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Sergey Lysenko

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TWISTED WHITTAKER MODELS FOR METAPLECTIC GROUPS

S. LYSENKO

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INTRODUCTION

0.0.1. In this paper inspired by [20] we study the twisted Whittaker categories for metaplectic groups (in the sense of [22]). This is a part of the quantum geometric Langlands program [27], ([17], Section 6.3).

Let G be a connected reductive group over an algebraically closed field k . The definition of the twisted Whittaker category for G from [20] extends to our (a bit more general) setting of G equipped with the metaplectic data (in the sense of [22]). We expect an analog of Lurie's conjecture ([20], Conjecture 0.4) to hold in our setting. One of the main ideas of [20] was the construction of the functor $G_n : \text{Whit}_n^c \rightarrow \text{FS}_n^c$ from the twisted Whittaker category of G to the category of factorizable sheaves assuming that the quantum parameter c is irrational (i.e., $q = \exp(\pi ic)$ is not a root of unity). Recall that the main result of [5] identified the category of factorizable sheaves with the category $\text{Rep}(u_q(\check{G}))$ of representations of the corresponding small quantum group $u_q(\check{G})$. When q is not a root of unity, the latter coincides with the big quantum group $U_q(\check{G})$.

In the metaplectic case, corresponding to q being a root of unity, $u_q(\check{G})$ and $U_q(\check{G})$ are substantially different, and the construction of G_n breaks down. One of our main results is a construction of a corrected version of the functor G_n in our metaplectic case. The definitions of the twisted Whittaker category Whit_n^κ and the category $\widetilde{\text{FS}}_n^\kappa$ of factorizable sheaves are given in Sections 2 and 3. Our Theorem 4.11.5 provides a functor

$$\overline{\mathbb{F}} : \text{Whit}_n^\kappa \rightarrow \widetilde{\text{FS}}_n^\kappa$$

exact for the perverse t-structures and commuting with the Verdier duality. It is constructed under the assumption that our metaplectic parameter, the quadratic form ϱ , satisfies what we call the *subtop cohomology property*. This is a local property that we prove for all the simple simply-connected reductive groups and most of parameters ϱ in Theorem 1.1.6 (and Remark 1.1.7), which is one of our main results. We formulate Conjecture 1.1.2 describing those quadratic forms ϱ for which we expect the subtop cohomology property to hold. These are precisely those ϱ for which our construction of $\overline{\mathbb{F}}$ makes sense.

To construct the functor $\overline{\mathbb{F}}$, we introduce natural compactifications of Zastava spaces (see Section 4.4) in Section 4.4. To prove that $\overline{\mathbb{F}}$ commutes with the Verdier duality, we introduce a new notion of the *universal local acyclicity with respect to a diagram* (as opposed to the ULA property for a morphism from [13]), see Definition 4.8.2. This property is studied in Section 4.8. Our proof also essentially uses the description of the twisted IC-sheaves of Drinfeld compactifications $\overline{\text{Bun}}_B$ from [23].

Let X be a smooth projective connected curve over k . For $x \in X$ an irreducible object of the twisted Whittaker category Whit_x^κ is of the form $\mathcal{F}_{x,\lambda}$ for some dominant coweight λ . Assuming the subtop cohomology property we show that

$$\overline{\mathbb{F}}(\mathcal{F}_{x,\lambda}) \xrightarrow{\sim} \bigoplus_{\mu \leq \lambda} \mathcal{L}_{x,\mu} \otimes V_\mu^\lambda,$$

where $\mathcal{L}_{x,\mu}$ are the irreducible objects of $\widetilde{\text{FS}}_x^\kappa$, and V_μ^λ are some multiplicity vector spaces (cf. Corollary 4.9.2 and Proposition 4.11.4). One of our main results is a description of the space V_μ^λ in Theorem 4.12.5. We show that V_μ^λ admits a canonical base, which is naturally a subset of $B(\lambda)$. Here $B(\lambda)$ is the crystal of the canonical base of the irreducible \check{G} -representation \mathbb{V}^λ of highest weight λ .

In [22] we associated to G and its metaplectic data a connected reductive group \check{G}_ζ , this is an analog of the Langlands dual group in the metaplectic setting. The dominant coweights of \check{G}_ζ form naturally a subset of the set Λ^+ of G -dominant coweights. Our Theorem 4.12.11 shows that if λ is a dominant coweight of \check{G}_ζ then V_μ^λ identifies with the μ -weight space in the irreducible representation $V(\lambda)$ of \check{G}_ζ of highest weight λ .

Write Bun_G for the moduli stack of G -torsors on X . In Section 5 we define the action of the category $\text{Rep}(\check{G}_\zeta)$ of representations of \check{G}_ζ by Hecke functors on the twisted derived category $\text{D}_\zeta(\overline{\text{Bun}}_G)$ of Bun_G , and on the twisted Whittaker category DWhit_x^κ . The main result of this Section is Theorem 5.3.1. It shows that the Hecke functors are exact for the perverse t-structure on the twisted Whittaker category. It also shows that acting on the basic object of Whit_x^κ by the Hecke functor corresponding to an irreducible representation of \check{G}_ζ , one gets the corresponding irreducible object of Whit_x^κ . This is an analog of ([18], Theorem 4) in the metaplectic setting.

Finally, in Appendix B we prove Proposition 2.7.1, which reformulates the subtop cohomology property as some categorical property of Whit_x^κ saying that Ext^1 in this category between some irreducible objects vanish.

0.0.2. *Notation.* Work over an algebraically closed ground field k of characteristic $p > 0$. Let X be a smooth projective connected curve. Let Ω denote the canonical line bundle on X . We fix a square root $\Omega^{\frac{1}{2}}$ of Ω . Set $\mathcal{O} = k[[t]] \subset F = k((t))$.

Let G be a connected reductive group over k with $[G, G]$ simply-connected. Let $B \subset G$ be a Borel subgroup, $B^- \subset G$ its opposite and $T = B \cap B^-$ a maximal torus. Let U (resp., U^-) denote the unipotent radical of B (resp., of B^-). Let Λ denote the coweights of T , $\check{\Lambda}$ the weights of G . The canonical pairing between the two is denoted by $\langle \cdot, \cdot \rangle$. By Λ^+ (resp., $\check{\Lambda}^+$) we denote the semigroup of dominant coweights (resp., dominant weights) for G . Let ρ be the half-sum of positive coroots of G . Let Λ^{pos} denote the \mathbb{Z}_+ -span of positive coroots in Λ .

Set $G_{ab} = G/[G, G]$, let Λ_{ab} (resp., $\check{\Lambda}_{ab}$) denote the coweights (resp., weights) of G_{ab} . Let J denote the set of connected components of the Dynkin diagram of G . For $j \in J$ write \mathcal{J}_j for the set of vertices of the j -th connected component of the Dynkin diagram, $\mathcal{J} = \cup_{j \in J} \mathcal{J}_j$. For $j \in \mathcal{J}$ let α_j (resp., $\check{\alpha}_j$) denote the corresponding simple coroot (resp., simple root). One has $G_{ad} = \prod_{j \in J} G_j$, where G_j is a simple adjoint group. Let $\mathfrak{g}_j = \text{Lie } G_j$. For $j \in J$ let $\kappa_j : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ be the Killing form for G_j , so

$$\kappa_j = \sum_{\check{\alpha} \in \check{R}_j} \check{\alpha} \otimes \check{\alpha},$$

where \check{R}_j is the set of roots of G_j . For a standard Levi subgroup M of G we have the corresponding semigroup Λ_M^{pos} . Our notation $\mu \leq_M \lambda$ for $\lambda, \mu \in \Lambda$ means that $\lambda - \mu \in \Lambda_M^{pos}$. For $M = G$ we write \leq instead of \leq_G .

By a super line we mean a $\mathbb{Z}/2\mathbb{Z}$ -graded line. As in [22], we denote by $\mathcal{E}^s(T)$ the groupoid of pairs: a symmetric bilinear form $\kappa : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$, and a central super extension $1 \rightarrow k^* \rightarrow \tilde{\Lambda}^s \rightarrow \Lambda \rightarrow 1$ whose commutator is $(\gamma_1, \gamma_2)_c = (-1)^{\kappa(\gamma_1, \gamma_2)}$.

Let Sch/k denote the category of k -schemes of finite type with Zarisky topology. The n -th Quillen K -theory group of a scheme form a presheaf on Sch/k . As in [8], K_n will denote the associated sheaf on Sch/k for the Zariski topology.

Pick a prime ℓ invertible in k . We work with (perverse) $\bar{\mathbb{Q}}_\ell$ -sheaves on k -stacks for the étale topology. Pick an injective character $\psi : \mathbb{F}_p \rightarrow \bar{\mathbb{Q}}_\ell^*$, let \mathcal{L}_ψ be the corresponding Artin-Schreier sheaf on \mathbb{A}^1 . The trivial G -torsor over some base is denoted \mathcal{F}_G^0 .

0.0.3. *Input data.* We fix the following data as in ([22], Section 2.3). Write $\text{Gr}_G = G(F)/G(\mathcal{O})$ for the affine grassmanian of G . For $j \in J$ let \mathcal{L}_j denote the ($\mathbb{Z}/2\mathbb{Z}$ -graded purely of parity zero) line bundle on Gr_G with fibre $\det(\mathfrak{g}_j(\mathcal{O}) : \mathfrak{g}_j(\mathcal{O})^g)$ at $gG(\mathcal{O})$ (the definition of this relative determinant is found in [16]). Let E_j^a be the punctured total space of the pull-back of \mathcal{L}_j to $G(F)$. This is a central extension

$$1 \rightarrow \mathbb{G}_m \rightarrow E_j^a \rightarrow G(F) \rightarrow 1.$$

It splits canonically over $G(\mathcal{O})$. Write $(\cdot, \cdot)_{st} : F^* \times F^* \rightarrow k^*$ for the tame symbol map ([22], Section 2.3). Pick a central extension

$$(1) \quad 1 \rightarrow K_2 \rightarrow \mathcal{V}_\beta \rightarrow G_{ab} \rightarrow 1$$

of sheaf of groups on Sch/k as in [8]. Let

$$(2) \quad 1 \rightarrow \mathbb{G}_m \rightarrow E_\beta \rightarrow G_{ab}(F) \rightarrow 1$$

be a central extension in the category of ind-schemes whose commutator $(\cdot, \cdot)_c : G_{ab}(F) \times G_{ab}(F) \rightarrow \mathbb{G}_m$ satisfies

$$(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)_{st}^{-\beta(\lambda_1, \lambda_2)}$$

for $\lambda_i \in \Lambda_{ab}, f_i \in F^*$. Here $\beta : \Lambda_{ab} \otimes \Lambda_{ab} \rightarrow \mathbb{Z}$ is an even symmetric bilinear form. The pull-back of (2) under $G(F) \rightarrow G_{ab}(F)$ is also denoted by E_β by abuse of notation. We assume that passing to F -points in (1) and further taking the push-out by the tame symbol $(\cdot, \cdot)_{st} : K_2(F) \rightarrow \mathbb{G}_m$ yields the extension (2).

Recall that $\mathcal{V}_\beta(\mathcal{O}) \rightarrow G_{ab}(\mathcal{O})$ is surjective, and the composition of the tame symbol with $K_2(\mathcal{O}) \rightarrow K_2(F)$ is trivial. For this reason (2) is equipped with a canonical section over $G_{ab}(\mathcal{O})$.

Let $N \geq 1$ be invertible in k . Let $\zeta : \mu_N(k) \rightarrow \bar{\mathbb{Q}}_\ell^*$ be an injective character, we write \mathcal{L}_ζ for the canonical rank one local system on $B(\mu_N)$ such that $\mu_N(k)$ acts on it by ζ . We have a map $s_N : \mathbb{G}_m \rightarrow B(\mu_N)$ corresponding to the μ_N -torsor $\mathbb{G}_m \rightarrow \mathbb{G}_m, z \mapsto z^N$. The local system $s_N^* \mathcal{L}_\zeta$ is sometimes also denoted by \mathcal{L}_ζ . For each $j \in J$ pick $c_j \in \mathbb{Z}$. To these data we associate the even symmetric bilinear form $\bar{\kappa} : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ given by

$$\bar{\kappa} = -\beta - \sum_{j \in J} c_j \kappa_j$$

and the quadratic form $\varrho : \Lambda \rightarrow \mathbb{Q}$ given by $\varrho(\mu) = \frac{\bar{\kappa}(\mu, \mu)}{2N}$. The true parameter in our quantum setting is rather ϱ instead of $(\bar{\kappa}, N)$.

The sum of the extensions $(E_j^a)^{c_j}, j \in J$ and the extension E_β is the central extension denoted

$$(3) \quad 1 \rightarrow \mathbb{G}_m \rightarrow \mathbb{E} \rightarrow G(F) \rightarrow 1.$$

It is equipped with the induced section over $G(\mathcal{O})$. Let

$$(4) \quad 1 \rightarrow \mathbb{G}_m \rightarrow V_{\mathbb{E}} \rightarrow \Lambda \rightarrow 1$$

be the pull-back of (3) under $\Lambda \rightarrow G(F), \lambda \mapsto t^\lambda$. The commutator in (4) is given by

$$(\lambda_1, \lambda_2)_c = (-1)^{\bar{\kappa}(\lambda_1, \lambda_2)}$$

Set $\text{Gra}_G = \mathbb{E}/G(\mathcal{O})$. Let $\widetilde{\text{Gr}}_G$ be the stack quotient of Gra_G under the \mathbb{G}_m -action such that $z \in \mathbb{G}_m$ acts as z^N . Let $\text{Perv}_{G, \zeta}$ be the category of $G(\mathcal{O})$ -equivariant perverse sheaves on $\widetilde{\text{Gr}}_G$ on which $\mu_N(k)$ acts by ζ .

0.0.4. *Metaplectic dual group.* In [22] we equipped $\text{Perv}_{G, \zeta}$ with a structure of a symmetric monoidal category, we introduced a symmetric monoidal category $\text{Perv}_{G, \zeta}^\sharp$ obtained from $\text{Perv}_{G, \zeta}$ by some modification of the commutativity constraint.

Set $\Lambda^\sharp = \{\lambda \in \Lambda \mid \bar{\kappa}(\lambda) \in N\check{\Lambda}\}$. Let $\check{T}_\zeta = \text{Spec } k[\Lambda^\sharp]$ be the torus whose weights lattice is Λ^\sharp . Let \check{G}_ζ be the reductive group over $\bar{\mathbb{Q}}_\ell$ defined in ([22], Theorem 2.1), it is equipped with canonical inclusions $\check{T}_\zeta \subset \check{B}_\zeta \subset \check{G}_\zeta$, where \check{T}_ζ is a maximal torus, and \check{B}_ζ is a Borel subgroup dual to $T \subset B \subset G$.

To get a fibre functor on $\text{Perv}_{G, \zeta}^\sharp$ one needs to pick an additional input datum. We make this choice as in [22]. Namely, let $\bar{V}_{\mathbb{E}}$ be the stack quotient of $V_{\mathbb{E}}$ by the \mathbb{G}_m -action,

where $z \in \mathbb{G}_m$ acts as z^N . It fits into an exact sequence of group stacks

$$(5) \quad 1 \rightarrow B(\mu_N) \rightarrow \bar{V}_{\mathbb{E}} \rightarrow \Lambda \rightarrow 1$$

We pick a morphism of group stacks $\mathfrak{t}_{\mathbb{E}} : \Lambda^{\sharp} \rightarrow \bar{V}_{\mathbb{E}}$, which is a section of (5) over Λ^{\sharp} . This yields as in ([22], Theorem 2.1) an equivalence of tensor categories $\mathbb{P}\text{erv}_{G,\zeta}^{\natural} \xrightarrow{\sim} \text{Rep}(\check{G}_{\zeta})$.

Let $\widetilde{\text{Gr}}_T$ be obtained from $\widetilde{\text{Gr}}_G$ by the base change $\text{Gr}_T \rightarrow \text{Gr}_G$. Write $\mathbb{P}\text{erv}_{T,G,\zeta}$ for the category of $T(\mathcal{O})$ -equivariant perverse sheaves on $\widetilde{\text{Gr}}_T$ on which $\mu_N(k)$ acts by ζ . As in ([22], Section 3.2), the datum of $\mathfrak{t}_{\mathbb{E}}$ yields an equivalence $\text{Loc}_{\zeta} : \text{Rep}(\check{T}_{\zeta}) \xrightarrow{\sim} \mathbb{P}\text{erv}_{T,G,\zeta}$.

0.0.5. *Line bundles.* For a reductive group H we denote by Bun_H the stack of H -torsors on X . Let

$$(6) \quad 1 \rightarrow \mathbb{G}_m \rightarrow V_{\beta} \rightarrow \Lambda_{ab} \rightarrow 1$$

be the restriction of (2) under $\Lambda_{ab} \rightarrow G_{ab}(F)$, $\lambda \mapsto t^{\lambda}$. It is given for each $\gamma \in \Lambda_{ab}$ by a line ϵ^{γ} over k together with isomorphisms

$$\epsilon^{\gamma_1, \gamma_2} : \epsilon^{\gamma_1} \otimes \epsilon^{\gamma_2} \xrightarrow{\sim} \epsilon^{\gamma_1 + \gamma_2}$$

for $\gamma_i \in \Lambda_{ab}$ (cf. [22], Section 2.3). As in ([22], Section 2.6) we associate to the pair $((6), -\beta) \in \mathcal{E}^s(G_{ab})$ a line bundle \mathcal{L}_{β} on $\text{Bun}_{G_{ab}}$. For $\mu \in \Lambda_{ab}$ consider the map $i_{\mu} : X \rightarrow \text{Bun}_{G_{ab}}$, $x \mapsto \mathcal{O}(-\mu x)$. Recall that one has canonically

$$i_{\mu}^* \mathcal{L}_{\beta} \xrightarrow{\sim} \Omega^{\frac{\beta(\mu, \mu)}{2}} \otimes \epsilon^{\mu}$$

For $j \in J$ let $\mathcal{L}_{j, \text{Bun}_G}$ be the line bundle on Bun_G whose fibre at $\mathcal{F} \in \text{Bun}_G$ is

$$\det \text{R}\Gamma(X, (\mathfrak{g}_j)_{\mathcal{F}_G^0}) \otimes \det \text{R}\Gamma(X, (\mathfrak{g}_j)_{\mathcal{F}})^{-1}$$

Denote by $\mathcal{L}^{\bar{\kappa}}$ the line bundle $\mathcal{L}_{\beta} \otimes (\otimes_{j \in J} \mathcal{L}_{j, \text{Bun}_G}^{c_j})$ on Bun_G .

For $x \in X$ let $\text{Gr}_{G,x}$ denote the affine grassmanian classifying a G -torsor \mathcal{F} on X with a trivialization $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_G^0|_{X-x}$. The restriction of $\mathcal{L}^{\bar{\kappa}}$ (with zero section removed) under the forgetful map $\text{Gr}_{G,x} \rightarrow \text{Bun}_G$ identifies with Gra_G (once we pick an isomorphism $D_x \xrightarrow{\sim} \text{Spec } \mathcal{O}$ for the formal disk D_x around x).

Let Ω^{ρ} denote the T -torsor on X obtained from $\Omega^{\frac{1}{2}}$ via the extension of scalars for $2\rho : \mathbb{G}_m \rightarrow T$. We denote by ${}^{\omega}\mathcal{L}^{\bar{\kappa}}$ the line bundle on Bun_G whose fibre at $\mathcal{F} \in \text{Bun}_G$ is $\mathcal{L}_{\mathcal{F}}^{\bar{\kappa}} \otimes (\mathcal{L}_{\Omega^{\rho}}^{\bar{\kappa}})^{-1}$. From ([24], Proposition 4.1) one gets the following.

Lemma 0.0.6. *Let $D = \sum_x \mu_x x$ be a Λ -valued divisor on X . The fibre of \mathcal{L}_{β} at $\Omega^{\rho}(-D)$ identifies canonically with*

$$(\mathcal{L}_{\beta})_{\Omega^{\rho}} \otimes (\otimes_{x \in X} (\Omega_x^{\frac{1}{2}})^{\beta(\mu_x, \mu_x + 2\rho)} \otimes \epsilon^{\bar{\mu}_x}),$$

where $\bar{\mu}_x \in \Lambda_{ab}$ is the image of μ_x .

0.0.7. *Langlands program for metaplectic groups.* Let $\widetilde{\text{Bun}}_G$ be the gerb of N -th roots of ${}^\omega \mathcal{L}^{\bar{\kappa}}$ over Bun_G . Let $\text{D}_\zeta(\widetilde{\text{Bun}}_G)$ denote the derived category of \mathbb{Q}_ℓ -sheaves on $\widetilde{\text{Bun}}_G$, on which $\mu_N(k)$ acts by ζ .

As in [23], where the case of G simple simply-connected was considered, we define an action of $\mathbb{P}\text{erv}_{G,\zeta}^h$ on $\text{D}_\zeta(\widetilde{\text{Bun}}_G)$ by Hecke functors (see Section 5.1). From our point of view, the geometric Langlands program for metaplectic groups is the problem of finding a spectral decomposition of $\text{D}_\zeta(\widetilde{\text{Bun}}_G)$ under this action. Our study of the twisted Whittaker model in this setting is motivated by this problem.

1. LOCAL PROBLEM: SUBTOP COHOMOLOGY

1.1. In this Section we formulate and partially prove Conjecture 1.1.2 that will be used in Proposition 4.11.2.

For a free \mathcal{O} -module M write $M_{\bar{e}} = M \otimes_{\mathcal{O}} k$. For $\mu \in \Lambda$ let Gr_B^μ (resp., $\text{Gr}_{B^-}^\mu$) denote the $U(F)$ -orbit (resp., $U^-(F)$ -orbit) in Gr_G through t^μ . For μ is in the coroots lattice, the \mathbb{G}_m -torsor $\text{Gra}_G \times_{\text{Gr}_G} \text{Gr}_B^\mu \rightarrow \text{Gr}_B^\mu$ is constant with fibre $\Omega_{\bar{e}}^{-\bar{\kappa}(\mu,\mu)} - 0$, and $T(\mathcal{O})$ acts on it by the character $T(\mathcal{O}) \rightarrow T \xrightarrow{-\bar{\kappa}(\mu)} \mathbb{G}_m$. The \mathbb{G}_m -torsor $\text{Gra}_G \times_{\text{Gr}_G} \text{Gr}_{B^-}^\mu \rightarrow \text{Gr}_{B^-}^\mu$ is constant with fibre $\Omega_{\bar{e}}^{-\bar{\kappa}(\mu,\mu)} - 0$, and $T(\mathcal{O})$ acts on it by $T(\mathcal{O}) \rightarrow T \xrightarrow{-\bar{\kappa}(\mu)} \mathbb{G}_m$.

As in ([18], Section 7.1.4), for $\eta \in \Lambda$ we will write $\chi_\eta : U(F) \rightarrow \mathbb{A}^1$ for an additive character of conductor $\bar{\eta}$, where $\bar{\eta}$ is the image of η in the coweights lattice of G_{ad} . For $\eta + \nu \in \Lambda^+$ we also write $\chi_\eta^\nu : \text{Gr}_B^\nu \rightarrow \mathbb{A}^1$ for any $(U(F), \chi_\eta)$ -equivariant function.

For $\mu \in \Lambda$ let $\widetilde{\text{Gr}}_B^\mu = \text{Gr}_B^\mu \times_{\text{Gr}_G} \widetilde{\text{Gr}}_G$. Pick $\chi_0 : U(F) \rightarrow \mathbb{A}^1$ and define $\chi_0^0 : \text{Gr}_B^0 \rightarrow \mathbb{A}^1$ by $\chi_0^0(uG(\mathcal{O})) = \chi_0(u)$ for $u \in U(F)$. Set $ev = \chi_0^0$. Using the canonical trivialization $\widetilde{\text{Gr}}_B^0 \xrightarrow{\sim} \text{Gr}_B^0 \times B(\mu_N)$, we consider $\mathcal{L}_G := ev^* \mathcal{L}_\psi \boxtimes \mathcal{L}_\zeta$ as a local system on $\widetilde{\text{Gr}}_B^0$.

For μ is the coroots lattice any trivialization of $\Omega_{\bar{e}}^{-\bar{\kappa}(\mu,\mu)}$ yields a section $s_\eta : \text{Gr}_{B^-}^\mu \rightarrow \widetilde{\text{Gr}}_{B^-}^\mu$. Recall that $\text{Gr}_B^0 \cap \text{Gr}_{B^-}^{-\lambda}$ is empty unless $\lambda \geq 0$, and for $\lambda \geq 0$ this is a scheme of finite type and pure dimension $\langle \lambda, \bar{\rho} \rangle$ by ([10], Section 6.3).

Definition 1.1.1. *We will say that the subtop cohomology property is satisfied for ϱ if for any $\lambda > 0$, which is not a simple coroot,*

$$(7) \quad \text{R}\Gamma_c(\text{Gr}_B^0 \cap \text{Gr}_{B^-}^{-\lambda}, s_{-\lambda}^* \mathcal{L}_G)$$

is placed in degrees $\leq \text{top} - 2$, where $\text{top} = \langle \lambda, 2\bar{\rho} \rangle$.

Conjecture 1.1.2. *Assume that $\varrho(\alpha_i) \notin \mathbb{Z}$ for any simple coroot α_i . Then the subtop cohomology property is satisfied for ϱ .*

This conjecture is motivated by our definition of the functor $\overline{\mathbb{F}}$ in Section 4.6.1, this is precisely the local property needed in Proposition 4.11.2. The assumption $\varrho(\alpha_i) \notin \mathbb{Z}$ is used in the construction of $\overline{\mathbb{F}}$ to get the correct answer over \mathring{X}^μ (see Proposition 4.3.4).

Remark 1.1.3. *i) The input data of Section 0.0.3 are functorial in a suitable sense. In particular, we may restrict them from G to $[G, G]$. Then $\bar{\kappa}$ gets replaced by its restriction to the coroots lattice. The subtop cohomology property holds for $[G, G]$ (with the induced input data) if and only if it holds for G .*

ii) We may pick a torus T_1 and an inclusion $Z([G, G]) \hookrightarrow T_1$, where $Z([G, G])$ is the center of $[G, G]$. Then $G_1 := ([G, G] \times T_1)/Z([G, G])$ has a connected center, here $Z([G, G])$ is included diagonally in the product. One may also extend the input data of Section 0.0.3 to G_1 and assume, if necessary, that G has a connected center.

