get a left action

$$(8.10) \quad \lambda^B : \mathcal{M}_B(k) \otimes \mathcal{Z}_{B,C}(l_1) \otimes \dots \otimes \mathcal{Z}_{B,C}(l_k) \rightarrow \mathcal{Z}_{B,C}(l_1 + \dots + l_k)$$

defined by

$$\lambda^B(f, (g_1; c_1), \dots, (g_k; c_k)) := (\gamma(f; g_1, g_2 \triangleright z_{c_1}^{(1)}, g_3 \triangleright z_{c_2}^{(1)} z_{c_1}^{(2)}, \dots, g_k \triangleright z_{c_{k-1}}^{(1)} \dots z_{c_1}^{(k-1)}); c_k c_{k-1}^{(2)} \dots c_1^{(k)}).$$

For  $s = 0$  above, we set an inclusion

$$(8.11) \quad \iota^B : \mathcal{M}_B(k) \rightarrow \mathcal{Z}_{B,C}(k) \\ f \mapsto (f; \mathbf{1}).$$

The right action is given by

$$(8.12) \quad \rho^B : \mathcal{Z}_{B,C}(k) \otimes \mathcal{M}_B(l_1) \otimes \dots \otimes \mathcal{M}_B(l_k) \rightarrow \mathcal{Z}_{B,C}(l_1 + \dots + l_k) \\ ((f; c), g_1, \dots, g_k) \mapsto (\gamma^B(f; g_1, \dots, g_k); c),$$

for any  $(f; c) = \mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_k \otimes c \in \mathcal{Z}_{B,C}(k)$  and  $g_i = \mathbf{b}_1^i \otimes \dots \otimes \mathbf{b}_{l_i}^i \in \mathcal{M}_B(l_i)$ .

**Lemma 8.10.** *The maps  $\lambda'^B$ ,  $\iota^B$  and  $\rho^B$  endow  $\mathcal{Z}_{B,C}$  with a stable wide bimodule structure over  $\mathcal{M}_B$ . In particular, using the map  $\mathcal{A}_s \hookrightarrow \mathcal{M}_B$ , the module  $\mathcal{Z}_{B,C}$  is a wide bimodule over  $\mathcal{A}_s$ . With regard to the infinitesimal bimodule structure over  $\mathcal{A}_s$ , the  $\delta_{\mathbb{Z}}$ -totalization of the corresponding (via Lemma 6.1) cosimplicial complex of  $\mathcal{Z}_{B,C}$  is  $\underline{\Omega}_u(B, C)$ .*

*Proof.* One has to show that the equation

$$(8.13) \quad \lambda'^B(f_1; \underbrace{1, \dots, (g_1; \mathbf{n}_1), \dots, (g_s; \mathbf{n}_s), \dots, 1}_{i-1}, \lambda'^B(f_2; 1, \dots, (g_{s+1}; \mathbf{n}_{s+1}), \dots, (g_r; \mathbf{n}_r), \dots, 1), \dots, (g_t; \mathbf{n}_t), \dots, 1) \\ = \lambda'^B(f_1 \circ_i^B f_2; 1, \dots, (g_1; \mathbf{n}_1), \dots, (g_s; \mathbf{n}_s), \dots, (g_{s+1}; \mathbf{n}_{s+1}), \dots, (g_r; \mathbf{n}_r), \dots, (g_t; \mathbf{n}_t), \dots, 1)$$

holds for all  $f_1 \in \mathcal{M}_B(k)$ ,  $f_2 \in \mathcal{M}_B(l)$  and  $(g_j; \mathbf{n}_j) \in \mathcal{Z}_{B,C}$  and  $1 \leq i \leq k$ ,  $1 \leq t \leq k + l - 1$ . One easily show that a similar equation to (7.7) holds here. The compatibility of  $\lambda'^B$  with the operadic structure of  $\mathcal{M}_B$  follows from the facts that  $\gamma$  is associative (what is essentially due to the fact that  $B$  is a bialgebra) and that  $C$  is a  $B$ -comodule in the category of unital algebras. More explicitly, the last requirement is used to show that the ‘‘coefficients’’ (i.e. the terms in  $C$ ) of the two terms from (8.13) coincide:

$$c_t^{(w_t)} \dots c_{r+1}^{(w_{r+1})} (c_r^{(v_r)} \dots c_{s+1}^{(v_{s+1})})^{(k-i+1)} \cdot c_s^{(w_s)} \dots c_1^{(w_1)} \\ = c_t^{(w_t)} \dots c_{r+1}^{(w_{r+1})} c_r^{(w_r)} \dots c_{s+1}^{(w_{s+1})} c_s^{(w_s)} \dots c_1^{(w_1)},$$

where  $w_a$  denotes the distance from  $(g_a; \mathbf{n}_a)$  to the last term:  $w_a = k + l - p_a$  where  $p_a$  is the position (from the left) of  $(g_a; \mathbf{n}_a)$  and  $v_a = i + l - p_a$ .  $\square$

