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MINIMALITY OF p -ADIC RATIONAL MAPS WITH GOOD REDUCTION

AIHUA FAN, SHILEI FAN, LINGMIN LIAO, AND YUEFEI WANG

ABSTRACT. A rational map with good reduction in the field \mathbb{Q}_p of p -adic numbers defines a 1-Lipschitz dynamical system on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over \mathbb{Q}_p . The dynamical structure of such a system is completely described by a minimal decomposition. That is to say, $\mathbb{P}^1(\mathbb{Q}_p)$ is decomposed into three parts: finitely many periodic orbits; finite or countably many minimal subsystems each consisting of a finite union of balls; and the attracting basins of periodic orbits and minimal subsystems. For any prime p , a criterion of minimality for rational maps with good reduction is obtained. When $p = 2$, a condition in terms of the coefficients of the rational map is proved to be necessary for the map being minimal and having good reduction, and sufficient for the map being minimal and 1-Lipschitz. It is also proved that a rational map having good reduction of degree 2, 3 and 4 can never be minimal on the whole space $\mathbb{P}^1(\mathbb{Q}_2)$.

1. INTRODUCTION

The present study contributes to the theory of p -adic dynamical systems which has recently been intensively and widely developed. For this development, one can consult the books [3, 6, 27] and their bibliographies therein.

For a prime number p , let \mathbb{Q}_p be the field of p -adic numbers and \mathbb{Z}_p be the ring of integers in \mathbb{Q}_p . Ergodic theory on the ring \mathbb{Z}_p is extremely important for applications to automata theory, computer science and cryptology, especially in connection with pseudo-random numbers and uniform distribution of sequences. The minimality, or equivalently the ergodicity with respect to the Haar measure, of non-expanding dynamical systems on \mathbb{Z}_p are extensively studied in [2, 8, 9, 12, 13, 14, 17, 18, 19, 21, 22, 25], and so on.

The dynamical properties of the fixed points of the rational maps have been studied in the space \mathbb{C}_p of p -adic complex numbers [5, 23, 24, 28] and in the adelic space [10]. The Fatou and Julia theory of the rational maps on \mathbb{C}_p , and on the Berkovich space over \mathbb{C}_p , are also developed [7, 6, 20, 26, 27]. However, the global dynamical structure of rational maps on \mathbb{Q}_p remains unclear, though the rational maps of degree one are totally characterized in [15].

Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of degree $d \geq 2$. Then ϕ induces a dynamical system on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over \mathbb{Q}_p , denoted by $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$. Let $E \subset \mathbb{P}^1(\mathbb{Q}_p)$ be a subset such that $\phi(E) \subset E$. Then restricted to E , ϕ defines a subsystem $(E, \phi|_E)$. The subsystem $(E, \phi|_E)$ is called *minimal* if for any point $x \in E$, the orbit of x under ϕ is dense in E . In this article, we suppose that ϕ has good reduction (see the definition below). The minimality of $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ and its subsystems will be fully investigated. As we will see, any rational map ϕ having good reduction is 1-Lipschitz continuous on $\mathbb{P}^1(\mathbb{Q}_p)$ which is

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equipped with its spherical metric. This suggests that $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ shares many properties with the polynomial dynamics on \mathbb{Z}_p , which are 1-Lipschitz with respect to the metric induced by the p -adic absolute value.

Let $\bar{\cdot}$ denote the *reduction modulo p* from \mathbb{Z}_p to $\mathbb{Z}_p/p\mathbb{Z}_p$ such that $a \mapsto \bar{a}$ with $a \equiv \bar{a} \pmod{p}$. For a polynomial $f(z) = \sum_{i=0}^d a_i z^i \in \mathbb{Z}_p[z]$, the *reduction of f modulo p* is defined as

$$\bar{f}(z) = \sum_{i=0}^d \bar{a}_i z^i.$$

Note that ϕ can be written as a quotient of polynomials $f, g \in \mathbb{Z}_p[z]$ having no common factors, such that at least one coefficient of f or g has absolute value 1. Let $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$ be the finite field of p elements. The *degree* of a rational map ϕ , denoted by $\deg \phi$, is the maximum of the degrees of its denominator and numerator without common factors. The *reduction of ϕ modulo p* is the rational function (of degree at most $\deg \phi$)

$$\bar{\phi}(z) = \frac{\bar{f}(z)}{\bar{g}(z)} \in \mathbb{F}_p(z),$$

obtained by canceling common factors in the reductions $\bar{f}(z), \bar{g}(z)$. If $\deg \bar{\phi} = \deg \phi$, we say ϕ has *good reduction*. If $\deg \bar{\phi} < \deg \phi$, we say ϕ has *bad reduction*.

Assume that ϕ has good reduction. Let $\rho(\cdot, \cdot)$ be the *spherical metric* on $\mathbb{P}^1(\mathbb{Q}_p)$ (see the definition in Section 2). Then the map ϕ is 1-Lipschitz continuous (everywhere non-expanding [27, p.59]):

$$\rho(\phi(P_1), \phi(P_2)) \leq \rho(P_1, P_2) \quad \text{for all } P_1, P_2 \in \mathbb{P}^1(\mathbb{Q}_p).$$

The 1-Lipschitz continuity of ϕ leads us to investigate the minimality and minimal decomposition of the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$, as in the case of polynomial dynamical systems on \mathbb{Z}_p studied in [14].

Let $\mathbb{P}^1(\mathbb{F}_p)$ be the projective line over \mathbb{F}_p . The reduction $\bar{\phi}$ induces a transformation from $\mathbb{P}^1(\mathbb{F}_p)$ into itself. We denote by ϕ^k the k -th iteration of ϕ . A criterion of the minimality of the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is given in the following theorem.

Theorem 1.1. *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of $\deg \phi \geq 2$ with good reduction. Then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if the following conditions are satisfied:*

- (1) *The reduction $\bar{\phi}$ is transitive on $\mathbb{P}^1(\mathbb{F}_p)$.*
- (2) *$(\phi^{(p+1)})'(0) \equiv 1 \pmod{p}$ and $|\phi^{(p+1)}(0)|_p = 1/p$.*
- (3) *For the case $p = 2$ or 3 , additionally, $|\phi^{(p+1)^p}(0)|_p = 1/p^2$.*

Under the spherical metric $\rho(\cdot, \cdot)$, $\mathbb{P}^1(\mathbb{Q}_p)$ can be considered as an infinite symmetric tree (see Figures 1 and 2). This tree is in fact an infinite $(p+1)$ -Cayley tree. The centered vertex which is the root of the tree, is called *Gauss point*. Other vertices correspond to balls with radius strictly less than one (with respect to the spherical metric). The points in $\mathbb{P}^1(\mathbb{Q}_p)$ are the boundary points of the tree. A vertex is said to be *at level n* ($n \geq 1$), if there are n edges between the vertex and the Gauss point. Then for $n \geq 1$, there are $(p+1)p^{n-1}$ vertices at level n . Since a rational map with good reduction is 1-Lipschitz continuous with respect to the spherical metric, it will induce an action on the tree under which the Gauss point is fixed and the sets of vertices at the same level are invariant.

Remark that if we consider ϕ as action on the tree, the condition (1) in Theorem 1.1 means that ϕ is transitive on the set of vertices at level 1, the conditions (1) and (2) imply that ϕ is transitive on the set of vertices at level 2, while the condition (3) implies that ϕ

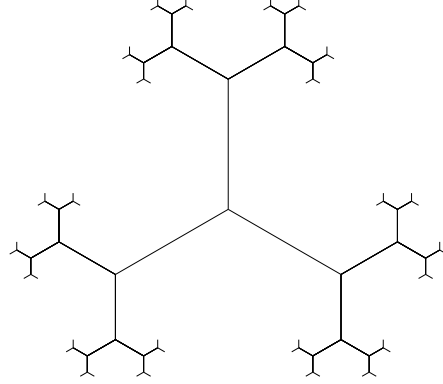


FIGURE 1. Tree structure of $\mathbb{P}^1(\mathbb{Q}_2)$. The points of $\mathbb{P}^1(\mathbb{Q}_2)$ are considered as the boundary points of the infinite tree.

is transitive on the set of vertices at level 3 if conditions (1) and (2) are also satisfied. We will see in Theorem 4.1 that the transitivity of ϕ on these finite levels is sufficient for the minimality of ϕ on the whole space. The conclusion of Theorem 1.1 can be interpreted as follows.

The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if ϕ is transitive on the set of vertices at level 2 for $p \geq 5$, and at level 3 for $p = 2$ or 3.

When the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is not minimal, we describe the dynamical structure of the system by showing all its minimal subsystems.

Theorem 1.2. *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of $\deg \phi \geq 2$ with good reduction. We have the following decomposition*

$$\mathbb{P}^1(\mathbb{Q}_p) = \mathcal{P} \bigsqcup \mathcal{M} \bigsqcup \mathcal{B}$$

where \mathcal{P} is the finite set consisting of all periodic points of ϕ , $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each \mathcal{M}_i is a finite union of balls and each subsystem $\phi : \mathcal{M}_i \rightarrow \mathcal{M}_i$ is minimal, and points in \mathcal{B} lie in the attracting basin of a periodic orbit or of a minimal subsystem. Moreover, the length of a periodic orbit has one of the following forms:

$$k \text{ or } k\ell, \quad \text{if } p \geq 5,$$

$$k \text{ or } k\ell \text{ or } kp, \quad \text{if } p = 2 \text{ or } 3,$$

where $1 \leq k \leq p + 1$ and $\ell \mid (p - 1)$.

The decomposition in Theorem 1.2 will be referred to as the *minimal decomposition* of the system $\phi : \mathbb{P}^1(\mathbb{Q}_p) \rightarrow \mathbb{P}^1(\mathbb{Q}_p)$.

We remark that the possible lengths of periodic orbits were investigated in [29], see also [27, pp.62-64].

