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# Uniform boundedness and long-time asymptotics for the two-dimensional Navier-Stokes equations in an infinite cylinder

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## Abstract

The incompressible Navier-Stokes equations are considered in the two-dimensional strip  $\mathbb{R} \times [0, L]$ , with periodic boundary conditions and no exterior forcing. If the initial velocity is bounded, it is shown that the solution remains uniformly bounded for all time, and that the vorticity distribution converges to zero as  $t \rightarrow \infty$ . This implies, after a transient period, the emergence of a laminar regime in which the solution rapidly converges to a shear flow described by the one-dimensional heat equation in an appropriate Galilean frame. The approach is constructive and provides explicit estimates on the size of the solution and the lifetime of the turbulent period in terms of the initial Reynolds number.

## 1 Introduction

We are interested in understanding the dynamics of the incompressible Navier-Stokes equations in large or unbounded spatial domains. In particular, for initial data with bounded energy density, we would like to estimate the kinetic energy of the solution in a small subdomain at a given time, independently of the total initial energy which may be infinite if the domain is unbounded. In other words, we are looking for *uniformly local* energy estimates that would control how much energy can be transferred from one region to another in the system.

This question is already interesting in the relatively simple situation where the fluid is supposed to evolve in a bounded two-dimensional domain  $\Omega \subset \mathbb{R}^2$ , with no-slip boundary condition and no exterior forcing. In that case, if the initial data are bounded, it is well known that the solution of the Navier-Stokes equations is globally defined in the energy space and converges to zero, at an exponential rate, as  $t \rightarrow +\infty$  [7]. This certainly implies that the fluid velocity  $u(x, t)$  is uniformly bounded for all time, but all estimates we are aware of depend on the size of the domain  $\Omega$  or on the total initial energy, and not only on the initial energy *density*. Indeed, although the total energy of the fluid is a decreasing function of time, the fluid velocity  $u(x, t)$  may temporarily increase in some regions due to energy redistribution in the system.

To control these fluctuations, it is rather natural to begin with the idealized situation where the Navier-Stokes equations are considered in the whole space  $\mathbb{R}^2$ , with initial data that are merely bounded. In that case, it is possible to prove the existence of a unique global solution [14], provided the pressure is defined in an appropriate way [17]. The corresponding velocity  $u(x, t)$  belongs to  $L^\infty(\mathbb{R}^2)$  for all  $t \geq 0$ , but it is not known whether the norm  $\|u(\cdot, t)\|_{L^\infty}$  stays

uniformly bounded for all time. Early results gave pessimistic estimates on that quantity [14, 20], but a substantial progress was recently made by S. Zelik [23] who showed that  $\|u(\cdot, t)\|_{L^\infty}$  cannot grow faster than  $t^2$  as  $t \rightarrow \infty$ , see Section 7 for a more precise statement. Still, we do not have any example of unbounded solution, and it is therefore unclear whether the above result is optimal.

The aim of the present paper is to address these issues in the the simplified setting where the fluid velocity and the pressure are supposed to be *periodic* in one space direction. In other words, we consider the incompressible Navier-Stokes equations in the two-dimensional strip  $\Omega_L = \mathbb{R} \times [0, L]$ , with periodic boundary conditions. Our system reads

$$\partial_t u + (u \cdot \nabla)u = \nu \Delta u - \frac{1}{\rho} \nabla p, \quad \operatorname{div} u = 0, \quad (1.1)$$

where  $u(x, t) \in \mathbb{R}^2$  is the velocity field and  $p(x, t) \in \mathbb{R}$  the associated pressure. We denote the space variable by  $x = (x_1, x_2)$ , where  $x_1 \in \mathbb{R}$  will be called the “horizontal” coordinate and  $x_2 \in [0, L]$  the “vertical” coordinate. The physical parameters in (1.1) are the kinematic viscosity  $\nu > 0$  and the fluid density  $\rho > 0$ , which are both supposed to be constant. Besides the pressure, an important quantity derived from the velocity  $u$  is the vorticity  $\omega = \partial_1 u_2 - \partial_2 u_1$ , which satisfies the advection-diffusion equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega. \quad (1.2)$$

As was already explained, we want to consider infinite-energy solutions of (1.1), for which the velocity field is merely bounded. We thus assume that, for any  $t \geq 0$ , the velocity  $u(\cdot, t)$  belongs to  $\text{BUC}(\Omega_L)$ , the space of all bounded and uniformly continuous functions  $u : \Omega_L \rightarrow \mathbb{R}^2$  that satisfy the periodicity condition  $u(x_1, 0) = u(x_1, L)$  for all  $x_1 \in \mathbb{R}$ . It is clear that  $\text{BUC}(\Omega_L)$  is a Banach space when equipped with the uniform norm

$$\|u\|_{L^\infty(\Omega_L)} = \sup_{x \in \Omega_L} |u(x)|, \quad \text{where } |u| = (u_1^2 + u_2^2)^{1/2}.$$

If  $u \in \text{BUC}(\Omega_L)$  is divergence-free and if the associated vorticity  $\omega$  is bounded, one can show that the elliptic equation  $-\Delta p = \rho \operatorname{div}(u \cdot \nabla)u$  has a *bounded* solution  $p : \Omega_L \rightarrow \mathbb{R}$  such that  $p(x_1, 0) = p(x_1, L)$  for all  $x_1 \in \mathbb{R}$ . Moreover, there exists  $C > 0$  such that

$$\|p\|_{L^\infty(\Omega_L)} \leq C \rho L^2 \|\omega\|_{L^\infty(\Omega_L)}^2,$$

see [13, Lemma 2.3] and Section 2.4 below. This is the canonical definition of the pressure, which coincides (up to a constant) with the choice made in [14, 17] for arbitrary solutions of (1.1) in  $\text{BUC}(\mathbb{R}^2)$ . In the nonperiodic case, however, the pressure is only known to belong to  $\text{BMO}(\mathbb{R}^2)$ .

Given divergence-free initial data  $u_0 \in \text{BUC}(\Omega_L)$  with associated vorticity distribution  $\omega_0$ , we introduce the Reynolds numbers

$$R_u = \frac{L}{\nu} \|u_0\|_{L^\infty(\Omega_L)}, \quad R_\omega = \frac{L^2}{\nu} \|\omega_0\|_{L^\infty(\Omega_L)}. \quad (1.3)$$

The following (already known) result shows that the Cauchy problem for equation (1.1) is globally well-posed in the space  $\text{BUC}(\Omega_L)$ .

**Theorem 1.1.** [14, 1, 13] *For any initial data  $u_0 \in \text{BUC}(\Omega_L)$  with  $\operatorname{div} u_0 = 0$ , the Navier-Stokes equations (1.1) with the canonical choice of the pressure have a unique global mild solution  $u \in C^0([0, +\infty), \text{BUC}(\Omega_L))$  such that  $u(0) = u_0$ . Moreover, there exists a constant  $C > 0$ , depending only on the initial Reynolds number  $R_u$ , such that*

$$\frac{L}{\nu} \|u(\cdot, t)\|_{L^\infty(\Omega_L)} \leq C \left(1 + \frac{\sqrt{\nu t}}{L}\right), \quad \text{for all } t \geq 0. \quad (1.4)$$

Existence of a unique global mild solution to (1.1) in  $BUC(\Omega_L)$  is ensured by the general results of Giga, Matsui, and Sawada [14], which apply to the Navier-Stokes equations in the whole plane  $\mathbb{R}^2$  with initial data in  $BUC(\mathbb{R}^2)$ . The specific situation where the flow is periodic in one space direction was considered by Afendikov and Mielke [1], with the motivation of understanding the transition to instability in Kolmogorov flows. In the particular case where no exterior forcing is applied, the results of [1] give an upper bound on  $\|u(\cdot, t)\|_{L^\infty}$  which grows linearly in time, and can be improved with little extra effort to provide estimate (1.4), see [13]. A further progress was made in [13], where the authors proved that  $\|u(\cdot, t)\|_{L^\infty}$  cannot grow faster than  $t^{1/6}$  as  $t \rightarrow \infty$ . Moreover, several results were obtained in [13, Theorem 1.3] which strongly suggest that solutions of (1.1) in  $\Omega_L$  should stay uniformly bounded for all time. For instance, for all  $T > 0$ , we have the following estimate

$$\sup_{x_1 \in \mathbb{R}} \int_0^T \int_0^L |u(x_1, x_2, t)|^2 dx_2 dt \leq C \frac{\nu^2 T}{L},$$

for some  $C > 0$  depending only on the initial Reynolds number  $R_u$ . In addition, for all  $R > 0$  and all  $T > 0$ , one finds

$$\int_{B_R} |u(x, T)|^2 dx + \nu \int_0^T \int_{B_R} |\nabla u(x, t)|^2 dx dt \leq C \frac{\nu^2}{L} (R + \sqrt{\nu T}), \quad (1.5)$$

where  $B_R = \{(x_1, x_2) \in \Omega_L \mid |x_1| \leq R\}$ . Estimate (1.5) shows in particular that the energy dissipation rate  $\nu |\nabla u(x, t)|^2$  converges to zero on average as  $t \rightarrow \infty$ , and this in turn implies that the solution  $u(x, t)$  approaches for “almost all” time the family of spatially homogeneous steady states of (1.1), uniformly on compact subsets of  $\Omega_L$ , see [13, Section 8] for a precise statement.

In this paper, we complement and substantially improve the results of [13] by showing that any solution of (1.1) in  $BUC(\Omega_L)$  remains *uniformly bounded* for all time, and converges as  $t \rightarrow \infty$  *uniformly on  $\Omega_L$*  to a simple shear flow of the form

$$u_\infty(x, t) = \begin{pmatrix} c \\ m(x_1, t) \end{pmatrix}, \quad p(x, t) = 0, \quad (1.6)$$

where  $c \in \mathbb{R}$  is a constant and  $m(x_1, t)$  is an approximate solution of the one-dimensional advection-diffusion equation  $\partial_t m + c \partial_1 m = \nu \partial_1^2 m$ . Our main result can be stated as follows:

**Theorem 1.2.** *For any divergence-free initial data  $u_0 \in BUC(\Omega_L)$  with bounded vorticity distribution  $\omega_0$ , the solution of the Navier-Stokes equations (1.1) given by Theorem 1.1 has the following properties:*

1. (Uniform boundedness of the velocity) *There exists  $C > 0$  such that, for all  $t \geq 0$ ,*

$$\frac{L}{\nu} \|u(\cdot, t)\|_{L^\infty(\Omega_L)} \leq C \left( R_u + R_\omega + (1 + R_\omega)(R_u^2 + R_\omega^2) \right), \quad (1.7)$$

where  $R_u, R_\omega$  are given by (1.3).

2. (Uniform decay of the vorticity) *There exists  $C > 0$  such that, for all  $t > 0$ ,*

$$\left( \frac{L^2}{\nu} \|\omega(\cdot, t)\|_{L^\infty(\Omega_L)} \right)^2 \leq C(1 + R_\omega)(R_u^2 + R_\omega^2) \frac{L}{\sqrt{\nu t}}. \quad (1.8)$$

3. (Exponential convergence to a shear flow) *For any  $\gamma < 2\pi^2$ , we have*

$$\frac{L}{\nu} \|u(\cdot, t) - u_\infty(\cdot, t)\|_{L^\infty(\Omega_L)} = \mathcal{O}\left(\exp\left(-\frac{\gamma \nu t}{L^2}\right)\right), \quad t \rightarrow \infty, \quad (1.9)$$

where  $u_\infty(x, t)$  is given by (1.6) with  $c \in \mathbb{R}$  and  $\partial_t m + c \partial_1 m - \nu \partial_1^2 m = \mathcal{O}(e^{-2\gamma \nu t/L^2})$  as  $t \rightarrow \infty$ .

**Remarks 1.3.**

1. The constant  $C$  in estimates (1.7), (1.8) is universal: the dependence of both members upon the initial data  $u_0$  and the physical parameters  $\nu, \rho, L$  is entirely explicit. Note also that inequalities (1.7)–(1.9) involve only dimensionless quantities, such as the initial Reynolds numbers  $R_u$  and  $R_\omega$ .

2. The assumption that the initial vorticity  $\omega_0 = \text{curl } u_0$  be bounded is by no means essential. Indeed, due to parabolic regularization, any solution of (1.1) given by Theorem 1.1 is smooth for positive time, and has a bounded vorticity distribution for  $t > 0$ . More quantitatively, according to the existence proof given in [13], there exists a positive constant  $C$  such that  $\|u(\cdot, t_0)\|_{L^\infty} \leq 2\|u_0\|_{L^\infty}$  and  $\nu\|\nabla u(\cdot, t_0)\|_{L^\infty} \leq C\|u_0\|_{L^\infty}^2$  if  $t_0 = C^{-2}\nu\|u_0\|_{L^\infty}^{-2}$ . The Reynolds numbers (1.3) computed at time  $t_0$  thus satisfy  $R_u(t_0) \leq 2R_u$  and  $R_\omega(t_0) \leq CR_u^2$ .

