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Ai-Hua Fan, Shilei Fan, Lingmin Liao, Yuefei Wang. MINIMALITY OF p -ADIC RATIONAL MAPS WITH GOOD REDUCTION. 2015. hal-01227597v2

HAL Id: hal-01227597

<https://hal.science/hal-01227597v2>

Preprint submitted on 15 Nov 2015 (v2), last revised 5 Dec 2016 (v3)

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MINIMALITY OF p -ADIC RATIONAL MAPS WITH GOOD REDUCTION

AIHUA FAN, SHILEI FAN, LINGMIN LIAO, AND YUEFEI WANG

ABSTRACT. A rational map with good reduction in the field \mathbb{Q}_p of p -adic numbers defines a 1-Lipschitz dynamical system on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over \mathbb{Q}_p . The dynamical structure of such a system is completely described by a minimal decomposition. That is to say, $\mathbb{P}^1(\mathbb{Q}_p)$ is decomposed into three parts: finitely many periodic orbits; finite or countably many minimal subsystems each consisting of a finite union of balls; and the attracting basins of the periodic orbits and minimal subsystems. For any prime p , a criterion of minimality for rational maps with good reduction is obtained. When $p = 2$, a complete characterization of minimal rational maps with good reduction is given in terms of their coefficients. It is also proved that a rational map of degree 2 or 3 can never be minimal on the whole space $\mathbb{P}^1(\mathbb{Q}_2)$.

1. INTRODUCTION

The present study contributes to the theory of p -adic dynamical systems which has recently been intensively and widely developed. For this development, one can consult the books [3, 6, 27] and their bibliographies therein.

For a prime number p , let \mathbb{Q}_p be the field of p -adic numbers and \mathbb{Z}_p be the ring of integers in \mathbb{Q}_p . Ergodic theory on the ring \mathbb{Z}_p is extremely important for applications to automata theory, computer science and cryptology, especially in connection with pseudo-random numbers and uniform distribution of sequences. The minimality, or equivalently the ergodicity with respect to the Haar measure, of non-expanding dynamical systems on \mathbb{Z}_p are extensively studied in [2, 8, 9, 12, 13, 14, 19, 21, 22], and so on.

The dynamical properties of the fixed points of the rational maps have been studied in the space \mathbb{C}_p of p -adic complex numbers [5, 23, 24, 28] and in the adelic space [10]. The Fatou and Julia theory of the rational maps on \mathbb{C}_p , and on the Berkovich space over \mathbb{C}_p , are also developed [7, 6, 20, 26, 27]. However, the global dynamical structure of rational maps on \mathbb{Q}_p remains unclear, though the rational maps of degree one are totally characterized in [16].

Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of degree $d \geq 2$. Then ϕ induces a dynamical system on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over \mathbb{Q}_p , denoted by $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$. Let $E \subset \mathbb{P}^1(\mathbb{Q}_p)$ be a subset such that $\phi^{-1}(E) \subset E$. Then restricted to E , ϕ defines a subsystem $(E, \phi|_E)$. The subsystem $(E, \phi|_E)$ is called *minimal* if for any point $x \in E$, the orbit of x under ϕ is dense in E . In this article, we suppose that ϕ has good reduction (see the definition below). The minimality of $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ and its subsystems will be fully investigated. As we will see, any rational map ϕ having good reduction is 1-Lipschitz continuous on $\mathbb{P}^1(\mathbb{Q}_p)$ which is equipped with its spherical metric. This suggests that $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ shares many

2010 *Mathematics Subject Classification*. Primary 37P05; Secondary 11S82, 37B05.

Key words and phrases. p -adic dynamical system, minimal decomposition, projective line, good reduction, rational map.

A. H. FAN was supported by NSF of China (Grant No. 11471132); S. L. FAN was supported by NSF of China (Grant No. 11401236); L. M. LIAO was supported by 12R03191A-MUTADIS; Y. F. WANG was supported by NSF of China (Grant No. 11231009).

properties with the polynomial dynamics on \mathbb{Z}_p , which are 1-Lipschitz with respect to the metric induced by the p -adic absolute value.

Let $\bar{\cdot}$ denote the *reduction modulo p* from \mathbb{Z}_p to $\mathbb{Z}_p/p\mathbb{Z}_p$ such that $a \mapsto \bar{a}$ with $a \equiv \bar{a} \pmod{p}$. For a polynomial $f(z) = \sum_{i=0}^d a_i z^i \in \mathbb{Z}_p[z]$, the *reduction of f modulo p* is defined as

$$\bar{f}(z) = \sum_{i=0}^d \bar{a}_i z^i.$$

Note that ϕ can be written as a quotient of polynomials $f, g \in \mathbb{Z}_p[z]$ having no common factors, such that at least one coefficient of f or g has absolute value 1. Let $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$ be the finite field of p elements. The *degree* of a rational map ϕ , denoted by $\deg \phi$, is the maximum of the degrees of its denominator and numerator without common factors. The *reduction of ϕ modulo p* is the rational function (of degree at most $\deg \phi$)

$$\bar{\phi}(z) = \frac{\bar{f}(z)}{\bar{g}(z)} \in \mathbb{F}_p(z),$$

obtained by canceling common factors in the reductions $\bar{f}(z), \bar{g}(z)$. If $\deg \bar{\phi} = \deg \phi$, we say ϕ has *good reduction*. If $\deg \bar{\phi} < \deg \phi$, we say ϕ has *bad reduction*.

Assume that ϕ has good reduction. Let $\rho(\cdot, \cdot)$ be the *spherical metric* on $\mathbb{P}^1(\mathbb{Q}_p)$ (see the definition in Section 2). Then the map ϕ is 1-Lipschitz continuous (everywhere non-expanding [27, p.59]):

$$\rho(\phi(P_1), \phi(P_2)) \leq \rho(P_1, P_2) \quad \text{for all } P_1, P_2 \in \mathbb{P}^1(\mathbb{Q}_p).$$

The 1-Lipschitz continuity of ϕ leads us to investigate the minimality and minimal decomposition of the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$, as in the case of polynomial dynamical systems on \mathbb{Z}_p studied in [14].

Let $\mathbb{P}^1(\mathbb{F}_p)$ be the projective line over \mathbb{F}_p . The reduction $\bar{\phi}$ induces a transformation from $\mathbb{P}^1(\mathbb{F}_p)$ into itself. We denote by $\phi^{\circ k}$ the k -th iteration of ϕ . A criterion of the minimality of the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is given in the following theorem.

Theorem 1.1. *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of $\deg \phi \geq 2$ with good reduction. Then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if the following conditions are satisfied:*

- (1) *The reduction $\bar{\phi}$ is transitive on $\mathbb{P}^1(\mathbb{F}_p)$.*
- (2) *$(\phi^{\circ(p+1)})'(0) \equiv 1 \pmod{p}$ and $|\phi^{\circ(p+1)}(0)|_p = 1/p$.*
- (3) *For the case $p = 2$ or 3 , additionally, $|\phi^{(p+1)p}(0)|_p = 1/p^2$.*

Under the spherical metric $\rho(\cdot, \cdot)$, $\mathbb{P}^1(\mathbb{Q}_p)$ can be considered as an infinite symmetric tree (see Figures 1 and 2). This tree is in fact an infinite $(p+1)$ -Cayley tree. The centered vertex which is the root of the tree, is called *Gauss point*. Other vertices correspond to balls with radius strictly less than one (with respect to the spherical metric). The points in $\mathbb{P}^1(\mathbb{Q}_p)$ are the boundary points of the tree. A vertex is said to be *at level n* ($n \geq 1$), if there are n edges between the vertex and the Gauss point. Then for $n \geq 1$, there are $(p+1)p^{n-1}$ vertices at level n . Since a rational map with good reduction is 1-Lipschitz continuous with respect to the spherical metric, it will induce an action on the tree under which the Gauss point is fixed and the sets of vertices at the same level are invariant.

Remark that if we consider ϕ as action on the tree, the condition (1) in Theorem 1.1 means that ϕ is transitive on the set of vertices at level 1, the conditions (1) and (2) imply that ϕ is transitive on the set of vertices at level 2, while the condition (3) implies that ϕ is transitive on the set of vertices at level 3 if conditions (1) and (2) are also satisfied. We

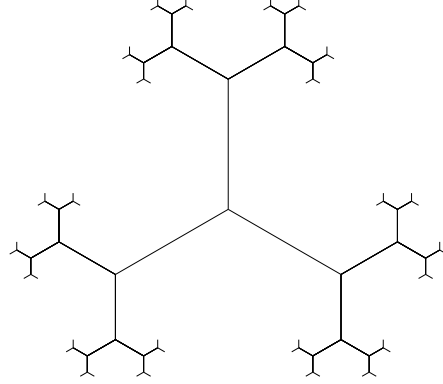


FIGURE 1. Tree structure of $\mathbb{P}^1(\mathbb{Q}_2)$. The points of $\mathbb{P}^1(\mathbb{Q}_2)$ are considered as the boundary points of the infinite tree.

will see in Corollary 4.1 that the transitivity of ϕ on these finite levels is sufficient for the minimality of ϕ on the whole space. The conclusion of Theorem 1.1 can be interpreted as follows.

The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only ϕ is transitive on the set of vertices at level 2 for $p \geq 5$, and at level 3 for $p = 2$ or 3.

