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Alexandre Quesney

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Swiss Cheese type operads and models for relative loop spaces

Alexandre Quesney

November 18, 2015

Abstract

We construct a (coloured) operad \mathcal{RL} in the category of sets that may be thought of as a combinatorial model for the Swiss Cheese operad. By adapting Batanin-Berger's condensation process we obtain a topological (resp. chain) operad weakly equivalent to the topological (resp. chain) Swiss Cheese operad. As an application, we exhibit models for relative loop spaces acted on by Swiss Cheese type operads (in dimension 2).

Introduction

In [BB09], Batanin and Berger introduce the notion of *condensation* of a coloured operad. By applying this condensation to the *lattice path operad* \mathcal{L} they obtain a model for the little cubes operad. More precisely, this means that, in the category of topological spaces \mathbf{Top} (resp. the category of chain complexes $\mathbf{Ch}(\mathbb{Z})$), the *condensation operad* of \mathcal{L} is weakly equivalent to the topological (resp. chain) little cubes operad.

We construct a coloured operad \mathcal{RL} in the category of sets that may be thought of as a combinatorial model for the Swiss Cheese operad.

Let us fix a closed monoidal symmetric category \mathbf{C} and a simplicial object $\delta : \Delta \rightarrow \mathbf{C}$. By adapting Batanin-Berger's method, we obtain a functor

$$F : \mathcal{RL}\text{-algebra} \longrightarrow \text{Coend}_{\mathcal{RL}(\delta)}\text{-algebra}$$

that sends algebras over \mathcal{RL} into algebras over the *condensation (2-coloured) operad* $\text{Coend}_{\mathcal{RL}(\delta)}$ in \mathbf{C} .

The operad \mathcal{RL} is filtered by suboperads \mathcal{RL}_m , $m \geq 1$ and we have the corresponding condensed operads $\text{Coend}_{\mathcal{RL}_m(\delta)}$, $m \geq 1$.

We are interested by two choices for δ :

$$\delta_{\text{Top}} : \Delta \xrightarrow{\delta_{\text{yon}}} \mathbf{Set}^{\Delta^{\text{op}}} \xrightarrow{|_|_} \mathbf{Top}$$

and

$$\delta_{\mathbb{Z}} : \Delta \xrightarrow{\delta_{yon}} \mathbf{Set}^{\Delta^{op}} \xrightarrow{C_*(-; \mathbb{Z})} \mathbf{Ch}(\mathbb{Z})$$

where $\delta_{yon}([n]) = \mathrm{Hom}_{\Delta}(-, [n])$ is the Yoneda functor. In this manner, the condensation of \mathcal{RL}_m leads to a topological operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathrm{Top}})$ and a chain operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$.

The Swiss Cheese operad that we consider is denoted by \mathcal{SC}_m , $m \geq 1$, and is the augmented (cubical) version of Voronov's Swiss Cheese operad $\mathcal{SC}_m^{\mathrm{vor}}$ defined in [Vor98].

For each $m \geq 1$, we construct a cellular decomposition of the Swiss Cheese operad \mathcal{SC}_m . This provides a recognition principle that we use afterwards to show the following.

Theorem 0.1. *Let $m \geq 1$. Then, the operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathrm{Top}})$ is weakly equivalent to the topological Swiss-Cheese operad \mathcal{SC}_m and, the operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ is weakly equivalent to the chain Swiss-Cheese operad $C_*(\mathcal{SC}_m)$.*

For each $m \geq 1$, the operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ admits a weakly equivalent sub-operad \mathcal{RS}_m . This operad \mathcal{RS}_m may be thought of as a "Swiss Cheese version" (or "relative version") of the *surjection operad* \mathcal{S}_m studied in [MS02, MS03, BF04].

Our method in providing a cellular decomposition of the Swiss Cheese operad \mathcal{SC}_m mainly uses ideas developed in [Ber97] and gives rise to a recognition principle. The cells are indexed by a poset operad \mathcal{RK}_m which is a Swiss Cheese (or relative) version of the *extended complete graph operad*. Such a decomposition provides a zig-zag of weak homotopy equivalences between the Swiss Cheese operad \mathcal{SC}_m and the classifying operad of \mathcal{RK}_m . The latter arise as a comparison object: for any topological \mathcal{RK}_m -cellular operad \mathcal{O} (see Definition 2.7) there is a zig-zag of weak equivalences linking \mathcal{O} to the classifying operad of \mathcal{RK}_m . Thus, we prove the following.

Theorem 0.2. *Let $m \geq 1$. Then any topological \mathcal{RK}_m -cellular operad is weakly equivalent to the Swiss Cheese operad \mathcal{SC}_m .*

The second objective of this paper is to provide models for relative loop spaces. We focus on the second stage filtration operad \mathcal{RL}_2 . This operad encodes couples $(\mathcal{M}, \mathcal{Z})$ together with a map $\mathcal{M} \rightarrow \mathcal{Z}$ satisfying some properties. In particular, \mathcal{M} is a multiplicative operad and \mathcal{Z} is both a weak-left-module and a left-module over \mathcal{M} together with compatibilities (see Section 5 for complete definition). These structures naturally endow \mathcal{M} and \mathcal{Z} with a cosimplicial structure. And it turns out that the functor F sends such a couple $(\mathcal{M}, \mathcal{Z})$ to the couple of totalizations $(\mathrm{Tot}_{\delta} \mathcal{M}, \mathrm{Tot}_{\delta} \mathcal{Z})$. Thus, the condensed operad $\mathrm{Coend}_{\mathcal{RL}_2}(\delta)$ acts on the couple of $(\mathrm{Tot}_{\delta} \mathcal{M}, \mathrm{Tot}_{\delta} \mathcal{Z})$.

Given a pair of topological spaces (X, Y) pointed at $*$ and such that $* \subset Y \subset X$, there exists a cosimplicial space $\omega(X, Y)$ such that its totalization

$Tot_{\delta_{Top}} \omega(X, Y)$ is homeomorphic to $\Omega(X, Y)$. In particular, $\omega(X) = \omega(X, *)$ is a model for the loop space $\Omega(X, *) = \Omega X$. We have the following.

Theorem 0.3. *Let (M, N) be a pair of topological monoids pointed at the unit such that N is a submonoid of M . Let δ be δ_{Top} (resp. $\delta_{\mathbb{Z}}$). Then there exists an operad $Coend_{\mathcal{RL}_2}(\delta)$ which is weakly equivalent to the topological operad \mathcal{SC}_2 (resp. to the chain operad $C_*(\mathcal{SC}_2)$) and which acts on the totalization $(Tot_{\delta} \omega(M), Tot_{\delta} \omega(M, N))$.*

A model for loop spaces can be provided by Adams' cobar construction [Ada56]. Given a 1-connected topological space X , Adams' quasi-isomorphism of dg-algebras takes the form

$$\Phi : \underline{\Omega} C_*^1 X \rightarrow C_*^{\square}(\Omega_M X).$$

Here, $\Omega_M X$ denotes the associative Moore loop space of X and $C_*^{\square}(-)$ stands for the cubical normalized chain functor. The presence of the cubical chain on the right side makes natural asking if the cobar construction $\underline{\Omega} C_*^1 X$ can be thought of as the chain complex of a cubical set with a monoidal structure. This is the point of view developed in [KS05]. It leads to an easy proof that Adams' morphism Φ is a morphism of dg-bialgebras. Such a compatibility plays an essential role for iterating the cobar construction, see [Bau81]. We define the *relative cobar construction* $\underline{\Omega}(-, -)$ and, interpreting it as a cubical set, we obtain the following.

Proposition 0.4. *Let (X, Y) be a pair of two 1-connected spaces pointed at $*$ such that $* \subset Y \subset X$. Then $\underline{\Omega}(C_*^1 X, C_*^1 Y)$ is naturally a coalgebra and an $\underline{\Omega} C_*^1 X$ -module. Moreover, there exists a Φ -equivariant morphism $\Psi : \underline{\Omega}(C_*^1 X, C_*^1 Y) \rightarrow C_*^{\square}(\Omega(X, Y))$ which is a quasi-isomorphism of dg-coalgebras.*

With regard to the Φ -equivariance of the above morphism Ψ , a similar statement was obtained in [FHT92]. However, both the coalgebra structure on $\underline{\Omega}(C_*^1 X, C_*^1 Y)$ and the coalgebra compatibility of Ψ are a direct consequence of the adopted point of view. In particular, Proposition 0.4 allows us to iterate the relative cobar construction by taking the cobar construction of the relative cobar construction, *i.e.*

$$\underline{\Omega} \underline{\Omega}(C_*^1 X, C_*^1 Y),$$

provided that X and Y are 2-connected. Note that another way to proceed is by taking the relative cobar of the cobar constructions, *i.e.*

$$\underline{\Omega}(\underline{\Omega} C_*^1 X, \underline{\Omega} C_*^1 Y).$$

The second choice is well-adapted to the following consideration. We explain how to obtain an action of \mathcal{RS}_2 on the couple $(\underline{\Omega} B, \underline{\Omega}(B, C))$ whenever B is a 1-reduced bialgebra and C a B -comodule in the category of unital algebras. As a by-product, taking $B = C_*^1 M$ and $C = C_*^1 N$ whenever (M, N) is a pair of monoids, one obtains algebraic counterpart to the fact that $\Omega(M, N) \cong \Omega(\Omega \mathcal{B} M, \Omega \mathcal{B} N)$ is a relative double loop space (where $\mathcal{B} G$ denotes the classifying space of the monoid G).

Outline of the paper. We begin by explaining how we *condense* a particular type of coloured operads (*SC-split operads*) to obtain 2-coloured operads. This result will be use in Section 3.

In section 2 we consider the (cubical) Swiss Cheese operad \mathcal{SC} . For each non zero natural number m , we construct a cellular decomposition of \mathcal{SC}_m indexed by a Swiss Cheese version of the *extended complete graph operad*, say \mathcal{RK}_m .

With the third section 3, we describe our main operad \mathcal{RL} which is a splitting operad. Using the condensation process developed in Section 1, one obtains the 2-coloured operad $\text{Coend}_{\mathcal{RL}}(\delta)$. We use results of Section 2 to prove the following. In the topological setting ($\delta = \delta_{\text{Top}}$) and the chain setting ($\delta = \delta_{\mathbb{Z}}$), we show the equivalence of $\text{Coend}_{\mathcal{RL}_m}(\delta)$ with the topological (resp. chain) Swiss Cheese operad \mathcal{SC}_m . We postpone a technical proof to Section 8.

In Section 4 we exhibit the sub-operad \mathcal{RS}_m and show that the inclusion $\mathcal{RS}_m \hookrightarrow \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ is a weak equivalence.

In Section 5 we focus on the operad \mathcal{RL}_2 and its representations.

The remaining Section 6 and Section 7 are devoted to models for relative loop spaces.

In Section 6 we show that the pair of cosimplicial spaces $(\omega(M), \omega(M, N))$ is a representation of \mathcal{RL}_2 whenever (M, N) are monoids. This implies that $(\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M), \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N))$ is an algebra over $\text{Coend}_{\mathcal{RL}_2}(\delta_{\mathbb{Z}})$, and so is a representation of \mathcal{RS}_2 by restriction. We make explicit the latter action of the sub-operad \mathcal{RS}_2 .

Finally, Section 7 is devoted to the relative cobar construction.

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1 Preliminaries

In [BB09] Batanin and Berger introduced the notion of condensation of a coloured operad. It consists of a realization followed by a totalization what "condenses" all the colours into a single one.

We consider particular coloured operads that we call *SC-split operads*. Roughly speaking, the set of colours of an SC-split operad can be split into two subsets that yield two sub-operads. We modify Batanin-Berger's condensation process for the SC-split operads. Our modification consists in condensing separately the colours of each of the two subsets of colours into one colour. This provides, in particular, 2-coloured operads.

1.1 SC functor-operads

Let \mathbf{C} be a closed symmetric monoidal category. Let \mathbf{A} and \mathbf{B} be two \mathbf{C} -categories. We denote by $\mathbf{A} \otimes \mathbf{B}$ the category with the pairs (a, b) for $a \in \mathbf{A}$ and $b \in \mathbf{B}$ as objects and

$$\mathrm{Hom}_{\mathbf{A} \otimes \mathbf{B}}((a, b), (a', b')) := \mathrm{Hom}_{\mathbf{A}}(a, a') \otimes \mathrm{Hom}_{\mathbf{B}}(b, b')$$

as hom-objects, where the tensor on the right hand side is the tensor of \mathbf{C} .

Definition 1.1. Let $k \geq 0$ and $j \geq 1$ be two integers. For a (k, j) -shuffle $\tau \in \mathrm{Sh}(k, j)$ we define the category

$$\mathbf{D}_{k,j}^\tau = \mathbf{U}_1 \otimes \dots \otimes \mathbf{U}_{k+j},$$

where $\mathbf{U}_i = \mathbf{A}$ if $i = \tau^{-1}(s)$ for a $1 \leq s \leq k$ and $\mathbf{U}_i = \mathbf{B}$ if $i = \tau^{-1}(s)$ for a $k+1 \leq s \leq k+j$. We call such a category $\mathbf{D}_{k,j}^\tau$ the (k, j) -shuffled category of (\mathbf{A}, \mathbf{B}) .

Definition 1.2. For a \mathbf{C} -functor $\zeta_k : \mathbf{A}^{\otimes k} \rightarrow \mathbf{A}$ and a permutation $\sigma \in \Sigma_k$, we denote by $\zeta_k^\sigma : \mathbf{A}^{\otimes k} \rightarrow \mathbf{A}$ the functor

$$\zeta_k^\sigma(X_1, \dots, X_k) = \zeta_k(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(k)}).$$

A \mathbf{C} -functor $\zeta_k : \mathbf{A}^{\otimes k} \rightarrow \mathbf{A}$ is called *twisting symmetric* if there exists \mathbf{C} -natural transformations $\phi_\sigma : \zeta_k \rightarrow \zeta_k^\sigma$, $\sigma \in \Sigma_k$, such that $\phi_{\sigma_1 \sigma_2} = (\phi_{\sigma_1})^{\sigma_2} \phi_{\sigma_2}$ and such that ϕ_{id} is the identity transformation where $id \in \Sigma_k$ denotes the neutral element.

For a family of \mathbf{C} -functors $\{\zeta_{\tau,k,j} : \mathbf{D}_{k,j}^\tau \rightarrow \mathbf{B}\}_{k \geq 0, j \geq 0, \tau \in \mathrm{Sh}(k,j)}$ and a permutation $\sigma \in \Sigma_{k+j}$, we denote by $\zeta_{\tau,k,j}^\sigma : \mathbf{D}_{k,j}^\tau \rightarrow \mathbf{B}$ the functor

$$\zeta_{\tau,k,j}^\sigma(X_1, \dots, X_{k+j}) = \zeta_{\tau_\sigma, k, j}(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(k+j)}),$$

where $\tau_\sigma \in \mathrm{Sh}(k, j)$ is the shuffle such that $X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(k+j)} \in \mathbf{D}_{k,j}^{\tau_\sigma}$. A family of \mathbf{C} -functor $\{\zeta_{\tau,k,j} : \mathbf{D}_{k,j}^\tau \rightarrow \mathbf{B}\}_{k \geq 0, j \geq 0, \tau \in \mathrm{Sh}(k,j)}$ is called *twisting symmetric* if there exists \mathbf{C} -natural transformations $\phi_\sigma : \zeta_{\tau,k,j} \rightarrow \zeta_{\tau,k,j}^\sigma$, $\sigma \in \Sigma_{k+j}$, such that $\phi_{\sigma_1 \sigma_2} = (\phi_{\sigma_1})^{\sigma_2} \phi_{\sigma_2}$ and such that ϕ_{id} is the identity transformation where id denotes the neutral element of Σ_{k+j} .

Definition 1.3. An *SC functor-operad* $\zeta = \{\zeta_k, \zeta_{\tau,k',j}\}_{k,k',j \geq 0, \tau \in \mathrm{Sh}(k',j)}$ over (\mathbf{A}, \mathbf{B}) is a family of twisted symmetric \mathbf{C} -functors

$$\zeta_k : \mathbf{A}^{\otimes k} \rightarrow \mathbf{A}$$

and a twisted symmetric family

$$\{\zeta_{\tau,k',j} : \mathbf{D}_{k',j}^\tau \rightarrow \mathbf{B}\}_{k' \geq 0, j \geq 0, \tau \in \mathrm{Sh}(k',j)}$$

of \mathbf{C} -functors, together with natural transformations

$$\mu_{i_1, \dots, i_k} : \zeta_k \circ (\zeta_{i_1} \otimes \dots \otimes \zeta_{i_k}) \rightarrow \zeta_{i_1 + \dots + i_k} \quad i_1, \dots, i_k \geq 0$$

$$\mu_{\tau', s_1, \dots, s_{k+j}, t_1, \dots, t_j} : \zeta_{\tau, k, j} \circ (\zeta(1) \otimes \dots \otimes \zeta(k+j)) \rightarrow \zeta_{\tau', s_1 + \dots + s_{k+j}, t_1 + \dots + t_j}$$

1.1 SC functor-operads

for $s_1, \dots, s_{k+j}, t_1, \dots, t_j \geq 0$, where $\tilde{\zeta}(i) = \zeta_{s_l}$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\tilde{\zeta}(i) = \zeta_{\tau_l, s_l, t_{l-k}}$ for $i = \tau^{-1}(s)$, $k+1 \leq l \leq k+j$; and, $\tau' = \tau(\tau_1, \dots, \tau_{k+j})$ where the shuffle τ_l associated to ζ_{s_l} is the identity. These natural transformations have to satisfy the following conditions.

1. ζ_1 is the identity functor and $\zeta_1 \circ \zeta_k = \zeta_k = \zeta_k \circ (\zeta_1 \otimes \dots \otimes \zeta_1)$ where the equalities are obtained via the μ_k and $\mu_{1, \dots, 1}$ respectively.
2. $\zeta_{id, 0, 1}$ is the identity functor and $\zeta_{id, 0, 1} \circ \zeta_{\tau, k, j} = \zeta_{\tau, k, j} = \zeta_{\tau, k, j} \circ (\tilde{\zeta}(1) \otimes \dots \otimes \tilde{\zeta}(k+j))$ where $\tilde{\zeta}(i) = \zeta_1$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\tilde{\zeta}(i) = \zeta_{id, 0, l}$ for $i = \tau^{-1}(s)$, $k+1 \leq l \leq k+j$ and where the equalities are obtained via the $\mu_{\tau, k, j}$ and $\mu_{id, 1, \dots, 1}$ respectively.
3. The μ_{i_1, \dots, i_k} and the $\mu_{\tau, s_1, \dots, s_{k+j}, t_1, \dots, t_j}$ are associative.
4. All the diagrams of the following forms commute:

$$\begin{array}{ccc}
 \zeta_k \circ (\zeta_{i_1} \otimes \dots \otimes \zeta_{i_k}) & \xrightarrow{\mu_{i_1, \dots, i_k}} & \zeta_{i_1 + \dots + i_k} \\
 \downarrow \phi_\sigma \circ (\phi_{\sigma_1} \otimes \dots \otimes \phi_{\sigma_{k+j}}) & & \downarrow \phi_{\sigma(\sigma_1, \dots, \sigma_{k+j})} \\
 \zeta_k^\sigma \circ (\zeta_{i_1}^\sigma \otimes \dots \otimes \zeta_{i_k}^\sigma) & \xrightarrow{\mu_{i_1, \dots, i_k}} & \zeta_{i_1 + \dots + i_k}^{\sigma(\sigma_1, \dots, \sigma_{k+j})}
 \end{array}$$

$$\begin{array}{ccc}
 \zeta_{\tau, k, j} \circ (\tilde{\zeta}(1) \otimes \dots \otimes \tilde{\zeta}(k+j)) & \xrightarrow{\mu_{\tau', s_1, \dots, s_{k+j}, t_1, \dots, t_j}} & \zeta_{\tau', s_1 + \dots + s_{k+j}, t_1 + \dots + t_j} \\
 \downarrow \phi_\sigma \circ (\phi_{\sigma_1} \otimes \dots \otimes \phi_{\sigma_{k+j}}) & & \downarrow \phi_{\sigma(\sigma_1, \dots, \sigma_{k+j})} \\
 \zeta_{\tau, k, j}^\sigma \circ (\tilde{\zeta}^\sigma(1) \otimes \dots \otimes \tilde{\zeta}^\sigma(k+j)) & \xrightarrow{\mu_{\tau', s_1, \dots, s_{k+j}, t_1, \dots, t_j}} & \zeta_{\tau', s_1 + \dots + s_{k+j}, t_1 + \dots + t_j}^{\sigma(\sigma_1, \dots, \sigma_{k+j})}
 \end{array}$$

where $\tilde{\zeta}^\sigma(i) = \zeta_{s_l}^{\sigma_l}$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\tilde{\zeta}^\sigma(i) = \zeta_{\tau_l, s_l, t_{l-k}}^{\sigma_l}$ for $i = \tau^{-1}(s)$, $k+1 \leq l \leq k+j$; $\tau' = \tau(\tau_1, \dots, \tau_{k+j})$.

Definition 1.4. Let $\zeta = \{\zeta_k, \zeta_{\tau, k, j}\}_{k, k', j \geq 0, \tau \in \text{Sh}(k', j)}$ be an SC functor-operad over (\mathbf{A}, \mathbf{B}) . A ζ -algebra X is a pair of objects $(X_A, X_B) \in \mathbf{A} \otimes \mathbf{B}$ equipped with a sequence of morphisms

$$\begin{aligned}
 \alpha_k &: \zeta_k(X_A, \dots, X_A) \rightarrow X_A, \quad k \geq 0, \\
 \alpha_{\tau, k, j} &: \zeta_{\tau, k, j}(\tilde{X}(1), \dots, \tilde{X}(k+j)) \rightarrow X_B, \quad k, j \geq 0, \tau \in \text{Sh}(k, j),
 \end{aligned}$$

where $\tilde{X}(i) = X_A$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\tilde{X}(i) = X_B$ for $i = \tau^{-1}(l)$, $k+1 \leq l \leq k+j$, and subject to the following conditions.

1. $\alpha_1 = 1_{X_A}$;
2. $\alpha_{id,0,1} = 1_{X_B}$;
3. $\alpha_k \circ \phi_\sigma = \alpha_k$, for all $\sigma \in \Sigma_k$;
4. $\alpha_{\tau,k,j} \circ \phi_\sigma = \alpha_{\tau,k,j}$, for all $\sigma \in \Sigma_{k+j}$;
5. for all $i_1, \dots, i_k \geq 0$, the following diagram commute

$$\begin{array}{ccc}
 \zeta_k \circ (\zeta_{i_1}(X_A, \dots, X_A) \otimes \dots \otimes \zeta_{i_k}(X_A, \dots, X_A)) & \xrightarrow{\mu_{i_1, \dots, i_k}} & \zeta_{i_1 + \dots + i_k}(X_A, \dots, X_A) \\
 \downarrow \zeta_k(\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k}) & & \downarrow \alpha_{i_1 + \dots + i_k} \\
 \zeta_k(X_A, \dots, X_A) & \xrightarrow{\alpha_k} & X_A
 \end{array}$$

6. for all $s_1, \dots, s_{k+j}, t_1, \dots, t_j \geq 0$, the following diagram commute

$$\begin{array}{ccc}
 \zeta_{\tau,k,j} \circ (\tilde{\zeta}(1)(X, \dots, X) \otimes \dots \otimes \tilde{\zeta}(k+j)(X, \dots, X)) & \xrightarrow{\mu_{\tau', s_1, \dots, s_{k+j}, t_1, \dots, t_j}} & \zeta_{\tau', s_1 + \dots + s_{k+j}, t_1 + \dots + t_j}(X, \dots, X) \\
 \downarrow \zeta_{\tau,k,j}(\tilde{\alpha}(1) \otimes \dots \otimes \tilde{\alpha}(k+j)) & & \downarrow \alpha_{\tau', s_1 + \dots + s_{k+j}, t_1 + \dots + t_j} \\
 \zeta_{\tau,k,j}(X, \dots, X) & \xrightarrow{\alpha_{\tau,k,j}} & X_B
 \end{array}$$

where, $\tilde{\alpha}(i) = \alpha_{s_l}$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\tilde{\alpha}(i) = \alpha_{\tau_l, s_l, t_{l-k}}$ for $i = \tau^{-1}(l)$, $k+1 \leq l \leq k+j$ and X stands here for X_A or X_B whenever it makes sense.

1.2 SC-split operads

Let us fix an arbitrary coloured operad \mathcal{O} in \mathbf{C} . We denote Col its set of colours, so that \mathcal{O} consists of objects

$$\mathcal{O}(n_1, \dots, n_k; n) \in \mathbf{C},$$

for $n_1, \dots, n_k \in Col$ and $k \geq 0$ together with the unit $I_C \rightarrow \mathcal{O}(n; n)$ and substitution maps

$$\mathcal{O}(n_1, \dots, n_k; n) \otimes \mathcal{O}(m_1, \dots, m_l; n_i) \rightarrow \mathcal{O}(n_1, \dots, n_{i-1}, m_1, \dots, m_l, n_{i+1}, \dots, n_k; n)$$

satisfying the natural unit, associativity and equivariance axioms.

1.2 SC-split operads

The underlying category of \mathcal{O} is the category \mathcal{O}_u with the colours $n \in Col$ as objects and the unary operations $\mathcal{O}_u(m, n) = \mathcal{O}(m; n)$ as morphisms. This way, we have functors

$$\mathcal{O}(\underbrace{-, \dots, -}_k; -) : (\mathcal{O}_u^{\text{op}})^{\otimes k} \otimes \mathcal{O}_u \rightarrow \mathbf{C}, \quad k \geq 1.$$

From now, we suppose \mathbf{C} endowed with a zero object *i.e.* an element $0 \in \mathbf{C}$ such that $0 \otimes X = X$ for all $X \in \mathbf{C}$. Let us suppose now that \mathcal{O} satisfies the following hypothesis.

- H1. $Col = Col_c \sqcup Col_o$.
- H2. The collection of the $\mathcal{O}(n_1, \dots, n_k; n)$ for $n_i, n \in Col_c, k \geq 1$ forms a sub-operad of \mathcal{O} .
- H3. The collection of the $\mathcal{O}(n_1, \dots, n_j; n)$ for $n_i, n \in Col_o, j \geq 1$ forms a sub-operad of \mathcal{O} .
- H4. The $\mathcal{O}(n_1, \dots, n_j; n)$ are the zero object for any $n \in Col_c$ if there exists an $1 \leq i \leq j$, such that $n_i \in Col_o, j \geq 1$.

We call such an operad an *SC-split operad*. The sub-operad in H2 is called the *closed part of \mathcal{O}* ; the sub-operad in H3 is called the *open part of \mathcal{O}* .