Definition 1.1.4. *If the center $Z(G)$ of G is not connected, replace G by the group G_1 as in Remark 1.1.3, so we may assume $Z(G)$ connected. Then pick fundamental coweights $\omega_i \in \Lambda$ of \check{G} corresponding to $\check{\alpha}_i$ for $i \in \mathcal{J}$. Say that ϱ satisfies the property (C) if the following holds. For any $i \in \mathcal{J}$, $\lambda > \alpha_i$ such that $\omega_i - \lambda$ appears as a weight of the fundamental representation \mathbb{V}^{ω_i} of \check{G} , $\bar{\kappa}(\lambda - \alpha_i)$ is not divisible by N in $\check{\Lambda}$.*

Here is the main result of this section.

Theorem 1.1.5. *If ϱ satisfies the property (C) then the subtop cohomology property is satisfied for ϱ .*

The proof of the following is given case by case in Appendix A.

Theorem 1.1.6. *The quadratic form ϱ satisfies the property (C), and hence the subtop cohomology property, in the following cases:*

- G is of type C_2 or A_n for $n \geq 1$, and $\varrho(\alpha_i) \notin \mathbb{Z}$ for any simple coroot α_i .
- G is of type B_n, C_n, D_n for $n \geq 1$ or G_2 , and $\varrho(\alpha_i) \notin \frac{1}{2}\mathbb{Z}$ for any simple coroot α_i .
- G is of type F_4 , and $\varrho(\alpha_i) \notin \frac{1}{2}\mathbb{Z}$, $\varrho(\alpha_i) \notin \frac{1}{3}\mathbb{Z}$ for any simple coroot α_i .

Remark 1.1.7. *Let G be of type E_n with $6 \leq n \leq 8$. As in the proof of Theorem 1.1.6, one shows that there is a collection of positive integers d_1, \dots, d_r (depending on n) with the following property. If $\varrho(\alpha_i) \notin \frac{1}{d_1}\mathbb{Z}, \dots, \frac{1}{d_r}\mathbb{Z}$ for any simple coroot α_i then the property (C) is satisfied for ϱ . This collection can be found in principle in a way similar to the one we use for other types, however, this requires a lot of explicit calculations. They could certainly be done with a suitable computer program (like [14]).*

In Section A.2 of Appendix A, we consider G of type E_8 and establish a necessary condition for the property (C). Namely, one needs at least that $\varrho(\alpha_i) \notin \frac{1}{10}\mathbb{Z}, \frac{1}{8}\mathbb{Z}, \frac{1}{6}\mathbb{Z}$ for the property (C) to hold for ϱ in this case.

1.2. Proof of Theorem 1.1.5.

1.2.1. Over $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_B^{-\lambda}$ we get two different trivializations of the \mathbb{G}_m -torsor $\mathrm{Gra}_G \rightarrow \mathrm{Gr}_G$, the first coming from Gr_B^0 , the second one from that over $\mathrm{Gr}_B^{-\lambda}$. The discrepancy between the two trivializations is a map $\gamma_G : \mathrm{Gr}_B^0 \cap \mathrm{Gr}_B^{-\lambda} \rightarrow \mathbb{G}_m$ that intertwines the natural $T(\mathcal{O})$ -action on the source with the $T(\mathcal{O})$ -action on \mathbb{G}_m by the character $T(\mathcal{O}) \rightarrow T \xrightarrow{\bar{\kappa}(\lambda)} \mathbb{G}_m$. To be precise, for the corresponding sections $s_B^0 : \mathrm{Gr}_B^0 \rightarrow \mathrm{Gra}_G$ and $s_B^{-\lambda} : \mathrm{Gr}_B^{-\lambda} \rightarrow \mathrm{Gra}_G$ one has $s_B^{-\lambda} = \gamma_G s_B^0$. Note that $s_{-\lambda}^* \mathcal{L}_G \xrightarrow{\sim} ev^* \mathcal{L}_\psi \otimes \gamma_G^* \mathcal{L}_\zeta$.

Recall that the restriction of $ev : \mathrm{Gr}_B^0 \cap \mathrm{Gr}_B^{-\lambda} \rightarrow \mathbb{A}^1$ to each irreducible component of $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_B^{-\lambda}$ is dominant ([20], Section 5.6). So, (7) is placed in degrees $\leq \mathrm{top} - 1$.

1.2.2. *Recollections on crystals.* As in [9], write $B_{\mathfrak{g}}(\lambda)$ for the set of irreducible components of $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^{-\lambda}$. One has the structure of a crystal on $B_{\mathfrak{g}} = \cup_{\lambda \geq 0} B_{\mathfrak{g}}(\lambda)$ defined in ([9], Sections 13.3-13.4). We recall the part of this crystal structure used in our proof.

For a standard parabolic $P \subset G$ with Levi quotient M let $\mathfrak{q}_P : \mathrm{Gr}_P \rightarrow \mathrm{Gr}_M$ be the natural map. Write $B(M)$ and $B^-(M)$ for the corresponding Borel subgroups of M . For $\lambda \geq 0$ the scheme $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^{-\lambda}$ is stratified by locally closed subschemes $\mathrm{Gr}_B^0 \cap \mathfrak{q}_P^{-1}(\mathrm{Gr}_{B^-(M)}^{-\mu}) \cap \mathrm{Gr}_{B^-}^{-\lambda}$ indexed by $0 \leq_M \mu \leq \lambda$. For such μ and any $g \in \mathrm{Gr}_{B^-(M)}^{-\mu}$ one has an isomorphism

$$(8) \quad \mathrm{Gr}_B^0 \cap \mathfrak{q}_P^{-1}(\mathrm{Gr}_{B^-(M)}^{-\mu}) \cap \mathrm{Gr}_{B^-}^{-\lambda} \xrightarrow{\sim} (\mathrm{Gr}_{B(M)}^0 \cap \mathrm{Gr}_{B^-(M)}^{-\mu}) \times (\mathfrak{q}_P^{-1}(g) \cap \mathrm{Gr}_{B^-}^{-\lambda})$$

Denote by $B_{\mathfrak{g}}^{\mathfrak{m},*}(\lambda - \mu)$ the set of irreducible components of $\mathfrak{q}_P^{-1}(g) \cap \mathrm{Gr}_{B^-}^{-\lambda}$ of (maximal possible) dimension $\langle \lambda - \mu, \check{\rho} \rangle$. This set is independent of $g \in \mathrm{Gr}_{B^-(M)}^{-\mu}$ in a natural sense (see *loc.cit.*). One gets the bijection

$$B_{\mathfrak{g}}(\lambda) \xrightarrow{\sim} \cup_{\mu} B_{\mathfrak{g}}^{\mathfrak{m},*}(\lambda - \mu) \times B_{\mathfrak{m}}(\mu)$$

sending an irreducible component b of $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^{-\lambda}$ to the pair (b_1, b_2) defined as follows. First, there is a unique $\mu \in \Lambda$ with $0 \leq_M \mu \leq \lambda$ such that $b \cap \mathfrak{q}_P^{-1}(\mathrm{Gr}_{B^-(M)}^{-\mu})$ is dense in b . Then $b \cap \mathfrak{q}_P^{-1}(\mathrm{Gr}_{B^-(M)}^{-\mu})$ corresponds via (8) to (b_1, b_2) .

For $i \in \mathcal{J}$ the operation $f_i : B_{\mathfrak{g}} \rightarrow B_{\mathfrak{g}} \cup 0$ is defined as follows. Let P_i be the standard parabolic whose Levi M_i has a unique simple coroot α_i . Our convention is that $f_i : B_{\mathfrak{m}_i} \rightarrow B_{\mathfrak{m}_i} \cup 0$ sends the unique element of $B_{\mathfrak{m}_i}(\nu)$ to the unique element of $B_{\mathfrak{m}_i}(\nu - \alpha_i)$ for $\nu \geq_{M_i} \alpha_i$ (resp., to 0 for $\nu = 0$). For the corresponding decomposition

$$B_{\mathfrak{g}}(\lambda) \xrightarrow{\sim} \cup_{\mu} B_{\mathfrak{g}}^{\mathfrak{m}_i,*}(\lambda - \mu) \times B_{\mathfrak{m}_i}(\mu)$$

write $b \in B_{\mathfrak{g}}(\lambda)$ as (b_1, b_2) . Then $f_i(b_1, b_2) = (b_1, f_i(b_2))$ by definition.

For $i \in \mathcal{J}$, $b \in B_{\mathfrak{g}}(\nu)$ set $\phi_i(b) = \max\{m \geq 0 \mid f_i^m b \neq 0\}$.

Let $B(-\infty)$ denote the standard crystal of the canonical base in $U(\check{\mathfrak{u}})$, here $\check{\mathfrak{u}}$ is the Lie algebra of the unipotent radical of the Borel $\check{B} \subset \check{G}$. It coincides with the crystal introduced in ([21], Remark 8.3). A canonical isomorphism $B_{\mathfrak{g}} \xrightarrow{\sim} B(-\infty)$ is established in [9]. For $\lambda \in \Lambda$ denote by T_{λ} the crystal with the unique element of weight λ , the notation from ([21], Example 7.3) and ([6], Section 2.2). For $\lambda \in \Lambda^+$ denote by $B(\lambda)$ the crystal of the canonical base of the irreducible \check{G} -representation \mathbb{V}^{λ} of highest weight λ . We identify it canonically with the crystal denoted by $B^G(\lambda)$ in ([11], Section 3.1). So, an element of $B(\lambda)$ is an irreducible component of $\mathrm{Gr}_B^{\nu} \cap \mathrm{Gr}_G^{\lambda}$ for some $\nu \in \Lambda$ appearing as a weight of \mathbb{V}^{λ} . Recall from ([6], Section 2.2) that for $\lambda \in \Lambda^+$ there is a canonical embedding $B(\lambda) \hookrightarrow T_{w_0(\lambda)} \otimes B(-\infty)$ whose image is

$$(9) \quad \{t_{w_0(\lambda)} \otimes b \mid b \in B(-\infty), \phi_i(b^*) \leq -\langle w_0(\check{\alpha}_i), \lambda \rangle \text{ for all } i \in \mathcal{J}\}$$

Here $B(-\infty) \rightarrow B(-\infty), b \mapsto b^*$ is the involution defined in ([6], Section 2.2). This inclusion is described in the geometric terms in ([6], Proposition 4.3). The involution $*$ is also described in geometric terms as the one coming from an automorphism of G in ([6], Section 4.1, p. 100).

1.2.3. Let $\bar{\mu} = \{\mu_i\}_{i \in \mathcal{J}}$ with $\mu_i \in \Lambda$, $\lambda \geq \mu_i \geq_{M_i} 0$. We have the corresponding maps $\mathfrak{q}_{P_i} : \mathrm{Gr}_{P_i} \rightarrow \mathrm{Gr}_{M_i}$. Set

$$Y^{\bar{\mu}} = \left(\bigcap_{i \in \mathcal{J}} \mathfrak{q}_{P_i}^{-1}(\mathrm{Gr}_{B^-(M_i)}^{-\mu_i}) \right) \cap \mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^{-\lambda}.$$

The scheme $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^{-\lambda}$ is stratified by locally closed subschemes $Y^{\bar{\mu}}$ for the collections $\bar{\mu}$ as above (some strata could be empty). Our strategy is to show that each stratum $Y^{\bar{\mu}}$ does not contribute to $\mathrm{top} - 1$ cohomology in (7).

Set $Z^{\bar{\mu}} = \prod_{i \in \mathcal{J}} \mathrm{Gr}_{B(M_i)}^0 \cap \mathrm{Gr}_{B^-(M_i)}^{-\mu_i}$. Let

$$\mathfrak{q}^{\bar{\mu}} : Y^{\bar{\mu}} \rightarrow Z^{\bar{\mu}}$$

be the product of the maps \mathfrak{q}_{P_i} . Write $U(M_i)$ for the unipotent radical of $B(M_i)$. For each $i \in \mathcal{J}$ define $ev_i : \mathrm{Gr}_{B(M_i)}^0 \rightarrow \mathbb{A}^1$ by $ev_i(uM_i(\mathcal{O})) = \chi_0(u)$ for $u \in U(M_i)(F)$. We have used here some section $M_i \hookrightarrow P_i$. For $ev^{\bar{\mu}} : Z^{\bar{\mu}} \rightarrow \mathbb{A}^1$ given by $ev^{\bar{\mu}} = \sum_{i \in \mathcal{J}} ev_i$ the restriction $ev|_{Y^{\bar{\mu}}}$ equals $ev^{\bar{\mu}} \mathfrak{q}^{\bar{\mu}}$.

By Definition 1.1.4, we assume $Z(G)$ connected and pick fundamental coweights ω_i of \check{G} . Note that $\gamma_G^* \mathcal{L}_\zeta$ is equivariant under the action of $\mathrm{Ker}(T(\mathcal{O}) \rightarrow T)$. If there is $i \in \mathcal{J}$ such that $\mu_i \geq_{M_i} 2\alpha_i$ then under the action of $\mathrm{Ker}(\mathcal{O}^* \xrightarrow{\omega_i} T(\mathcal{O}) \rightarrow T)$ the sheaf $ev_i^* \mathcal{L}_\psi$ on $\mathrm{Gr}_{B(M_i)}^0 \cap \mathrm{Gr}_{B^-(M_i)}^{-\mu_i}$ will change by a nontrivial additive character. Therefore, $ev^* \mathcal{L}_\psi \otimes \gamma_G^* \mathcal{L}_\zeta$ on $Y^{\bar{\mu}}$ will also change by a nontrivial additive character under the action of this group. So, the integral over this stratum vanishes by ([25], Lemma 3.3).

Assume from now on that each μ_i is either α_i or zero. The stratum $Y^{\bar{\mu}}$, where $\mu_i = 0$ for all i , is of dimension $< \langle \lambda, \check{\rho} \rangle$ by ([20], Section 5.6).

Consider a stratum $Y^{\bar{\mu}}$ such that $\mu_i \neq 0$ for precisely m different elements $i \in \mathcal{J}$ with $m \geq 2$. Recall that $\mathrm{Gr}_{B(M_i)}^0 \cap \mathrm{Gr}_{B^-(M_i)}^{-\alpha_i} \xrightarrow{\sim} \mathbb{G}_m$. The group T acts transitively on $Z^{\bar{\mu}}$. Since $\mathfrak{q}^{\bar{\mu}}$ is $T(\mathcal{O})$ -equivariant, the dimensions of the fibres of $\mathfrak{q}^{\bar{\mu}}$ are $\leq \langle \lambda, \check{\rho} \rangle - m$. Our claim in this case is reduced to the following. For any $T(\mathcal{O})$ -equivariant constructible sheaf F on $Z^{\bar{\mu}}$, the complex $\mathrm{R}\Gamma_c(Z^{\bar{\mu}}, F \otimes (ev^{\bar{\mu}})^* \mathcal{L}_\psi)$ is placed in degrees $\leq m$. This is easy to check.

The only remaining case is the stratum $Y^{\bar{\mu}}$ such that there is $i \in \mathcal{J}$ with $\mu_i = \alpha_i$ and $\mu_j = 0$ for $j \neq i$. In particular, $\lambda \geq \alpha_i$. We may assume that $Y^{\bar{\mu}}$ contains an irreducible component b of dimension $\langle \lambda, \check{\rho} \rangle$, otherwise this stratum does not contribute to $\mathrm{top} - 1$ cohomology in (7). The closure of b in $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^{-\lambda}$ is an element $\bar{b} \in B_{\mathfrak{g}}(\lambda)$ such that $f_j \bar{b} = 0$ for $j \neq i$ and $f_i^2 \bar{b} = 0$. The following is derived from ([21], Proposition 8.2, Section 8.3), see the formula (9).

Proposition 1.2.4. *Pick $i \in I$. If $\nu > 0$ and $\bar{b} \in B_{\mathfrak{g}}(\nu)$ such that $f_j \bar{b} = 0$ for all $j \neq i$, and $f_i^2 \bar{b} = 0$ then $\omega_i - \nu$ appears in the fundamental representation \mathbb{V}^{ω_i} of \check{G} with highest weight ω_i . In other words, $w(\omega_i - \nu) \leq \omega_i$ for all $w \in W$.*

We conclude that $\omega_i - \lambda$ appears in \mathbb{V}^{ω_i} (for other λ the proof is already finished).

For $P = P_i$ and $g = t^{-\alpha_i}$ the isomorphism (8) becomes

$$(10) \quad \mathrm{Gr}_B^0 \cap \mathfrak{q}_{P_i}^{-1}(\mathrm{Gr}_{B^-(M_i)}^{-\alpha_i}) \cap \mathrm{Gr}_{B^-}^{-\lambda} \xrightarrow{\sim} (\mathrm{Gr}_{B(M_i)}^0 \cap \mathrm{Gr}_{B^-(M_i)}^{-\alpha_i}) \times (\mathfrak{q}_{P_i}^{-1}(t^{-\alpha_i}) \cap \mathrm{Gr}_{B^-}^{-\lambda})$$

We let $T(\mathcal{O})$ act on the right hand side of (10) as the product of the natural actions of $T(\mathcal{O})$ on the two factors. Then (10) is $T(\mathcal{O})$ -equivariant (see Section 1.2.5). The \mathbb{G}_m -torsor $\text{Gra}_G \rightarrow \text{Gr}_G$ is constant over $\mathfrak{q}_{P_i}^{-1}(t^{-\alpha_i})$ with fibre $\Omega_{\bar{c}}^{-\bar{\kappa}(\alpha_i, \alpha_i)} - 0$, and $T(\mathcal{O})$ acts on it by the character

$$T(\mathcal{O}) \rightarrow T \xrightarrow{\bar{\kappa}(\alpha_i)} \mathbb{G}_m$$

Pick any trivialization of $\Omega_{\bar{c}}^{-\bar{\kappa}(\alpha_i, \alpha_i)}$, let $\bar{s}_i : \mathfrak{q}_{P_i}^{-1}(t^{-\alpha_i}) \rightarrow \text{Gra}_G$ be the corresponding section of the \mathbb{G}_m -torsor. We get the discrepancy function $\gamma_i : \mathfrak{q}_{P_i}^{-1}(t^{-\alpha_i}) \cap \text{Gr}_{B^-}^{-\lambda} \rightarrow \mathbb{G}_m$ such that $s_{B^-}^{-\lambda} = \gamma_i \bar{s}_i$ over $\mathfrak{q}_{P_i}^{-1}(t^{-\alpha_i}) \cap \text{Gr}_{B^-}^{-\lambda}$. The map γ_i intertwines the natural $T(\mathcal{O})$ -action on $\mathfrak{q}_{P_i}^{-1}(t^{-\alpha_i}) \cap \text{Gr}_{B^-}^{-\lambda}$ with the action on \mathbb{G}_m by $T(\mathcal{O}) \rightarrow T \xrightarrow{\bar{\kappa}(\lambda - \alpha_i)} \mathbb{G}_m$.

Let Gra_{M_i} be the restriction of Gra_G under $\text{Gr}_{M_i} \rightarrow \text{Gr}_G$. As for G , one defines the discrepancy function $\gamma_{M_i} : \text{Gr}_{B(M_i)}^0 \cap \text{Gr}_{B^-(M_i)}^{-\alpha_i} \rightarrow \mathbb{G}_m$. The map

$$(\text{Gr}_{B(M_i)}^0 \cap \text{Gr}_{B^-(M_i)}^{-\alpha_i}) \times (\mathfrak{q}_{P_i}^{-1}(t^{-\alpha_i}) \cap \text{Gr}_{B^-}^{-\lambda}) \xrightarrow{\gamma_{M_i} \gamma_i} \mathbb{G}_m$$

coincides with the restriction of γ_G .

There is a $T(\mathcal{O})$ -invariant subscheme $\mathcal{Y} \subset \mathfrak{q}_{P_i}^{-1}(t^{-\alpha_i}) \cap \text{Gr}_{B^-}^{-\lambda}$ such that (10) restricts to an isomorphism

$$Y^{\bar{\mu}} \xrightarrow{\sim} (\text{Gr}_{B(M)}^0 \cap \text{Gr}_{B^-(M)}^{-\alpha_i}) \times \mathcal{Y}$$

The contribution of $Y^{\bar{\mu}}$ becomes

$$\text{R}\Gamma_c(\text{Gr}_{B(M)}^0 \cap \text{Gr}_{B^-(M)}^{-\alpha_i}, ev_i^* \mathcal{L}_\psi \otimes \gamma_{M_i}^* \mathcal{L}_\zeta) \otimes \text{R}\Gamma_c(\mathcal{Y}, \gamma_i^* \mathcal{L}_\zeta)$$

We have $\dim(\mathcal{Y}) \leq \langle \lambda, \check{\rho} \rangle - 1$. To finish the proof it suffices to show that $\gamma_i^* \mathcal{L}_\zeta$ is nonconstant on each irreducible component of \mathcal{Y} of dimension $\langle \lambda, \check{\rho} \rangle - 1$. This is the case, because the character $\bar{\kappa}(\lambda - \alpha_i)$ is not divisible by N in $\check{\Lambda}$, so that $\gamma_i^* \mathcal{L}_\zeta$ changes under the $T(\mathcal{O})$ -action by a nontrivial character. Theorem 1.1.5 is proved.

1.2.5. Equivariant decomposition. If G is a group scheme, and $f : Y \rightarrow Z$ is a G -equivariant map such that G acts transitively on Z , assume that for any $y \in Y$, the inclusion $\text{Stab}_G(y, Y) \subset \text{Stab}_G(f(y), Z)$ is an equality. Then a choice of $z \in Z$ yields an isomorphism $\xi : Z \times f^{-1}(z) \xrightarrow{\sim} Y$. Namely, let $S = \text{Stab}_G(z, Z)$. The map $(G/S) \times f^{-1}(z) \rightarrow Y$, $(gS, y) \mapsto gy$ is well defined and gives this isomorphism.

Assume in addition we have a semi-direct product $1 \rightarrow G \rightarrow \tilde{G} \rightarrow H \rightarrow 1$ with a section $H \hookrightarrow \tilde{G}$ as a subgroup. Assume f is in addition \tilde{G} -equivariant. Assume $z \in Z$ is fixed by H . Then SH is a subgroup of \tilde{G} equal to $\text{Stab}_{\tilde{G}}(z, Z)$. So, H acts on S by conjugation. If we identify $G/S \xrightarrow{\sim} Z$, $gS \mapsto gz$ then the action of $h \in H$ on $gS \in G/S \xrightarrow{\sim} Z$ sends gS to $hgh^{-1}S$. Now $\xi : Z \times f^{-1}(z) \xrightarrow{\sim} Y$ becomes H -equivariant if we let $h \in H$ act on $Z \times f^{-1}(z)$ as the product of the actions, that is, $h \in H$ acts on $(z_1, y) \in Z \times f^{-1}(z)$ as (hz_1, hy) .

2. THE TWISTED WHITTAKER CATEGORY

2.1. The definition of the twisted Whittaker category from ([20], Section 2) naturally extends to our setting, we give the detailed exposition. For $\lambda \in \Lambda^+$ denote by \mathcal{V}^λ the

corresponding Weyl module for G as in ([12], Section 0.4.1). For $n \geq 0$ let \mathfrak{M}_n be the stack classifying:

- $(x_1, \dots, x_n) \in X^n$, a G -torsor \mathcal{F} on X ,
- for each $\check{\lambda} \in \check{\Lambda}^+$ a non-zero map

$$(11) \quad \kappa^{\check{\lambda}} : \Omega^{\langle \check{\lambda}, \rho \rangle} \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}},$$

which is allowed to have any poles at x_1, \dots, x_n . The maps $\kappa^{\check{\lambda}}$ are required to satisfy the Plücker relations as in [12].

For $n = 0$ the stack \mathfrak{M}_n is rather denoted by \mathfrak{M}_{\emptyset} . Let $\mathfrak{p} : \mathfrak{M}_n \rightarrow \text{Bun}_G$ be the map sending the above point to \mathcal{F} .

Let $\mathcal{P}^{\check{\kappa}}$ denote the line bundle $\mathfrak{p}^*(\omega \mathcal{L}^{\check{\kappa}})$ on \mathfrak{M}_n . By $\widetilde{\mathfrak{M}}_n$ we denote the gerb of N -th roots of $\mathcal{P}^{\check{\kappa}}$ over \mathfrak{M}_n . Let $D_{\zeta}(\mathfrak{M}_n)$ denote the derived category of \mathbb{Q}_{ℓ} -sheaves on $\widetilde{\mathfrak{M}}_n$, on which $\mu_N(k)$ acts by ζ . This category does not change (up to an equivalence) if $\check{\kappa}$ and N are multiplied by the same integer, so essentially depends only on ϱ .

2.2. Pick $y \in X$. Write D_y (resp., D_y^*) for the formal disk (resp., punctured formal disk) around $y \in X$. Let Ω_B^{ρ} be the B -torsor on X obtained from Ω^{ρ} via extension of scalars $T \rightarrow B$. Let ${}^{\omega}\mathcal{N}$ be the group scheme over X of automorphisms of Ω_B^{ρ} acting trivially on the induced T -torsor. Let $\mathcal{N}_y^{\text{reg}}$ (resp., $\mathcal{N}_y^{\text{mer}}$) be the group scheme (resp., group ind-scheme) of sections of ${}^{\omega}\mathcal{N}$ over D_y (resp., D_y^*). Recall that

$$\mathcal{N}_y^{\text{mer}} / [\mathcal{N}_y^{\text{mer}}, \mathcal{N}_y^{\text{mer}}] \xrightarrow{\sim} \Omega|_{D_y^*} \times \dots \times \Omega|_{D_y^*},$$

the product taken over simple roots of G . Taking the sum of residues in this product, one gets the character $\chi_y : \mathcal{N}_y^{\text{mer}} \rightarrow \mathbb{A}^1$.

As in ([20], Section 2.3) for a collection of distinct points $\bar{y} := y_1, \dots, y_m$ let $\mathcal{N}_{\bar{y}}^{\text{reg}}$ (resp., $\mathcal{N}_{\bar{y}}^{\text{mer}}$) denote the product of the corresponding groups $\mathcal{N}_{y_i}^{\text{reg}}$ (resp., $\mathcal{N}_{y_i}^{\text{mer}}$). The sum of the corresponding characters gives the character $\chi_{\bar{y}} : \mathcal{N}_{\bar{y}}^{\text{mer}} \rightarrow \mathbb{A}^1$.