When the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is not minimal, we describe the dynamical structure of the system by showing all its minimal subsystems.

Theorem 1.2. *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of $\deg \phi \geq 2$ with good reduction. We have the following decomposition*

$$\mathbb{P}^1(\mathbb{Q}_p) = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B}$$

where \mathcal{P} is the finite set consisting of all periodic points of ϕ , $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each \mathcal{M}_i is a finite union of balls and each subsystem $\phi : \mathcal{M}_i \rightarrow \mathcal{M}_i$ is minimal, and points in \mathcal{B} lie in the attracting basin of a periodic orbit or of a minimal subsystem. Moreover, the length of a periodic orbit has one of the following forms:

$$k \text{ or } k\ell, \quad \text{if } p \geq 5,$$

$$k \text{ or } k\ell \text{ or } kp, \quad \text{if } p = 2 \text{ or } 3,$$

where $1 \leq k \leq p + 1$ and $\ell \mid (p - 1)$.

The decomposition in Theorem 1.2 will be referred to as the *minimal decomposition* of the system $\phi : \mathbb{P}^1(\mathbb{Q}_p) \rightarrow \mathbb{P}^1(\mathbb{Q}_p)$.

We remark that the possible lengths of periodic orbits were investigated in [29], see also [27, pp.62-64].

A finite periodic orbit of ϕ is by definition a minimal set. But for the convenience of the presentation of the paper, only the sets \mathcal{M}_i in the above decomposition are called *minimal components*. What kind of set can be a minimal component of a good reduction rational map? In a recent work [8], the authors showed that for any 1-Lipschitz continuous map, a minimal component must be a Legendre set and that any Legendre set is a minimal component of some 1-Lipschitz continuous map. Recall that $E \subset \mathbb{P}^1(\mathbb{Q}_p)$ is a *Legendre*

set (see [8, p. 778]) if for every integer $n \geq 0$, every non-empty intersection of E with a ball of radius p^{-n} contains the same number of balls of radius $p^{-(n+1)}$.

We will show that for a rational map with good reduction, the minimal components \mathcal{M}_i are Legendre sets of special forms. Further, the dynamics of the minimal subsystems are adding machines. Let $(p_s)_{s \geq 1}$ be a sequence of positive integers such that $p_s | p_{s+1}$ for every $s \geq 1$. We denote by $\mathbb{Z}_{(p_s)}$ the inverse limit of $\mathbb{Z}/(p_s \mathbb{Z})$, called an *odometer*. The sequence $(p_s)_{s \geq 1}$ is called the *structure sequence* of the odometer. The map $z \rightarrow z + 1$ on $\mathbb{Z}_{(p_s)}$ is the *adding machine* on $\mathbb{Z}_{(p_s)}$. The structures of minimal components are determined as follows.

Theorem 1.3. *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of $\deg \phi \geq 2$ with good reduction. Assume that $E \subset \mathbb{P}^1(\mathbb{Q}_p)$ is a minimal component of ϕ . Then $\phi : E \rightarrow E$ is conjugate to the adding machine on an odometer $\mathbb{Z}_{(p_s)}$, where*

$$(p_s) = (k, k\ell, k\ell p, k\ell p^2, \dots)$$

with integers k and ℓ satisfying that $1 \leq k \leq p + 1$ and $\ell \mid (p - 1)$.

In applications, one usually would like to construct or determine a minimal rational (polynomial) map by its coefficients. However, this is far from easy, although a criterion of the minimality of rational maps with good reduction is given in Theorem 1.1. For rational maps of degree one, a complete description is given in [16]. Here we give the description of rational maps of degree at least two with good reduction, but only for $p = 2$. It seems to be a much more harder task for $p \geq 3$.

For a rational map $\phi \in \mathbb{Q}_p(z)$ of degree at least 2, it is easy to check that there exists a $z_0 \in \mathbb{Q}_p$ such that $z_0, \phi(z_0), \phi^{\circ 2}(z_0)$ are distinct. Consider the linear fraction

$$g(z) = \frac{(z - z_0)(\phi^{\circ 2}(z_0) - \phi(z_0))}{(z - \phi(z_0))(\phi^{\circ 2}(z_0) - z_0)}.$$

Then $g(z_0) = 0, g(\phi(z_0)) = \infty$ and $g(\phi^{\circ 2}(z_0)) = 1$. Let $\psi = g \circ \phi \circ g^{-1}$ be the conjugation of ϕ by g . Then we have $\psi(0) = \infty$ and $\psi(\infty) = 1$ and the rational function ψ can be written as

$$\psi(z) = \frac{a_0 + a_1 z + \dots + a_{d-1} z^{d-1} + z^d}{b_1 z + \dots + b_{d-1} z^{d-1} + z^d}$$

where and $a_i, b_j \in \mathbb{Q}_p$ and $d \geq 2$ is the degree of ϕ . Therefore, without loss of generality, we can always assume that $\phi(0) = \infty$ and $\phi(\infty) = 1$.

The following theorem provides, in some sense, a principle for constructing minimal rational maps on $\mathbb{P}^1(\mathbb{Q}_2)$. For a degree $d \geq 2$ rational map of form

$$\phi(z) = \frac{a_0 + a_1 z + \dots + a_{d-1} z^{d-1} + a_d z^d}{b_1 z + \dots + b_{d-1} z^{d-1} + b_d z^d} \quad (1.1)$$

with $a_i, b_j \in \mathbb{Q}_2$ and $a_d = b_d = 1$, we set $A_\phi := \sum_{i \geq 0} a_i, B_\phi := \sum_{j \geq 1} b_j, A_{\phi,1} := \sum_{i \geq 0} a_{2i+1}, A_{\phi,2} := \sum_{i \geq 0} a_{4i+1}$ and $A_{\phi,3} := \sum_{i \geq 0} a_{4i+3}$.

Theorem 1.4. *Let ϕ be defined as (1.1). Then ϕ has good reduction and is minimal on $\mathbb{P}^1(\mathbb{Q}_2)$ if and only if*

$$\left\{ \begin{array}{l} a_i, b_j \in \mathbb{Z}_2, \text{ for } 0 \leq i \leq d-1 \text{ and } 1 \leq j \leq d-1, \\ a_0 \equiv 1 \pmod{2}, \\ B_\phi \equiv 1 \pmod{2}, \\ A_\phi \equiv 2 \pmod{4}, \\ A_{\phi,1} \equiv 1 \pmod{2}, \\ b_1 \equiv 1 \pmod{2}, \\ a_{d-1} - b_{d-1} \equiv 1 \pmod{2}, \\ a_0 b_1 (a_{d-1} - b_{d-1}) (A_{\phi,2} - A_{\phi,3}) B_\phi + \\ 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1} + A_{\phi,3}) \equiv 1 \pmod{4}. \end{array} \right. \quad (1.2)$$

A complete characterization of minimal polynomial maps on \mathbb{Z}_p in terms of their coefficients are given in [22] for $p = 2$ and in [12] for $p = 3$. However, there is still no complete description for $p \geq 5$. On the other hand, the characterizations of minimal Mahler series and van der Put series by their coefficients are investigated in [1, 3] and in [4, 21] respectively.

The above condition (1.2) could be quickly checked by computer, since there are only arithmetic operations ‘+’, ‘-’, ‘ \times ’ in the equations of (1.2). Thus, using computer, we obtain that a rational map of degree 2 or 3 with good reduction is never minimal on $\mathbb{P}^1(\mathbb{Q}_2)$.

Corollary 1.5. *Let $\phi \in \mathbb{Q}_2(z)$ be a rational map of degree 2 or 3 with good reduction. Then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is not minimal.*

We also obtain all the minimal rational maps in $\mathbb{Q}_2(z)$ of degree 4 with good reduction. See the table in Section 5.2. Furthermore, as action on the infinite 3-Cayley tree, the orbits of such rational maps of the 12 vertices at the third level of the tree are determined.

The paper is organized as follows. In Section 2, we give some preliminaries of $\mathbb{P}^1(\mathbb{Q}_p)$, including the spherical metric and the tree structure of $\mathbb{P}^1(\mathbb{Q}_p)$. Section 3 is devoted to the induced dynamical systems on the vertices. The proofs of Theorems 1.1–1.3 are given in Section 4. In Section 5, we characterize the minimal rational maps with good reduction in terms of their coefficients. To illustrate our result, in the last section, we present two rational maps with good reduction whose exact minimal decompositions are described when $p = 3$.

2. PROJECTIVE LINE

Any point in the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ may be given in homogeneous coordinates by a pair $[x_1 : x_2]$ of points in \mathbb{Q}_p which are not both zero. Two such pairs are equal if they differ by an overall (nonzero) factor $\lambda \in \mathbb{Q}_p^*$:

$$[x_1 : x_2] = [\lambda x_1 : \lambda x_2].$$

The field \mathbb{Q}_p may be identified with the subset of $\mathbb{P}^1(\mathbb{Q}_p)$ given by

$$\{[x : 1] \in \mathbb{P}^1(\mathbb{Q}_p) \mid x \in \mathbb{Q}_p\}.$$

This subset contains all points in $\mathbb{P}^1(\mathbb{Q}_p)$ except one: the point of infinity, which may be given as $\infty = [1 : 0]$.