**Theorem 8.11.** *Let  $B$  be a unital/counital bialgebra and let  $C$  be a  $B$ -comodule in the category of unital algebras. Then the couple  $(\underline{\Omega}_u B, \underline{\Omega}_u(B, C))$  is an algebra over  $\text{Coend}_{\mathcal{R}\mathcal{L}_2}(\delta_{\mathbb{Z}})$ .*

### 8.3 Action of $\mathcal{RS}_2$

Let  $(M, N)$  be a pair of 1-connected topological monoids pointed at the unit 1 such that  $N$  is a sub-monoid of  $M$ . We explain how the following couple  $(\underline{\Omega}C_*^1M, \underline{\Omega}(C_*^1M, C_*^1N))$  can be endowed with an action of  $\mathcal{RS}_2$ . The operation are similar to the ones described in the previous section. However, the absence of a unit in the reduced dg-bialgebra  $(C_*^1M)^+$  constrains us to write down the operations "manually".

Let us fix an unital/counital dg-bialgebra  $(B, \nabla_B)$  with counit  $\epsilon$ , such that  $B = B^+ \oplus \mathbb{Z}$  and with  $B_1 = 0$ ; and  $(C, \nabla_C)$  be a left  $B$ -dg-comodule in the category of unital algebras. For example  $B = C_*^1M$  and  $C = C_*^1N$ . We recall that the cobar construction  $\underline{\Omega}B$  is, in particular, given as the free tensor algebra over  $s^{-1}B^+$ . We adopt the same notations as in Section 8.2. We denote the reduced coproducts  $\overline{\nabla}_B(b) = \nabla_B(b) - 1 \otimes b - b \otimes 1$ ,  $\overline{\nabla}_C(c) = \nabla_C(c) - 1 \otimes c$ . We denote

$$[b_1, \dots, b_k] := (s^{-1})^{\otimes k}(b_1 \otimes \dots \otimes b_k).$$

The closed part of  $\mathcal{RS}_2$  acts on it, see [Kad05] for an example with  $\mathbb{Z}_2$  as the field of coefficients. The multiplication is the concatenation, and, for  $k \geq 1$ ,

$$(8.14) \quad E'_{1,k} : \underline{\Omega}B \otimes (\underline{\Omega}B)^{\otimes k} \rightarrow \underline{\Omega}B$$

is given by

$$E'_{1,k}(f; g_1, \dots, g_k) = \sum \pm [a_1, \dots, a_{i_1} \triangleleft g_1, \dots, a_{i_k} \triangleleft g_k, \dots, a_n]$$

where  $f = [a_1, \dots, a_n]$  and  $g_i \in \underline{\Omega}B$ .

In the same way, we can define a multiplication

$$(8.15) \quad \begin{aligned} \mu'_0 : \underline{\Omega}(B, C) \otimes \underline{\Omega}(B, C) &\rightarrow \underline{\Omega}(B, C) \\ [a_1, \dots, a_k]c \otimes [b_1, \dots, b_l]c' &\mapsto \sum_{(c)} \pm [a_1, \dots, a_k, (b_1, \dots, b_l) \triangleright z_c^{(1)}]c'c^{(2)}, \end{aligned}$$

where, we recall, for  $c \in C$ , the element  $\nabla_C(c)$  is denoted by  $\sum_{(c)} z_c^{(1)} \otimes c^{(2)}$  so that  $1 \otimes c$  is a term of this sum; and, a family of operations

$$(8.16) \quad E'_j : (\underline{\Omega}B) \otimes (\underline{\Omega}(B, C))^{\otimes j} \rightarrow \underline{\Omega}(B, C)$$

given by

$$\begin{aligned} E'_j(f; (g_1; c_1), \dots, (g_j; c_j)) &= \sum \pm [a_1, \dots, a_{i_1} \triangleleft g_1, a_{i_1+1} z_{c_1}^{(1)}, \dots \\ &\quad \dots, a_{i_j} \triangleleft g_j \triangleright (z_{c_{j-1}}^{(1)} \cdots z_{c_1}^{(j-1)}), \dots, a_n (z_{c_{j-1}}^{(1)} \cdots z_{c_1}^{(j-1)})] c_j c_{j-1}^{(1)} \cdots c_1^{(j)}, \end{aligned}$$

for any  $f = [a_1, \dots, a_n]$  and  $(g_i; c_i) \in \underline{\Omega}(B, C)$ .

We claim that, for appropriate signs, the operations (8.14), (8.15), (8.16) and the inclusion of  $\underline{\Omega}B$  into  $\underline{\Omega}(B, C)$  define an  $\mathcal{RS}_2$ -algebra structure on  $(\underline{\Omega}B, \underline{\Omega}(B, C))$ .

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