Let $(\mathfrak{M}_n)_{\text{good at } \bar{y}} \subset \mathfrak{M}_n$ be the open substack given by the property that all x_i are different from the points of \bar{y} , and $\kappa^{\check{\lambda}}$ have no zeros at \bar{y} . A point of $(\mathfrak{M}_n)_{\text{good at } \bar{y}}$ defines a B -torsor \mathcal{F}_B over $D_{\bar{y}} = \prod_{j=1}^m D_{y_j}$ equipped with a trivialization $\epsilon_B : \mathcal{F}_B \times_B T \xrightarrow{\sim} \Omega^{\rho}$ over $D_{\bar{y}}$.

Let ${}_{\bar{y}}\mathfrak{M}_n$ denote the $\mathcal{N}_{\bar{y}}^{\text{reg}}$ -torsor over $(\mathfrak{M}_n)_{\text{good at } \bar{y}}$ classifying a point of $(\mathfrak{M}_n)_{\text{good at } \bar{y}}$ as above together with a trivialization $\mathcal{F}_B \xrightarrow{\sim} \Omega_B^{\rho}|_{D_{\bar{y}}}$ compatible with ϵ_B .

Now ${}_{\bar{y}}\mathfrak{M}_n$ can be seen as the stack classifying: $(x_1, \dots, x_n) \in X^n$ different from \bar{y} , a G -torsor \mathcal{F} over $X - \bar{y}$ with a trivialization $\epsilon_{\mathcal{F}} : \mathcal{F} \xrightarrow{\sim} \Omega_B^{\rho} \times_B G|_{D_{\bar{y}}^*}$, for $\check{\lambda} \in \check{\Lambda}^+$ non-zero maps (11) over $X - \bar{y} - \bar{x}$ satisfying the Plücker relations and compatible with the trivialization $\epsilon_{\mathcal{F}}$. Here we denoted $D_{\bar{y}}^* \xrightarrow{\sim} \prod_{j=1}^m D_{y_j}^*$.

The group $\mathcal{N}_{\bar{y}}^{\text{mer}}$ acts on ${}_{\bar{y}}\mathfrak{M}_n$ by changing the trivialization $\epsilon_{\mathcal{F}}$ via its action on $\Omega_B^{\rho}|_{D_{\bar{y}}^*}$. The composition ${}_{\bar{y}}\mathfrak{M}_n \rightarrow \mathfrak{M}_n \xrightarrow{\mathfrak{p}} \text{Bun}_G$ sends the above point to the gluing of $\mathcal{F}|_{X-\bar{y}}$ with $\Omega_B^{\rho} \times_B G|_{D_{\bar{y}}}$ via $\epsilon_{\mathcal{F}} : \mathcal{F} \xrightarrow{\sim} \Omega_B^{\rho} \times_B G|_{D_{\bar{y}}^*}$.

Denote by ${}_{\bar{y}}\mathcal{P}^{\check{\kappa}}$ the restriction of $\mathcal{P}^{\check{\kappa}}$ to ${}_{\bar{y}}\mathfrak{M}_n$. As in ([20], Lemma 2.4), the action of $\mathcal{N}_{\bar{y}}^{\text{mer}}$ on ${}_{\bar{y}}\mathfrak{M}_n$ lifts naturally to an action on ${}_{\bar{y}}\mathcal{P}^{\check{\kappa}}$.

Let $\widetilde{\mathfrak{M}}_n$ (resp., $_{\bar{y}}\widetilde{\mathfrak{M}}_n, (\widetilde{\mathfrak{M}}_n)_{\text{good at } \bar{y}}$) be the gerb of N -th roots of the corresponding line bundle $\mathcal{P}^{\bar{\kappa}}$ (resp., its restriction). We denote by $\text{Perv}_{\zeta}((\widetilde{\mathfrak{M}}_n)_{\text{good at } \bar{y}})$ the category of perverse sheaves on $(\widetilde{\mathfrak{M}}_n)_{\text{good at } \bar{y}}$, on which $\mu_N(k)$ acts by zeta. Write $(\text{Whit}_n^{\kappa})_{\text{good at } \bar{y}}$ for the full subcategory of $\text{Perv}_{\zeta}((\widetilde{\mathfrak{M}}_n)_{\text{good at } \bar{y}})$ consisting of perverse sheaves, whose restriction to $_{\bar{y}}\widetilde{\mathfrak{M}}_n$ is $(\mathcal{N}_{\bar{y}}^{\text{mer}}, \chi_{\bar{y}}^* \mathcal{L}_{\psi})$ -equivariant (as in [20], Section 2.5).

If \bar{y}' and \bar{y}'' are two collections of points, set $\bar{y} = \bar{y}' \cup \bar{y}''$. Over $(\widetilde{\mathfrak{M}}_n)_{\text{good at } \bar{y}}$ one gets the corresponding torsors with respect to each of the groups

$$\mathcal{N}_{\bar{y}'}^{\text{reg}}, \mathcal{N}_{\bar{y}''}^{\text{reg}}, \mathcal{N}_{\bar{y}}^{\text{reg}}$$

As in ([20], Section 2.5), the three full subcategories of $\text{Perv}_{\zeta}((\widetilde{\mathfrak{M}}_n)_{\text{good at } \bar{y}})$ given by the equivariance condition with respect to one of these groups are equal.

Let $\text{Whit}_n^{\kappa} \subset \text{Perv}_{\zeta}(\widetilde{\mathfrak{M}}_n)$ be the full subcategory of $F \in \text{Perv}_{\zeta}(\widetilde{\mathfrak{M}}_n)$ such that for any \bar{y} as above, the restriction of F to $(\widetilde{\mathfrak{M}}_n)_{\text{good at } \bar{y}}$ lies in $(\text{Whit}_n^{\kappa})_{\text{good at } \bar{y}}$. As in ([19], Lemma 4.8), the full subcategory $\text{Whit}_n^{\kappa} \subset \text{Perv}_{\zeta}(\widetilde{\mathfrak{M}}_n)$ is stable under sub-quotients and extensions, and is therefore a Serre subcategory. So, we also define the full triangulated subcategory $\text{DWhit}_n^{\kappa} \subset \text{D}_{\zeta}(\widetilde{\mathfrak{M}}_n)$ of complexes whose all perverse cohomologies lie in Whit_n^{κ} .

The Verdier duality preserves Whit_n^{κ} (up to replacing ψ by ψ^{-1} and ζ by ζ^{-1}), because the corresponding action maps are smooth (as in [19], Section 4.7).

2.3. For a n -tuple $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$ of dominant coweights of G let $\mathfrak{M}_{n, \leq \bar{\lambda}} \subset \mathfrak{M}_n$ be the closed substack given by the property that for each $\check{\lambda} \in \check{\Lambda}^+$ the map

$$(12) \quad \kappa^{\check{\lambda}} : \Omega^{\langle \rho, \check{\lambda} \rangle} \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \left(\sum_i \langle \lambda_i x_i, \check{\lambda} \rangle \right)$$

is regular over X . For $\bar{x} = (x_1, \dots, x_n) \in X^n$ fixed let $\mathfrak{M}_{\bar{x}}$ denote the fibre of \mathfrak{M}_n over this point of X^n . Write $\text{Whit}_{\bar{x}}^{\kappa}$ for the corresponding version of the Whittaker category of twisted perverse sheaves on $\mathfrak{M}_{\bar{x}}$. (By a twisted perverse sheaf on a base we mean a perverse sheaf on some gerb over this base).

Assume (x_1, \dots, x_n) pairwise different. Define the closed substack $\mathfrak{M}_{\bar{x}, \leq \bar{\lambda}} \subset \mathfrak{M}_{\bar{x}}$ as above. The irreducible objects of $\text{Whit}_{\bar{x}}^{\kappa}$ are as follows. Let $\mathfrak{M}_{\bar{x}, \bar{\lambda}} \subset \mathfrak{M}_{\bar{x}, \leq \bar{\lambda}}$ be the open substack given by the property that for each $\check{\lambda} \in \check{\Lambda}^+$ the map (12) has no zeros over X . Let

$$j_{\bar{x}, \bar{\lambda}} : \mathfrak{M}_{\bar{x}, \bar{\lambda}} \hookrightarrow \mathfrak{M}_{\bar{x}, \leq \bar{\lambda}}$$

be the corresponding open immersion. Recall that $j_{\bar{x}, \bar{\lambda}}$ is affine ([18], Proposition 3.3.1).

In the same way, one defines the version of the Whittaker category of twisted perverse sheaves on $\mathfrak{M}_{\bar{x}, \bar{\lambda}}$. As in ([20], Lemma 2.7), this category is non-canonically equivalent to that of vector spaces. Let $\mathcal{F}_{\bar{x}, \bar{\lambda}}$ denote the unique (up to a non-canonical scalar automorphism) irreducible object of this category. As in ([18], Section 4.2.1), one defines a canonical evaluation map $ev_{\bar{x}, \bar{\lambda}} : \mathfrak{M}_{\bar{x}, \bar{\lambda}} \rightarrow \mathbb{A}^1$. The restriction of the line bundle $\mathcal{P}^{\bar{\kappa}}$ to $\mathfrak{M}_{\bar{x}, \bar{\lambda}}$ is constant with fibre

$$(13) \quad \omega \mathcal{L}_{\Omega^{\rho}(-\sum_i \lambda_i x_i)}^{\bar{\kappa}}$$

Any trivialization of (13) yields a trivialization $\widetilde{\mathfrak{M}}_{\bar{x},\bar{\lambda}} \xrightarrow{\sim} \mathfrak{M}_{\bar{x},\bar{\lambda}} \times B(\mu_N)$ of the gerb $\widetilde{\mathfrak{M}}_{\bar{x},\bar{\lambda}} \rightarrow \mathfrak{M}_{\bar{x},\bar{\lambda}}$. There is an isomorphism

$$\mathcal{F}_{\bar{x},\bar{\lambda}} \xrightarrow{\sim} ev_{\bar{x},\bar{\lambda}}^* \mathcal{L}_\psi \boxtimes \mathcal{L}_\zeta[\dim \mathfrak{M}_{\bar{x},\bar{\lambda}}]$$

For $\bar{\lambda} = 0$ the line (13) is canonically trivialized. So, $\mathcal{F}_{\bar{x},0}$ is defined up to a canonical isomorphism.

Let $\mathcal{F}_{\bar{x},\bar{\lambda},!}$ (resp., $\mathcal{F}_{\bar{x},\bar{\lambda},*}$, $\mathcal{F}_{\bar{x},\bar{\lambda}}$) denote the extension of $\mathcal{F}_{\bar{x},\bar{\lambda}}$ by $j_{\bar{x},\bar{\lambda},!}$ (resp., $j_{\bar{x},\bar{\lambda},*}$, $j_{\bar{x},\bar{\lambda},!*}$). Since $j_{\bar{x},\bar{\lambda}}$ is affine, these are perverse sheaves. As in ([18], Proposition 6.2.1), one checks that all of three are objects of $\text{Whit}_{\bar{x}}^\kappa$, and the version of ([20], Lemma 2.8) holds:

Lemma 2.3.1. (a) *Every irreducible object in $\text{Whit}_{\bar{x}}^\kappa$ is of the form $\mathcal{F}_{\bar{x},\bar{\lambda}}$ for some n -tuple of dominant coweights $\bar{\lambda}$.*

(b) *The cones of the canonical maps*

$$(14) \quad \mathcal{F}_{\bar{x},\bar{\lambda},!} \rightarrow \mathcal{F}_{\bar{x},\bar{\lambda}} \rightarrow \mathcal{F}_{\bar{x},\bar{\lambda},*}$$

are extensions of objects $\mathcal{F}_{\bar{x},\bar{\lambda}'}$ for $\bar{\lambda}' < \bar{\lambda}$.

Here the notation $\bar{\lambda}' < \bar{\lambda}$ means that $\lambda'_i \leq \lambda_i$ for all $1 \leq i \leq n$ and for at least one i the inequality is strict. Recall that the maps (14) are not isomorphisms in general. Let $D\text{Whit}_{\bar{x}}^\kappa \subset D_\zeta(\widetilde{\mathfrak{M}}_{\bar{x}})$ denote the full subcategory of objects whose all perverse cohomologies lie in $\text{Whit}_{\bar{x}}^\kappa$.

Remark 2.3.2. *Let $n = 1$. One may define a version of Kazhdan-Lusztig's polynomials expressing for $\mu < \lambda$ the $*$ -restriction of $\mathcal{F}_{x,\lambda}$ to $\mathfrak{M}_{x,\mu}$ via $\mathcal{F}_{x,\mu}$. In other words, expressing the relation between the two bases in the Grothendieck group of Whit_x^κ , the first constings of $\mathcal{F}_{x,\lambda,!}$, the second constings of the irreducible objects. To the best of our knowledge, they are not found in the published literature. According to Lurie's conjecture ([19], Conjecture 0.4), $\mathcal{F}_{x,\lambda}$ should correspond to the irreducible representations of the quantum group, and $\mathcal{F}_{x,\lambda,!}$ should correspond to the Verma modules. So, these polynomials will then give a relation between the two corresponding bases of the Grothendieck group of the category of certain representations of the big quantum group.*

2.4. The basic object of the category $\text{Whit}_\emptyset^\kappa$ is denoted \mathcal{F}_\emptyset . Recall the open substack $\mathfrak{M}_{\emptyset,0} \subset \mathfrak{M}_\emptyset$ given by the property that the maps (11) have neither zeros nor poles over X . Since there are no dominant weights < 0 , from Lemma 2.3.1 we learn that the canonical maps

$$j_{\emptyset,0,!}(\mathcal{F}_{\emptyset,0}) \xrightarrow{\sim} j_{\emptyset,0,!}(\mathcal{F}_{\emptyset,0}) \xrightarrow{\sim} j_{\emptyset,0,*}(\mathcal{F}_{\emptyset,0})$$

are isomorphisms.

2.5. For $n \geq 0$ and $\mu \in \Lambda$ let X_n^μ be the ind-scheme classifying $(x_1, \dots, x_n) \in X^n$, and a Λ -valued divisor D on X of degree μ which is anti-effective away from x_1, \dots, x_n . This means that for any $\bar{\lambda} \in \bar{\Lambda}^+$, $\langle \bar{\lambda}, D \rangle$ is anti-effective away from x_1, \dots, x_n .

For $n = 0$ we rather use the notation X_0^μ or X^μ instead of X_\emptyset^μ . If $\mu = -\sum_{i \in \mathcal{J}} m_i \alpha_i$ with $m_i \geq 0$ then $X^\mu = \prod_i X^{(m_i)}$.

For a n -tuple $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$ of elements of Λ denote by $X_{n, \leq \bar{\lambda}}^\mu \subset X_n^\mu$ the closed subscheme classifying $(x_1, \dots, x_n, D) \in X_n^\mu$ such that

$$D - \sum_{i=1}^n \lambda_i x_i$$

is anti-effective over X . We have an isomorphism $X^n \times X^{\mu - \lambda_1 - \dots - \lambda_n} \xrightarrow{\sim} X_{n, \leq \bar{\lambda}}^\mu$ sending (x_1, \dots, x_n, D') to $D' + \sum_{i=1}^n \lambda_i x_i$. For another collection $\bar{\lambda}' = (\lambda'_1, \dots, \lambda'_n)$ with $\lambda'_i \geq \lambda_i$ one has a natural closed embedding $X_{n, \leq \bar{\lambda}}^\mu \hookrightarrow X_{n, \leq \bar{\lambda}'}^\mu$, and

$$X_n^\mu = \lim_{\substack{\longrightarrow \\ \bar{\lambda}}} X_{n, \leq \bar{\lambda}}^\mu$$

2.5.1. By abuse of notation, the restriction of ${}^\omega \mathcal{L}^{\bar{\kappa}}$ under $\text{Bun}_T \rightarrow \text{Bun}_G$ is still denoted by ${}^\omega \mathcal{L}^{\bar{\kappa}}$. Let $AJ : X_n^\mu \rightarrow \text{Bun}_T$ be the Abel-Jacobi map sending (x_1, \dots, x_n, D) to $\Omega^\rho(-D)$. The line bundle $AJ^*({}^\omega \mathcal{L}^{\bar{\kappa}})$ is denoted by $\mathcal{P}^{\bar{\kappa}}$ by abuse of notations.

Denote by ${}^\omega \mathcal{L}_{j, \text{Bun}_G}$ the line bundle on Bun_G whose fibre at $\mathcal{F} \in \text{Bun}_G$ is $(\mathcal{L}_{j, \text{Bun}_G})_{\mathcal{F}} \otimes (\mathcal{L}_{j, \text{Bun}_G})_{\Omega^\rho}^{-1}$. For $D = \sum_x \mu_x x \in X_n^\mu$ one has

$$({}^\omega \mathcal{L}_{j, \text{Bun}_G})_{\Omega^\rho(-D)} \xrightarrow{\sim} \otimes_{x \in X} (\Omega_x^{\frac{1}{2}})^{\kappa_j(\mu_x, \mu_x + 2\rho)}$$

This isomorphism uses a trivialization of all the positive root spaces of \mathfrak{g} that we fix once and for all (they yield also trivializations of all the negative root spaces).

Lemma 2.5.2. *For $D = \sum_x \mu_x x \in X_n^\mu$ one has*

$$({}^\omega \mathcal{L}^{\bar{\kappa}})_{\Omega^\rho(-D)} \xrightarrow{\sim} \otimes_{x \in X} (\Omega_x^{\frac{1}{2}})^{-\bar{\kappa}(\mu_x, \mu_x + 2\rho)} \otimes \epsilon^{\bar{\mu}_x} \xrightarrow{\sim} (\otimes_{x \in X} (\Omega_x^{\frac{1}{2}})^{-\bar{\kappa}(\mu_x, \mu_x + 2\rho)}) \otimes (\otimes_{i=1}^n \epsilon^{\bar{\mu}_{x_i}})$$

where $\bar{\mu}_x \in \Lambda_{ab}$ is the image of μ_x .

Proof. Use Lemma 0.0.6 and the fact that ϵ^0 is trivialized. \square

Let \tilde{X}_n^μ denote the gerb of N -th roots of $\mathcal{P}^{\bar{\kappa}}$ over X_n^μ . Write $\text{Perv}_\zeta(X_n^\mu)$ for the category of perverse sheaves on \tilde{X}_n^μ , on which $\mu_N(k)$ acts by ζ . Similarly, one has the derived category $\text{D}_\zeta(X_n^\mu)$.

2.6. For $\mu \in \Lambda$ denote by ${}_\mu \mathfrak{M}_n \subset \mathfrak{M}_n$ the ind-substack classifying $(x_1, \dots, x_n, D) \in X_n^\mu$, a B -torsor \mathcal{F}_B on X with an isomorphism $\mathcal{F}_B \times_B T \xrightarrow{\sim} \Omega^\rho(-D)$. As μ varies in Λ this ind-stacks form a stratification of \mathfrak{M}_n . Let $\pi_{\mathfrak{M}} : {}_\mu \mathfrak{M}_n \rightarrow X_n^\mu$ be the map sending the above point to (x_1, \dots, x_n, D) .

For a collection $\bar{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Lambda^n$ let ${}_\mu \mathfrak{M}_{n, \leq \bar{\lambda}}$ be obtained from ${}_\mu \mathfrak{M}_n$ by the base change $\mathfrak{M}_{n, \leq \bar{\lambda}} \rightarrow \mathfrak{M}_n$. The map $\pi_{\mathfrak{M}}$ restricts to a morphism still denoted $\pi_{\mathfrak{M}} : {}_\mu \mathfrak{M}_{n, \leq \bar{\lambda}} \rightarrow X_{n, \leq \bar{\lambda}}^\mu$.

By the same token, one defines the version of the Whittaker category $\text{Whit}^\kappa({}_\mu \mathfrak{M}_n) \subset \text{Perv}_\zeta({}_\mu \tilde{\mathfrak{M}}_n)$ and its derived version $\text{DWhit}^\kappa({}_\mu \mathfrak{M}_n) \subset \text{D}_\zeta({}_\mu \tilde{\mathfrak{M}}_n)$.

Let ${}^+ X_n^\mu \hookrightarrow X_n^\mu$ be the closed subscheme given by the condition $\langle D, \check{\alpha} \rangle \geq 0$ for any simple root $\check{\alpha}$ of G . Let ${}^+ {}_\mu \mathfrak{M}_n$ be the preimage of ${}^+ X_n^\mu$ in ${}_\mu \mathfrak{M}_n$. As above, we have the natural evaluation map $ev : {}^+ {}_\mu \mathfrak{M}_n \rightarrow \mathbb{A}^1$. The derived category $\text{D}_\zeta({}^+ X_n^\mu)$ is defined as

in Section 2.5.1. Since the map $\pi_{\mathfrak{M}} : {}_{\mu}\mathfrak{M}_n \rightarrow X_n^{\mu}$ has contractible fibres, as in ([19], Proposition 4.13), one gets the following.

Lemma 2.6.1. *Each object of $\mathrm{DWhit}^{\kappa}({}_{\mu}\mathfrak{M}_n)$ is the extension by zero from ${}_{\mu}^+\mathfrak{M}_n$. The functor $\mathrm{D}_{\zeta}({}_{\mu}^+X_n^{\mu}) \rightarrow \mathrm{DWhit}^{\kappa}({}_{\mu}\mathfrak{M}_n)$ sending K to $\pi_{\mathfrak{M}}^*K \otimes ev^*\mathcal{L}_{\psi}$ is an equivalence.*

As in ([19], Lemma 4.11), one gets the following.

Lemma 2.6.2. *i) Let $\mu \in \Lambda$. The $*$ and $!$ restrictions send $\mathrm{DWhit}_n^{\kappa}$ to $\mathrm{DWhit}^{\kappa}({}_{\mu}\mathfrak{M}_n)$.
ii) The $*$ and $!$ direct images send $\mathrm{DWhit}^{\kappa}({}_{\mu}\mathfrak{M}_n)$ to $\mathrm{DWhit}_n^{\kappa}$.
iii) An object $K \in \mathrm{D}_{\zeta}(\mathfrak{M}_n)$ lies in $\mathrm{DWhit}_n^{\kappa}$ if and only if its $*$ -restrictions (or, equivalently, $!$ -restrictions) to all ${}_{\mu}\mathfrak{M}_n$ belong to $\mathrm{DWhit}^{\kappa}({}_{\mu}\mathfrak{M}_n)$.*

Remark 2.6.3. *i) Consider a point $(x_1, \dots, x_n, D) \in {}_{\mu}^+X_n^{\mu}$. Assume $(y_1, \dots, y_m) \in X^m$ pairwise different such that $\{y_1, \dots, y_m\} = \{x_1, \dots, x_n\}$. Then there is a collection of G -dominant coweights (μ_1, \dots, μ_m) such that $D = \sum_{i=1}^m \mu_i y_i$ with $\sum_{i=1}^m \mu_i = \mu$. In particular, ${}_{\mu}^+X_n^{\mu}$ is empty unless μ is G -dominant.*

ii) Let $\bar{x} = (x_1, \dots, x_n) \in X^n$ be a k -point with x_i pairwise different. Define ${}_{\bar{x}}^+X_n^{\mu}$ as the fibre of ${}_{\mu}^+X_n^{\mu}$ over $\bar{x} \in X^n$. Let $\bar{\lambda} \in \Lambda^n$ with $\mu \leq \sum_i \lambda_i$. Define the closed subscheme ${}_{\bar{x}, \leq \bar{\lambda}}^+X_n^{\mu}$ by the condition $D \leq \sum_i \lambda_i x_i$. Then ${}_{\bar{x}, \leq \bar{\lambda}}^+X_n^{\mu}$ is a discrete finite set of points.

2.7. Let $x \in X$. In Appendix B we show that the subtop cohomology property admits the following reformulation in terms of Whit_x^{κ} .

Proposition 2.7.1. *The following properties are equivalent.*

- i) The subtop cohomology property is satisfied for ϱ .*
- ii) Let $\lambda > 0$, which is not a simple coroot. For $\mu \in \Lambda^{\sharp}$ deep enough in the dominant chamber the complex $j_{x, \mu - \lambda}^* \mathcal{F}_{x, \mu}$ over $\widetilde{\mathfrak{M}}_{x, \mu - \lambda}$ is placed in perverse degrees ≤ -2 .*
- iii) Let $\lambda > 0$, which is not a simple coroot. For $\mu \in \Lambda^{\sharp}$ deep enough in the dominant chamber one has $\mathrm{Ext}^1(\mathcal{F}_{x, \mu - \lambda}, \mathcal{F}_{x, \mu}) = 0$ in Whit_x^{κ} .*

Based on this proposition, we propose the following.

Conjecture 2.7.2. *Let $\mu < \mu'$ be dominant coweights such that $\mu' - \mu$ is not a simple coroot. Then $\mathrm{Ext}^1(\mathcal{F}_{x, \mu}, \mathcal{F}_{x, \mu'}) = 0$ in Whit_x^{κ} .*

3. THE FS CATEGORY

3.1. The definition of the category of factorizable sheaves from ([20], Section 3) extends to our setting, we give a detailed exposition for the convenience of the reader.

For a partition $n = n_1 + n_2$, $\mu = \mu_1 + \mu_2$ with $\mu_i \in \Lambda$, let

$$\mathrm{add}_{\mu_1, \mu_2} : X_{n_1}^{\mu_1} \times X_{n_2}^{\mu_2} \rightarrow X_n^{\mu}$$

be the addition map. Given n_1 -tuple $\bar{\lambda}_1$, n_2 -tuple $\bar{\lambda}_2$ of coweights let

$$(X_{n_1, \leq \bar{\lambda}_1}^{\mu_1} \times X_{n_2, \leq \bar{\lambda}_2}^{\mu_2})_{\mathrm{dis}j}$$

be the open part of the product given by the property that the supports of the two divisors do not intersect. The restriction of $\mathrm{add}_{\mu_1, \mu_2}$ to the above scheme is an étale map to $X_{n, \leq \bar{\lambda}_1 \cup \bar{\lambda}_2}^{\mu}$.