3. It is well-known that all bounded solutions of the linear equation  $\partial_t m + c\partial_1 m = \nu\partial_1^2 m$  converge uniformly on compact sets of  $\mathbb{R}$  toward the family of spatially homogeneous equilibria, because  $\|\partial_1 m(\cdot, t)\|_{L^\infty} = \mathcal{O}((\nu t)^{-1/2})$  as  $t \rightarrow \infty$ . The same result holds for the nonlinear equation satisfied by the average vertical flow  $m(x_1, t)$  in (1.6), because the additional term decays exponentially as  $t \rightarrow \infty$ . We thus deduce from (1.9) that all solutions of (1.1) given by Theorem 1.1 converge uniformly on compact sets of  $\Omega_L$  toward the family of spatially homogeneous equilibria of the form  $u = (c_1, c_2)^\top$ , with  $c_1, c_2 \in \mathbb{R}$ . It is interesting to note that this conclusion is *stronger* than what one typically expects for general extended dissipative systems, see [12]. It also follows from (1.9) that the Navier-Stokes equations in  $\text{BUC}(\Omega_L)$  have no other equilibria, and that nontrivial time-periodic solutions or recurrent orbits do not exist.

4. By the parabolic maximum principle, the vorticity bound  $\|\omega(\cdot, t)\|_{L^\infty}$  is nonincreasing with time, and (1.8) shows that this quantity converges to zero as  $t \rightarrow \infty$ . As we shall see in Section 6, when the vorticity Reynolds number  $L^2\|\omega(\cdot, t)\|_{L^\infty}/\nu$  becomes smaller than a universal constant (related to the Poincaré inequality), the system enters a laminar regime where the solution rapidly converges to a shear flow. Thus, for large initial data, we can identify two different stages in the evolution of the system: a long transient period, in which turbulence can develop, and an asymptotic laminar regime described by (1.9). In view of (1.8), the lifetime  $T$  of the turbulent period satisfies  $\nu T/L^2 \leq CR^6$  for some  $C > 0$ , where  $R = \max(R_u, R_\omega)$ .

5. Although our motivation for using periodic boundary conditions is to shed some light on the behavior of solutions to the Navier-Stokes equations in the whole plane  $\mathbb{R}^2$ , it is natural at this point to ask what happens if we consider (1.1) in the strip  $\Omega_L = \mathbb{R} \times [0, L]$  with other conditions at the boundary. If we assume that the velocity  $u$  vanishes on  $\partial\Omega$  (*no-slip boundary conditions*), then the solution of (1.1) decays exponentially to zero as  $t \rightarrow \infty$ , see [22, 3]. We thus have the analog of (1.9) with  $u_\infty = 0$ . Another interesting possibility is to suppose that  $u_2 = \partial_2 u_1 = 0$  on  $\partial\Omega$  (*perfect slip boundary conditions*). In that case, if we extend the solution  $u(x, t)$  to the larger strip  $\mathbb{R} \times [-L, L]$  in such a way that  $u_1$  (resp.  $u_2$ ) is an even (resp. odd) function of the vertical coordinate  $x_2$ , it is easy to verify that the extended velocity field satisfies periodic boundary conditions on  $\mathbb{R} \times [-L, L]$ . In addition, the vertical velocity is by construction an odd function of  $x_2$ , hence has a zero vertical average. It follows that (1.7), (1.8) hold, as well as (1.9) with  $u_\infty = (c, 0)^\top$  and  $\gamma < \pi^2/2$ . Finally, it is also possible to consider *Navier friction conditions*, but this intermediate case has not been studied so far, and our approach does not apply directly due to the lack of a priori estimate on the vorticity.

Combining the existence part in Theorem 1.1 with the uniform bounds and the decay estimates in Theorem 1.2, we obtain the following final result which summarizes the main properties of the solutions to the Navier-Stokes equations in the two-dimensional strip  $\Omega_L = \mathbb{R} \times [0, L]$  with periodic boundary conditions.

**Corollary 1.4.** *For any  $u_0 \in \text{BUC}(\Omega_L)$  with  $\text{div } u_0 = 0$ , the Navier-Stokes equations (1.1) with the canonical choice of the pressure have a unique global mild solution  $u \in C^0([0, +\infty), \text{BUC}(\Omega_L))$  such that  $u(0) = u_0$ . Moreover, there exists a constant  $C > 0$ , depending only on the initial Reynolds number  $R_u$ , such that*

$$\frac{L}{\nu} \left( \|u(\cdot, t)\|_{L^\infty(\Omega_L)} + (\nu t)^{1/2} \|\omega(\cdot, t)\|_{L^\infty(\Omega_L)} \right) \leq C, \quad \text{for all } t > 0, \quad (1.10)$$

where  $\omega = \partial_1 u_2 - \partial_2 u_1$ .

**Remark 1.5.** The proof shows that the constant in (1.10) satisfies  $C \leq C'(R_u + R_u^6)$  for some universal constant  $C' > 0$ , but we do not claim that this dependence upon the Reynolds number is optimal. It is important to note that the decay rate of the vorticity  $\omega$  is faster in (1.10) than in (1.8). This is because, in the asymptotic laminar regime described in Remark 1.3.4, we can use estimate (1.9) and smoothing properties of the parabolic equation satisfied by  $m(x, t)$  to show that  $\omega(x, t)$  decays actually like  $t^{-1/2}$  as  $t \rightarrow \infty$ , see also Remark 1.3.3.

The rest of this paper is devoted to the proof of Theorem 1.2, which relies on previous results from [13] and is also strongly inspired by our recent work on extended dissipative systems [12]. In Section 2 below, we recall a few basic facts about equation (1.1) which were already established in [1, 13]. In particular, we single out the important role played by the vertical average of the velocity field, which cannot be estimated in a simple way using the Biot-Savart law and the a priori bound on the vorticity. We also give an explicit formula for the pressure in (1.1). In Section 3, we study in some detail the linear advection-diffusion equation (1.2) in  $\Omega_L$ , with periodic boundary conditions, assuming that the velocity field  $u(x, t)$  is given. Using ideas that date back from the pioneering work of Nash [19, 10], we establish an accurate upper bound on the fundamental solution of (1.2) which shows that, if  $\text{div } u = 0$  and if the first component  $u_1$  has zero vertical average, solutions of (1.2) spread diffusively as  $t \rightarrow \infty$ . The core of the proof begins in Section 4, where we control the evolution of the velocity and vorticity fields using weighted energy estimates. This is strongly related to the approach developed in [13], although the new formulation that we propose here is completely self-contained. Combining the weighted energy and enstrophy estimates of Section 4 with the results of Section 3, we prove in Section 5 the first two assertions of Theorem 1.2, namely the uniform bound on the velocity field and the decay estimate for the vorticity. In Section 6, we study the time evolution of small solutions of (1.1) and show that they converge to shear flows as  $t \rightarrow \infty$ , thereby concluding the proof of Theorem 1.2. We also give a short proof of Corollary 1.4. Finally, some conclusions and perspectives are presented in Section 7, while Section 8 is an appendix which contains the proof of a technical lemma stated in Section 3.

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## 2 Preliminaries

In this section we recall some basic properties of equation (1.1) which were already established in [1, 13], and we prepare the proof of Theorem 1.2 by performing a few preliminary steps.

### 2.1 Nondimensionalization

Our system contains three physical parameters: the kinematic viscosity  $\nu$ , the fluid density  $\rho$ , and the width  $L$  of the spatial domain. All of them can be eliminated if we introduce the new

variables  $\tilde{x} = x/L \in \mathbb{R} \times [0, 1]$ ,  $\tilde{t} = \nu t/L^2 \geq 0$ , and the new functions  $\tilde{u}, \tilde{\omega}, \tilde{p}$  defined by the relations

$$u(x, t) = \frac{\nu}{L} \tilde{u}\left(\frac{x}{L}, \frac{\nu t}{L^2}\right), \quad \omega(x, t) = \frac{\nu}{L^2} \tilde{\omega}\left(\frac{x}{L}, \frac{\nu t}{L^2}\right), \quad p(x, t) = \frac{\rho \nu^2}{L^2} \tilde{p}\left(\frac{x}{L}, \frac{\nu t}{L^2}\right). \quad (2.1)$$

In what follows, we work exclusively with the rescaled variables  $\tilde{x}, \tilde{t}$  and the dimensionless functions  $\tilde{u}, \tilde{\omega}, \tilde{p}$ , but we drop the tildes for notational simplicity. We thus consider the nondimensionalized Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla) u = \Delta u - \nabla p, \quad \operatorname{div} u = 0, \quad (2.2)$$

as well as the corresponding vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega. \quad (2.3)$$

In both systems, since we impose periodic boundary conditions, it is mathematically convenient to assume that the space variable  $x = (x_1, x_2)$  lies in the cylinder  $\Omega = \mathbb{R} \times \mathbb{T}$ , where  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  is the one-dimensional torus. Using (2.1), it is straightforward to translate Theorems 1.1 and 1.2 into their equivalent, nondimensional form. In particular, we observe that the Reynolds numbers defined in (1.3) are now simply given by  $R_u = \|u_0\|_{L^\infty(\Omega)}$  and  $R_\omega = \|\omega_0\|_{L^\infty(\Omega)}$ .

## 2.2 Decomposition of the velocity

Let  $u(x, t)$  be a solution of the Navier-Stokes equation (2.2) given by Theorem 1.1. Due to the incompressibility condition, the vertical average of the horizontal velocity

$$\langle u_1 \rangle(x_1, t) := \int_{\mathbb{T}} u_1(x_1, x_2, t) dx_2 \quad (2.4)$$

satisfies  $\partial_1 \langle u_1 \rangle = 0$ , and using (2.2) together with our definition of the pressure, which will be presented in Section 2.4 below, one can also show that  $\partial_t \langle u_1 \rangle = 0$ , see [1, 13]. Thus  $\langle u_1 \rangle$  is a constant which can be eliminated using an appropriate Galilean transformation, without affecting our results in any essential way. In what follows, we assume therefore that  $\langle u_1 \rangle = 0$ , so that the velocity  $u(x, t)$  has the following decomposition:

$$u(x, t) = \begin{pmatrix} 0 \\ m(x_1, t) \end{pmatrix} + \hat{u}(x, t), \quad \text{where } m(x_1, t) = \int_{\mathbb{T}} u_2(x_1, x_2, t) dx_2. \quad (2.5)$$

By construction we have  $\langle \hat{u} \rangle = \int_{\mathbb{T}} \hat{u} dx_2 = 0$ , and it is straightforward to verify that the mean vertical velocity  $m$  and the oscillating part  $\hat{u} = (\hat{u}_1, \hat{u}_2)^\top$  satisfy the evolution equations

$$\partial_t m + \partial_1 \langle \hat{u}_1 \hat{u}_2 \rangle = \partial_1^2 m, \quad (2.6)$$

$$\partial_t \hat{u}_1 + \hat{u}_1 \partial_1 \hat{u}_1 + (m + \hat{u}_2) \partial_2 \hat{u}_1 = \Delta \hat{u}_1 - \partial_1 p, \quad (2.7)$$

$$\partial_t \hat{u}_2 + \hat{u}_1 \partial_1 \hat{u}_2 + (m + \hat{u}_2) \partial_2 \hat{u}_2 + \hat{u}_1 \partial_1 m - \partial_1 \langle \hat{u}_1 \hat{u}_2 \rangle = \Delta \hat{u}_2 - \partial_2 p, \quad (2.8)$$

where the brackets  $\langle \cdot \rangle$  denote the vertical average, as in (2.4). In a similar way, we can decompose the vorticity as  $\omega(x, t) = \partial_1 m(x_1, t) + \hat{\omega}(x, t)$ , where  $\langle \hat{\omega} \rangle = 0$ .

### 2.3 The Biot-Savart law

As is explained in [1, 13], the oscillating part of the velocity can be reconstructed from the vorticity via the Biot-Savart formula  $\widehat{u} = \nabla^\perp K * \widehat{\omega}$ , where  $\nabla^\perp = (-\partial_2, \partial_1)^\top$  and  $K$  is the fundamental solution of the Laplace operator in  $\Omega = \mathbb{R} \times \mathbb{T}$ :

$$K(x_1, x_2) = \frac{1}{4\pi} \log\left(2 \cosh(2\pi x_1) - 2 \cos(2\pi x_2)\right), \quad x \in \mathbb{R}^2. \quad (2.9)$$

Explicitly, we have

$$\widehat{u}(x) = \int_{\Omega} \nabla^\perp K(x - y) \widehat{\omega}(y) dy, \quad x \in \Omega. \quad (2.10)$$

In contrast, the mean vertical velocity  $m(x_1, t)$  cannot be fully reconstructed from the vorticity, and we only know that  $\partial_1 m = \langle \omega \rangle$ . Using the fact that  $\partial_2 K \in L^1(\Omega)$  and  $\partial_1(K - |x_1|/2) \in L^1(\Omega)$ , one can deduce from (2.10) that

$$\|\widehat{u}\|_{L^\infty(\Omega)} \leq C_1 \|\widehat{\omega}\|_{L^\infty(\Omega)} \leq 2C_1 \|\omega\|_{L^\infty(\Omega)}, \quad (2.11)$$

for some universal constant  $C_1 > 0$ , see e.g. [13, Section 9.3].