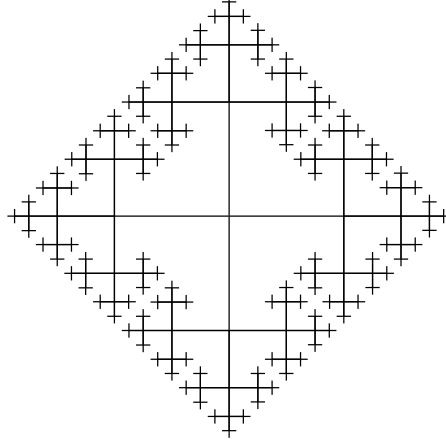


FIGURE 2. Tree structure of $\mathbb{P}^1(\mathbb{Q}_3)$. The points of $\mathbb{P}^1(\mathbb{Q}_3)$ are considered as the boundary points of the infinite tree.

The spherical metric defined on $\mathbb{P}^1(\mathbb{Q}_p)$ is analogous to the standard spherical metric on the Riemann sphere. If $P = [x_1, y_1]$ and $Q = [x_2, y_2]$ are two points in $\mathbb{P}^1(\mathbb{Q}_p)$, we define

$$\rho(P, Q) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}$$

or, viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $\mathbb{Q}_p \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{Q}_p \cup \{\infty\}$ we define

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}} \quad \text{if } z_1, z_2 \in \mathbb{Q}_p,$$

and

$$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \leq 1, \\ 1/|z|_p, & \text{if } |z|_p > 1. \end{cases}$$

Remark that the restriction of the spherical metric on the ring $\mathbb{Z}_p := \{x \in \mathbb{Q}_p, |x| \leq 1\}$ of p -adic integers is same as the metric induced by the absolute value $|\cdot|_p$.

A rational map $\phi \in \mathbb{Q}_p(z)$ induces a transformation on $\mathbb{P}^1(\mathbb{Q}_p)$. Rational maps are always Lipschitz continuous on $\mathbb{P}^1(\mathbb{Q}_p)$ with respect to the spherical metric (see [27, Theorem 2.14.]). In particular, rational maps with good reduction are 1-Lipschitz continuous.

Lemma 2.1 ([27], p.59). *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map with good reduction. Then the map $\phi : \mathbb{P}^1(\mathbb{Q}_p) \mapsto \mathbb{P}^1(\mathbb{Q}_p)$ is 1-Lipschitz continuous:*

$$\rho(\phi(P), \phi(Q)) \leq \rho(P, Q), \quad \forall P, Q \in \mathbb{P}^1(\mathbb{Q}_p).$$

For $a \in \mathbb{P}^1(\mathbb{Q}_p)$ and an integer $n \geq 1$, denote by

$$B_n(a) := \{x \in \mathbb{P}^1(\mathbb{Q}_p) : \rho(x, a) \leq p^{-n}\}$$

the ball of radius p^{-n} centered at a . The projective line $\mathbb{P}^1(\mathbb{Q}_p)$ consists of $p + 1$ disjoint balls of radius p^{-1} ,

$$\mathbb{P}^1(\mathbb{Q}_p) = B_1(\infty) \bigsqcup \left(\bigsqcup_{0 \leq i \leq p-1} B_1(i) \right).$$

For each integer $n \geq 1$, every ball of radius p^{-n} consists of p disjoint balls of radius p^{-n-1} .

As mentioned in Section 1, the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over \mathbb{Q}_p could be considered as an infinite $(p+1)$ -Cayley tree: the branch index of each vertex is $p+1$, i.e. each vertex is an endpoint of $p+1$ edges. There is a centered vertex which is called Gauss point. Excluding Gauss point, each vertex corresponds to a ball in $\mathbb{P}^1(\mathbb{Q}_p)$ of radius strictly less than one (with respect to the spherical metric). The Gauss point can be considered as the ball of radius 1 which is actually the whole space $\mathbb{P}^1(\mathbb{Q}_p)$. The set of edges of the tree consists of the pairs of balls (B, B') of radius ≤ 1 such that

$$B' \subset B, r(B) = p \cdot r(B')$$

where $r(B)$ and $r(B')$ are the radii of balls B and B' respectively.

The points in $\mathbb{P}^1(\mathbb{Q}_p)$ are the boundary points of the tree. Remind that a vertex is said to be at level n ($n \geq 1$) if there are n edges between Gauss point and the vertex. A vertex at level n corresponds a ball of radius p^{-n} . The projective line $\mathbb{P}^1(\mathbb{Q}_p)$ consists of $(p+1)p^{n-1}$ disjoint balls of radius p^{-n} . There are $(p+1)p^{n-1}$ vertices at level n . For example, see Figures 1 and 2 for the tree structures of $\mathbb{P}^1(\mathbb{Q}_2)$ and $\mathbb{P}^1(\mathbb{Q}_3)$.

3. INDUCED DYNAMICS

For each positive integer n , $\mathbb{P}^1(\mathbb{Q}_p)$ consists of $(p+1)p^{n-1}$ disjoint balls of radius p^{-n} . Denote by \mathfrak{B}_n the set of the $(p+1)p^{n-1}$ disjoint balls of radius p^{-n} . In general, any 1-Lipschitz continuous map $\phi : \mathbb{P}^1(\mathbb{Q}_p) \rightarrow \mathbb{P}^1(\mathbb{Q}_p)$ induces a transformation on \mathfrak{B}_n . Denote by ϕ_n the induced map of ϕ on \mathfrak{B}_n , i.e.

$$\phi_n(B_n(x)) = B_n(\phi(x)), \quad \forall x \in \mathbb{P}^1(\mathbb{Q}_p).$$

As $\mathbb{P}^1(\mathbb{Q}_p)$ is considered as an infinite tree, for each positive integer n , there is an one to one correspondence between \mathfrak{B}_n and the vertices of the tree at level n . The 1-Lipschitz continuous map ϕ induces a transformation on the tree under which the set of vertices of each level is invariant.

Many properties of the dynamics ϕ on $\mathbb{P}^1(\mathbb{Q}_p)$ are linked to those of ϕ_n on \mathfrak{B}_n .

Proposition 3.1. *Let $\phi : \mathbb{P}^1(\mathbb{Q}_p) \rightarrow \mathbb{P}^1(\mathbb{Q}_p)$ be a 1-Lipschitz continuous map. Then the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if the finite system (\mathfrak{B}_n, ϕ_n) is minimal for all integers $n \geq 1$.*

The equivalence in Proposition 3.1 is similar to that for 1-Lipschitz continuous maps on the ring \mathbb{Z}_p or on a discrete valuation domain (see [1, p. 111], [2, Theorem 1.2] and [8, Corollary 4]).

Proof. The “only if” part of the statement is obvious, we prove only the “if” part. Suppose that $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is not minimal. Then there exist an open set S and a point $z \in \mathbb{P}^1(\mathbb{Q}_p)$ such that $\phi^{\circ k}(z) \notin S$ for all integers $k \geq 1$. Since the set S is open, there exists a ball $B_{n_0}(z_0) \subset S$ for some $z_0 \in \mathbb{P}^1(\mathbb{Q}_p)$ and some integer n_0 large enough. Recall that (\mathfrak{B}_n, ϕ_n) is minimal for all integers $n > 0$. So there exists an integer k_0 such that $\phi_{n_0}^{k_0}(B_{n_0}(z_0)) = B_{n_0}(z_0)$. This implies $\rho(\phi^{k_0}(z), z_0) \leq p^{-n_0}$, which contradicts the fact that $\phi^{k_0}(z) \notin S$. \square

Recall that a rational map with good reduction is 1-Lipschitz continuous. For a rational map $\phi \in \mathbb{Q}_p(z)$ with good reduction, to study the minimality of the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$, we shall study the induced dynamics (\mathfrak{B}_n, ϕ_n) for all integers $n \geq 1$.

In particular, each point in $\mathbb{P}^1(\overline{\mathbb{F}_p})$ corresponds to a ball in \mathfrak{B}_1 . The induced map ϕ_1 of ϕ can be considered as the reduction $\overline{\phi}$ acting on $\mathbb{P}^1(\overline{\mathbb{F}_p})$. The map $\overline{\phi} : \mathbb{P}^1(\overline{\mathbb{F}_p}) \rightarrow \mathbb{P}^1(\overline{\mathbb{F}_p})$ admits some periodic orbits. A periodic orbit is called a *cycle* of $\overline{\phi}$. The points outside any cycle will be mapped into some cycle after several iterations.

Let $(\overline{x}_1, \dots, \overline{x}_k) \subset \mathbb{P}^1(\overline{\mathbb{F}_p})$ be a cycle of $\overline{\phi}$. To simplify notation, we find a transformation $f \in \text{PGL}_2(\mathbb{Z}_p)$ with $f(x_1) = 0$, where x_1 is a point in the ball corresponding to \overline{x}_1 . Replacing x_1 and ϕ with 0 and $\phi^f = f \circ \phi \circ f^{-1}$ respectively, we may assume that $\overline{x}_1 = 0$.