From Lemma 2.5.2 we obtain the following factorization property

$$(15) \quad \text{add}_{\mu_1, \mu_2}^* \mathcal{P}^{\bar{\kappa}} |_{(X_{n_1, \leq \bar{\lambda}_1}^{\mu_1} \times X_{n_2, \leq \bar{\lambda}_2}^{\mu_2})_{disj}} \xrightarrow{\sim} \mathcal{P}^{\bar{\kappa}} \boxtimes \mathcal{P}^{\bar{\kappa}} |_{(X_{n_1, \leq \bar{\lambda}_1}^{\mu_1} \times X_{n_2, \leq \bar{\lambda}_2}^{\mu_2})_{disj}}$$

compatible with refinements of partitions.

Let $(X^{\mu_1} \times X_n^{\mu_2})_{disj}$ denote the ind-subscheme of $X^{\mu_1} \times X_n^{\mu_2}$ consisting of points

$$(D_1 \in X^{\mu_1}, (\bar{x}, D_2) \in X_n^{\mu_2})$$

such that D_1 is disjoint from both \bar{x} and D_2 . Let $\text{add}_{\mu_1, \mu_2, disj} : (X^{\mu_1} \times X_n^{\mu_2})_{disj} \rightarrow X_n^\mu$ denote the restriction of $\text{add}_{\mu_1, \mu_2}$. For a n -tuple $\bar{\lambda}$ the restriction is étale

$$\text{add}_{\mu_1, \mu_2, disj} : (X^{\mu_1} \times X_{n, \leq \bar{\lambda}}^{\mu_2})_{disj} \rightarrow X_{n, \leq \bar{\lambda}}^\mu.$$

Over $(X^{\mu_1} \times X_n^{\mu_2})_{disj}$ we get an isomorphism

$$(16) \quad \text{add}_{\mu_1, \mu_2, disj}^* \mathcal{P}^{\bar{\kappa}} \xrightarrow{\sim} \mathcal{P}^{\bar{\kappa}} \boxtimes \mathcal{P}^{\bar{\kappa}}$$

3.2. For $\mu \in -\Lambda^{pos}$ let $\mathring{X}^\mu \subset X^\mu$ be the open subscheme classifying divisors of the form $D = \sum_k \mu_k y_k$ with y_k pairwise different and each μ_k being a minus simple coroot. Denote by $j^{diag} : \mathring{X}^\mu \subset X^\mu$ the open immersion.

If α is a simple coroot then $\bar{\kappa}(-\alpha, -\alpha + 2\rho) = 0$. Therefore, $\mathcal{P}^{\bar{\kappa}} |_{\mathring{X}^\mu}$ is canonically trivialized. We get a canonical equivalence

$$\text{Perv}(\mathring{X}^\mu) \xrightarrow{\sim} \text{Perv}_\zeta(\mathring{X}^\mu)$$

Let $\mathring{\mathcal{L}}_\emptyset^\mu \in \text{Perv}_\zeta(\mathring{X}^\mu)$ be the object corresponding via the above equivalence to the sign local system on \mathring{X}^μ . If $\mu = -\sum m_i \alpha_i$ with $m_i \geq 0$ then the sign local system on \mathring{X}^μ is by definition the product of sign local systems on $\mathring{X}^{(m_i)}$ for all i . Set

$$\mathcal{L}_\emptyset^\mu = j_{!*}^{diag}(\mathring{\mathcal{L}}_\emptyset^\mu),$$

the intermediate extension being taken in $\text{Perv}_\zeta(X^\mu)$.

Note that for $\mu = \mu_1 + \mu_2$ with $\mu_i \in -\Lambda^{pos}$ we have a canonical isomorphism

$$(17) \quad \text{add}_{\mu_1, \mu_2, disj}^*(\mathcal{L}_\emptyset^\mu) \xrightarrow{\sim} \mathcal{L}_\emptyset^{\mu_1} \boxtimes \mathcal{L}_\emptyset^{\mu_2}$$

3.3. As in ([20], Section 3.5), we first define $\widetilde{\text{FS}}_n^\kappa$ as the category, whose objects are collections $\mathcal{L}_n^\mu \in \text{Perv}_\zeta(X_n^\mu)$ for each $\mu \in \Lambda$ equipped with the factorization isomorphisms: for any partition $\mu = \mu_1 + \mu_2$ with $\mu_2 \in \Lambda$, $\mu_1 \in -\Lambda^{pos}$ for the map

$$\text{add}_{\mu_1, \mu_2, disj} : (X^{\mu_1} \times X_n^{\mu_2})_{disj} \rightarrow X_n^\mu$$

we must be given an isomorphism

$$(18) \quad \text{add}_{\mu_1, \mu_2, disj}^* \mathcal{L}_n^\mu \xrightarrow{\sim} \mathcal{L}_\emptyset^{\mu_1} \boxtimes \mathcal{L}_n^{\mu_2}$$

compatible with refinements of partitions with respect to (17).

For $\mu_0, \mu_1 \in -\Lambda^{pos}$, $\mu_2 \in \Lambda$ let $(X^{\mu_0} \times X^{\mu_1} \times X_n^{\mu_2})_{disj}$ be the open subscheme classifying $(D_0, D_1, x_1, \dots, x_n, D_2) \in X^{\mu_0} \times X^{\mu_1} \times X_n^{\mu_2}$ such that D_0, D_1 are mutually disjoint

and disjoint with \bar{x}, D_2 . Compatibility with refinements of partitions means that for $\mu = \mu_1 + \mu_2$ the diagram

$$\begin{array}{ccc} (X^{\mu_0} \times X^{\mu_1} \times X_n^{\mu_2})_{disj} & \rightarrow & (X^{\mu_0+\mu_1} \times X_n^{\mu_2})_{disj} \\ \downarrow & & \downarrow \\ (X^{\mu_0} \times X_n^\mu)_{disj} & \rightarrow & X_n^{\mu_0+\mu} \end{array}$$

yields the commutative diagram of isomorphisms over $(X^{\mu_0} \times X^{\mu_1} \times X_n^{\mu_2})_{disj}$

$$\begin{array}{ccc} \mathcal{L}_n^{\mu_0+\mu} & \xrightarrow{\cong} & \mathcal{L}_\emptyset^{\mu_0} \boxtimes \mathcal{L}_n^\mu \\ \downarrow & & \downarrow \\ \mathcal{L}_\emptyset^{\mu_0+\mu_1} \boxtimes \mathcal{L}_n^{\mu_2} & \xrightarrow{(17)} & \mathcal{L}_\emptyset^{\mu_0} \boxtimes \mathcal{L}_\emptyset^{\mu_1} \boxtimes \mathcal{L}_n^{\mu_2}, \end{array}$$

where to simplify the notations we omitted the corresponding functors add^* .

A morphism from a collection $\{\mathcal{L}_n^\mu\}$ to another collection $\{\mathcal{L}_n^\mu\}$ is a collection of maps ${}^1\mathcal{L}_n^\mu \rightarrow {}^2\mathcal{L}_n^\mu$ in $\text{Perv}_\zeta(X_n^\mu)$ compatible with the isomorphisms (18).

Let $j^{poles} : \dot{X}^n \hookrightarrow X^n$ be the complement to all the diagonals. For $\mu \in \Lambda$ set $X_n^\mu = X_n^\mu \times_{X^n} \dot{X}^n$. By the same token, one defines the category $\widetilde{\text{FS}}_n^\kappa$ consisting of collections $\mathcal{L}_n^\mu \in \text{Perv}_\zeta(X_n^\mu)$ with factorization isomorphisms. Both $\widetilde{\text{FS}}_n^\kappa$ and $\widetilde{\text{FS}}_{\dot{n}}^\kappa$ are abelian categories.

We have the restriction functor $(j^{poles})^* : \widetilde{\text{FS}}_n^\kappa \rightarrow \widetilde{\text{FS}}_{\dot{n}}^\kappa$ and its left adjoint

$$j_{\dot{n}}^{poles} : \widetilde{\text{FS}}_{\dot{n}}^\kappa \rightarrow \widetilde{\text{FS}}_n^\kappa$$

well-defined because j^{poles} is an affine open embedding.

If $\bar{n} = n_1 + \dots + n_k$ is a partition of n , let $\Delta_{\bar{n}} : X^k \rightarrow X^n$ and $\dot{\Delta}_{\bar{n}} : \dot{X}^k \rightarrow X^n$ be the corresponding diagonal and its open subscheme. We have the natural functors

$$(\Delta_{\bar{n}})! : \widetilde{\text{FS}}_k^\kappa \rightarrow \widetilde{\text{FS}}_n^\kappa \quad \text{and} \quad (\dot{\Delta}_{\bar{n}})! : \widetilde{\text{FS}}_{\dot{k}}^\kappa \rightarrow \widetilde{\text{FS}}_n^\kappa$$

The corresponding restriction functors are well-defined on the level of derived categories (the latter are understood as the derived categories of the corresponding abelian categories):

$$(\Delta_{\bar{n}})^* : \text{D}(\widetilde{\text{FS}}_n^\kappa) \rightarrow \text{D}(\widetilde{\text{FS}}_k^\kappa) \quad \text{and} \quad (\dot{\Delta}_{\bar{n}})^* : \text{D}(\widetilde{\text{FS}}_n^\kappa) \rightarrow \text{D}(\widetilde{\text{FS}}_{\dot{k}}^\kappa)$$

They coincide with the same named functors on the level of derived categories of \mathbb{Q}_ℓ -sheaves on the corresponding gerbs.

3.4. For a k -scheme Y and $F \in \text{D}(Y)$ we denote by $\text{SS}(F)$ the singular support of F in the sense of Beilinson [3]. Define the full subcategory $\text{FS}_n^\kappa \subset \widetilde{\text{FS}}_n^\kappa$ as follows. A collection $\mathcal{L}_n \in \widetilde{\text{FS}}_n^\kappa$ lies in FS_n^κ if the following conditions are satisfied:

- (i) \mathcal{L}_n^μ may be nonzero only for μ belonging to finitely many cosets in $\pi_1(G)$. For each $\tau \in \pi_1(G)$ there is a collection $\bar{\nu} = (\nu_1, \dots, \nu_n) \in \Lambda^n$ with $\sum_i \nu_i = \tau \in \pi_1(G)$ such that for any $\mu \in \Lambda$ over τ the perverse sheaf \mathcal{L}_n^μ is the extension by zero from $X_{n, \leq \bar{\nu}}^\mu$.
- (ii) The second condition is first formulated over \dot{X}^n , that is, we first define the subcategory $\text{FS}_{\dot{n}}^\kappa \subset \widetilde{\text{FS}}_{\dot{n}}^\kappa$. Let $\mathcal{L}_{\dot{n}} \in \widetilde{\text{FS}}_{\dot{n}}^\kappa$, $\mu \in \Lambda$ and $\bar{\nu} \in \Lambda^n$ with $\sum_i \nu_i = \mu \in \pi_1(G)$ such that $\mathcal{L}_{\dot{n}}^\mu$ is the extension by zero from $\widetilde{X}_{\dot{n}, \leq \bar{\nu}}^\mu$. Then there are only

finitely many collections $(\mu_1, \dots, \mu_n) \in \Lambda^n$ with $\sum_i \mu_i = \mu$ such that $\text{SS}(\mathcal{L}_n^\mu)$ contains the conormal to the subscheme $\dot{X}^n \hookrightarrow X_{n, \leq \bar{\nu}}^\mu$, $(x_1, \dots, x_n) \mapsto \sum_i \mu_i x_i$.

Now the condition (ii) over X^n is that for any partition $n = n_1 + \dots + n_k$ each of the cohomologies of $(\dot{\Delta}_{\bar{n}})^*(\mathcal{L}_n)$, which is an object of $\widetilde{\text{FS}}_k^\kappa$, belongs to FS_n^κ .

3.5. For $\bar{x} = (x_1, \dots, x_n) \in X^n$ fixed let $X_{\bar{x}}^\mu$ denote the fibre of X_n^μ over $\bar{x} \in X^n$. In a similar way, one introduces the abelian category $\widetilde{\text{FS}}_{\bar{x}}^\kappa$. We define $\text{FS}_{\bar{x}}^\kappa$ as the full subcategory of objects of finite length in $\widetilde{\text{FS}}_{\bar{x}}^\kappa$. As in Section 3.2, one defines the category $\text{Perv}_\zeta(X_{\bar{x}}^\mu)$.

Pick $\bar{x} \in X^n$ with x_i pairwise distinct. Let $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a n -tuple of elements of Λ . For $\mu \in \Lambda$ with $(\sum_i \lambda_i) - \mu \in \Lambda^{\text{pos}}$ consider the closed subscheme $X_{\bar{x}, \leq \bar{\lambda}}^\mu = X_{\bar{x}}^\mu \cap X_{n, \leq \bar{\lambda}}^\mu$. Let $X_{\bar{x}, = \bar{\lambda}}^\mu \subset X_{\bar{x}, \leq \bar{\lambda}}^\mu$ be the open subscheme classifying divisors of the form

$$\left(\sum_{i=1}^n \lambda_i x_i \right) - D',$$

where D' is Λ^{pos} -valued divisor on X of degree $(\sum_i \lambda_i) - \mu$, and x_i is not in the support of D' for any $1 \leq i \leq n$. One similarly defines the categories $\text{Perv}_\zeta(X_{\bar{x}, \leq \bar{\lambda}}^\mu)$ and $\text{Perv}_\zeta(X_{\bar{x}, = \bar{\lambda}}^\mu)$. Let

$$\overset{\circ}{X}_{\bar{x}, \leq \bar{\lambda}}^\mu \subset X_{\bar{x}, = \bar{\lambda}}^\mu$$

be the open subscheme given by requiring that D' is of the form $D' = \sum \mu_k y_k$, where y_k are pairwise distinct, and each μ_k is a simple coroot of G . Here, of course, y_i is different from all the x_i . Denote the corresponding open immersions by

$$\overset{\circ}{X}_{\bar{x}, \leq \bar{\lambda}}^\mu \xrightarrow{''j^{\text{poles}}} X_{\bar{x}, = \bar{\lambda}}^\mu \xrightarrow{''j^{\text{poles}}} X_{\bar{x}, \leq \bar{\lambda}}^\mu$$

Lemma 3.5.1. *The restriction of $\mathcal{P}^{\bar{\kappa}}$ to $\overset{\circ}{X}_{\bar{x}, \leq \bar{\lambda}}^\mu$ is constant with fibre*

$$(19) \quad \otimes_{i=1}^n (\Omega_{\bar{x}}^{\frac{1}{2}})^{-\bar{\kappa}(\lambda_i, \lambda_i + 2\rho)} \otimes \epsilon^{\bar{\lambda}_i},$$

where $\bar{\lambda}_i \in \Lambda_{ab}$ is the image of λ_i .

If $(\sum_i \lambda_i) - \mu = \sum_{j \in \mathcal{J}} m_j \alpha_j$ then $\prod_{j \in \mathcal{J}} X^{(m_j)} \xrightarrow{\sim} X^{\mu - \sum_i \lambda_i}$ via the map sending $\{D_j\}_{j \in \mathcal{J}}$ to $-\sum_{j \in \mathcal{J}} D_j \alpha_j$.

We have an open immersion $j_{\bar{\lambda}}^\mu : X_{\bar{x}, = \bar{\lambda}}^\mu \hookrightarrow X^{\mu - \sum_i \lambda_i}$ sending D to $D - \sum_{i=1}^n \lambda_i x_i$. The line bundle $\mathcal{P}^{\bar{\kappa}}$ over $X_{\bar{x}, = \bar{\lambda}}^\mu$ identifies with the tensor product of $(j_{\bar{\lambda}}^\mu)^* \mathcal{P}^{\bar{\kappa}}$ with (19). So, for any trivialization of the line (19), we get the restriction functor

$$\text{Perv}_\zeta(X^{\mu - \sum_i \lambda_i}) \rightarrow \text{Perv}_\zeta(X_{\bar{x}, = \bar{\lambda}}^\mu)$$

We denote by $\overset{\circ}{\mathcal{L}}_{\bar{x}, \bar{\lambda}}^\mu$ the image of $\mathcal{L}_\emptyset^{\mu - \sum_i \lambda_i}$ under the latter functor. So, $\overset{\circ}{\mathcal{L}}_{\bar{x}, \bar{\lambda}}^\mu$ is defined up to a non-unique scalar automorphism. Set

$$\mathcal{L}_{\bar{x}, \bar{\lambda}, !}^\mu = {}''j_!^{\text{poles}}(\overset{\circ}{\mathcal{L}}_{\bar{x}, \bar{\lambda}}^\mu), \quad \mathcal{L}_{\bar{x}, \bar{\lambda}}^\mu = {}''j_{!*}^{\text{poles}}(\overset{\circ}{\mathcal{L}}_{\bar{x}, \bar{\lambda}}^\mu), \quad \mathcal{L}_{\bar{x}, \bar{\lambda}, * }^\mu = {}''j_*^{\text{poles}}(\overset{\circ}{\mathcal{L}}_{\bar{x}, \bar{\lambda}}^\mu)$$

Define the collection $\mathcal{L}_{\bar{x}, \bar{\lambda}, !} = \{\mathcal{L}_{\bar{x}, \bar{\lambda}, !}^\mu\}_{\mu \in \Lambda}$ by the property

$$\mathcal{L}_{\bar{x}, \bar{\lambda}, !}^\mu = \begin{cases} \mathcal{L}_{\bar{x}, \bar{\lambda}, !}^\mu, & \mu \in (\sum_i \lambda_i) - \Lambda^{pos} \\ 0, & \text{otherwise} \end{cases}$$

It is understood that we use the same trivialization of (19) for all μ in the above formula. One similarly defines the collections $\mathcal{L}_{\bar{x}, \bar{\lambda}}, \mathcal{L}_{\bar{x}, \bar{\lambda}, *}$. All the three are objects of $\widetilde{\text{FS}}_{\bar{x}}^\kappa$.

Lemma 3.5.2. *i) For any irreducible object F of $\widetilde{\text{FS}}_{\bar{x}}^\kappa$ there is a collection $\bar{\lambda} \in \Lambda^n$ such that it is isomorphic to $\mathcal{L}_{\bar{x}, \bar{\lambda}}$.*

ii) The kernels and cokernels of the natural maps

$$\mathcal{L}_{\bar{x}, \bar{\lambda}, !} \rightarrow \mathcal{L}_{\bar{x}, \bar{\lambda}} \rightarrow \mathcal{L}_{\bar{x}, \bar{\lambda}, *}$$

in $\widetilde{\text{FS}}_{\bar{x}}^\kappa$ are extensions of objects of the form $\mathcal{L}_{\bar{x}, \bar{\lambda}'}$ for $\bar{\lambda}' < \bar{\lambda}$.

Proof. i) Let $\bar{\lambda} \in \Lambda^n$ be such that the $*$ -fibre of F at $\sum_{i=1}^n \lambda_i x_i \in X_{\bar{x}}^\mu$ is nonzero for some $\mu \in \Lambda$. We may assume (changing $\bar{\lambda}$ if necessary) that for any $\nu \in \Lambda$ with $\nu = \mu$ in $\pi_1(G)$ the twisted perverse sheaf $F^\nu \in \text{Perv}_\zeta(X_{\bar{x}}^\nu)$ is the extension by zero from $X_{\bar{x}, \leq \lambda}^\nu$. Then from the factorization property we see that we must have $F \xrightarrow{\sim} \mathcal{L}_{\bar{x}, \bar{\lambda}}$. \square

Lemma 3.5.3. *Let $\bar{x} = (x_1, \dots, x_n)$ with x_i pairwise different, $\bar{\lambda} \in \Lambda^n$. Then the objects $\mathcal{L}_{\bar{x}, \bar{\lambda}, !}, \mathcal{L}_{\bar{x}, \bar{\lambda}, *}$ in $\widetilde{\text{FS}}_{\bar{x}}^\kappa$ are of finite length.*

Proof. Set $\bar{\kappa} = -\sum_{j \in J} c_j \kappa_j$. Write $D \in X_{\bar{x}, \leq \bar{\lambda}}^\mu$ as $D = (\sum_{y \in X} \mu_y y) + \sum_{i=1}^n \lambda_i x_i$ with $\mu_y \in -\Lambda^{pos}$ for all $y \in X$. Denote by $\mathcal{P}^{\bar{\kappa}}$ the line bundle on $X_{\bar{x}, \leq \bar{\lambda}}^\mu$ whose fibre at the above point D is

$$\otimes_{y \in X} (\Omega_y^{\frac{1}{2}})^{-\bar{\kappa}(\mu_y, \mu_y + 2\rho)}$$

The line bundle $\mathcal{P}^{\bar{\kappa}} \otimes (\mathcal{P}^{\bar{\kappa}})^{-1}$ on the scheme $X_{\bar{x}, \leq \bar{\lambda}}^\mu$ is constant. So, it suffices to prove our claim under the assumption $\beta = 0$. The latter is done in ([20], Lemma 3.8(b)). \square

4. ZASTAVA SPACES

4.1. Our purpose is to construct an exact functor $\text{Whit}_n^\kappa \rightarrow \widetilde{\text{FS}}_n^\kappa$. We first adopt the approach from ([20], Section 4) to our setting, it produces an approximation of the desired functor. We will further correct it to get the desired one.

For $\mu \in \Lambda$ let $\text{Bun}_{B^-}^\mu$ denote the connected component of Bun_{B^-} classifying B^- -torsors on X such that the induced T -torsor is of degree $(2g - 2)\rho - \mu$. Recall that a point of $\text{Bun}_{B^-}^\mu$ can be seen as a collection: a G -torsor \mathcal{F} on X , a T -torsor \mathcal{F}_T on X of degree $(2g - 2)\rho - \mu$, a collection of surjective maps of coherent sheaves

$$\kappa^{\bar{\lambda}, -} : \mathcal{V}_{\mathcal{F}}^{\bar{\lambda}} \rightarrow \mathcal{L}_{\mathcal{F}_T}^{\bar{\lambda}}, \quad \bar{\lambda} \in \check{\Lambda}^+$$

satisfying the Plücker relations. Define $\mathfrak{p}^-, \mathfrak{q}^-$ as the projections in the diagram

$$\text{Bun}_G \xleftarrow{\mathfrak{p}^-} \text{Bun}_{B^-}^\mu \xrightarrow{\mathfrak{q}^-} \text{Bun}_T$$

The line bundle $(\mathfrak{p}^-)^*(\omega \mathcal{L}^{\bar{\kappa}})$ is denoted by $\mathcal{P}^{\bar{\kappa}}$ by abuse of notations. One has naturally $\mathcal{P}^{\bar{\kappa}} \xrightarrow{\sim} (\mathfrak{q}^-)^*(\omega \mathcal{L}^{\bar{\kappa}})$.

Denote by $\mathcal{Z}_n^\mu \subset \mathfrak{M}_n \times_{\text{Bun}_G} \text{Bun}_{B^-}^\mu$ the open substack given by the property that for each G -dominant weight $\check{\lambda}$ the composition

$$(20) \quad \Omega^{\langle \check{\lambda}, \rho \rangle} \xrightarrow{\kappa^{\check{\lambda}}} \mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \xrightarrow{\kappa^{\check{\lambda}, -}} \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}},$$

which is a map over $X - \cup_i x_i$, is not zero. Let $'\mathfrak{p}$, $'\mathfrak{p}_B$ denote the projections in the diagram

$$\mathfrak{M}_n \xleftarrow{'\mathfrak{p}} \mathcal{Z}_n^\mu \xrightarrow{'\mathfrak{p}_B} \text{Bun}_{B^-}^\mu$$

Let $\pi^\mu : \mathcal{Z}_n^\mu \rightarrow X_n^\mu$ be the map sending the above point to (x_1, \dots, x_n, D) such that the maps (20) induce an isomorphism $\Omega^\rho(-D) \xrightarrow{\sim} \mathcal{F}_T$.

For any n -tuple $\bar{\lambda} \in \Lambda^n$ define the closed substack $\mathcal{Z}_{n, \leq \bar{\lambda}}^\mu$ by the base change $\mathfrak{M}_{n, \leq \bar{\lambda}} \hookrightarrow \mathfrak{M}_n$. The map π^μ restricts to a map

$$\pi^\mu : \mathcal{Z}_{n, \leq \bar{\lambda}}^\mu \rightarrow X_{n, \leq \bar{\lambda}}^\mu$$

However, the preimage of $X_{n, \leq \bar{\lambda}}^\mu$ under $\pi^\mu : \mathcal{Z}_n^\mu \rightarrow X_n^\mu$ is not $\mathcal{Z}_{n, \leq \bar{\lambda}}^\mu$.

Remark 4.1.1. For $\mu \in \Lambda$ let $\text{Gr}_{\omega_{\mathcal{N}^-}, X_n^\mu}$ be the ind-scheme classifying $(x_1, \dots, x_n, D) \in X_n^\mu$, a B^- -torsor \mathcal{F} on X with compatible isomorphisms $\mathcal{F} \times_{B^-} T \xrightarrow{\sim} \Omega^\rho(-D)$ over X and $\mathcal{F} \xrightarrow{\sim} \Omega^\rho \times_T B^-|_{X-D-\cup_i x_i}$. We have a closed immersion $\mathcal{Z}_n^\mu \hookrightarrow \text{Gr}_{\omega_{\mathcal{N}^-}, X_n^\mu}$ given by the property that the corresponding maps

$$\Omega^{\langle \rho, \check{\lambda} \rangle} \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}}$$

for $\check{\lambda} \in \check{\Lambda}^+$ are regular over $X - \cup_i x_i$. Since the projection $\text{Gr}_{\omega_{\mathcal{N}^-}, X_n^\mu} \rightarrow X_n^\mu$ is ind-affine, the map $\pi^\mu : \mathcal{Z}_n^\mu \rightarrow X_n^\mu$ is also ind-affine.