The underlying category \mathcal{O}_u contains two particular categories:

- \mathcal{O}_u^c the sub-category of \mathcal{O}_u with objects the colours in Col_c and morphisms the $\mathcal{O}_u(n, m)$ for $n, m \in Col_c$;
- \mathcal{O}_u^o the sub-category of \mathcal{O}_u with objects the colours in Col_o and morphisms the $\mathcal{O}_u(n, m)$ for $n, m \in Col_o$.

By H2 and H3 both \mathcal{O}_u^c and \mathcal{O}_u^o are \mathbf{C} -categories. The category $\mathbf{C}^{\mathcal{O}_u^c}$ (resp. $\mathbf{C}^{\mathcal{O}_u^o}$) of \mathbf{C} -functors from \mathcal{O}_u^c (resp. from \mathcal{O}_u^o) to \mathbf{C} is a \mathbf{C} -category.

For $k \geq 0$, we define the \mathbf{C} -functor

$$\tilde{\zeta}(\mathcal{O})_k : \underbrace{\mathbf{C}^{\mathcal{O}_u^c} \otimes \dots \otimes \mathbf{C}^{\mathcal{O}_u^c}}_k \rightarrow \mathbf{C}^{\mathcal{O}_u^c}$$

as the coend

$$\tilde{\zeta}(\mathcal{O})_k(X_1, \dots, X_k)(n) = \mathcal{O}(\underbrace{-, \dots, -}_k; n) \otimes_{(\mathcal{O}_u^c)^{\otimes k}} \underbrace{X_1(-) \otimes \dots \otimes X_k(-)}_k;$$

and for $k, j \geq 0$ and $\tau \in \text{Sh}(k, j)$ we define the \mathbf{C} -functor

$$\zeta(\mathcal{O})_{\tau, k, j} : (\mathbf{C}^{\mathbf{O}})_{k, j}^{\tau} \rightarrow \mathbf{C}^{\mathcal{O}_u^o}$$

by

$$\tilde{\zeta}(\mathcal{O})_{\tau,k,j}(X_1, \dots, X_{k+j})(n) = \mathcal{O}(\underbrace{-, \dots, -}_{k+j}; n) \otimes_{\mathbf{O}_{k,j}^{\tau}} \underbrace{X_1(-) \otimes \dots \otimes X_{k+j}(-)}_k,$$

where $\mathbf{O}_{k,j}^{\tau}$ is the (k, j) -shuffled category of $(\mathcal{O}_u^c, \mathcal{O}_u^o)$ and $(\mathbf{C}^{\mathcal{O}})_{k,j}^{\tau}$ is the (k, j) -shuffled category of $(\mathbf{C}^{\mathcal{O}_u^c}, \mathbf{C}^{\mathcal{O}_u^o})$.

An \mathcal{O} -algebra X is a family $\{X(n)\}_{n \in \text{Col}}$ of objects $X(n) \in \mathbf{C}$ equipped with morphisms

$$(1.1) \quad \mathcal{O}(n_1, \dots, n_k; n) \otimes X(n_1) \otimes \dots \otimes X(n_k) \rightarrow X(n), \quad n_1, \dots, n_k, n \in \text{Col}$$

subject to the natural unit, associative and equivariance axioms. In particular, from the hypothesis on \mathcal{O} , X can be seen as a pair (X_c, X_o) where X_c is the sub-family $\{X_c(n)\}_{n \in \text{Col}_c}$ and X_o is the sub-family $\{X_o(n)\}_{n \in \text{Col}_o}$. We have an SC analogue to [BB09, Proposition 1.8] or [DS03]:

Proposition 1.5. *The functors $\tilde{\zeta}(\mathcal{O})_k$ and $\tilde{\zeta}(\mathcal{O})_{\tau,k,j}$ extend to an SC functor-operad $\tilde{\zeta}(\mathcal{O})$, such that the category of \mathcal{O} -algebras and the category of $\tilde{\zeta}(\mathcal{O})$ -algebras are isomorphic.*

Proof. Let us first show that $\{\tilde{\zeta}(\mathcal{O})_k\}_{k \geq 0}$ and $\{\tilde{\zeta}(\mathcal{O})_{\tau,k,j}\}_{k,j \geq 0, \tau \in \text{Sh}(k,j)}$ form an SC functor-operad. Let us describe the map $\mu_{\tau,k,j}$ with an example. To simplify the exposition, we choose τ as the neutral element $id \in \Sigma_{k+j}$ and the j other shuffles τ_l involved in $\tilde{\zeta}(\mathcal{O})_{\tau_l, s_{k+l}, t_l}$ as the neutral element $id \in \Sigma_{s_{k+l}+t_l}$; we set $s(l) = s_1 + \dots + s_l$ and $t(l) = t_1 + \dots + t_l$. We denote by $\mathbf{n}_{a,b}$ the list $n_{a,1}, \dots, n_{a,b}$ of objects in \mathcal{O}_u^c and by $\mathbf{m}_{a,b}$ the list $m_{a,1}, \dots, m_{a,b}$ of objects in \mathcal{O}_u^o . In this case, the map $\mu_{id,k,j}$ is

$$\begin{aligned} & \tilde{\zeta}(\mathcal{O})_{id,k,j} \circ (\tilde{\zeta}(\mathcal{O})_{s_1}(X_1, \dots, X_{s_1}), \dots, \tilde{\zeta}(\mathcal{O})_{id, s_{k+j}, t_j}(X_{s(k+j-1)+t(j-1)+1}, \dots, X_{s(k+j)+t(j)})) \\ &= \int^{\mathbf{n}_{0,k}, \mathbf{m}_{0,j}} \mathcal{O}(\mathbf{n}_{0,k}, \mathbf{m}_{0,j}; -) \otimes \int^{\mathbf{n}_{1,s_1}} \mathcal{O}(\mathbf{n}_{1,s_1}; n_{0,1}) \otimes X_1(n_{1,1}) \otimes \dots \\ & \quad \dots \otimes \int^{\mathbf{n}_{k+j, s_{k+j}}, \mathbf{m}_{j, t_j}} \mathcal{O}(\mathbf{n}_{k+j, s_{k+j}}, \mathbf{m}_{j, t_j}; m_{0,j}) \otimes X_{s(k+j)+t(j)}(m_{j, t_j}) \\ & \cong \int^{\mathbf{n}_{0,k}, \mathbf{m}_{0,j}, \mathbf{n}_{1,s_1}, \dots, \mathbf{n}_{k+j, s_{k+j}}, \mathbf{m}_{j, t_j}} \mathcal{O}(\mathbf{n}_{0,k}, \mathbf{m}_{0,j}; -) \otimes \mathcal{O}(\mathbf{n}_{1,s_1}; n_{0,1}) \otimes \dots \\ & \quad \dots \otimes \mathcal{O}(\mathbf{n}_{k+j, s_{k+j}}, \mathbf{m}_{j, t_j}; m_{0,j}) \otimes X_1(n_{1,1}) \otimes \dots \otimes X_{s(k+j)+t(j)}(m_{j, t_j}) \\ & \rightarrow \int^{\mathbf{n}_{1,s_1}, \dots, \mathbf{n}_{k+j, s_{k+j}}, \mathbf{m}_{j, t_j}} \mathcal{O}(\mathbf{n}_{1,s_1}, \dots, \mathbf{n}_{k+j, s_{k+j}}, \mathbf{m}_{j, t_j}; -) \otimes X_1(n_{1,1}) \otimes \dots \otimes X_{s(k+j)+t(j)}(m_{j, t_j}) \\ & =: \mathcal{O}(\underbrace{-, \dots, -}_{s_1+\dots+s_{k+j}}, \underbrace{-, \dots, -}_{t_1+\dots+t_j}) \otimes_{(\mathcal{O}_u^c)^{\otimes s_1} \otimes \dots \otimes (\mathcal{O}_u^c)^{\otimes s_{k+j}} \otimes (\mathcal{O}_u^o)^{\otimes t_j}} X_1 \otimes \dots \otimes X_{s(k+j)+t(j)}. \end{aligned}$$

The last map is induced by the composition map of \mathcal{O} . The associativity property of the latter implies that $\tilde{\zeta}$ satisfies the associativity axiom. The unit axioms 1 and 2 of Definition 1.3 are due to the Yoneda lemma. The twisted symmetric condition is obtained from the equivariance of the operad.

1.3 Condensation

Via the hypothesis H2 and H3, X_c and X_o can be seen as functors $X_c : \mathcal{O}_u^c \rightarrow \mathbf{C}$ and $X_o : \mathcal{O}_u^o \rightarrow \mathbf{C}$ respectively. The maps (1.1) gives (for $\tau = id$):

$$(1.2) \quad \mathcal{O}(n_1, \dots, n_{k+j}; n) \otimes X_c(n_1) \otimes \dots \otimes X_o(n_{k+j}) \rightarrow X_o(n),$$

for any $n_1, \dots, n_k \in Col_c$ and $n_{k+1}, \dots, n_{k+j}, n \in Col_o$. Since these maps satisfy the associativity and unit axioms, they induce a map

$$\begin{aligned} \xi_{id,k,j}(\mathcal{O})(X_c, \dots, X_o) &= \int^{n_1, \dots, n_{k+j}} \mathcal{O}(n_1, \dots, n_{k+j}; n) \otimes X_c(n_1) \otimes \dots \otimes X_o(n_{k+j}) \\ &\rightarrow X_o(n). \end{aligned}$$

This way, we obtain

$$\begin{aligned} \alpha_k &: \xi(\mathcal{O})_k(X_c, \dots, X_c) \rightarrow X_c, \quad k \geq 0, \\ \alpha_{\tau,k,j} &: \xi(\mathcal{O})_{\tau,k,j}(\tilde{X}(1), \dots, \tilde{X}(k+j)) \rightarrow X_o, \quad k, j \geq 0, \tau \in \text{Sh}(k, j), \end{aligned}$$

where $\tilde{X}(i) = X_c$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\tilde{X}(i) = X_o$ for $i = \tau^{-1}(l)$, $k+1 \leq l \leq k+j$. We conclude that X is a $\xi(\mathcal{O})$ -algebra because of the unit, associativity and equivariance properties of maps (1.2). Conversely, H4 says that the $\xi(\mathcal{O})_k$'s and the $\xi(\mathcal{O})_{\tau,k,j}$'s recover all maps in (1.1). \square

1.3 Condensation

Let $\delta^c : \mathcal{O}_u^c \rightarrow \mathbf{C}$ and $\delta^o : \mathcal{O}_u^o \rightarrow \mathbf{C}$ be two functors. We set $\delta = (\delta^c, \delta^o)$. We define the coendomorphism operad $\text{Coend}_{\xi(\mathcal{O})}(\delta)$ as the operad in \mathbf{C} with objects:

$$\text{Coend}_{\xi(\mathcal{O})}(\delta)(k) = \text{Hom}_{\mathbf{C}^{\mathcal{O}_u^c}}(\delta^c, \underbrace{\xi(\mathcal{O})_k(\delta^c, \dots, \delta^c)}_k) \quad k \geq 0;$$

$$\text{Coend}_{\xi(\mathcal{O})}(\delta)(\tau, k, j) = \text{Hom}_{\mathbf{C}^{\mathcal{O}_u^o}}(\delta^o, \xi(\mathcal{O})_{\tau,k,j}(\tilde{\delta}(1), \dots, \tilde{\delta}(k+j))) \quad k, j \geq 0, \tau \in \text{Sh}(k, j),$$

where $\tilde{\delta}(i) = \delta^c$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\tilde{\delta}(i) = \delta^o$ for $i = \tau^{-1}(l)$, $k+1 \leq l \leq k+j$; the composition maps are given as follows.

The composition maps

$$\begin{aligned} \text{Coend}_{\xi(\mathcal{O})}(\delta)(k) \otimes \text{Coend}_{\xi(\mathcal{O})}(\delta)(i_1) \otimes \dots \otimes \text{Coend}_{\xi(\mathcal{O})}(\delta)(i_k) \\ \rightarrow \text{Coend}_{\xi(\mathcal{O})}(\delta)(i_1 + \dots + i_k) \end{aligned}$$

are given by sending maps $f \otimes g_1 \otimes \dots \otimes g_k$ to the composite

$$\begin{array}{ccc} \delta^c & & \xi(\mathcal{O})_{i_1+\dots+i_k}(\delta^c) \\ \downarrow f & & \uparrow \alpha_{i_1, \dots, i_k} \\ \xi(\mathcal{O})_k(\underbrace{\delta^c, \dots, \delta^c}_k) & \xrightarrow{\xi(\mathcal{O})_k(g_1, \dots, g_k)} & \xi(\mathcal{O})_k(\xi(\mathcal{O})_{i_1}(\delta^c), \dots, \xi(\mathcal{O})_{i_k}(\delta^c)) \end{array}$$

And similarly for the composition maps

$$\begin{aligned} \text{Coend}_{\xi(\mathcal{O})}(\delta)(\tau, k, j) \otimes \widetilde{\text{Coend}}_{\xi(\mathcal{O})}(\delta)(1) \otimes \dots \otimes \widetilde{\text{Coend}}_{\xi(\mathcal{O})}(\delta)(k+j) \\ \rightarrow \text{Coend}_{\xi(\mathcal{O})}(\delta)(\tau', s_1 + \dots + s_{k+j}, t_1 + \dots + t_j), \end{aligned}$$

where $\widetilde{\text{Coend}}_{\xi(\mathcal{O})}(\delta)(l) = \text{Coend}_{\xi(\mathcal{O})}(\delta^c)(s_l)$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\widetilde{\text{Coend}}_{\xi(\mathcal{O})}(\delta)(l) = \text{Coend}_{\xi(\mathcal{O})}(\delta)(\tau_{l-k}, s_l, t_{l-k})$ for $i = \tau^{-1}(l)$, $k+1 \leq l \leq k+j$.

Given an \mathcal{O} -algebra $X = (X_c, X_o)$, we denote by

$$\begin{aligned} \text{Tot}_{\delta^c} X_c &:= \text{Hom}_{\mathcal{C}^{\mathcal{O}_i^c}}(\delta^c, X_c); \\ \text{Tot}_{\delta^o} X_o &:= \text{Hom}_{\mathcal{C}^{\mathcal{O}_i^o}}(\delta^o, X_o). \end{aligned}$$

In virtue of Proposition 1.5, the couple $(\text{Tot}_{\delta^c} X_c, \text{Tot}_{\delta^o} X_o)$ is a $\text{Coend}_{\xi(\mathcal{O})}(\delta)$ -algebra. The action maps

$$\text{Coend}_{\xi(\mathcal{O})}(\delta)(k) \otimes (\text{Tot}_{\delta^c} X_c)^{\otimes k} \rightarrow \text{Tot}_{\delta^c} X_c$$

are given by sending maps $f \otimes g_1 \otimes \dots \otimes g_k$ to the composite

$$\begin{array}{ccc} \begin{array}{c} \delta^c \\ \downarrow f \\ \xi(\mathcal{O})_k(\delta^c, \dots, \delta^c) \end{array} & \xrightarrow{\xi(\mathcal{O})_k(g_1, \dots, g_k)} & \begin{array}{c} X_c \\ \uparrow \alpha_k \\ \xi(\mathcal{O})_k(X_c, \dots, X_c) \end{array} \end{array}$$

And similarly for action maps

$$\text{Coend}_{\xi(\mathcal{O})}(\delta)(\tau, k, j) \otimes \widetilde{\text{Tot}}_{\delta} X(1) \otimes \dots \otimes \widetilde{\text{Tot}}_{\delta} X(k+j) \rightarrow \text{Tot}_{\delta^o} X_o,$$

where $\widetilde{\text{Tot}}_{\delta} X(l) = \widetilde{\text{Tot}}_{\delta^c} X_c$ for $i = \tau^{-1}(l)$, $1 \leq l \leq k$ and $\widetilde{\text{Tot}}_{\delta} X(l) = \widetilde{\text{Tot}}_{\delta^o} X_o$ for $i = \tau^{-1}(l)$, $k+1 \leq l \leq k+j$. Unit, associative and equivariance axioms are deduced from the SC functor-operad properties of $\xi(\mathcal{O})$.

2 A cellular decomposition of the Swiss Cheese operad

The little cubes operad \mathcal{C} has a cellular decomposition indexed by the extended complete graph operad \mathcal{K} , see [Ber97] and [BFV07, 4.1]. We extend this result to the Swiss Cheese operads \mathcal{SC}_m , $m \geq 1$ what provides a recognition principle for Swiss Cheese type operads. In particular, we construct a poset operad \mathcal{RK}_m that indexes the cells $(\mathcal{SC}_m)^{(\alpha)}$ of \mathcal{SC}_m . This leads to a zig-zag of weak equivalences of operads

$$\mathcal{SC}_m \xleftarrow{\sim} \text{hocolim}_{\alpha \in \mathcal{RK}_m} (\mathcal{SC}_m)^{(\alpha)} \xrightarrow{\sim} \mathcal{BRK}_m,$$

between the Swiss Cheese operad \mathcal{SC}_m and the classifying operad of \mathcal{RK}_m .

2.1 The Swiss Cheese operad

The Swiss Cheese operad that we use is the cubical version of the one defined in [Kon99].

Let $m \geq 1$. Let $Sym : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be the reflection $Sym(x_1, \dots, x_m) = (x_1, \dots, -x_m)$, and let $Half_+$ be the upper half space

$$Half_+ = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_m > 0\}.$$

The standard cube C_0 in \mathbb{R}^m is $C_0 = [-1, 1]^{\times m}$. A cube C in the standard cube is of the form $C = [x_1, y_1] \times [x_2, y_2] \times \dots \times [x_m, y_m]$ with $-1 < x_j < y_j < 1$ for $1 \leq j \leq m$.

Definition 2.1. Let $\mathfrak{C} := \{\mathfrak{c}, \mathfrak{o}\}$. For $n \geq 0$ and $c_i, c \in \mathfrak{C}$ we define a topological Σ_n -space $\mathcal{SC}_m(c_1, \dots, c_n; c)$ as the empty-set if both $c = \mathfrak{c}$ and there exists $1 \leq i \leq n$ such that $c_i = \mathfrak{o}$; for the other cases, it is

- the space of the little m -cubes operad $\mathcal{C}^{(m)}(n)$ defined in [May72] for $c = \mathfrak{c}$;
- the empty set if $n = 0$;
- the one-point space if $n = 1$;
- in the case $s + t = n \geq 2$ with $s, t \geq 0$ such that s colours c_i are \mathfrak{c} and t colours c_j are \mathfrak{o} , the space of configuration of $2s + t$ disjoint cubes (C_1, \dots, C_{2s+t}) in the standard cube $C_0 \in \mathbb{R}^m$ such that $Sym(C_i) = C_{i+s}$ for $1 \leq i \leq s$ and $Sym(C_i) = C_i$ for $2s + 1 \leq i \leq 2s + t$ and such that all the cubes (C_1, \dots, C_s) are in the upper half space.

Remark 2.2. Because of the symmetry conditions imposed by Sym , we may thought of $\mathcal{SC}_m(c_1, \dots, c_n; \mathfrak{o})$ as the configuration space of cubes (C_1, \dots, C_s) and semi-cubes $(C_{s+1}, \dots, C_{s+t})$ lying into the standard semi-cube $\{(x_1, \dots, x_m) \in C_0 \mid x_m > 0\}$.

Similarly to the little m -cubes operad $\mathcal{C}^{(m)}$ the composition maps

$$\circ_i : \mathcal{SC}_m(c_1, \dots, c_n; c) \times \mathcal{SC}_m(d_1, \dots, d_r; c_i) \rightarrow \mathcal{SC}_m(c_1, \dots, c_{i-1}, d_1, \dots, d_r, c_{i+1}, \dots, c_n; c)$$

are defined as substitutions of cubes. We denote the resulting 2-colours operad \mathcal{SC}_m .

Remark 2.3. Up to passage from cubes to discs we have an inclusion of operads $\mathcal{SC}_m^{\text{vor}} \subset \mathcal{SC}_m$ where $\mathcal{SC}_m^{\text{vor}}$ denotes the Swiss Cheese operad originally defined by A. Voronov in [Vor98].

2.2 The SC extended complete graph operad

We define the *SC (or relative) extended complete graph* operad \mathcal{RK} . It is a 2-coloured poset operad with filtration $\{\mathcal{RK}_m\}_{m \geq 1}$. Its closed part is \mathcal{K}_m , its

open part is \mathcal{K}_{m-1} , where $\{\mathcal{K}_m\}_{m \geq 1}$ denotes the extended complete graph operad defined in [BFV07, Section 4.1].

Given n colours $c_i \in \mathfrak{C} = \{\mathfrak{c}, \mathfrak{o}\}$, we denote by $\{\tilde{c}_1, \dots, \tilde{c}_n\}$ the set with

$$\tilde{c}_i = \begin{cases} i & \text{if } c_i = \mathfrak{c}; \\ \underline{i} & \text{if } c_i = \mathfrak{o}. \end{cases}$$

A colouring and an orientation on a complete graph on $\{\tilde{c}_1, \dots, \tilde{c}_n\}$ is, for each edge between \tilde{c}_i and \tilde{c}_j , an orientation $\sigma_{i,j}$ (that is, $\tilde{c}_i \rightarrow \tilde{c}_j$ or $\tilde{c}_i \leftarrow \tilde{c}_j$) and a strict positive natural number $\mu_{i,j} \in \mathbb{N}^{>0}$ as the colour. A *monochromatic acyclic orientation* of a complete graph is a colouring and orientation such that there exist no oriented cycles with the same colour, *i.e.* there are no configurations of the form $\tilde{c}_{i_1} \rightarrow \tilde{c}_{i_2} \rightarrow \dots \rightarrow \tilde{c}_{i_k} \rightarrow \tilde{c}_{i_1}$ with $\mu_{i_1, i_2} = \mu_{i_2, i_3} = \dots = \mu_{i_{k-1}, i_k} = \mu_{i_k, i_1}$.

If there exists an i such that $c_i = \mathfrak{o}$, we set $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$ as the empty set. Else, the poset $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$ is the set of pair $(\mu, \sigma)^{\mathfrak{c}}$ of monochromatic acyclic orientations of the complete graph on $\{\tilde{c}_1, \dots, \tilde{c}_n\}$. The colouring μ is a collection of $\mu_{i,j}$ for $1 \leq i < j \leq n$ and the orientation σ is a collection of orientations $\sigma_{i,j}$ for $1 \leq i < j \leq n$.

The poset structure is given by

$$(\mu, \sigma)^{\mathfrak{c}} \leq (\mu', \sigma')^{\mathfrak{c}} \Leftrightarrow \forall i < j, \text{ either } (\mu_{i,j}, \sigma_{i,j}) = (\mu'_{i,j}, \sigma'_{i,j}) \text{ or } \mu_{i,j} < \mu'_{i,j}.$$

The filtration $(\mathcal{RK}_m)_{m \geq 1}$ is as follows.

For $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$ with $c_i = \mathfrak{c}$ for all i , we set

$$\mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{c}) = \{(\mu, \sigma)^{\mathfrak{c}} \in \mathcal{RK}(c_1, \dots, c_n; \mathfrak{c}) \mid \mu_{i,j} \leq m \forall i < j\}$$

For $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{o})$, we set

$$(2.1) \quad \mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{o}) = \{(\mu, \sigma)^{\mathfrak{o}} \in \mathcal{RK}(c_1, \dots, c_n; \mathfrak{o}) \mid \begin{array}{ll} \mu_{i,j} \leq m & \text{if } c_i = c_j = \mathfrak{c}, \\ \mu_{i,j} \leq m - 1 & \text{if } c_i = c_j = \mathfrak{o}, \\ \mu_{i,j} \leq m & \text{if } \underline{i} \rightarrow j, \\ \mu_{i,j} \leq m - 1 & \text{if } \underline{i} \leftarrow j, \\ \mu_{i,j} \leq m - 1 & \text{if } i \rightarrow \underline{j}, \\ \mu_{i,j} \leq m & \text{if } i \leftarrow \underline{j}. \end{array}\}$$

Given a permutation $\sigma \in \Sigma_n$ and an element $(\mu, \tau)^{\mathfrak{c}} \in \mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$, the resulting element $\sigma \cdot (\mu, \tau)^{\mathfrak{c}} \in \mathcal{RK}(c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)}; \mathfrak{c})$ is given by permuting the numbers i by σ without changing neither the underline nor the orientation nor the colouring. For example, the edges $\underline{i} \rightarrow j$ of $(\mu, \tau)^{\mathfrak{c}}$ with colours $\mu_{i,j}$ become the edges $\underline{\sigma(i)} \rightarrow \sigma(j)$ with the same colours $\mu_{i,j}$.

The compositions

$$\begin{aligned} \mathcal{RK}(c_1, \dots, c_n; \mathfrak{c}) \times \mathcal{RK}(c_{1,1}, \dots, c_{1,k_1}; c_1) \times \dots \times \mathcal{RK}(c_{n,1}, \dots, c_{n,k_n}; c_n) \\ \rightarrow \mathcal{RK}(c_{1,1}, \dots, c_{n,k_n}; \mathfrak{c}) \end{aligned}$$

2.3 Cellular decomposition of SC type operads

send a tuple of $\mathcal{RK}(c_1, \dots, c_n; c) \times \mathcal{RK}(c_{1,1}, \dots, c_{1,k_1}; c_1) \times \dots \times \mathcal{RK}(c_{n,1}, \dots, c_{n,k_n}; c_n)$ to an element in $\mathcal{RK}(c_{1,1}, \dots, c_{n,k_n}; c)$ obtained as follows. The sub complete graphs with vertices in the same block $\{c_{i,1}, \dots, c_{i,k_i}\}$ is oriented and coloured as in $\mathcal{RK}(c_{i,1}, \dots, c_{i,k_i}; c_i)$; the edges with vertices in two different blocks are oriented and coloured as the edges between the corresponding vertices in $\mathcal{RK}(c_1, \dots, c_n; c)$.

Remark 2.4. For $m = 1$ the conditions where $\mu_{i,j} \leq m - 1$ cannot be satisfied. It follows that, in $\mathcal{RK}_1(c_1, \dots, c_n; \mathfrak{o})$, the tuple (c_1, \dots, c_n) has at most one open colour.

2.3 Cellular decomposition of SC type operads

Definition 2.5 ([Ber97]). Let X be a topological space and \mathcal{A} be a poset. We say that X admits an \mathcal{A} -cellulation if there is a functor $\Theta : \mathcal{A} \rightarrow \mathbf{Top}$ such that:

1. $\text{colim}_{\alpha \in \mathcal{A}} \Theta(\alpha) \cong X$;
2. $\beta \leq \alpha \in \mathcal{A} \Leftrightarrow \Theta(\beta) \subseteq \Theta(\alpha)$;
3. the inclusions $\Theta(\beta) \subseteq \Theta(\alpha)$ are a closed cofibration;
4. for each $\alpha \in \mathcal{A}$, the ‘‘cell’’ $\Theta(\alpha)$ is contractible.