For two rational maps $\phi, \psi \in \mathbb{Q}_p(z)$ with good reduction, the composition $\phi \circ \psi$ has good reduction, and $\overline{\phi \circ \psi} = \overline{\phi} \circ \overline{\psi}$ (see [27, p.59, Theorem 2.18]). So $\phi^{\circ k}$ has good reduction and 0 is a fixed point of $\overline{\phi^{\circ k}}$. We write $\phi^{\circ k}$ in the form

$$\phi^{\circ k}(x) = \frac{a_0 + a_1 z + \dots + a_d z^d}{b_0 + b_1 z + \dots + b_d z^d}$$

with coefficients $a_0, \dots, a_d, b_0, \dots, b_d \in \mathbb{Z}_p$ and at least one coefficient in $\mathbb{Z}_p \setminus p\mathbb{Z}_p$. The fact that 0 is a fixed point of $\overline{\phi^{\circ k}}$ implies that $\phi^{\circ k}(0) = a_0/b_0 \equiv 0 \pmod{p}$. Since $\phi^{\circ k}$ has good reduction, we have $a_0 \in p\mathbb{Z}_p$ and $b_0 \in \mathbb{Z}_p$. Otherwise, the numerator and denominator of the reduction have the common factor $(z - 0)$, which implies $\deg \overline{\phi^{\circ k}} < \deg \phi^{\circ k}$. Multiplying numerator and denominator by b_0^{-1} , we may thus write $\phi^{\circ k}$ in the form

$$\phi^{\circ k}(x) = \frac{a_0 + a_1 z + \dots + a_d z^d}{1 + b_1 z + \dots + b_d z^d}.$$

The Taylor expansion of $\phi^{\circ k}$ around $z = 0$, which in this case can be obtained by simple long division, is

$$\phi^{\circ k}(z) = a_0 + \lambda z + \frac{A(z)}{1 + zB(z)} z^2$$

with $A(z), B(z) \in \mathbb{Z}_p[[z]]$ and $\lambda = (\phi^{\circ k})'(0) \in \mathbb{Z}_p$. Observe that

$$\frac{1}{1 + zB(z)} = 1 - zB(z) + z^2(B(z))^2 - z^3(B(z))^3 + \dots$$

on $p\mathbb{Z}_p$. So we can write $\phi^{\circ k}(z)$ as a power series with coefficients in \mathbb{Z}_p which converges on $p\mathbb{Z}_p$, i.e.

$$\phi^{\circ k}(z) = a_0 + \lambda z + \lambda_2 z^2 + \lambda_3 z^3 + \dots \quad (3.1)$$

with $\lambda_i \in \mathbb{Z}_p$ for all $i \geq 2$. Obviously, $\phi^{\circ k}$ induces a 1-Lipschitz map from $p\mathbb{Z}_p$ to itself.

Now we study the dynamical system $(p\mathbb{Z}_p, \phi^{\circ k})$. For simplicity, we write ϕ instead of $\phi^{\circ k}$. Note that every power series

$$\phi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$$

converges on $p\mathbb{Z}_p$. If $\lambda_0 \in p\mathbb{Z}_p$, then $\phi(p\mathbb{Z}_p) \subset p\mathbb{Z}_p$ and ϕ induces a 1-Lipschitz map from $p\mathbb{Z}_p$ to itself.

In the remainder of this section we assume $\phi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ with $\lambda_0 \in p\mathbb{Z}_p$. The dynamical system $(p\mathbb{Z}_p, \phi)$ is conjugate to a system (\mathbb{Z}_p, ϕ') by the transformation

$f(z) = z/p$, i.e.

$$\begin{array}{ccc} p\mathbb{Z}_p & \xrightarrow{\phi} & p\mathbb{Z}_p \\ z/p \downarrow & & \downarrow z/p \\ \mathbb{Z}_p & \xrightarrow{\varphi} & \mathbb{Z}_p, \end{array}$$

where $\varphi(z) = \sum_{i=0}^{\infty} p^{i-1} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ converges on \mathbb{Z}_p and induces a map from \mathbb{Z}_p to itself.

The dynamical system of φ on \mathbb{Z}_p and its induced dynamics on $\mathbb{Z}_p/p^n\mathbb{Z}_p$ was studied in [17]. Actually, the convergent series with coefficient in the integral ring \mathcal{O}_K of a finite extension K of \mathbb{Q}_p were studied as dynamical systems on \mathcal{O}_K in [17]. The system (\mathbb{Z}_p, φ) is a special case when $K = \mathbb{Q}_p$.

For the convenience of the reader we repeat the relevant materials from [14, 17] without proofs.

For each $n \geq 1$, denote by ϕ_n the induced map of ϕ on $p\mathbb{Z}_p/p^n\mathbb{Z}_p$, i.e.,

$$\phi(x \bmod p^n) \equiv \phi(x) \bmod p^n,$$

for all $x \in p\mathbb{Z}_p$. In the spirit of Proposition 3.1, the minimality and minimal decomposition of the system $(p\mathbb{Z}_p, \phi)$ can be deduced from its induced dynamics $(p\mathbb{Z}_p/p^n\mathbb{Z}_p, \phi_n)$. So we need to study the finite systems $(p\mathbb{Z}_p/p^n\mathbb{Z}_p, \phi_n)$. To this end, the main idea, which comes from [11, 29], is to study the behaviour of systems $(p\mathbb{Z}_p/p^n\mathbb{Z}_p, \phi_n)$ by induction. We refer the reader to [14, 17] for the application of this idea to give a minimal decomposition theorem for any convergent power series in $\mathbb{Z}_p[[z]]$.

A collection $\sigma = (x_1, \dots, x_k)$ of k distinct points in $p\mathbb{Z}_p/p^n\mathbb{Z}_p$ is called a *cycle* of ϕ_n of length k or a *k -cycle* at level n , if

$$\phi_n(x_1) = x_2, \dots, \phi_n(x_i) = x_{i+1}, \dots, \phi_n(x_k) = x_1.$$

Set

$$X_\sigma := \bigsqcup_{i=1}^k X_i \text{ where } X_i := \{x_i + p^n t; t = 0, 1, \dots, p-1\} \subset p\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p.$$

Then

$$\phi_{n+1}(X_i) \subset X_{i+1} \text{ (} 1 \leq i \leq k-1 \text{) and } \phi_{n+1}(X_k) \subset X_1.$$

In the following we are concerned with the behavior of the finite dynamics ϕ_{n+1} on the set X_σ and determine all cycles in X_σ of ϕ_{n+1} , which will be called *lifts* of σ . Remark that the length of any lift $\tilde{\sigma}$ of σ is a multiple of k .

Let $\mathbb{X}_i = x_i + p^n\mathbb{Z}_p := \{x \in p\mathbb{Z}_p : x \equiv x_i \pmod{p^n}\}$ be the closed disk of radius p^{-n} corresponding to $x_i \in \sigma$ and

$$\mathbb{X}_\sigma := \bigsqcup_{i=1}^k \mathbb{X}_i$$

be the clopen set corresponding to the cycle σ .

Let $\psi := \phi^{\circ k}$ be the k -th iteration of ϕ . Then, any point in σ is fixed by ψ , the n -th induced map of ψ . For any point $x \in \mathbb{X}_\sigma$, we have $(\psi(x) - x)/p^n \in \mathbb{Z}_p$. Let

$$\alpha_n(x) := \psi'(x) = \prod_{j=0}^{k-1} \phi'(\phi^{\circ j}(x)) \quad (3.2)$$

$$\beta_n(x) := \frac{\psi(x) - x}{p^n} = \frac{\phi^{\circ k}(x) - x}{p^n}. \quad (3.3)$$

One can check that $\alpha_n(x) \pmod{p}$ is always constant on \mathbb{X}_σ . We distinguish the following four behaviours of ϕ_{n+1} on X_σ :

- (a) If $\alpha_n \equiv 1 \pmod{p}$ and $\beta_n \not\equiv 0 \pmod{p}$, then ϕ_{n+1} restricted to X_σ preserves a single cycle of length pk . In this case we say σ *grows*.
- (b) If $\alpha_n \equiv 1 \pmod{p}$ and $\beta_n \equiv 0 \pmod{p}$, then ϕ_{n+1} restricted to X_σ preserves p cycles of length k . In this case we say σ *splits*.
- (c) If $\alpha_n \equiv 0 \pmod{p}$, then ϕ_{n+1} restricted to X_σ preserves one cycle of length k and the remaining points of X_σ are mapped into this cycle. In this case we say σ *grows tails*.
- (d) If $\alpha_n \not\equiv 0, 1 \pmod{p}$, then ϕ_{n+1} restricted to X_σ preserves one cycle of length k and $\frac{p-1}{\ell}$ cycles of length $k\ell$, where ℓ is the order of α_n in the multiplicative group $\mathbb{F}_p \setminus \{0\}$. In this case we say σ *partially splits*.

For $n \geq 1$, let $\sigma = (x_1, \dots, x_k) \subset p\mathbb{Z}_p/p^n\mathbb{Z}_p$ be a k -cycle and let $\tilde{\sigma}$ be a lift of σ . To find the change of nature from a cycle to its lifts, we shall show the relation between (α_n, β_n) and $(\alpha_{n+1}, \beta_{n+1})$. For the calculation of $(\alpha_{n+1} \pmod{p}, \beta_{n+1} \pmod{p})$ from $(\alpha_n \pmod{p}, \beta_n \pmod{p})$, the interested reader are referred to [14, Lemma 2] and [17, p.1636]. The following propositions predict the behaviours of the lifts of a cycle σ by the properties of σ .