4.2. The ind-scheme \mathcal{Z}_0^μ is rather denoted \mathcal{Z}^μ . Recall that for $\mu_1 \in -\Lambda^{\text{pos}}$, $\mu_2 \in \Lambda$ and $\mu = \mu_1 + \mu_2$ we have the factorization property ([20], Proposition 4.7)

$$(21) \quad (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{X_n^\mu} \mathcal{Z}_n^\mu \xrightarrow{\sim} (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\mathcal{Z}^{\mu_1} \times \mathcal{Z}_n^{\mu_2})$$

Recall that the diagram commutes

$$(22) \quad \begin{array}{ccc} \mathfrak{M}_n & \xleftarrow{'\mathfrak{p}} & \mathcal{Z}_n^\mu & \xrightarrow{'\mathfrak{p}_B} & \text{Bun}_{B^-}^\mu \\ & & \downarrow \pi^\mu & & \downarrow \mathfrak{q}^- \\ & & X_n^\mu & \xrightarrow{A^J} & \text{Bun}_T \end{array}$$

and $('\mathfrak{p})^* \mathcal{P}^{\bar{\kappa}} \xrightarrow{\sim} (\pi^\mu)^* \mathcal{P}^{\bar{\kappa}}$ canonically, this line bundle is also denoted $\mathcal{P}^{\bar{\kappa}}$. Let $\tilde{\mathcal{Z}}_n^\mu$ denote the gerb of N -th roots of $\mathcal{P}^{\bar{\kappa}}$ over \mathcal{Z}_n^μ , $\text{D}_\zeta(\mathcal{Z}_n^\mu)$ the corresponding derived category of twisted $\bar{\mathbb{Q}}_\ell$ -sheaves.

This allows to define the following functors. First, we have the functor $F^\mu : \text{D}_\zeta(\mathfrak{M}_n) \rightarrow \text{D}_\zeta(\mathcal{Z}_n^\mu)$ given by

$$F^\mu(K) = ('\mathfrak{p})^* K[\dim. \text{rel}(' \mathfrak{p})]$$

As in ([20], Section 4.8), this functor commutes with the Verdier duality for μ satisfying $\langle \mu, \check{\alpha} \rangle < 0$ for any simple root $\check{\alpha}$. Using the factorization property, we will be able to assume that μ satisfies the latter inequality, so this functor *essentially always* commutes with the Verdier duality. We get the functor $\mathbb{F} : \text{D}_\zeta(\mathfrak{M}_n) \rightarrow \text{D}_\zeta(X_n^\mu)$ given by

$$\mathbb{F}(K) = \pi_!^\mu ('\mathfrak{p})^*(K)[\dim. \text{rel}(' \mathfrak{p})]$$

4.3. The analog of ([20], Proposition 4.13) holds in our setting:

Proposition 4.3.1. *Let $\mu_1 \in -\Lambda^{pos}$, $\mu_2 \in \Lambda$, $\mu = \mu_1 + \mu_2$ and $\mathcal{F} \in \text{Whit}_n^\kappa$. Under the isomorphism (21), the complex*

$$\text{add}_{\mu_1, \mu_2, \text{disj}}^* F^\mu(\mathcal{F}) \in \text{D}_\zeta((X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{X_n^\mu} \mathcal{Z}_n^\mu)$$

identifies with

$$F^{\mu_1}(\mathcal{F}_\emptyset) \boxtimes F^{\mu_2}(\mathcal{F}) \in \text{D}_\zeta((X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\mathcal{Z}^{\mu_1} \times \mathcal{Z}_n^{\mu_2}))$$

Proof. We write down the complete proof for the convenience of the reader and to correct some misprints in ([20], proof of Proposition 4.13). Set $\mathring{\mathcal{Z}}^{\mu_1} = \mathcal{Z}^{\mu_1} \times_{\mathfrak{M}_{\emptyset, 0}} \mathfrak{M}_{\emptyset, 0}$. Let $(\mathfrak{M}_n)_{\text{good at } \mu_1} \subset X^{\mu_1} \times \mathfrak{M}_n$ be the open substack given by the property that $D \in X^{\mu_1}$ does not contain pole points (x_1, \dots, x_n) , and all κ^λ are morphisms of vector bundles in a neighbourhood of $\text{supp}(D)$.

Let $\mathcal{N}_{\mu_1}^{reg}$ (resp., $\mathcal{N}_{\mu_1}^{mer}$) be the group scheme (resp., group ind-scheme) over X^{μ_1} , whose fibre at D is the group scheme (resp., group ind-scheme) of sections of $\omega\mathcal{N}$ over the formal neighbourhood of D (resp., the punctured formal neighbourhood of D). As in Section 2.2, we have the character $\chi_{\mu_1} : \mathcal{N}_{\mu_1}^{mer} \rightarrow \mathbb{A}^1$.

For a point of $(\mathfrak{M}_n)_{\text{good at } \mu_1}$ we get a B -torsor \mathcal{F}_B over the formal neighbourhood \bar{D} of D with a trivialization $\epsilon_B : \mathcal{F}_B \times_B T \xrightarrow{\sim} \Omega^\rho$ over \bar{D} . Let ${}_{\mu_1}\mathfrak{M}_n$ denote the $\mathcal{N}_{\mu_1}^{reg}$ -torsor over $(\mathfrak{M}_n)_{\text{good at } \mu_1}$ classifying a point of $(\mathfrak{M}_n)_{\text{good at } \mu_1}$ together with a trivialization $\mathcal{F}_B \xrightarrow{\sim} \Omega_B^\rho|_{\bar{D}}$ compatible with ϵ_B . The group ind-scheme $\mathcal{N}_{\mu_1}^{mer}$ acts on ${}_{\mu_1}\mathfrak{M}_n$ over X^{μ_1} , this action lifts naturally to an action on $\mathcal{P}^{\bar{\kappa}}$. Let

$$\text{act}_{\mu_1} : \mathcal{N}_{\mu_1}^{mer} \times \mathcal{N}_{\mu_1}^{reg} ({}_{\mu_1}\mathfrak{M}_n) \rightarrow (\mathfrak{M}_n)_{\text{good at } \mu_1}$$

be the action map. For each $\mathcal{F} \in \text{Whit}_n^\kappa$ one has an isomorphism of twisted perverse sheaves

$$\text{act}_{\mu_1}^*(\mathcal{F}) \xrightarrow{\sim} \chi_{\mu_1}^* \mathcal{L}_\psi \boxtimes \mathcal{F}$$

As the fibre $\mathcal{N}_{\mu_1}^{mer}/\mathcal{N}_{\mu_1}^{reg}$ at $D \in X^{\mu_1}$ can be written as an inductive system of affine spaces, the above system of isomorphisms makes sense, see ([19], Section 4).

The preimage of $(\mathfrak{M}_n)_{\text{good at } \mu_1}$ under the map

$$(X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{X_n^\mu} \mathcal{Z}_n^\mu \xrightarrow{\text{p}} X^{\mu_1} \times \mathfrak{M}_n$$

goes over under the isomorphism (21) to

$$(23) \quad (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\mathring{\mathcal{Z}}^{\mu_1} \times \mathcal{Z}_n^{\mu_2})$$

Note that $\mathcal{N}_{\mu_1}^{mer}/\mathcal{N}_{\mu_1}^{reg}$ can be seen as the ind-scheme classifying $D \in X^{\mu_1}$, a B -torsor \mathcal{F} on X with compatible isomorphisms $\mathcal{F} \times_B T \xrightarrow{\sim} \Omega^\rho$ over X and $\mathcal{F} \xrightarrow{\sim} \Omega_B^\rho|_{X-D}$. The character χ_{μ_1} decomposes as

$$\mathcal{N}_{\mu_1}^{mer}/\mathcal{N}_{\mu_1}^{reg} \rightarrow \mathfrak{M}_{\emptyset, 0} \xrightarrow{\text{ev}_{\emptyset, 0}} \mathbb{A}^1$$

We have a locally closed embedding over X^{μ_1}

$$\mathring{\mathcal{Z}}^{\mu_1} \hookrightarrow \mathcal{N}_{\mu_1}^{mer}/\mathcal{N}_{\mu_1}^{reg}$$

given by the property that for each $\check{\lambda} \in \check{\Lambda}^+$ the map $\kappa^{\check{\lambda}, -} : \mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \rightarrow \mathcal{L}_{\Omega^{\rho}(-D)}^{\check{\lambda}}$, initially defined over $X - D$, is regular over X and surjective.

For $\mathcal{F} \in \text{Whit}_n^{\kappa}$ its pull-back to

$$(X^{\mu_1} \times \mathfrak{M}_n) \times_{(X^{\mu_1} \times X^n)} (X^{\mu_1} \times X^n)_{disj}$$

is the extension by $*$ and also by $!$ from $(\mathfrak{M}_n)_{\text{good at } \mu_1}$, because there are no dominant coweight strictly smaller than 0 (see Section 2.4). So, it suffices to prove the desired isomorphism over the open substack (23).

The composition

$$(X^{\mu_1} \times X_n^{\mu_2})_{disj} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\overset{\circ}{\mathcal{Z}}^{\mu_1} \times \mathcal{Z}_n^{\mu_2}) \rightarrow (X^{\mu_1} \times X_n^{\mu_2})_{disj} \times_{X_n^{\mu_2}} \mathcal{Z}_n^{\mu_2} \rightarrow X^{\mu_1} \times \mathfrak{M}_n$$

factors as

$$\begin{aligned} & (X^{\mu_1} \times X_n^{\mu_2})_{disj} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\overset{\circ}{\mathcal{Z}}^{\mu_1} \times \mathcal{Z}_n^{\mu_2}) \rightarrow \\ & (X^{\mu_1} \times X_n^{\mu_2})_{disj} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\mathcal{N}_{\mu_1}^{mer} / \mathcal{N}_{\mu_1}^{reg} \times \mathcal{Z}_n^{\mu_2}) \\ & \xrightarrow{\sim} (X^{\mu_1} \times X_n^{\mu_2})_{disj} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\mathcal{N}_{\mu_1}^{mer} \times \mathcal{N}_{\mu_1}^{reg} (\mu_1 \mathfrak{M}_n \times_{\mathfrak{M}_n} \mathcal{Z}_n^{\mu_2})) \\ & \rightarrow \mathcal{N}_{\mu_1}^{mer} \times \mathcal{N}_{\mu_1}^{reg} \times_{\mu_1} \mathfrak{M}_n \xrightarrow{\text{act } \mu_1} (\mathfrak{M}_n)_{\text{good at } \mu_1} \hookrightarrow X^{\mu_1} \times \mathfrak{M}_n, \end{aligned}$$

where the second arrow used the trivialization of the $\mathcal{N}_{\mu_1}^{reg}$ -torsor

$$(\mu_1 \mathfrak{M}_n \times_{\mathfrak{M}_n} \mathcal{Z}_n^{\mu_2}) \times_{(X^{\mu_1} \times X_n^{\mu_2})} (X^{\mu_1} \times X_n^{\mu_2})_{disj}$$

(see Remark 4.1.1). □

Corollary 4.3.2. *For $\mathcal{F} \in \text{Whit}_n^{\kappa}$, $\mu_1 \in -\Lambda^{pos}$, $\mu_2 \in \Lambda$ and $\mu = \mu_1 + \mu_2$ one has*

$$\text{add}_{\mu_1, \mu_2, disj}^* \mathbb{F}(\mathcal{F}) \xrightarrow{\sim} \mathbb{F}(\mathcal{F}_{\emptyset}) \boxtimes \mathbb{F}(\mathcal{F})$$

in $D_{\zeta}((X^{\mu_1} \times X_n^{\mu_2})_{disj})$. These isomorphisms are compatible with refinements of partitions.

We will use the following.

Remark 4.3.3. *Let $M \subset G$ be a standard Levi, Λ_M^{pos} the \mathbb{Z}_+ -span of M -positive coroots in Λ . For $\mu \in -\Lambda^{pos}$ let Z_G^{μ} denote the Zastava space classifying $D \in X^{\mu}$, U^- -torsor \mathcal{F} on X , a trivialization $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_{U^-}^0|_{X-D}$ that gives rise to a generalized B -structure on $\mathcal{F}_G := \mathcal{F} \times_{U^-} G$ over X with the corresponding T -torsor $\mathcal{F}_T^0(D)$. That is, for each $\check{\lambda} \in \check{\Lambda}^+$ the natural map*

$$\kappa^{\check{\lambda}} : \mathcal{O}(\langle D, \check{\lambda} \rangle) \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}}$$

is regular over X . Assume in addition $\mu \in -\Lambda_M^{pos}$. Then we have the similarly defined ind-scheme Z_M^{μ} for M . The natural map $Z_M^{\mu} \rightarrow Z_G^{\mu}$ is an isomorphism over X^{μ} .

Proposition 4.3.4. *Assume $\varrho(\alpha_i) \notin \mathbb{Z}$ for any simple coroot α_i . Then for $\mu \in -\Lambda^{pos}$ we have a (non-canonical) isomorphism $\mathcal{L}_{\emptyset}^{\mu} \xrightarrow{\sim} \mathbb{F}(\mathcal{F}_{\emptyset})$ in $D_{\zeta}(\check{X}^{\mu})$.*

Proof. Consider first the case $\mu = -\alpha$, where α is a simple coroot of G . Then $X^\mu = X$. Applying Remark 4.3.3 for the corresponding subminimal Levi, we get $\mathcal{Z}^{-\alpha} \xrightarrow{\sim} X \times \mathbb{A}^1$, and $\mathring{\mathcal{Z}}^{-\alpha} \xrightarrow{\sim} X \times \mathbb{G}_m$ is the complement to the zero section. The line bundle $\mathcal{P}^{\bar{\kappa}}$ over X^μ is trivialized canonically. However, over $\mathring{\mathcal{Z}}^{-\alpha}$ we get another trivialization of $\mathcal{P}^{\bar{\kappa}}$ inherited from the trivialization of $\mathcal{P}^{\bar{\kappa}}|_{\mathfrak{M}_{\theta,0}}$. The discrepancy between the two trivializations is the map

$$\mathring{\mathcal{Z}}^{-\alpha} \xrightarrow{\sim} X \times \mathbb{G}_m \xrightarrow{\text{pr}} \mathbb{G}_m \xrightarrow{z \mapsto z^d} \mathbb{G}_m,$$

where $d = \frac{-\bar{\kappa}(\alpha, \alpha)}{2}$. Since our answer here is different from that of ([20], Section 5.1), we give more details. Let M be the standard subminimal Levi corresponding to the coroot α , M_0 be the derived group of M , so $M_0 \xrightarrow{\sim} \text{SL}_2$. Pick $x \in X$. Let \mathbb{P} denote the projective line classifying lattices \mathcal{M} included into

$$(24) \quad \Omega^{-\frac{1}{2}}(-x) \oplus \Omega^{\frac{1}{2}} \subset \mathcal{M} \subset \Omega^{-\frac{1}{2}} \oplus \Omega^{\frac{1}{2}}(x)$$

such that $\mathcal{M}/(\Omega^{-\frac{1}{2}}(-x) \oplus \Omega^{\frac{1}{2}})$ is 1-dimensional. This defines a map $\mathbb{P} \rightarrow \text{Bun}_{M_0}$ sending \mathcal{M} to \mathcal{M} viewed as a M_0 -torsor on X . Let \mathcal{L} denote the line bundle on \mathbb{P} with fibre

$$\frac{\det \text{R}\Gamma(X, \Omega^{\frac{1}{2}}) \otimes \det \text{R}\Gamma(X, \Omega^{-\frac{1}{2}})}{\det \text{R}\Gamma(X, \mathcal{M})}$$

at \mathcal{M} . The restriction of ${}^\omega \mathcal{L}^{\bar{\kappa}}$ under the composition $\mathbb{P} \rightarrow \text{Bun}_{M_0} \rightarrow \text{Bun}_G$ identifies with $\mathcal{L}^{\frac{-\bar{\kappa}(\alpha, \alpha)}{2}}$. The fibre $\mathcal{Z}^{-\alpha}$ over $D = -\alpha x$ is the open subscheme of \mathbb{P} given by the property that $\Omega^{-\frac{1}{2}}(-x) \subset \mathcal{M}$ is a subbundle. The formula for d follows from the fact that $\mathcal{L} \xrightarrow{\sim} \mathcal{O}(1)$ on \mathbb{P} .

So, if $\varrho(\alpha) \notin \mathbb{Z}$ then $\mathbb{F}(\mathcal{F}_\theta) \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell[1]$ non-canonically in $\text{D}_\zeta(X^{-\alpha})$.

Let now $\mu = -\sum m_i \alpha_i \in -\Lambda^{\text{pos}}$ with $m_i \geq 0$. Applying Corollary 4.3.2 and the above computation, one gets the desired isomorphism after the pull-back to $\prod_i X^{m_i} - \Delta$, where Δ is the diagonal divisor. From the Künneth formula one sees that the product of the corresponding symmetric groups $\prod_i S_{m_i}$ acts by the sign character because the Gauss sum $\text{R}\Gamma_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\zeta^d)$ is concentrated in the degree 1 for $d \notin N\mathbb{Z}$. \square

The isomorphism of Proposition 4.3.4 does not hold in $\text{D}_\zeta(X^\mu)$. This is already seen in the following special case.

Lemma 4.3.5. *Assume $G = \text{SL}_2$ and $\varrho(\alpha) \notin \mathbb{Z}$ for the simple coroot α . Then for $\mu \in -\Lambda^{\text{pos}}$, $\mathbb{F}(\mathcal{F}_\theta) \in \text{D}_\zeta(X^\mu)$ is the extension by zero from \mathring{X}^μ .*

Proof. Take $\mu = -m\alpha$, $m \geq 0$. So, $X^{(m)} \xrightarrow{\sim} X^\mu$ via the map $D \mapsto -D\alpha$. The scheme \mathcal{Z}^μ is a vector bundle over X^μ with fibre

$$\text{Ext}^1(\Omega^{\frac{1}{2}}(D)/\Omega^{\frac{1}{2}}, \Omega^{-\frac{1}{2}}(-D)) = \Omega^{-1}(-D)/\Omega^{-1}(-2D)$$

at $-D\alpha$. A point of \mathcal{Z}^μ is given by $D \in X^{(m)}$ and a diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \Omega^{-\frac{1}{2}}(-D) & \rightarrow & M & \rightarrow & \Omega^{\frac{1}{2}}(D) \rightarrow 0 \\ & & & & \swarrow & & \uparrow \\ & & & & & & \Omega^{\frac{1}{2}} \end{array}$$

The line bundle $\mathcal{P}^{\bar{\kappa}}$ over $X^{(m)}$ identifies canonically with $\mathcal{O}(-4c_j \Delta)$, where $\Delta \subset X^{(m)}$ is the divisor of the diagonals.

For a line bundle L on X and an $D \in X^{(m)}$ let $(L(D)/L)_{max} \subset L(D)/L$ be the open subscheme consisting of those $v \in L(D)/L$ such that for any $0 \leq D' < D$, $v \notin L(D')/L$. Note that $(L(D)/L)_{max}$ identifies canonically with $(L^{-1}(-D)/L^{-1}(-2D))_{max}$.

The fibre of \mathring{Z}^μ over $D \in X^{(m)}$ is $(\Omega^{-1}(-D)/\Omega^{-1}(-2D))_{max} \xrightarrow{\sim} (\Omega(D)/\Omega)_{max}$. Let $D = \sum_k m_k x_k \in X^{(m)}$. Then $(\Omega(D)/\Omega)_{max} \xrightarrow{\sim} \prod_k (\Omega(m_k x_k)/\Omega)_{max}$. The fibre of $\mathcal{P}^{\bar{\kappa}}$ at $-D\alpha \in X^\mu$ is

$$(\otimes_k \Omega_{x_k}^{m_k^2 - m_k})^{4c_j}$$

Write a point of $\prod_k (\Omega(m_k x_k)/\Omega)_{max}$ as $v = (v_k)$, $v_k \in (\Omega(m_k x_k)/\Omega)_{max}$. Let \bar{v}_k be the image of v_k in the geometric fibre $(\Omega(m_k x_k))_{x_k} = \Omega_{x_k}^{1-m_k}$. The canonical section of $\pi^{\mu*} \mathcal{P}^{\bar{\kappa}}$ over \mathring{Z}^μ sends v to $(\otimes_k \bar{v}_k^{-m_k})^{4c_j}$. So, the $*$ -fibre of $\mathbb{F}(\mathcal{F}_\emptyset)$ at $-D\alpha \in X^\mu$ identifies (up to a shift) with the tensor product over k of the complexes

$$(25) \quad \mathrm{R}\Gamma_c((\Omega(m_k x_k)/\Omega)_{max}, ev^* \mathcal{L}_\psi \otimes \eta_k^* \mathcal{L}_{\zeta^{4c_j m_k}}),$$

where η_k is the map

$$\eta_k : (\Omega(m_k x_k)/\Omega)_{max} \rightarrow (\Omega(m_k x_k))_{x_k} \xrightarrow{\tau_k} \mathbb{G}_m$$

for some isomorphisms τ_k . Calculate (25) via the composition $(\Omega(m_k x_k)/\Omega)_{max} \rightarrow (\Omega(m_k x_k))_{x_k} \rightarrow \mathrm{Spec} k$. If $m_k > 1$ for some k then the sheaf $ev^* \mathcal{L}_\psi$ on $(\Omega(m_k x_k)/\Omega)_{max}$ changes under the action of the vector space $\Omega((m_k - 1)x_k)/\Omega$ by the Artin-Schreier character, so (25) vanishes for this k . Our claim follows. \square

Remark 4.3.6. Assume that $\varrho(\alpha_i) \notin \mathbb{Z}$ for any simple coroot α_i . For $G = \mathrm{SL}_2$ the fibres of $\mathcal{L}_\emptyset^\mu$ are calculated in [5], it is not the extension by zero from \mathring{X}^μ . As in ([20], Proposition 4.10), one may show that for any $K \in \mathrm{Whit}_n^\kappa$ the object $\mathbb{F}(K)$ is placed in perverse cohomological degree zero (this is essentially done in Proposition 4.11.4). However, Lemma 4.3.5 shows that the functor \mathbb{F} does not produce an object of $\widetilde{\mathrm{FS}}_n^\kappa$, and should be corrected.

4.4. Compactified Zastava. For $\mu \in \Lambda$ let $\overline{\mathrm{Bun}}_{B^-}^\mu$ be the Drinfeld compactification of $\mathrm{Bun}_{B^-}^\mu$. Namely, this is the stack classifying a G -torsor \mathcal{F} on X , a T -torsor \mathcal{F}_T on X of degree $(2g - 2)\rho - \mu$, and a collection of nonzero maps of coherent sheaves for $\check{\lambda} \in \check{\Lambda}^+$

$$\kappa^{\check{\lambda}, -} : \mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \rightarrow \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}}$$

satisfying the Plücker relations. This means that for any $\check{\lambda}, \check{\mu} \in \check{\Lambda}^+$ the composition

$$\mathcal{V}_{\mathcal{F}}^{\check{\lambda} + \check{\mu}} \rightarrow (\mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \otimes \mathcal{V}_{\mathcal{F}}^{\check{\mu}})_{\mathcal{F}} \xrightarrow{\kappa^{\check{\lambda}, -} \otimes \kappa^{\check{\mu}, -}} \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda} + \check{\mu}}$$

coincides with $\kappa^{\check{\lambda} + \check{\mu}, -}$, and $\kappa^{0, -} : \mathcal{O} \rightarrow \mathcal{O}$ is the identity map. Let $\bar{q}^- : \overline{\mathrm{Bun}}_{B^-}^\mu \rightarrow \mathrm{Bun}_T$ be the map sending the above point to \mathcal{F}_T .

For $n \geq 0$ denote by $\widetilde{\mathcal{Z}}_n^\mu$ the open substack of $\mathfrak{M}_n \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_{B^-}^\mu$ given by the property that for each $\check{\lambda} \in \check{\Lambda}^+$ the composition

$$(26) \quad \Omega^{(\check{\lambda}, \rho)} \xrightarrow{\kappa^{\check{\lambda}}} \mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \xrightarrow{\kappa^{\check{\lambda}, -}} \mathcal{L}_{\mathcal{F}_T}^{\check{\lambda}},$$

which is regular over $X - \cup_i x_i$, is not zero. Define the projections by the diagram

$$\mathfrak{M}_n \xleftarrow{\prime\bar{p}} \overline{\mathcal{Z}}_n^\mu \xrightarrow{\prime\bar{p}_B} \overline{\text{Bun}}_{B-}^\mu$$

Let $\bar{\pi}^\mu : \overline{\mathcal{Z}}_n^\mu \rightarrow X_n^\mu$ be the map sending the above point to (x_1, \dots, x_n, D) such that the maps (26) induce an isomorphism $\Omega^\rho(-D) \xrightarrow{\sim} \mathcal{F}_T$. Note that $\mathcal{Z}_n^\mu \subset \overline{\mathcal{Z}}_n^\mu$ is open.

For a n -tuple $\bar{\lambda} \in \Lambda^n$ define the closed substack $\overline{\mathcal{Z}}_{n, \leq \bar{\lambda}}^\mu$ by the base change $\mathfrak{M}_{n, \leq \bar{\lambda}} \rightarrow \mathfrak{M}_n$. The map $\bar{\pi}^\mu$ restricts to a map

$$(27) \quad \bar{\pi}^\mu : \overline{\mathcal{Z}}_{n, \leq \bar{\lambda}}^\mu \rightarrow X_{n, \leq \bar{\lambda}}^\mu$$

The stack $\overline{\mathcal{Z}}_0^\mu$ will be rather denoted $\overline{\mathcal{Z}}^\mu$. As in ([20], Proposition 4.5), one gets the following.

Lemma 4.4.1. *Let $(\bar{x}, \mathcal{F}, \mathcal{F}_T, (\kappa^{\check{\lambda}}), (\kappa^{\check{\lambda}, -}))$ be a point of $\overline{\mathcal{Z}}_n^\mu$, whose image under $\bar{\pi}^\mu$ is (\bar{x}, D) . Then the restriction of \mathcal{F} to $X - D - \cup_i x_i$ is equipped with an isomorphism $\mathcal{F} \xrightarrow{\sim} \Omega^\rho \times_T G$ with the tautological maps $\kappa^{\check{\lambda}}, \kappa^{\check{\lambda}, -}$. In particular, $\overline{\mathcal{Z}}_n^\mu$ is an ind-scheme over k .*

Let $\text{Gr}_{\omega_G, X_n^\mu}$ denote the ind-scheme classifying $(x_1, \dots, x_n, D) \in X_n^\mu$, a G -torsor \mathcal{F} on X , a trivialization $\mathcal{F} \xrightarrow{\sim} \Omega^\rho \times_T G$ over $X - D - \cup_i x_i$. The projection $\text{Gr}_{\omega_G, X_n^\mu} \rightarrow X_n^\mu$ is ind-proper.