### 2.4 Definition of the pressure

The pressure satisfies the following elliptic equation in  $\Omega = \mathbb{R} \times \mathbb{T}$ :

$$-\Delta p = \operatorname{div}(u \cdot \nabla)u = \Delta(u_1^2) + 2\partial_2(\omega u_1), \quad (2.12)$$

where the second equality is an identity that holds for any divergence-free vector field  $u$  with  $\omega = \partial_1 u_2 - \partial_2 u_1$ . Inverting the Laplace operator with the help of the fundamental solution (2.9), we obtain the formula

$$p = -u_1^2 - 2\partial_2 K * (\omega u_1), \quad (2.13)$$

which is our choice of the pressure in (2.2). Note that (2.13) gives, up to an irrelevant additive constant, the unique solution of (2.12) that is bounded and periodic in the vertical direction. (Other solutions, including for instance a nonzero pressure gradient at infinity, could be considered as well, but they would correspond to rather different physical situations.) As  $u_1 = \widehat{u}_1$ , it follows from (2.11) and (2.13) that

$$\|p\|_{L^\infty(\Omega)} \leq C_2 \|\omega\|_{L^\infty(\Omega)}^2, \quad (2.14)$$

for some universal constant  $C_2 > 0$ .

**Remark 2.1.** Estimates (2.11) and (2.14) will play a crucial role in our analysis. Indeed, the parabolic maximum principle applied to (2.3) implies that the  $L^\infty$  norm of the vorticity  $\omega(\cdot, t)$  is under control for all time. By (2.11) and (2.14), we thus have a uniform bound on the pressure  $p$  and on the oscillating part  $\widehat{u}$  of the velocity. But we recall that, by our choice of the Galilean frame, the horizontal velocity  $u_1$  has vanishing vertical average, so that  $u_1 = \widehat{u}_1$ , see (2.5). So, we have in particular a *uniform a priori estimate on the horizontal velocity*  $u_1$ , in terms of the initial vorticity  $\omega_0$ . This information will allow us to derive sharp estimates on the solutions of the vorticity equation in the following section.



### 3 Estimates for the vorticity equation

In this section, we assume that we are given a smooth divergence-free vector field  $u(x, t)$  on the two-dimensional cylinder  $\Omega = \mathbb{R} \times \mathbb{T}$ , and we study the linear advection-diffusion equation

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega, \quad x \in \Omega, \quad t \geq 0. \quad (3.1)$$

Our goal is to establish a priori estimates on the solutions of (3.1), in the spirit of the fundamental work of Nash [19]. These estimates are well known when Eq. (3.1) is considered in the whole space  $\mathbb{R}^n$ , but the case of the product manifold  $\Omega = \mathbb{R} \times \mathbb{T}$  is apparently less documented in the literature (see however [9, 15, 16]). In any case, the proofs are rather standard, and we reproduce them below for the reader's convenience.

#### 3.1 The Nash inequality in $\mathbb{R} \times \mathbb{T}$

In the whole space  $\mathbb{R}^n$ , it was shown by Nash [19] that there exists a constant  $C_n > 0$  such that

$$\|f\|_{L^2(\mathbb{R}^n)}^{1+2/n} \leq C_n \|\nabla f\|_{L^2(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}^{2/n}, \quad (3.2)$$

for any  $f \in H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ . In the cylinder  $\Omega = \mathbb{R} \times \mathbb{T}$  inequality (3.2) does not hold, but we have the following estimate, which can be interpreted as a combination of (3.2) with  $n = 1$  and  $n = 2$ .

**Lemma 3.1.** *There exists a constant  $C > 0$  such that, for all  $f \in H^1(\Omega) \cap L^1(\Omega)$ ,*

$$\|f\|_{L^2(\Omega)} \leq C \max \left\{ \|\nabla f\|_{L^2(\Omega)}^{1/3} \|f\|_{L^1(\Omega)}^{2/3}, \|\nabla f\|_{L^2(\Omega)}^{1/2} \|f\|_{L^1(\Omega)}^{1/2} \right\}. \quad (3.3)$$

**Proof.** We mimic the proof of the classical Nash inequality [19]. Given a nonzero  $f \in H^1(\Omega) \cap L^1(\Omega)$ , we use the Fourier representation

$$f(x) = \int_{\hat{\Omega}} \hat{f}(\xi) e^{i\xi \cdot x} d\mu(\xi), \quad \hat{f}(\xi) = \int_{\Omega} f(x) e^{-i\xi \cdot x} dx,$$

where  $\hat{\Omega} = \mathbb{R} \times (2\pi\mathbb{Z})$  is the dual manifold and  $\mu$  is the positive measure on  $\hat{\Omega}$  defined by

$$\int_{\hat{\Omega}} \phi(\xi) d\mu(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \phi(k, 2\pi n) dk,$$

for any continuous function  $\phi : \hat{\Omega} \rightarrow \mathbb{C}$  with compact support. Given  $R > 0$ , the Parseval identity implies

$$\int_{\Omega} |f(x)|^2 dx = \int_{\hat{\Omega}} |\hat{f}(\xi)|^2 d\mu(\xi) = \int_{\hat{\Omega}_R} |\hat{f}(\xi)|^2 d\mu(\xi) + \int_{\hat{\Omega}_R^c} |\hat{f}(\xi)|^2 d\mu(\xi), \quad (3.4)$$

where  $\hat{\Omega}_R = \{\xi \in \hat{\Omega} \mid |\xi| \leq R\}$ . To estimate the right-hand side of (3.4), we observe that

$$\begin{aligned} \int_{\hat{\Omega}_R} |\hat{f}(\xi)|^2 d\mu(\xi) &\leq \|\hat{f}\|_{L^\infty(\hat{\Omega})}^2 \mu(\hat{\Omega}_R) \leq \|f\|_{L^1(\Omega)}^2 \mu(\hat{\Omega}_R), \\ \int_{\hat{\Omega}_R^c} |\hat{f}(\xi)|^2 d\mu(\xi) &\leq \int_{\hat{\Omega}_R^c} \frac{|\xi|^2}{R^2} |\hat{f}(\xi)|^2 d\mu(\xi) \leq \frac{1}{R^2} \|\nabla f\|_{L^2(\Omega)}^2, \end{aligned}$$

and it is easy to verify that  $\mu(\hat{\Omega}_R) \leq \max(R, R^2)$  for any  $R > 0$ . We thus have

$$\|f\|_{L^2(\Omega)}^2 \leq \|f\|_{L^1(\Omega)}^2 \max(R, R^2) + \frac{1}{R^2} \|\nabla f\|_{L^2(\Omega)}^2. \quad (3.5)$$

If we now choose

$$R = \begin{cases} \|\nabla f\|_{L^2(\Omega)}^{2/3} \|f\|_{L^1(\Omega)}^{-2/3} & \text{if } \|\nabla f\|_{L^2(\Omega)} \leq \|f\|_{L^1(\Omega)}, \\ \|\nabla f\|_{L^2(\Omega)}^{1/2} \|f\|_{L^1(\Omega)}^{-1/2} & \text{if } \|\nabla f\|_{L^2(\Omega)} \geq \|f\|_{L^1(\Omega)}, \end{cases}$$

we see that (3.3) follows from (3.5).  $\square$

We use below an equivalent form of (3.3), which is called a  $\psi$ -Nash inequality in [9]:

**Corollary 3.2.** *There exists a constant  $C > 0$  such that, for all nonzero  $f \in H^1(\Omega) \cap L^1(\Omega)$ ,*

$$\|\nabla f\|_{L^2(\Omega)} \geq C \|f\|_{L^2(\Omega)} \min \left\{ \frac{\|f\|_{L^2(\Omega)}}{\|f\|_{L^1(\Omega)}}, \frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{L^1(\Omega)}^2} \right\}. \quad (3.6)$$

### 3.2 $L^p - L^q$ estimates

Using Nash's inequality, we next derive  $L^p - L^q$  estimates for solutions of (3.1).

**Proposition 3.3.** *Given  $1 \leq p \leq q \leq \infty$ , there exists a constant  $K_1 > 0$  (independent of  $u$ ) such that any solution of (3.1) with initial data  $\omega_0 \in L^p(\Omega)$  satisfies*

$$\|\omega(t)\|_{L^q(\Omega)} \leq \frac{K_1}{V(t)^{\frac{1}{p} - \frac{1}{q}}} \|\omega_0\|_{L^p(\Omega)}, \quad t > 0, \quad (3.7)$$

where  $V(t) = \min(t, \sqrt{t})$ .

**Remark 3.4.** From now on we simply denote  $\omega(t), u(t)$  instead of  $\omega(\cdot, t), u(\cdot, t)$ .

**Remark 3.5.** We observe that  $V(t)$  is, up to inessential constants, the volume of a ball of radius  $\sqrt{t}$  in the manifold  $\Omega = \mathbb{R} \times \mathbb{T}$ . The fact that estimate (3.7) holds with a constant  $K_1$  independent of  $u$  is intuitively clear, because the advection term  $u \cdot \nabla \omega$  in (3.1) does not affect the time evolution of  $L^p$  norms of  $\omega$ .

**Proof.** Since the velocity field  $u(x, t)$  in (3.1) is divergence-free, it is well-known that, for any  $p \in [1, \infty]$ , the  $L^p$  norm of any solution of (3.1) is a nonincreasing function of time. This means that (3.7) holds with  $K_1 = 1$  and  $q = p$  for any  $p \in [1, \infty]$ . Thus it remains to prove (3.7) for  $p = 1, q = \infty$ , and the other cases will follow by interpolation, see [6, Section 1.1].

We argue as in [8, Proposition II.1]. Let  $\omega(x, t)$  be a solution of (3.1) with nonzero initial data  $\omega_0 \in L^1(\Omega)$ . Since the velocity field  $u(x, t)$  is divergence-free, a direct calculation shows that

$$\frac{d}{dt} \int_{\Omega} \omega(x, t)^2 dx = -2 \int_{\Omega} |\nabla \omega(x, t)|^2 dx \leq 0, \quad t > 0.$$

To estimate the right-hand side, we use Nash's inequality (3.6) which gives

$$\|\nabla \omega(t)\|_{L^2(\Omega)}^2 \geq C \|\omega(t)\|_{L^2(\Omega)}^2 \min \left\{ \frac{\|\omega(t)\|_{L^2(\Omega)}^2}{\|\omega_0\|_{L^1(\Omega)}^2}, \frac{\|\omega(t)\|_{L^2(\Omega)}^4}{\|\omega_0\|_{L^1(\Omega)}^4} \right\}, \quad t > 0,$$

since  $\|\omega(t)\|_{L^1(\Omega)} \leq \|\omega_0\|_{L^1(\Omega)}$ . Thus, if we define

$$N(t) = \frac{\|\omega(t)\|_{L^2(\Omega)}^2}{\|\omega_0\|_{L^1(\Omega)}^2}, \quad \text{and} \quad \psi(x) = \min(x^2, x^3),$$

we obtain the differential inequality  $N'(t) \leq -c\psi(N(t))$ , for some constant  $c > 0$ . It follows that  $\Psi(N(t)) \geq ct$  for all  $t > 0$ , where  $\Psi : (0, \infty) \rightarrow (0, \infty)$  is the one-to-one function defined by

$$\Psi(x) = \int_x^\infty \frac{1}{\psi(y)} dy = \begin{cases} \frac{1}{x} & x \geq 1, \\ \frac{x^2+1}{2x^2} & x < 1, \end{cases} \quad \Psi^{-1}(t) = \begin{cases} \frac{1}{t} & t \leq 1, \\ \frac{1}{\sqrt{2t-1}} & t > 1. \end{cases}$$

Since  $\Psi$  is decreasing, we conclude that  $N(t) \leq \Psi^{-1}(ct)$  for all  $t > 0$ , hence

$$\|\omega(t)\|_{L^2(\Omega)}^2 \leq \|\omega_0\|_{L^1(\Omega)}^2 \Psi^{-1}(ct) \leq \|\omega_0\|_{L^1(\Omega)}^2 V(ct)^{-1}, \quad t > 0,$$

where  $V(t) = \min(t, \sqrt{t})$ . This shows that (3.7) holds for  $p = 1, q = 2$ .

To complete the proof, we use a classical duality argument. Fix  $T > 0$  and let  $w(x, t)$  be the solution of the adjoint equation

$$\partial_t w - u \cdot \nabla w = \Delta w, \quad x \in \Omega, \quad 0 \leq t \leq T, \quad (3.8)$$

with initial data  $w_0 \in L^1(\Omega)$ . By construction the quantity  $\int_\Omega \omega(x, t)w(x, T-t) dx$  is independent of time, so that

$$\int_\Omega \omega(x, T)w_0(x) dx = \int_\Omega \omega_0(x)w(x, T) dx.$$

Applying (3.7) with  $p = 1, q = 2$  to the adjoint equation (3.8), we thus obtain

$$\left| \int_\Omega \omega(x, T)w_0(x) dx \right| \leq \|\omega_0\|_{L^2(\Omega)} \|w(T)\|_{L^2(\Omega)} \leq C \|\omega_0\|_{L^2(\Omega)} \|w_0\|_{L^1(\Omega)} V(T)^{-1/2},$$

and since  $w_0 \in L^1(\Omega)$  was arbitrary we conclude that  $\|\omega(T)\|_{L^\infty(\Omega)} \leq C \|\omega_0\|_{L^2(\Omega)} V(T)^{-1/2}$ . This proves (3.7) for  $p = 2, q = \infty$ .