Proposition 3.2 ([14] Proposition 1, and [17] Proposition 4.4). *Let σ be a cycle of ϕ_n .*

- (1) *If σ grows or splits, then any lift $\tilde{\sigma}$ grows or splits.*
- (2) *If σ grows tails, then the single lift $\tilde{\sigma}$ also grows tails.*
- (3) *If σ partially splits, then the lift $\tilde{\sigma}$ of the same length as σ partially splits, and the other lifts of length $k\ell$ grow or split.*

Proposition 3.3 ([14, 17]). *Let σ be a growing cycle of ϕ_n and $\tilde{\sigma}$ be the unique lift of σ .*

- (1) *If $p > 3$ and $n \geq 1$, then $\tilde{\sigma}$ also grows.*
- (2) *If $p = 3$ and $n \geq 2$, then $\tilde{\sigma}$ also grows.*
- (3) *If $p = 2$ and $\tilde{\sigma}$ grows, then the lift of $\tilde{\sigma}$ grows.*

For more details of Proposition 3.3, we refer the reader to [14, Proposition 2] for the first two assertion, and to [14, Corollary 1] for the third assertion. Remark that the results in Proposition 3.3 are presented only for polynomials in $\mathbb{Z}_p[z]$. Actually, these results also hold for convergent series in $\mathbb{Z}_p[[z]]$ (see [17, Propositions 4.7, 4.8, 4.10, 4.11] for a more general setting).

Let $E \subset p\mathbb{Z}_p$ be a ϕ -invariant compact open set. Let

$$E/p^n\mathbb{Z}_p := \{x \in \mathbb{Z}_p/p^n\mathbb{Z}_p : \exists y \in E \text{ such that } x \equiv y \pmod{p^n}\}.$$

It is now well known that the subsystem (E, ϕ) is minimal if and only if the induced map $\phi_n : E/p^n\mathbb{Z}_p \rightarrow E/p^n\mathbb{Z}_p$ is minimal for any $n \geq 1$ (see [1, p. 111], [2, Theorem 1.2] and [8, Corollary 4]). So by Proposition 3.3, a criterion of the minimality of the system ϕ on $p\mathbb{Z}_p$ can be obtained.

Corollary 3.4. *The dynamical system $(p\mathbb{Z}_p, \phi)$ is minimal if and only if*

- (1) *the finite system $(p\mathbb{Z}_p/p^3\mathbb{Z}_p, \phi_3)$ is minimal for $p = 2$ or 3 ;*
- (2) *the finite system $(p\mathbb{Z}_p/p^2\mathbb{Z}_p, \phi_2)$ is minimal for $p \geq 5$.*

As we investigate the induced dynamical systems level by level, the possible periods of ϕ on $p\mathbb{Z}_p$ can also be obtained, as a consequence of Proposition 3.3.

Corollary 3.5. *Let $\phi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ with $\lambda_0 \in p\mathbb{Z}_p$ and consider the dynamical system $(p\mathbb{Z}_p, \phi)$.*

- (1) *If $p > 3$, the periods of periodic orbit must be factors of $p - 1$.*
- (2) *If $p = 3$, the periods of periodic orbit must be 1, 2 or 3.*
- (3) *If $p = 2$, the periods of periodic orbit must be 1 or 2.*

Furthermore, the dynamical structure of ϕ on \mathbb{Z}_p is fully illustrated by the following minimal decomposition.

Theorem 3.6 ([17], Theorem 1.1). *Let $\phi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ with $\lambda_0 \in p\mathbb{Z}_p$. Suppose $\phi^n \neq id$ for all $n \geq 1$. We have the following decomposition*

$$p\mathbb{Z}_p = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},$$

where \mathcal{P} is the finite set consisting of all periodic points of f , $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each \mathcal{M}_i is a finite union of balls and each subsystem $f : \mathcal{M}_i \rightarrow \mathcal{M}_i$ is minimal, and the points in \mathcal{B} lie in the attracting basin of $\mathcal{P} \sqcup \mathcal{M}$.

The sets \mathcal{M}_i in the above decomposition are called *minimal components*. To completely characterize the dynamical system $(p\mathbb{Z}_p, \phi)$, each minimal component is described as the adding machine on an odometer by giving the structure sequence of the odometer.

Theorem 3.7 ([17], Theorem 1.1). *Let $\phi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ with $\lambda_0 \in p\mathbb{Z}_p$. If $E \subset p\mathbb{Z}_p$ is a minimal clopen invariant set of ϕ , then $\phi : E \rightarrow E$ is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$, where*

$$(p_s) = (\ell, \ell p, \ell p^2, \dots)$$

with integer $\ell \mid (p - 1)$.

4. PROOF OF THEOREMS 1.1–1.3

Proof of Theorem 1.1. We start proving the “if” part of the theorem. The reduction $\bar{\phi}$ is minimal on $\mathbb{P}^1(\mathbb{F}_p)$ which implies that ϕ_1 is minimal on \mathfrak{B}_1 . Let $\psi = \phi^{\circ(p+1)}$ be the $(p + 1)$ -th iteration of ϕ . Then the ball $B_1(0) = p\mathbb{Z}_p$ is an invariant set of ψ . By the argument of Section 3, ψ can be written as a power series with coefficients in \mathbb{Z}_p which converges on $p\mathbb{Z}_p$, i.e.

$$\psi(z) = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \dots$$

with $\lambda_i \in \mathbb{Z}_p$ for all $i \geq 1$ and $\lambda_0 \in p\mathbb{Z}_p$. The minimality of the dynamical system $(p\mathbb{Z}_p, \psi)$ implies the minimality of the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$. It suffices to show that the dynamical system $(p\mathbb{Z}_p, \psi)$ is minimal.

For $n = 1$, the cycle (0) is the unique cycle of ψ_1 . The condition $(\phi^{\circ(p+1)})'(0) = 1 \pmod{p}$ implies that the cycle (0) grows or splits. By the condition $|\phi^{\circ(p+1)}(0)|_p = 1/p$, the cycle (0) grows which then means that the dynamical system $(p\mathbb{Z}_p/p^2\mathbb{Z}_p, \psi_2)$ is minimal.

For the cases $p > 3$, it follows from Corollary 3.4 that the dynamical system $(p\mathbb{Z}_p, \psi)$ is minimal.

For the cases $p = 2$ or 3 , the additional condition implies that the unique lift of (0) at level 2 grows. Thus the system $(p\mathbb{Z}_p/p^3\mathbb{Z}_p, \psi_n)$ is minimal. By Corollary 3.4, the dynamical system $(p\mathbb{Z}_p, \psi)$ is minimal.

Now we prove the ‘‘only if’’ part. Let ϕ be a minimal rational map with good reduction. By Proposition 3.1, the system (\mathfrak{B}_n, ϕ_n) is minimal for all integers $n \geq 1$. The minimality of system (\mathfrak{B}_1, ϕ_1) implies that the reduction $\bar{\phi}$ is minimal on $\mathbb{P}^1(\mathbb{F}_p)$, while the minimality of system (\mathfrak{B}_2, ϕ_2) implies that the cycle (0) of $\psi_1 = \phi_1^p$ grows. Hence we obtain $(\phi^{\circ(p+1)})'(0) = 1 \pmod{p}$ and $|\phi^{\circ(p+1)}(0) - 0|_p = 1/p$. Furthermore, by the minimality of system (\mathfrak{B}_3, ϕ_3) , the unique lifts of (0) of $\phi^{\circ(p+1)}$ at level 2 grows. So we have $|\phi^{\circ p(p+1)}(0)|_p = 1/p^2$. \square

By the arguments in the proof of Theorem 1.1, we have the following consequence.

Corollary 4.1. *Let $\phi : \mathbb{P}^1(\mathbb{Q}_p) \mapsto \mathbb{P}^1(\mathbb{Q}_p)$ be a rational map of $\deg \phi \geq 2$ with good reduction. Then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if the following condition is satisfied:*

- (1) *the system (\mathfrak{B}_3, ϕ_3) is minimal for $p = 2$ or 3 ,*
- (2) *the system (\mathfrak{B}_2, ϕ_2) is minimal for $p \geq 5$.*

Proof of Theorem 1.2. The space $\mathbb{P}^1(\mathbb{Q}_p)$ is a union of $p + 1$ balls with radius p^{-1} . Each ball is identified with a point in $\mathbb{P}^1(\mathbb{F}_p)$. The reduction map $\bar{\phi}$ on $\mathbb{P}^1(\mathbb{F}_p)$ admits some cycles. By iteration of $\bar{\phi}$, the points outside the cycles are attracted by the cycles. The ball corresponding to such a point will be put into the third part \mathcal{B} in the minimal decomposition.

Let $(\bar{x}_1, \dots, \bar{x}_k) \subset \mathbb{P}^1(\mathbb{F}_p)$ be a cycle of $\bar{\phi}$. Without loss of generality, we assume $\bar{x}_1 = 0$. Let $\psi = \phi^{\circ k}$ be the k -th iteration of ϕ . Then the ball $B_1(0) = p\mathbb{Z}_p$ is an invariant set of ψ . By Theorem 3.6, the dynamical system $(p\mathbb{Z}_p, \psi)$ has a minimal decomposition which then gives a minimal decomposition of the system $(\bigsqcup_{i=1}^k B_1(x_i), \phi)$.

Applying the same argument to all the cycles of $\bar{\phi}$, we obtain the minimal decomposition of the whole system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$.