We have a closed immersion $\overline{\mathcal{Z}}_n^\mu \hookrightarrow \text{Gr}_{\omega_G, X_n^\mu}$ given by the property that for each $\check{\lambda} \in \check{\Lambda}$ the natural map $\kappa^{\check{\lambda}, -} : \mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \rightarrow \Omega^{\langle \rho, \check{\lambda} \rangle}(-\langle D, \check{\lambda} \rangle)$ is regular over X , and

$$\kappa^{\check{\lambda}} : \Omega^{\langle \rho, \check{\lambda} \rangle} \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}}$$

is regular over $X - \cup_i x_i$. So, $\bar{\pi}^\mu : \overline{\mathcal{Z}}_n^\mu \rightarrow X_n^\mu$ is ind-proper.

Lemma 4.4.2. *For $\mu_1 \in -\Lambda^{\text{pos}}, \mu_2 \in \Lambda$ and $\mu = \mu_1 + \mu_2$ we have the following factorization property*

$$(28) \quad (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{X_n^\mu} \overline{\mathcal{Z}}_n^\mu \xrightarrow{\sim} (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\overline{\mathcal{Z}}^{\mu_1} \times \overline{\mathcal{Z}}_n^{\mu_2})$$

compatible with (21).

The diagram (22) extends to the diagram

$$(29) \quad \begin{array}{ccc} \mathfrak{M}_n & \xleftarrow{\prime\bar{p}} & \overline{\mathcal{Z}}_n^\mu & \xrightarrow{\prime\bar{p}_B} & \overline{\text{Bun}}_{B-}^\mu \\ & & \downarrow \bar{\pi}^\mu & & \downarrow \bar{q}^- \\ & & X_n^\mu & \xrightarrow{A^J} & \text{Bun}_T \end{array}$$

Now we face the difficulty that the line bundles $\prime\bar{p}^* \mathcal{P}^{\bar{\kappa}}$ and $(\bar{\pi}^\mu)^* \mathcal{P}^{\bar{\kappa}}$ are not isomorphic over $\overline{\mathcal{Z}}_n^\mu$, but only over its open part \mathcal{Z}_n^μ .

4.5. Description of fibres. Let \mathcal{O}_x denote the completed local ring of X at x , F_x its fraction field. For $\mu \in \Lambda$ we have the point $t^\mu \in \text{Gr}_{G, x} = G(F_x)/G(\mathcal{O}_x)$. Recall that Gr_B^μ is the $U(F_x)$ -orbit in $\text{Gr}_{G, x}$ through t^μ . We also have the closed ind-subscheme $\overline{\text{Gr}}_B^\mu \subset \text{Gr}_{G, x}$ defined in ([18], Section 7.1.1). It classifies a G -torsor \mathcal{F} on X with a trivialization $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_G^0|_{X-x}$ such that for each $\check{\lambda} \in \check{\Lambda}^+$ the map

$$\kappa^{\check{\lambda}} : \mathcal{O}(-\langle \mu, \check{\lambda} \rangle) \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}}$$

is regular over X . This is a scheme-theoretical version of the closure of Gr_B^μ .

Recall that $\text{Gr}_{B^-}^\mu$ is the $U^-(F_x)$ -orbit through t^μ in $\text{Gr}_{G,x}$. Similarly, one defines $\overline{\text{Gr}}_{B^-}^\mu \subset \text{Gr}_{G,x}$. To be precise, $\overline{\text{Gr}}_{B^-}^\mu$ classifies a G -torsor \mathcal{F} on X with a trivialization $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_G^0|_{X-x}$ such that for any $\check{\lambda} \in \check{\Lambda}^+$ the map

$$\kappa^{\check{\lambda}, -} : \mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \rightarrow \mathcal{O}(-\langle \mu, \check{\lambda} \rangle)$$

is regular over X . Note that if $\text{Gr}_{B^-}^\nu \subset \overline{\text{Gr}}_{B^-}^\mu$ for some $\nu \in \Lambda$ then $\nu \geq \mu$. If $\text{Gr}_{B^-}^\nu \subset \overline{\text{Gr}}_{B^-}^\mu$ then $\nu \leq \mu$.

Let $\mu \in -\Lambda^{\text{pos}}$. The fibre $\overline{\mathcal{Z}}_{loc,x}^\mu$ of $\overline{\mathcal{Z}}^\mu$ over $\mu x \in X^\mu$ identifies naturally with

$$(30) \quad (\overline{\text{Gr}}_B^0 \cap \overline{\text{Gr}}_{B^-}^\mu) \times^{T(\mathcal{O}_x)} \Omega^\rho|_{D_x},$$

where $\Omega^\rho|_{D_x}$ denotes the corresponding $T(\mathcal{O}_x)$ -torsor.

Lemma 4.5.1. *If $\mu \in -\Lambda^{\text{pos}}$ then (30) is a projective scheme of finite type and of dimension $\leq -\langle \mu, \check{\rho} \rangle$ (and not just an ind-scheme).*

Proof. Let $\nu \in \Lambda$ be such that $\text{Gr}_{B^-}^\nu \subset \overline{\text{Gr}}_{B^-}^\mu$, so $\nu \geq \mu$. We know from ([10], Section 6.3) that $\overline{\text{Gr}}_B^0 \cap \text{Gr}_{B^-}^\nu$ can be nonempty only for $\nu \leq 0$, and in this case it is a scheme of finite type and of dimension $\leq -\langle \nu, \check{\rho} \rangle$. Since the set of $\nu \in \Lambda$ satisfying $\mu \leq \nu \leq 0$ is finite, we are done. \square

Lemma 4.5.1 implies that $\bar{\pi}^\mu : \overline{\mathcal{Z}}^\mu \rightarrow X^\mu$ is proper, its fibres are projective schemes of finite type of dimension $\leq -\langle \mu, \check{\rho} \rangle$.

Let $\mu \in \Lambda$. The fibre of $\overline{\mathcal{Z}}_1^\mu$ over μx_1 identifies naturally with $\overline{\text{Gr}}_{B^-}^\mu \times^{T(\mathcal{O}_x)} \Omega^\rho|_{D_x}$. For $n \geq 1$ the fibre of $\bar{\pi}^\mu : \overline{\mathcal{Z}}_n^\mu \rightarrow X_n^\mu$ over (\bar{x}, D) is only an ind-scheme (not a scheme). Let also $\lambda \in \Lambda$. Then the fibre of $\overline{\mathcal{Z}}_{1, \leq \lambda}^\mu$ over μx_1 identifies naturally with

$$(\overline{\text{Gr}}_B^\lambda \cap \overline{\text{Gr}}_{B^-}^\mu) \times^{T(\mathcal{O}_x)} \Omega^\rho|_{D_x}$$

This could be non-empty only for $\mu \leq \lambda$, and in that case this is a projective scheme of dimension $\leq \langle \lambda - \mu, \check{\rho} \rangle$.

Now if $\bar{\lambda} \in \Lambda^n$ from the factorization property we see that the map (27) is proper, its fibres are projective schemes of finite type.

4.6. In Section 0.0.7 we defined $\widetilde{\text{Bun}}_G$ as the gerb of N -th roots of ${}^\omega \mathcal{L}^{\bar{\kappa}}$ over Bun_G , similarly for $\widetilde{\text{Bun}}_T$.

Let $\overline{\text{Bun}}_{B^-, \check{G}} = \overline{\text{Bun}}_{B^-} \times_{\text{Bun}_G} \widetilde{\text{Bun}}_G$ and $\overline{\text{Bun}}_{\check{B}^-} = \overline{\text{Bun}}_{B^-, \check{G}} \times_{\text{Bun}_T} \widetilde{\text{Bun}}_T$. Set also $\text{Bun}_{B^-, \check{G}} = \text{Bun}_{B^-} \times_{\text{Bun}_G} \widetilde{\text{Bun}}_G$. Let $\text{Bun}_{\check{B}^-}$ be the preimage of Bun_{B^-} in $\overline{\text{Bun}}_{\check{B}^-}$.

A point of $\overline{\text{Bun}}_{\check{B}^-}$ is given by $(\mathcal{F}, \mathcal{F}_T, \kappa^{\check{\lambda}, -})$ and lines $\mathcal{U}, \mathcal{U}_G$ equipped with isomorphisms

$$\mathcal{U}^N \xrightarrow{\sim} ({}^\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}_T}, \quad \mathcal{U}_G^N \xrightarrow{\sim} ({}^\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}}$$

Let $\text{D}_{\zeta^{-1}, \zeta}(\overline{\text{Bun}}_{B^-})$ denote the derived category of \mathbb{Q}_ℓ -sheaves on $\overline{\text{Bun}}_{B^-}$ on which $\mu_N(k) \subset \text{Aut}(\mathcal{U})$ acts by ζ , and $\mu_N(k) \subset \text{Aut}(\mathcal{U}_G)$ acts by ζ^{-1} . We define the irreducible

perverse sheaf $\mathrm{IC}_\zeta \in \mathrm{Perv}_{\zeta^{-1}, \zeta}(\overline{\mathrm{Bun}}_{B^-})$ as follows (see [23], Definition 3.1). One has the isomorphism

$$(31) \quad B(\mu_N) \times \mathrm{Bun}_{B^-, \tilde{G}} \xrightarrow{\sim} \mathrm{Bun}_{\tilde{B}^-}$$

sending $(\mathcal{F}_{B^-}, \mathcal{U}_G, \mathcal{U}_0 \in B(\mu_N))$ with $\mathcal{U}_0^N \xrightarrow{\sim} k$ to $(\mathcal{F}_{B^-}, \mathcal{U}_G, \mathcal{U})$ with $\mathcal{U} = \mathcal{U}_G \otimes \mathcal{U}_0$. View $\mathcal{L}_\zeta \boxtimes \mathrm{IC}(\mathrm{Bun}_{B^-, \tilde{G}})$ as a perverse sheaf on $\mathrm{Bun}_{\tilde{B}^-}$ via (31). Let IC_ζ be its intermediate extension to $\overline{\mathrm{Bun}}_{\tilde{B}^-}$.

4.6.1. Let $\tilde{\mathcal{Z}}_n^\mu$ denote the gerb of N -th roots of $(\bar{\pi}^\mu)^* \mathcal{P}^{\bar{\kappa}}$, $\mathrm{D}_\zeta(\tilde{\mathcal{Z}}_n^\mu)$ denote the derived category of $\tilde{\mathcal{Q}}_\ell$ -sheaves on $\tilde{\mathcal{Z}}_n^\mu$, on which $\mu_N(k)$ acts by ζ . For $\mu \in \Lambda$ define the functor $\bar{F}^\mu : \mathrm{D}_\zeta(\mathfrak{M}_n) \rightarrow \mathrm{D}_\zeta(\tilde{\mathcal{Z}}_n^\mu)$ by

$$\bar{F}^\mu(K) = {}' \bar{\mathfrak{p}}^* K \otimes ({}' \bar{\mathfrak{p}}_B)^* \mathrm{IC}_\zeta[-\dim \mathrm{Bun}_G]$$

We will write $\bar{F}_\zeta^\mu := \bar{F}^\mu$ if we need to express the dependence on ζ . Define the functor $\bar{\mathbb{F}} : \mathrm{D}_\zeta(\mathfrak{M}_n) \rightarrow \mathrm{D}_\zeta(X_n^\mu)$ by

$$\bar{\mathbb{F}}(K) = (\bar{\pi}^\mu)_! \bar{F}^\mu(K)$$

We will see below that the functor $\bar{F}^\mu : \mathrm{Whit}_n^\kappa \rightarrow \mathrm{D}_\zeta(\tilde{\mathcal{Z}}_n^\mu)$ commutes with the Verdier duality (up to replacing ζ by ζ^{-1}).

4.7. For $\mu \in -\Lambda^{\mathrm{pos}}$ set $\overset{\circ}{\mathcal{Z}}^\mu = \bar{\mathcal{Z}}^\mu \times_{\mathfrak{M}_0} \mathfrak{M}_{0,0}$.

Proposition 4.7.1. *Let $\mu_1 \in -\Lambda^{\mathrm{pos}}$, $\mu_2 \in \Lambda$, $\bar{\mu} = \mu_1 + \mu_2$ and $\mathcal{F} \in \mathrm{Whit}_n^\kappa$. Under the isomorphism (28) the complex*

$$\mathrm{add}_{\mu_1, \mu_2, \mathrm{disj}}^* \bar{F}^\mu(\mathcal{F}) \in \mathrm{D}_\zeta((X^{\mu_1} \times X^{\mu_2})_{\mathrm{disj}} \times_{X_n^\mu} \bar{\mathcal{Z}}_n^\mu)$$

identifies with

$$\bar{F}^{\mu_1}(\mathcal{F}_\emptyset) \boxtimes \bar{F}^{\mu_2}(\mathcal{F}) \in \mathrm{D}_\zeta((X^{\mu_1} \times X^{\mu_2})_{\mathrm{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\bar{\mathcal{Z}}^{\mu_1} \times \bar{\mathcal{Z}}_n^{\mu_2}))$$

Proof. The preimage of $(\mathfrak{M}_n)_{\mathrm{good at } \mu_1}$ under the map

$$(X^{\mu_1} \times X^{\mu_2})_{\mathrm{disj}} \times_{X_n^\mu} \bar{\mathcal{Z}}_n^\mu \xrightarrow{{}' \bar{\mathfrak{p}}} X^{\mu_1} \times \mathfrak{M}_n$$

goes over under the isomorphism (28) to

$$(32) \quad (X^{\mu_1} \times X^{\mu_2})_{\mathrm{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\overset{\circ}{\mathcal{Z}}^{\mu_1} \times \bar{\mathcal{Z}}_n^{\mu_2})$$

Recall that $\mathcal{N}_{\mu_1}^{\mathrm{mer}} / \mathcal{N}_{\mu_1}^{\mathrm{reg}}$ is the ind-scheme classifying $D \in X^{\mu_1}$, a B -torsor \mathcal{F} on X with compatible isomorphisms $\mathcal{F} \times_B T \xrightarrow{\sim} \Omega^\rho$ over X and $\mathcal{F} \xrightarrow{\sim} \Omega_B^\rho|_{X-D}$. We have the closed embedding over X^{μ_1}

$$\overset{\circ}{\mathcal{Z}}^{\mu_1} \hookrightarrow \mathcal{N}_{\mu_1}^{\mathrm{mer}} / \mathcal{N}_{\mu_1}^{\mathrm{reg}}$$

given by the property that for each $\check{\lambda} \in \check{\Lambda}^+$ the map $\kappa^{\check{\lambda}, -} : \mathcal{V}_{\mathcal{F}}^{\check{\lambda}} \rightarrow \mathcal{L}_{\Omega^\rho(-D)}^{\check{\lambda}}$, initially defined over $X - D$, is regular over X .

The two complexes we want to identify are extensions by zero from the open substack (32), so, it suffices to establish the desired isomorphism over (32). By ([23],

Theorem 4.1), the complex $\text{add}_{\mu_1, \mu_2, \text{disj}}^*(\bar{\mathfrak{p}}_B^* \text{IC}_\zeta)$ goes over under (28) to the complex $\bar{\mathfrak{p}}_B^* \text{IC}_\zeta \boxtimes (\bar{\mathfrak{p}}_B)^* \text{IC}_\zeta$ up to a shift.

The composition

$$(X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\bar{\mathcal{Z}}^{\mu_1} \times \bar{\mathcal{Z}}_n^{\mu_2}) \rightarrow (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{X_n^\mu} \bar{\mathcal{Z}}_n^\mu \rightarrow X^{\mu_1} \times \mathfrak{M}_n$$

factors as

$$\begin{aligned} & (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\bar{\mathcal{Z}}^{\mu_1} \times \bar{\mathcal{Z}}_n^{\mu_2}) \rightarrow \\ & \quad (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\mathcal{N}_{\mu_1}^{\text{mer}} / \mathcal{N}_{\mu_1}^{\text{reg}} \times \bar{\mathcal{Z}}_n^{\mu_2}) \\ & \quad \xrightarrow{\sim} (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}} \times_{(X^{\mu_1} \times X_n^{\mu_2})} (\mathcal{N}_{\mu_1}^{\text{mer}} \times \mathcal{N}_{\mu_1}^{\text{reg}} (\mu_1 \mathfrak{M}_n \times \mathfrak{M}_n \bar{\mathcal{Z}}_n^{\mu_2})) \\ & \quad \rightarrow \mathcal{N}_{\mu_1}^{\text{mer}} \times \mathcal{N}_{\mu_1}^{\text{reg}} \mu_1 \mathfrak{M}_n \xrightarrow{\text{act}_{\mu_1}} (\mathfrak{M}_n)_{\text{good at } \mu_1} \hookrightarrow X^{\mu_1} \times \mathfrak{M}_n, \end{aligned}$$

where the second arrow used the trivialization of the $\mathcal{N}_{\mu_1}^{\text{reg}}$ -torsor

$$(\mu_1 \mathfrak{M}_n \times \mathfrak{M}_n \bar{\mathcal{Z}}_n^{\mu_2}) \times_{(X^{\mu_1} \times X_n^{\mu_2})} (X^{\mu_1} \times X_n^{\mu_2})_{\text{disj}}$$

as in Proposition 4.3.1. One finishes the proof as in Proposition 4.3.1. \square

4.8. Generalizing the ULA property. Let S_1 be a smooth equidimensional stack. Let $p_1 : Y_1 \rightarrow S_1$ and $q_1 : S \rightarrow S_1$ be morphisms of stacks locally of finite type. Let $Y = Y_1 \times_{S_1} S$. Let $p : Y \rightarrow S$ and $q : Y \rightarrow Y_1$ denote the projections. Denote by $g : Y \rightarrow Y_1 \times S$ the map (q, p) . For $L \in \text{D}(Y_1)$ consider the functor $\mathcal{F}_L : \text{D}(S) \rightarrow \text{D}(Y)$ given by

$$\mathcal{F}_L(K) = p^* K \otimes q^* L \langle -\frac{\dim S_1}{2} \rangle,$$

where $\langle d \rangle = [2d](d)$.

Lemma 4.8.1. *i) For $K \in \text{D}(Y_1 \times S)$ there is a canonical morphism functorial in K*

$$(33) \quad g^* K \langle -\frac{\dim S_1}{2} \rangle \rightarrow g^! K \langle \frac{\dim S_1}{2} \rangle,$$

ii) There is a canonical morphism functorial in $K \in \text{D}(S), L \in \text{D}(Y_1)$

$$(34) \quad \mathcal{F}_{\mathbb{D}L}(\mathbb{D}K) \rightarrow \mathbb{D}(\mathcal{F}_L(K))$$

Proof. i) We have a diagram, where the squares are cartesian

$$\begin{array}{ccc} S_1 & \xrightarrow{\Delta} & S_1 \times S_1 \\ \uparrow q_1 & & \uparrow \text{id} \times q_1 \\ S & \rightarrow & S_1 \times S \\ \uparrow p & & \uparrow p_1 \times \text{id} \\ Y & \xrightarrow{g} & Y_1 \times S \end{array}$$

One has $\Delta^! \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell \langle -\dim S_1 \rangle$, because S_1 is smooth. By ([2], XVII 2.1.3), one has the base change morphism $p^* q_1^* \Delta^! \rightarrow g^!(p_1 \times q_1)^*$. Applying it to the previous isomorphism, one gets a canonical map $\text{can} : \bar{\mathbb{Q}}_\ell \langle -\dim S_1 \rangle \rightarrow g^! \bar{\mathbb{Q}}_\ell$.

According to ([12], Section 5.1.1), there is a canonical morphism $g^*K \otimes g^!K' \rightarrow g^!(K \otimes K')$ functorial in $K, K' \in \mathbb{D}(Y_1 \times S)$. Taking $K' = \bar{\mathbb{Q}}_\ell$ we define (33) as the composition

$$g^*K \langle -\dim S_1 \rangle \xrightarrow{\text{id} \otimes \text{can}} g^*K \otimes g^!\bar{\mathbb{Q}}_\ell \rightarrow g^!K$$

ii) Apply (33) to $\mathbb{D}L \boxtimes \mathbb{D}K$. \square

Definition 4.8.2. Let $\mathring{Y} \subset Y$ be an open substack. Say that $L \in \mathbb{D}(Y_1)$ is locally acyclic with respect to the diagram $S \xleftarrow{p} \mathring{Y} \xrightarrow{q} Y_1$ if for any $K \in \mathbb{D}(S)$ the map (34) is an isomorphism over \mathring{Y} . Say that $L \in \mathbb{D}(Y_1)$ is universally locally acyclic with respect to the diagram $S \xleftarrow{p} \mathring{Y} \xrightarrow{q} Y_1$ if the same property holds after any smooth base change $S'_1 \rightarrow S_1$.

4.8.3. Here are some properties of the above ULA condition:

- 1) If $S_1 = \text{Spec } k$ then any $L \in \mathbb{D}(Y_1)$ is ULA with respect to the diagram $S \xleftarrow{p} \mathring{Y} \xrightarrow{q} Y_1$.
- 2) If $r_1 : V_1 \rightarrow Y_1$ is smooth of fixed relative dimension, and $L \in \mathbb{D}(Y_1)$ is ULA with respect to $S \xleftarrow{p} \mathring{Y} \xrightarrow{q} Y_1$ then r_1^*L is ULA with respect to the diagram $S \leftarrow \mathring{V} \rightarrow V_1$. Here we defined $r : V \rightarrow Y$ as the base change of $r_1 : V_1 \rightarrow Y_1$ by $q : Y \rightarrow Y_1$, and \mathring{V} is the preimage of \mathring{Y} in V . Conversely, if $r_1 : V_1 \rightarrow Y_1$ is smooth and surjective, and r_1^*L is ULA with respect to the diagram $S \leftarrow \mathring{V} \rightarrow V_1$, then $L \in \mathbb{D}(Y_1)$ is ULA with respect to $S \xleftarrow{p} \mathring{Y} \xrightarrow{q} Y_1$.
- 3) Assume given a diagram as above $S \xleftarrow{p} Y \xrightarrow{q} Y_1$ such that both S_1 and S are smooth and equidimensional. Assume $L \in \mathbb{D}(Y_1)$, and the natural map $q^*L \langle \dim S - \dim S_1 \rangle \rightarrow q^!L$ is an isomorphism. Then $\mathbb{D}(q^*L)$ is locally acyclic with respect to $p : \mathring{Y} \rightarrow S$ if and only if L is locally acyclic with respect to the diagram $S \xleftarrow{p} \mathring{Y} \xrightarrow{q} Y_1$.

Proof. 3) Let $\bar{p} : Y \rightarrow Y \times S$ be the graph of $p : Y \rightarrow S$. By ([12], Section 5.1.1), we have a canonical morphism, say $\alpha : \bar{p}^*(\cdot) \langle -\dim S \rangle \rightarrow \bar{p}^!$. Since S and S_1 are smooth, $q_1^!\bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell \langle \dim S - \dim S_1 \rangle$. As in Section 4.8, since the map $q \times \text{id} : Y \times S \rightarrow Y_1 \times S$ is obtained from q_1 by base change, the above isomorphism yields a canonical map $\text{can} : \bar{\mathbb{Q}}_\ell \langle \dim S - \dim S_1 \rangle \rightarrow (q \times \text{id})^!\bar{\mathbb{Q}}_\ell$. For $K \in \mathbb{D}(Y_1 \times S)$ we get a canonical map

$$\beta : (q \times \text{id})^*K \langle \dim S - \dim S_1 \rangle \rightarrow (q \times \text{id})^!K$$

defined as the composition $(q \times \text{id})^*K \langle \dim S - \dim S_1 \rangle \xrightarrow{\text{id} \otimes \text{can}} (q \times \text{id})^*K \otimes (q \times \text{id})^!\bar{\mathbb{Q}}_\ell \rightarrow (q \times \text{id})^!K$. The composition $Y \xrightarrow{\bar{p}} Y \times S \xrightarrow{q \times \text{id}} Y_1 \times S$ equals g . For $K \in \mathbb{D}(Y_1 \times S)$ the map (33) equals the composition

$$\bar{p}^*(q \times \text{id})^*K \langle -\dim S_1 \rangle \xrightarrow{\beta} \bar{p}^*(q \times \text{id})^!K \langle -\dim S \rangle \xrightarrow{\alpha} \bar{p}^!(q \times \text{id})^!K$$

Let now $K \in \mathbb{D}(S)$. By our assumptions, the map $\beta : (q \times \text{id})^*(\mathbb{D}L \boxtimes \mathbb{D}K) \langle \dim S - \dim S_1 \rangle \xrightarrow{\sim} (q \times \text{id})^!(\mathbb{D}L \boxtimes \mathbb{D}K)$ is an isomorphism. The map $\mathbb{D}(q^*L)$ is locally acyclic

with respect to $p : \overset{\circ}{Y} \rightarrow S$ if and only if the map $\alpha : \bar{p}^*(\mathbb{D}(q^*L) \boxtimes \mathbb{D}K)\langle -\dim S \rangle \rightarrow \bar{p}^!(\mathbb{D}(q^*L) \boxtimes \mathbb{D}K)$ is an isomorphism over $\overset{\circ}{Y}$ for any $K \in D(S)$. Our claim follows. \square

4.8.4. We will say that for a morphism $p_1 : Y_1 \rightarrow S_1$ an object $L \in D(Y_1)$ is ULA with respect to p_1 if it satisfies ([13], Definition 2.12). One may check that this definition is equivalent to ([12], Definition 5.1). In the latter one requires that local acyclicity holds after any smooth base change, whence in the former one requires it to hold after any base change $q_1 : S \rightarrow S_1$.

Assume given a cartesian square as in Section 4.8

$$(35) \quad \begin{array}{ccc} Y & \xrightarrow{q} & Y_1 \\ \downarrow p & & \downarrow p_1 \\ S & \xrightarrow{q_1} & S_1 \end{array}$$

with S_1 smooth equidimensional.

Proposition 4.8.5. *Assume q_1 representable. Let $L \in D(Y_1)$ be ULA with respect to p_1 . Then L is ULA with respect to the diagram $S \xleftarrow{p} Y \xrightarrow{q} Y_1$.*

To establish Proposition 4.8.5 we need the following.