Given a cell $\Theta(\alpha)$, we denote its boundary $(\bigcup_{\beta < \alpha} \Theta(\beta))$ by $\partial\Theta(\alpha)$; we denote its interior $\Theta(\alpha) \setminus \partial\Theta(\alpha)$ by $\overset{\circ}{\Theta}(\alpha)$. A cell with a non empty interior is called *proper*; else, it is called *improper*.

Lemma 2.6. [Ber97, Lemma 1.7] Let X be a topological space X with an \mathcal{A} -cellulation. Then we have the weak equivalences

$$X \cong \text{colim}_{\alpha \in \mathcal{A}} \Theta(\alpha) \xleftarrow{\sim} \text{hocolim}_{\alpha \in \mathcal{A}} \Theta(\alpha) \xrightarrow{\sim} \text{hocolim}_{\alpha \in \mathcal{A}} (*) \cong \mathcal{BA},$$

where \mathcal{BA} denote the realization of the nerve of the category \mathcal{A} .

Proof. Items 2 and 3 of Definition 2.5 give the left hand equivalence (see [BFSV03, Proposition 6.9] for details); the item 4 gives the right one. \square

We recall the notion of *symmetric* functors and *twisted symmetric* family of functors, Definition 1.2. A tuple $(c_1, \dots, c_n; \mathfrak{o})$ with $c_i \in \mathfrak{C} = \{c, \mathfrak{o}\}$ is bijectively determined by natural numbers k, j and $\tau \in \text{Sh}(k, j)$; the number k is the number of closed colours and j the number of open ones.

Let $k \geq 0$, and $\Theta_n : \mathcal{RK}(c_1, \dots, c_n; c) \rightarrow \mathbf{Top}$ be a functor. For $\sigma \in \Sigma_n$, we set $\Theta_n^\sigma(\alpha) := \Theta_n(\sigma \cdot \alpha)$.

Let $\Theta_{\tau,k,j} : \mathcal{RK}(c_1, \dots, c_n; \mathfrak{o}) \rightarrow \mathbf{Top}$ be a family of functors indexed by k, j and $\tau \in \text{Sh}(k, j)$. For $\sigma \in \Sigma_n$, we set $\Theta_{\tau,k,j}^\sigma(\alpha) := \Theta_{\tau_\sigma,k,j}(\sigma \cdot \alpha)$, where τ_σ is as in Definition 1.2.

Definition 2.7. A topological 2-coloured operad \mathcal{O} with colours $\mathfrak{C} = \{\mathfrak{c}, \mathfrak{o}\}$ is called an *SC type operad* if $\mathcal{O}(c_1, \dots, c_n; \mathfrak{c})$ is empty whenever there is an i such that $c_i = \mathfrak{o}$. For such an operad \mathcal{O} , suppose we have given a family of $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{c})$ -cellulation

$$\Theta_n : \mathcal{RK}(c_1, \dots, c_n; \mathfrak{c}) \rightarrow \mathbf{Top}$$

of $\mathcal{O}(c_1, \dots, c_n; \mathfrak{c})$ and a family of $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{o})$ -cellulation

$$\Theta_{\tau, k, j} : \mathcal{RK}(c_1, \dots, c_n; \mathfrak{o}) \rightarrow \mathbf{Top}$$

of $\mathcal{O}(c_1, \dots, c_n; \mathfrak{o})$. To stress the dependence of the cells upon $(c_1, \dots, c_n; c)$, we denote $\mathcal{O}(c_1, \dots, c_n; c)^{(\alpha)}$ for either the cell $\Theta_n(\alpha)$ or the cell $\Theta_{\tau, k, j}(\alpha)$ according to c . With this notation, the families of cellulations $\Theta_n, \Theta_{\tau, k, j}$ are said *compatible with the operad structure* of \mathcal{O} if

$$\begin{aligned} \gamma^{\mathcal{O}}(\mathcal{O}(c_1, \dots, c_n; c)^{(\alpha)} \times \mathcal{O}(c_{1,1}, \dots, c_{1,k_1}; c_1)^{(\alpha_1)} \times \dots \times \mathcal{O}(c_{n,1}, \dots, c_{n,k_n}; c_n)^{(\alpha_n)}) \\ \subseteq \mathcal{O}(c_{1,1}, \dots, c_{n,k_n}; c)^{(\gamma^{\mathcal{RK}}(\alpha; \alpha_1, \dots, \alpha_n))}, \end{aligned}$$

for all variables $c, c_i, c_{i,j}, \alpha, \alpha_i$.

Definition 2.8. Let $m \geq 1$. A topological SC type operad \mathcal{O} is called an *\mathcal{RK}_m -cellular operad* if there is, for each $n \geq 0$, a family of $\mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{c})$ -cellulations

$$\Theta_n : \mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{c}) \rightarrow \mathbf{Top}$$

of $\mathcal{O}(c_1, \dots, c_n; \mathfrak{c})$ and a family of $\mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{o})$ -cellulations

$$\Theta_{\tau, k, j} : \mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{o}) \rightarrow \mathbf{Top}$$

of $\mathcal{O}(c_1, \dots, c_n; \mathfrak{o})$, subject to the following two compatibilities.

1. The cellulations Θ_n and $\Theta_{\tau, k, j}$ are compatible with the Σ_n -action. This means that:
 - (a) $\Theta_n(\sigma \cdot \alpha) = \sigma \cdot \Theta_n(\alpha)$ for all $\sigma \in \Sigma_n$; and,
 - (b) the family $\Theta_{\tau, k, j} : \mathcal{RK}_m(c_1, \dots, c_n; \mathfrak{o}) \rightarrow \mathbf{Top}$ is twisting symmetric such that $\Theta_{\tau, k, j}^{\sigma}(\alpha) = \sigma \cdot \Theta_{\tau, k, j}(\alpha)$.
2. The cellulations are compatible with the operadic structure of \mathcal{O} .

We have the "Swiss Cheese analogue" to Theorem 1.16 [Ber97]:

Theorem 2.9. *Let $m \geq 1$. Any two topological \mathcal{RK}_m -cellular operads are equivalent. Moreover, the Swiss Cheese operad SC_m has a structure of an \mathcal{RK}_m -cellular operad.*

Proof. Let \mathcal{O} be a cellular SC type operad. Analogue to [BFSV03, Lemma 6.11] is the fact that $\{\text{hocolim}_{\alpha \in \mathcal{RK}(c_1, \dots, c_n; c)} \Theta_n(\alpha)\}_{n \geq 0}$ forms an operad. Moreover, operad structures are compatible with the equivalences of Lemma 2.6.

2.3 Cellular decomposition of SC type operads

We show that, for each $m \geq 1$, the operad \mathcal{SC}_m has a structure of a cellular SC type operad indexed by \mathcal{RK}_m . The "close" part of \mathcal{SC}_m , that is the little n -cube operad \mathcal{C}_n , is already shown to have a structure of a cellular operad (indexed by \mathcal{K}_m), cf. [BFV07, Ber97].

We use the description of \mathcal{SC}_m via cubes and semi-cubes given in Remark 2.2. The number $m \geq 1$ is fixed. For C_1 either a cube or a semi-cube and C_2 either a cube or a semi-cube, we note $C_1 \square_{\mu} C_2$ if there are separated by a hyperplane H_i orthogonal to the i -th coordinate axis for some $i \leq \mu$, such that whenever there is no separating hyperplane H_i for $i < \mu$, the left element C_1 lies in the negative side of H_{μ} and C_2 lies in the positive side of H_{μ} .

Note that, whenever C_1 is a semi-cube and C_2 is a cube, if H_m exists, then C_1 lies in the negative side of H_m .

For $\alpha = (\mu, \sigma) \in \mathcal{RK}(c_1, \dots, c_k; \mathfrak{o})$, we set $\mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{o})^{(\alpha)}$ the cell

$$\{(C_1, \dots, C_k) \in \mathcal{SC}_m(c_1, \dots, c_k; \mathfrak{o}) \mid C_i \square_{\mu_{i,j}} C_j \text{ if } \tilde{c}_i \rightarrow \tilde{c}_j \text{ and } C_j \square_{\mu_{i,j}} C_i \text{ if } \tilde{c}_i \leftarrow \tilde{c}_j\}.$$

To see that $\mathcal{SC}_m(c_1, \dots, c_k; c)$ is the colimit of its cells, the only delicate point is to show that if

$$x \in \mathcal{SC}_m(c_1, \dots, c_k; c)^{(\alpha)} \cap \mathcal{SC}_m(c_1, \dots, c_k; c)^{(\beta)},$$

with neither $\alpha \leq \beta$ nor $\alpha \geq \beta$ then

$$x \in \partial \mathcal{SC}_m(c_1, \dots, c_k; c)^{(\alpha)} \cap \partial \mathcal{SC}_m(c_1, \dots, c_k; c)^{(\beta)}.$$

Here $\partial \mathcal{SC}_m(c_1, \dots, c_k; c)^{(\alpha)}$ denotes the boundary $\bigcup_{\gamma < \alpha} \mathcal{SC}_m(c_1, \dots, c_k; c)^{(\gamma)}$. For such an x , we construct $\gamma \in \mathcal{RK}(c_1, \dots, c_k; c)$ such that $\gamma \leq \alpha$ and $\gamma \leq \beta$ as follows. For each \tilde{c}_i and \tilde{c}_j we define a colouring and an orientation as the minimum among $(\mu_{i,j}^{\alpha}, \sigma_{i,j}^{\alpha})$ and $(\mu_{i,j}^{\beta}, \sigma_{i,j}^{\beta})$. This minimum exists since α and β represent the same configuration x . This defines an element $\gamma \in \mathcal{RK}(c_1, \dots, c_k; c)$.

The compatibility with the operadic structure of \mathcal{SC}_m is clear.

Let us explain how works the contractibility of the cells. Let us fix a tuple of colours $\mathbf{c} = (c_1, \dots, c_k; c)$. Recall that the interior of a cell $\mathcal{SC}_m(\mathbf{c})^{(\alpha)}$ is $\mathcal{SC}_m(\mathbf{c})^{(\alpha)} \setminus (\bigcup_{\beta < \alpha} \mathcal{SC}_m(\mathbf{c})^{(\beta)})$. Contractibility of proper cells (cell with a non empty interior) onto an interior point is obtained by coordinate-wise contractions starting from the last coordinate, see [Ber97, Theorem 1.16] for details. For improper cells we remark the following. If a cell $\mathcal{SC}_m(\mathbf{c})^{(\alpha)}$ is improper then at least three cubes/semi-cubes are involved *i.e.* $k \geq 3$.

Two elements \tilde{c}_i and \tilde{c}_j of α are said related by a *positive (resp. negative) monochromatic path of colour v* if there exist an $l \geq 2$ and indices $i =: i_0, i_1, \dots, i_{l-1}, i_l := j$ such that $\tilde{c}_{i_r} \rightarrow \tilde{c}_{i_{r+1}}$ (resp. $\tilde{c}_{i_r} \leftarrow \tilde{c}_{i_{r+1}}$) and $\mu_{i_r, i_{r+1}} = v$ for all $0 \leq r \leq l-1$. Two elements \tilde{c}_i and \tilde{c}_j are called an *improper pair* if there are related by at least one monochromatic path of colour v such that $\mu_{i,j} > v$.

Then, an improper cell is exactly a cell indexed by an elements α which contains at least one improper pair.

Let α be such an element indexing an improper cell. To such an improper pair \tilde{c}_i and \tilde{c}_j of α one assigns the colour $\mu'_{i,j}$ defined as the minimal v among

all the monochromatic paths of colour ν satisfying $\mu_{i,j} > \nu$; also, one assigns the following orientation $\tilde{c}_i \rightarrow \tilde{c}_j$ if the¹ path of the minimal colour $\nu = \mu'_{i,j}$ is positive, and $\tilde{c}_i \leftarrow \tilde{c}_j$ if this path is negative.

Let β be the element obtained by applying such an assignment for each improper pair of α . Then β is such that $\beta \leq \alpha$ and it has no improper pairs, so that its corresponding cell is proper. Moreover, by construction it is unique and maximal for α . Thus, any improper cell has a unique maximal proper cell and then is contractible. \square

Remark 2.10. The inclusion $\mathcal{SC}_m \subset \mathcal{SC}_{m+1}$ is not compatible with the cellular decomposition.

Remark 2.11. We do not claim our cellular decomposition to be the "finest" one.

3 The operad \mathcal{RL}

3.1 Definition of the operad \mathcal{RL}

We describe an SC-split operad \mathcal{RL} in the category of sets, **Set**.

The operad \mathcal{RL} has a natural filtration by sub operads \mathcal{RL}_m for $m \geq 1$. For each $m \geq 1$, \mathcal{RL}_m can be thought of as a mix between the sub operads \mathcal{L}_m and \mathcal{L}_{m-1} of the Lattice paths operad \mathcal{L} introduced in [BB09].

For $m = 2$, a description of \mathcal{RL}_2 using planar trees is given in Section 5.

Definition of \mathcal{RL} . Let $\mathbf{Cat}_{*,*}$ be the category of bi-pointed small categories and functors preserving the two distinguished objects. An ordinal $[i]$ defines a category freely generated by the linear graph $l_i = \{0 \rightarrow 1 \rightarrow \dots \rightarrow i\}$. For two ordinals $[i]$ and $[j]$, the tensor product $[i] \otimes [j]$ is the category freely generated by the graph $l_i \otimes l_j$. The category $[i]$ is bi-pointed in $(0, i)$; the tensor product $[i] \otimes [j]$ is bi-pointed in $((0, 0), (i, j))$.

The set of colours of \mathcal{RL} is

$$Col = Col_c \sqcup Col_o,$$

where Col_c is the set \mathbb{N} of natural numbers and Col_o is the set of natural numbers decorated with an underline. Hence, $n \in Col_c$ whereas $\underline{n} \in Col_o$. In general a colour in Col is denoted by \tilde{n} , so that it is either n or \underline{n} .

The set $\mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ is defined as:

$$\mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \emptyset$$

if $\tilde{n} = n \in Col_c$ and if there is an i such that $\tilde{n}_i = \underline{n}_i \in Col_o$;

$$\mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \mathbf{Cat}_{*,*}([\tilde{n} + 1], [\tilde{n}_1 + 1] \otimes [\tilde{n}_2 + 1] \otimes \dots \otimes [\tilde{n}_k + 1])$$

¹More precisely, such a path is not necessary unique. However, if \tilde{c}_i and \tilde{c}_j are related by two monochromatic paths with the same colour ν then both have the same direction, either positive or negative.

3.1 Definition of the operad \mathcal{RL}

else.

The substitutions maps are given by tensor and composition in $\mathbf{Cat}_{*,*}$.
For instance, an element $x \in \mathcal{RL}(n_1, \underline{n_2}; \underline{n})$ is a functor

$$(3.1) \quad x : [\underline{n} + 1] \rightarrow [n_1 + 1] \otimes [\underline{n_2} + 1]$$

that sends $(0, \underline{n} + 1)$ on $((0, 0), (n_1 + 1, \underline{n_2} + 1))$ and is determined by the image of the n remaining objects of $[\underline{n} + 1]$ and the morphisms into the lattice $[n_1 + 1] \otimes [\underline{n_2} + 1]$.

Example 3.1. The following lattice x belongs to $\mathcal{RL}(3, \underline{2}; \underline{3})$:

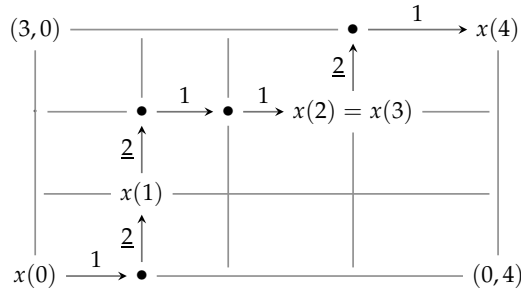


Figure 3.1: Lattice paths of $(\underline{12}|\underline{211}||\underline{21})^\circ$.

The elements of $\mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ correspond bijectively to a string of (decorated) natural numbers separated by vertical bars. Indeed, let us consider an $x \in \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$. The relative lattice x is a path from $x(0)$ to $x(n + 1)$ made of edges in the grid $l_1 \otimes \dots \otimes l_k$. By running through x from $x(0)$ to $x(n + 1)$ we construct the integer-string with vertical bars as follows. To each parallel edge to the i -axis of the grid $l_1 \otimes \dots \otimes l_k$ we assign i if $\tilde{n}_i = n_i$ or \underline{i} if $\tilde{n}_i = \underline{n}_i$; to each object $x(s)$ for $1 \leq s \leq n$ we assign a vertical bar. Additionally, we put an extra labelled according to the nature of the output colour. Example 3.1 gives

$$x = (\underline{12}|\underline{211}||\underline{21})^\circ.$$

Example 3.2.

$$(121)^\circ \in \mathcal{RL}(1, 0; \underline{0}) \text{ whereas } (121) \in \mathcal{RL}(1, 0; 0).$$

Let us expose the corresponding composition on integer-string representations via an example.

Example 3.3.

$$(\underline{12}|\underline{14231}||\underline{24})^\circ \circ_2 (13|213|31)^\circ = (\underline{124}|\underline{1632451}||\underline{426})^\circ$$

The composition is at $\underline{2}$ and the second term has 3 outputs. Then one has renumbered the integer-string $(\underline{12}|\underline{14231}||\underline{24})^\circ$ by increasing by $2 = 3 - 1$ the

numbers greater than $\underline{2}$; one gets $(\underline{12}|16\underline{2}51||\underline{26})^\circ$. One has increased the numbers of the second integer-string $(\underline{13}|21\underline{3}|\underline{31})^\circ$ by $1 = 2 - 1$: $(\underline{24}|32\underline{4}|\underline{42})^\circ$. Finally, one has substituted the three occurrences of $\underline{2}$ by the three sub-sequences $\underline{24}$, $3\underline{24}$ and $\underline{42}$.

We use left action for the symmetric group: for $\sigma \in \Sigma_k$ and $x \in \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$, the string-integer representation of $\sigma \cdot x \in \mathcal{RL}(\tilde{n}_{\sigma^{-1}(1)}, \dots, \tilde{n}_{\sigma^{-1}(k)}; \tilde{n})$ is obtained by permuting the number i (resp. \underline{i}) of the string-integer representation of x by the number $\sigma(i)$ (resp. $\underline{\sigma(i)}$).

Example 3.4. For $x = (\underline{12}|3\underline{2}11||\underline{21})^\circ$ and $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$ we have:

$$\sigma \cdot x = (\underline{23}|1\underline{3}22||\underline{32})^\circ.$$

The underlying category of \mathcal{RL} . Let \mathbf{Cat} be the category of small categories. Via *Joyal-duality*:

$$\mathbf{Cat}_{*,*}([n+1], [m+1]) \cong \mathbf{Cat}([m], [n]).$$

The bijection is given by $(\phi : [n+1] \rightarrow [m+1]) \leftrightarrow (\psi : [m] \rightarrow [n])$ given by $\psi(i) + 1 = \min\{j | \phi(j) > i\}$ and $\phi(j) - 1 = \max\{i | \psi(i) < j\}$. We have immediately that:

$$\begin{aligned} \mathcal{RL}(n; m) &= \text{Hom}_\Delta([n], [m]); \\ \mathcal{RL}(\underline{n}; \underline{m}) &= \text{Hom}_\Delta([n], [m]); \\ \mathcal{RL}(n; \underline{m}) &= \text{Hom}_\Delta([n], [m]), \end{aligned}$$

where Δ is simplicial category. Thus, the underlying category $(\mathcal{RL})_u$ of \mathcal{RL} is the category with objects

$$\begin{aligned} [n] &= \{0 < 1 < \dots < n\} \text{ for } n \in \mathbb{N}; \text{ and} \\ \underline{[n]} &= \{\underline{0} < \underline{1} < \dots < \underline{n}\} \text{ for } n \in \mathbb{N}; \end{aligned}$$

and with hom-sets

$$\begin{aligned} \text{Hom}([n], [m]) &= \text{Hom}_\Delta([n], [m]); \\ \text{Hom}(\underline{[n]}, \underline{[m]}) &= \text{Hom}_\Delta([n], [m]); \\ \text{Hom}([n], \underline{[m]}) &= \text{Hom}_\Delta([n], [m]); \\ \text{Hom}(\underline{[n]}, [m]) &= \emptyset. \end{aligned}$$

The two sub-categories $(\mathcal{RL})_u^c$ and $(\mathcal{RL})_u^o$ are (canonically isomorphic to) the category Δ .

This implies that, for each $k \geq 0$, the functor

$$(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) \mapsto \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$$

is a multisimplicial/cosimplicial set.

3.1 Definition of the operad \mathcal{RL}

Filtration by sub operads \mathcal{RL}_m . Let us define two maps

$$c_{i,j}, c'_{i,j} : \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) \rightarrow \mathbb{N}.$$

For $1 \leq i < j \leq k$, we denote by

$$\phi_{ij} : \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) \rightarrow \mathcal{RL}(\tilde{n}_i, \tilde{n}_j; \tilde{n})$$

the projection induced by the canonical projection

$$p_{ij} : [\tilde{n}_1 + 1] \otimes \dots \otimes [\tilde{n}_k + 1] \rightarrow [\tilde{n}_i + 1] \otimes [\tilde{n}_j + 1].$$

For $x \in \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ and $1 \leq i < j \leq k$, we define $c_{ij}(x)$ as the number of changes of directions in the lattice paths $\phi_{ij}(x)$.

The second number $c'_{i,j}(x)$ is defined as follows. Since $x \in \mathcal{RL}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$, its integer-string representation is in particular a sequence of numbers (underlined or not) between 1 and k . For $1 \leq i \leq k$, we set i^- (resp. \underline{i}^-) the first occurrence of i (resp. \underline{i}) in the integer-string representation. Equivalently, i^- (resp. \underline{i}^-) is the first edge of the lattice x which is in the i -th direction. We write $n^- < m^-$ if the element n^- precedes m^- .

For $1 \leq i < j \leq k$, we set:

$$(3.2) \quad c'_{i,j}(x) = \begin{cases} c_{i,j}(x) & \text{if } i^- < \underline{j}^-; \\ c_{i,j}(x) + 1 & \text{if } i^- > \underline{j}^-; \\ c_{i,j}(x) + 1 & \text{if } \underline{i}^- < j^-; \\ c_{i,j}(x) & \text{if } \underline{i}^- > j^-. \end{cases}$$

For $m \geq 1$, we define $\mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ as the set of elements $x \in \mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ satisfying the three conditions:

$$\begin{aligned} \max_{(i,j)} c_{i,j}(x) &\leq m; \\ \max_{(\underline{i},j)} c_{i,j}(x) &\leq m - 1; \text{ and,} \\ \max_{(\underline{i},j) \text{ or } (i,j)} c'_{i,j}(x) &\leq m. \end{aligned}$$

Remark 3.5. Changing the filtration defined in (3.2) by:

$$(3.3) \quad c''_{i,j}(x) = \begin{cases} c_{i,j}(x) + 1 & \text{if } i^- < \underline{j}^-; \\ c_{i,j}(x) & \text{if } i^- > \underline{j}^-; \\ c_{i,j}(x) & \text{if } \underline{i}^- < j^-; \\ c_{i,j}(x) + 1 & \text{if } \underline{i}^- > j^-. \end{cases}$$

we get another filtration of \mathcal{RL} by sub-operads \mathbf{scL}_m . These operads seem to be more adapted for proving the Swiss Cheese version of Deligne's conjecture, at least for $m = 2$.

3.2 The operad $\text{Coend}_{\mathcal{RL}_m}(\delta)$ as a Swiss Cheese operad

We apply the method developed in [BB09, Sections 3.5-3.6]. More precisely, given a functor $\delta : \Delta \rightarrow \mathbf{C}$ where \mathbf{C} is a *monoidal model category*, we construct a zig-zag of weak equivalences of operads

$$(3.4) \quad \text{Coend}_{\mathcal{RL}_m}(\delta) \xleftarrow{\sim} \text{Coend}_{\widehat{\mathcal{RL}}_m}(\delta) \xrightarrow{\sim} \mathcal{B}_\delta \mathcal{RK}_m,$$

whenever δ satisfies some conditions. Here, $\mathcal{B}_\delta \mathcal{A}$ denote the δ -realization of the nerve of the category \mathcal{A} . The intermediate operad $\widehat{\mathcal{RL}}_m$ is defined using homotopy colimits in \mathbf{C} applied on a decomposition of \mathcal{RL}_m indexed by \mathcal{RK}_m . Such a decomposition is a consequence of the following lemma.

Lemma 3.6. *There is a morphism $q : \mathcal{RL} \rightarrow \mathcal{RK}$ of filtered operads.*

Proof. Let us recall that if $x \in \mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_k : \tilde{n})$ then:

$$\begin{array}{ll} c_{i,j}(x) \leq m & \text{if } \tilde{n}_i = n_i, \tilde{n}_j = n_j; \\ c_{i,j}(x) \leq m - 1 & \text{if } \tilde{n}_i = \underline{n}_i, \tilde{n}_j = \underline{n}_j; \\ c_{i,j}(x) \leq m & \text{if } i^- < j^-; \\ c_{i,j}(x) \leq m - 1 & \text{if } i^- > j^-; \\ c_{i,j}(x) \leq m - 1 & \text{if } \underline{i}^- < \underline{j}^-; \\ c_{i,j}(x) \leq m & \text{if } \underline{i}^- > \underline{j}^-, \end{array}$$

for $1 \leq i < j \leq k$.

The element $q(x) = (\mu, \sigma) \in \mathcal{RK}$ is defined, for $1 \leq i < j \leq n$, by:

$$\begin{aligned} \mu_{i,j} &= c_{i,j}(x) \\ \tilde{c}_i &\rightarrow \tilde{c}_j \text{ for } \tilde{i}^- > \tilde{j}^- \\ \tilde{c}_i &\leftarrow \tilde{c}_j \text{ for } \tilde{i}^- < \tilde{j}^-. \end{aligned}$$

The fact that q preserves the filtration is clear from its definition. Let us sketch the proof that q is a morphism of operad by using similar arguments to [BB09, Proposition 3.4]. It is straightforward to check that the image of q is contained in a sub-operad \mathcal{RK}_m^- of \mathcal{RK}_m . This operad \mathcal{RK}_m^- consists of colouring and orientation on complete graphs that are acyclic, that is, no (polychromatic) cyclic orientations are allowed. Such acyclic orientations on a complete graph on n elements correspond to the choice of a permutation in Σ_n . Then $\mathcal{RK}_m^-(c_1, \dots, c_n; c)$ is the set of pair (μ, σ) of $\{1, \dots, m\}^{\binom{n}{2}} \times \Sigma_n$ submitted to the same conditions than in (2.1) (i.e. $\tilde{c}_i \rightarrow \tilde{c}_j$ corresponds to $\sigma(i) < \sigma(j)$ and $\tilde{c}_i \leftarrow \tilde{c}_j$ corresponds to $\sigma(i) > \sigma(j)$). An explicit formula for operadic composition is given in [BB09, Proposition 3.2] for the non Swiss Cheese case; its the same formula in our context. In particular, permutations are composed as in the (Swiss Cheese version of the) Symmetric operad \mathcal{RS} , and we have an operadic map $\mathcal{RK}_m^- \rightarrow \mathcal{RS}$.