Moreover, by Corollary 3.5, the possible lengths of periodic orbits are obtained. \square

Proof of Theorem 1.3. Let k denote the length of the induced periodic orbit of ϕ on the minimal set E at the first level. Each point at this first level corresponds to a point in $\mathbb{P}^1(\mathbb{F}_p)$. So $k \leq p + 1$.

By Theorem 3.7, we deduce that $\phi : E \rightarrow E$ is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$, where

$$(p_s) = (k, k\ell, k\ell p, k\ell p^2, \dots)$$

with $\ell \mid (p - 1)$. \square

5. MINIMAL RATIONAL MAP FOR THE CASE $p = 2$

5.1. Minimal Conditions. Let

$$\phi(z) = \frac{a_0 + a_1 z + \dots + a_{d-1} z^{d-1} + a_d z^d}{b_1 z + \dots + b_{d-1} z^{d-1} + b_d z^d}$$

with $a_i, b_j \in \mathbb{Q}_2$ and $a_d = b_d = 1$ be a rational map of degree $d \geq 2$. Recall that $A_\phi = \sum_{i \geq 0} a_i$, $B_\phi = \sum_{j \geq 1} b_j$, $A_{\phi,1} = \sum_{i \geq 0} a_{2i+1}$, $A_{\phi,2} = \sum_{i \geq 0} a_{4i+1}$ and $A_{\phi,3} = \sum_{i \geq 0} a_{4i+3}$.

Lemma 5.1. *The rational map ϕ has good reduction and the system (\mathfrak{B}_1, ϕ_1) is minimal if and only if*

$$\begin{cases} a_i, b_j \in \mathbb{Z}_2, \text{ for } 0 \leq i \leq d-1 \text{ and } 1 \leq j \leq d-1, \\ a_0 \equiv 1 \pmod{2}, \\ A_\phi \equiv 0 \pmod{2}, \\ B_\phi \equiv 1 \pmod{2}. \end{cases}$$

Proof. We start proving the ‘‘only if’’ part of the lemma. Since ϕ has good reduction, the coefficients a_i and b_j are in \mathbb{Z}_2 . Otherwise, the degree of the reduction map of ϕ is strictly less than d , which implies that ϕ has bad reduction.

Let

$$\bar{\phi}(z) = \frac{\bar{f}(z)}{\bar{g}(z)} = \frac{\bar{a}_0 + \bar{a}_1 z + \cdots + \bar{a}_{d-1} z^{d-1} + z^d}{\bar{b}_1 z + \cdots + \bar{b}_{d-1} z^{d-1} + z^d}$$

be the reduction of ϕ modulo p . The condition that ϕ has good reduction implies that the polynomials \bar{f}, \bar{g} have no common zero. Since $\bar{g}(0) = 0$, it follows $\bar{f}(0) = \bar{a}_0 \neq 0$. So we have $a_0 \equiv 1 \pmod{2}$. Observe that $\bar{\phi}(0) = \infty$ and $\bar{\phi}(\infty) = 1$. The minimality of system (\mathfrak{B}_1, ϕ_1) means that $\bar{\phi}(1) = 0$. So we have $\bar{f}(1) = 0$ which implies $A_\phi \equiv 0 \pmod{2}$. Since the polynomials \bar{f}, \bar{g} have no common zero, we have $\bar{g}(1) \neq 0$ which means that $B_\phi \equiv 1 \pmod{2}$.

Now we prove the ‘‘if’’ part. The argument in the previous paragraph shows that the polynomials \bar{f} and \bar{g} have no common zero. Thus $\deg \bar{\phi} = d$. Since $\bar{f}(1) = 0$, it follows that $\bar{\phi}$ is minimal on $\mathbb{P}^1(\mathbb{F}_2)$. Hence ϕ has good reduction and (\mathfrak{B}_1, ϕ_1) is minimal. \square

In the following lemma, we will calculate the derivative of $\phi^{\circ 3}$ at 0. Since $\phi(0) = \infty$, for the convenience of calculation, we shall select a suitable coordinate. Let

$$\psi(z) := \frac{1}{\phi(z)} = \frac{b_1 z + \cdots + b_{d-1} z^{d-1} + b_d z^d}{a_0 + a_1 z + \cdots + a_{d-1} z^{d-1} + a_d z^d}$$

and

$$\varphi(z) := \phi\left(\frac{1}{z}\right) = \frac{a_d + a_{d-1} z + \cdots + a_1 z^{d-1} + a_0 z^d}{b_d + b_{d-1} z + \cdots + b_1 z^{d-1}}.$$

Then we have $\phi^{\circ 3} = \phi \circ \varphi \circ \psi$. Using the chain rule, we obtain

$$(\phi^{\circ 3})'(0) = \psi'(0) \cdot \varphi'(0) \cdot \phi'(1). \quad (5.1)$$

For simplicity, we denote $A'_\phi := \sum_{i \geq 1} i a_i$ and $B'_\phi := \sum_{i \geq 1} i b_i$. By calculation, we have

$$\alpha_1 := \psi'(0) = b_1/a_0,$$

$$\alpha_2 := \varphi'(0) = a_{d-1} - b_{d-1},$$

and

$$\alpha := \phi'(1) = \frac{A'_\phi B_\phi - B'_\phi A_\phi}{B_\phi^2}.$$

Lemma 5.2. *The rational map ϕ has good reduction and the system (\mathfrak{B}_2, ϕ_2) is minimal if and only if*

$$\begin{cases} a_i, b_j \in \mathbb{Z}_2, \text{ for } 0 \leq i \leq d-1 \text{ and } 1 \leq j \leq d-1, \\ a_0 \equiv 1 \pmod{2}, \\ B_\phi \equiv 1 \pmod{2}, \\ A_\phi \equiv 2 \pmod{4} \\ A_{\phi,1} \equiv 1 \pmod{2}, \\ b_1 \equiv 1 \pmod{2}, \\ a_{d-1} - b_{d-1} \equiv 1 \pmod{2}. \end{cases}$$

Proof. The rational map ϕ has good reduction and (\mathfrak{B}_2, ϕ_2) is minimal implies that (\mathfrak{B}_1, ϕ_1) is also minimal. So $p\mathbb{Z}_p$ is $\phi^{\circ 3}$ -invariant. By the classification of the lifts of the cycles, (\mathfrak{B}_2, ϕ_2) is minimal if and only if

$$(\mathfrak{B}_1, \phi_1) \text{ is minimal, } (\phi^{\circ 3})'(0) \equiv 1 \pmod{2}, \text{ and } \phi(1) \equiv 2 \pmod{4} \quad (5.2)$$

under the condition that ϕ has good reduction.

By Lemma 5.1, ϕ has good reduction and (\mathfrak{B}_1, ϕ_1) is minimal if and only if $a_0 \equiv 1 \pmod{2}$, $B_\phi \equiv 1 \pmod{2}$ and $A_\phi \equiv 0 \pmod{2}$.

Recall that

$$(\phi^{\circ 3})'(0) = \psi'(0) \cdot \varphi'(0) \cdot \phi'(1) = \frac{b_1}{a_0}(a_{d-1} - b_{d-1}) \frac{A'_\phi B_\phi - B'_\phi A_\phi}{B_\phi^2}.$$

The second condition $(\phi^{\circ 3})'(0) \equiv 1 \pmod{2}$ of (5.2) implies that

$$b_1 \equiv 1 \pmod{2},$$

$$(a_{d-1} - b_{d-1}) \equiv 1 \pmod{2},$$

and

$$A'_\phi \equiv 1 \pmod{2}.$$

Since

$$A'_\phi = \sum_{i \geq 0} (2i+1)a_{2i+1} + \sum_{i \geq 1} 2ia_{2i} \equiv \sum_{i \geq 0} a_{2i+1} \pmod{2},$$

we have $A_{\phi,1} \equiv 1 \pmod{2}$.

By the last condition $\phi(1) \equiv 2 \pmod{4}$ in (5.2), we immediately get

$$A_\phi \equiv 2 \pmod{4}.$$

□

For the proof of Theorem 1.4, we need calculate the second derivative of $\phi^{\circ 3}$ at 0. For simplicity, denote $A''_\phi := \sum_{i \geq 2} i(i-1)a_i$ and $B''_\phi := \sum_{i \geq 2} i(i-1)b_i$. Before the proof let us first calculate the second derivatives $\psi''(0)$, $\varphi''(0)$ and $\phi''(1)$:

$$\beta_1 := \psi''(0) = \frac{2b_2a_0^2 - 2a_1b_1a_0}{a_0^3},$$

$$\beta_2 := \varphi''(0) = 2(a_{d-2} - b_{d-2}) + 2(b_{d-1}^2 - a_{d-1}b_{d-1}),$$

$$\beta := \phi''(1) = \frac{A''_\phi B_\phi^2 - B''_\phi A_\phi B_\phi + 2(A_\phi(B'_\phi)^2 - A'_\phi B'_\phi B_\phi)}{B_\phi^3}.$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Under the condition that ϕ has good reduction, Corollary 4.1 implies that $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if (\mathfrak{B}_3, ϕ_3) is minimal which is equivalent to the following three conditions

$$(\mathfrak{B}_2, \phi_2) \text{ is minimal, } (\phi^{\circ 6})'(0) \equiv 1 \pmod{2}, \text{ and } \phi^{\circ 6}(0) \equiv 4 \pmod{8}. \quad (5.3)$$

In the following, we will characterize the three conditions of (5.3).