Finally, combining both estimates above and using the semigroup property for the evolution operator defined by (3.1), we obtain for any  $t > 0$ :

$$\|\omega(t)\|_{L^\infty(\Omega)} \leq \frac{C}{V(t/2)^{1/2}} \|\omega(t/2)\|_{L^2(\Omega)} \leq \frac{C^2}{V(t/2)} \|\omega_0\|_{L^1(\Omega)},$$

which proves (3.7) for  $p = 1, q = \infty$ . □

### 3.3 Bounds on the fundamental solution

The solution of (3.1) with initial data  $\omega_0$  can be represented as

$$\omega(x, t) = \int_\Omega \Gamma_u(x, y; t, 0) \omega_0(y) dy, \quad x \in \Omega, \quad t > 0, \quad (3.9)$$

where  $\Gamma_u(x, y; t, t_0)$  is the *fundamental solution* of the advection-diffusion equation (3.1). The strong maximum principle implies that  $\Gamma_u(x, y; t, t_0) > 0$  whenever  $t > t_0$ , and it is also known that

$$\int_\Omega \Gamma_u(x, y; t, t_0) dy = 1, \quad \text{and} \quad \int_\Omega \Gamma_u(x, y; t, t_0) dx = 1,$$

for all  $x, y \in \Omega$  and all  $t > t_0$  (the last relation uses the assumption that  $\operatorname{div} u = 0$ ). Finally, the semigroup property  $\Gamma_u(x, y; t, t_0) = \int_{\Omega} \Gamma_u(x, z; t, s) \Gamma_u(z, y; s, t_0) dz$  holds for all  $x, y \in \Omega$  whenever  $t > s > t_0$ . We are interested in pointwise upper bounds on the fundamental solution  $\Gamma_u$ , in the spirit of Aronson [5].

We assume henceforth that the first component of the advection field  $u = (u_1, u_2)^\top$  in (3.1) has zero vertical average:

$$\int_{\mathbb{T}} u_1(x_1, x_2, t) dx_2 = 0, \quad x_1 \in \mathbb{R}, \quad t \geq 0, \quad (3.10)$$

and is uniformly bounded:

$$\sup_{t \geq 0} \|u_1(\cdot, t)\|_{L^\infty(\Omega)} \leq \mathcal{M} < \infty. \quad (3.11)$$

These assumptions are of course motivated by the application to the original equation (2.3), in which the velocity field is obtained from the vorticity via the Biot-Savart law, see Remark 2.1. They allow us to prove the following Gaussian upper bound on the fundamental solution.

**Proposition 3.6.** *Assume that  $u$  is a divergence-free vector field satisfying (3.10) and (3.11), for some  $\mathcal{M} \geq 0$ . Then, for any  $\lambda \in (0, 1)$ , there exists a constant  $K_2 > 0$  (independent of  $u$ ) such that*

$$\Gamma_u(x, y; t, 0) \leq \frac{K_2}{V(t)} \exp\left(-\lambda \frac{|x_1 - y_1|^2}{4(1+\mathcal{M}^2)t}\right), \quad x, y \in \Omega, \quad t > 0, \quad (3.12)$$

where  $V(t) = \min(t, \sqrt{t})$ .

**Remark 3.7.** It is clear that (3.12) implies estimate (3.7) for  $p = 1$ ,  $q = \infty$ , which was the only nontrivial step in the proof of Proposition 3.3. However, as we shall see, the proof of Proposition 3.6 is more complicated and requires stronger assumptions on the advection field  $u$  than the rather elementary argument used in Section 3.2.

**Proof.** We follow the approach of Fabes and Stroock [10]. Let  $\omega_0 : \Omega \rightarrow \mathbb{R}$  be continuous and compactly supported, and assume moreover that  $\omega_0 \geq 0$  and  $\omega_0 \not\equiv 0$ . By the maximum principle, the solution of (3.1) with initial data  $\omega_0$  satisfies  $\omega(x, t) > 0$  for all  $x \in \Omega$  and all  $t > 0$ . Given any  $\alpha \in \mathbb{R}$ , we define

$$w(x, t) = e^{-\alpha x_1} \omega(x, t), \quad x = (x_1, x_2) \in \Omega, \quad t \geq 0. \quad (3.13)$$

The new function  $w(x, t)$  satisfies the modified equation

$$\partial_t w + u \cdot \nabla w + \alpha u_1 w = \Delta w + 2\alpha \partial_1 w + \alpha^2 w, \quad (3.14)$$

where  $u_1$  is the horizontal component of the velocity field  $u$ . Since  $u$  is divergence-free, a direct calculation shows that, for any positive integer  $p \in \mathbb{N}^*$ ,

$$\frac{d}{dt} \int_{\Omega} w(x, t)^{2p} dx = -2p(2p-1) \int_{\Omega} w^{2p-2} |\nabla w|^2 dx + 2p\alpha^2 \int_{\Omega} w^{2p} dx - 2p\alpha \int_{\Omega} u_1 w^{2p} dx.$$

As  $p \geq 1$ , we have

$$2p(2p-1) \int_{\Omega} w^{2p-2} |\nabla w|^2 dx \geq 2p^2 \int_{\Omega} w^{2p-2} |\nabla w|^2 dx = 2 \int_{\Omega} |\nabla w^p|^2 dx.$$

Moreover, it follows from that (3.10) that  $u_1 = \partial_2 v_1$  for some  $v_1 : \Omega \rightarrow \mathbb{R}$  which satisfies the uniform bound  $\|v_1\|_{L^\infty(\Omega)} \leq \mathcal{M}/2$ . Thus, integrating by parts, we obtain

$$\begin{aligned} -2p\alpha \int_{\Omega} u_1 w^{2p} dx &= -2p\alpha \int_{\Omega} \partial_2 v_1 (w^p)^2 dx = 4p\alpha \int_{\Omega} v_1 w^p \partial_2 w^p dx \\ &\leq 2p|\alpha|\mathcal{M} \int_{\Omega} |w^p| |\partial_2 w^p| dx \leq \int_{\Omega} \left( |\nabla w^p|^2 + p^2 \alpha^2 \mathcal{M}^2 w^{2p} \right) dx . \end{aligned}$$

Combining these estimates, we arrive at

$$\frac{d}{dt} \int_{\Omega} w(x, t)^{2p} dx \leq - \int_{\Omega} |\nabla w(x, t)^p|^2 dx + \alpha^2 (2p + p^2 \mathcal{M}^2) \int_{\Omega} w(x, t)^{2p} dx . \quad (3.15)$$

To simplify the notations we define, for all  $p \geq 1$ ,

$$w_p(t) = \|w(\cdot, t)\|_{L^p(\Omega)} = \left( \int_{\Omega} |w(x, t)|^p dx \right)^{1/p}, \quad t \geq 0 .$$

Applying Nash's inequality (3.6) to the function  $f = w(\cdot, t)^p > 0$ , we obtain the lower bound

$$\int_{\Omega} |\nabla w(x, t)^p|^2 dx \geq C w_{2p}(t)^{2p} \min \left\{ \frac{w_{2p}(t)^{2p}}{w_p(t)^{2p}}, \frac{w_{2p}(t)^{4p}}{w_p(t)^{4p}} \right\} ,$$

for some universal constant  $C > 0$ . Thus it follows from (3.15) that

$$w'_{2p}(t) \leq -\frac{C}{2p} \min_{\beta=2,4} \left\{ \left( \frac{w_{2p}(t)}{w_p(t)} \right)^{\beta p} \right\} w_{2p}(t) + \alpha^2 \left( 1 + \frac{p}{2} \mathcal{M}^2 \right) w_{2p}(t), \quad t > 0 . \quad (3.16)$$

**Remark 3.8.** In [10, Section 1] the authors obtain a differential inequality of the form (3.16) with  $\beta = 4/n$ , where  $n \in \mathbb{N}^*$  is the space dimension. Here we have a combination of  $\beta = 4$  ( $n = 1$ ) and  $\beta = 2$  ( $n = 2$ ) because we consider the cylinder  $\Omega = \mathbb{R} \times \mathbb{T}$ . This makes the inequalities (3.16) less homogeneous and more cumbersome to integrate.

**Lemma 3.9.** *Assume that inequalities (3.16) hold for all  $p \in \{1\} \cup S$ , where  $S = \{2^k \mid k \in \mathbb{N}^*\}$ . Then for any  $\epsilon > 0$  there exists a constant  $C_\epsilon > 0$  such that*

$$w_p(t) \leq C_\epsilon \frac{e^{(1+\epsilon)\alpha^2(1+\mathcal{M}^2)t}}{V(t)^{\frac{p-2}{2p}}} w_2(0), \quad t > 0, \quad p \in S, \quad (3.17)$$

where  $V(t) = \min(t, \sqrt{t})$ .

The proof of Lemma 3.9 being somewhat technical, we postpone it to Section 8, and assuming that (3.17) holds we now conclude the proof of Proposition 3.6. Taking the limit  $p \rightarrow \infty$  in (3.17), we obtain

$$\|w(t)\|_{L^\infty(\Omega)} \leq \frac{C_\epsilon}{V(t)^{1/2}} e^{(1+\epsilon)\alpha^2(1+\mathcal{M}^2)t} \|w(0)\|_{L^2(\Omega)}, \quad t > 0 .$$

As in the proof of Proposition 3.3, a duality argument gives the same bound for  $\|w(t)\|_{L^2(\Omega)}$  in terms of  $\|w_0\|_{L^1(\Omega)}$ , so altogether we obtain

$$\|w(t)\|_{L^\infty(\Omega)} \leq \frac{\tilde{C}_\epsilon}{V(t)} e^{(1+\epsilon)\alpha^2(1+\mathcal{M}^2)t} \|w(0)\|_{L^1(\Omega)}, \quad t > 0 . \quad (3.18)$$

Finally, we return to the original equation (3.1). If we take a sequence of initial data  $\omega_0$  approaching a Dirac mass at some point  $y \in \Omega$ , the corresponding solutions  $\omega(x, t)$  converge

by definition to the fundamental solution  $\Gamma_u(x, y; t, 0)$ . In view of (3.13), estimate (3.18) then implies

$$\Gamma_u(x, y; t, 0) \leq \frac{\tilde{C}_\epsilon}{V(t)} e^{(1+\epsilon)\alpha^2(1+\mathcal{M}^2)t} e^{\alpha(x_1-y_1)} , \quad (3.19)$$

for all  $x, y \in \Omega$ , all  $t > 0$ , and all  $\alpha \in \mathbb{R}$ . If we now choose

$$\alpha = -\frac{x_1 - y_1}{2(1+\epsilon)(1+\mathcal{M}^2)t} ,$$

we obtain from (3.19)

$$\Gamma_u(x, y; t, 0) \leq \frac{\tilde{C}_\epsilon}{V(t)} \exp\left(-\frac{|x_1 - y_1|^2}{4(1+\epsilon)(1+\mathcal{M}^2)t}\right) , \quad x, y \in \Omega , \quad t > 0 . \quad (3.20)$$

This gives (3.12) if  $\epsilon > 0$  is taken sufficiently small.  $\square$

**Remark 3.10.** It does not seem possible to obtain estimate (3.20) for all time using the simpler argument given in the proof of Proposition 3.3. The reason is that, when  $\alpha \neq 0$ , we do not have a good a priori estimate on the  $L^1$  norm of  $w(x, t)$ . The best we can deduce from (3.14) is

$$\|w(t)\|_{L^1(\Omega)} \leq \|w_0\|_{L^1(\Omega)} e^{(\alpha^2 + |\alpha|\mathcal{M})t} , \quad t > 0 ,$$

which does not take into account the crucial assumption (3.10), and therefore cannot be used to derive estimate (3.12) for large time.

## 4 Weighted energy estimates

We now arrive at the core proof of Theorem 1.2. In what follows, we consider a global solution of the Navier-Stokes equations (2.2) in  $\Omega = \mathbb{R} \times \mathbb{T}$  given by Theorem 1.1. Without loss of generality, we suppose that the velocity field satisfies (3.10) for all  $t \geq 0$ , so that  $u(x, t)$  can be decomposed as in (2.5). We also assume that the initial vorticity  $\omega_0 = \text{curl } u_0$  is bounded, and we denote  $M = \|\omega_0\|_{L^\infty}$ . It then follows from the maximum principle that  $\|\omega(t)\|_{L^\infty} \leq M$  for all  $t \geq 0$ . Finally, we recall that the pressure in (2.2) is defined by (2.13). In view of (2.11) and (2.14), we thus have  $\|u_1(t)\|_{L^\infty} \leq 2C_1M$  and  $\|p(t)\|_{L^\infty} \leq C_2M^2$  for all  $t \geq 0$ .