A finite periodic orbit of ϕ is by definition a minimal set. But for the convenience of the presentation of the paper, only the sets \mathcal{M}_i in the above decomposition are called *minimal components*. What kind of set can be a minimal component of a good reduction rational map? In a recent work [8], the authors showed that for any 1-Lipschitz continuous map, a minimal component must be a Legendre set and that any Legendre set is a minimal

component of some 1-Lipschitz continuous map. Recall that $E \subset \mathbb{P}^1(\mathbb{Q}_p)$ is a *Legendre set* (see [8, p. 778]) if for every integer $n \geq 0$, every non-empty intersection of E with a ball of radius p^{-n} contains the same number of balls of radius $p^{-(n+1)}$.

We will show that for a rational map with good reduction, the minimal components \mathcal{M}_i are Legendre sets of special forms. Further, the dynamics of the minimal subsystems are adding machines. Let $(p_s)_{s \geq 1}$ be a sequence of positive integers such that $p_s | p_{s+1}$ for every $s \geq 1$. We denote by $\mathbb{Z}_{(p_s)}$ the inverse limit of $\mathbb{Z}/(p_s\mathbb{Z})$, called an *odometer*. The sequence $(p_s)_{s \geq 1}$ is called the *structure sequence* of the odometer. The map $z \rightarrow z + 1$ on $\mathbb{Z}_{(p_s)}$ is the *adding machine* on $\mathbb{Z}_{(p_s)}$. The structures of minimal components are determined as follows.

Theorem 1.3. *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of $\deg \phi \geq 2$ with good reduction. Assume that $E \subset \mathbb{P}^1(\mathbb{Q}_p)$ is a minimal component of ϕ . Then $\phi : E \rightarrow E$ is conjugate to the adding machine on an odometer $\mathbb{Z}_{(p_s)}$, where*

$$(p_s) = (k, k\ell, k\ell p, k\ell p^2, \dots)$$

with integers k and ℓ satisfying that $1 \leq k \leq p + 1$ and $\ell \mid (p - 1)$.

In applications, one usually would like to construct or determine a minimal rational (polynomial) map by its coefficients. However, this is far from easy, although a criterion of the minimality of rational maps with good reduction is given in Theorem 1.1. For rational maps of degree one, a complete description is given in [15]. Here we give the description of rational maps of degree at least two with good reduction, but only for $p = 2$. It seems to be a much more harder task for $p \geq 3$.

For a rational map $\phi \in \mathbb{Q}_p(z)$ of degree at least 2, the number of periodic points of a fixed period must be finite. Hence, there exists a $z_0 \in \mathbb{Q}_p$ such that $z_0, \phi(z_0), \phi^2(z_0)$ are distinct. Consider the linear fraction

$$g(z) = \frac{(z - z_0)(\phi^2(z_0) - \phi(z_0))}{(z - \phi(z_0))(\phi^2(z_0) - z_0)}.$$

Then $g(z_0) = 0$, $g(\phi(z_0)) = \infty$ and $g(\phi^2(z_0)) = 1$. Let $\psi = g \circ \phi \circ g^{-1}$ be the conjugation of ϕ by g . Then we have $\psi(0) = \infty$ and $\psi(\infty) = 1$ and the rational function ψ can be written as

$$\psi(z) = \frac{a_0 + a_1 z + \dots + a_{d-1} z^{d-1} + z^d}{b_1 z + \dots + b_{d-1} z^{d-1} + z^d}$$

where $a_i, b_j \in \mathbb{Q}_p$ ($0 \leq i < d, 1 \leq j < d$) and $d \geq 2$ is the degree of ϕ . Therefore, without loss of generality, we can always assume that $\phi(0) = \infty$ and $\phi(\infty) = 1$.

The following theorem provides, in some sense, a principle for constructing minimal rational maps on $\mathbb{P}^1(\mathbb{Q}_2)$. For a degree $d \geq 2$ rational map of form

$$\phi(z) = \frac{a_0 + a_1 z + \dots + a_{d-1} z^{d-1} + a_d z^d}{b_1 z + \dots + b_{d-1} z^{d-1} + b_d z^d} \quad (1.1)$$

with $a_i, b_j \in \mathbb{Q}_2$ and $a_d = b_d = 1$, we set $A_\phi := \sum_{i \geq 0} a_i$, $B_\phi := \sum_{j \geq 1} b_j$, $A_{\phi,1} := \sum_{i \geq 0} a_{2i+1}$, $A_{\phi,2} := \sum_{i \geq 0} a_{4i+1}$ and $A_{\phi,3} := \sum_{i \geq 0} a_{4i+3}$.

Theorem 1.4. *Let ϕ be defined as (1.1). If ϕ has good reduction and is minimal on $\mathbb{P}^1(\mathbb{Q}_2)$, then*

$$\left\{ \begin{array}{l} a_i, b_j \in \mathbb{Z}_2, \text{ for } 0 \leq i \leq d-1 \text{ and } 1 \leq j \leq d-1, \\ a_0 \equiv 1 \pmod{2}, \\ B_\phi \equiv 1 \pmod{2}, \\ A_\phi \equiv 2 \pmod{4}, \\ A_{\phi,1} \equiv 1 \pmod{2}, \\ b_1 \equiv 1 \pmod{2}, \\ a_{d-1} - b_{d-1} \equiv 1 \pmod{2}, \\ a_0 b_1 (a_{d-1} - b_{d-1}) (A_{\phi,2} - A_{\phi,3}) B_\phi + \\ \quad 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1} + A_{\phi,3}) \equiv 1 \pmod{4}. \end{array} \right. \quad (1.2)$$

Conversely, the condition (1.2) implies that ϕ is 1-Lipschitz continuous and minimal on $\mathbb{P}^1(\mathbb{Q}_2)$.

Corollary 1.5. *Let $\phi \in \mathbb{Q}_2(z)$ be a rational map of degree 2, 3 or 4 with good reduction. Then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is not minimal.*

A complete characterization of minimal polynomial maps on \mathbb{Z}_p in terms of their coefficients are given in [22] for $p = 2$ and in [12] for $p = 3$. However, there is still no complete description for $p \geq 5$. On the other hand, the characterizations of minimal Mahler series and van der Put series by their coefficients are investigated in [1, 3] and in [4, 21] respectively.

The above condition (1.2) could be quickly checked by computer, since there are only arithmetic operations ‘+’, ‘−’, ‘×’ in the equations of (1.2). By condition (1.2) and a quick computer computation, we obtain a series of minimal rational maps in $\mathbb{Q}_2(z)$ of degree 4 which is 1-Lipschitz but do not have good reduction. See the table in Section 5.2.

The paper is organized as follows. In Section 2, we give some preliminaries of $\mathbb{P}^1(\mathbb{Q}_p)$, including the spherical metric and the tree structure of $\mathbb{P}^1(\mathbb{Q}_p)$. Section 3 is devoted to the induced dynamical systems on the vertices. The proofs of Theorems 1.1–1.3 are given in Section 4. In Section 5, we characterize the minimal rational maps with good reduction in terms of their coefficients for the case $p = 2$. Theorem 1.4 and Corollary 1.5 will be proved in this section. To illustrate our result, in the last section, we present two rational maps with good reduction whose exact minimal decompositions are described when $p = 3$.

2. PROJECTIVE LINE

Any point in the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ may be given in homogeneous coordinates by a pair $[x_1 : x_2]$ of points in \mathbb{Q}_p which are not both zero. Two such pairs are equal if they differ by an overall (nonzero) factor $\lambda \in \mathbb{Q}_p^*$:

$$[x_1 : x_2] = [\lambda x_1 : \lambda x_2].$$

The field \mathbb{Q}_p may be identified with the subset of $\mathbb{P}^1(\mathbb{Q}_p)$ given by

$$\{[x : 1] \in \mathbb{P}^1(\mathbb{Q}_p) \mid x \in \mathbb{Q}_p\}.$$

This subset contains all points in $\mathbb{P}^1(\mathbb{Q}_p)$ except one: the point of infinity, which may be given as $\infty = [1 : 0]$.

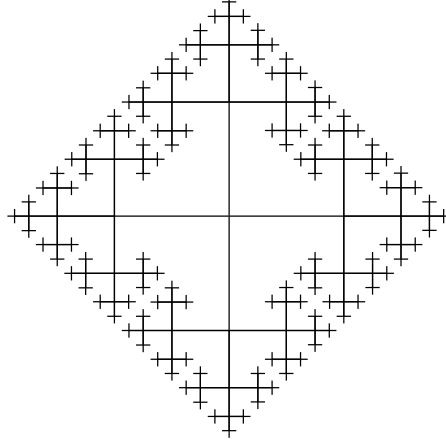


FIGURE 2. Tree structure of $\mathbb{P}^1(\mathbb{Q}_3)$. The points of $\mathbb{P}^1(\mathbb{Q}_3)$ are considered as the boundary points of the infinite tree.

The spherical metric defined on $\mathbb{P}^1(\mathbb{Q}_p)$ is analogous to the standard spherical metric on the Riemann sphere. If $P = [x_1, y_1]$ and $Q = [x_2, y_2]$ are two points in $\mathbb{P}^1(\mathbb{Q}_p)$, we define

$$\rho(P, Q) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}$$

or, viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $\mathbb{Q}_p \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{Q}_p \cup \{\infty\}$ we define

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}} \quad \text{if } z_1, z_2 \in \mathbb{Q}_p,$$

and

$$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \leq 1, \\ 1/|z|_p, & \text{if } |z|_p > 1. \end{cases}$$

Remark that the restriction of the spherical metric on the ring $\mathbb{Z}_p := \{x \in \mathbb{Q}_p, |x|_p \leq 1\}$ of p -adic integers is same as the metric induced by the absolute value $|\cdot|_p$.

A rational map $\phi \in \mathbb{Q}_p(z)$ induces a transformation on $\mathbb{P}^1(\mathbb{Q}_p)$. Rational maps are always Lipschitz continuous on $\mathbb{P}^1(\mathbb{Q}_p)$ with respect to the spherical metric (see [27, Theorem 2.14.]). In particular, rational maps with good reduction are 1-Lipschitz continuous.