3.2 The operad $\text{Coend}_{\mathcal{RL}_m}(\delta)$ as a Swiss Cheese operad

Then, it is sufficient to show that the composite $\mathcal{RL}_m \rightarrow \mathcal{RK}_m^- \rightarrow \mathcal{R}\Sigma$ is a morphism of operads. This follows almost directly from the definition of q . Since q "reverses" the orientations we have to check that the morphisms $Rev_n : \Sigma_n \rightarrow \Sigma_n$, that send σ to $Rev_n \circ (\sigma)$ where $Rev_n = \begin{pmatrix} 1 & 2 & \dots & n \\ n & \dots & 2 & 1 \end{pmatrix}$, induce a morphism of operad $Rev : \mathcal{R}\Sigma \rightarrow \mathcal{R}\Sigma$. The Swiss Cheese symmetric operad $\mathcal{R}\Sigma$ consists of sets $\mathcal{R}\Sigma(c_1, \dots, c_n; c)$ to be either Σ_n or empty according to the usual conditions on the colours; the operadic compositions are defined as in the classical non colour case. \square

Remark 3.7. Our morphism "reverses" the orientations. This is due to the choice of cellulation of \mathcal{SC}_m we have made. However, if $\text{sc}\mathcal{L}_m$ is the operad as in Remark 3.5, the similar morphism $q' : \text{sc}\mathcal{L}_m \rightarrow \mathcal{RK}_m$ dose not "reverse" the orientations.

Let us recall that for $\delta^c : \Delta \rightarrow \mathbf{C}$, the functor

$$\xi(\mathcal{RL}_m)_k(\delta^c) : \Delta \rightarrow \mathbf{C}$$

denotes the realization of $\mathcal{RL}_m(k)$:

$$\xi(\mathcal{RL}_m)_k(\delta^c)(n) = \mathcal{RL}_m(k) \left(\underbrace{-, \dots, -}_k ; n \right) \otimes_{\Delta^k} \underbrace{\delta^c(-) \otimes \dots \otimes \delta^c(-)}_k,$$

where we use implicitly the strong monoidal functor

$$\begin{aligned} \mathbf{Set} &\rightarrow \mathbf{C} \\ E &\mapsto \prod_{e \in E} 1_{\mathbf{C}}. \end{aligned}$$

And similarly, for $\delta = (\delta^c, \delta^o)$, with $\delta^c, \delta^o : \Delta \rightarrow \mathbf{C}$, the functor

$$\xi(\mathcal{RL}_m)_{\tau, k, j}(\delta) : \Delta \rightarrow \mathbf{C}$$

denotes the realization of $\mathcal{RL}_m(k, j)$.

We use the same functor $\delta^c = \delta^o$ that we denote by δ . We fix two functors

$$\delta_{\mathbf{Top}} : \Delta \xrightarrow{\delta_{yon}} \mathbf{Set}^{\Delta^{\text{op}}} \xrightarrow{|\ - \ |} \mathbf{Top}$$

and

$$\delta_{\mathbf{Z}} : \Delta \xrightarrow{\delta_{yon}} \mathbf{Set}^{\Delta^{\text{op}}} \xrightarrow{C_*(-; \mathbf{Z})} \mathbf{Ch}(\mathbf{Z})$$

where:

- $\delta_{yon}([n]) = \text{Hom}_{\Delta}(-, [n])$ is the Yoneda functor;
- $|\ - \ | : \mathbf{Set}^{\Delta^{\text{op}}} \rightarrow \mathbf{Top}$ is the geometric realization; and,

- $C_*(-; \mathbb{Z}) : \mathbf{Set}^{\Delta^{\text{op}}} \rightarrow \mathbf{Ch}(\mathbb{Z})$ is the normalized chain complex.

For $\alpha \in \mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c)$ and $\tilde{n}_i \in \text{Col}_{c_i}$, $\tilde{n} \in \text{Col}_c$, we set

$$(\mathcal{R}\mathcal{L}_m)_\alpha(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \{x \in \mathcal{R}\mathcal{L}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) \mid q(x) \leq \alpha\}.$$

Then we have:

$$\mathcal{R}\mathcal{L}_m(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \text{colim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c)} (\mathcal{R}\mathcal{L}_m)_\alpha(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}),$$

for all $(c_1, \dots, c_k; c)$ and $\tilde{n}_i \in \text{Col}_{c_i}$, $\tilde{n} \in \text{Col}_c$. Thus

$$\xi(\mathcal{R}\mathcal{L}_m)_k(\delta) = \text{colim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c)} \xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta)$$

whenever $c_i = c$ for all $1 \leq i \leq k$; and

$$\xi(\mathcal{R}\mathcal{L}_m)_{\tau, k, j}(\delta) = \text{colim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_{k+j}; \mathfrak{o})} \xi(\mathcal{R}\mathcal{L}_m)_\alpha(\delta),$$

where $\tau \in \text{Sh}(k, j)$ is the shuffle determined by $(c_1, \dots, c_{k+j}; \mathfrak{o})$.

We suppose \mathbf{C} endowed with a *monoidal model structure*, cf. [Hov99]. The categories $\mathbf{Set}^{\Delta^{\text{op}}}$ and \mathbf{Top} are considered with the Quillen model structure; the category $\mathbf{Ch}(\mathbb{Z})$ is considered with its projective model structure.

In [BB09, Sections 3.5-3.6] it is proved that, given a *standard system of simplices* δ , the operad $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta)$ is weakly equivalent to $\mathcal{B}_\delta \mathcal{R}\mathcal{K}$, provided that operad $\mathcal{R}\mathcal{L}_m$ is *strongly δ -reductive*.

A standard system of simplices $\delta : \Delta \rightarrow \mathbf{C}$ (cf. [BM06, Definition A.6]) provides, in particular, a "monoidal symmetric" cosimplicial frame $(-) \otimes_\Delta \delta$ so that homotopy colimits are compatible with the symmetric monoidal structure of \mathbf{C} . Moreover, for such a δ , the realisation functor $(-) \otimes_\Delta \delta$ preserves and reflects weak equivalences.

It is proved in [BM06, A.13, A.16] that the functors δ_{yon} , δ_{Top} and $\delta_{\mathbb{Z}}$ are such standard system of simplicies.

The *strong δ -reductivity* allows us to show that the zig-zag of weak equivalence (3.4) follows essentially from the weak equivalences $\xi(\mathcal{R}\mathcal{L}_m)_k(\delta)^n \rightarrow \xi(\mathcal{R}\mathcal{L}_m)_k(\delta)^0$ and $\xi(\mathcal{R}\mathcal{L}_m)_{\tau, k, j}(\delta)^n \rightarrow \xi(\mathcal{R}\mathcal{L}_m)_{\tau, k, j}(\delta)^0$, for all $n \geq 0$, $k \geq 0$ and $j \geq 0$.

We define the *strong δ -reductivity* condition for $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta)$ similarly to [BB09, Definition 3.7] by extending it to the functors

$$\xi(\mathcal{R}\mathcal{L}_m)_{\tau, k, j}(\delta)^n \rightarrow \xi(\mathcal{R}\mathcal{L}_m)_{\tau, k, j}(\delta)^0, \text{ for all } n \geq 0, k \geq 0 \text{ and } j \geq 0,$$

see next definitions.

Definition 3.8. A weak equivalence in \mathbf{C} is called *universal* if any pullback of it is again a weak equivalence.

Definition 3.9. Let δ be a standard system of simplices in \mathbf{C} . The operad $\mathcal{R}\mathcal{L}_m$ is called *δ -reductive* if:

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- for any $n \geq 0$ and $k \geq 0$, the map $\zeta(\mathcal{R}\mathcal{L}_m)_k(\delta)^n \rightarrow \zeta(\mathcal{R}\mathcal{L}_m)_k(\delta)^0$ is a universal weak equivalence; and,
- for any $n \geq 0, k \geq 0$ and $j \geq 0$, the map $\zeta(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta)^n \rightarrow \zeta(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta)^0$ is a universal weak equivalence.

The operad $\mathcal{R}\mathcal{L}_m$ is called *strongly δ -reductive* if in addition the induced maps $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta)(k) \rightarrow \zeta(\mathcal{R}\mathcal{L}_m)_k(\delta)^0$ and $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta)(\tau, k, j) \rightarrow \zeta(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta)^0$ are universal weak equivalence in \mathbf{C} .

The proof of [BB09, Theorem 3.8] can be applied *mutatis mutandis* to the functors $\zeta(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta)$. So, we have almost for free the analogue to [BB09, Theorem 3.8]:

Theorem 3.10. *Let δ be a standard system of simplices in a model monoidal category \mathbf{C} with a zero object. If the operad $\mathcal{R}\mathcal{L}_m$ is strongly δ -reductive, then the operad $\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta)$ is weakly equivalent to $\mathcal{B}_\delta \mathcal{R}\mathcal{K}$.*

Proof. We outline the proof.

We construct a zig-zag of weak equivalences of operads

$$\text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta) \xleftarrow{\sim} \text{Coend}_{\widehat{\mathcal{R}\mathcal{L}_m}}(\delta) \xrightarrow{\sim} \mathcal{B}_\delta \mathcal{R}\mathcal{K}_m.$$

The intermediate operad $\widehat{\mathcal{R}\mathcal{L}_m}$ is defined as

$$\widehat{\mathcal{R}\mathcal{L}_m}(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}) = \text{hocolim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_k; c)} (\mathcal{R}\mathcal{L}_m)_\alpha(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n}),$$

for all $(c_1, \dots, c_k; c)$ and $\tilde{n}_i \in \text{Col}_{c_i}$, $\tilde{n} \in \text{Col}_c$. The properties of the standard system of simplicies δ imply that $\text{Coend}_{\widehat{\mathcal{R}\mathcal{L}_m}}(\delta)$ is an operad.

We denote by $\widehat{\zeta}(\mathcal{R}\mathcal{L}_m)$ the corresponding SC functor-operad of $\widehat{\mathcal{R}\mathcal{L}_m}$, so that

$$\widehat{\zeta}(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta) = \text{hocolim}_{\mathcal{R}\mathcal{K}_m(c_1, \dots, c_{k+j}; \mathfrak{o})} \zeta(\mathcal{R}\mathcal{L}_m)_\alpha(\delta),$$

for $(c_1, \dots, c_k; \mathfrak{o})$ where $\tau \in \text{Sh}(k, j)$ is the shuffle determined by $(c_1, \dots, c_{k+j}; \mathfrak{o})$. The left hand map $\text{Coend}_{\widehat{\mathcal{R}\mathcal{L}_m}}(\delta) \rightarrow \text{Coend}_{\mathcal{R}\mathcal{L}_m}(\delta)$ in the zig-zag is induced by the maps $\widehat{\zeta}(\mathcal{R}\mathcal{L}_m)_k(\delta) \rightarrow \zeta(\mathcal{R}\mathcal{L}_m)_k(\delta)$ and $\widehat{\zeta}(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta) \rightarrow \zeta(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta)$. Because of the strong δ -reductivity condition, it is sufficient to show that

$$\widehat{\zeta}(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta)^0 \rightarrow \zeta(\mathcal{R}\mathcal{L}_m)_{\tau,k,j}(\delta)^0,$$

is a weak equivalence (and similarly for $\widehat{\zeta}(\mathcal{R}\mathcal{L}_m)_k(\delta)^0$). This follows from general properties of standard system of simplicies.

To show the other weak equivalence

$$\text{Coend}_{\widehat{\mathcal{R}\mathcal{L}_m}}(\delta) \rightarrow \mathcal{B}_\delta \mathcal{R}\mathcal{K}_m,$$

we first remark that $\mathcal{B}_\delta \mathcal{RK}_m(c_1, \dots, c_{k+j}; c) \cong \text{hocolim}_{\mathcal{RK}_m(c_1, \dots, c_{k+j}; c)}(cc\delta^0) \cong \text{Hom}(\delta, \widehat{\xi}(\mathcal{RL}_m)_{\tau, k, j}(cc\delta^0))$ where $cc\delta^0$ is the constant cosimplicial object at $\delta^0 = 1_{\mathcal{C}}$. Then it is sufficient to prove that

$$\text{Hom}_{\mathcal{C}^\Delta}(\delta, \widehat{\xi}(\mathcal{RL}_m)_{\tau, k, j}(cc\delta^0)) \rightarrow \text{Hom}_{\mathcal{C}^\Delta}(\delta, \widehat{\xi}(\mathcal{RL}_m)_{\tau, k, j}(\delta^0))$$

is a weak equivalence (and the same for $\widehat{\xi}(\mathcal{RL}_m)_k(\delta^0)$). Since $\text{Hom}_{\mathcal{C}^\Delta}(-, -)$ preserves weak equivalences, this is satisfied if

$$\widehat{\xi}(\mathcal{RL}_m)_{\tau, k, j}(\delta)^n \rightarrow \widehat{\xi}(\mathcal{RL}_m)_{\tau, k, j}(cc\delta^0)^n$$

is a weak equivalence for all $n \geq 0$. Since we have

$$\widehat{\xi}(\mathcal{RL}_m)_{\tau, k, j}(\delta)^n \cong \text{hocolim}_{\mathcal{RK}_m(c_1, \dots, c_{k+j}; \delta)} \xi(\mathcal{RL}_m)_\alpha(\delta),$$

it remains to show that

$$\xi(\mathcal{RL}_m)_\alpha(\delta)^n \rightarrow \xi(\mathcal{RL}_m)_\alpha(cc\delta^0)^n = 1_{\mathcal{C}}$$

is a weak equivalence for all α and $n \geq 0$. By δ -reductivity the left vertical arrow in

$$\begin{array}{ccc} \xi(\mathcal{RL}_m)_\alpha(\delta)^n & \longrightarrow & \xi(\mathcal{RL}_m)_{\tau, k, j}(\delta)^n \\ \downarrow & & \downarrow \\ \xi(\mathcal{RL}_m)_\alpha(\delta)^0 & \longrightarrow & \xi(\mathcal{RL}_m)_{\tau, k, j}(\delta)^0 \end{array}$$

is a weak equivalence. Moreover, for each $(c_1, \dots, c_k; c)$ and $\alpha \in \mathcal{RK}_m(c_1, \dots, c_k; c)$, the object $\xi(\mathcal{RL}_m)_\alpha(\delta)^0$ is weakly contractible. The latter is due to the following fact. Properties of the standard system of simplices imply that: the realisation functor $(-) \otimes_{\Delta} \delta$ preserves and reflect weak equivalences; the two objects $\xi(\mathcal{RL}_m)_\alpha(\delta)^0$ and $\xi(\mathcal{RL}_m)_\alpha(\delta_{\text{yon}})^0 \otimes_{\Delta} \delta$ are weakly equivalent; so that, it is sufficient to prove that for $\delta = \delta_{\text{Top}}$, the space $\xi(\mathcal{RL}_m)_\alpha(\delta_{\text{Top}})^0$ is weakly contractible. \square

In Theorem 3.10 we have used our version of [MS04, Lemma 14.8] (see also [BB09, Lemma 3.9]):

Lemma 3.11. *For each $m \geq 1$, $(c_1, \dots, c_k; c)$ and $\alpha \in \mathcal{RK}_m(c_1, \dots, c_k; c)$, the space $\xi(\mathcal{RL}_m)_\alpha(\delta_{\text{Top}})^0$ is weakly contractible.*

Proof. The proof is quite similar to [MS04, Lemma 14.8]; it is sufficient to check that the arguments are compatible with our operads \mathcal{RK} and \mathcal{RL} . We outline the proof for convenience.

For each $\alpha \in \mathcal{RK}(c_1, \dots, c_k; c)$, we construct a retraction

$$\iota : \xi(\mathcal{RL}_m)_\alpha(\text{Cone}(\delta_{\text{Top}}), \dots, \text{Cone}(\delta_{\text{Top}}))^0 \rightrightarrows \xi(\mathcal{RL}_m)_\alpha(\delta_{\text{Top}}, \dots, \delta_{\text{Top}})^0 : \rho$$

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such that $\rho \circ \iota = \text{id}$. The functor $\text{Cone}(\delta_{\text{Top}}) : \Delta \rightarrow \mathbf{Top}$ is defined as follows.

$$\begin{aligned} \text{Cone}(\delta_{\text{Top}})^n &= \delta_{\text{Top}}^{n+1} \\ \text{Cone}(\delta_{\text{Top}})(f : [l] \rightarrow [n]) &= \delta_{\text{Top}}(\tilde{f} : [l+1] \rightarrow [n+1]), \end{aligned}$$

where $\tilde{f} : [l+1] \rightarrow [n+1]$ is defined as $\tilde{f}(0) = 0$ and $\tilde{f}(k) = f(k-1) + 1$ for $1 \leq k \leq l+1$.

The map ρ sends the class of (x, u_1, \dots, u_k) to the class of $(x, d^0 u_1, \dots, d^0 u_k)$ where $d^0 : \delta_{\text{Top}}^n \rightarrow \delta_{\text{Top}}^{n+1}$ is the zeroth cosimplicial face operator.

For $x \in \mathcal{RL}_\alpha(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$, with $(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ according to $(c_1, \dots, c_k; c)$, the map ι assigns to the class of (x, u_1, \dots, u_k) the class of $(\tilde{x}, u_1, \dots, u_k)$ where \tilde{x} is the lattice obtained from x by doubling the first occurrence of i (or \bar{i}) for $1 \leq i \leq k$ in its integer-string representation. Thus the lattice $\tilde{x} \in \mathcal{RL}_\alpha(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$.

Since $\text{Cone}(\delta_{\text{Top}})$ is contractible, it provides a map

$$\xi(\mathcal{RL}_m)_\alpha(\text{Cone}(\delta_{\text{Top}}), \dots, \text{Cone}(\delta_{\text{Top}}))^0 \rightarrow \xi(\mathcal{RL}_m)_\alpha(*, \dots, *)^0 = *$$

which is a weak equivalence. \square

Once again, the proof of the following is completely similar to [BB09, Examples 3.10].

Proposition 3.12. *For δ being δ_{Top} or $\delta_{\mathbb{Z}}$, the operad \mathcal{RL}_m is strongly δ -reductive. Consequently, the operad $\text{Coend}_{\mathcal{RL}_m}(\delta)$ is equivalent to the topological (resp. chain) Swiss Cheese operad \mathcal{SC}_m (resp. $C_*\mathcal{SC}_m$) for δ being δ_{Top} (resp. $\delta_{\mathbb{Z}}$).*

Proof. A) The topological case $\delta = \delta_{\text{Top}}$. Let us recall from [MS04, Proposition 13.4] that, for all $k \geq 0$, we have an isomorphism of cosimplicial spaces

$$(3.5) \quad \theta_{\text{Top}}^* : \xi(\mathcal{RL}_m)_k(\delta_{\text{Top}})^* \rightarrow \delta_{\text{Top}}^* \times \xi(\mathcal{RL}_m)_k(\delta_{\text{Top}})^0.$$

The same argument² applies to show that, for all $k \geq 0, j \geq 0$ and $\tau \in \text{Sh}(k, j)$,

$$(3.6) \quad \theta_{\text{Top}}^* : \xi(\mathcal{RL}_m)_{\tau, k, j}(\delta_{\text{Top}})^* \rightarrow \delta_{\text{Top}}^* \times \xi(\mathcal{RL}_m)_{\tau, k, j}(\delta_{\text{Top}})^0$$

is an isomorphism of cosimplicial spaces. Moreover, both the cosimplicial isomorphisms (3.5) and (3.6) are compatible with the projection onto the second factor so that one obtains trivial fibrations θ_{Top}^n and θ'_{Top}^n for each $n \geq 0$. Moreover θ' induces a homeomorphism on $\text{Coend}_{\mathcal{L}_m}(\delta_{\text{Top}})(\tau, k, j) \cong \text{Hom}_\Delta(\delta_{\text{Top}}, \delta_{\text{Top}}) \times \xi(\mathcal{RL}_m)_{\tau, k, j}(\delta_{\text{Top}})^0$. Using the contractibility of $\text{Hom}_\Delta(\delta_{\text{Top}}, \delta_{\text{Top}})$ one gets a weak equivalence

$$(3.7) \quad \text{Coend}_{\mathcal{RL}_m}(\delta_{\text{Top}})(\tau, k, j) \rightarrow \xi(\mathcal{RL}_m)_{\tau, k, j}(\delta_{\text{Top}})^0,$$

² In Proposition 12.7 and Proposition 13.4 from [MS04] it is sufficient to consider elements

$\bar{k} \sqcup \bar{j} \xleftarrow{f} T \xrightarrow{h} s$ with $\bar{k} \sqcup \bar{j}$ instead of \bar{k} .

which is also universal.

B) *The chain complex case* $\delta = \delta_{\mathbb{Z}}$. We will show that the maps

$$(3.8) \quad \theta_{\mathbb{Z}}^* : \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^* \rightarrow \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^0$$

and

$$(3.9) \quad \theta'_{\mathbb{Z}} : \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^* \rightarrow \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^0$$

are trivial Reedy fibrations in $\mathbf{Ch}(\mathbb{Z})^{\Delta}$. Indeed, the trivial Reedy fibrations are objectwise trivial fibrations what implies the $\delta_{\mathbb{Z}}$ -reductivity. Moreover, since $\delta_{\mathbb{Z}}$ is a standard system of simplicies, the $\delta_{\mathbb{Z}}$ -totalization functor $\mathrm{Hom}_{\mathbf{Ch}(\mathbb{Z})}(\delta_{\mathbb{Z}}, -)$ is a right Quillen functor. Then, the induced maps

$$(3.10) \quad \mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})(k) \rightarrow \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^0$$

and

$$(3.11) \quad \mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})(\tau, k, j) \rightarrow \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^0$$

are trivial fibrations in $\mathbf{Ch}(\mathbb{Z})$.

Recall that the realization functors $| - |_{\delta_{\mathrm{Top}}}$ and $| - |_{\delta_{\mathbb{Z}}}$ preserve and reflect the weak equivalences. Then, since θ_{Top}^n and θ'_{Top}^n are weak equivalences for each $n \geq 0$, we deduce that for $n \geq 0$, the maps

$$(3.12) \quad \theta_{\mathbb{Z}}^n : \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^n \rightarrow \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^0$$

and

$$(3.13) \quad \theta'_{\mathbb{Z}} : \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^n \rightarrow \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^0$$

so are. It remains to show that $\theta_{\mathbb{Z}}^*$ and $\theta'_{\mathbb{Z}}^*$ are Reedy fibrations in $\mathbf{Ch}(\mathbb{Z})^{\Delta}$. Let us denote by $M_n X$ the matching object of X . Since $M_n \mathrm{cc} \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^0 = \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^0$, it is sufficient to show that

Lemma 3.13. *The maps*

$$(3.14) \quad j_n : \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^n \rightarrow M_n \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^*$$

and

$$(3.15) \quad j'_n : \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^n \rightarrow M_n \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^*,$$

induced by $\theta_{\mathbb{Z}}^n$ and $\theta'_{\mathbb{Z}}^n$ respectively, are surjective for each $n \geq 1$.

We postpone the technical proof of this lemma to Section 8.

4 The relative surjection operad \mathcal{RS}_m

For $m \geq 1$, we define an operad \mathcal{RS}_m , sub-operad of $\text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$. We show that the inclusion $\mathcal{RS}_m \hookrightarrow \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ is a weak equivalence.

Let $C = \{\mathfrak{c}, \mathfrak{o}\}$ be the set of colours of \mathcal{RS}_m , $m \geq 1$. Recall that a tuple (c_1, \dots, c_{k+j}) of colours of C determines a shuffle $\tau \in \text{Sh}(k, j)$.

As complexes, we set:

1. $\mathcal{RS}_m(c_1, \dots, c_k; \mathfrak{c}) := \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^0$, if $c_i = \mathfrak{c}$ for all $1 \leq i \leq k$;
2. $\mathcal{RS}_m(c_1, \dots, c_{k+j}; \mathfrak{o}) := \zeta(\mathcal{RL}_m)_{\tau, k, j}(\delta_{\mathbb{Z}})^0$.

For (c_1, \dots, c_k) where $c_i \in \mathfrak{C} = \{\mathfrak{c}, \mathfrak{o}\}$, we set

$$\tilde{c}_i = \begin{cases} i & \text{if } c_i = \mathfrak{c}; \\ \underline{i} & \text{if } c_i = \mathfrak{o}. \end{cases}$$

A surjection $f : \{1, \dots, k+r\} \rightarrow \{\tilde{c}_1, \dots, \tilde{c}_k\}$ is called *degenerate* if there exists an i such that $f(i) = f(i+1)$.

Then, an element in $\mathcal{RS}_m(c_1, \dots, c_k; \mathfrak{c})$ is a sum of *non degenerate* surjections $f : \{1, \dots, k+r\} \rightarrow \{\tilde{c}_1, \dots, \tilde{c}_k\}$. Indeed, a generator of the complex

$$\zeta(\mathcal{RL}_m)_{\tau, k, j}(\delta_{\mathbb{Z}})^0 = \mathcal{RL}_m(-, \dots, -; \underline{0}) \otimes_{\Delta^{k+j}} \delta_{\mathbb{Z}} \otimes \dots \otimes \delta_{\mathbb{Z}}$$

is represented by an element in $\mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_{k+j}; \underline{0})$ without repetitions (that is without doubles ii or \underline{ii} in the integer-string representation); and similarly for a generator of $\zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^0$. The degree of $f : \{1, \dots, k+r\} \rightarrow \{\tilde{c}_1, \dots, \tilde{c}_k\}$ is the total degree $r = n_1 + \dots + n_k$ of the corresponding generator in

$$\mathcal{RL}_m(\tilde{n}_1, \dots, \tilde{n}_{k+j}; \underline{0}) \otimes (\delta_{\mathbb{Z}}^{\tilde{n}_1})_{n_1} \otimes \dots \otimes (\delta_{\mathbb{Z}}^{\tilde{n}_k})_{n_k}.$$

Following [BB09] we define maps

$$\vartheta_n : \zeta_{\tau, k, j}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0 \otimes \delta_{\mathbb{Z}}^n \rightarrow \zeta_{\tau, k, j}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^n, \quad n \geq 0,$$

as follows. Let $T \in \mathcal{RS}_m = \zeta_{\tau, k, j}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0$ be a generator. We denote by $T_{\#} \in \zeta_{\tau, k, j}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^n$ the element represented by the relative lattice obtained by adding n vertical bars in its integer-string representation. Let us denote by e_n the generator of $(\delta_{\mathbb{Z}}^n)_n = C_n(\text{Hom}_{\Delta}(-, [n]); \mathbb{Z})$, so that $(\delta_{\mathbb{Z}}^n)_n \cong \mathbb{Z}[e_n]$ for $n \geq 0$. Let $T \in \zeta_{\tau, k, j}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0$ be a generator so that T is the class of $T \otimes e_{n_1} \otimes \dots \otimes e_{n_{k+j}}$; we denote $[T \otimes e_{n_1} \otimes \dots \otimes e_{n_{k+j}}]$ such a class. We define $\vartheta_n(T \otimes e_n)$ to be the signed sum of elements $[T_{\#} \otimes e_{n_1}^{\#} \otimes \dots \otimes e_{n_{k+j}}^{\#}]$; the sum being over all possible relative lattices path of the form $T_{\#}$. We extend ϑ_n on $\zeta_{\tau, k, j}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0 \otimes \delta_{\mathbb{Z}}^n$ as a cosimplicial map.