First, the minimality of the system (\mathfrak{B}_2, ϕ_2) by coefficients has already been characterized in Lemma 5.2. Then we obtain all the conditions in (1.2) except the last one.

Second, if the first condition of (5.3) is satisfied then by (5.2),

$$(\phi^{\circ 6})'(0) \equiv ((\phi^{\circ 3})'(0))^2 \equiv 1 \pmod{2}.$$

Thus the second condition of (5.3) is in fact included in the first condition of (5.3).

Last, we are going to investigate the relation among the coefficients by using the last condition $\phi^{\circ 6}(0) \equiv 4 \pmod{8}$ of (5.3). By such an investigation, we will obtain the last condition of (1.2).

Applying Taylor's expansion, we have for $z \in 2\mathbb{Z}_2$

$$\phi^{\circ 3}(z) = \phi(1) + (\phi^{\circ 3})'(0)z + \frac{(\phi^{\circ 3})''(0)}{2}z^2 \pmod{8}.$$

Thus

$$\phi^{\circ 6}(0) = \phi(1) + (\phi^{\circ 3})'(0)\phi(1) + \frac{(\phi^{\circ 3})''(0)}{2}\phi(1)^2 \pmod{8}. \quad (5.4)$$

Notice that $\phi(1) \equiv 2 \pmod{4}$ and $(\phi^{\circ 3})'(0) \equiv 1 \pmod{2}$. Then dividing both sides of (5.4) by $\phi(1)$, we deduce that the condition $\phi^{\circ 6}(0) \equiv 4 \pmod{8}$ is equivalent to

$$1 + (\phi^{\circ 3})'(0) + \frac{(\phi^{\circ 3})''(0)}{2}\phi(1) \equiv 2 \pmod{4},$$

which implies $(\phi^{\circ 3})'' \equiv 0 \pmod{2}$. Therefore, the last condition $\phi^{\circ 6}(0) \equiv 4 \pmod{8}$ of (5.3) is equivalent to

$$(\phi^{\circ 3})'(0) + (\phi^{\circ 3})''(0) \equiv 1 \pmod{4}. \quad (5.5)$$

To continue, let us calculate $(\phi^{\circ 3})'(0) \pmod{4}$, $(\phi^{\circ 3})''(0) \pmod{4}$ and simplify their expressions. Using $a_0 \equiv 1 \pmod{2}$ and $B_\phi \equiv 1 \pmod{2}$, we have $1/a_0 \equiv a_0 \pmod{4}$ and $B_\phi^2 \equiv 1 \pmod{4}$. Then

$$(\phi^{\circ 3})'(0) \equiv a_0 b_1 (a_{d-1} - b_{d-1})(A'_\phi B_\phi - A_\phi B'_\phi) \pmod{4}.$$

Since $A_\phi \equiv 2 \pmod{4}$ and $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod{2}$, we have

$$(\phi^{\circ 3})'(0) \equiv a_0 b_1 (b_{d-1} - b_{d-1})A'_\phi B_\phi - A_\phi B'_\phi \pmod{4}.$$

For the second derivative of $\phi^{\circ 3}$ at 0, by the chain rule, we have

$$(\phi^{\circ 3})''(0) = (\psi'(0) \cdot \varphi'(0) \cdot \phi'(1))' = \alpha\alpha_2\beta_1 + \alpha\alpha_1^2\beta_2 + \beta\alpha_1^2\alpha_2^2. \quad (5.6)$$

Since $\alpha \equiv \alpha_1 \equiv \alpha_2 \equiv 1 \pmod{2}$, we have $\alpha^2 \equiv \alpha_1^2 \equiv \alpha_2^2 \equiv 1 \pmod{4}$. Equation (5.6) can be simplified as

$$(\phi^{\circ 3})''(0) \equiv \alpha\alpha_2\beta_1 + \alpha\beta_2 + \beta \pmod{4}. \quad (5.7)$$

Apply the expressions of β, β_1 and β_2 to Equation (5.7). Using the facts that $a_0 \equiv b_1 \equiv a_{d-1} - b_{d-1} \equiv 1 \pmod{2}$ and $B_\phi \equiv 1 \pmod{2}$, we deduce

$$\begin{aligned} (\phi^{\circ 3})''(0) &\equiv 2\alpha\alpha_2(a_0b_2 - a_1b_1) + 2\alpha(a_{d-2} - b_{d-2} + b_{d-1}^2 - a_{d-1}b_{d-1}) + \\ &\quad + A''_\phi B_\phi - 2A'_\phi B'_\phi - A_\phi B''_\phi \pmod{4} \\ &\equiv 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1}) + A''_\phi B_\phi - 2A'_\phi B'_\phi - A_\phi B''_\phi \pmod{4}. \end{aligned}$$

Observing that $B''_\phi \equiv 0 \pmod{2}$ and $A_\phi \equiv 0 \pmod{2}$, we have

$$\begin{aligned} &(\phi^{\circ 3})'(0) + (\phi^{\circ 3})''(0) \\ &\equiv a_0b_1(a_{d-1} - b_{d-1})A'_\phi B_\phi - A_\phi B'_\phi + 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1}) \\ &\quad + A''_\phi B_\phi - 2A'_\phi B'_\phi \pmod{4}. \end{aligned}$$

Since $A_\phi \equiv 2 \pmod{4}$ and $A'_\phi \equiv A'_{\phi,1} \equiv 1 \pmod{2}$, we get

$$\begin{aligned} &(\phi^{\circ 3})'(0) + (\phi^{\circ 3})''(0) \\ &\equiv a_0b_1(a_{d-1} - b_{d-1})A'_\phi B_\phi + 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1}) + A''_\phi B_\phi \pmod{4}. \end{aligned}$$

Note that

$$A'_\phi \equiv \sum_{i \geq 0} a_{4i+1} + 2 \sum_{i \geq 0} a_{4i+2} - \sum_{i \geq 0} a_{4i+3} \pmod{4}$$

and

$$A''_\phi \equiv 2 \sum_{i \geq 0} a_{4i+2} + 2 \sum_{i \geq 0} a_{4i+3} \pmod{4}.$$

Then

$$\begin{aligned} &(\phi^{\circ 3})'(0) + (\phi^{\circ 3})''(0) \\ &\equiv a_0b_1(a_{d-1} - b_{d-1})(A_{\phi,2} - A_{\phi,3})B_\phi + 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1}) \\ &\quad + 2 \sum_{i \geq 0} a_{4i+2} + 2 \left(\sum_{i \geq 0} a_{4i+2} + A_{\phi,3} \right) B_\phi \pmod{4} \\ &\equiv a_0b_1(a_{d-1} - b_{d-1})(A_{\phi,2} - A_{\phi,3})B_\phi \\ &\quad + 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1} + A_{\phi,3}) \pmod{4}. \end{aligned}$$

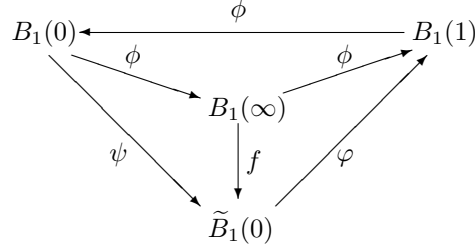
By (5.5), we obtain the last condition of (1.2). This achieves the proof. \square

5.2. Minimal rational maps of degree 4. Any rational map of degree 2 with good reduction is not minimal on $\mathbb{P}(\mathbb{Q}_2)$, since the point ∞ either has 0 or 2 pre-images. By computation, the system of equations (1.2) has no solution for $d = 3$. By Theorem 1.4, there is no rational map of degree 3 with good reduction which is minimal on $\mathbb{P}^1(\mathbb{Q}_2)$. The remaining of the section is devoted to investigate the minimal rational maps of degree 4 with good reduction and their induced orbits on \mathfrak{B}_3 .

For a rational map

$$\phi(z) = \frac{a_0 + a_1z + \cdots + a_{d-1}z^{d-1} + a_dz^d}{b_1z + \cdots + b_{d-1}z^{d-1} + b_dz^d}$$

of degree d with $a_i, b_j \in \mathbb{Q}_2$ and $a_d = b_d = 1$. Assume that ϕ has good reduction and is minimal on $\mathbb{P}^1(\mathbb{Q}_2)$. The orbit of ϕ_1 on \mathfrak{B}_1 is $0 \rightarrow \infty \rightarrow 1 \rightarrow 0$. For the convenience of investigating the induced orbit on \mathfrak{B}_n with $n \geq 2$, we choose the local coordinate around ∞ by $f(z) = 1/z$. Denote by $\tilde{B}_1(0)$ the image of $B_1(\infty)$ under f . we have the following communicating graph.



Actually, the two balls $B_1(0)$ and $\tilde{B}_1(0)$ have no difference. To avoid confusion, the elements in $\tilde{B}_1(0)$ are denoted by \tilde{z} , such as $\tilde{1}, \tilde{2}$.