### 4.1 Energy density, energy flux, energy dissipation

As in [13], our approach relies on a careful study of the local energy dissipation in the system. For any  $x_1 \in \mathbb{R}$  and  $t \geq 0$ , we define

$$e(x_1, t) = \frac{1}{2} \int_{\mathbb{T}} |u(x_1, x_2, t)|^2 dx_2 + \frac{M^2}{2} , \quad (4.1)$$

$$h(x_1, t) = \int_{\mathbb{T}} \left( p(x_1, x_2, t) + \frac{1}{2} |u(x_1, x_2, t)|^2 \right) u_1(x_1, x_2, t) dx_2 , \quad (4.2)$$

$$d(x_1, t) = \int_{\mathbb{T}} |\nabla u(x_1, x_2, t)|^2 dx_2 , \quad (4.3)$$

as well as  $f(x_1, t) = \partial_1 e(x_1, t) - h(x_1, t)$ . The quantities  $e, f, d$  will be referred to as the energy density, the energy flux, and the energy dissipation rate, respectively. It is clear that  $e(x_1, t) \geq 0$  and  $d(x_1, t) \geq 0$ . Moreover, a direct calculation shows that the following local energy dissipation law holds:

$$\partial_t e(x_1, t) = \partial_1 f(x_1, t) - d(x_1, t) , \quad x_1 \in \mathbb{R} , \quad t > 0 . \quad (4.4)$$

Finally, the initial energy density is uniformly bounded, and we have

$$e_*(0) := \sup_{x_1 \in \mathbb{R}} e(x_1, 0) \leq \frac{1}{2} \|u_0\|_{L^\infty}^2 + \frac{M^2}{2} . \quad (4.5)$$

The reason for including the constant  $M^2/2$  in the definition (4.1) of the energy density will become clear in the proof of the following lemma, which provides a crucial estimate for the energy flux in terms of the energy density and the energy dissipation.

**Lemma 4.1.** *There exists a constant  $C_3 > 0$  such that*

$$|f|^2 \leq C_3(1 + M)^2 ed , \quad |\partial_1 e|^2 \leq 2ed . \quad (4.6)$$

**Proof.** We first estimate the inviscid flux  $h$  defined by (4.2). Since  $\langle u_1 \rangle = 0$ , the Poincaré-Wirtinger inequality implies that

$$\int_{\mathbb{T}} |u_1(x_1, x_2, t)|^2 dx_2 \leq \frac{1}{4\pi^2} \int_{\mathbb{T}} |\partial_2 u_1(x_1, x_2, t)|^2 dx_2 \leq \frac{d(x_1, t)}{4\pi^2} .$$

Using (2.14) and Hölder's inequality, we thus find

$$\left| \int_{\mathbb{T}} p u_1 dx_2 \right| \leq C_2 M^2 \int_{\mathbb{T}} |u_1| dx_2 \leq \frac{C_2 M^2}{2\pi} d^{1/2} \leq \frac{C_2 M}{2\pi} (2ed)^{1/2} , \quad (4.7)$$

where in the last inequality we used the fact that  $e \geq M^2/2$ . On the other hand, we know from (2.5) and (2.10) that  $u_1 = \widehat{u}_1 = \partial_2 v_1$ , where  $v_1 = -K * \widehat{\omega}$ . In particular, we deduce from (2.11) that  $\|v_1\|_{L^\infty} \leq \frac{1}{2} \|\widehat{u}_1\|_{L^\infty} \leq C_1 M$ . Therefore

$$\frac{1}{2} \int_{\mathbb{T}} |u|^2 u_1 dx_2 = - \int_{\mathbb{T}} (u \cdot \partial_2 u) v_1 dx_2 ,$$

and using Hölder's inequality again we obtain

$$\left| \frac{1}{2} \int_{\mathbb{T}} |u|^2 u_1 dx_2 \right| \leq \|v_1\|_{L^\infty} \int_{\mathbb{T}} |u| |\partial_2 u| dx_2 \leq C_1 M (2ed)^{1/2} . \quad (4.8)$$

Inequalities (4.7) and (4.8) imply that  $|h|^2 \leq CM^2 ed$  for some  $C > 0$ . On the other hand, we have  $\partial_1 e = \int_{\mathbb{T}} u \cdot \partial_1 u dx_2$ , hence  $|\partial_1 e|^2 \leq 2ed$  by Hölder's inequality. Since  $f = \partial_1 e - h$ , this proves (4.6).  $\square$

## 4.2 Localized energy estimate

In what follows we denote  $\beta = C_3(1 + M)^2$ , where  $C_3 > 0$  is as in (4.6). By Lemma 4.1, the energy flux satisfies

$$|f(x_1, t)|^2 \leq \beta e(x_1, t) d(x_1, t) , \quad x_1 \in \mathbb{R} , \quad t > 0 . \quad (4.9)$$

Our goal is to control the solution (2.2) using localized energy estimates. Given  $\rho > 0$ , we introduce the localization function  $\chi_\rho(x_1) = \exp(-\rho|x_1|)$ , and we define

$$E_\rho(t) = \int_{\mathbb{R}} \chi_\rho(x_1) e(x_1, t) dx_1 , \quad D_\rho(t) = \int_{\mathbb{R}} \chi_\rho(x_1) d(x_1, t) dx_1 , \quad (4.10)$$

for all  $t \geq 0$ . We then have the following estimate on the localized energy  $E_\rho(t)$ :

**Proposition 4.2.** Fix  $T > 0$ , and let  $\rho = 1/\sqrt{\beta T}$  where  $\beta > 0$  is as in (4.9). Then

$$E_\rho(T) + \frac{1}{2} \int_0^T D_\rho(t) dt \leq 4e_*(0)\sqrt{\beta T}, \quad (4.11)$$

where  $e_*(0)$  is given by (4.5).

**Proof.** Differentiating  $E_\rho(t)$  with respect to time and using (4.4), we obtain

$$E'_\rho(t) = \int_{\mathbb{R}} \chi_\rho \partial_t e dx_1 = \int_{\mathbb{R}} \chi_\rho (\partial_1 f - d) dx_1 = - \int_{\mathbb{R}} (\chi'_\rho f + \chi_\rho d) dx_1. \quad (4.12)$$

Since  $|\chi'_\rho| \leq \rho \chi_\rho$ , it follows from (4.9) that

$$\left| \int_{\mathbb{R}} \chi'_\rho f dx_1 \right| \leq \rho \int_{\mathbb{R}} \chi_\rho (\beta e d)^{1/2} dx_1 \leq \frac{1}{2} \int_{\mathbb{R}} \chi_\rho d dx_1 + \frac{\rho^2 \beta}{2} \int_{\mathbb{R}} \chi_\rho e dx_1.$$

Thus (4.12) leads to the differential inequality  $E'_\rho(t) + \frac{1}{2} D_\rho(t) \leq \frac{1}{2} \rho^2 \beta E_\rho(t)$ , which can be integrated using Gronwall's lemma to give

$$E_\rho(T) + \frac{1}{2} \int_0^T D_\rho(t) dt \leq E_\rho(0) \exp\left(\frac{1}{2} \rho^2 \beta T\right).$$

Since  $E_\rho(0) \leq e_*(0) \int_{\mathbb{R}} \chi_\rho(x_1) dx_1 = 2e_*(0)/\rho$ , choosing  $\rho = (\beta T)^{-1/2}$  yields the desired result.  $\square$

**Remark 4.3.** Together with (4.4), Lemma 4.1 implies that the Navier-Stokes equations in the domain  $\Omega$  define a one-dimensional "extended dissipative system", in the sense of [12]. This point of view was thoroughly exploited in [13], where results similar to Proposition 4.2 were obtained using a slightly different approach. In particular, one can verify that estimate (1.5) with  $R = \sqrt{\nu T}$  is equivalent to (4.11).

### 4.3 Localized enstrophy estimate

We now perform a similar analysis at the level of the vorticity equation (2.3). In analogy with (4.1)–(4.3) we define, for all  $x_1 \in \mathbb{R}$  and all  $t \geq 0$ ,

$$\varepsilon(x_1, t) = \frac{1}{2} \int_{\mathbb{T}} |\omega(x_1, x_2, t)|^2 dx_2, \quad (4.13)$$

$$\zeta(x_1, t) = \frac{1}{2} \int_{\mathbb{T}} \omega(x_1, x_2, t)^2 u_1(x_1, x_2, t) dx_2, \quad (4.14)$$

$$\delta(x_1, t) = \int_{\mathbb{T}} |\nabla \omega(x_1, x_2, t)|^2 dx_2, \quad (4.15)$$

as well as  $\phi(x_1, t) = \partial_1 \varepsilon(x_1, t) - \zeta(x_1, t)$ . Using (2.3) we obtain the local enstrophy dissipation law

$$\partial_t \varepsilon(x_1, t) = \partial_1 \phi(x_1, t) - \delta(x_1, t), \quad x_1 \in \mathbb{R}, \quad t > 0, \quad (4.16)$$

and we have the analog of Lemma 4.1 :

**Lemma 4.4.** *There exists a constant  $C_4 > 0$  such that*

$$|\phi|^2 \leq C_4(1 + M)^2 \varepsilon \delta, \quad |\partial_1 \varepsilon|^2 \leq 2\varepsilon \delta. \quad (4.17)$$



**Proof.** We proceed as in the proof of Lemma 4.1. As  $u_1 = \partial_2 v_1$  with  $\|v_1\|_{L^\infty} \leq C_1 M$ , we have

$$|\zeta| = \left| \int_{\mathbb{T}} \omega(\partial_2 \omega) v_1 \, dx_2 \right| \leq \|v_1\|_{L^\infty} \int_{\mathbb{T}} |\omega| |\partial_2 \omega| \, dx_2 \leq C_1 M (2\varepsilon\delta)^{1/2} .$$

Since  $\phi = \partial_1 \varepsilon - \zeta$  and  $|\partial_1 \varepsilon|^2 \leq 2\varepsilon\delta$  by Hölder's inequality, we obtain (4.17).  $\square$

As in (4.10) we define the localized enstrophy and the corresponding dissipation by

$$\mathcal{E}_\rho(t) = \int_{\mathbb{R}} \chi_\rho(x_1) \varepsilon(x_1, t) \, dx_1 , \quad \mathcal{D}_\rho(t) = \int_{\mathbb{R}} \chi_\rho(x_1) \delta(x_1, t) \, dx_1 . \quad (4.18)$$

We then have the following estimate:

**Proposition 4.5.** *There exists  $C_5 > 0$  such that, if  $T > 0$  and  $\rho = 1/\sqrt{\beta T}$ , then*

$$\mathcal{E}_\rho(T) + \frac{1}{2} \int_{T/2}^T \mathcal{D}_\rho(t) \, dt \leq C_5 (1 + M) e_*(0) \frac{1}{\sqrt{T}} . \quad (4.19)$$

**Proof.** Proceeding as in the proof of Proposition 4.2, we obtain for  $\mathcal{E}_\rho(t)$  the differential inequality

$$\mathcal{E}'_\rho(t) + \frac{1}{2} \mathcal{D}_\rho(t) \leq \frac{C_4}{2} (1 + M)^2 \rho^2 \mathcal{E}_\rho(t) , \quad t > 0 . \quad (4.20)$$

Since  $\omega^2 \leq 2|\nabla u|^2$ , we have  $\varepsilon(x_1, t) \leq d(x_1, t)$ , hence  $\mathcal{E}_\rho(t) \leq \mathcal{D}_\rho(t)$  for all  $t \geq 0$ . Therefore (4.11) implies

$$\int_0^T \mathcal{E}_\rho(t) \, dt \leq \int_0^T \mathcal{D}_\rho(t) \, dt \leq 8e_*(0) \sqrt{\beta T} . \quad (4.21)$$

In particular, there exists  $t_0 \in [0, T/2]$  such that  $\mathcal{E}_\rho(t_0) \leq 16e_*(0) \sqrt{\beta/T}$ . Integrating (4.20) between  $t_0$  and  $T$  and using (4.21), we thus obtain

$$\mathcal{E}_\rho(T) + \frac{1}{2} \int_{t_0}^T \mathcal{D}_\rho(t) \, dt \leq \mathcal{E}_\rho(t_0) + 4C_4 (1 + M)^2 \rho^2 e_*(0) \sqrt{\beta T} . \quad (4.22)$$

This gives the desired result since  $t_0 \leq T/2$ ,  $\rho = 1/\sqrt{\beta T}$ , and  $\beta = C_3(1 + M)^2$ .  $\square$

## 5 Uniform estimates for the velocity and the vorticity

Combining the weighted energy estimates of the previous section with the bounds on the fundamental solution of the vorticity equation obtained in Section 3, we are now able to prove assertions 1) and 2) in Theorem 1.2. We keep the same notations as in Section 4. In particular, we assume that  $u(x, t)$  is a solution of (2.2) with bounded initial velocity  $u_0$  and vorticity  $\omega_0$ , which satisfies (3.10), and we denote  $M = \|\omega_0\|_{L^\infty}$ . In view of (2.11), the horizontal velocity  $u_1$  satisfies the bound (3.11) with  $\mathcal{M} = 2C_1 M$ .

### 5.1 Uniform decay of the vorticity

**Proposition 5.1.** *There exists a constant  $C_6 > 0$  such that*

$$\|\omega(t)\|_{L^\infty(\Omega)}^2 \leq C_6 (1 + M) e_*(0) \frac{1}{\sqrt{t}} , \quad t > 0 , \quad (5.1)$$

where  $M = \|\omega_0\|_{L^\infty(\Omega)}$  and  $e_*(0)$  is defined in (4.5).