And similarly, we have maps

$$\vartheta_n : \zeta_k(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^0 \otimes \delta_{\mathbb{Z}}^n \rightarrow \zeta_k(\mathcal{RL}_m)(\delta_{\mathbb{Z}})^n.$$

Let us describe the partial compositions of \mathcal{RS}_m . For two surjections $f \in \mathcal{RS}_m(c_1, \dots, c_k; c)$ and $g \in \mathcal{RS}_m(d_1, \dots, d_j; c_i)$ we set

$$(4.1) \quad f \circ_{\tilde{c}_i}^{\mathcal{RS}} g = f \circ_i \vartheta_{n(i)}(g \otimes e_{n(i)}),$$

where $n(i)$ denotes the number of occurrences of \tilde{c}_i in the surjection f (i.e. $n(i)$ is the cardinal of $f^{-1}(\tilde{c}_i)$) and \circ_i denotes the partial composition of $\text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$. We extend the composition by linearity. Such partial compositions give an operadic structure on \mathcal{RS}_m .

Example 4.1.

$$(\underline{121})^\circ \circ_1^{\mathcal{RS}} (12) = (\underline{1312})^\circ + (\underline{1231})^\circ.$$

Proposition 4.2. *The inclusion $\mathcal{RL}_m \hookrightarrow \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ is a weak equivalence of operads.*

Proof. The maps $\{\vartheta_n\}_n$ induce, by adjunction, a map $\vartheta' : \mathcal{RS}_m \hookrightarrow \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$. Except for signs, the fact that ϑ' is compatible with the operadic structures is straightforward from the definition. Such a compatibility implies (and then defines) the signs involved in the definition of ϑ_n . In particular, signs can be determined by an induction process on the dimension of the generators of \mathcal{RS}_m .

Moreover, let us denote by

$$\pi : \text{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}}) \rightarrow \mathcal{RS}_m$$

the weak equivalence coming from both (3.10) and (3.11); it satisfies $\pi\vartheta' = id$. Thus, ϑ' is a weak equivalence. \square

From the description of \mathcal{RL}_2 in term of planar trees given in the next section, we obtain a similar description for \mathcal{RS}_2 . More precisely, elements in $\mathcal{RS}_2(c_1, \dots, c_k; c)$ are sums of planar trees as described in Section 5 with no terminal vertices.

Proposition 4.3. *As an operad, \mathcal{RS}_2 is generated by the following elements*

$$\begin{aligned} \mu_c &= (12) \\ T_k &= (1213 \cdots 1k1), \quad k \geq 2 \\ \mu_o &= (\underline{12})^\circ \\ T_j &= (\underline{1213} \cdots \underline{1j1})^\circ, \quad j \geq 2 \\ inc &= (1)^\circ, \end{aligned}$$

and the two unit elements $id_c = (1)$ and $id_o = (\underline{1})^\circ$.

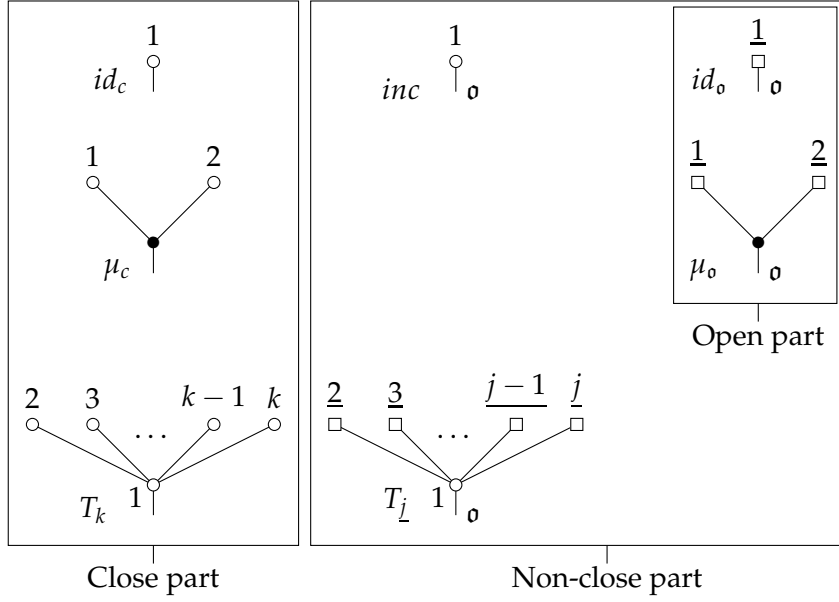


Figure 4.1: Generators of \mathcal{RS}_2 .

5 The operad \mathcal{RL}_2

5.1 The operad \mathcal{RL}_2 in term of trees

We describe an SC-split operad \mathcal{RL}_2 in the category of sets, **Set**. Algebras over this operads that we will consider are (family of) objects in **Top** or **Ch**(\mathbb{Z}). This is made possible by using the strong monoidal functor

$$\begin{aligned} \mathbf{Set} &\rightarrow \mathbf{C} \\ E &\mapsto \coprod_{e \in E} 1_{\mathbf{C}} \end{aligned}$$

where $1_{\mathbf{C}}$ denotes the unit for the monoidal structure \otimes of the cocomplete, closed monoidal symmetric category \mathbf{C} .

The operad \mathcal{RL}_2 can be thought of as a mix between the Lattice paths operads \mathcal{L}_2 and \mathcal{L}_1 introduced in [BB09]. The close part of \mathcal{RL}_2 is the sub-operad \mathcal{L}_2 of multiplicative operads. The open part of \mathcal{RL}_2 is the sub-operad \mathcal{L}_1 of cosimplicial \square -monoids (cf. [MS04, Definition 2.1]). In Section 5.2 we define the notion of *stable wide* bimodules over an operad so that a cosimplicial \square -monoids is a wide bimodule over the non-symmetric operad of associative algebras $ASSOC$.

A *multiplicative operad* \mathcal{M} is a (non-symmetric) operad endowed with an inclusion $ASSOC \hookrightarrow \mathcal{M}$ of non-symmetric operads.

The operad \mathcal{RL}_2 encodes the couples $(\mathcal{M}, \mathcal{Z})$ subject to the following conditions.

- I. \mathcal{M} is a (non-symmetric) multiplicative operad.
- II. \mathcal{Z} is a stable wide **left**-module over \mathcal{M} ; in particular, using $ASSOC \hookrightarrow \mathcal{M}$, it is a wide **left**-module over $ASSOC$; this is a part of the following structure III.
- III. \mathcal{Z} is a wide **bimodule** over $ASSOC$.

A stable wide bimodule is, in particular, an infinitesimal bimodule.

Lemma 5.1. [Tur10, Lemma 4.2] *The structure of a cosimplicial vector space is equivalent to the structure of an infinitesimal-bimodule over $ASSOC$.*

Our operad \mathcal{RL}_2 is closed to the operad \emptyset constructed in [DTT11, section 3.1]. The operad \emptyset was constructed in order to obtain an action of a Swiss Cheese operad on the pair $(CC^*(A, A), A)$ formed by an associative algebra A and its Hochschild cochain complex $CC^*(A, A)$. We enlarge the (non-closed part of the) operad \emptyset in order to obtain an action of a Swiss Cheese operad on the couple $(Tot_{\delta_{\mathcal{Z}}}\mathcal{M}, Tot_{\delta_{\mathcal{Z}}}\mathcal{Z})$ where $(\mathcal{M}, \mathcal{Z})$ is a couple as above and $Tot_{\delta_{\mathcal{Z}}}(-)$ denotes the $\delta_{\mathcal{Z}}$ -totalization. In particular, our open part takes into account the cosimplicial structure of \mathcal{Z} .

Recall from Section 3.1 that the set of colours of \mathcal{RL}_2 is

$$Col = Col_c \sqcup Col_o,$$

where Col_c is the set of natural numbers and Col_o is the set of natural numbers decorated with an underline.

By an abuse of notation, we call *planar rooted tree* its isomorphism class of planar rooted trees.

Definition 5.2. Let T be a planar rooted tree. Let v be a vertex of T . We denote by T_v the maximal sub-tree of T such that v is the root vertex of T_v .

For $n_i, n \in Col_c$, the set $\mathcal{RT}(n_1, \dots, n_k; n)$ is the set of equivalence classes of planar rooted trees T satisfying:

- a subset of the set of vertices of a tree T is indexed by the set $\{1, \dots, k\} \sqcup \{1, \dots, n\}$ in such a way that:
 - the vertices indexed by $\{1, \dots, n\}$ are only terminal vertices,
- the ordered set of edges originating at the vertex indexed by $s \in \{1, \dots, k\}$ is identified with $[n_i - 1]$.

The subset of vertices identified with $\{1, \dots, n\}$ benefits of an order given by $\{1 < \dots < n\}$. We require that this order coincides with the order which is given by turning around the tree in the clockwise direction starting from the root vertex. The equivalence class of equivalence is the same as in [DTT11,

3.2.1]. Explicitly, it is the finest one in which two trees are equivalent if one of them can be obtained from the other by either:

- the contraction of an edge with unmarked ends; or,
- removing an unmarked vertex with only one edge originating from it and joining the two edges adjacent to this vertex into one edge.

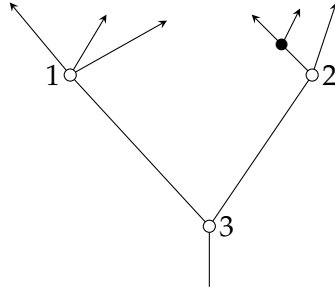


Figure 5.1: Element in $\mathcal{RL}_2(3, 2, 2; 6)$.

Let $(\tilde{n}_1, \dots, \tilde{n}_{k+j})$ be a tuple of colour in Col . We set $\{s_1, \dots, s_{k+j}\}$ the set such that

$$s_i = \begin{cases} i & \text{if } \tilde{n}_i = n_i; \\ \underline{i} & \text{if } \tilde{n}_i = \underline{n}_i. \end{cases}$$

The set $\mathcal{RT}(\tilde{n}_1, \dots, \tilde{n}_{k+j}; \underline{n})$ is the set of equivalence classes of planar rooted trees T satisfying:

- a subset of the set of vertices of a tree T is indexed by the set $\{s_1, \dots, s_{k+j}\} \sqcup \{1, \dots, n\}$ in such a way that:
 - given a vertex v of T indexed by $\underline{i} \in \{s_1, \dots, s_{k+j}\}$ then, in the tree T_v there is no vertex different of v indexed by an element $s_j \in \{s_1, \dots, s_{k+j}\}$,
 - the vertices indexed by $\{1, \dots, n\}$ are only terminal vertices,
- the ordered set of edges originating at the vertex indexed by $s_i \in \{s_1, \dots, s_{k+j}\}$ is identified with $[\tilde{n}_i - 1]$;
- the root is decorated by an σ .

The subset of vertices identified with $\{1, \dots, n\}$ benefits of an order given by $\{1 < \dots < n\}$. We require that this order coincides with the order which is given by turning around the tree in the clockwise direction starting from the root vertex. The equivalence class of equivalence is the same as the previous one.

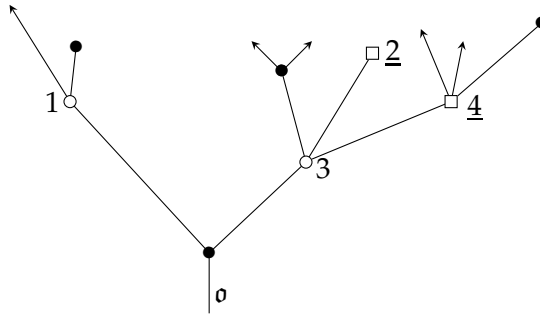


Figure 5.2: Element in $\mathcal{RT}(2, \underline{0}, 3, \underline{3}; \underline{5})$.

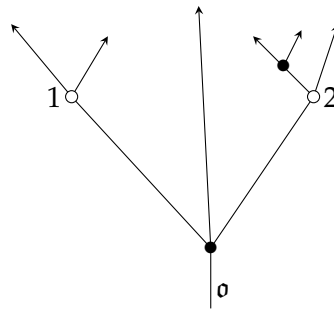


Figure 5.3: Element in $\mathcal{RT}(2, 2; \underline{6})$.

The composition maps in \mathcal{RT} are defined by substitution of trees into marked vertices.

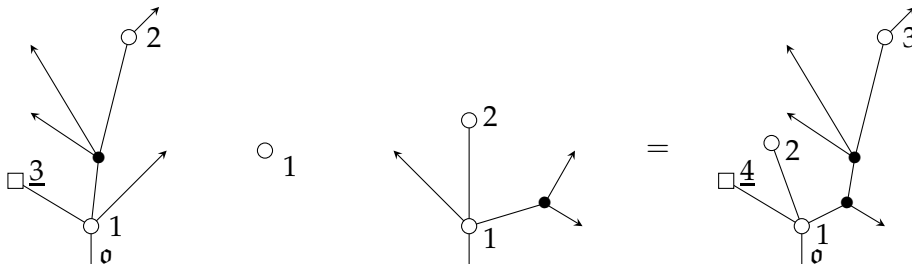


Figure 5.4: Example of composition.

Proposition 5.3. *The two operads \mathcal{RL}_2 and \mathcal{RT} are isomorphic.*

Proof. Recall that $x \in \mathcal{RL}_2(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ has an integer-string representation. For each i , the number s_i (which is i or \bar{i}) appears $n_i + 1$ times. We denote by s_i^a the a -th occurrence of s_i in this integer-string representation, so that $1 \leq a \leq n_i + 1$.

Given a vertex v in a tree T , we denote by T_v^a the maximal sub tree of T which has the a -th output of v as the root; outputs are ordered by the clockwise order.

We proceed as in [BB09, Proposition 2.14]. For $x \in \mathcal{RL}_2(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ we construct a labelled tree T_x as follows. For each i , the tree T_x has one vertex v_i indexed by s_i . The number of outputs of v_i is n_i and is one less than the number of occurrences of s_i in the integer-string representation. We suppose $s_i^1 < s_j^1$. Then, by filtration hypothesis: either there exists a $1 \leq a \leq n_i + 1$ such that $s_i^a < s_j^1 < \dots < s_j^{n_j+1} < s_i^{a+1}$ and then $v_j \in T_{v_i}^a$; or $s_i^{n_i+1} < s_j^1$ and then there exists an unmarked vertex \bullet such that $v_i \in T_{\bullet}^p$ and $v_j \in T_{\bullet}^q$ for some $p < q$.

For $n \geq 1$, if s_i^a and s_i^{a+1} are separated by n -vertical bars, then $T_{v_i}^a$ is the corolla C_n with n terminal vertices and an unmarked vertex. If $s_i^a < s_j^b$ are adjacent and separated by n -vertical bars: either there exist an unmarked vertex \bullet and a r such that $T_{\bullet} \subset T_{v_i}^a$ and $v_j \in T_{\bullet}^r$, then T_{\bullet}^{r-1} is the corolla C_n ; or, there exist a vertex \bullet and a p such that $v_i \in T_{\bullet}^p$ and $v_j \in T_{\bullet}^{p+2}$, then T_{\bullet}^{p+1} is the corolla C_n . Finally, if the integer-string has both $n \geq 0$ vertical bars preceding all the $s_i^{a_i}$ (for all $1 \leq i \leq k$ and all $1 \leq a_i \leq n_i + 1$), and $m \geq 0$ vertical bars succeeding all the $s_i^{a_i}$ (for all $1 \leq i \leq k$ and all $1 \leq a_i \leq n_i + 1$), then there exists an unmarked vertex \bullet with 3 outputs such that T_{\bullet}^1 is the corolla C_n , T_{\bullet}^3 is the corolla C_m and T_{\bullet}^2 contains all the vertices v_i . The corolla C_0 is defined as the rooted tree with one unmarked vertex and no other vertices.

The tree T_x is the minimal tree satisfying all these properties.

Note that filtration's conditions tell us that for $s_i^1 < s_j^1$ with $s_i = \underline{i}$ then $s_i^{n_i+1} < s_j^1$. Thus, in this case, the tree T_{v_i} does not contain any marked vertex.

Conversely, let us take a labelled tree $T \in \mathcal{RT}$. One runs through the tree T in clockwise direction starting from the root in such a way that one passes exactly two times on each edges (once per sense). One assigns the number s_i each time one meets the vertex v_i and one assigns a vertical bar each times one meets a terminal vertex.

This way one obtains an isomorphism of operads formed by these two assignments. \square

5.2 The algebras over \mathcal{RL}_2

Let us fix a cocomplete, closed monoidal symmetric category $(\mathbf{C}, \otimes, 1_{\mathbf{C}}, \tau_{\mathbf{C}})$.

We refer to [AT14, Definition 2.4-2.8] for definitions of right module, left module, infinitesimal left module, bimodule, infinitesimal bimodule all over an operad.

Definition 5.4. Let \mathcal{M} be a (1-coloured) operad in \mathbf{C} . A wide left module \mathcal{Z} over \mathcal{M} is both a left module over \mathcal{M} with a structure map

$$\lambda_{\alpha} : \mathcal{M}(B) \otimes \bigotimes_{b \in B} \mathcal{Z}(\alpha^{-1}(b)) \rightarrow \mathcal{Z}(A),$$

for any sets A and B and $\alpha : A \rightarrow B$ map of sets; a weak-left module over \mathcal{M} with a structure map

$$\lambda_a : \mathcal{M}(A) \otimes \mathcal{Z}(B) \rightarrow \mathcal{Z}(A \cup_a B),$$

for any sets A and B and $a \in A$, such that this is a part of the following structure. For any two sets A and B , and $\alpha : A \rightarrow B$ map of sets, there is a map

$$\lambda'_\alpha : \mathcal{M}(B) \otimes \bigotimes_{b \in B} \mathcal{Z}(\alpha^{-1}(b)) \rightarrow \mathcal{Z}(A \sqcup B_\alpha),$$

where $B_\alpha := B \setminus \text{Im}(\alpha)$, such that:

1. if α is surjective, then $\lambda'_\alpha = \lambda_\alpha$;
2. if $\text{Im}(\alpha) = b \in B$, then $\lambda'_\alpha = \lambda_b$; and,
3. all the diagrams of the following form commute

$$\begin{array}{ccc} \mathcal{M}(A) \otimes \mathcal{M}(B) \otimes \bigotimes_{b \in B} \mathcal{Z}(\alpha^{-1}(b)) & \xrightarrow{\lambda'_\alpha} & \mathcal{M}(A) \otimes \mathcal{Z}(C \sqcup B_\alpha) \\ \downarrow - \circ_a - & & \downarrow \lambda_a \\ \mathcal{M}(A \cup_a B) \otimes \bigotimes_{b \in B} \mathcal{Z}(\alpha^{-1}(b)) & \xrightarrow{\lambda'_{\tilde{\alpha}}} & \mathcal{Z}(C \sqcup (A \cup_a B_\alpha)), \end{array}$$

where $\tilde{\alpha} : C \xrightarrow{\alpha} B \hookrightarrow A \cup_a B$.

Definition 5.5. A wide left module \mathcal{Z} over an operad \mathcal{M} is called *stable* if there are maps in \mathbf{C}

$$\iota_A : \mathcal{M}(A) \rightarrow \mathcal{Z}(A)$$

for each set A , such that all the diagrams of the following form commute

$$\begin{array}{ccc} \mathcal{M}(B) \otimes \bigotimes_{b \in B} \mathcal{M}(\alpha^{-1}(b)) & \xrightarrow{id \otimes \bigotimes_{b \in B} \iota_{\alpha^{-1}(b)}} & \mathcal{M}(B) \otimes \bigotimes_{b \in B} \mathcal{Z}(\alpha^{-1}(b)) \\ \downarrow - \circ_\alpha - & & \downarrow \lambda'_\alpha \\ \mathcal{M}(A \sqcup B_\alpha) & \xrightarrow{\iota_{A \sqcup B_\alpha}} & \mathcal{Z}(A \sqcup B_\alpha), \end{array}$$

where $\alpha : A \rightarrow B$.

Definition 5.6. A (*resp.* *stable*) wide bimodule over \mathcal{M} is a bimodule and an infinitesimal bimodule all two over \mathcal{M} such that their underlying left module and infinitesimal left module structures are a part of a (*resp.* *stable*) wide left module structure over \mathcal{M} .

Definition 5.7. The non-symmetric operad \mathcal{ASSOC} in \mathbf{C} is given as $\mathcal{ASSOC}(n) = 1_{\mathbf{C}}$ for $n \geq 0$.

Let \mathbf{E} be the category with objects the couples $(\mathcal{M}, \mathcal{Z}) \in \mathbf{C} \otimes \mathbf{C}$ satisfying the three conditions I, II and III of section 5; and, with morphisms the pair $(f, g) : (\mathcal{M}, \mathcal{Z}) \rightarrow (\mathcal{M}', \mathcal{Z}')$ where $f : \mathcal{M} \rightarrow \mathcal{M}'$ is a morphism of multiplicative operads and $g : \mathcal{Z} \rightarrow \mathcal{Z}'$ is a morphism of stable wide left modules over \mathcal{M} and wide bimodules over \mathcal{ASSOC} .

Lemma 5.8. *The category of cosimplicial \square -monoids in \mathbf{C} is isomorphic to the category of wide bimodules over \mathcal{ASSOC} in \mathbf{C} .*

Proposition 5.9. [BB09, Proposition 2.14] *The category of \mathcal{L}_1 -algebras (resp. of \mathcal{L}_2 -algebras) in \mathbf{C} is isomorphic to the category of cosimplicial \square -monoids (resp. of multiplicative operads) in \mathbf{C} .*

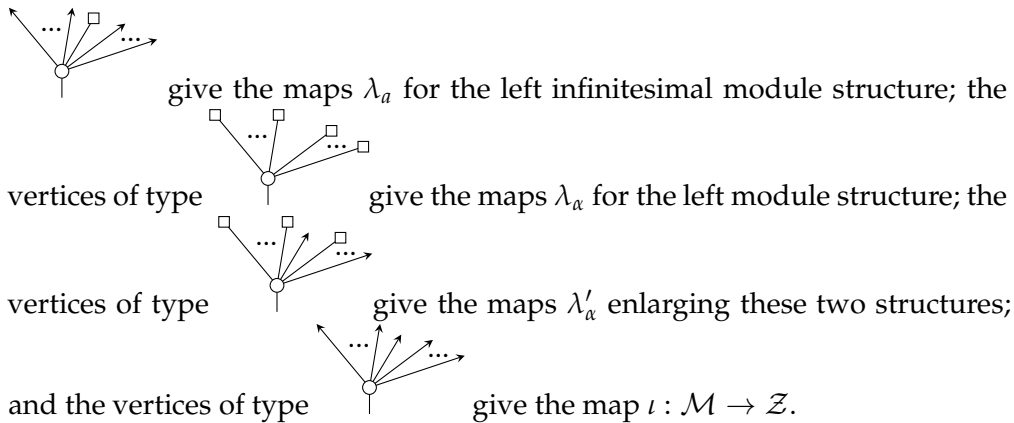
Proposition 5.10. *Let \mathbf{C} be a cocomplete, closed monoidal symmetric category with a zero object. The category of \mathcal{RL}_2 -algebras in \mathbf{C} is isomorphic to the category \mathbf{E} .*

Proof. We use the interpretation of \mathcal{RL}_2 in terms of planar trees, see Proposition 5.3. Given a tree in $\mathcal{RL}_2(\tilde{n}_1, \dots, \tilde{n}_k; \tilde{n})$ we recall that $\{s_1, \dots, s_k\}$, where

$$s_i = \begin{cases} i & \text{if } \tilde{n}_i = n_i; \\ \underline{i} & \text{if } \tilde{n}_i = \underline{n}_i, \end{cases}$$

denotes the set that labels "open" and "close" vertices. The action of \mathcal{RL}_2 on a object $(\mathcal{M}, \mathcal{Z})$ of \mathbf{E} is given as follows. One decorates the close marked vertex indexed by i with an element $x_i \in \mathcal{M}([n_i - 1])$ and one decorates the open marked vertex indexed by \underline{j} with an element $y_j \in \mathcal{Z}([\underline{n}_j - 1])$; the resulting element in $\mathcal{Z}(\{1, \dots, n\})$ is obtained by composing the decorating elements along the tree, using the module structures of \mathcal{M} and \mathcal{Z} .

Conversely, the vertices of type



One checks that the operadic structure of \mathcal{RL}_2 and the action maps of \mathcal{RL}_2 on $(\mathcal{M}, \mathcal{Z})$ give the compatibilities so that one has a stable wide left module structure on \mathcal{Z} . \square

6 Cosimplicial relative loop space

In this section we define a cosimplicial model $\omega(X, Y)$ for the relative loop spaces $\Omega(X, Y)$, that is, a cosimplicial space such that its δ_{Top} -totalization $Tot_{\delta_{Top}}\omega(X, Y)$ is homeomorphic to $\Omega(X, Y)$. For (M, N) a pair of monoids pointed at the unit $*$ and such that N is a sub-monoid of M , we show that $\omega(M, N)$ is endowed with an additional structure. More precisely, we show that there exists an $\mathcal{R}\mathcal{L}_2$ -algebra $(\underline{\omega}(M), \underline{\omega}(M, N))$ associated to $(\omega(M), \omega(M, N))$ such that, for δ being δ_{Top} or $\delta_{\mathbb{Z}}$, the totalization $(Tot_{\delta}\omega(M), Tot_{\delta}\omega(M, N))$ is an algebra over $\text{Coend}_{\mathcal{R}\mathcal{L}_2}(\delta)$.

Let us start by a definition.

Definition 6.1. Let (X, Y) be two topological spaces pointed at $*$ and such that $* \subset Y \subset X$. The relative loop space of (X, Y) , $\Omega(X, Y)$, is the space of continuous maps $\gamma : [0, 1] \rightarrow X$ satisfying $\gamma(0) = *$ and $\gamma(1) \in Y$.