Let $\tilde{\mathbb{P}}^1(\mathbb{Q}_2)$ be the disjoint union of $B_1(0)$, $\tilde{B}_1(0)$ and $B_1(1)$, i.e.

$$\tilde{\mathbb{P}}^1(\mathbb{Q}_2) = B_1(0) \sqcup \tilde{B}_1(0) \sqcup B_1(1).$$

Then we can define a map $\tilde{\phi}$ from $\tilde{\mathbb{P}}^1(\mathbb{Q}_2)$ to itself as follows:

$$\tilde{\phi}(z) := \begin{cases} \psi(z), & \text{if } z \in B_1(0); \\ \varphi(z), & \text{if } z \in \tilde{B}_1(0); \\ \phi(z), & \text{if } z \in B_1(1). \end{cases}$$

Instead of investigating the dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$, we study the dynamical system $(\tilde{\mathbb{P}}^1(\mathbb{Q}_2), \tilde{\phi})$, so that we can do modulo calculation by computer. We obtain all rational maps of degree 4 with coefficients in $\{0, 1, 2, 3\}$ which have good reduction and are minimal on $\mathbb{P}^1(\mathbb{Q}_p)$. We associate each of the 12 vertices at level 3 a label

$$\begin{cases} i, & \text{where } i \in \{0, 1, 2, \dots, 7\}, \text{ if the vertex corresponds to the ball } B_n(i), \\ \tilde{i}, & \text{where } i \in \{0, 2, 4, 6\}, \text{ if the vertex corresponds to the ball } B_n(1/i). \end{cases}$$

The rational maps and their induced orbits at level 3 are showed in the following table.

Coefficients $a_0, a_1, a_2, a_3, b_1, b_2, b_3$	Induced periodic orbits at level 3
1, 0, 1, 3, 3, 1, 0	$0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 6 \rightarrow \tilde{6} \rightarrow 3$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 2 \rightarrow \tilde{2} \rightarrow 7$
1, 1, 1, 2, 3, 2, 3	
1, 2, 1, 1, 3, 3, 2	
1, 2, 3, 3, 3, 3, 0	
1, 3, 1, 0, 3, 0, 1	
1, 3, 3, 2, 3, 0, 3	
3, 2, 1, 3, 1, 1, 0	
3, 3, 1, 2, 1, 2, 3	
1, 0, 1, 3, 1, 1, 2	$0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 6 \rightarrow \tilde{2} \rightarrow 3$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 2 \rightarrow \tilde{6} \rightarrow 7$
1, 1, 1, 2, 1, 2, 1	
1, 2, 1, 1, 1, 3, 0	
1, 2, 3, 3, 1, 3, 2	
1, 3, 1, 0, 1, 0, 3	
1, 3, 3, 2, 1, 0, 1	
3, 2, 1, 3, 3, 1, 2	
3, 3, 1, 2, 3, 2, 1	

Coefficients $a_0, a_1, a_2, a_3, b_1, b_2, b_3$	Induced periodic orbits at level 3
1, 0, 1, 3, 1, 1, 0 1, 1, 1, 2, 1, 2, 3 1, 2, 1, 1, 1, 3, 2 1, 2, 3, 3, 1, 3, 0 1, 3, 1, 0, 1, 0, 1 1, 3, 3, 2, 1, 0, 3 3, 2, 1, 3, 3, 1, 0 3, 3, 1, 2, 3, 2, 3	$0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 2 \rightarrow \tilde{6} \rightarrow 3$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 6 \rightarrow \tilde{2} \rightarrow 7$
1, 0, 1, 3, 3, 1, 2 1, 1, 1, 2, 3, 2, 1 1, 2, 1, 1, 3, 3, 0 1, 2, 3, 3, 3, 3, 2 1, 3, 1, 0, 3, 0, 3 1, 3, 3, 2, 3, 0, 1 3, 2, 1, 3, 1, 1, 2 3, 3, 1, 2, 1, 2, 1	$0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 2 \rightarrow \tilde{2} \rightarrow 3$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 6 \rightarrow \tilde{6} \rightarrow 7$
1, 0, 3, 1, 3, 1, 0 1, 1, 3, 0, 3, 2, 3 3, 0, 1, 1, 1, 3, 0 3, 0, 3, 3, 1, 3, 2 3, 1, 1, 0, 1, 0, 3 3, 1, 3, 2, 1, 0, 1 3, 2, 3, 1, 1, 1, 0 3, 3, 3, 0, 1, 2, 3	$0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 6 \rightarrow \tilde{6} \rightarrow 7$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 2 \rightarrow \tilde{2} \rightarrow 3$
1, 0, 3, 1, 1, 1, 2 1, 1, 3, 0, 1, 2, 1 3, 0, 1, 1, 3, 3, 2 3, 0, 3, 3, 3, 3, 0 3, 1, 1, 0, 3, 0, 1 3, 1, 3, 2, 3, 0, 3 3, 2, 3, 1, 3, 1, 2 3, 3, 3, 0, 3, 2, 1	$0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 6 \rightarrow \tilde{2} \rightarrow 7$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 2 \rightarrow \tilde{6} \rightarrow 3$
1, 0, 3, 1, 1, 1, 0 1, 1, 3, 0, 1, 2, 3 3, 0, 1, 1, 3, 3, 0 3, 0, 3, 3, 3, 3, 2 3, 1, 1, 0, 3, 0, 3 3, 1, 3, 2, 3, 0, 1 3, 2, 3, 1, 3, 1, 0 3, 3, 3, 0, 3, 2, 3	$0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 2 \rightarrow \tilde{6} \rightarrow 7$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 6 \rightarrow \tilde{2} \rightarrow 3$
1, 0, 3, 1, 3, 1, 2 1, 1, 3, 0, 3, 2, 1 3, 0, 1, 1, 1, 3, 2 3, 0, 3, 3, 1, 3, 0 3, 1, 1, 0, 1, 0, 1 3, 1, 3, 2, 1, 0, 3 3, 2, 3, 1, 1, 1, 2 3, 3, 3, 0, 1, 2, 1	$0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 2 \rightarrow \tilde{2} \rightarrow 7$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 6 \rightarrow \tilde{6} \rightarrow 3$

The other minimal rational maps of degree 4 with good reduction can be obtained by perturbing the coefficients in their small neighborhoods.

6. SOME EXAMPLE

Example 6.1. Let $p = 3$ and $\phi(x) = -\frac{2z^2+2z+1}{z^3-3z^2+z+1}$. The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal.

The reduction of ϕ is

$$\bar{\phi}(z) = \frac{z^2 + z + 2}{z^3 + z + 1}.$$

So ϕ has good reduction. Consider the map $\bar{\phi} : \mathbb{P}^1(\mathbb{F}_3) \rightarrow \mathbb{P}^1(\mathbb{F}_3)$. By a simple calculation, we get $\bar{\phi}(1) = \infty$, $\bar{\phi}(\infty) = 0$, $\bar{\phi}(0) = 2$, $\bar{\phi}(2) = 1$. Thus $\bar{\phi}$ is transitive on $\mathbb{P}^1(\mathbb{F}_3)$. Further, using the software Mathematica, we can check $(\phi^{\circ 4})'(0) = 1 \pmod{3}$. Let $\sigma = (1, \infty, 0, -1)$ be the cycle at level 1. Since $\phi^{\circ 4}(1) = 7 \pmod{3^3}$, σ grows. Let $\hat{\sigma}$ be the lift of σ at level 2. By calculating $\phi^{\circ 12}(1) = 10 \pmod{3^3}$, we deduce that $\hat{\sigma}$ grows which then implies that $\hat{\sigma}$ grows forever. Therefore, the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal.

Example 6.2. Let $p = 3$ and $\phi(z) = \frac{2z+3}{(z-1)(z-2)}$. The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is not minimal and we decompose $\mathbb{P}^1(\mathbb{Q}_p)$ as

$$\mathbb{P}^1(\mathbb{Q}_p) = B_1(0) \bigsqcup (\mathbb{P}^1(\mathbb{Q}_p) \setminus B_1(0)),$$

where $B_1(0)$ is a minimal component of ϕ and the points in $\mathbb{P}^1(\mathbb{Q}_p) \setminus B_1(0)$ are attracted into $B_1(0)$.

We can check that $\bar{\phi}(z) = \frac{2z}{z^2+2}$. Then $\deg \bar{\phi} = \deg \phi$, which implies that ϕ has good reduction. First, we consider the map $\bar{\phi} : \mathbb{P}^1(\mathbb{F}_3) \rightarrow \mathbb{P}^1(\mathbb{F}_3)$. Since $\bar{\phi}(1) = \infty$, $\bar{\phi}(2) = \infty$, $\bar{\phi}(\infty) = 0$ and $\bar{\phi}(0) = 0$, it follows that

$$\begin{aligned} \phi(B_1(1)) &\subset B_1(\infty), \phi(B_1(2)) \subset B_1(\infty), \\ \phi(B_1(\infty)) &\subset B_1(0) \text{ and } \phi(B_1(0)) \subset B_1(0). \end{aligned}$$

So $\mathbb{P}^1(\mathbb{Q}_p) \setminus B_1(0)$ lies in the attracting basin of $B_1(0)$. It suffices to study the subsystem $(B_1(0), \phi)$. We shall show that the system $(B_1(0), \phi)$ is minimal. In fact, the derivative of ϕ at 0 is $13/4$. Thus $\phi'(0) = 1 \pmod{3}$. Let $\sigma = (0)$ be the cycle at level 1. Then σ grows or splits. A simple calculation shows that

$$\phi(0) = 15 \pmod{3^3}, \phi(15) = 3 \pmod{3^3}, \phi(3) = 18 \pmod{3^3}.$$

So σ grows, and the lift of σ grows too. It then follows that σ grows forever. Hence, the system $(B_1(0), \phi)$ is minimal.

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