Lemma 2.1 ([27], p.59). *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map with good reduction. Then the map $\phi : \mathbb{P}^1(\mathbb{Q}_p) \mapsto \mathbb{P}^1(\mathbb{Q}_p)$ is 1-Lipschitz continuous:*

$$\rho(\phi(P), \phi(Q)) \leq \rho(P, Q), \quad \forall P, Q \in \mathbb{P}^1(\mathbb{Q}_p).$$

For $a \in \mathbb{P}^1(\mathbb{Q}_p)$ and an integer $n \geq 1$, denote by

$$B_n(a) := \{x \in \mathbb{P}^1(\mathbb{Q}_p) : \rho(x, a) \leq p^{-n}\}$$

the ball of radius p^{-n} centered at a . The projective line $\mathbb{P}^1(\mathbb{Q}_p)$ consists of $p + 1$ disjoint balls of radius p^{-1} ,

$$\mathbb{P}^1(\mathbb{Q}_p) = B_1(\infty) \bigsqcup \left(\bigsqcup_{0 \leq i \leq p-1} B_1(i) \right).$$

For each integer $n \geq 1$, every ball of radius p^{-n} consists of p disjoint balls of radius p^{-n-1} . For example, $B_1(\infty)$ consists of $B_2(1/p), B_2(2/p), \dots, B_2((p-1)/p), B_2(\infty)$.

As mentioned in Section 1, the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over \mathbb{Q}_p could be considered as an infinite $(p+1)$ -Cayley tree: the branch index of each vertex is $p+1$, i.e. each vertex is an endpoint of $p+1$ edges. There is a centered vertex which is called Gauss point. Excluding Gauss point, each vertex corresponds to a ball in $\mathbb{P}^1(\mathbb{Q}_p)$ of radius strictly less than one (with respect to the spherical metric). The Gauss point can be considered as the ball of radius 1 centered at the origin which is actually the whole space $\mathbb{P}^1(\mathbb{Q}_p)$. The set of edges of the tree consists of the pairs of balls (B, B') of radius ≤ 1 such that

$$B' \subset B, r(B) = p \cdot r(B')$$

where $r(B)$ and $r(B')$ are the radii of balls B and B' respectively.

The points in $\mathbb{P}^1(\mathbb{Q}_p)$ are the boundary points of the tree. Remind that a vertex is said to be at level n ($n \geq 1$) if there are n edges between Gauss point and the vertex. A vertex at level n corresponds a ball of radius p^{-n} . The projective line $\mathbb{P}^1(\mathbb{Q}_p)$ consists of $(p+1)p^{n-1}$ disjoint balls of radius p^{-n} . There are $(p+1)p^{n-1}$ vertices at level n . For example, see Figures 1 and 2 for the tree structures of $\mathbb{P}^1(\mathbb{Q}_2)$ and $\mathbb{P}^1(\mathbb{Q}_3)$.

3. INDUCED DYNAMICS

For each positive integer n , $\mathbb{P}^1(\mathbb{Q}_p)$ consists of $(p+1)p^{n-1}$ disjoint balls of radius p^{-n} . Denote by \mathfrak{B}_n the set of the $(p+1)p^{n-1}$ disjoint balls of radius p^{-n} . In general, any 1-Lipschitz continuous map $\phi : \mathbb{P}^1(\mathbb{Q}_p) \rightarrow \mathbb{P}^1(\mathbb{Q}_p)$ induces a transformation on \mathfrak{B}_n . Denote by ϕ_n the induced map of ϕ on \mathfrak{B}_n , i.e.

$$\phi_n(B_n(x)) = B_n(\phi(x)), \quad \forall x \in \mathbb{P}^1(\mathbb{Q}_p).$$

As $\mathbb{P}^1(\mathbb{Q}_p)$ is considered as an infinite tree, for each positive integer n , there is a one-to-one correspondence between \mathfrak{B}_n and the vertices of the tree at level n . The 1-Lipschitz continuous map ϕ induces a transformation on the tree under which the set of vertices of each level is invariant.

Many properties of the dynamics ϕ on $\mathbb{P}^1(\mathbb{Q}_p)$ are linked to those of ϕ_n on \mathfrak{B}_n .

Proposition 3.1. *Let $\phi : \mathbb{P}^1(\mathbb{Q}_p) \rightarrow \mathbb{P}^1(\mathbb{Q}_p)$ be a 1-Lipschitz continuous map. Then the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if the finite system (\mathfrak{B}_n, ϕ_n) is minimal for all integers $n \geq 1$.*

The equivalence in Proposition 3.1 is similar to that for 1-Lipschitz continuous maps on the ring \mathbb{Z}_p or on a discrete valuation domain (see [1, p. 111], [2, Theorem 1.2] and [8, Corollary 4]).

Proof. The “only if” part of the statement is obvious, we prove only the “if” part. Suppose that $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is not minimal. Then there exist an open set S and a point $z \in \mathbb{P}^1(\mathbb{Q}_p)$ such that $\phi^k(z) \notin S$ for all integers $k \geq 1$. Since the set S is open, there exists a ball $B_{n_0}(z_0) \subset S$ for some $z_0 \in \mathbb{P}^1(\mathbb{Q}_p)$ and some integer n_0 large enough. Recall that (\mathfrak{B}_n, ϕ_n) is minimal for all integers $n > 0$. So there exists an integer k_0 such that $\phi_{n_0}^{k_0}(B_{n_0}(z_0)) = B_{n_0}(z_0)$. This implies $\rho(\phi^{k_0}(z), z_0) \leq p^{-n_0}$, which contradicts the fact that $\phi^{k_0}(z) \notin S$. \square

Recall that a rational map with good reduction is 1-Lipschitz continuous. For a rational map $\phi \in \mathbb{Q}_p(z)$ with good reduction, to study the minimality of the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$, it suffices to study the induced dynamics (\mathfrak{B}_n, ϕ_n) for all integers $n \geq 1$.

Now let us first study the dynamics (\mathfrak{B}_1, ϕ_1) at level 1. Note that each point in $\mathbb{P}^1(\mathbb{F}_p)$ corresponds to a ball in \mathfrak{B}_1 . The induced map ϕ_1 of ϕ can be considered as the reduction $\bar{\phi}$ acting on $\mathbb{P}^1(\mathbb{F}_p)$. The map $\bar{\phi} : \mathbb{P}^1(\mathbb{F}_p) \rightarrow \mathbb{P}^1(\mathbb{F}_p)$ admits some periodic orbits. A periodic orbit is called a *cycle* of $\bar{\phi}$. The points outside any cycle will be mapped into some cycle after several iterations.

Let $(\bar{x}_1, \dots, \bar{x}_k) \subset \mathbb{P}^1(\mathbb{F}_p)$ be a cycle of $\bar{\phi}$. To simplify notation, we consider a transformation $f \in \text{PGL}_2(\mathbb{Z}_p)$ with $f(x_1) = 0$, where x_1 is a point in the ball corresponding to \bar{x}_1 . Replacing x_1 and ϕ with 0 and $f \circ \phi \circ f^{-1}$ respectively, we may assume that $\bar{x}_1 = \bar{0}$. Under this assumption, we will see in the following that ϕ^k acting on $p\mathbb{Z}_p$ is conjugate to a power series acting on \mathbb{Z}_p .

For two rational maps $\theta, \omega \in \mathbb{Q}_p(z)$ with good reduction, the composition $\theta \circ \omega$ has good reduction, and $\theta \circ \omega = \bar{\theta} \circ \bar{\omega}$ (see [27, p.59, Theorem 2.18]). So ϕ^k has good reduction and $\bar{0}$ is a fixed point of $\bar{\phi}^k$. We write ϕ^k in the form

$$\phi^k(z) = \frac{a_0 + a_1z + \dots + a_nz^n}{b_0 + b_1z + \dots + b_nz^n}$$

with coefficients $a_0, \dots, a_d, b_0, \dots, b_d \in \mathbb{Z}_p$ and at least one coefficient in $\mathbb{Z}_p \setminus p\mathbb{Z}_p$. The fact that $\bar{0}$ is a fixed point of $\bar{\phi}^k$ implies that $\phi^k(0) = a_0/b_0 \equiv 0 \pmod{p}$. Since ϕ^k has good reduction, we have $a_0 \in p\mathbb{Z}_p$ and $b_0 \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$. Otherwise, the numerator and denominator of the reduction have the common factor z , which implies $\deg \bar{\phi}^k < \deg \phi^k$. Multiplying numerator and denominator by b_0^{-1} , we may thus write ϕ^k in the form

$$\phi^k(z) = \frac{a_0 + a_1z + \dots + a_nz^n}{1 + b_1z + \dots + b_nz^n}.$$

Then by the following Lemma 3.2, ϕ^k is 1-Lipschitz from $p\mathbb{Z}_p$ to itself and

$$\phi^k(z) = a_0 + \lambda z + \lambda_2 z^2 + \lambda_3 z^3 + \dots,$$

with $\lambda_i \in \mathbb{Z}_p$ for all $i \geq 2$.

Lemma 3.2. *Let $\phi \in \mathbb{Q}_p(z)$ be a rational map of the form*

$$\frac{a_0 + a_1z + \dots + a_nz^n}{1 + b_1z + \dots + b_nz^n}, \quad \text{with } a_j, b_j \in \mathbb{Z}_p.$$

Then ϕ is 1-Lipschitz on $p\mathbb{Z}_p$. Furthermore,

$$\phi(z) = a_0 + \lambda z + \lambda_2 z^2 + \lambda_3 z^3 + \dots \quad (3.1)$$

with $\lambda_i \in \mathbb{Z}_p$ for all $i \geq 2$.