**Proof.** Since  $\|\omega(t)\|_{L^\infty} \leq M$  for all  $t \geq 0$  and  $e_*(0) \geq M^2/2$ , it is clear that (5.1) holds with  $C_6 = 2\sqrt{2}$  whenever  $t \leq 2(1+M)^2$ . Thus we assume henceforth that  $t \geq 2(1+M)^2$ , and given such a time  $t$  we denote  $T = t - 1 \geq t/2 \geq 1$ . We also define  $A = \sqrt{\beta T} = 1/\rho$ , where  $\beta = C_3(1+M)^2$  is as in (4.9). Using the fundamental solution  $\Gamma_u$  introduced in Section 3, we decompose

$$\omega(x, t) = \int_{\Omega_1} \Gamma_u(x, y; t, T)\omega(y, T) dy + \int_{\Omega_2} \Gamma_u(x, y; t, T)\omega(y, T) dy = \omega_1(x, t) + \omega_2(x, t),$$

where  $\Omega_1 = \{x \in \Omega \mid |x_1| \leq A\}$  and  $\Omega_2 = \{x \in \Omega \mid |x_1| > A\}$ . In view of Proposition 3.3, we have

$$\sup_{x \in \Omega} |\omega_1(x, t)|^2 \leq K_1^2 \int_{\Omega_1} |\omega(y, T)|^2 dy \leq K_1^2 e \int_{\Omega} \chi_\rho(y_1) |\omega(y, T)|^2 dy,$$

because  $\chi_\rho(y_1) = \exp(-\rho|y_1|) \geq e^{-1}$  when  $y \in \Omega_1$ . Using Proposition 4.5, we thus find

$$\sup_{x \in \Omega} |\omega_1(x, t)|^2 \leq 2K_1^2 e \mathcal{E}_\rho(T) \leq (2K_1^2 e) C_5(1+M)e_*(0) \frac{1}{\sqrt{T}}. \quad (5.2)$$

To bound  $\omega_2$ , we use Proposition 3.6 which gives, for any  $\lambda \in (0, 1)$ ,

$$|\omega_2(x, t)| \leq K_2 \int_{\Omega_2} \exp\left(-\lambda \frac{|x_1 - y_1|^2}{4(1+\mathcal{M})^2}\right) |\omega(y, T)| dy, \quad x \in \Omega. \quad (5.3)$$

In particular, if  $|x_1| \leq A/2$ , we have  $|x_1 - y_1| \geq A/2$  whenever  $y \in \Omega_2$ , hence using the a priori estimate  $|\omega(y, T)| \leq M$  we find

$$|\omega_2(x, t)| \leq 2K_2 M \int_{A/2}^\infty \exp\left(-\frac{\lambda z^2}{4(1+\mathcal{M})^2}\right) dz \leq 8K_2 M \frac{(1+\mathcal{M})^2}{A\lambda} \exp\left(-\frac{\lambda A^2}{16(1+\mathcal{M})^2}\right).$$

Since  $A^2 = \beta T = C_3(1+M)^2 T$ ,  $\mathcal{M} = 2C_1 M$ , and  $T \geq (1+M)^2$ , we conclude that

$$\sup_{|x_1| \leq A/2} |\omega_2(x, t)| \leq CM(1+M) \frac{1}{\sqrt{T}} e^{-T/C} \leq \tilde{C} \frac{M}{\sqrt{T}}, \quad (5.4)$$

for some  $C, \tilde{C} > 0$ . Combining (5.2), (5.4) and using the fact that  $T \geq t/2 \geq 1$ , we obtain

$$\sup_{|x_1| \leq A/2} |\omega(x, t)|^2 \leq C_6(1+M)e_*(0) \frac{1}{\sqrt{t}}, \quad t \geq 2(1+M)^2, \quad (5.5)$$

for some  $C_6 > 0$ . Now, it is clear that estimate (5.5) still holds if we replace the vorticity  $\omega(x_1, x_2, t)$  by  $\omega(x_1 - a, x_2, t)$  for any  $a \in \mathbb{R}$ , because equations (2.2), (2.3) are translation invariant in the horizontal direction, and the right-hand side of (5.5) involves only translation invariant quantities. Thus in (5.5) we can take the supremum over all  $x \in \Omega$ , and this proves that (5.1) holds for all  $t \geq 2(1+M)^2$ .  $\square$

## 5.2 Uniform bound on the velocity field

**Proposition 5.2.** *There exists a constant  $C_7 > 0$  such that*

$$\|u(t)\|_{L^\infty(\Omega)} \leq C_7 \left( \|u_0\|_{L^\infty(\Omega)} + M + (1+M)e_*(0) \right), \quad t \geq 0, \quad (5.6)$$

where  $M = \|\omega_0\|_{L^\infty(\Omega)}$  and  $e_*(0)$  is defined in (4.5).

**Proof.** If  $u(x, t)$  is decomposed as in (2.5), we already know that  $\|\widehat{u}(t)\|_{L^\infty} \leq 2C_1\|\omega(t)\|_{L^\infty} \leq 2C_1M$  for all  $t \geq 0$ . Thus it remains to bound the mean vertical velocity  $m(x_1, t)$ . The integral equation corresponding to (2.6) reads

$$m(t) = S_1(t)m(0) - \int_0^t \partial_1 S_1(t-s) \langle \widehat{u}_1(s) \widehat{u}_2(s) \rangle ds, \quad (5.7)$$

where  $m(t) = m(\cdot, t)$  and  $S_1(t) = e^{t\partial_1^2}$  is the heat semigroup on  $\mathbb{R}$ . By Proposition 5.1, we have

$$\|\widehat{u}(t)\|_{L^\infty}^2 \leq 4C_1^2\|\omega(t)\|_{L^\infty}^2 \leq 4C_1^2C_6(1+M)e_*(0)\frac{1}{\sqrt{t}}, \quad t > 0,$$

hence

$$\|m(t)\|_{L^\infty} \leq \|m(0)\|_{L^\infty} + \int_0^t \frac{\|\widehat{u}(s)\|_{L^\infty}^2}{\sqrt{\pi(t-s)}} ds \leq \|m(0)\|_{L^\infty} + 4\sqrt{\pi}C_1^2C_6(1+M)e_*(0).$$

We conclude that, for all  $t \geq 0$ ,

$$\|u(t)\|_{L^\infty} \leq \|\widehat{u}(t)\|_{L^\infty} + \|m(t)\|_{L^\infty} \leq 2C_1M + \|u_0\|_{L^\infty} + 4\sqrt{\pi}C_1^2C_6(1+M)e_*(0),$$

and (5.6) follows.  $\square$

**Remark 5.3.** According to (2.1), to translate our results back into the original variables we have to replace  $\|u(t)\|_{L^\infty}$  by  $L\|u(t)\|_{L^\infty}/\nu$ ,  $\|\omega(t)\|_{L^\infty}$  by  $L^2\|\omega(t)\|_{L^\infty}/\nu$ , and  $t$  by  $\nu t/L^2$ . Thus  $\|u_0\|_{L^\infty}$  is replaced by  $R_u$ , and  $M = \|\omega_0\|_{L^\infty}$  by  $R_\omega$ . Since  $e_*(0) \leq \frac{1}{2}M^2 + \frac{1}{2}\|u_0\|_{L^\infty}^2$ , we see that (1.7), (1.8) follow from (5.6), (5.1) respectively.

## 6 Exponential decay in the laminar regime

Finally we prove assertion 3) in Theorem 1.2. As is clear from Proposition 5.1, any solution of (2.2), (2.13) with bounded initial data satisfies  $\|\omega(t)\|_{L^\infty} < 4\pi^2$  for  $t$  sufficiently large. In this section, we assume that the velocity field satisfies (3.10), and that the initial vorticity  $\omega_0$  is small enough so that

$$\kappa := \frac{\|\omega_0\|_{L^\infty}}{4\pi^2} < 1. \quad (6.1)$$

If the solution  $u(x, t)$  is decomposed as in (2.5), we define, in analogy with (4.1)–(4.3),

$$\widehat{e}(x_1, t) = \frac{1}{2} \int_{\mathbb{T}} |\widehat{u}(x_1, x_2, t)|^2 dx_2, \quad (6.2)$$

$$\widehat{h}(x_1, t) = \int_{\mathbb{T}} \left( p + \frac{1}{2} |\widehat{u}(x_1, x_2, t)|^2 \right) \widehat{u}_1(x_1, x_2, t) dx_2, \quad (6.3)$$

$$\widehat{d}(x_1, t) = \int_{\mathbb{T}} |\nabla \widehat{u}(x_1, x_2, t)|^2 dx_2, \quad (6.4)$$

as well as  $\widehat{f}(x_1, t) = \partial_1 \widehat{e}(x_1, t) - \widehat{h}(x_1, t)$ . Using (2.7), (2.8), it is not difficult to establish the modified energy dissipation law

$$\partial_t \widehat{e}(x_1, t) = \partial_1 \widehat{f}(x_1, t) - \widehat{d}(x_1, t) - \widehat{g}(x_1, t), \quad x_1 \in \mathbb{R}, \quad t > 0, \quad (6.5)$$

where  $\widehat{g} = (\partial_1 m) \langle \widehat{u}_1 \widehat{u}_2 \rangle$ . As in Lemma 4.1, we then have

**Lemma 6.1.** *There exists a constant  $C_8 > 0$  such that*

$$\widehat{e} \leq \frac{\widehat{d}}{8\pi^2}, \quad |\widehat{f}|^2 \leq C_8 \kappa^2 \widehat{d}, \quad |\widehat{g}| \leq \kappa \widehat{d}. \quad (6.6)$$

**Proof.** The first and the last estimate in (6.6) follow immediately from the Poincaré-Wirtinger inequality, if we observe in addition that  $|\partial_1 m| = |\langle \omega \rangle| \leq \|\omega_0\|_{L^\infty}$ . To bound  $\widehat{h}$ , we proceed as in (4.7) and (4.8). We find

$$\left| \int_{\mathbb{T}} p \widehat{u}_1 dx_2 \right| \leq C_2 \|\omega_0\|_{L^\infty}^2 \int_{\mathbb{T}} |\widehat{u}_1| dx_2 \leq C \kappa^2 \widehat{d}^{1/2},$$

and

$$\left| \frac{1}{2} \int_{\mathbb{T}} |\widehat{u}|^2 \widehat{u}_1 dx_2 \right| \leq \|v_1\|_{L^\infty} \int_{\mathbb{T}} |\widehat{u}| |\partial_2 \widehat{u}| dx_2 \leq C \kappa^2 \widehat{d}^{1/2},$$

In the last inequality, we used the fact that both  $\|v_1\|_{L^\infty}$  and  $\|\widehat{u}\|_{L^\infty}$  are bounded by  $C_1 \|\omega_0\|_{L^\infty}$ . Thus we have  $|\widehat{h}|^2 \leq C \kappa^4 \widehat{d}$ , and we also know that  $|\partial_1 \widehat{e}|^2 \leq 2\widehat{e} \widehat{d} \leq C \kappa^2 \widehat{d}$ . Combining these estimates and using the assumption that  $\kappa < 1$ , we obtain  $|\widehat{f}|^2 \leq C_8 \kappa^2 \widehat{d}$  for some  $C_8 > 0$ .  $\square$

**Proposition 6.2.** *If the initial data satisfy (6.1), then for any  $\epsilon > 0$  there exists a constant  $C_9 > 0$  such that*

$$\sup_{a \in \mathbb{R}} \int_{a-1}^{a+1} \widehat{e}(x_1, t) dx_1 \leq \frac{C_9 \kappa^2}{1 - \kappa} e^{-\gamma t/2}, \quad t \geq 0, \quad (6.7)$$

where  $\gamma = 8\pi^2(1 - \epsilon)(1 - \kappa)$ .

**Proof.** Following the approach developed in [12, 13], we first establish a differential inequality for the energy density  $\widehat{e}$  defined in (6.2). Using (6.5) and (6.6), we find

$$\partial_t \widehat{e}(x_1, t) \leq \partial_1 \widehat{f}(x_1, t) - (1 - \kappa) \widehat{d}(x_1, t) \leq \partial_1 \widehat{f}(x_1, t) - \eta \widehat{f}(x_1, t)^2 - \gamma \widehat{e}(x_1, t), \quad (6.8)$$

where  $\eta, \gamma > 0$  satisfy  $C_8 \kappa^2 \eta + \gamma / (8\pi^2) = 1 - \kappa$ . For definiteness, we take  $\epsilon \in (0, 1)$  and choose

$$\eta = \frac{\epsilon(1 - \kappa)}{C_8 \kappa^2}, \quad \gamma = 8\pi^2(1 - \epsilon)(1 - \kappa). \quad (6.9)$$

Inequality (6.8) can be written in the equivalent form

$$\partial_t \left( \widehat{e}(x_1, t) e^{\gamma t} \right) \leq e^{\gamma t} \left( \partial_1 \widehat{f}(x_1, t) - \eta \widehat{f}(x_1, t)^2 \right), \quad x_1 \in \mathbb{R}, \quad t > 0. \quad (6.10)$$

To exploit (6.10), we fix  $T > 0$  and we introduce the integrated flux

$$F(x_1) = \int_0^T \widehat{f}(x_1, t) e^{\gamma t} dt, \quad x_1 \in \mathbb{R},$$

which can be estimated as follows:

$$F(x_1)^2 \leq \left( \int_0^T e^{\gamma t} dt \right) \left( \int_0^T \widehat{f}(x_1, t)^2 e^{\gamma t} dt \right) \leq \frac{e^{\gamma T}}{\gamma} \left( \int_0^T \widehat{f}(x_1, t)^2 e^{\gamma t} dt \right).$$