Definition 6.2. Let (X, Y) be a pair of topological spaces pointed at $*$ such that $* \subset Y \subset X$. The *cosimplicial relative loop space* $\omega(X, Y)$ is the cosimplicial space such that $\omega(X, Y)^0 = Y$, and $\omega(X, Y)^k = X^{\times k} \times Y$ for $k \geq 1$, with

$$\begin{aligned} d^0(x_1, \dots, x_k, y) &= (*, x_1, \dots, x_k, y) \\ d^i(x_1, \dots, x_k, y) &= (x_1, \dots, x_i, x_i, \dots, x_k, y), 1 \leq i \leq k \\ d^{k+1}(x_1, \dots, x_k, y) &= (x_1, \dots, x_k, y, y) \\ s^i(x_1, \dots, x_k, y) &= (x_1, \dots, x_i, x_{i+2}, \dots, x_k, y), 0 \leq i \leq k. \end{aligned}$$

Remark 6.3. For $Y = *$, the cosimplicial space $\omega(X, Y)$ is the cosimplicial space $\omega X \times \{*\} \cong \omega X$ that is a model for the loop space ΩX described in [Sal09].

Proposition 6.4. *The maps*

$$\begin{aligned} \Omega(X, Y) \times \Delta^k &\rightarrow X^k \times Y \\ (\gamma, (t_1, \dots, t_k)) &\mapsto (\gamma(t_1), \dots, \gamma(t_k), \gamma(1)), \quad k \in \mathbb{N}, \end{aligned}$$

induce, by adjunction, a homeomorphism $\Omega(X, Y) \cong Tot(\omega(X, Y))$.

6.1 Cosimplicial relative loop space of monoids

Given a topological monoid M and a sub-monoid N , the totalization of the cosimplicial relative loop space $\omega(M, N)$ is homeomorphic to $\Omega(M, N) \cong \Omega(\Omega \mathcal{B} M, \Omega \mathcal{B} N) \cong \Omega^2(\mathcal{B} M, \mathcal{B} N)$ where $\mathcal{B} G$ denotes the classifying space of the monoid G . From [Sal09] we know that $\omega(M)$ can be seen as the cosimplicial space coming from a multiplicative operad $\underline{\omega}(M)$. This property implies that the totalization of $\omega(M)$ is an E_2 -algebra.

In the same spirit we show that $\omega(M, N)$ comes from $\underline{\omega}(M, N)$, which is both a wide **bimodule** over \mathcal{ASSOC} and a stable wide **left module** over the multiplicative operad $\underline{\omega}(M)$. Then, one obtains that the couple of totalizations

6.1 Cosimplicial relative loop space of monoids

$(Tot_\delta \omega(M), Tot_\delta \omega(M, N))$ is a Swiss Cheese algebra for $\delta = \delta_{Top}$ or $\delta = \delta_Z$.

The structural map of the operad $\underline{\omega}(M)$ is denoted by

$$\begin{aligned} \gamma : \underline{\omega}(M)(k) \times \underline{\omega}(M)(l_1) \times \cdots \times \underline{\omega}(M)(l_k) &\rightarrow \underline{\omega}(M)(l_1 + \cdots + l_k) \\ (f, g_1, \dots, g_k) &\mapsto \gamma(f; g_1, \dots, g_k). \end{aligned}$$

Since $\underline{\omega}(M)$ has a unit, γ is equivalent to infinitesimal maps

$$\begin{aligned} \circ_i : \underline{\omega}(M)(k) \times \underline{\omega}(M)(l) &\rightarrow \underline{\omega}(M)(l_1 + \cdots + l_k) \\ (f, g) &\mapsto f \circ_i g. \end{aligned}$$

Explicitly, if $f = (x_1, \dots, x_k)$ and $g = (y_1, \dots, y_l)$, then

$$(6.1) \quad f \circ_i g := (x_1, \dots, x_{i-1}, x_i y_1, \dots, x_i y_l, x_{i+1}, \dots, x_k),$$

see [Sal09].

The family $\underline{\omega}(M, N) = \{\underline{\omega}(M, N)(l)\}_{l \geq 0}$ has, as underlying spaces, the spaces $\underline{\omega}(M, N)(l) := \omega(M, N)_l$. A typical element in $\underline{\omega}(M, N)(l)$ is denoted by a couple $(g; n)$ so that $g \in M^{\times l}$ and $n \in N$.

Let us define a stable wide left action. For $k \geq 1$, $1 \leq s \leq k$, $l_i \geq 1$ and an order preserving map $\beta : \{1 < \cdots < s\} \rightarrow \{1 < \cdots < k\}$, we define

$$(6.2) \quad \begin{aligned} \zeta' : \underline{\omega}(M)(k) \times \underline{\omega}(M, N)(l_1) \times \cdots \times \underline{\omega}(M, N)(l_s) \\ \rightarrow \underline{\omega}(M, N)(l_1 + \cdots + l_s + k - s) \end{aligned}$$

by

$$\begin{aligned} \zeta'(f, (g_1; n_1), \dots, (g_s; n_s)) = \\ (\gamma(f; \mathbf{1}^{\beta(1)-1}, g_1, \mathbf{n}_1^{\beta(2)-\beta(1)-1}, g_2 \triangleright n_1, \mathbf{n}_2 \mathbf{n}_1^{\beta(3)-\beta(2)-1}, \dots \\ \dots, g_s \triangleright n_{s-1} \cdots n_1, \mathbf{n}_s \cdots \mathbf{n}_1^{k-\beta(s)-1}); n_s \cdots n_1), \end{aligned}$$

where, for $n, m \in N$ and $j \geq 0$, $\mathbf{n}^j := \underbrace{n, \dots, n}_j$ and $\mathbf{nm}^j := \underbrace{nm, \dots, nm}_j$.

For $s = 1$ above, we get an infinitesimal left action

$$(6.3) \quad \begin{aligned} \zeta_i : \underline{\omega}(M)(k) \times \underline{\omega}(M, N)(l) &\rightarrow \underline{\omega}(M, N)(k + l - 1) \\ (f, (g; n)) &\mapsto (\gamma(f; \underbrace{1, \dots, 1}_{i-1}, g, n, \dots, n); n), \end{aligned}$$

for $1 \leq i \leq k$.

For $s = k$ above, we get a left action

$$(6.4) \quad \begin{aligned} \zeta : \underline{\omega}(M)(k) \times \underline{\omega}(M, N)(l_1) \times \cdots \times \underline{\omega}(M, N)(l_k) &\rightarrow \underline{\omega}(M, N)(l_1 + \cdots + l_k) \\ (f, (g_1; n_1), \dots, (g_k; n_k)) &\mapsto (\gamma(f; g_1, g_2 \triangleright n_1, g_3 \triangleright n_2 n_1, \dots, g_k \triangleright n_{k-1} \cdots n_1); n_k \cdots n_1), \end{aligned}$$

for all $k \geq 1$ and $l_i \geq 1$.

For $s = 0$ above, we set the inclusion

$$(6.5) \quad \begin{aligned} \iota : \underline{\omega}(M)(k) &\rightarrow \underline{\omega}(M, N)(k) \\ f &\mapsto (f; 1), \end{aligned}$$

for all $k \geq 1$.

The right action is given by

$$(6.6) \quad \begin{aligned} \rho : \underline{\omega}(M, N)(k) \times \underline{\omega}(M)(l_1) \times \cdots \times \underline{\omega}(M)(l_k) &\rightarrow \underline{\omega}(M, N)(l_1 + \cdots + l_k) \\ ((f; n), g_1, \dots, g_k) &\mapsto (\gamma(f; g_1, \dots, g_k); n), \end{aligned}$$

where γ is the structural map of the operad $\underline{\omega}(M)$.

Lemma 6.5. *The maps $\zeta', \zeta, \{\zeta_i\}$ and ρ endow $\underline{\omega}(M, N)$ with a stable wide bimodule structure over $\underline{\omega}(M)$. In particular, using the map $\mathcal{ASSOC} \hookrightarrow \underline{\omega}(M)$, the module $\underline{\omega}(M, N)$ is a wide bimodule over \mathcal{ASSOC} . With regard to the infinitesimal bimodule structure over \mathcal{ASSOC} , the corresponding (via Lemma 5.1) cosimplicial space is $\omega(M, N)$.*

In virtue of the above Lemme 6.5, Proposition 5.10, Theorem 3.12 and Section 1.3, we have

Theorem 6.6. *Let (M, N) be a pair of topological monoids pointed at the unit $*$ such that N is a submonoid of M . Then the operad $\text{Coend}_{\mathcal{RL}_2}(\delta_{\mathbb{Z}})$, which is weakly equivalent to the Swiss-Cheese operad $C_*(\mathcal{SC}_2)$, acts on the couple of totalizations $(\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M), \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N))$.*

6.2 Action of $H_*(\mathcal{SC}_2)$ on $(H_*\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M), H_*\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N))$

We write down a few operations on $(\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M), \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N))$ when (M, N) is a pair of monoids pointed at the unit $*$ with N submonoid of M .

The described operations are the sufficient to ensure the existence of an $H_*(\mathcal{SC}_2)$ -algebra structure on $(H_*\text{Tot}_{\delta_{\mathbb{Z}}}C_*(M), H_*\text{Tot}_{\delta_{\mathbb{Z}}}(C_*(M), C_*(N)))$. They form a part of the \mathcal{RS}_2 -algebra structure we make explicit in the next section.

Recall that an algebra over the homology operad $H_*(\mathcal{SC}_2)$ is a triple (A, G, f) where A is an associative algebra; G is a Gerstenhaber algebra; and, $f : G \rightarrow A$ is an algebra morphism such that $f(G)$ belongs to the center of A , for example see [HL13, Proposition 3.2.1].

For a cosimplicial space K , the totalization $\text{Tot}_{\delta_{\mathbb{Z}}}(K)$ is equal to $\prod_{n \geq 0} \mathbb{Z}[K_n]$ with the differential usually given as the sum of the cosimplicial face maps. We denote by $K(n)$ the n -th component $\mathbb{Z}[K_n]$; the degree of an element $f \in K(n)$ is n and it is denoted by $|f|$.

Let us recall a consequence of McClure and Smith's work.

Theorem 6.7 ([MS04]). *Let \mathcal{O} be a multiplicative operad. Then $\text{Tot}_{\delta_{\mathbb{Z}}}\mathcal{O}$ is an E_2 -algebra.*

In particular, $\text{Tot}_{\delta_{\mathbb{Z}}}\mathcal{O}$ is endowed with a product, \cup , commutative up to a chain homotopy $E_{1,1}$.

The product \cup on $\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M)$ is given by

$$(6.7) \quad f \cup g := (\mu \circ_2 g) \circ_1 f,$$

for $f, g \in \omega(M)$. We define an associative dg-product \sqcup on $\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$ by

$$(6.8) \quad u \sqcup v := \zeta(\mu; u, v),$$

for $u, v \in \omega(M, N)$ where ζ is the map defined in (6.4).

In other words, for $u = (f; m)$ and $v = (g; n)$ we have

$$u \sqcup v = (f, g \triangleright m; nm).$$

The inclusion $inc : \text{Tot}_{\delta_{\mathbb{Z}}}\omega M \hookrightarrow \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$ is given on its components by

$$(6.9) \quad \begin{aligned} inc : \omega(M)(k) &\rightarrow \omega(M, N)(k) \\ (a_1, \dots, a_k) &\mapsto (a_1, \dots, a_k; 1) \end{aligned}$$

and it is a chain map.

From this, we easily deduce that

$$inc(f \cup g - (-1)^{|f||g|}g \cup f) \sqcup u = inc(\partial E_{1,1}(f; g)) \sqcup u = \partial(inc(E_{1,1}(f; g)) \sqcup u),$$

for any homogeneous elements $f, g \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M)$ and $u \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$. Finally, we define a chain homotopy H such that

$$inc(f) \sqcup u - (-1)^{|f||u|}u \sqcup inc(f) = \partial H(f, u)$$

for any two homogeneous elements $f \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M)$ and $u \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$. We recall the infinitesimal left action ζ_i defined in (6.3). The homotopy H is defined as

$$(6.10) \quad H(f, (g; n)) = \sum_{1 \leq i \leq k} (-1)^{i+|g|+|f||g|} \zeta_i(f; (g; n)),$$

for any $f \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M)$ of degree k and $(g; n) \in \text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N)$.

We have shown, in particular, the following.

Proposition 6.8. *Let (M, N) be a pair of 1-connected monoids with N a sub-monoid of M . Then the above operations (6.7), (6.8) and (6.9), induce an $H_*(\mathcal{SC}_2)$ -algebra structure on the pair $(H_*\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M), H_*\text{Tot}_{\delta_{\mathbb{Z}}}\omega(M, N))$.*

Geometric interpretation. Let us give a geometric interpretation of the above operations by describing, on $(\Omega M, \Omega(M, N))$, the corresponding ones.

Given two paths in τ and τ' in M , we denote by $\tau \cdot \tau'$ the path $\tau \cdot \tau'(t) = \tau(t)\tau'(t)$ given by the product of M . For $x \in M$, we denote by c_x the constant path at x .

The (associative up to homotopy) product in $\Omega(M, N)$ is given by $\mu_o(\gamma, \gamma') = \gamma(\gamma' \cdot c_{\gamma(1)})$.

The (associative and commutative all two up to homotopy) product in $\Omega(M)$ is given by $\mu_c(\gamma, \gamma') = \gamma\gamma'$. The inclusion of $\Omega M \hookrightarrow \Omega(M, N)$ is given by the inclusion of loops.

The inclusion and the multiplication in $\Omega(M, N)$ provide a left action $l : \Omega M \times \Omega(M, N) \rightarrow \Omega(M, N)$ given by concatenation $l(\gamma, \tau) = \gamma\tau$. Since, M, N are monoids, it is easy to see that the right action $r : \Omega(M, N) \times \Omega M \rightarrow \Omega(M, N)$ given by translation/concatenation $r(\tau, \gamma) = \tau(\gamma \cdot c_{\tau(1)})$ is homotopic to the left one l .

Clearly, these operations induce an $H_*(\mathcal{SC}_2)$ -algebra structure on the pair $(H_*\Omega(M), H_*\Omega(M, N))$.

6.3 Action of the whole operad \mathcal{RS}_2

The inclusion $\mathcal{RS}_2 \hookrightarrow \text{Coend}_{\mathcal{RL}_2}(\delta_{\mathbb{Z}})$ implies an action of the operad \mathcal{RS}_2 on $(\widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega(M), \widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega(M, N))$. We describe this action.

To do that it is sufficient to write down explicitly the operations corresponding to the generators of \mathcal{RS}_2 .

The close part of \mathcal{RS}_2 acts as described in [Kad05]. We recall from Proposition 4.3 that \mathcal{RS}_2 is generated by μ_c, T_k, μ_o, T_j and inc . By a slight abuse of notation, for an element $T \in \mathcal{RS}_2(c_1, \dots, c_{k+j}; c)_d$ we write also T the corresponding operation

$$T : (\widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega)_*^{\otimes k+j} \rightarrow (\widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega)_{*-d},$$

where $\widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega$ is either $\widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega M$ or $\widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega(M, N)$ according to the colours $(c_1, \dots, c_{k+j}; c)$.

The multiplication $\mu_c \in \mathcal{RS}_2(c, c; c)_0$ acts as the product \cup in (6.7).

The trees $T_k \in \mathcal{RS}_2(\underbrace{c, \dots, c}_k; c)_{2k+1}$ act as follows. For any $f \in \omega(M)(n)$ and $g_1, \dots, g_{k-1} \in \widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega(M, N)$, we have

$$T_k(f; g_1, \dots, g_{k-1}) = \sum_{\substack{0 \leq i_1 + \dots + i_{k-1} \leq n-k+1 \\ i_s \geq 0}} \pm \gamma(f; 1^{i_1}, g_1, 1^{i_2}, g_2, \dots, g_{k-1}, 1^{n-i_1+\dots+i_{k-1}}).$$

The generator $\mu_o \in \mathcal{RS}_2(o, o; o)_0$ acts as the product \sqcup in (6.8).

The generator $T_j \in \mathcal{RS}_2(c, \underbrace{o, \dots, o}_{j-1}; o)_{2j+1}$ acts as follows. For each homogeneous elements $f \in \widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega(M)$ and $(h_1; n_1), \dots, (h_{j-1}; n_{j-1}) \in \widetilde{\text{Tot}}_{\delta_{\mathbb{Z}}}\omega(M, N)$,

the element $T_j(f; (h_1; n_1), \dots, (h_{j-1}; n_{j-1}))$ is given as

$$\sum_{\substack{0 \leq i_1 + i_2 + \dots + i_j \leq n-j \\ i_s \geq 0}} \pm \zeta(f; 1^{i_1}, h_1, 1^{i_2}, h_2, \dots, h_j, 1^{n-i_1+\dots+i_j}),$$

where ζ is the map defined in (6.4). In other words, $T_j(f; (h_1; n_1), \dots, (h_{j-1}; n_{j-1}))$ is the sum over $\sum_{\substack{0 \leq i_1 + i_2 + \dots + i_j \leq n-j \\ i_s \geq 0}}$ of

$$(\gamma(f; 1^{i_1}, h_1, \underbrace{n_1, \dots, n_1}_{i_2}, h_2 \triangleright n_1, \dots, h_j \triangleright (n_{j-1} \cdots n_2 n_1), \underbrace{n_j \cdots n_2 n_1, \dots, n_j \cdots n_2 n_1}_{n-i_1+\dots+i_j}; n_j \cdots n_2 n_1)$$

The generator $inc \in \mathcal{RS}_2(\mathfrak{c}; \mathfrak{o})_0$ is the canonical inclusion of $Tot_{\delta_{\mathbb{Z}}} \omega(M)$ into $Tot_{\delta_{\mathbb{Z}}} \omega(M, N)$.

7 Relative cobar construction

Definition 7.1. Let (C, N) be a pair of dg-module such that C is a 1-reduced dg-coalgebra (coaugmented, counital, coassociative) and N is a left C -comodule. The *relative cobar construction* of (C, N) is the free graded $\underline{\Omega}C$ -module on N , $\underline{\Omega}C \otimes N$, with the following differential D .

The differential is given by: $D = d_0 + d_1$, where d_0 is the Koszul differential of the module $\underline{\Omega}C \otimes N$ induces by d_C and d_N ; and d_1 is the quadratic part induces by the quadratic part of the cobar construction $\underline{\Omega}C$ and a twist with the reduced C -comodule structure of N . We write the reduced coproducts $\overline{\nabla}_C(c) = \sum c^1 \otimes c^2 = \nabla_C(c) - c \otimes 1 - 1 \otimes c$ and $\overline{\nabla}_N(n) = \sum z^1 \otimes n^2 = \nabla_N(n) - 1 \otimes n$. We have:

$$\begin{aligned} d_1(s^{-1}c) &= \sum (-1)^{c^1} s^{-1} c^1 \otimes s^{-1} c^2, \quad c \in C^{>1} \\ d_1(n) &= \sum s^{-1} z^1 \otimes n^2, \quad n \in N. \end{aligned}$$

The augmentation $\epsilon : \underline{\Omega}C \otimes N \rightarrow k$ is given by $\epsilon = \epsilon_{\underline{\Omega}C} \cdot \epsilon_N$. The relative cobar construction $(\underline{\Omega}C \otimes N, D, \epsilon)$ is denoted by $\underline{\Omega}(C, N)$.

The action of $\underline{\Omega}C$ is given by concatenation

$$\begin{aligned} a_{\underline{\Omega}(C, N)} : \underline{\Omega}C \otimes \underline{\Omega}(C, N) &\rightarrow \underline{\Omega}(C, N) \\ (s^{-1}x_1 \otimes \dots \otimes s^{-1}x_k) \otimes (s^{-1}x_{k+1} \otimes \dots \otimes s^{-1}x_r \otimes n) &\mapsto s^{-1}x_1 \otimes \dots \otimes s^{-1}x_r \otimes n, \end{aligned}$$

and yields on $\underline{\Omega}(C, N)$ a $\underline{\Omega}C$ -module structure in the category of dg-modules.

Data 7.2. Let us fix a pair of module maps $(f : C \rightarrow A, g : N \rightarrow M)$, where:

- C is a 1-connected coaugmented dg-coalgebra;

- A is an augmented dg-algebra;
- N is a left dg-comodule over C with coaction $c_N : N \rightarrow C \otimes N$;
- M is a left dg-module over A with action $a_M : A \otimes M \rightarrow M$.

We denote by $\bar{f} : \underline{\Omega}C \rightarrow A$ the induced algebra morphism of f . Let

$$(7.1) \quad \overline{(f, g)} : \underline{\Omega}C \otimes N \rightarrow M$$

be the map $\bar{f} \cdot g$ whose the k -th component is

$$C^{\otimes k} \otimes N \xrightarrow{\bar{f} \otimes g} A \otimes M \xrightarrow{a_M} M.$$

As an immediate consequence, $\overline{(f, g)} a_{\underline{\Omega}(C, N)} = a_M(\bar{f} \otimes g)$, so that $\overline{(f, g)}$ is \bar{f} -equivariant.

Now, we define a relative version of twisting cochain. We recall first that the set of twisting cochains $\text{Tw}(C, A)$ is the set of maps $f : C \rightarrow A$ satisfying $f \cup f = \partial f$, where $f_1 \cup f_2 = \mu_A(f_1 \otimes f_2) \nabla_C$ and ∂ denotes the usual differential in $\text{Hom}(C, A)$.

Definition 7.3. Let $(f : C \rightarrow A, g : N \rightarrow M)$ a pair as in [Data 7.2](#). The pair (f, g) is called *relative twisting* if f is a twisting cochain and if

$$(7.2) \quad \partial g = a_M(f \otimes g) c_N.$$

Proposition 7.4. Let (f, g) be a pair as in [Data 7.2](#). Suppose f is a twisting cochain. Then the following propositions are equivalent.

- The pair (f, g) is a relative twisting pair.
- The \bar{f} -equivariant map $\overline{(f, g)} : \underline{\Omega}C \otimes N \rightarrow M$ is a morphism of dg-modules.

7.1 Cubical model for relative loop spaces

In this section we define a model for Moore relative loop spaces. Given a pair (X, Y) of topological spaces, pointed at $*$ and such that $* \subset Y \subset X$, the Moore relative loop space $\Omega_M(X, Y)$ is defined as

$$\Omega_M(X, Y) = \{(\gamma, r) \in X^{[0, +\infty)} \times [0, +\infty) \mid \gamma(0) = * \text{ and } \gamma(t) = \gamma(r) \in Y \forall t \geq r\}.$$

We denote simply $\Omega_M(X, *)$ by $\Omega_M X$. The Moore path space $P_M X$ is defined as

$$P_M X = \{(\gamma, r) \in X^{[0, +\infty)} \times [0, +\infty) \mid \gamma(0) = * \text{ and } \gamma(t) = \gamma(r) \forall t \geq r\}.$$

Let us start with a few definitions about cubical sets. There are extracted from [\[KS05, Section 2.4\]](#).

Definition 7.5. A cubical set Q is a graded set $Q = \{Q_n\}_{n \geq 0}$ with face operators $d_i^\epsilon : Q_n \rightarrow Q_{n-1}$, $n \geq 1$, $1 \leq i \leq n$, $\epsilon = 0, 1$, and degeneracy operators $\eta_i : Q_n \rightarrow Q_{n+1}$, $n \geq 0$, $1 \leq i \leq n+1$, satisfying:

$$\begin{aligned} d_j^\epsilon d_i^{\epsilon'} &= d_i^{\epsilon'} d_{j+1}^\epsilon \quad i \leq j \\ d_i^\epsilon \eta_j &= \begin{cases} \eta_{j-1} d_i^\epsilon & i < j \\ 1 & i = j \\ \eta_j d_{i-1}^\epsilon & i > j \end{cases} \\ \eta_i \eta_j &= \eta_{j+1} \eta_i \quad i \leq j. \end{aligned}$$

The product of two cubical sets Q and Q' , is

$$(7.3) \quad Q \times Q' = \{(Q \times Q')_n = \bigcup_{p+q=n} Q_p \times Q'_q\} / \sim$$

where $(\eta_{p+1}(x), y) \sim (x, \eta_1(y))$ for $(x, y) \in Q_p \times Q'_q$. The face and degeneracy operators are induced by those of Q and Q' in the obvious way.

For $n \geq 0$, let I^n be the cube of dimension n given as the cartesian product of the interval $I = [0, 1]$; then the cube I^0 is a point. Let X be a pointed connected topological space. The cubical set $Sing^\square X = \{Sing_n^\square X\}_{n \geq 0}$ is formed by the continuous maps $I^n \rightarrow X$.

Let Q be a cubical set. We denote by $C_*^\square Q$ the normalized chain complex of Q . For a topological space X , $C_*^\square X$ denotes the normalized chain complex of $Sing^\square X$.

Definition 7.6. A *monoidal cubical set* to be a cubical set Q with an associative cubical multiplication $\mu : Q \times Q \rightarrow Q$ for which a distinguished element $e \in Q_0$ is the unit.

The chain complex $C_*^\square Q$ of a monoidal cubical set is a dg-bialgebra.

Definition 7.7. Let Q be a monoidal cubical set. A cubical set P is a Q -module if there is an associative cubical map $Q \times P \rightarrow P$ with the unit of Q acting as identity.

The chain complex $C_*^\square P$ of a Q -monoidal cubical set P is a $C_*^\square Q$ -module.

Definition 7.8. For $n \geq 0$, a simplicial set K is n -reduced if $K_0 = \dots = K_n = \{*\}$.

Let X be a 1-connected space pointed at $*$. Let $Sing^1 X = \{Sing_n^1 X\}_{n \geq 0}$ be the simplicial set formed by the singular simplexes $\sigma_n : \Delta^n \rightarrow X$ such that σ_n sends the 1-skeleton of Δ^n to the base point $* \in X$. For any topological space Z , let us denote by $C_*^1 Z$ the normalized chain complex generated by singular simplexes whose the 1-skeleton is sent to the base point of Z ; that is $C_*^1 Z = C_*(Sing^1 Z)$.