Proof. The Taylor expansion of ϕ around $z = 0$, which in this case can be obtained by simple long division, gives

$$\phi(z) = a_0 + \lambda z + \frac{A(z)}{1 + zB(z)} z^2$$

with $A(z), B(z) \in \mathbb{Z}_p[z]$ and $\lambda = \phi'(0) \in \mathbb{Z}_p$. Observe that on $p\mathbb{Z}_p$, we have

$$\frac{1}{1 + zB(z)} = 1 - zB(z) + z^2(B(z))^2 - z^3(B(z))^3 + \dots.$$

Hence we can write $\phi(z)$ as the form (3.1). Obviously, ϕ induces a 1-Lipschitz map on $p\mathbb{Z}_p$. \square

Now we study the dynamical system $(p\mathbb{Z}_p, \phi^k)$. For simplicity, we write φ instead of ϕ^k . Note that every power series

$$\varphi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$$

converges on $p\mathbb{Z}_p$. If $\lambda_0 \in p\mathbb{Z}_p$, then $\varphi(p\mathbb{Z}_p) \subset p\mathbb{Z}_p$ and φ induces a 1-Lipschitz map from $p\mathbb{Z}_p$ to itself.

In the remainder of this section we assume $\varphi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ with $\lambda_0 \in p\mathbb{Z}_p$. The dynamical system $(p\mathbb{Z}_p, \varphi)$ is conjugate to a system (\mathbb{Z}_p, χ) by the transformation $f(z) = z/p$, i.e.

$$\begin{array}{ccc} p\mathbb{Z}_p & \xrightarrow{\varphi} & p\mathbb{Z}_p \\ z/p \downarrow & & \downarrow z/p \\ \mathbb{Z}_p & \xrightarrow{\chi} & \mathbb{Z}_p, \end{array}$$

where $\chi(z) = \sum_{i=0}^{\infty} p^{i-1} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ converges on \mathbb{Z}_p and induces a map from \mathbb{Z}_p to itself.

The dynamical system of χ on \mathbb{Z}_p and its induced dynamics on $\mathbb{Z}_p/p^n\mathbb{Z}_p$ was studied in [16]. Actually, the convergent series with coefficient in the integral ring \mathcal{O}_K of a finite extension K of \mathbb{Q}_p were studied as dynamical systems on \mathcal{O}_K in [16]. The system (\mathbb{Z}_p, θ) is a special case when $K = \mathbb{Q}_p$.

In the following we shall translate results about θ in the language of results about φ . For more details on this subject, the reader may consult [14, 16].

For each $n \geq 1$, denote by φ_n the induced map of φ on $p\mathbb{Z}_p/p^n\mathbb{Z}_p$, i.e.,

$$\varphi(x \bmod p^n) \equiv \varphi(x) \pmod{p^n},$$

for all $x \in p\mathbb{Z}_p$. In the spirit of Proposition 3.1, the minimality and minimal decomposition of the system $(p\mathbb{Z}_p, \varphi)$ can be deduced from its induced dynamics $(p\mathbb{Z}_p/p^n\mathbb{Z}_p, \varphi_n)$. So we need to study the finite systems $(p\mathbb{Z}_p/p^n\mathbb{Z}_p, \varphi_n)$. To this end, the main idea, which comes from [11, 29], is to study the behaviour of systems $(p\mathbb{Z}_p/p^n\mathbb{Z}_p, \varphi_n)$ by induction. We refer the reader to [14, 16] for the application of this idea to give a minimal decomposition theorem for any convergent power series in $\mathbb{Z}_p[[z]]$.

A collection $\sigma = (x_1, \dots, x_k)$ of k distinct points in $p\mathbb{Z}_p/p^n\mathbb{Z}_p$ is called a *cycle* of φ_n of length k or a *k-cycle* at level n , if

$$\varphi_n(x_1) = x_2, \dots, \varphi_n(x_i) = x_{i+1}, \dots, \varphi_n(x_k) = x_1.$$

Set

$$X_\sigma := \bigsqcup_{i=1}^k X_i \text{ where } X_i := \{x_i + p^n t + p^{n+1}\mathbb{Z}_p; t = 0, 1, \dots, p-1\} \subset p\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p.$$

Then

$$\varphi_{n+1}(X_i) \subset X_{i+1} \ (1 \leq i \leq k-1) \text{ and } \varphi_{n+1}(X_k) \subset X_1.$$

In the following we are concerned with the behavior of the finite dynamics φ_{n+1} on the set X_σ and determine all cycles in X_σ of φ_{n+1} , which will be called *lifts* of σ . Remark that the length of any lift $\tilde{\sigma}$ of σ is a multiple of k .

Let $\mathbb{X}_i = x_i + p^n \mathbb{Z}_p$ be the closed disk of radius p^{-n} corresponding to $x_i \in \sigma$ and

$$\mathbb{X}_\sigma := \bigsqcup_{i=1}^k \mathbb{X}_i$$

be the clopen set corresponding to the cycle σ .

Let $\psi := \varphi^k$ be the k -th iteration of φ . Then, any point in σ is fixed by ψ_n , the n -th induced map of ψ . For any point $x \in \mathbb{X}_\sigma$, we have $(\psi(x) - x)/p^n \in \mathbb{Z}_p$. Let

$$\alpha_n(x) := \psi'(x) = \prod_{j=0}^{k-1} \varphi'(\varphi^j(x)) \quad (3.2)$$

$$\beta_n(x) := \frac{\psi(x) - x}{p^n} = \frac{\varphi^k(x) - x}{p^n}. \quad (3.3)$$

One can check that $\alpha_n(x) \pmod{p}$ is always constant on \mathbb{X}_σ . We should remark that under the condition that $\alpha_n \equiv 1 \pmod{p}$, $\beta_n \pmod{p}$ is also constant on \mathbb{X}_σ [14, 16]. We distinguish the following four behaviours of φ_{n+1} on X_σ :

(a) If $\alpha_n \equiv 1 \pmod{p}$ and $\beta_n \not\equiv 0 \pmod{p}$, then φ_{n+1} restricted to X_σ preserves a single cycle of length pk . In this case we say σ *grows*.

(b) If $\alpha_n \equiv 1 \pmod{p}$ and $\beta_n \equiv 0 \pmod{p}$, then φ_{n+1} restricted to X_σ preserves p cycles of length k . In this case we say σ *splits*.

(c) If $\alpha_n \equiv 0 \pmod{p}$, then φ_{n+1} restricted to X_σ preserves one cycle of length k and the remaining points of X_σ are mapped into this cycle. In this case we say σ *grows tails*.

(d) If $\alpha_n \not\equiv 0, 1 \pmod{p}$, then φ_{n+1} restricted to X_σ preserves one cycle of length k and $\frac{p-1}{\ell}$ cycles of length $k\ell$, where ℓ is the order of α_n in the multiplicative group $\mathbb{F}_p \setminus \{0\}$. In this case we say σ *partially splits*.

For $n \geq 1$, let $\sigma = (x_1, \dots, x_k) \subset p\mathbb{Z}_p/p^n\mathbb{Z}_p$ be a k -cycle and let $\tilde{\sigma}$ be a lift of σ which is a of φ_{n+1} contained in X_σ . To illustrate the change of nature for a cycle to its lifts, we shall show the relationship between (α_n, β_n) and $(\alpha_{n+1}, \beta_{n+1})$. For the calculation of $(\alpha_{n+1} \pmod{p}, \beta_{n+1} \pmod{p})$ from $(\alpha_n \pmod{p}, \beta_n \pmod{p})$, the interested reader may consult to [14, Lemma 2] and [16, p.1636]. The following propositions predict the behaviours of the lifts of a cycle σ .

Proposition 3.3 ([14] Proposition 1, and [16] Proposition 4.4). *Let σ be a cycle of φ_n .*

- (1) *If σ grows or splits, then any lift $\tilde{\sigma}$ grows or splits.*
- (2) *If σ grows tails, then the single lift $\tilde{\sigma}$ also grows tails.*
- (3) *If σ partially splits, then the lift $\tilde{\sigma}$ of the same length as σ partially splits, and the other lifts of length $k\ell$ grow or split.*

Proposition 3.4 ([14, 16]). *Let σ be a growing cycle of φ_n and $\tilde{\sigma}$ be the unique lift of σ .*

- (1) *If $p > 3$ and $n \geq 1$, then $\tilde{\sigma}$ also grows.*
- (2) *If $p = 3$ and $n \geq 2$, then $\tilde{\sigma}$ also grows.*
- (3) *If $p = 2$ and $\tilde{\sigma}$ grows, then the lift of $\tilde{\sigma}$ grows.*

For more details of Proposition 3.4, we refer the reader to [14, Proposition 2] for the first two assertion, and to [14, Corollary 1] for the third assertion. Remark that the results in Proposition 3.4 are presented only for polynomials in $\mathbb{Z}_p[z]$. Actually, these results also hold for convergent series in $\mathbb{Z}_p[[z]]$ (see [16, Propositions 4.7, 4.8, 4.10, 4.11] for a more general setting).

Let $E \subset p\mathbb{Z}_p$ be a φ -invariant compact open set. Let

$$E/p^n\mathbb{Z}_p := \{x \in \mathbb{Z}_p/p^n\mathbb{Z}_p : \exists y \in E \text{ such that } x \equiv y \pmod{p^n}\}.$$

It is now well known that the subsystem (E, φ) is minimal if and only if the induced map $\varphi_n : E/p^n\mathbb{Z}_p \rightarrow E/p^n\mathbb{Z}_p$ is minimal for any $n \geq 1$ (see [1, p. 111], [2, Theorem 1.2] and [8, Corollary 4]). Then by Proposition 3.4, a criterion of the minimality of the system φ on $p\mathbb{Z}_p$ can be obtained.

Corollary 3.5. *The dynamical system $(p\mathbb{Z}_p, \varphi)$ is minimal if and only if*

- (1) *the finite system $(p\mathbb{Z}_p/p^3\mathbb{Z}_p, \varphi_3)$ is minimal for $p = 2$ or 3 ;*
- (2) *the finite system $(p\mathbb{Z}_p/p^2\mathbb{Z}_p, \varphi_2)$ is minimal for $p \geq 5$.*

As we investigate the induced dynamical systems level by level, the possible periods of φ on $p\mathbb{Z}_p$ can also be obtained, as a consequence of Proposition 3.4.