Integrating both sides of (6.10) over  $t \in [0, T]$ , we thus obtain

$$\widehat{e}(x_1, T) e^{\gamma T} - \widehat{e}(x_1, 0) \leq F'(x_1) - \eta \gamma e^{-\gamma T} F(x_1)^2, \quad x_1 \in \mathbb{R}. \quad (6.11)$$

In particular, we see that the integrated flux  $F(x_1)$  satisfies the differential inequality

$$F'(x_1) \geq -\widehat{e}_*(0) + \eta\gamma e^{-\gamma T} F(x_1)^2, \quad x_1 \in \mathbb{R}, \quad (6.12)$$

where  $\widehat{e}_*(0) = \sup_{x_1 \in \mathbb{R}} \widehat{e}(x_1, 0) \leq 2C_1^2 \|\omega_0\|_{L^\infty}^2$ . Now, it is easy to verify that any solution of (6.12) that is globally defined on  $\mathbb{R}$  necessarily satisfies

$$F(x_1)^2 \leq \frac{e^{\gamma T}}{\eta\gamma} \widehat{e}_*(0), \quad \text{for all } x_1 \in \mathbb{R}, \quad (6.13)$$

see [12, Proposition 3.1] for a similar argument. So, if we integrate (6.11) over  $x_1 \in [a-1, a+1]$  and then use (6.13), we arrive at the inequality

$$e^{\gamma T} \int_{a-1}^{a+1} \widehat{e}(x_1, T) dx_1 \leq 2\widehat{e}_*(0) + F(a+1) - F(a-1) \leq 2\widehat{e}_*(0) + 2\frac{e^{\gamma T/2}}{(\eta\gamma)^{1/2}} \widehat{e}_*(0)^{1/2},$$

which gives (6.7) since  $\eta, \gamma$  are given by (6.9) and  $\widehat{e}_*(0) \leq C\kappa^2$  with  $\kappa < 1$ .  $\square$

Proposition 6.2 shows that the oscillating part  $\widehat{u}(x, t)$  of the velocity converges exponentially to zero as  $t \rightarrow +\infty$  in the uniformly local space  $L_{\text{ul}}^2(\Omega)$ , whose norm is defined as follows:

$$\|\widehat{u}\|_{L_{\text{ul}}^2(\Omega)}^2 = \sup_{a \in \mathbb{R}} \int_{a-1}^{a+1} \int_{\mathbb{T}} |\widehat{u}(x_1, x_2)|^2 dx_2 dx_1,$$

see e.g. [4, 11]. To conclude the proof of Theorem 1.2, it remains to verify that we also have exponential decay in  $L^\infty(\Omega)$ . This follows directly from the following result:

**Lemma 6.3.** *Assume that  $u(x, t)$  is a solution of (2.2), (2.13) satisfying*

$$\sup_{t \geq 0} \left( \|u(t)\|_{L^\infty(\Omega)} + \|\omega(t)\|_{L^\infty(\Omega)} \right) \leq N, \quad (6.14)$$

for some  $N > 0$ . Then there exist positive constants  $\tau, C_{10}$  such that, for all  $t \geq 0$ ,

$$\|\widehat{u}(t + \tau)\|_{L^\infty(\Omega)} \leq C_{10} \|\widehat{u}(t)\|_{L_{\text{ul}}^2(\Omega)}. \quad (6.15)$$

**Proof.** We use equations (2.7), (2.8), which can be written in the compact form

$$\partial_t \widehat{u} + \partial_1 A_1(u) + \partial_2 A_2(u) + B(u) = \Delta \widehat{u}, \quad (6.16)$$

where

$$A_1 = \begin{pmatrix} \widehat{u}_1^2 + p \\ \widehat{u}_1 \widehat{u}_2 - \langle \widehat{u}_1 \widehat{u}_2 \rangle \end{pmatrix}, \quad A_2 = \begin{pmatrix} (m + \widehat{u}_2) \widehat{u}_1 \\ (m + \widehat{u}_2) \widehat{u}_2 + p \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \widehat{u}_1 \partial_1 m \end{pmatrix}.$$

Here it is understood that the velocity field  $u$  is decomposed as in (2.5), and that the pressure  $p$  is given by (2.13). The integral equation associated to (6.16) is

$$\widehat{u}(t) = S(t - t_0) \widehat{u}(t_0) - \int_{t_0}^t \left( \nabla \cdot S(t - s) A(u(s)) + S(t - s) B(u(s)) \right) ds, \quad (6.17)$$

where  $A = (A_1, A_2)^\top$  and  $S(t) = e^{t\Delta}$  is the heat semigroup in  $\Omega$ . We have the following smoothing estimate

$$\|S(t)f\|_{L^\infty(\Omega)} \leq \frac{C}{\min\{1, \sqrt{t}\}} \|f\|_{L_{\text{ul}}^2(\Omega)}, \quad t > 0, \quad (6.18)$$

which is easily established if we extend the function  $f$  by periodicity and use the  $L^2_{\text{ul}} - L^\infty$  bound that is known for the heat semigroup in the whole plane  $\mathbb{R}^2$  [4]. On the other hand, in view of (6.14) and (2.13), we have the following bound on the nonlinear terms in (6.17) :

$$\|A(u)\|_{L^\infty(\Omega)} + \|B(u)\|_{L^\infty(\Omega)} \leq CN\|\widehat{u}\|_{L^\infty(\Omega)} . \quad (6.19)$$

Now, we fix  $t_0 \geq 0$  and assume that  $t_0 < t \leq t_0 + 1$ . Using (6.18), (6.19), we obtain the following estimate for the solution of (6.17) :

$$\|\widehat{u}(t)\|_{L^\infty} \leq \frac{C}{(t-t_0)^{1/2}} \|\widehat{u}(t_0)\|_{L^2_{\text{ul}}} + \int_{t_0}^t \frac{C'N}{(t-s)^{1/2}} \|\widehat{u}(s)\|_{L^\infty} ds ,$$

for some positive constants  $C, C'$  which do not depend on  $t_0$ . As a consequence, if we denote

$$\Phi(t) = \sup\{(s-t_0)^{1/2} \|\widehat{u}(s)\|_{L^\infty} \mid t_0 < s \leq t\} ,$$

we see that

$$\Phi(t) \leq C\|\widehat{u}(t_0)\|_{L^2_{\text{ul}}} + \pi C'N(t-t_0)^{1/2}\Phi(t) , \quad t_0 < t \leq t_0 + 1 . \quad (6.20)$$

We now choose  $\tau \in (0, 1]$  such that  $\pi C'N\tau^{1/2} \leq 1/2$ . It then follows from inequality (6.20) that  $\Phi(t) \leq 2C\|\widehat{u}(t_0)\|_{L^2_{\text{ul}}}$  for  $t_0 < t \leq t_0 + \tau$ , hence in particular

$$\|\widehat{u}(t_0 + \tau)\|_{L^\infty} \leq \frac{\Phi(t_0 + \tau)}{\tau^{1/2}} \leq \frac{2C}{\tau^{1/2}} \|\widehat{u}(t_0)\|_{L^2_{\text{ul}}} .$$

Since  $t_0 \geq 0$  was arbitrary and  $\tau > 0$  does not depend on  $t_0$ , this proves (6.15).  $\square$

**Corollary 6.4.** *Assume that  $u(x, t)$  is a solution of (2.2), (2.13) in  $\Omega$  with bounded initial data satisfying (6.1). Then there exists a constant  $c \in \mathbb{R}$  such that*

$$u(x, t) = \begin{pmatrix} c \\ m(x_1, t) \end{pmatrix} + \widehat{u}(x, t) , \quad x \in \Omega , \quad t \geq 0 ,$$

where the mean vertical velocity evolves according to

$$\partial_t m + c\partial_1 m + \partial_1 \langle \widehat{u}_1 \widehat{u}_2 \rangle = \partial_1^2 m , \quad x_1 \in \mathbb{R} ,$$

and the oscillating part satisfies  $\|\widehat{u}(t)\|_{L^\infty(\Omega)} = \mathcal{O}(e^{-\gamma t})$  as  $t \rightarrow \infty$  for any  $\gamma < 2\pi^2$ .

**Proof.** Fix  $\gamma < 2\pi^2$  and let  $c = \int_{\mathbb{T}} u_1(x_1, x_2, 0) dx_2$ , for some (arbitrary)  $x_1 \in \mathbb{R}$ . If  $c = 0$ , then (3.10) holds for all  $t \geq 0$ , and the velocity field can be decomposed as in (2.5), where  $m(x_1, t)$  satisfies (2.6) and  $\|\widehat{u}(t)\|_{L^\infty} = \mathcal{O}(e^{-\gamma t})$  as  $t \rightarrow \infty$  by Proposition 6.2 and Lemma 6.3. This gives the desired conclusion if  $c = 0$ , and the general case can be reduced to that particular situation by a simple Galilean transformation.  $\square$

If we now return to the original variables, we see that Corollary 6.4 implies (1.9). The proof of Theorem 1.2 is thus complete.

**Proof of Corollary 1.4.** In terms of nondimensional variables, we have to show that

$$\|u(t)\|_{L^\infty(\Omega)} + t^{1/2}\|\omega(t)\|_{L^\infty(\Omega)} \leq C , \quad \text{for all } t > 0 , \quad (6.21)$$

where the constant  $C > 0$  depends only on  $\|u_0\|_{L^\infty(\Omega)}$ . To prove (6.21), we split the time interval  $(0, \infty)$  into three (slightly overlapping) regions that we consider separately.

1) *Short times.* As was already mentioned in Remark 1.3.2, the local existence theory for equation (2.2) with initial data  $u_0 \in \text{BUC}(\Omega)$  shows that  $\|u(t)\|_{L^\infty} + t^{1/2}\|\omega(t)\|_{L^\infty} \leq C_0\|u_0\|_{L^\infty}$  for  $0 < t \leq 6t_0$ , where  $t_0 = C_0^{-2}\|u_0\|_{L^\infty}^{-2}$  and  $C_0 > 0$  is a universal constant. This yields (6.21) for  $t \leq 6t_0$ , and for later use we observe that  $\|u(t_0)\|_{L^\infty} \leq C_0\|u_0\|_{L^\infty}$  and  $\|\omega(t_0)\|_{L^\infty} \leq C_0^2\|u_0\|_{L^\infty}^2$ .

2) *Intermediate times.* For  $t > t_0$  we can use the results of Section 5 because both  $u(t_0)$  and  $\omega(t_0)$  are bounded. Estimate (5.6) shows that  $\|u(t)\|_{L^\infty} \leq K$  for all  $t > t_0$ , where  $K$  depends only on  $\|u(t_0)\|_{L^\infty}$  and  $\|\omega(t_0)\|_{L^\infty}$ , which are both functions of  $\|u_0\|_{L^\infty}$ . This gives the desired bound (6.21) for the velocity field, so from now on we concentrate on the vorticity. From (5.1) we know that  $(t - t_0)^{1/2}\|\omega(t)\|_{L^\infty}^2 \leq K$  for  $t > t_0$ , with a similar constant  $K$ , but this does not imply (6.21) for large times. We thus define  $t_1 = t_0 + K^2$ , so that  $\|\omega(t_1)\|_{L^\infty} \leq 1$ , and we claim that (6.21) holds at least for  $t \leq 3t_1$ . Indeed, if  $t_0 \geq K^2$ , then  $6t_0 \geq 3t_1$  hence the claim was already proved in the previous step. If  $t_0 < K^2$ , we have from (5.1)

$$t^{1/2}\|\omega(t)\|_{L^\infty} \leq \frac{(Kt)^{1/2}}{(t - t_0)^{1/4}} \leq 2K, \quad 3t_0 < t \leq 3t_1,$$

which proves the claim since the bound for  $t \leq 3t_0$  was obtained in the previous step.

3) *Large times.* Finally, for  $t > t_1$  we can use the results of Section 6 because  $\|\omega(t_1)\|_{L^\infty} \leq 1$ . We decompose  $\omega = \partial_1 m + \widehat{\omega}$ . From Proposition 6.2 and Lemma 6.3, we know that the oscillating part  $\widehat{u}$  of the velocity converges exponentially to zero as  $t \rightarrow \infty$ , and using standard parabolic estimates it is straightforward to prove a similar result for the derivative  $\nabla \widehat{u}$ . We thus obtain

$$\|\widehat{u}(t)\|_{L^\infty} + \|\nabla \widehat{u}(t)\|_{L^\infty} \leq C e^{-\gamma(t-2t_1)}, \quad t \geq 2t_1, \quad (6.22)$$

where (for instance)  $\gamma = \pi^2$  and  $C > 0$  is a universal constant. Estimate (6.22) clearly implies that  $t^{1/2}\|\widehat{\omega}(t)\|_{L^\infty}$  is bounded by a universal constant for  $t \geq 3t_1$ . To treat the remaining part  $\langle \omega \rangle = \partial_1 m$ , we use the integral equation (5.7) for  $t > 2t_1$ , which gives (after differentiating both sides with respect to  $x_1$ )

$$\partial_1 m(t) = \partial_1 S_1(t - 2t_1)m(2t_1) - \int_{2t_1}^t \partial_1 S_1(t - s) \partial_1 \langle \widehat{u}_1(s) \widehat{u}_2(s) \rangle ds, \quad t > 2t_1. \quad (6.23)$$

Since  $\|m(2t_1)\|_{L^\infty} \leq \|\omega(2t_1)\|_{L^\infty} \leq 1$  and  $\|\partial_1 S_1(t - s)f\|_{L^\infty} \leq (\pi(t - s))^{-1/2}\|f\|_{L^\infty}$  for any function  $f \in L^\infty(\mathbb{R})$ , it follows easily from (6.22), (6.23) that  $t^{1/2}\|\partial_1 m(t)\|_{L^\infty}$  is bounded by a universal constant for  $t \geq 3t_1$ . This concludes the proof of (6.21).  $\square$

## 7 Conclusion and perspectives

In this final section, we briefly present some results obtained by S. Zelik [23] for the Navier-Stokes equations in the whole plane  $\mathbb{R}^2$ , and we compare them to the conclusions of Theorem 1.2 which hold when periodicity is assumed in one space direction. We first mention that, in [23], the following more general equation is considered :

$$\partial_t u + (u \cdot \nabla)u = \Delta u - \alpha u - \nabla p + g, \quad \text{div } u = 0,$$

which includes an additional dissipation term  $-\alpha u$  with constant coefficient  $\alpha \geq 0$ , as well as a divergence-free external force  $g(x)$ . However, in the spirit of the present work, we only discuss here the results of [23] in the particular case where  $\alpha = 0$  and  $g = 0$ .