Lemma 4.8.6. *Assume given a diagram (35), where S, S_1 are smooth of dimensions d, d_1 respectively, and q_1 is representable. If $L \in D(Y_1)$ is ULA with respect to p_1 then the natural map $\eta : q^*L\langle \frac{d-d_1}{2} \rangle \rightarrow q^!L\langle \frac{d_1-d}{2} \rangle$ is an isomorphism.*

Proof. One has canonical maps $p^*q_1^!\bar{\mathbb{Q}}_\ell \rightarrow q^!\bar{\mathbb{Q}}_\ell$ and $q^*L \otimes q^!\bar{\mathbb{Q}}_\ell \rightarrow q^!L$, the second one is defined in ([12], Section 5.1.1). One has $q_1^!\bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell\langle d-d_1 \rangle$ canonically. Recall that η is defined as the composition $q^*L\langle d-d_1 \rangle \rightarrow q^*L \otimes q^!\bar{\mathbb{Q}}_\ell \rightarrow q^!L$.

If q_1 is smooth then our claim is well known. If q_1 is a closed immersion then this follows from ([12], Lemma B.3). In general, write q_1 as the composition $S \xrightarrow{\text{id} \times q_1} S \times S_1 \xrightarrow{\text{pr}_2} S_1$. Localizing on S_1 in smooth topology, we may assume S_1 is a smooth affine scheme. Then $\text{id} \times q_1$ is a closed immersion. \square

Proof of Proposition 4.8.5. Let $K \in D(S)$. Localizing on S_1 in smooth topology we may assume S_1 is a smooth affine scheme of dimension d_1 . Let $i_1 : S_0 \rightarrow S$ be a locally closed smooth subscheme with $\dim S_0 = d_0$, E a local system on S_0 . Decomposing K in the derived category, it is enough to treat the case of $K = (i_1)_*E$. We must show that for this K the map (34) is an isomorphism over Y . Let $i : Y_0 \hookrightarrow Y$ be obtained from i_1 by the base change $p : Y \rightarrow S$. Let $p_0 : Y_0 \rightarrow S_0$ be the projection. By Lemma 4.8.6,

$$i^*q^*L\langle d_0 - d_1 \rangle \xrightarrow{\sim} i^!q^!L$$

Since i^*q^*L is ULA over S_0 , by 3) of Section 4.8.3, L is locally acyclic with respect to the diagram $S_0 \xleftarrow{p_0} Y_0 \xrightarrow{q_0^i} Y_1$. That is, one has an isomorphism over Y_0

$$(36) \quad \mathbb{D}(p_0^*E \otimes i^*q^*L) \xrightarrow{\sim} p_0^*(\mathbb{D}E) \otimes i^*q^*(\mathbb{D}L)\langle -d_1 \rangle$$

We must show that the natural map

$$(37) \quad q^*(\mathbb{D}L) \otimes p^*(i_1)_*E^*\langle d_0 - d_1 \rangle \rightarrow \mathbb{D}(q^*L \otimes p^*(i_1)_*E)$$

is an isomorphism over Y . By ([15], Theorem 7.6.9), $q^*L \otimes p^*(i_1)_*E \xrightarrow{\sim} i_*(i^*q^*L \otimes p_0^*E)$. So, both sides of (37) are extensions by zero under i , and over Y_0 the desired isomorphism reduces to (36). \square

4.9. The above notion of ULA was introduced, because we hoped that for $\mu \in \Lambda$, $\bar{\lambda} \in \Lambda^n$ the perverse sheaf $\mathrm{IC}_\zeta \in \mathrm{Perv}_{\zeta^{-1}, \zeta}(\overline{\mathrm{Bun}}_{B^-}^\mu)$ is ULA with respect to the diagram

$$\mathfrak{M}_{n, \leq \lambda} \xleftarrow{\bar{p}} \overline{\mathcal{Z}}_{n, \leq \lambda}^\mu \xrightarrow{\bar{p}_B} \overline{\mathrm{Bun}}_{B^-}^\mu$$

Unfortunately, this claim is not literally true. However, it will be used in the proof of following result. For $\mu \in \Lambda$, $K \in \mathrm{D}_\zeta(\mathfrak{M}_n)$ the map (34) defines a canonical morphism

$$(38) \quad \bar{F}_{\zeta^{-1}}^\mu(\mathbb{D}K) \rightarrow \mathbb{D}(\bar{F}^\mu(K))$$

Proposition 4.9.1. *For any $K \in \mathrm{Whit}_n^\zeta$ the map (38) is an isomorphism.*

Proof. Pick a collection of dominant coweights $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\mu \in \Lambda$ with $\mu \leq \sum_i \lambda_i$. We assume K is the extension by zero from $\mathfrak{M}_{n, \leq \bar{\lambda}}$. We must show that (38) is an isomorphism over $\overline{\mathcal{Z}}_{n, \leq \bar{\lambda}}^\mu$. The question is local with respect to X , so we may and do assume X of genus zero.

For $\theta \in \pi_1(G)$ write Bun_G^θ for the corresponding connected component of Bun_G . Let θ be the image of $(2g-2)\rho - \mu$ in $\pi_1(G)$. Write $\mathrm{Bun}_G^{\theta, 0} \subset \mathrm{Bun}_G^\theta$ for the open Shatz stratum in the component Bun_G^θ . Write $q : \overline{\mathrm{Bun}}_{B^-}^\mu \rightarrow \mathrm{Bun}_G$ for the projection. By ([13], [Th. finitude], Theorem 2.13), the restriction of IC_ζ is ULA with respect to $q^{-1}(\mathrm{Bun}_G^{\theta, 0}) \rightarrow \mathrm{Bun}_G^{\theta, 0}$. Let U^μ denote the preimage of $\mathrm{Bun}_G^{\theta, 0}$ under the composition

$$\overline{\mathcal{Z}}_{n, \leq \bar{\lambda}}^\mu \xrightarrow{\bar{p}_B} \overline{\mathrm{Bun}}_{B^-}^\mu \xrightarrow{q} \mathrm{Bun}_G$$

By Proposition 4.8.5, (38) is an isomorphism over U^μ .

Recall that if $\langle \mu, \check{\alpha} \rangle < 0$ for any simple coroot $\check{\alpha}$ then $\mathrm{Bun}_{B^-}^\mu \rightarrow \mathrm{Bun}_G^\theta$ is smooth. Recall also that for each simple coroot α one has $\overline{\mathcal{Z}}^{-\alpha} \xrightarrow{\sim} X \times \mathbb{P}^1$. Now for $\nu \in -\Lambda^{\mathrm{pos}}$ consider the diagram

$$\overline{\mathcal{Z}}_n^\mu \xleftarrow{a} (\overset{\circ}{X}^\nu \times X_n^\mu)_{\mathrm{disj}} \times_{X_n^{\nu+\mu}} \overline{\mathcal{Z}}_n^{\nu+\mu} \xrightarrow{b} \overline{\mathrm{Bun}}_{B^-}^{\nu+\mu} \xrightarrow{q} \mathrm{Bun}_G,$$

where the projection a is obtained from the factorization property (28), and b is the projection on the second factor composed with \bar{p}_B .

Let η be a k -point of $\overline{\mathcal{Z}}_{n, \leq \bar{\lambda}}^\mu$. For this η there exists $\nu \in -\Lambda^{\mathrm{pos}}$ such that η lies in $a((qb)^{-1}(\mathrm{Bun}_G^{\theta, 0}))$. Pick a k -point $\eta' \in (qb)^{-1}(\mathrm{Bun}_G^{\theta, 0})$ over η . By Proposition 4.7.1, it suffices to show that the canonical map

$$\bar{F}_{\zeta^{-1}}^{\mu+\nu}(\mathbb{D}K) \rightarrow \mathbb{D}\bar{F}^{\mu+\nu}(K)$$

is an isomorphism in a neighbourhood of η' . This is the case because $\eta' \in U^{\mu+\nu}$. \square

Let $\overline{\mathcal{Z}}_{\bar{x}, \bar{\lambda}}^\mu \subset \overline{\mathcal{Z}}_n^\mu$ (resp., $\overline{\mathcal{Z}}_{\bar{x}, \leq \bar{\lambda}}^\mu \subset \overline{\mathcal{Z}}_n^\mu$) be the substack obtained from $\overline{\mathcal{Z}}_n^\mu$ by the base change $\mathfrak{M}_{\bar{x}, \bar{\lambda}} \rightarrow \mathfrak{M}_n$ (resp., $\mathfrak{M}_{\bar{x}, \leq \bar{\lambda}} \rightarrow \mathfrak{M}_n$). Let $\mathcal{Z}_{\bar{x}, \bar{\lambda}}^\mu$ be the preimage of $\mathrm{Bun}_{B^-}^\mu$ in $\overline{\mathcal{Z}}_{\bar{x}, \bar{\lambda}}^\mu$.

Corollary 4.9.2. *i) If $\mu \in -\Lambda^{pos}$ then $\bar{F}^\mu(\mathcal{F}_\emptyset)$ is an irreducible perverse sheaf, the extension by zero from $\bar{\mathcal{Z}}^\mu$.*

ii) Let $\bar{x} = (x_1, \dots, x_n) \in X^n$ be pairwise different, $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i \in \Lambda^+$, $\mu \in \Lambda$ with $\mu \leq \sum_i \lambda_i$. Then $\bar{F}^\mu(\mathcal{F}_{\bar{x}, \bar{\lambda}, !})$ is perverse, and $\mathbb{D}\bar{F}^\mu(\mathcal{F}_{\bar{x}, \bar{\lambda}, !}) \xrightarrow{\sim} \bar{F}_{\zeta^{-1}}^\mu(\mathbb{D}\mathcal{F}_{\bar{x}, \bar{\lambda}, !})$.

iii) The complex $\bar{F}^\mu(\mathcal{F}_{\bar{x}, \bar{\lambda}})$ is an irreducible perverse sheaf, the intermediate extension from $\bar{\mathcal{Z}}_{\bar{x}, \bar{\lambda}}^\mu$. So, $\bar{\mathbb{F}}(\mathcal{F}_{\bar{x}, \bar{\lambda}})$ is a direct sum of (shifted) irreducible perverse sheaves.

Proof. i) and ii). The fact that $\bar{F}^\mu(\mathcal{F}_{\bar{x}, \bar{\lambda}, !})$ is an irreducible perverse sheaf over $\bar{\mathcal{Z}}_{\bar{x}, \bar{\lambda}}^\mu$ is essentially explained in [10] (see also [23]). Our claim follows now from Proposition 4.9.1 and the fact that \mathcal{F}_\emptyset is self-dual (up to replacing ψ by ψ^{-1}).

iii) For each collection of dominant coweights $\bar{\lambda}' < \bar{\lambda}$ the $*$ -restriction of $\mathcal{F}_{\bar{x}, \bar{\lambda}}$ to $\widetilde{\mathfrak{M}}_{\bar{x}, \bar{\lambda}'}$ is placed in perverse degrees < 0 . Therefore, the $*$ -restriction of $\bar{F}^\mu(\mathcal{F}_{\bar{x}, \bar{\lambda}})$ to $\bar{\mathcal{Z}}_{\bar{x}, \bar{\lambda}'}^\mu$ is placed in perverse degrees < 0 by ii). Our claim follows. \square

Remark 4.9.3. *Let us precise some dimensions in Corollary 4.9.2. As in ([10], Section 5.2) one checks that $\bar{\mathcal{Z}}_{\bar{x}, \bar{\lambda}}^\mu$ is irreducible of dimension $\langle -\mu + \sum_i \lambda_i, 2\check{\rho} \rangle$. The stack $\mathfrak{M}_{\bar{x}, \bar{\lambda}}$ is smooth irreducible of dimension $(g-1)\dim U - \langle (2g-2)\rho - \sum_i \lambda_i, 2\check{\rho} \rangle$, and $\dim \overline{\text{Bun}}_{B^-}^\mu = (g-1)\dim B + \langle 2\check{\rho}, (2g-2)\rho - \mu \rangle$.*

The $$ -restriction of $\bar{F}^\mu(\mathcal{F}_{\bar{x}, \bar{\lambda}, !})$ to $\bar{\mathcal{Z}}_{\bar{x}, \bar{\lambda}}^\mu$ is a local system placed in the usual degree $\langle \mu - \sum_i \lambda_i, 2\check{\rho} \rangle$.*

4.10. The $*$ -restrictions of IC_ζ to a natural stratification have been calculated in ([23], Theorem 4.1) under the additional assumption that G is simple, simply-connected, but the answer and the argument hold also in our case of $[G, G]$ simply-connected. This way one gets the following description.

Let $\check{\mathfrak{u}}_\zeta^-$ denote the Lie algebra of the unipotent radical of the Borel subgroup $\check{B}_\zeta^- \subset \check{G}_\zeta$ corresponding to B^- . For $\nu \in \Lambda^\sharp$ and $V \in \text{Rep}(\check{T}_\zeta)$ write V_ν for the direct summand of V , on which \check{T}_ζ acts by ν .

Let $\theta \in -\Lambda^{pos}$. We will write $\mathfrak{U}(\theta)$ for an element of the free abelian semigroup generated by $-\Lambda^{pos} - 0$. In other words, $\mathfrak{U}(\theta)$ is a way to write

$$(39) \quad \theta = \sum_m n_m \theta_m,$$

where $\theta_m \in -\Lambda^{pos} - 0$ are pairwise different, and $n_m \geq 0$. Set $|\mathfrak{U}(\theta)| = \sum_m n_m$. We denote by $X^{\mathfrak{U}(\theta)}$ the corresponding partially symmetrized power of the curve $X^{\mathfrak{U}(\theta)} = \prod_m X^{(n_m)}$. Let $\mathring{X}^{\mathfrak{U}(\theta)} \subset X^{\mathfrak{U}(\theta)}$ be the complement to all the diagonals in $X^{\mathfrak{U}(\theta)}$. We view $\mathring{X}^{\mathfrak{U}(\theta)}$ as a locally closed subscheme of X^θ via the map $\mathring{X}^{\mathfrak{U}(\theta)} \rightarrow X^\theta, (D_m) \mapsto \sum_m D_m \theta_m$.

Set ${}_{\mathfrak{U}(\theta)}\overline{\text{Bun}}_{B^-} = \text{Bun}_{B^-} \times \mathring{X}^{\mathfrak{U}(\theta)}$. We get locally closed immersions ${}_{\mathfrak{U}(\theta)}\overline{\text{Bun}}_{B^-} \hookrightarrow \text{Bun}_{B^-} \times X^\theta \hookrightarrow \overline{\text{Bun}}_{B^-}$, the second one sending $(\mathcal{F}, \mathcal{F}_T, \kappa^-, D)$ to $(\mathcal{F}, \mathcal{F}_T(-D), \kappa^-)$. Let ${}_{\mathfrak{U}(\theta)}\overline{\text{Bun}}_{\check{B}^-}$ be obtained from ${}_{\mathfrak{U}(\theta)}\overline{\text{Bun}}_{B^-}$ by the base change $\overline{\text{Bun}}_{\check{B}^-} \rightarrow \overline{\text{Bun}}_{B^-}$.

Let $\mathcal{H}_T^{+, \mathfrak{U}(\theta)}$ be the stack classifying $\mathcal{F}_T \in \text{Bun}_T, D \in \mathring{X}^{\mathfrak{U}(\theta)}$ viewed as a point of X^θ . Let $\mathcal{H}_T^{+, \mathfrak{U}(\theta)}$ be the stack classifying a point of $\mathcal{H}_T^{+, \mathfrak{U}(\theta)}$ as above, and lines $\mathcal{U}, \mathcal{U}_G$

equipped with

$$\mathcal{U}^N \xrightarrow{\sim} (\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}_T(-D)}, \quad \mathcal{U}_G^N \xrightarrow{\sim} (\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}_T}.$$

As in ([23], Section 4.4.1), we have an isomorphism

$$(40) \quad \mathfrak{u}(\theta) \overline{\text{Bun}}_{\tilde{B}^-} \xrightarrow{\sim} \text{Bun}_{B^-} \times_{\text{Bun}_T} \mathcal{H}_T^{+, \mathfrak{u}(\theta)},$$

where to define the fibred product we used the map $\mathcal{H}_T^{+, \mathfrak{u}(\theta)} \rightarrow \text{Bun}_T$ sending the above point to \mathcal{F}_T .

Consider the line bundle on $\mathring{X}^{\mathfrak{u}(\theta)}$, whose fibre at D is $\mathcal{L}_{\mathcal{F}_T^0(-D)}^{\bar{\kappa}}$, here we view $\mathring{X}^{\mathfrak{u}(\theta)} \subset X^\theta$ as a subscheme. Let $\widetilde{\text{Gr}}_T^{+, \mathfrak{u}(\theta)}$ be the gerb of N -th roots of this line bundle. Call $V \in \text{Rep}(\tilde{T}_\zeta)$ negative if each \tilde{T}_ζ -weight appearing in V lies in $-\Lambda^{\text{pos}}$. Actually, such a weight is in $-\Lambda^{\sharp, \text{pos}}$, where $\Lambda^{\sharp, \text{pos}} = \Lambda^\sharp \cap \Lambda^{\text{pos}}$.

For $V \in \text{Rep}(\tilde{T}_\zeta)$ negative we get a perverse sheaf $\text{Loc}_\zeta^{\mathfrak{u}(\theta)}(V)$ on $\widetilde{\text{Gr}}_T^{+, \mathfrak{u}(\theta)}$ on which $\mu_N(k)$ acts by ζ , and such that for $D = \sum_k \theta_k x_k \in \mathring{X}^{\mathfrak{u}(\theta)}$ its restriction to

$$\prod_k \widetilde{\text{Gr}}_{T, x_k}^{\theta_k}$$

is $(\boxtimes_k \text{Loc}_\zeta(V_{\theta_k}))[\mathfrak{u}(\theta)]$. Here $\text{Gr}_{T, x}^\theta$ is the connected component of $\text{Gr}_{T, x}$ containing $t_x^\theta T(0)$, in other words, corresponding to $\mathcal{F}_T^0(-\theta x)$ with the evident trivialization off x . The functor Loc_ζ was defined in Section 0.0.4. Note that $\text{Loc}_\zeta^{\mathfrak{u}(\theta)}(V)$ vanishes unless in the decomposition (39) each term lies in $-\Lambda^{\sharp, \text{pos}}$.

For $V \in \text{Rep}(\tilde{T}_\zeta)$ negative define a perverse sheaf $\text{Loc}_{\text{Bun}_T, \zeta}^{\mathfrak{u}(\theta)}(V)$ on $\mathcal{H}_T^{+, \mathfrak{u}(\theta)}$ as follows. Let $\text{Bun}_{T, \mathfrak{u}(\theta)}$ denote the stack classifying $\mathcal{F}_T \in \text{Bun}_T$, $D \in \mathring{X}^{\mathfrak{u}(\theta)}$, and a trivialization of \mathcal{F}_T over the formal neighbourhood of D . Let $\widetilde{\text{Bun}}_{T, \mathfrak{u}(\theta)} = \text{Bun}_{T, \mathfrak{u}(\theta)} \times_{\text{Bun}_T} \widetilde{\text{Bun}}_T$. Let $T_{\mathfrak{u}(\theta)}$ be the scheme classifying $D \in \mathring{X}^{\mathfrak{u}(\theta)}$ and a section of T over the formal neighbourhood of D , this is a group scheme over $\mathring{X}^{\mathfrak{u}(\theta)}$. For $(\mathcal{F}_T, D) \in \text{Bun}_{T, \mathfrak{u}(\theta)}$ we have a natural isomorphism $(\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}_T} \otimes (\mathcal{L}^{\bar{\kappa}})_{\mathcal{F}_T^0(-D)} \xrightarrow{\sim} (\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}_T(-D)}$. So, as in ([23], Section 4.4.2), we get a $T_{\mathfrak{u}(\theta)}$ -torsor

$$\widetilde{\text{Bun}}_{T, \mathfrak{u}(\theta)} \times_{\mathring{X}^{\mathfrak{u}(\theta)}} \widetilde{\text{Gr}}_T^{+, \mathfrak{u}(\theta)} \rightarrow \mathcal{H}_T^{+, \mathfrak{u}(\theta)}$$

For $\mathcal{T} \in \text{D}(\widetilde{\text{Bun}}_T)$ and a $T_{\mathfrak{u}(\theta)}$ -equivariant perverse sheaf S on $\widetilde{\text{Gr}}_T^{+, \mathfrak{u}(\theta)}$ we may form their twisted product $\mathcal{T} \boxtimes S$ on $\mathcal{H}_T^{+, \mathfrak{u}(\theta)}$ using the above torsor. The perverse sheaf $\text{Loc}_\zeta^{\mathfrak{u}(\theta)}(V)$ on $\widetilde{\text{Gr}}_T^{+, \mathfrak{u}(\theta)}$ is naturally $T_{\mathfrak{u}(\theta)}$ -equivariant. For $V \in \text{Rep}(\tilde{T}_\zeta)$ negative define

$$\text{Loc}_{\text{Bun}_T, \zeta}^{\mathfrak{u}(\theta)}(V) = \text{IC}(\widetilde{\text{Bun}}_T) \boxtimes \text{Loc}_\zeta^{\mathfrak{u}(\theta)}(V)$$

For the map $\mathfrak{q}^- : \text{Bun}_{B^-} \rightarrow \text{Bun}_T$ on (40) we get the perverse sheaf denoted

$$\text{Loc}_{\text{Bun}_{B^-}, \zeta}^{\mathfrak{u}(\theta)}(V) = (\mathfrak{q}^-)^* \text{Loc}_{\text{Bun}_T, \zeta}^{\mathfrak{u}(\theta)}(V)[\dim. \text{rel}(\mathfrak{q}^-)]$$

Theorem 4.10.1 ([23], Theorem 4.1). *The $*$ -restriction of IC_ζ to $\mathfrak{U}(\theta) \overline{\mathrm{Bun}}_{\tilde{B}^-}$ vanishes unless in the decomposition (39) each term lies in $-\Lambda^{\sharp, \mathrm{pos}}$. In the latter case it is isomorphic to*

$$\mathrm{Loc}_{\mathrm{Bun}_{B, \zeta}}^{\mathfrak{U}(\theta)} \left(\bigoplus_{i \geq 0} \mathrm{Sym}^i(\check{\mathfrak{u}}_\zeta^-)[2i] \right) \otimes \bar{\mathbb{Q}}_\ell[-|\mathfrak{U}(\theta)|],$$

where $\bigoplus_{i \geq 0} \mathrm{Sym}^i(\check{\mathfrak{u}}_\zeta^-)[2i]$ is viewed as a cohomologically graded \check{T}_ζ -module.

4.11. Our purpose now is to improve Proposition 4.3.4 as follows.

Proposition 4.11.1. *i) Assume $\varrho(\alpha) \notin \mathbb{Z}$ for any simple coroot α . Then for $\mu \in -\Lambda^{\mathrm{pos}}$ we have a (non-canonical) isomorphism $\mathcal{L}_\emptyset^\mu \xrightarrow{\sim} \overline{\mathbb{F}}(\mathcal{F}_\emptyset)$ in $\mathrm{D}_\zeta(\overset{\circ}{X}^\mu)$.*

ii) The complex $\overline{\mathbb{F}}(\mathcal{F}_\emptyset)$ is perverse. If in addition the subtop cohomology property is satisfied for ϱ then we have a (non-canonical) isomorphism $\mathcal{L}_\emptyset^\mu \xrightarrow{\sim} \overline{\mathbb{F}}(\mathcal{F}_\emptyset)$ in $\mathrm{D}_\zeta(X^\mu)$.

Proof. i) If $-\mu$ is a simple coroot of G then, by Theorem 4.10.1, $\bar{F}^\mu(\mathcal{F}_\emptyset)$ is the extension by zero under $\mathcal{Z}^\mu \hookrightarrow \check{\mathcal{Z}}^\mu$. Therefore, over $\overset{\circ}{X}^\mu$ the desired isomorphism follows from the factorization property combined with Proposition 4.3.4.

ii) Denote by $\overline{\mathbb{F}}(\mathcal{F}_\emptyset)_{\mu x}$ the $*$ -fibre of $\overline{\mathbb{F}}(\mathcal{F}_\emptyset)$ at $\mu x \in X^\mu$. If $D = \sum_k \mu_k x_k \in X^\mu$ with x_k pairwise different, the $*$ -fibre of $\overline{\mathbb{F}}(\mathcal{F}_\emptyset)$ at D , by factorization property, identifies with

$$\boxtimes_k \overline{\mathbb{F}}(\mathcal{F}_\emptyset)_{\mu_k x_k}$$

Our claim is reduced to the following Proposition 4.11.2. □

Proposition 4.11.2. *Let $x \in X$ and $\mu < 0$.*

i) The complex $\overline{\mathbb{F}}(\mathcal{F}_\emptyset)_{\mu x}$ is placed in degree ≤ -1 .

ii) Assume in addition that the subtop cohomology property is satisfied for ϱ . Then $\overline{\mathbb{F}}(\mathcal{F}_\emptyset)_{\mu x}$ is placed in degree < -1 unless $-\mu$ is a simple coroot.

Proof. We are integrating over the fibre, say Y , of $\check{\mathcal{Z}}^\mu$ over μx . From (30), Y identifies with $(\mathrm{Gr}_B^0 \cap \overline{\mathrm{Gr}}_{B^-}^\mu) \times^{T(\theta_x)} \Omega^\rho |_{D_x}$. The restriction of $\bar{F}^\mu(\mathcal{F}_\emptyset)$ to the stratum

$$(\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^\mu) \times^{T(\theta_x)} \Omega^\rho |_{D_x}$$

is a local system placed in usual degree $\langle \mu, 2\check{\rho} \rangle$.

Denote by $ev_x : \mathrm{Gr}_B^0 \times^{T(\theta_x)} \Omega^\rho |_{D_x} \rightarrow \mathbb{A}^1$ the restriction of the canonical map $ev : \mathfrak{M}_{\emptyset, 0} \rightarrow \mathbb{A}^1$. As is explained in ([20], Section 5.6), the local system $ev_x^* \mathcal{L}_\psi$ is nonconstant on each irreducible component of $(\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^\mu) \times^{T(\theta_x)} \Omega^\rho |_{D_x}$ of dimension $-\langle \mu, \check{\rho} \rangle$. So, the restriction of $\bar{F}^\mu(\mathcal{F}_\emptyset)$ to each such irreducible component is also nonconstant. Thus, the contribution of the stratum $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^\mu$ is placed in the usual degree ≤ -1 .