In [KS05] the cubical cobar construction $\Omega^\square(\text{Sing}^1 X)$ on the simplicial set $\text{Sing}^1 X$ is constructed. It is a monoidal cubical set; it is shown to be homotopically equivalent to the monoidal cubical set $\text{Sing}^\square \Omega_M X$, cf. [KS05, Theorem 5.1]. In particular, taking the cubical chain complexes, one obtains Adams' morphism [Ada56]

$$\Phi : C_*^\square \Omega^\square(\text{Sing}^1 X) = \underline{\Omega} C_*^1 X \rightarrow C_*^\square \Omega_M X,$$

that is a quasi-isomorphism of dg-algebras. In fact, since Φ arise from a map of cubical sets

$$(7.4) \quad \phi : \Omega^\square(\text{Sing}^1 X) \rightarrow \text{Sing}^\square \Omega_M X,$$

it is immediate that Φ is a morphism of dg-coalgebras. Indeed, given a cubical set Q , its cubical chain complex $C^\square(Q)$ is a dg-coalgebra; the coproduct we consider is the Serre diagonal defined using the face operator of Q (see [KS05, 2.4 equation (3)]). The compatibility of ϕ with the monoidal structures implies that Adams' morphism is a quasi-isomorphism of dg-bialgebras.

The construction of ϕ is realised by an induction process involving another map $p : P^\square \text{Sing}^1 X \rightarrow \text{Sing}^1 P X$ between the *cubical path construction* and the cubical set of the path space $P X$. For a simplicial set K , the cubical set $P^\square K$ is defined as the *twisted cartesian product* ([KS05, Definition 4.2])

$$(7.5) \quad P^\square(K) := \Omega^\square K \times_{\tau_U} K,$$

where $\tau_U : K \rightarrow \Omega^\square K$ is the *universal truncating twisting function* [KS05, Section 4]. The maps p and ϕ satisfy $p(\sigma', \sigma) = \phi(\sigma') \cdot p(e, \sigma)$ where \cdot stands for the left action $\Omega_M X \times P_M X \rightarrow P_M X$ and where $e \in \Omega^\square X$ is the unit of the monoidal cubical set, see [KS05, Proof of Theorem 5.1].

For a pair (K, L) of 1-reduced simplicial sets such that L is a sub simplicial set of K , we define the *relative cubical cobar construction* $\Omega^\square(K, L)$ as the twisted cartesian product

$$(7.6) \quad \Omega^\square(K, L) := \Omega^\square K \times_{\tau} L,$$

where $\tau : L \rightarrow \Omega^\square K$ is the inclusion $L \subset K$ composed with the universal truncating twisting function $\tau_U : K \rightarrow \Omega^\square K$.

It is easy to show that the concatenation $\Omega^\square K \times \Omega^\square(K, L) \rightarrow \Omega^\square(K, L)$ makes $\Omega^\square(K, L)$ into a cubical $\Omega^\square K$ -module.

Let (X, Y) be a pair of 1-connected pointed spaces such that $Y \subset X$. We defined a map of cubical sets

$$(7.7) \quad \psi : \Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) \rightarrow C_*^\square \Omega_M(X, Y)$$

as follows. The inclusion $Y \subset X$ gives rise to an inclusion of cubical sets $\Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) \subset P^\square(\text{Sing}^1 Y)$. The map ψ is defined as the restriction

7.2 Unreduced (relative) cobar construction

$p|_{\Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y)}$, and satisfies $\psi(\sigma', \sigma) = \phi(\sigma') \cdot \psi(e, \sigma)$ where \cdot stands for the left action induced by $\Omega_M X \times \Omega_M(X, Y) \rightarrow \Omega_M(X, Y)$. The map ψ is then a map of cubical modules. The fact that ψ is a homotopy equivalence can be deduced from the long exact sequences in homotopy induced by

$$\begin{aligned} \Omega_M X &\rightarrow \Omega_M(X, Y) \rightarrow Y, \text{ and} \\ |\Omega^\square \text{Sing}^1 X| &\rightarrow |\Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y)| \rightarrow |\text{Sing}^1 Y|, \end{aligned}$$

using that ϕ is a homotopy equivalence.

By construction (see [KS05, (4) iii]), we have

$$C_* \Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) = \underline{\Omega}(C_*^1 X, C_*^1 Y).$$

Consequently, we obtain

Proposition 7.9. *Let (X, Y) be a pair of 1-connected topological spaces pointed at $*$ such that $* \subset Y \subset X$. The map $\psi : \Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) \rightarrow \text{Sing}^1 \Omega_M(X, Y)$ induces a Φ -equivariant morphism $\Psi : C_* \Omega^\square(\text{Sing}^1 X, \text{Sing}^1 Y) = \underline{\Omega}(C_*^1 X, C_*^1 Y) \rightarrow C_*^1 \Omega_M(X, Y)$ that is a quasi-isomorphism of dg-coalgebras.*

7.2 Unreduced (relative) cobar construction

With regard to its structure, this section is analogous to Section 6.1: we exhibit a couple $(\mathcal{M}_B, \mathcal{Z}_{B,C})$ giving rise to the both *unreduced* cobar and relative cobar constructions over a couple of a coalgebra/comodule (B, C) in the category of algebras; the couple $(\mathcal{M}_B, \mathcal{Z}_{B,C})$ is shown to be an algebra over $\mathcal{R}\mathcal{L}_2$.

Let (B, ∇_B) be an (ungraded) unital/counital bialgebra with counit ϵ (i.e. B is a counital coalgebra in the category of unital algebras) and let (C, ∇_C) be a left B -comodule in the category of unital algebras.

Let us consider the *unreduced cobar construction* $\underline{\Omega}_u B$ given by

$$\underline{\Omega}_u B = (TB, D)$$

where D is the differential

$$\begin{aligned} D(b_1 \otimes \dots \otimes b_n) &= 1 \otimes b_1 \otimes \dots \otimes b_n + \sum_{i=1}^n (-1)^i b_1 \otimes \dots \otimes \nabla_B(b_i) \otimes \dots \otimes b_n \\ &\quad + (-1)^{n+1} b_1 \otimes \dots \otimes b_n \otimes 1, \end{aligned}$$

for $b_i \in B$. In the same fashion, we define $\underline{\Omega}_u(B, C)$ to be the *unreduced relative cobar construction*

$$\underline{\Omega}_u(B, C) = (TB \otimes C, D')$$

with differential

$$\begin{aligned} D'(b_1 \otimes \dots \otimes b_n \otimes c) &= 1 \otimes b_1 \otimes \dots \otimes b_n \otimes c + \\ &+ \sum_{i=1}^n (-1)^i b_1 \otimes \dots \otimes \nabla_B(b_i) \otimes \dots \otimes b_n \otimes c + (-1)^{n+1} b_1 \otimes \dots \otimes b_n \otimes \nabla_C(c), \end{aligned}$$

for $b_i \in B, c \in C$.

Let us fix some notations. For $k \geq 0$, $B^{\otimes k}$ is an algebra for the product $(a_1 \otimes \dots \otimes a_k) \cdot (b_1 \otimes \dots \otimes b_k) = a_1 b_1 \otimes \dots \otimes a_k b_k$. We denote by $\nabla_B^{(0)} = \epsilon$, $\nabla_B^{(1)} = \nabla_B$, and $\nabla_B^{(k)} = (\nabla_B \otimes id) \nabla_B^{(k-1)}$ for $k \geq 2$. Since B is a bialgebra, the tensor product $B^{\otimes k}$, $k \geq 0$, is: a left B -module with $b \triangleleft (b_1 \otimes \dots \otimes b_k) = \nabla_B^{(k-1)}(b) \cdot (b_1 \otimes \dots \otimes b_k)$ where the dot \cdot stands for the multiplication in $B^{\otimes k}$; and, a right B -module for $(b_1 \otimes \dots \otimes b_k) \triangleright b = (b_1 \otimes \dots \otimes b_k) \cdot \nabla_B^{(k-1)}(b)$. For $c \in C$, let us denote $\nabla_C(c) = \sum_{(c)} z_c^{(1)} \otimes c^{(2)}$, and more generally, $(\nabla_B^{(k-1)} \otimes id) \nabla_C(c) = \sum_{(c)} z_c^{(1)} \otimes \dots \otimes z_c^{(k)} \otimes c^{(k+1)}$.

Let us define \mathcal{M}_B to be the following multiplicative operad (for instance see [Men04]). For $k \geq 0$, we set $\mathcal{M}_B(k) = B^{\otimes k}$.

The partial composition maps

$$\circ_i^B : \mathcal{M}_B(k) \otimes \mathcal{M}_B(l) \rightarrow \mathcal{M}_B(k+l-1),$$

for $1 \leq i \leq k$, are given by

$$(a_1 \otimes \dots \otimes a_k) \circ_i^B (b_1 \otimes \dots \otimes b_l) := (a_1 \otimes \dots \otimes a_{i-1} \otimes a_i \triangleleft (b_1 \otimes \dots \otimes b_l) \otimes a_{i+1} \otimes \dots \otimes a_k).$$

We denote by γ^B the corresponding structural map of this operad. The multiplication $\mu \in \mathcal{M}_B(2)$ is $\mu = 1 \otimes 1$. Its $\delta_{\mathcal{Z}}$ -totalization is the unreduced cobar construction $\underline{\Omega}_{\mu} B$.

We define $\mathcal{Z}_{B,C}$ as $\mathcal{Z}_{B,C}(k) = B^{\otimes k} \otimes C$ for $k \geq 0$.

Let us define a wide left action of \mathcal{M}_B on $\mathcal{Z}_{B,C}$.

For $k \geq 1$, $1 \leq s \leq k$, $l_i \geq 1$ and an order preserving map $\beta : \{1 < \dots < s\} \rightarrow \{1 < \dots < k\}$, we define

$$(7.8) \quad \lambda_{\beta}^B : \mathcal{M}_B(k) \otimes \mathcal{Z}_{B,C}(l_1) \otimes \dots \otimes \mathcal{Z}_{B,C}(l_s) \rightarrow \mathcal{Z}_{B,C}(l_1 + \dots + l_s + k - s)$$

by

$$\begin{aligned} \lambda_{\beta}^B(f, (g_1; c_1), \dots, (g_s; c_s)) = & (\gamma(f; \mathbf{1}^{\beta(1)-1}, g_1, z_{c_1}^{(1)}, \dots, z_{c_1}^{(\beta(2)-\beta(1)-1)}, g_2 \triangleright z_{c_1}^{(\beta(2)-\beta(1))}, z_{c_2}^{(1)} z_{c_1}^{(\beta(2)-\beta(1)+1)}, \dots \\ & \dots, g_s \triangleright z_{c_{s-1}}^{(1)} \dots z_{c_1}^{(\beta(s)-1)}, \dots, z_{c_s}^{(k-\beta(s))} \dots z_{c_1}^{(k-\beta(1))}); \\ & c_s^{(k-\beta(s)+1)} c_{s-1}^{(k-\beta(s)+2)} \dots c_1^{(k-\beta(1)+1)}), \end{aligned}$$

where $\mathbf{1}^{\beta(1)-1} := \underbrace{1, \dots, 1}_{\beta(1)-1}$.

For $s = 1$ above, we get an infinitesimal left action

$$(7.9) \quad \lambda_i^B : \mathcal{M}_B(k) \otimes \mathcal{Z}_{B,C}(l) \rightarrow \mathcal{Z}_{B,C}(k+l-1)$$

given by

$$\lambda_i^B \left((a_1 \otimes \dots \otimes a_k) \otimes (b_1 \otimes \dots \otimes b_l \otimes c) \right) = \sum_{(c)} (a_1 \otimes \dots \otimes a_{i-1} \otimes a_i \triangleleft (b_1 \otimes \dots \otimes b_l) \otimes (a_{i+1} \otimes \dots \otimes a_k) \triangleright z' \otimes c''),$$

for $1 \leq i \leq k$.

For $s = k$ above, we get a left action

$$(7.10) \quad \lambda^B : \mathcal{M}_B(k) \otimes \mathcal{Z}_{B,C}(l_1) \otimes \dots \otimes \mathcal{Z}_{B,C}(l_k) \rightarrow \mathcal{Z}_{B,C}(l_1 + \dots + l_k)$$

defined by

$$\lambda^B(f, (g_1; c_1), \dots, (g_k; c_k)) := (\gamma(f; g_1, g_2 \triangleright z_{c_1}^{(1)}, g_3 \triangleright z_{c_2}^{(1)} z_{c_1}^{(2)}, \dots, g_k \triangleright z_{c_{k-1}}^{(1)} \dots z_{c_1}^{(k-1)}); c_k c_{k-1}^{(2)} \dots c_1^{(k)}).$$

For $s = 0$ above, we set an inclusion

$$(7.11) \quad \begin{aligned} \iota^B : \mathcal{M}_B(k) &\rightarrow \mathcal{Z}_{B,C}(k) \\ f &\mapsto (f; 1) \end{aligned}$$

The right action is given by

$$(7.12) \quad \begin{aligned} \rho^B : \mathcal{Z}_{B,C}(k) \otimes \mathcal{M}_B(l_1) \otimes \dots \otimes \mathcal{M}_B(l_k) &\rightarrow \mathcal{Z}_{B,C}(l_1 + \dots + l_k) \\ ((f; c), g_1, \dots, g_k) &\mapsto (\gamma^B(f; g_1, \dots, g_k); c), \end{aligned}$$

for any $(f; c) = b_1 \otimes \dots \otimes b_k \otimes c \in \mathcal{Z}_{B,C}(k)$ and $g_i = b_1^i \otimes \dots \otimes b_{l_i}^i \in \mathcal{M}_B(l_i)$.

Lemma 7.10. *The maps λ'^B , λ^B , $\{\lambda_i^B\}_i$, ι^B and ρ^B endow $\mathcal{Z}_{B,C}$ with a stable wide bimodule structure over \mathcal{M}_B . In particular, using the map $\text{ASSOC} \hookrightarrow \mathcal{M}_B$, the module $\mathcal{Z}_{B,C}$ is a wide bimodule over ASSOC . With regard to the infinitesimal bimodule structure over ASSOC , the $\delta_{\mathbb{Z}}$ -totalization of the corresponding (via Lemma 5.1) cosimplicial complex of $\mathcal{Z}_{B,C}$ is $\underline{\Omega}_u(B, C)$.*

Proof. Tedious but straightforward. The compatibility of λ'^B with the operadic structure of \mathcal{M} follows from the fact that C is a B -comodule in the category of unital algebras. \square

Finally, we obtain

Theorem 7.11. *Let B be a unital/counital bialgebra and let C be a B -comodule in the category of unital algebras. Then the couple $(\underline{\Omega}_u B, \underline{\Omega}_u(B, C))$ is an algebra over $\text{Coend}_{\mathcal{R}\mathcal{L}_2}(\delta_{\mathbb{Z}})$.*

7.3 Action of \mathcal{RS}_2

Let (M, N) be a pair of 1-connected topological monoids pointed at the unit 1 such that N is a sub-monoid of M . We explain how the following couple $(\underline{\Omega}C_*^1M, \underline{\Omega}(C_*^1M, C_*^1N))$ can be endowed with an action of \mathcal{RS}_2 . The operations are similar to the ones described in the previous section. However, the absence of a unit in the reduced dg-bialgebra $(C_*^1M)^+$ constrains us to write down the operations "manually".

Let us fix an unital/counital dg-bialgebra (B, ∇_B) with counit ϵ , such that $B = B^+ \oplus \mathbb{Z}$ and with $B_1 = 0$; and (C, ∇_C) be a left B -dg-comodule in the category of unital algebras. For example $B = C_*^1M$ and $C = C_*^1N$. We recall that the cobar construction $\underline{\Omega}B$ is, in particular, given as the free tensor algebra over $s^{-1}B^+$. We adopt the same notations as in Section 7.2. We recall the Koszul sign rule $(f \otimes g)(a \otimes b) = (-1)^{|g||a|}f(a) \otimes g(b)$. We denote the reduced coproducts $\overline{\nabla}_B(b) = \nabla_B(b) - 1 \otimes b - b \otimes 1$, $\overline{\nabla}_C(c) = \nabla_C(c) - 1 \otimes c$. We denote

$$[b_1, \dots, b_k] := (s^{-1})^{\otimes k}(b_1 \otimes \dots \otimes b_k)$$

The closed part of \mathcal{RS}_2 acts on it, see [Kad05] for an example in \mathbb{Z}_2 coefficients. The multiplication is the concatenation, and, for $k \geq 1$,

$$(7.13) \quad E'_{1,k} : \underline{\Omega}B \otimes (\underline{\Omega}B)^{\otimes k} \rightarrow \underline{\Omega}B$$

is given by

$$E'_{1,k}(f; g_1, \dots, g_k) = \sum \pm [a_1, \dots, a_{i_1} \triangleleft g_1, \dots, a_{i_k} \triangleleft g_k, \dots, a_n]$$

where $f = [a_1, \dots, a_n]$ and $g_i \in \underline{\Omega}B$.

In the same way, we can defined a multiplication

$$(7.14) \quad \mu'_o : \underline{\Omega}(B, C) \otimes \underline{\Omega}(B, C) \rightarrow \underline{\Omega}(B, C)$$

$$[a_1, \dots, a_k]c \otimes [b_1, \dots, b_l]c' \mapsto \sum_{(c)} \pm [a_1, \dots, a_k, (b_1, \dots, b_l) \triangleright z_c^{(1)}]c'c^{(2)},$$

we recall that for $c \in C$, $\sum_{(c)} z_c^{(1)} \otimes c^{(2)}$ denotes the coproduct $\nabla_C(c)$ so that $1 \otimes c$ is a term of this sum; and, a family of operation

$$(7.15) \quad E'_j : (\underline{\Omega}B) \otimes (\underline{\Omega}(B, C))^{\otimes j} \rightarrow \underline{\Omega}(B, C)$$

given by

$$E'_j(f; (g_1; c_1), \dots, (g_j; c_j)) = \sum \pm [a_1, \dots, a_{i_1} \triangleleft g_1, a_{i_1+1} z_{c_1}^{(1)}, \dots, \dots, a_{i_j} \triangleleft g_j \triangleright (z_{c_{j-1}}^{(1)} \cdots z_{c_1}^{(j-1)}), \dots, a_n (z_{c_{j-1}}^{(1)} \cdots z_{c_1}^{(j-1)})] c_j c_{j-1}^{(1)} \cdots c_1^{(j)},$$

for any $f = [a_1, \dots, a_n]$ and $(g_i; c_i) \in \underline{\Omega}(B, C)$.

We claim that, for appropriate signs, the operations (7.13), (7.14), (7.15) and the inclusion of $\underline{\Omega}B$ into $\underline{\Omega}(B, C)$ define an \mathcal{RS}_2 -algebra structure on $(\underline{\Omega}B, \underline{\Omega}(B, C))$.

8 Proof of Lemma 3.13

A straightforward computation shows that, for $n \geq 1$,

$$M_n \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^* = \bigcap_{0 \leq a \leq b \leq n-2} \text{Ker}(-s^b p_{a+1} + s^a p_{b+2}),$$

where $p_c : (\zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^{n-1})^{\oplus n} \rightarrow \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^{n-1}$ denotes the projection on the c -th factor and $s^c : \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^{n-1} \rightarrow \zeta(\mathcal{RL}_m)_{\tau,k,j}(\delta_{\mathbb{Z}})^n$ denotes the c -th degeneracy map.

Moreover, the maps j_n and j'_n are given by $s^0 \oplus \dots \oplus s^{n-1}$.

The proofs for the surjectivity of j_n and j'_n are quite similar; we focus on j_n .

Let $k \geq 0$ and $n \geq 1$ and $x_1 \oplus \dots \oplus x_n \in M_n \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^*$ be an arbitrary but fixed element. Let us fix $I = \{a_1, a_2, \dots, a_s\}$ with $1 \leq a_1 < a_2 < \dots < a_s \leq n$.

The family $\mathcal{F} = \{x_a\}_{a \in I}$ is called *complete* if:

1. $s^{a_v-2} x_{a_u} = s^{a_u-1} x_{a_v} \neq 0$ for $r \leq u < v \leq s$; and,
2. $s^{b-2} x_a = s^{a-1} x_b = 0$, for $1 \leq a < b \leq n$ whenever only one element either x_a or x_b belongs to \mathcal{F} .

Note that a complete family \mathcal{F} is *maximal*. Indeed, if \mathcal{F} is a strict subfamily of a complete family, then \mathcal{F} is not complete since the condition 2 above is not satisfied.

Recall that an element $x \in \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^{n-1}$ is represented by $T \otimes f_1 \otimes \dots \otimes f_k$ where T is a (non-degenerate) integer-string with numbers between 1 and k , and the coefficients $f_i \in \delta_{\mathbb{Z}}$ for each number $1 \leq i \leq k$.

Suppose that, in the integer-string T , the $(a-1)$ -th vertical bar is between the $(k_{a-1}+1)$ -th and the $(k_{a-1}+2)$ -th occurrences of the number i_{a-1} . Then $s^{a-1}x$ is represented by $s^{a-1}T \otimes f_1 \otimes \dots \otimes s^{k_{a-1}} f_{i_{a-1}} \otimes \dots \otimes f_k$ where $s^{a-1}T$ is the integer-string obtained from T by removing the $(a-1)$ -th vertical bar.

Note that if $s^{a-1}x = 0$ then $s^{k_{a-1}} f_{i_{a-1}} = 0$.

For an element $x_1 \oplus \dots \oplus x_n \in M_n \zeta(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^*$ we write $T_a \otimes f_1^a \otimes \dots \otimes f_k^a$ the non-degenerate representative element of x_a .

If $s^{b-2}x_a = s^{a-1}x_b \neq 0$ for some $1 \leq a < b \leq n$, then the two coefficients f_i^a and f_i^b satisfy one of the following properties R1 - R4.

R1. They are equal.

$$\text{R2. } s^{k_{b-2}} f_i^a = s^{k_{a-1}} f_i^b.$$

$$\text{R3. } s^{k_{b-2}} f_i^a = f_i^b.$$

$$\text{R4. } f_i^a = s^{k_{a-1}} f_i^b.$$

Remark 8.1. If f_i^a and f_i^b satisfy R3 then:

- there is a $j \neq i$ such that f_j^a and f_j^b satisfy R4; and,
- f_l^a and f_l^b satisfy R1 for all $l \neq i, j$;

If f_i^a and f_i^b satisfy R2 then f_j^a and f_j^b satisfy R1 for all $j \neq i$.

We say that the coefficients f_i^a and f_i^b are related by:

- an equality, $f_i^a = f_i^b$, in case R1;
- a cycle, $f_i^a \Leftrightarrow f_i^b$, in case R2;
- an arrow, $f_i^a \rightarrow f_i^b$, from f_i^a to f_i^b in case R3;
- an arrow, $f_i^b \rightarrow f_i^a$, from f_i^b to f_i^a in case R4.

For instance see Figure 8.3.

Definition 8.2. A coefficient is called *maximal* if it is related to any other element either by an equality or by an outgoing arrow.

Lemma 8.3. Let $k \geq 0$ and $n \geq 1$ and $x_1 \oplus \dots \oplus x_n \in M_n \xi(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^*$. Then, for each complete family $\mathcal{F} = \{x_a\}_{a \in I}$, we have either:

a) there exists a unique $1 \leq i_0 \leq k$ such that, for any two indices $a \neq b$ in I , the coefficients f_i^a and f_i^b are:

- related by a cycle if $i = i_0$,
- equal if $i \neq i_0$;

or,

b) for each $1 \leq i \leq k$ there exists a coefficient f_i^a which is maximal. Moreover, any two maximal elements f_i^a and f_i^b are equal.

Proof. Let us remark that, for degree reasons, given an element f_i^a for $a \in I$, then the following configuration $f_i^b \rightarrow f_i^a \rightarrow f_i^c$ is not possible for any $b, c \in I$. One shows that a cycle cannot be maximal (that is, none of the coefficients involved in a cycle can be related to another one uniquely by an outgoing arrow or by an equality). Then, together with the previous remark, this implies that if two elements f_i^a and f_i^b are maximal then there are equal. Also, from the non maximality of the cycles, one deduces that either there are only cycles or, there exists a maximal element. \square

Let $\mathcal{F} = \{x_a\}$ be a family with only one element; it is a complete family. Let us fix i_0 .

C1. Suppose there exists β_0 being the maximal number $\leq a - 1$ such that the β_0 -th vertical bar is attached to i_0 . This means that there exists k_{β_0} such that the β_0 -th vertical bar lies between the $k_{\beta_0} + 1$ and the $k_{\beta_0} + 2$ occurrences of i_0 . Let $\beta_0, \beta_1, \dots, \beta_q$ the maximal sequence of consecutive vertical

bars *i.e.* the maximal sequence such that, for $1 \leq v \leq q$, the β_v -th vertical bar lies between the $(k_{\beta_v} + 1)$ -th and the $(k_{\beta_v} + 2)$ -th occurrence of i_0 with $k_{\beta_v} + 2 = k_{\beta_{v-1}} + 1$.

- (a) Suppose $\beta_0 < a - 1$. We denote by DT_a the integer-string obtained from T_a by adding both a vertical bar at the position $(a - 1)$ and an occurrence of i_0 in such a way that the $(a - 1)$ -th additional vertical bar lies between the $(k_{\beta_0} + 2)$ -th and the $(k_{\beta_0} + 3)$ -th occurrences of i_0 .

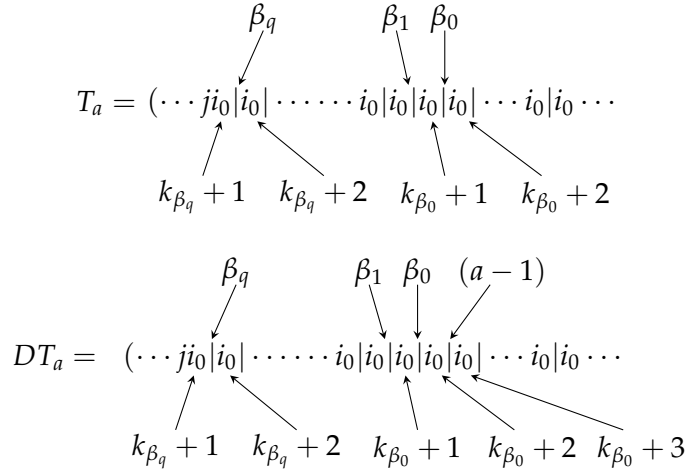


Figure 8.1: T_a and DT_a in case $\beta_0 < a - 1$.

Let y_a be the element represented by

$$DT_a \otimes f_1^a \otimes \cdots \otimes f_{i_0-1}^a \otimes F_{i_0}^a \otimes f_{i_0+1}^a \otimes \cdots \otimes f_k^a,$$

where $F_{i_0}^a = \sum_{p=0}^{q+1} (-1)^{\beta_0 - \beta_p} d^{k_{\beta_0} + 1 - p} f_{i_0}^a$.

- (b) Suppose $\beta_0 = a - 1$. We denote by DT_a the integer-string obtained from T_a by adding both a vertical bar at the position $(a - 1)$ and an occurrence of i_0 in such a way that the $(a - 1)$ -th additional vertical bar lies between the k_{β_0} -th and the $(k_{\beta_0} + 1)$ -th occurrences of i_0 .