Corollary 3.6. *Let $\varphi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ with $\lambda_0 \in p\mathbb{Z}_p$ and consider the dynamical system $(p\mathbb{Z}_p, \varphi)$.*

- (1) *If $p > 3$, the period of a periodic orbit must be a factor of $p - 1$.*
- (2) *If $p = 3$, the period of a periodic orbit must be 1, 2 or 3.*
- (3) *If $p = 2$, the period of a periodic orbit must be 1 or 2.*

Furthermore, the dynamical structure of φ on $p\mathbb{Z}_p$ is fully illustrated by the following minimal decomposition.

Theorem 3.7 ([16], Theorem 1.1). *Let $\varphi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ with $\lambda_0 \in p\mathbb{Z}_p$. Suppose $\varphi^n \neq \text{id}$ for all $n \geq 1$. We have the following decomposition*

$$p\mathbb{Z}_p = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},$$

where \mathcal{P} is the finite set consisting of all periodic points of φ , $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each \mathcal{M}_i is a finite union of balls and each subsystem $\varphi : \mathcal{M}_i \rightarrow \mathcal{M}_i$ is minimal, and the points in \mathcal{B} lie in the attracting basin of $\mathcal{P} \sqcup \mathcal{M}$.

The sets \mathcal{M}_i in the above decomposition are called *minimal components*. To completely characterize the dynamical system $(p\mathbb{Z}_p, \varphi)$, each minimal component is described as the adding machine on an odometer by giving the structure sequence of the odometer.

Theorem 3.8 ([16], Theorem 1.1). *Let $\varphi(z) = \sum_{i=0}^{\infty} \lambda_i z^i \in \mathbb{Z}_p[[z]]$ with $\lambda_0 \in p\mathbb{Z}_p$. If $E \subset p\mathbb{Z}_p$ is a minimal clopen invariant set of φ , then $\varphi : E \rightarrow E$ is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$, where*

$$(p_s) = (\ell, \ell p, \ell p^2, \dots)$$

where $\ell \geq 1$ is an integer dividing $p - 1$.

4. PROOF OF THEOREMS 1.1–1.3

Proof of Theorem 1.1. We begin with proving the “if” part of the theorem. The reduction $\bar{\phi}$ is minimal on $\mathbb{P}^1(\mathbb{F}_p)$ which implies that ϕ_1 is minimal on \mathfrak{B}_1 . Let $\psi = \phi^{p+1}$ be the $p+1$ -th iteration of ϕ . Then the ball $B_1(0) = p\mathbb{Z}_p$ is an invariant set of ψ . By the argument of Section 3, ψ can be written as a power series with coefficients in \mathbb{Z}_p which converges on $p\mathbb{Z}_p$, i.e.

$$\psi(z) = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \dots$$

with $\lambda_i \in \mathbb{Z}_p$ for all $i \geq 1$ and $\lambda_0 \in p\mathbb{Z}_p$. Noting that ϕ is 1-Lipschitz, ψ is also 1-Lipschitz. The minimality of the dynamical system $(p\mathbb{Z}_p, \psi)$ leads to ψ is isometric on $p\mathbb{Z}_p$, see [3, 8]. So ϕ is isometric on $\mathbb{P}^1(\mathbb{Q}_p)$ with respect to the spherical metric. Hence,

ψ is minimal on each ball of radius $1/p$ which implies the minimality of $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$. It suffices to show that the dynamical system $(p\mathbb{Z}_p, \psi)$ is minimal.

For $n = 1$, the cycle (0) is the unique cycle of ψ_1 . The condition $(\phi^{(p+1)})'(0) \equiv 1 \pmod{p}$ implies that the cycle (0) grows or splits. By the condition $|\phi^{(p+1)}(0)|_p = 1/p$, the cycle (0) grows which then means that the dynamical system $(p\mathbb{Z}_p/p^2\mathbb{Z}_p, \psi_2)$ is minimal.

For the cases $p > 3$, it follows from Corollary 3.5 that the dynamical system $(p\mathbb{Z}_p, \psi)$ is minimal.

Note that $(\phi^{p(p+1)})'(0) \equiv (\phi^{(p+1)})'(0) \equiv 1 \pmod{p}$. For the cases $p = 2$ or 3 , the additional condition implies that the unique lift of (0) at level 2 grows. Thus the system $(p\mathbb{Z}_p/p^3\mathbb{Z}_p, \psi_n)$ is minimal. By Corollary 3.5, the dynamical system $(p\mathbb{Z}_p, \psi)$ is minimal.

Now we prove the ‘‘only if’’ part. Let ϕ be a minimal rational map with good reduction. By Proposition 3.1, the system (\mathfrak{B}_n, ϕ_n) is minimal for all integers $n \geq 1$. The minimality of system (\mathfrak{B}_1, ϕ_1) implies that the reduction $\bar{\phi}$ is minimal on $\mathbb{P}^1(\mathbb{F}_p)$, while the minimality of system (\mathfrak{B}_2, ϕ_2) implies that the cycle (0) of $\psi_1 = \phi_1^{p+1}$ grows. Hence we obtain $(\phi^{(p+1)})'(0) \equiv 1 \pmod{p}$ and $|\phi^{(p+1)}(0) - 0|_p = 1/p$. Furthermore, by the minimality of system (\mathfrak{B}_3, ϕ_3) , the unique lift of (0) of $\phi^{(p+1)}$ at level 2 grows. So we have $|\phi^{p(p+1)}(0)|_p = 1/p^2$. \square

In the above proof of Theorem 1.1, we have indeed established the following result.

Theorem 4.1. *Let $\phi : \mathbb{P}^1(\mathbb{Q}_p) \mapsto \mathbb{P}^1(\mathbb{Q}_p)$ be a rational map of $\deg \phi \geq 2$ with good reduction. Then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if the following condition is satisfied:*

- (1) *the system (\mathfrak{B}_3, ϕ_3) is minimal for $p = 2$ or 3 ,*
- (2) *the system (\mathfrak{B}_2, ϕ_2) is minimal for $p \geq 5$.*

Proof of Theorem 1.2. The space $\mathbb{P}^1(\mathbb{Q}_p)$ is a union of $p + 1$ balls with radius p^{-1} . Each ball can be identified with a point in $\mathbb{P}^1(\mathbb{F}_p)$. The reduction map $\bar{\phi}$ on $\mathbb{P}^1(\mathbb{F}_p)$ admits some cycles. By iteration of $\bar{\phi}$, the points outside the cycles are attracted by the cycles. The ball corresponding to such a point will be put into the third part \mathcal{B} in the minimal decomposition.

Let $(\bar{x}_1, \dots, \bar{x}_k) \subset \mathbb{P}^1(\mathbb{F}_p)$ be a cycle of $\bar{\phi}$. Without loss of generality, we assume $\bar{x}_1 = 0$. Let $\psi = \phi^k$ be the k -th iteration of ϕ . Then the ball $B_1(0) = p\mathbb{Z}_p$ is an invariant set of ψ . Noting that ϕ is a rational map of $\deg \phi \geq 2$, ϕ^n is a rational map of degree $(\deg \phi)^n$. So $\phi^n \neq id$ for all $n \geq 1$. By Theorem 3.7, the dynamical system $(p\mathbb{Z}_p, \psi)$ has a minimal decomposition which then gives a minimal decomposition of the system $(\bigsqcup_{i=1}^k B_1(x_i), \phi)$.

Applying the same argument to all the cycles of $\bar{\phi}$, we obtain the minimal decomposition of the whole system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$.

Moreover, by Corollary 3.6, the possible lengths of periodic orbits are obtained. \square

Proof of Theorem 1.3. Let k denote the length of the induced periodic orbit of ϕ on the minimal set E at the first level. Each point at this first level corresponds to a point in $\mathbb{P}^1(\mathbb{F}_p)$. So $k \leq p + 1$.

Without loss of generality, we assume that $0 \in E$. The set $B_1(0) \cap E$ is invariant by ϕ^k . Consider the dynamical system $(B_1(0) \cap E, \phi^k)$. Noting ϕ have good reduction, by (3.1) and Theorem 3.8, we deduce that $\phi^k : B_1(0) \cap E \rightarrow B_1(0) \cap E$ is conjugate to the

adding machine on the odometer $\mathbb{Z}_{(p_s)}$, where

$$(p_s) = (\ell, \ell p, \ell p^2, \dots)$$

with $\ell \mid (p-1)$, which implies $\phi : E \rightarrow E$ is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$, where

$$(p_s) = (k, k\ell, k\ell p, k\ell p^2, \dots).$$

□

5. MINIMAL RATIONAL MAP FOR THE CASE $p = 2$

5.1. Minimal Conditions. Let

$$\phi(z) = \frac{a_0 + a_1 z + \dots + a_{d-1} z^{d-1} + a_d z^d}{b_1 z + \dots + b_{d-1} z^{d-1} + b_d z^d}$$

be a rational map of degree $d \geq 2$ with $a_i, b_j \in \mathbb{Q}_2$ and $a_d = b_d = 1$. Recall that $A_\phi = \sum_{i \geq 0} a_i$, $B_\phi = \sum_{j \geq 1} b_j$, $A_{\phi,1} = \sum_{i \geq 0} a_{2i+1}$, $A_{\phi,2} = \sum_{i \geq 0} a_{4i+1}$ and $A_{\phi,3} = \sum_{i \geq 0} a_{4i+3}$.

Since $\phi(0) = \infty$, for the convenience of calculation, we shall select a suitable coordinate. Let

$$\psi(z) := \frac{1}{\phi(z)} = \frac{b_1 z + \dots + b_{d-1} z^{d-1} + b_d z^d}{a_0 + a_1 z + \dots + a_{d-1} z^{d-1} + a_d z^d} \quad (5.1)$$

and

$$\varphi(z) := \phi\left(\frac{1}{z}\right) = \frac{a_d + a_{d-1} z + \dots + a_1 z^{d-1} + a_0 z^d}{b_d + b_{d-1} z + \dots + b_1 z^{d-1}}. \quad (5.2)$$

Then we have $\phi^3 = \phi \circ \varphi \circ \psi$.