Let  $L_{\text{ul}}^2(\mathbb{R}^2)$  be the uniformly local  $L^2$  space on  $\mathbb{R}^2$  defined by the norm

$$\|f\|_{L_{\text{ul}}^2} = \sup_{x \in \mathbb{R}^2} \left( \int_{|y-x| \leq 1} |f(y)|^2 dy \right)^{1/2}.$$

If  $u_0 \in L_{\text{ul}}^2(\mathbb{R}^2)^2$  is divergence-free, it is known that the Navier-Stokes equations (2.2) have a unique global mild solution with initial data  $u_0$ , provided the pressure  $p$  is defined by

$$p = \sum_{i,j=1}^2 R_i R_j (u_i u_j), \quad (7.1)$$

where  $R_1, R_2$  are the Riesz transforms on  $\mathbb{R}^2$  [14, 18]. This solution is smooth for positive time, and in particular the vorticity  $\omega = \partial_1 u_2 - \partial_2 u_1$  is bounded for all  $t > 0$ . Since we are mainly interested in the long-time behavior, we may thus assume without loss of generality that  $\omega_0 = \text{curl } u_0 \in L^\infty(\mathbb{R}^2)$ .

**Proposition 7.1.** [23] *Assume that  $u_0 \in L_{\text{ul}}^2(\mathbb{R}^2)^2$ ,  $\text{div } u_0 = 0$ , and  $\omega_0 = \text{curl } u_0 \in L^\infty(\mathbb{R}^2)$ . Then there exists a constant  $K \geq 1$  (depending only on  $\|u_0\|_{L_{\text{ul}}^2}$  and  $\|\omega_0\|_{L^\infty}$ ) such that the solution of (2.2), (7.1) in  $\mathbb{R}^2$  with initial data  $u_0$  satisfies, for all  $t \geq 1$ ,*

$$\sup_{x \in \mathbb{R}^2} \frac{1}{Kt^2} \left( \int_{|y-x| \leq Kt^2} |u(y,t)|^2 dy \right)^{1/2} \leq C \|u_0\|_{L_{\text{ul}}^2}, \quad (7.2)$$

where  $C > 0$  is a universal constant.

Remarkably enough, the proof of Proposition 7.1 given in [23, Section 7] does not use the viscous dissipation term  $\Delta u$  in the Navier-Stokes equation. This means that estimate (7.2) also holds for solutions of the Euler equations in  $\mathbb{R}^2$  with bounded velocity and vorticity [21, 2]. In contrast, we emphasize that the viscosity was used in the proof of Theorem 1.2, in particular in Section 4.3.

As was observed in [23], estimate (7.2) is in some sense optimal. For instance, if the initial velocity  $u_0$  is constant and nonzero, then  $u(x,t) = u_0$  for all  $x \in \mathbb{R}^2$  and all  $t > 0$ , hence (7.2) is sharp. However one should observe that, in the left-hand side of (7.2), averages are taken over very large disks of radius  $Kt^2$ , whereas in Sections 4.2 and 4.3 the corresponding domains (determined by the localization function  $\chi_\rho$ ) have a much smaller diameter, of order  $\sqrt{\beta t}$ . The reason for this discrepancy is that, in the cylinder  $\Omega = \mathbb{R} \times \mathbb{T}$ , it was easy to freeze the Galilean invariance of the system and to assume, as in Section 2.2, that the horizontal velocity has zero vertical average. As is shown in Section 3.3, this condition (3.10) allows us to prove that solutions of the vorticity equation (2.3) behave diffusively (in the horizontal direction) as  $t \rightarrow \infty$ , which suggests that the diffusion length  $\mathcal{O}(\sqrt{t})$  is appropriate to describe the spreading of solutions to the Navier-Stokes equation (2.2) in that particular case. In the whole plane  $\mathbb{R}^2$ , the situation is more complicated, and if we do not eliminate somehow the Galilean invariance we are forced to take averages over disks of radius at least  $\mathcal{O}(Ut)$ , where  $U$  is an upper bound on  $\|u\|_{L^\infty}$ . Since no a priori control on  $U$  is available, the proof of Proposition 7.1 is rather delicate and relies on a self-consistent argument which eventually gives (7.2). As  $Kt^2 \geq 1$  for  $t \geq 1$ , we immediately deduce from (7.2) that  $\|u(t)\|_{L_{\text{ul}}^2} \leq CKt^2 \|u_0\|_{L_{\text{ul}}^2}$ , but that estimate is certainly not optimal.

If we do use the viscous dissipation in the Navier-Stokes equations, then proceeding as in Section 4 it is possible to obtain the following result.



**Corollary 7.2.** *Under the assumptions of Proposition 7.1, the vorticity  $\omega = \partial_1 u_2 - \partial_2 u_1$  satisfies, for all  $t \geq 1$ ,*

$$\sup_{x \in \mathbb{R}^2} \frac{1}{Kt^2} \left( \int_{|y-x| \leq Kt^2} |\omega(y, t)|^2 dy \right)^{1/2} \leq \frac{C}{\sqrt{t}} \|u_0\|_{L^2_{\text{ul}}} . \quad (7.3)$$

Estimate (7.3) shows that the enstrophy of the solution, when averaged over sufficiently large disks, decays to zero as  $t \rightarrow \infty$ . This is the analog of Proposition 4.5, except for the important discrepancy regarding the size of the disks, which was already discussed. Unlike in the case of the cylinder, we are not able to convert (7.3) into a uniform decay estimate for the vorticity. Nevertheless, estimate (7.3) strongly suggests that the vorticity converges to zero in some sense as  $t \rightarrow \infty$ , so that the long time asymptotics of (2.2) in  $\mathbb{R}^2$  should be described by irrotational flows, as was proved (in a particular case) in Theorem 1.2. We hope to come back to this interesting question in a future work.

## 8 Appendix

**Proof of Lemma 3.9.** For any  $p \in \{1\} \cup S$ , it follows from (3.16) that

$$w'_{2p}(t) \leq -\frac{a}{p} \min_{\beta=2,4} \left\{ \left( \frac{w_{2p}(t)}{w_p(t)} \right)^{\beta p} \right\} w_{2p}(t) + pb\alpha^2 w_{2p}(t) , \quad t > 0 , \quad (8.1)$$

where  $a = C/2$  and  $b = 1 + \mathcal{M}^2/2$ . We shall prove inductively that

$$w_p(t) \leq \bar{w}_p(t) := \frac{A_p e^{B_p \alpha^2 t}}{V(t)^{\frac{p-2}{2p}}} , \quad t > 0 , \quad p \in S , \quad (8.2)$$

provided the constants  $A_p, B_p$  are chosen appropriately. First, applying (8.1) with  $p = 1$ , we see that  $w'_2(t) \leq \alpha^2 b w_2(t)$ , hence (8.2) obviously holds for  $p = 2$  if  $A_2 = w_2(0)$  and  $B_2 = b$ . Thus, it remains to show that, if (8.2) holds for some  $p \in S$ , then the same inequality remains true with  $p$  replaced by  $2p$ , provided  $A_{2p}$  and  $B_{2p}$  are chosen appropriately.

To do that, we first observe that, if (8.2) holds for some  $p \in S$ , then the function  $w_{2p}$  satisfies the differential inequality

$$w'_{2p}(t) \leq -\frac{a}{p} \min_{\beta=2,4} \left\{ \left( \frac{w_{2p}(t)}{\bar{w}_p(t)} \right)^{\beta p} \right\} w_{2p}(t) + pb\alpha^2 w_{2p}(t) , \quad t > 0 , \quad (8.3)$$

which is obtained from (8.1) by replacing  $w_p(t)$  with  $\bar{w}_p(t)$ . As we shall show, we can choose the constants  $A_{2p}, B_{2p}$  so that the function  $\bar{w}_{2p}$  defined by (8.2) satisfies the *reverse* inequality

$$\bar{w}'_{2p}(t) > -\frac{a}{p} \min_{\beta=2,4} \left\{ \left( \frac{\bar{w}_{2p}(t)}{\bar{w}_p(t)} \right)^{\beta p} \right\} \bar{w}_{2p}(t) + pb\alpha^2 \bar{w}_{2p}(t) , \quad t > 0 . \quad (8.4)$$

Since obviously  $\bar{w}_{2p}(t) \rightarrow +\infty$  as  $t \rightarrow 0+$ , it follows from (8.3), (8.4) that  $w_{2p}(t) \leq \bar{w}_{2p}(t)$  for all  $t > 0$ , which proves (8.2).

It remains to establish (8.4). Using the definition (8.2) of  $\bar{w}_p$  and  $\bar{w}_{2p}$ , we find by a direct calculation

$$\frac{\bar{w}'_{2p}(t)}{\bar{w}_{2p}(t)} = B_{2p} \alpha^2 - \frac{p-1}{2p} \frac{V'(t)}{V(t)} , \quad \text{and} \quad \left( \frac{\bar{w}_{2p}(t)}{\bar{w}_p(t)} \right)^{\beta p} = \left( \frac{A_{2p}}{A_p} \right)^{\beta p} \frac{e^{\beta p (B_{2p} - B_p) \alpha^2 t}}{V(t)^{\beta/2}} .$$

Thus (8.4) holds provided

$$B_{2p}\alpha^2 - \frac{p-1}{2p} \frac{V'(t)}{V(t)} > pb\alpha^2 - \frac{a}{p} \left( \frac{A_{2p}}{A_p} \right)^{\beta p} \frac{e^{\beta p(B_{2p}-B_p)\alpha^2 t}}{V(t)^{\beta/2}}, \quad t > 0, \quad (8.5)$$

for  $\beta = 2, 4$ . We now fix some  $\epsilon \in (0, 1)$  and choose  $N \geq 1$  such that  $2\epsilon aN \geq 1$ . We assume that the constants  $A_p, B_p$  in (8.2) satisfy the recursion relations

$$A_{2p} = A_p(Np^2)^{1/(2p)}, \quad \text{and} \quad B_{2p} = B_p(1 + \epsilon/p), \quad p \in S. \quad (8.6)$$

Then (8.5) is equivalent to

$$B_{2p}\alpha^2 + \frac{a}{p} \left( \frac{Np^2}{V(t)} \right)^{\beta/2} e^{\epsilon\beta B_p\alpha^2 t} > pb\alpha^2 + \frac{p-1}{2p} \frac{V'(t)}{V(t)}, \quad t > 0.$$

In fact, since  $B_p \geq B_2 = b$ , we have  $e^{\epsilon\beta B_p\alpha^2 t} \geq 1 + \epsilon\beta b\alpha^2 t$ , and it is thus sufficient to establish the stronger inequality

$$\frac{a}{p} \left( \frac{Np^2}{V(t)} \right)^{\beta/2} (1 + \epsilon\beta b\alpha^2 t) \geq pb\alpha^2 + \frac{V'(t)}{2V(t)}, \quad t > 0,$$

which is obviously satisfied if we can prove that

$$\frac{a}{p} \left( \frac{Np^2}{V(t)} \right)^{\beta/2} \geq \frac{V'(t)}{2V(t)}, \quad \text{and} \quad \frac{a}{p} \left( \frac{Np^2}{V(t)} \right)^{\beta/2} \epsilon\beta t \geq p, \quad t > 0. \quad (8.7)$$

But, since  $V(t) = \min(t, \sqrt{t})$ , it is clear that (8.7) holds for  $t > 0$  and  $\beta = 2, 4$  if  $N \geq 1$  and  $2\epsilon aN \geq 1$ . This concludes the proof of the upper bound (8.2).

Finally, we iterate the recursion relations (8.6) to show that the coefficients  $A_p, B_p$  are uniformly bounded. A direct calculation shows that

$$\sup_{p \in S} A_p = A_2 \prod_{p \in S} (p^2 N)^{1/(2p)} = 4A_2 N^{1/2}, \quad \text{and} \quad \sup_{p \in S} B_p = B_2 \prod_{p \in S} (1 + \epsilon/p) \leq B_2 e^\epsilon.$$

Since  $A_2 = w_2(0)$  and  $B_2 = b = 1 + \mathcal{M}^2/2 \leq 1 + \mathcal{M}^2$ , we see that (3.17) follows from (8.2).  $\square$

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