Let y_a be the element represented by

$$DT_a \otimes f_1^a \otimes \cdots \otimes f_{i_0-1}^a \otimes F_{i_0}^a \otimes f_{i_0+1}^a \otimes \cdots \otimes f_k^a,$$

where $F_{i_0}^a = \sum_{p=0}^q (-1)^{\beta_0 - \beta_p} d^{k_{\beta_0} - p} f_{i_0}^a$.

- C2. Suppose there is no maximal number $\beta \leq a - 1$ such that the β -th vertical bar is attached to i_0 . Then we define DT_a as the integer-string obtained from T_a by adding both a vertical bar at the position $(a - 1)$ and an occurrence of i_0 in such a way that the $(a - 1)$ -th additional vertical bar lies

between the first and the second occurrences of i_0 . We define y_a as the element represented by

$$DT_a \otimes f_1^a \otimes \cdots \otimes f_{i_0-1}^a \otimes F_{i_0}^a \otimes f_{i_0+1}^a \otimes \cdots \otimes f_k^a,$$

where $F_{i_0}^a = d^1 f_{i_0}^a$.

Lemma 8.4. . Let $\mathcal{F} = \{x_a\}$ be a family with only one element. With the notations above, for each case C1a, C1b or C2, the element y_a satisfies

$$s^{b-1}y_a = \begin{cases} x_a & \text{if } b = a; \\ 0 & \text{if } b \neq a. \end{cases}$$

Proof. Straightforward. □

Let us suppose a complete family $\mathcal{F} = \{x_a\}_{a \in I}$ with $\text{card}(I) \geq 2$, satisfying the property **a** of Lemma 8.3. For $a < b$ two indices with $a \in I$, let us denote by k_{b-2}^a the integer such that the element $s^{b-2}x_a$ is represented by $s^{b-2}T \otimes f_1^a \otimes \cdots \otimes s^{k_{b-2}^a} f_{i_{b-2}}^a \otimes \cdots \otimes f_k^a$. If $b \in I$ then $i_{b-2}^a =: i_0$ as in **a** of Lemma 8.3. Similarly k_{b-1}^a denotes the integer involved in $s^{b-1}x_a$ for $b < a$. For the sake of simplicity, since we are interested only on the coefficients $f_{i_0}^a$, we drop the index i_0 .

We remark that, for $a, b, c \in I$:

$$\begin{aligned} k_{b-2}^a &= k_{b-2}^c \text{ for } a, c < b \\ k_{b-1}^a &= k_{b-1}^c \text{ for } a, c > b \\ k_{b-2}^a &= k_{b-1}^c - 1 \text{ for } a < b < c. \end{aligned}$$

Recall that $I = \{a_1, \dots, a_s\}$. We set

$$\begin{aligned} k_1 &:= k_{a_1-1}^{a_2} - 1 \\ k_2 &:= k_{a_1-2}^{a_1} \\ k_3 &:= k_{a_2-2}^{a_1} \\ &\vdots \\ k_s &:= k_{a_s-2}^{a_1}. \end{aligned}$$

Then we get

$$(8.1) \quad s^{k_j} f^{a_i} = s^{k_i+1} f^{a_j} \text{ for all } 1 \leq i < j \leq s.$$

Moreover, if I has at least 3 elements, then $k_i < k_j$ whenever $1 \leq i < j \leq s$.

Let us suppose there exists a index $b \notin I$. We set

$$\begin{aligned} r^b &:= k_{b-1}^{a_1} - 1 \text{ for } b < a_1 \\ r^b &:= k_{b-2}^{a_1} \text{ for } a_1 < b. \end{aligned}$$

Lemma 8.5. *With the notations above, let $\mathcal{F} = \{x_a\}_{a \in I}$ be a complete family with $\text{card}(I) \geq 2$, satisfying the property *a* of Lemma 8.3. Then, there exists an element $g \in \delta_{\mathbb{Z}}^n$ such that:*

- $s^{k_i+1} = f^{a_i}$ for $1 \leq i \leq s$; and,
- if $b \notin I$, then $s^{b+1}g = 0$.

Proof. It is sufficient to prove the statement for the $f^a \in \delta_{\mathbb{Z}}^{n_a}$ considered as generators of the complex $\delta_{\mathbb{Z}}^{n_a}$, $a \in I$. Then, we have $f^a = d^{i_v^a} d^{i_{v-1}^a} \cdots d^{i_0^a} e_t$ for some $i_v^a \leq i_{v-1}^a \leq \cdots \leq i_0^a \leq t$. Note that (8.1) says that v and t above are invariant with $a \in I$. For $j \geq 2$, let us denote by α_j the maximal index among the indices $1 \leq u \leq v$ that satisfy

$$i_u = \begin{cases} k_j - (\alpha_j - 1); & \text{or,} \\ k_j - (\alpha_j - 1) + 1. \end{cases}$$

For each $j \geq 2$, the index α_j can be placed into the increasing sequence $i_v^a \leq i_{v-1}^a \leq \cdots \leq i_0^a \leq t$ in such a way that the obtained sequence is increasing. Following this way, we add $d^{i_{\alpha_2}}, d^{i_{\alpha_3}}, \dots, d^{i_{\alpha_s}}$ to the element $s^{k_2} s^{k_3} \cdots s^{k_s} f^a$; we define g as the resulting element. It is straightforward to check that g satisfies the statement. \square

In order to establish the following lemma, we need to perform a construction. Let $a < b$ and x_a, x_b related by $s^{a-1}x_b = s^{b-2}x_a \neq 0$. This implies that the integer-strings T_a and T_b satisfy $s^{a-1}T_b = s^{b-2}T_a$. In other words, once removed both the $(a-1)$ -th vertical bar of T_b and the $(k_{a-1}^b + 2)$ -th occurrence of i_{a-1}^b ; and, once removed both the $(b-2)$ -th vertical bar of T_a and the $(k_{b-2}^a + 2)$ -th occurrence of i_{b-2}^a one gets the same integer-string. We suppose

$$(8.2) \quad k_{b-2}^a \geq k_{a-1}^b.$$

Then, we define $T_{a \wedge b}$ to be the integer-string obtained from T_a by adding both a vertical bar at the position $(a-1)$ and an occurrence of i_{a-1}^b in such a way that the $(a-1)$ -th additional vertical bar lies between the $(k_{a-1}^b + 1)$ -th and the $(k_{a-1}^b + 2)$ -th occurrences of i_{a-1}^b . By construction, $s^{a-1}T_{a \wedge b} = T_a$ and $s^{b-1}T_{a \wedge b} = T_b$.

$$T_{a \wedge b} = (\cdots i_{a-1}^b | i_{a-1}^b | \cdots i_{b-2}^a | i_{b-2}^a \cdots)$$

Figure 8.2: $T_{a \wedge b}$ in case $k_{b-2}^a \geq k_{a-1}^b$ and $i_{b-2}^a \neq i_{a-1}^b$.

Lemma 8.6. Let $\mathcal{F} = \{x_a\}_{a \in I}$ be a complete family with $I = \{a_1, \dots, a_s\}$, $s \geq 2$.

- i) Suppose \mathcal{F} satisfies the property a) of Lemma 8.3. Suppose $s \geq 3$ or $I = \{a < b\}$ together with the property (8.2). Let y be the element represented by $T_{a_1 \wedge a_2} \otimes f_1^{a_1} \otimes \dots \otimes f_{i_0-1}^{a_1} \otimes g \otimes f_{i_0+1}^{a_1} \otimes \dots \otimes f_k^{a_1}$ where i_0 is as in a) of Lemma 8.3. Then $s^{a-1}y = x_a$ for all $a \in I$. Moreover, if $b \notin I$, then $s^b y = 0$.
- ii) Suppose \mathcal{F} satisfies the property b) of Lemma 8.3. For $1 \leq i \leq k$, let F_i be the maximal coefficient among the $f_i^{a_1}, \dots, f_i^{a_s}$. Let y be the element represented by $T_{a_1 \wedge a_2} \otimes F_1 \otimes \dots \otimes F_k$. Then, $s^{a-1}y = x_a$ for all $a \in I$. Moreover, for $b \notin I$, one has $s^b y = 0$.

Proof. Case i). Clear from Lemma 8.5.

Case ii). We observe first that, by completeness of the family, one has $i_{a_1-1}^b = i_{a_1-1}^c$ for all $b, c > a_1$ with $b, c \in I$. Then, for $b > a_1$ with $b \in I$, one has $F_i = f_i^{a_1}$ whenever $i \neq i_{a_1-1}^b$. This can be deduced from the observation that the total degree $\chi = \sum_{i=1}^k |f_i^{a_1}|$ of the coefficients of x_{a_1} is invariant under relations, that is here, $\chi = \sum_{i=1}^k |f_i^b|$ for all $b \in I$.

It results that, given any $b > a_1$ with $b \in I$, one has either:

1. $F_{i_{b-1}^y} = f_{i_{b-2}^{a_1}}^{a_1}$ and there is an arrow from $F_{i_{b-1}^y}$ to $f_{i_{b-2}^{a_1}}^b$ and $F_i = f_i^c$ for $i \neq i_{b-2}^{a_1}$; or,
2. $F_{i_{b-1}^y} = f_{i_{b-2}^c}^c$ if $c < b$ (or $F_{i_{b-1}^y} = f_{i_{b-1}^c}^c$ if $c > b$) and there is an arrow from $F_{i_{b-1}^y}$ to $f_{i_{b-1}^c}^b$ and $F_i = f_i^c$ for $i \neq i_{b-1}^c$.

The fact that $s^{b-1}y = 0$ follows from the existence of an $a \in I$ such that $s^{k_{b-1}^a} f_{i_{b-1}^a}^a = 0$ where $f_{i_{b-1}^a}^a$ is maximal. \square

Suppose that the complete family $\mathcal{F} = \{x_a\}_{a \in I}$ with $I = \{a < b\}$ satisfies the property a) of Lemma 8.3 and suppose $k_{b-2}^a < k_{a-1}^b$.

In this case we define $T_a^{a,b}$ to be the integer-string obtained from T_a by adding a vertical bar at the position $(a-1)$ together with an occurrence of i_{b-2}^a in such a way that the additional $(a-1)$ -th vertical bar lies between the $(k_{b-2}^a + 1)$ -th and the $(k_{b-2}^a + 2)$ -th occurrences of i_{b-2}^a . Thus, the $(a-1)$ -th and the $(b-1)$ -th vertical of $T_a^{a,b}$ are separated solely by the $(k_{b-2}^a + 2)$ -th occurrence of i_{b-2}^a .

Similarly, we define $T_b^{a,b}$ to be the integer-string obtained from T_b by adding a vertical bar at the position $(b-1)$ together with an occurrence of i_{a-1}^b in such a way that the additional $(b-1)$ -th vertical bar lies between the $(k_{a-2}^b + 2)$ -th and the $(k_{a-1}^b + 3)$ -th occurrences of i_{a-1}^b . Thus, the $(a-1)$ -th and the $(b-1)$ -th vertical of $T_b^{a,b}$ are separated solely by the $(k_{a-2}^b + 2)$ -th occurrence of i_{a-1}^b . A brief investigation of the properties of \mathcal{F} shows that such constructions are possible.

Lemma 8.7. Let $\mathcal{F} = \{x_a\}_{a \in I}$ be a complete family with $I = \{a < b\}$. Suppose \mathcal{F} satisfies the property a) of Lemma 8.3 and suppose $k_{b-2}^a < k_{a-1}^b$.

Let y_a be the element represented by

$$T_a^{a,b} \otimes f_1^a \otimes \cdots \otimes f_{i_0-1}^a \otimes d^{k_{b-2}^a+1} f_{i_0}^a \otimes f_{i_0+1}^a \otimes \cdots \otimes f_k^a,$$

and let y_b be the element represented by

$$T_b^{a,b} \otimes f_1^b \otimes \cdots \otimes f_{i_0-1}^b \otimes d^{k_{a-1}^b+1} f_{i_0}^b \otimes f_{i_0+1}^b \otimes \cdots \otimes f_k^b,$$

where i_0 is as in a) of Lemma 8.3 and $T_a^{a,b}, T_b^{a,b}$ are defined above.

Then $s^{c-1}y_c = x_c$ for all $a \in I$. Moreover, the family

$$\{x_{a'} = s^{a-1}y_b, x_{b'} = s^{b-1}y_a\}$$

satisfies the property a) of Lemma 8.3 together with $k_{b'-2}^{a'} \geq k_{a'-1}^{b'}$.

Proof. Straightforward. □

Example 8.8. Let $\mathcal{F} = \{x_a, x_b, x_c\}$ be complete family such that

$$\begin{aligned} s^{k_{b-2}^a} f_1^a &= f_1^b \text{ and } s^{k_{a-1}^b} f_2^b = f_2^a \\ s^{k_{c-2}^a} f_1^a &= f_1^c \text{ and } s^{k_{a-1}^c} f_2^c = f_2^a \\ s^{k_{c-2}^b} f_1^b &= s^{k_{b-1}^c} f_1^c. \end{aligned}$$

Pictorially, we represent \mathcal{F} as follows.

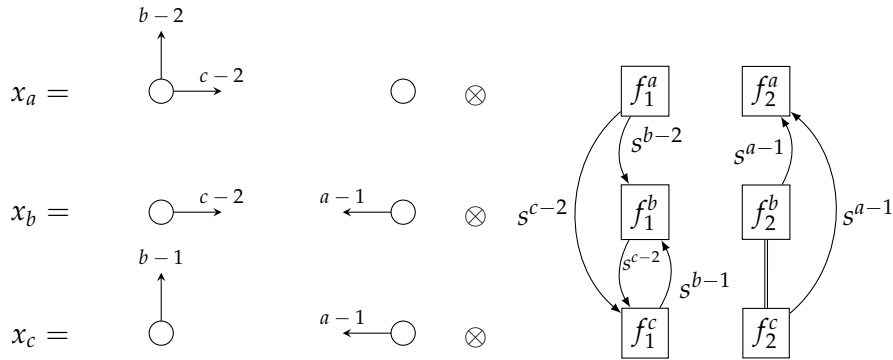


Figure 8.3: A representation of a complete family.

An integer-string T of k numbers with v ordered vertical bars can be (in part) represented by a sequence of k vertices together with v arrows and a label (e.g. k_{b-1}^a) for each arrow indicating its location T . In this example, we have drawn only the arrows involved in the relation among x_a, x_b, x_c without mentioning their label.

The solution y given by ii) of Lemma 8.6 is represented by

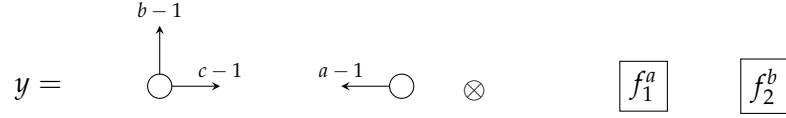


Figure 8.4: A representation of a solution for the complete family of Figure 8.3.

Definition 8.9. Let $\mathcal{F} = \{x_a\}_{a \in I}$ be a family. The graph of \mathcal{F} , denoted by $\mathcal{G}_{\mathcal{F}}$, is defined as follows. The vertices of $\mathcal{G}_{\mathcal{F}}$ are indexed by I ; two vertices a and b with $a < b$ are related by an edge if and only if $s^{b-1}x_a = s^{a-2}x_b \neq 0$.

Definition 8.10. A family is called *connected* if its graph so is. Two families \mathcal{F} and \mathcal{F}' are called *disjoint* if their graphs $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{G}_{\mathcal{F}'}$ are disjoint.

It is clear that an element $x_1 \oplus \cdots \oplus x_n \in M_n \xi(\mathcal{RL}_m)_k(\delta_{\mathbb{Z}})^*$ can be decomposed into complete families and maximal connected families which are 2 by 2 disjoint. Moreover, we have seen that, for a complete family $\mathcal{F} = \{x_a\}_{a \in I}$, there exists an element in the pre-image which vanishes outside *i.e.* there exists an element y such that: $s^{a-1}y = x_a$ for all $a \in I$; and, $s^b y = 0$ else. In fact, this statement is true for maximal non connected families as well.

We dedicate the remaining part of this section in proving this.

Definition 8.11. A family \mathcal{F} is *starred at a* if every other vertex of $\mathcal{G}_{\mathcal{F}}$ is related to the vertex a by an edge.

Lemma 8.12. Let $\mathcal{F} = \{x_a\}_{a \in I}$ be a maximal connected family which is not complete. Let J be the maximal subset of I such that the family is starred at each element of J . Then we have either the following Case 1 or Case 2.

Case 1: $J = \emptyset$. Then there exists, for each $a \in I$, an element y_a such that:

- (a) $s^{a-1}y_a = x_a$; and,
- (b) the graph of the connected family $\{s^{b-1}y_b\}_{b \in I}$ is a strict sub-graph of $\mathcal{G}_{\mathcal{F}}$.

Case 2: $J \neq \emptyset$. Then, there exists a y_J such that

$$s^{a-1}y_J = \begin{cases} \neq 0 & \text{for } a \in I; \\ = x_a & \text{for } a \in J; \\ = 0 & \text{for } a \notin I. \end{cases}$$

Moreover, there exists, for each $a \in I$, an element y_a such that:

- (a) $s^{a-1}y_a = x_a$; and,
- (b) the graph of the connected family $\{s^{b-1}y_b\}_{b \in I}$ is a strict sub-graph of $\mathcal{G}_{\mathcal{F}}$.

Proof. Case 1: Let $a \in I$. Since $J = \emptyset$, there exists at least one $b \in I$ non related to a by an edge. That is, at least one of the coefficients of a is involved in a vanishing relation (with x_b). Explicitly, by denoting by $f_{i_0}^a$ such a coefficient, we have $s^{k_{b-1}^a} f_{i_0}^a = 0$ if $b < a$ or $s^{k_{b-2}^a} f_{i_0}^a = 0$ if $b > a$.

Suppose $b > a$. We pick a j and we define y_a similarly to C1 and C2, as the element represented by

$$DT_a \otimes f_1^a \otimes \cdots \otimes f_{j-1}^a \otimes F_j^a \otimes f_{j+1}^a \otimes \cdots \otimes f_k^a.$$

The case $b < a$ is similar.

Case 2: The elements indexed by J generate a sub-graph \mathcal{G}_J which is complete³. Using similar constructions/arguments than those in Lemma 8.6 and Lemma 8.7, we get the existence of a solution y_J such that

$$s^{a-1} y_J = \begin{cases} x_a & \text{for } a \in J; \\ 0 & \text{for } a \notin I. \end{cases}$$

One checks that the given solution y_J satisfies also $s^{a-1} y_J \neq 0$ for $a \in I \setminus J$. Indeed, by construction, the coefficients of y_J satisfy either:

- they are as in a) of Lemma 8.3; or,
- they are maximal coefficients among the coefficients of the x_c for $c \in J$, as in b) of Lemma 8.3.

The second case is clear. Indeed, since $\mathcal{G}_{\mathcal{F}}$ is starred at each $a \in J$, all the coefficients of x_a are not involved in any vanishing relation (among the x_b , for $b \in I$). Thus, each coefficient f_i^c of each element x_c is related to f_i^a non trivially. For the first case, a straightforward verification shows that the solution given in i) of Lemma 8.6 and the solution given in Lemma 8.7 satisfy the required property.

For each $a \in I \setminus J$, the element y_a is defined as in *Case 1* of this proof. \square

Notation : let $\mathcal{F} = \{x_a\}_{a \in I}$ be a family; we denote by $||x_a||$ the number of edges in the graph $\mathcal{G}_{\mathcal{F}}$ starting from a .

Let $\mathcal{F} = \{x_a\}_{a \in I}$ be a maximal connected family which is not complete. Suppose $J = \emptyset$ as in Case 1 of Lemma 8.12. For each $a \in I$, let us denote by I_a the maximal subset of I such that $s^{b-1} y_a \neq 0$ for $b \in I_a$.

From the definition of y_a in Lemma 8.12 it is clear that, for each $a \in I$, the family $\mathcal{F}_a = \{s^{b-1} y_a\}_{b \in I_a}$ is starred at a .

Then, by connectivity, for each a there exists $b \in I$ such that $a, b \in I_a \cap I_b$. This allows us to define two families obtained by exchanging some elements.

³Note that the family $\{x_a\}_{a \in J}$ is not complete.

Precisely, we denote by $\mathcal{F}_{a|b}^1 = \{\tilde{x}_c\}_{c \in I_a \cup I_b}$ the family defined by

$$\tilde{x}_c = \begin{cases} s^{c-1}y_a & \text{for } c \in I_a \setminus (I_a \cap I_b); \\ s^{c-1}y_b & \text{for } c \in I_b \setminus (I_a \cap I_b); \\ s^{c-1}y_a & \text{for } c \in I_a \cap I_b \text{ if } \|s^{c-1}y_a\| > \|s^{c-1}y_b\| \text{ or if } s^{c-1}y_a = x_a; \\ s^{c-1}y_b & \text{for } c \in I_a \cap I_b \text{ if } \|s^{c-1}y_b\| > \|s^{c-1}y_a\| \text{ or if } s^{c-1}y_b = x_b, \end{cases}$$

and we denote by $\mathcal{F}_{a|b}^2 = \{\hat{x}_c\}_{c \in I_a \cap I_b}$ the family defined by

$$\hat{x}_c = \begin{cases} s^{c-1}y_b & \text{for } c \in I_a \cap I_b \text{ if } \|s^{c-1}y_a\| > \|s^{c-1}y_b\| \text{ or if } s^{c-1}y_a = x_a; \\ s^{c-1}y_a & \text{for } c \in I_a \cap I_b \text{ if } \|s^{c-1}y_b\| > \|s^{c-1}y_a\| \text{ or if } s^{c-1}y_b = x_b. \end{cases}$$

The family $\mathcal{F}_{a|b}^1$ contains x_a and x_b ; the family $\mathcal{F}_{a|b}^2$ is the by-product of this "exchange". It is straightforward to check that both $\mathcal{F}_{a|b}^1$ and $\mathcal{F}_{a|b}^2$ are connected.

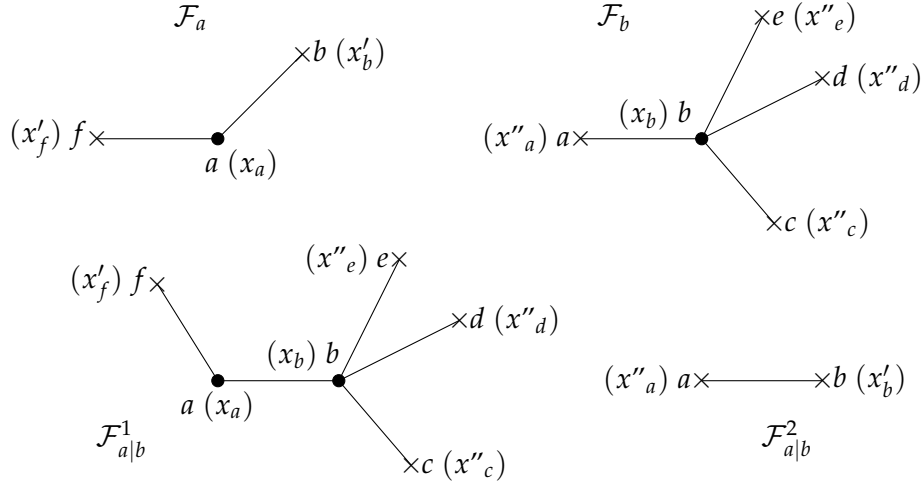


Figure 8.5: Example of graphs of \mathcal{F}_a and \mathcal{F}_b , and the corresponding ones of $\mathcal{F}_{a|b}^1$ and $\mathcal{F}_{a|b}^2$.

By connectivity, once formed $\mathcal{F}_{a|b}^1$, there is an index $c \in I$ such that $c \in (I_a \cup I_b) \cap I_c$ and $a \in I_a \cap I_c$ or $b \in I_b \cap I_c$. In the same way as above we define the family $\mathcal{F}_{a|b|c}^1 = \mathcal{F}_{(a|b)|c}^1$. We repeat this until one exhausts I . We denote by \mathcal{F}_I^1 the last (largest) family. Together with these families, come $\mathcal{F}_{a|b}^2$, $\mathcal{F}_{a|b|c}^2$, ..., and \mathcal{F}_I^2 .

Lemma 8.13. *Let $\mathcal{F} = \{x_a\}_{a \in I}$ be a maximal connected family which is not complete. With the notations above, suppose $J = \emptyset$. Then $\mathcal{F}_I^1 = \mathcal{F}$.*

Moreover, suppose that for each of the families $\mathcal{F}_{a|b}^2$, $\mathcal{F}_{a|b|c}^2$, ..., and \mathcal{F}_I^2 , there is an

element $z_a^2, z_{a|b}^2, \dots$, and z_I^2 , respectively, such that

$$s^{u-1}z_{a|b|\dots|q}^2 = \begin{cases} \hat{x}_u & \text{for } u \notin I_a \cap I_b \cap \dots \cap I_q; \\ 0 & \text{for } u \in I_a \cap I_b \cap \dots \cap I_q. \end{cases}$$

Then, the element

$$Y := \sum_{a \in I} y_a - (z_a^2 + z_{a|b}^2 + \dots + z_I^2),$$

where the y_a are defined in Lemma 8.12, satisfies

$$s^{b-1}Y = \begin{cases} x_b & \text{for } b \in I; \\ 0 & \text{for } b \notin I. \end{cases}$$

Proof. Straightforward. □

Lemma 8.14. Let $\mathcal{F} = \{x_a\}_{a \in I}$ be a maximal connected family which is not complete. Suppose $J \neq \emptyset$ as in Case 2 of Lemma 8.12. For each $a \in I \setminus J$, let us denote by I_a the maximal subset of I such that $s^{b-1}y_a \neq 0$ for $b \in I_a$. Then, the family $\mathcal{F} = \{s^{b-1}y_a, s^{a-1}y_J\}_{b \in I_a \setminus \{a\}}$ is connected.

Let suppose that, for each $a \in I \setminus J$ as in Case 2 of Lemma 8.12, there exists a z_a such that

$$s^{b-1}z_a = \begin{cases} 0 & \text{for } b \notin I_a; \\ s^{b-1}y_a & \text{for } b \in I_a \setminus \{a\}; \\ s^{a-1}y_J & \text{for } b = a. \end{cases}$$

Then, the element

$$Y := y_J + \sum_{a \in J} (y_a - z_a)$$

satisfies

$$s^{b-1}Y = \begin{cases} x_b & \text{for } b \in I; \\ 0 & \text{for } b \notin I. \end{cases}$$

Proof. Straightforward. □

We conclude the proof by induction on s (the length of I) that ensures the existence of the z_a^2 and z_a 's used in Lemma 8.13 and 8.14. □

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UNIVERSIDADE FEDERAL DO PARANÁ, DEPARTAMENTO DE MATEMÁTICA C.P. 019081, 81531-990 CURITIBA, PR - BRASIL

E-mail address: alexandre.quesney@univ-nantes.fr