Lemma 5.1. *If ϕ has good reduction and (\mathfrak{B}_1, ϕ_1) is minimal, then*

$$\begin{cases} a_i, b_j \in \mathbb{Z}_2, \text{ for } 0 \leq i \leq d-1 \text{ and } 1 \leq j \leq d-1, \\ a_0 \equiv 1 \pmod{2}, \\ A_\phi \equiv 0 \pmod{2}, \\ B_\phi \equiv 1 \pmod{2}. \end{cases} \quad (5.3)$$

Conversely, the condition (5.3) implies that ϕ is 1-Lipschitz continuous and (\mathfrak{B}_1, ϕ_1) is minimal. Moreover, the Taylor expansion of ϕ^3 at 0 is of the form

$$\phi^3(z) = \phi(1) + \lambda z + \lambda_2 z^2 + \lambda_3 z^3 + \dots,$$

where $\lambda, \lambda_2, \lambda_3, \dots \in \mathbb{Z}_2$.

Proof. Assume that ϕ has good reduction, the coefficients a_i and b_j are in \mathbb{Z}_2 . Otherwise, the degree of the reduction map of ϕ is strictly less than d , which implies that ϕ has bad reduction.

Let

$$\bar{\phi}(z) = \frac{\bar{f}(z)}{\bar{g}(z)} = \frac{\bar{a}_0 + \bar{a}_1 z + \dots + \bar{a}_{d-1} z^{d-1} + z^d}{\bar{b}_1 z + \dots + \bar{b}_{d-1} z^{d-1} + z^d}$$

be the reduction of ϕ modulo p . The condition that ϕ has good reduction implies that the polynomials \bar{f}, \bar{g} have no common zero. As we have already indicated above, we can assume $\phi(0) = \infty$ and $\phi(\infty) = 1$. So we have $a_0 \equiv 1 \pmod{2}$. The minimality of the system (\mathfrak{B}_1, ϕ_1) means that $\bar{\phi}(1) = 0$. So we have $\bar{f}(1) = 0$ which implies $A_\phi \equiv 0 \pmod{2}$. Since the polynomials \bar{f}, \bar{g} have no common zero, we have $\bar{g}(1) \neq 0$ which means that $B_\phi \equiv 1 \pmod{2}$.

Now we assume that the coefficients of ϕ satisfy the condition (5.3). By (5.3), we can check directly that (\mathfrak{B}_1, ϕ_1) is minimal.

By Lemma 3.2, the conditions $A_\phi \equiv 0 \pmod{2}$, and $B_\phi \equiv 1 \pmod{2}$ imply that $\phi(z+1)$ is 1-Lipschitz from $p\mathbb{Z}_p$ to $p\mathbb{Z}_p$ and can be written as the form of (3.1). Recall that $\phi^3 = \phi \circ \varphi \circ \psi$ and note that $\psi(0) = 0$ and $\varphi(0) = 1$. It suffices to prove that ψ and $\varphi - 1$ are both 1-Lipschitz from $p\mathbb{Z}_p$ to itself and have the form (3.1). These can be obtained by direct calculations and by applying Lemma 3.2. \square

In the following lemma, we will calculate the derivative of ϕ^3 at 0. Using the chain rule, we obtain

$$(\phi^3)'(0) = \psi'(0) \cdot \varphi'(0) \cdot \phi'(1). \quad (5.4)$$

For simplicity, we denote $A'_\phi := \sum_{i \geq 1} ia_i$ and $B'_\phi := \sum_{i \geq 1} ib_i$. By calculation, we have

$$\eta_1 := \psi'(0) = b_1/a_0,$$

$$\eta_2 := \varphi'(0) = a_{d-1} - b_{d-1},$$

and

$$\eta := \phi'(1) = \frac{A'_\phi B_\phi - B'_\phi A_\phi}{B_\phi^2}.$$

Lemma 5.2. *Assume that the rational map ϕ has good reduction. Then the system (\mathfrak{B}_2, ϕ_2) is minimal if and only if*

$$\left\{ \begin{array}{l} a_i, b_j \in \mathbb{Z}_2, \text{ for } 0 \leq i \leq d-1 \text{ and } 1 \leq j \leq d-1, \\ a_0 \equiv 1 \pmod{2}, \\ B_\phi \equiv 1 \pmod{2}, \\ A_\phi \equiv 2 \pmod{4} \\ A_{\phi,1} \equiv 1 \pmod{2}, \\ b_1 \equiv 1 \pmod{2}, \\ a_{d-1} - b_{d-1} \equiv 1 \pmod{2}. \end{array} \right.$$

Proof. The rational map ϕ has good reduction and (\mathfrak{B}_2, ϕ_2) is minimal implies that (\mathfrak{B}_1, ϕ_1) is also minimal. So $p\mathbb{Z}_p$ is ϕ^3 -invariant. By the classification of the lifts of the cycles, (\mathfrak{B}_2, ϕ_2) is minimal if and only if

$$(\mathfrak{B}_1, \phi_1) \text{ is minimal, } (\phi^3)'(0) \equiv 1 \pmod{2}, \text{ and } \phi^3(0)\phi(1) \equiv 2 \pmod{4} \quad (5.5)$$

under the condition that ϕ has good reduction.

By Lemma 5.1, (\mathfrak{B}_1, ϕ_1) is minimal if and only if $a_0 \equiv 1 \pmod{2}$, $B_\phi \equiv 1 \pmod{2}$ and $A_\phi \equiv 0 \pmod{2}$.

Recall that

$$(\phi^3)'(0) = \psi'(0) \cdot \varphi'(0) \cdot \phi'(1) = \frac{b_1}{a_0} (a_{d-1} - b_{d-1}) \frac{A'_\phi B_\phi - B'_\phi A_\phi}{B_\phi^2}.$$

The second condition $(\phi^3)'(0) \equiv 1 \pmod{2}$ of (5.5) implies that

$$b_1 \equiv 1 \pmod{2},$$

$$(a_{d-1} - b_{d-1}) \equiv 1 \pmod{2},$$

and

$$A'_\phi \equiv 1 \pmod{2}.$$

Since

$$A'_\phi = \sum_{i \geq 0} (2i+1)a_{2i+1} + \sum_{i \geq 1} 2ia_{2i} \equiv \sum_{i \geq 0} a_{2i+1} \pmod{2},$$

we have $A_{\phi,1} \equiv 1 \pmod{2}$.

By the last condition $\phi(1) \equiv 2 \pmod{4}$ in (5.5), we immediately get

$$A_\phi \equiv 2 \pmod{4}.$$

□

For the proof of Theorem 1.4, we need calculate the second derivative of ϕ^3 at 0. For simplicity, denote $A''_\phi := \sum_{i \geq 2} i(i-1)a_i$ and $B''_\phi := \sum_{i \geq 2} i(i-1)b_i$. Before the proof let us first calculate the second derivatives $\psi''(0)$, $\varphi''(0)$ and $\phi''(1)$:

$$\begin{aligned} \xi_1 := \psi''(0) &= \frac{2b_2a_0^2 - 2a_1b_1a_0}{a_0^3}, \\ \xi_2 := \varphi''(0) &= 2(a_{d-2} - b_{d-2}) + 2(b_{d-1}^2 - a_{d-1}b_{d-1}), \\ \xi := \phi''(1) &= \frac{A''_\phi B_\phi^2 - B''_\phi A_\phi B_\phi + 2(A_\phi (B'_\phi)^2 - A'_\phi B'_\phi B_\phi)}{B_\phi^3}. \end{aligned}$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Under the condition that ϕ has good reduction, Theorem 4.1 implies that $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if (\mathfrak{B}_3, ϕ_3) is minimal which is equivalent to the following three conditions

$$(\mathfrak{B}_2, \phi_2) \text{ is minimal, } (\phi^6)'(0) \equiv 1 \pmod{2}, \text{ and } \phi^6(0) \equiv 4 \pmod{8}. \quad (5.6)$$

In the following, we will characterize the three conditions of (5.6).

Firstly, the minimality of the system (\mathfrak{B}_2, ϕ_2) has already been characterized by coefficients in Lemma 5.2. Then we obtain all the conditions in (1.2) except the last one.

Secondly, if the first condition of (5.6) is satisfied then by (5.5),

$$(\phi^6)'(0) \equiv ((\phi^3)'(0))^2 \equiv 1 \pmod{2}.$$

Thus the second condition of (5.6) is in fact included in the first condition of (5.6).

Finally, we are going to investigate the relation among the coefficients by using the last condition $\phi^6(0) \equiv 4 \pmod{8}$ of (5.6) which will lead to the last condition of (1.2).

Note that all the Taylor's coefficients of ϕ^3 expanded at $z = 0$ belong to \mathbb{Z}_2 . Hence for $z \in 2\mathbb{Z}_2$, we have

$$\phi^3(z) \equiv \phi(1) + (\phi^3)'(0)z + \frac{(\phi^3)''(0)}{2}z^2 \pmod{8}.$$

Thus

$$\phi^6(0) \equiv \phi(1) + (\phi^3)'(0)\phi(1) + \frac{(\phi^3)''(0)}{2}\phi(1)^2 \pmod{8}. \quad (5.7)$$

Notice that $\phi(1) \equiv 2 \pmod{4}$ and $(\phi^3)'(0) \equiv 1 \pmod{2}$. Then dividing both sides of (5.7) by $\phi(1)$, we deduce that the condition $\phi^6(0) \equiv 4 \pmod{8}$ is equivalent to

$$1 + (\phi^3)'(0) + \frac{(\phi^3)''(0)}{2}\phi(1) \equiv 2 \pmod{4},$$

which in turn, by noting that $(\phi^3)''(0)/2 \in \mathbb{Z}_2$, is equivalent to

$$(\phi^3)'(0) + (\phi^3)''(0) \equiv 1 \pmod{4}. \quad (5.8)$$

To continue, let us calculate $(\phi^3)'(0) \pmod{4}$, $(\phi^3)''(0) \pmod{4}$ and simplify their expressions. Using $a_0 \equiv 1 \pmod{2}$ and $B_\phi \equiv 1 \pmod{2}$, we have $1/a_0 \equiv a_0 \pmod{4}$ and $B_\phi^2 \equiv 1 \pmod{4}$. Then

$$(\phi^3)'(0) \equiv a_0 b_1 (a_{d-1} - b_{d-1}) (A'_\phi B_\phi - A_\phi B'_\phi) \pmod{4}.$$

Since $A_\phi \equiv 2 \pmod{4}$ and $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod{2}$, we have

$$(\phi^3)'(0) \equiv a_0 b_1 (b_{d-1} - b_{d-1}) A'_\phi B_\phi - A_\phi B'_\phi \pmod{4}.$$

For the second derivative of ϕ^3 at 0, by the chain rule, we have

$$(\phi^3)''(0) = \eta \eta_2 \xi_1 + \eta \eta_1^2 \xi_2 + \xi \eta_1^2 \eta_2^2. \quad (5.9)$$

Since $\eta \equiv \eta_1 \equiv \eta_2 \equiv 1 \pmod{2}$, we have $\eta^2 \equiv \eta_1^2 \equiv \eta_2^2 \equiv 1 \pmod{4}$. Consequently, we obtain

$$(\phi^3)''(0) \equiv \eta \eta_2 \xi_1 + \eta \xi_2 + \xi \pmod{4}. \quad (5.10)$$

Apply the expressions of ξ , ξ_1 and ξ_2 to Equation (5.10). Using the facts that $a_0 \equiv b_1 \equiv a_{d-1} - b_{d-1} \equiv 1 \pmod{2}$ and $B_\phi \equiv 1 \pmod{2}$, we deduce

$$\begin{aligned} (\phi^3)''(0) &\equiv 2\eta \eta_2 (a_0 b_2 - a_1 b_1) + 2\eta (a_{d-2} - b_{d-2} + b_{d-1}^2 - a_{d-1} b_{d-1}) + \\ &\quad + A''_\phi B_\phi - 2A'_\phi B'_\phi - A_\phi B''_\phi \pmod{4} \\ &\equiv 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1}) + A''_\phi B_\phi - 2A'_\phi B'_\phi - A_\phi B''_\phi \pmod{4}. \end{aligned}$$

Observing that $B''_\phi \equiv 0 \pmod{2}$ and $A_\phi \equiv 0 \pmod{2}$, we have

$$\begin{aligned} &(\phi^3)'(0) + (\phi^3)''(0) \\ &\equiv a_0 b_1 (a_{d-1} - b_{d-1}) A'_\phi B_\phi - A_\phi B'_\phi + 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1}) \\ &\quad + A''_\phi B_\phi - 2A'_\phi B'_\phi \pmod{4}. \end{aligned}$$

Since $A_\phi \equiv 2 \pmod{4}$ and $A'_\phi \equiv A_{\phi,1} \equiv 1 \pmod{2}$, we get

$$\begin{aligned} &(\phi^3)'(0) + (\phi^3)''(0) \\ &\equiv a_0 b_1 (a_{d-1} - b_{d-1}) A'_\phi B_\phi + 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1}) + A''_\phi B_\phi \pmod{4}. \end{aligned}$$

Note that

$$A'_\phi \equiv \sum_{i \geq 0} a_{4i+1} + 2 \sum_{i \geq 0} a_{4i+2} - \sum_{i \geq 0} a_{4i+3} \pmod{4}$$

and

$$A''_\phi \equiv 2 \sum_{i \geq 0} a_{4i+2} + 2 \sum_{i \geq 0} a_{4i+3} \pmod{4}.$$

Then

$$\begin{aligned} &(\phi^3)'(0) + (\phi^3)''(0) \\ &\equiv a_0 b_1 (a_{d-1} - b_{d-1}) (A_{\phi,2} - A_{\phi,3}) B_\phi + 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1}) \\ &\quad + 2 \sum_{i \geq 0} a_{4i+2} + 2 \left(\sum_{i \geq 0} a_{4i+2} + A_{\phi,3} \right) \pmod{4} \\ &\equiv a_0 b_1 (a_{d-1} - b_{d-1}) (A_{\phi,2} - A_{\phi,3}) B_\phi \\ &\quad + 2(b_2 - a_1 + a_{d-2} - b_{d-2} + b_{d-1} + A_{\phi,3}) \pmod{4}. \end{aligned}$$

By (5.8), we obtain the last condition of (1.2). Hence, we obtain the first part of Theorem 1.4.

For the other part of Theorem 1.4, note that the first four conditions of (1.2) imply the 1-Lipschitz continuity of ϕ . Further, ϕ^3 can be written as a form of power series with coefficients in \mathbb{Z}_p . Thus from the proof of Theorem 1.1, we deduce that the minimality of ϕ is equivalent to the conditions (5.6). Hence the above proof of the first part leads to the conclusion of the second part of the theorem. \square

Proof of Corollary 1.5. Remark that the good reduction property and minimality of ϕ imply that ϕ is isometric on $\mathbb{P}^1(\mathbb{Q}_2)$.

For a rational map ϕ of degree 2 which has good reduction and is minimal on $\mathbb{P}^1(\mathbb{Q}_2)$, each point has either 0 or 2 pre-images since ϕ can not have critical point in $\mathbb{P}^1(\mathbb{Q}_2)$. This contradicts to the fact that ϕ is isometric.

For a rational map ϕ of degree 3 which is defined as (1.1), the good reduction property and minimality give the reduction $\bar{\phi}(z) = \frac{(z-1)f(z)}{zg(z)}$, where $f, g \in \mathbb{F}_2[z]$ are two quadratic irreducible polynomials. However, $1+z+z^2$ is the unique quadratic irreducible polynomial over \mathbb{F}_2 . Hence, $f = g$, which contradicts that ϕ has good reduction.

For a rational map ϕ of degree 4 which is defined as (1.1), the good reduction property and minimality give the reduction $\bar{\phi}(z) = \frac{(z-1)f(z)}{zg(z)}$, with $f, g \in \mathbb{F}_2[z]$ being two different cubic irreducible polynomials. It is known that $1+z+z^3$ and $1+z^2+z^3$ are the only two cubic irreducible polynomials over \mathbb{F}_2 . Hence, $\bar{\phi}(z)$ can only have two cases:

$$\bar{\phi}(z) = \frac{(z-1)(1+z+z^3)}{z(1+z^2+z^3)} \quad \text{or} \quad \bar{\phi}(z) = \frac{(z-1)(1+z^2+z^3)}{z(1+z+z^3)}.$$

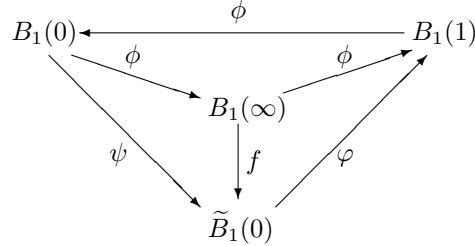
By simple calculations, in both cases, $a_3 - b_3 \equiv 0 \pmod{2}$, which contradicts the seventh equation in (1.2). \square

5.2. Minimal rational maps of degree 4. This section is devoted to investigate the minimal rational maps of degree 4 which are 1-Lipschitz continuous and their induced orbits on \mathfrak{B}_3 .

For a rational map

$$\phi(z) = \frac{a_0 + a_1z + \cdots + a_{d-1}z^{d-1} + a_dz^d}{b_1z + \cdots + b_{d-1}z^{d-1} + b_dz^d}$$

of degree d with $a_i, b_j \in \mathbb{Q}_2$ and $a_d = b_d = 1$. Assume that ϕ 1-Lipschitz continuous and minimal on $\mathbb{P}^1(\mathbb{Q}_2)$. The orbit of ϕ_1 on \mathfrak{B}_1 is $0 \rightarrow \infty \rightarrow 1 \rightarrow 0$. For the convenience of investigating the induced orbit on \mathfrak{B}_n with $n \geq 2$, we choose the local coordinate around ∞ by $f(z) = 1/z$. Denote by $\tilde{B}_1(0)$ the image of $B_1(\infty)$ under f . we have the following communicating graph.



Actually, the two balls $B_1(0)$ and $\tilde{B}_1(0)$ have no difference. To avoid confusion, the elements in $\tilde{B}_1(0)$ are denoted by \tilde{z} , such as $\tilde{1}, \tilde{2}$.

Let $\widetilde{\mathbb{P}^1}(\mathbb{Q}_2)$ be the disjoint union of $B_1(0)$, $\widetilde{B}_1(0)$ and $B_1(1)$, i.e.

$$\widetilde{\mathbb{P}^1}(\mathbb{Q}_2) = B_1(0) \sqcup \widetilde{B}_1(0) \sqcup B_1(1).$$

Then we can define a map $\widetilde{\phi}$ from $\widetilde{\mathbb{P}^1}(\mathbb{Q}_2)$ to itself as follows:

$$\widetilde{\phi}(z) := \begin{cases} \widetilde{\psi}(z), & \text{if } z \in B_1(0); \\ \varphi(z), & \text{if } z \in \widetilde{B}_1(0); \\ \phi(z), & \text{if } z \in B_1(1). \end{cases}$$

Instead of investigating the dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$, we study the dynamical system $(\widetilde{\mathbb{P}^1}(\mathbb{Q}_2), \widetilde{\phi})$, so that we can do modulo calculation by computer. We obtain all rational maps of degree 4 with coefficients in $\{0, 1, 2, 3\}$ which have good reduction and are minimal on $\mathbb{P}^1(\mathbb{Q}_p)$. We associate each of the 12 vertices at level 3 a label

$$\begin{cases} i, & \text{where } i \in \{0, 1, 2, \dots, 7\}, \text{ if the vertex corresponds to the ball } B_n(i), \\ \widetilde{i}, & \text{where } i \in \{0, 2, 4, 6\}, \text{ if the vertex corresponds to the ball } B_n(1/i). \end{cases}$$

The rational maps and their induced orbits at level 3 are showed in the following table.

| Coefficients $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ | Induced periodic orbits at level 3 |
|--|--|
| $1, 0, 1, 3, 3, 1, 0$ $1, 1, 1, 2, 3, 2, 3$ $1, 2, 1, 1, 3, 3, 2$ $1, 2, 3, 3, 3, 3, 0$ $1, 3, 1, 0, 3, 0, 1$ $1, 3, 3, 2, 3, 0, 3$ $3, 2, 1, 3, 1, 1, 0$ $3, 3, 1, 2, 1, 2, 3$ | $0 \rightarrow \widetilde{0} \rightarrow 1 \rightarrow 6 \rightarrow \widetilde{6} \rightarrow 3$ $\rightarrow 4 \rightarrow \widetilde{4} \rightarrow 5 \rightarrow 2 \rightarrow \widetilde{2} \rightarrow 7$ |
| $1, 0, 1, 3, 1, 1, 2$ $1, 1, 1, 2, 1, 2, 1$ $1, 2, 1, 1, 1, 3, 0$ $1, 2, 3, 3, 1, 3, 2$ $1, 3, 1, 0, 1, 0, 3$ $1, 3, 3, 2, 1, 0, 1$ $3, 2, 1, 3, 3, 1, 2$ $3, 3, 1, 2, 3, 2, 1$ | $0 \rightarrow \widetilde{0} \rightarrow 1 \rightarrow 6 \rightarrow \widetilde{2} \rightarrow 3$ $\rightarrow 4 \rightarrow \widetilde{4} \rightarrow 5 \rightarrow 2 \rightarrow \widetilde{6} \rightarrow 7$ |
| $1, 0, 1, 3, 1, 1, 0$ $1, 1, 1, 2, 1, 2, 3$ $1, 2, 1, 1, 1, 3, 2$ $1, 2, 3, 3, 1, 3, 0$ $1, 3, 1, 0, 1, 0, 1$ $1, 3, 3, 2, 1, 0, 3$ $3, 2, 1, 3, 3, 1, 0$ $3, 3, 1, 2, 3, 2, 3$ | $0 \rightarrow \widetilde{0} \rightarrow 1 \rightarrow 2 \rightarrow \widetilde{6} \rightarrow 3$ $\rightarrow 4 \rightarrow \widetilde{4} \rightarrow 5 \rightarrow 6 \rightarrow \widetilde{2} \rightarrow 7$ |

| Coefficients $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ | Induced periodic orbits at level 3 |
|--|--|
| 1, 0, 1, 3, 3, 1, 2 1, 1, 1, 2, 3, 2, 1 1, 2, 1, 1, 3, 3, 0 1, 2, 3, 3, 3, 3, 2 1, 3, 1, 0, 3, 0, 3 1, 3, 3, 2, 3, 0, 1 3, 2, 1, 3, 1, 1, 2 3, 3, 1, 2, 1, 2, 1 | $0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 2 \rightarrow \tilde{2} \rightarrow 3$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 6 \rightarrow \tilde{6} \rightarrow 7$ |
| 1, 0, 3, 1, 3, 1, 0 1, 1, 3, 0, 3, 2, 3 3, 0, 1, 1, 1, 3, 0 3, 0, 3, 3, 1, 3, 2 3, 1, 1, 0, 1, 0, 3 3, 1, 3, 2, 1, 0, 1 3, 2, 3, 1, 1, 1, 0 3, 3, 3, 0, 1, 2, 3 | $0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 6 \rightarrow \tilde{6} \rightarrow 7$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 2 \rightarrow \tilde{2} \rightarrow 3$ |
| 1, 0, 3, 1, 1, 1, 2 1, 1, 3, 0, 1, 2, 1 3, 0, 1, 1, 3, 3, 2 3, 0, 3, 3, 3, 3, 0 3, 1, 1, 0, 3, 0, 1 3, 1, 3, 2, 3, 0, 3 3, 2, 3, 1, 3, 1, 2 3, 3, 3, 0, 3, 2, 1 | $0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 6 \rightarrow \tilde{2} \rightarrow 7$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 2 \rightarrow \tilde{6} \rightarrow 3$ |
| 1, 0, 3, 1, 1, 1, 0 1, 1, 3, 0, 1, 2, 3 3, 0, 1, 1, 3, 3, 0 3, 0, 3, 3, 3, 3, 2 3, 1, 1, 0, 3, 0, 3 3, 1, 3, 2, 3, 0, 1 3, 2, 3, 1, 3, 1, 0 3, 3, 3, 0, 3, 2, 3 | $0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 2 \rightarrow \tilde{6} \rightarrow 7$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 6 \rightarrow \tilde{2} \rightarrow 3$ |
| 1, 0, 3, 1, 3, 1, 2 1, 1, 3, 0, 3, 2, 1 3, 0, 1, 1, 1, 3, 2 3, 0, 3, 3, 1, 3, 0 3, 1, 1, 0, 1, 0, 1 3, 1, 3, 2, 1, 0, 3 3, 2, 3, 1, 1, 1, 2 3, 3, 3, 0, 1, 2, 1 | $0 \rightarrow \tilde{0} \rightarrow 1 \rightarrow 2 \rightarrow \tilde{2} \rightarrow 7$ $\rightarrow 4 \rightarrow \tilde{4} \rightarrow 5 \rightarrow 6 \rightarrow \tilde{6} \rightarrow 3$ |

6. SOME EXAMPLES

Example 6.1. Let $p = 3$ and $\phi(x) = -\frac{2z^2+2z+1}{z^3-3z^2+z+1}$. The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal.

The reduction mod 3 of ϕ is

$$\bar{\phi}(z) = \frac{z^2 + z + 2}{z^3 + z + 1}.$$

So ϕ has good reduction. Consider the map $\bar{\phi} : \mathbb{P}^1(\mathbb{F}_3) \rightarrow \mathbb{P}^1(\mathbb{F}_3)$. By a simple calculation, we get $\bar{\phi}(1) = \infty$, $\bar{\phi}(\infty) = 0$, $\bar{\phi}(0) = 2$, $\bar{\phi}(2) = 1$. Thus $\bar{\phi}$ is transitive on $\mathbb{P}^1(\mathbb{F}_3)$. Further, using the software Mathematica, we can check $(\phi^4)'(1) \equiv (\phi^4)'(0) \equiv 1 \pmod{3}$. Let $\sigma = (1, \infty, 0, -1)$ be the cycle at level 1. Since $\phi^4(1) \equiv 7 \pmod{3^3}$, σ grows. Let $\hat{\sigma}$ be the lift of σ at level 2. By calculating $\phi^{12}(1) \equiv 10 \pmod{3^3}$, we deduce that $\hat{\sigma}$ grows which then implies that $\hat{\sigma}$ grows forever. Therefore, the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal.

Example 6.2. Let $p = 3$ and $\phi(z) = \frac{2z+3}{(z-1)(z-2)}$. The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is not minimal and we decompose $\mathbb{P}^1(\mathbb{Q}_p)$ as

$$\mathbb{P}^1(\mathbb{Q}_p) = B_1(0) \bigsqcup (\mathbb{P}^1(\mathbb{Q}_p) \setminus B_1(0)),$$

where $B_1(0)$ is a minimal component of ϕ and the points in $\mathbb{P}^1(\mathbb{Q}_p) \setminus B_1(0)$ are attracted into $B_1(0)$.

One can check that $\bar{\phi}(z) = \frac{2z}{z^2+2}$. Then $\deg \bar{\phi} = \deg \phi$, which implies that ϕ has good reduction. First, we consider the map $\bar{\phi} : \mathbb{P}^1(\mathbb{F}_3) \rightarrow \mathbb{P}^1(\mathbb{F}_3)$. Since $\bar{\phi}(1) = \infty$, $\bar{\phi}(2) = \infty$, $\bar{\phi}(\infty) = 0$ and $\bar{\phi}(0) = 0$, it follows that

$$\begin{aligned} \phi(B_1(1)) &\subset B_1(\infty), \phi(B_1(2)) \subset B_1(\infty), \\ \phi(B_1(\infty)) &\subset B_1(0) \text{ and } \phi(B_1(0)) \subset B_1(0). \end{aligned}$$

So $\mathbb{P}^1(\mathbb{Q}_p) \setminus B_1(0)$ lies in the attracting basin of $B_1(0)$. It suffices to study the subsystem $(B_1(0), \phi)$. We shall show that the system $(B_1(0), \phi)$ is minimal. In fact, the derivative of ϕ at 0 is $13/4$. Thus $\phi'(0) \equiv 1 \pmod{3}$. Let $\sigma = (0)$ be the cycle at level 1. Then σ grows or splits. A simple calculation shows that

$$\phi(0) \equiv 15 \pmod{3^3}, \phi(15) \equiv 3 \pmod{3^3}, \phi(3) \equiv 18 \pmod{3^3}.$$

So σ grows, by Proposition 3.4, the lift of σ grows too. It then follows that σ grows forever. Hence, the system $(B_1(0), \phi)$ is minimal.

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SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL CHINA NORMAL UNIVERSITY, 430079, WUHAN, CHINA & LAMFA, UMR 7352, CNRS, UNIVERSITÉ DE PICARDIE JULES VERNE, 33, RUE SAINT LEU, 80039 AMIENS CEDEX 1, FRANCE

E-mail address: ai-hua.fan@u-picardie.fr

SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL CHINA NORMAL UNIVERSITY, 430079, WUHAN, CHINA

E-mail address: slfan@mail.ccnu.edu.cn

LAMA, UMR 8050, CNRS, UNIVERSITÉ PARIS-EST CRÉTEIL, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX, FRANCE

E-mail address: lingmin.liao@u-pec.fr

INSTITUTE OF MATHEMATICS, AMSS, CHINESE ACADEMY OF SCIENCES, 100190 BEIJING, CHINA.

E-mail address: wangyf@math.ac.cn