For $\mu = \nu + \theta$ with $\nu, \theta < 0$ consider the stratum $Y_\nu := (\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^\nu) \times^{T(\theta_x)} \Omega^\rho |_{D_x}$ of Y . Let $\mathfrak{U}(\theta)$ be the trivial decomposition $\theta = \theta$, so $\overset{\circ}{X}^{\mathfrak{U}(\theta)} = X$. Pick some trivialization of the line $\mathcal{L}_{\mathcal{F}_T^0(-\theta x)}^{\check{\kappa}}$. This allows for $V \in \mathrm{Rep}(\check{T}_\zeta)$ to see $\mathrm{Loc}_\zeta(V_\theta)$ as a complex over $\mathrm{Spec} k$. Then the $*$ -restriction of $\bar{F}^\mu(\mathcal{F}_\emptyset)$ to Y_ν identifies with

$$\mathrm{Loc}_\zeta \left(\left(\bigoplus_{i \geq 0} \mathrm{Sym}^i(\check{\mathfrak{u}}_\zeta^-)[2i] \right)_\theta \right) \otimes ev_x^* \mathcal{L}_\psi \otimes \mathcal{E}[-\langle 2\check{\rho}, \nu \rangle],$$

where \mathcal{E} is a rank one tame local system. If $\nu \neq 0$ then $ev_x^* \mathcal{L}_\psi \otimes \mathcal{E}$ is nontrivial on each irreducible component of Y_ν of dimension $-\langle \tilde{\rho}, \nu \rangle$. Since $\text{Loc}_\zeta((\bigoplus_{i \geq 0} \text{Sym}^i(\tilde{u}_\zeta^-)[2i]))_\theta$ is placed in degrees < 0 , for $\nu \neq 0$ the contribution of Y_ν is placed in degrees ≤ -2 .

For $\nu = 0$ we get $Y_\nu = \text{Spec } k$. The $*$ -restriction of $\bar{F}^\mu(\mathcal{F}_\emptyset)$ to this point identifies with

$$\text{Loc}_\zeta((\bigoplus_{i \geq 0} \text{Sym}^i(\tilde{u}_\zeta^-)[2i])_\mu),$$

the latter is placed in degrees ≤ -2 . So, $\bar{\mathbb{F}}(\mathcal{F}_\emptyset)_{\mu x}$ is placed in degree ≤ -1 , and only the open stratum Y_μ may contribute to the cohomology group $H^{-1}(\bar{\mathbb{F}}(\mathcal{F}_\emptyset)_{\mu x})$.

ii) By definition of the subtop cohomology property, the open stratum Y_μ does not contribute to $H^{-1}(\bar{\mathbb{F}}(\mathcal{F}_\emptyset)_{\mu x})$. \square

Remark 4.11.3. *Conjecture 1.1.2 would imply the following. Assume $\varrho(\alpha) \notin \mathbb{Z}$ for any simple coroot α . Then $\mathcal{L}_\emptyset^\mu \xrightarrow{\sim} \bar{\mathbb{F}}(\mathcal{F}_\emptyset)$ in $D_\zeta(X^\mu)$.*

Proposition 4.11.4. *The functor $\bar{\mathbb{F}} : D\text{Whit}_n^\kappa \rightarrow D_\zeta(X_n^\mu)$ is exact for the perverse t -structures.*

Proof. Pick $K \in \text{Whit}_n^\kappa$. Let $\eta : \{1, \dots, n\} \rightarrow A$ be a surjection. Pick $\mu_a \in \Lambda$ for $a \in A$ with $\sum_a \mu_a = \mu$. Let $V \subset X_n^\mu$ be the subscheme classifying disjoint points $\{y_a \in X\}_{a \in A}$ such that $x_i = y_{\eta(i)}$ for each i , and $D = \sum_{a \in A} \mu_a y_a$. In view of the factorization property and Propositions 4.9.1, 4.11.1, it suffices to show that the $*$ -restriction of $\bar{\mathbb{F}}(K)$ to V is placed in perverse degrees ≤ 0 . Let $\bar{\mathcal{Z}}_V^\mu$ be the preimage of V under $\bar{\pi}^\mu : \bar{\mathcal{Z}}_n^\mu \rightarrow X_n^\mu$. The fibre of $\bar{\mathcal{Z}}_V^\mu$ over $\{y_a\}$ is

$$\prod_a \overline{\text{Gr}}_{B^-, y_a}^{\mu_a} \times^{T(\mathcal{O}_{y_a})} \Omega^\rho |_{D_{y_a}}$$

Pick a collection $\bar{\lambda} = \{\lambda_a\}_{a \in A}$ with $\lambda_a \in \Lambda^+$, $\mu_a \leq \lambda_a$. Let $\mathfrak{M}_{\eta, \bar{\lambda}} \subset \mathfrak{M}_n$ be the substack classifying a point of V as above (this defines x_i), and such that for each $\check{\lambda} \in \check{\Lambda}^+$ the map

$$\kappa^{\check{\lambda}} : \Omega^{\langle \rho, \check{\lambda} \rangle} \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}}(\sum_a \langle \lambda_a y_a, \check{\lambda} \rangle)$$

is regular over X and has no zeros over X . Let $\bar{\mathcal{Z}}_{V, \bar{\lambda}}^\mu$ be obtained from $\bar{\mathcal{Z}}_V^\mu$ by the base change $\mathfrak{M}_{\eta, \bar{\lambda}} \rightarrow \mathfrak{M}_n$. Let $\pi_\eta : \mathfrak{M}_{\eta, \bar{\lambda}} \rightarrow V$ be the projection, $ev_{\bar{\lambda}} : \mathfrak{M}_{\eta, \bar{\lambda}} \rightarrow \mathbb{A}^1$ the corresponding evaluation map (as in Section 2.3). Let $K^{\bar{\lambda}}$ be a complex on V placed in perverse degrees ≤ 0 such that the $*$ -restriction $K|_{\mathfrak{M}_{\eta, \bar{\lambda}}}$ identifies with

$$\pi_\eta^* K^{\bar{\lambda}} \otimes ev_{\bar{\lambda}}^* \mathcal{L}_\psi[\dim],$$

where $\dim = (g-1)\dim U - \langle (2g-2)\rho - \sum_a \lambda_a, 2\check{\rho} \rangle$. This is the relative dimension of π_η .

Only finite number of the strata $\bar{\mathcal{Z}}_{V, \bar{\lambda}}^\mu$ of $\bar{\mathcal{Z}}_V^\mu$ contribute to $\bar{\mathbb{F}}(K)|_V$. Let $K_{\bar{\lambda}}$ denote the $!$ -direct image under $\bar{\pi}^\mu : \bar{\mathcal{Z}}_{V, \bar{\lambda}}^\mu \rightarrow V$ of the $*$ -restriction $\bar{F}^\mu(K)|_{\bar{\mathcal{Z}}_{V, \bar{\lambda}}^\mu}$. It suffices to show that $K_{\bar{\lambda}}$ is placed in perverse degrees ≤ 0 . From Theorem 4.10.1 we conclude that

$K_{\bar{\lambda}} \xrightarrow{\sim} K^{\bar{\lambda}} \otimes M$, where M is a complex on V with locally constant cohomology sheaves. It remains to show that M is placed in degrees ≤ 0 .

The problem being local, we may and do assume that A is the one element set. Write $\mu = \mu_a$, $\lambda_a = \lambda$, $y_a = y$. Then the fibre Y of $\overline{\mathcal{Z}}_{V,\lambda}^\mu$ over y is

$$(\mathrm{Gr}_{B,y}^\lambda \cap \overline{\mathrm{Gr}}_{B-,y}^\mu) \times^{T(0_y)} \Omega^\rho |_{D_y}$$

For $\mu \leq \nu \leq \lambda$ let $Y_\nu = (\mathrm{Gr}_{B,y}^\lambda \cap \mathrm{Gr}_{B-,y}^\nu) \times^{T(0_y)} \Omega^\rho |_{D_y}$, they form a stratification of Y . For $\mu = \nu + \theta$ with $\nu \leq \lambda, \theta \leq 0$ let $\mathfrak{U}(\theta)$ be the trivial decomposition $\theta = \theta$, so $\overset{\circ}{X}^{\mathfrak{U}(\theta)} = X$. Pick some trivialization of the line $\mathcal{L}_{\mathcal{F}_T^{\bar{\zeta}}(-\theta x)}$. This allows for $V \in \mathrm{Rep}(\check{T}_{\bar{\zeta}})$ to see $\mathrm{Loc}_{\bar{\zeta}}(V_\theta)$ as a complex over $\mathrm{Spec} k$ (as in Proposition 4.11.2). The $*$ -restriction $\overline{F}^\mu(K) |_{Y_\nu}$ identifies with

$$\mathrm{Loc}_{\bar{\zeta}}\left(\left(\bigoplus_{i \geq 0} \mathrm{Sym}^i(\check{\mathfrak{u}}_{\bar{\zeta}}^-)[2i]\right)_\theta\right) \otimes ev_{\bar{\lambda}}^* \mathcal{L}_\psi \otimes \mathcal{E} \otimes K_y^{\bar{\lambda}}[\langle \lambda - \nu, 2\check{\rho} \rangle],$$

where \mathcal{E} is some rank one local system. Since $\dim Y_\nu \leq \langle \lambda - \nu, \check{\rho} \rangle$, we see that the contribution of Y_ν to the complex M_y is placed in degrees ≤ 0 . We are done. \square

Combining Propositions 4.7.1, 4.11.4, one gets the following.

Theorem 4.11.5. *Assume that ϱ satisfies the subtop cohomology property. Then $\overline{\mathbb{F}}$ gives rise to the functor $\overline{\mathbb{F}} : \mathrm{Whit}_n^\kappa \rightarrow \widehat{\mathrm{FS}}_n^\kappa$, which is exact for the perverse t -structures and commutes with the Verdier duality (up to replacing ψ by ψ^{-1} and ζ by ζ^{-1}).*

4.12. Multiplicity spaces.

4.12.1. For a topological space \mathcal{X} write $\mathrm{Irr}(\mathcal{X})$ for the set of irreducible components of \mathcal{X} . Recall for $\nu \geq 0$ the notation $B_{\mathfrak{g}}(\nu)$ and the functions ϕ_i on this crystal from Section 1.2.1.

Let $\mu \in \Lambda, \lambda \in \Lambda^+$ with $\mu \leq \lambda$. Let $b \subset \mathrm{Gr}_B^\lambda \cap \mathrm{Gr}_{B-}^\mu$ be an irreducible component. Denote by $\bar{b} \subset \mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B-}^{\mu-\lambda}$ the component $t^{-\lambda}b$, so $\bar{b} \in B_{\mathfrak{g}}(\lambda - \mu)$. By Andersen's theorem ([1], Proposition 3) we have a bijection

$$(41) \quad \{a \in \mathrm{Irr}(\mathrm{Gr}_{B-}^\mu \cap \mathrm{Gr}_B^\lambda) \mid a \subset \overline{\mathrm{Gr}}_G^\lambda\} \xrightarrow{\sim} \mathrm{Irr}(\mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_{B-}^\mu)$$

sending a to the closure of $a \cap \mathrm{Gr}_G^\lambda$.

Lemma 4.12.2. *Under the above assumptions the following are equivalent.*

- i) For all $i \in \mathcal{J}$, $\phi_i(\bar{b}) \leq \langle \lambda, \check{\alpha}_i \rangle$,
- ii) $b \subset \overline{\mathrm{Gr}}_G^\lambda$.

Proof. Recall the canonical inclusion $B(-w_0(\lambda)) \hookrightarrow T_{-\lambda} \otimes B(-\infty)$ from ([6], p. 87), see also Section 1.2.2. Its image is the set of $t_{-\lambda} \otimes a$ such that $a \in B(-\infty)$, and for each $i \in \mathcal{J}$, $\phi_i(a^*) \leq \langle \check{\alpha}_i, \lambda \rangle$. So, i) is equivalent to $t_{-\lambda} \otimes \bar{b}^* \in B(-w_0(\lambda))$. By ([1], Proposition 3), we have a canonical bijection of irreducible components (up to passing to the closure)

$$\mathrm{Irr}(t^\mu \mathrm{Gr}_G^{-w_0(\lambda)} \cap \mathrm{Gr}_B^0) \xrightarrow{\sim} \{a \in \mathrm{Irr}(\mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B-}^{\mu-\lambda}) \mid a \subset t^\mu \overline{\mathrm{Gr}}_G^{-w_0(\lambda)}\}$$

So, i) is equivalent to the property that $t^{-\mu}\bar{b}^* \in \text{Irr}(\text{Gr}_G^{-w_0(\lambda)} \cap \text{Gr}_B^{-\mu})$. Our claim follows now from the properties of the bijection $*$: $B(-\infty) \rightarrow B(-\infty)$ and (41). \square

4.12.3. *Additional input data.* Recall that the pull-back of the exact sequence (4) to Λ^\sharp is abelian. Pick a splitting $\mathfrak{t}_\mathbb{E}^0 : \Lambda^\sharp \rightarrow V_\mathbb{E}$ of the exact sequence (4) over Λ^\sharp . We assume $\mathfrak{t}_\mathbb{E}^0$ is compatible with the section $\mathfrak{t}_\mathbb{E}$ from Section 0.0.4.

For each $\bar{\lambda} \in \Lambda/\Lambda^\sharp$ we make the following choice. Pick compatible trivializations $\delta_\lambda : (V_\mathbb{E})_\lambda \xrightarrow{\sim} \mathbb{G}_m$ of the fibre of $\text{Gra}_G \rightarrow \text{Gr}_G$ at $t^\lambda G(\mathcal{O})$ for all $\lambda \in \Lambda$ over $\bar{\lambda}$. Here compatible means equivariant under the action of Λ^\sharp via $\mathfrak{t}_\mathbb{E}^0$.

4.12.4. For each $\lambda \in \Lambda$ the above trivialization δ_λ yields sections $s_B^\lambda : \text{Gr}_B^\lambda \rightarrow \text{Gra}_G$, $s_{B^-}^\mu : \text{Gr}_{B^-}^\mu \rightarrow \text{Gra}_G$ of the \mathbb{G}_m -torsor $\text{Gra}_G \rightarrow \text{Gr}_G$. The discrepancy between them is a map that we denote by

$$\gamma_\lambda^\mu : \text{Gr}_B^\lambda \cap \text{Gr}_{B^-}^\mu \rightarrow \mathbb{G}_m$$

and define by $s_{B^-}^\mu = \gamma_\lambda^\mu s_B^\lambda$. Note that if $\lambda - \mu \in \Lambda^\sharp$ then γ_λ^μ does not depend of the choice of δ (so depends only on $\mathfrak{t}_\mathbb{E}^0$).

Theorem 4.12.5. *Assume that ϱ satisfies the subtop cohomology property. Pick $\lambda \in \Lambda^+$ and $x \in X$. There is a decomposition*

$$(42) \quad \overline{\mathbb{F}}(\mathcal{F}_{x,\lambda}) \xrightarrow{\sim} \bigoplus_{\mu \leq \lambda, \lambda - \mu \in \Lambda^\sharp} \mathcal{L}_{x,\mu} \otimes V_\mu^\lambda$$

in $\widetilde{\text{FS}}_x^\kappa$, where V_μ^λ is the \mathbb{Q}_ℓ -vector space with a canonical base indexed by those $b \in \text{Irr}(\text{Gr}_{B,x}^\lambda \cap \text{Gr}_{B^-,x}^\mu)$ that satisfy the following two properties:

- $b \subset \overline{\text{Gr}}_{G,x}^\lambda$,
- the local system $(\gamma_\lambda^\mu)^* \mathcal{L}_\zeta$ is trivial on b .

In particular, we have $V_\lambda^\lambda = \mathbb{Q}_\ell$.

4.12.6. *Proof of Theorem 4.12.5.* Recall that $\mathcal{F}_{x,\lambda}$ is the extension by zero from $\widetilde{\mathfrak{M}}_{x,\leq\lambda}$. Since $\bar{\pi}^\mu$ factors through $\bar{\pi}^\mu : \bar{\mathfrak{Z}}_{x,\leq\lambda}^\mu \rightarrow X_{x,\leq\lambda}^\mu$, $\overline{\mathbb{F}}(\mathcal{F}_{x,\lambda})$ will be the extension by zero from $X_{x,\leq\lambda}^\mu$. The latter scheme is empty unless $\mu \leq \lambda$. So, the μ -component of $\overline{\mathbb{F}}(\mathcal{F}_{x,\lambda})$ vanishes unless $\mu \leq \lambda$.

By Corollary 4.9.2, since $\bar{\pi}^\mu$ is proper for each μ , there is a decomposition

$$(43) \quad \overline{\mathbb{F}}(\mathcal{F}_{x,\lambda}) \xrightarrow{\sim} \bigoplus_{\mu \leq \lambda} \mathcal{L}_{x,\mu} \otimes V_\mu^\lambda.$$

It remains to determine the spaces V_μ^λ . Pick $\mu \leq \lambda$. Set for brevity $\gamma = \gamma_\lambda^\mu$. Recall the notation $\chi_0^\lambda : \text{Gr}_{B,x}^\lambda \rightarrow \mathbb{A}^1$ from Section 1.1.

Lemma 4.12.7. *The space V_μ^λ in (43) has a canonical base consisting of those irreducible components of $\text{Gr}_{B,x}^\lambda \cap \text{Gr}_{B^-,x}^\mu$ over which the local system $(\chi_0^\lambda)^* \mathcal{L}_\psi \otimes \gamma^* \mathcal{L}_\zeta$ is constant.*

Proof. Since $\overline{\mathbb{F}}(\mathcal{F}_{x,\lambda}) \in \widetilde{\text{FS}}_x^\kappa$, it suffices to determine the fibre $K := \overline{\mathbb{F}}(\mathcal{F}_{x,\lambda})_{\mu x}$. By Proposition 4.11.4, K is placed in degrees ≤ 0 . Pick a trivialization of $\mathcal{P}^{\bar{\kappa}}$ at $\mu x \in X_{x,\leq\lambda}^\mu$.

This allows to see K as a complex over $\text{Spec } k$, it also determines $\mathcal{L}_{x,\mu}$ up to a unique isomorphism, so yields an isomorphism

$$V_\mu^\lambda \xrightarrow{\sim} \mathbf{H}^0(\overline{\mathbb{F}}(\mathcal{F}_{x,\lambda})_{\mu x})$$

The fibre of $\bar{\pi}^\mu : \bar{\mathcal{Z}}_{x,\leq\lambda}^\mu \rightarrow X_{x,\leq\lambda}^\mu$ over μx is

$$Y := (\overline{\text{Gr}}_{B,x}^\lambda \cap \overline{\text{Gr}}_{B^-,x}^\mu) \times^{T(\theta_x)} \Omega^\rho |_{D_x}$$

For $\eta \in \Lambda^+$, $\eta \leq \lambda$ let

$$Y_\eta = (\text{Gr}_{B,x}^\eta \cap \overline{\text{Gr}}_{B^-,x}^\mu) \times^{T(\theta_x)} \Omega^\rho |_{D_x}$$

Denote by K^η the constant complex over $\text{Spec } k$ such that $j_{x,\eta}^* \mathcal{F}_{x,\lambda} \xrightarrow{\sim} K^\eta \otimes \mathcal{F}_{x,\eta}$. Here K^η is placed in degrees < 0 for $\eta < \lambda$, and $K^\lambda = \overline{\mathbb{Q}}_\ell$.

Let K_η be the contribution of the $*$ -restriction $\mathcal{F}_{x,\lambda} |_{\bar{\mathfrak{M}}_{x,\eta}}$ to K . In other words,

$$K_\eta = \mathbf{R}\Gamma_c(Y_\eta, \bar{F}^\mu(\mathcal{F}_{x,\lambda}) |_{Y_\eta}),$$

where we used the $*$ -restriction to Y_η , and the above trivialization of $\mathcal{P}^{\bar{\kappa}}$ at $\mu x \in X_{x,\leq\lambda}^\mu$ to get rid of the corresponding gerb. By Proposition 4.11.4, if $\eta < \lambda$ then K_η is placed in degrees < 0 . So, it suffices to analyze K_λ .

For $\mu \leq \nu \leq \lambda$ let

$$Y_{\lambda,\nu} = (\text{Gr}_{B,x}^\lambda \cap \text{Gr}_{B^-,x}^\nu) \times^{T(\theta_x)} \Omega^\rho |_{D_x}.$$

The schemes $Y_{\lambda,\nu}$ with $\mu \leq \nu \leq \lambda$ form a stratification of Y_λ .

For $\mu = \nu + \theta$ with $\nu \leq \lambda$, $\theta \leq 0$ let $\mathfrak{U}(\theta)$ be the trivial decomposition $\theta = \theta$. Pick a trivialization of the line $\mathcal{L}_{\mathcal{F}_T^0(-\theta x)}^{\bar{\kappa}}$. As in the proof of Proposition 4.11.4 this allows for $V \in \text{Rep}(\tilde{T}_\zeta)$ to see $\text{Loc}_\zeta(V_\theta)$ as a complex over $\text{Spec } k$. The $*$ -restriction $\bar{F}^\mu(\mathcal{F}_{x,\lambda}) |_{Y_{\lambda,\nu}}$ identifies with

$$\text{Loc}_\zeta((\bigoplus_{i \geq 0} \text{Sym}^i(\check{\mathfrak{u}}_\zeta^-)[2i])_\theta) \otimes ev_{x,\lambda}^* \mathcal{L}_\psi \otimes \mathcal{E}[\langle \lambda - \nu, 2\check{\rho} \rangle],$$

where \mathcal{E} is some rank one local system. Recall that $Y_{\lambda,\nu}$ is of pure dimension $\langle \lambda - \nu, \check{\rho} \rangle$. So, the contribution $K_{\lambda,\nu}$ of $Y_{\lambda,\nu}$ to K_λ is

$$\text{Loc}_\zeta((\bigoplus_{i \geq 0} \text{Sym}^i(\check{\mathfrak{u}}_\zeta^-)[2i])_\theta) \otimes \mathbf{R}\Gamma_c(Y_{\lambda,\nu}, ev_{x,\lambda}^* \mathcal{L}_\psi \otimes \mathcal{E})[\langle \lambda - \nu, 2\check{\rho} \rangle]$$

It is placed in degrees ≤ 0 , and the inequality is strict unless $\theta = 0$. There remains to analyze the complex

$$K_{\lambda,\mu} = \mathbf{R}\Gamma_c(Y_{\lambda,\mu}, ev_{x,\lambda}^* \mathcal{L}_\psi \otimes \mathcal{E})[\langle \lambda - \mu, 2\check{\rho} \rangle]$$

We see that only the open part $\mathcal{Z}_{x,\lambda}^\mu \subset \bar{\mathcal{Z}}_{x,\leq\lambda}^\mu$ contributes to the 0-th cohomology of K . This allows to describe the local system \mathcal{E} over $Y_{\lambda,\mu}$. From the definitions we get $\gamma^* \mathcal{L}_\zeta \xrightarrow{\sim} \mathcal{E}$. So, $K_{\lambda,\mu}$ identifies with

$$\mathbf{R}\Gamma_c(\text{Gr}_{B,x}^\lambda \cap \text{Gr}_{B^-,x}^\mu, (\chi_0^\lambda)^* \mathcal{L}_\psi \otimes \gamma^* \mathcal{L}_\zeta)[\langle \lambda - \mu, 2\check{\rho} \rangle]$$

for some character $\chi_0 : U(F_x) \rightarrow \mathbb{A}^1$ of conductor zero. Our claim follows. \square

Lemma 4.12.8. *Let $\mu \leq \lambda$, $\lambda \in \Lambda^+$. Let $b \subset \mathrm{Gr}_B^\lambda \cap \mathrm{Gr}_{B^-}^\mu$ be an irreducible component. Denote by $\bar{b} \subset \mathrm{Gr}_B^0 \cap \mathrm{Gr}_{B^-}^{\mu-\lambda}$ the component $t^{-\lambda}b$, so $\bar{b} \in B_{\mathfrak{g}}(\lambda - \mu)$. The restriction $\chi_0^\lambda : b \rightarrow \mathbb{A}^1$ of χ_0^λ is dominant if and only if there is $i \in \mathcal{J}$ such that $\phi_i(\bar{b}) > \langle \lambda, \check{\alpha}_i \rangle$.*

Proof. For $i \in \mathcal{J}$ recall the maps $\mathfrak{q}_{P_i} : \mathrm{Gr}_{P_i} \rightarrow \mathrm{Gr}_{M_i}$. For $i \in \mathcal{J}$ let $\mu_i \leq \lambda$ be the unique element such that $\mathfrak{q}_{P_i}^{-1}(\mathrm{Gr}_{B^-}^{\mu_i}(M_i)) \cap b$ is dense in b . Note that $b \subset \mathrm{Gr}_B^\lambda \cap \mathrm{Gr}_{B^-}^\mu$ is a $T(\mathcal{O})$ -invariant subscheme. Let

$$b_0 = b \cap \left(\bigcap_{i \in \mathcal{J}} \mathfrak{q}_{P_i}^{-1}(\mathrm{Gr}_{B^-}^{\mu_i}(M_i)) \right),$$

it is a dense $T(\mathcal{O})$ -invariant subscheme of b . Set $\bar{\mu} = \{\mu_i\}_{i \in \mathcal{J}}$ and

$$Z^{\bar{\mu}} = \prod_{i \in \mathcal{J}} \mathrm{Gr}_{B(M_i)}^\lambda \cap \mathrm{Gr}_{B^-(M_i)}^{\mu_i}.$$

Let $\mathfrak{q}^{\bar{\mu}} : b_0 \rightarrow Z^{\bar{\mu}}$ be the product of the maps \mathfrak{q}_{P_i} . This map is $T(\mathcal{O})$ -equivariant. Since $T(\mathcal{O})$ acts transitively on $Z^{\bar{\mu}}$, the map $\mathfrak{q}^{\bar{\mu}}$ is surjective. For $i \in \mathcal{J}$ let ev_i be the composition

$$\mathrm{Gr}_{B(M_i)}^\lambda \cap \mathrm{Gr}_{B^-(M_i)}^{\mu_i} \hookrightarrow \mathrm{Gr}_{B(M_i)}^\lambda \rightarrow \mathrm{Gr}_B^\lambda \xrightarrow{\chi_0^\lambda} \mathbb{A}^1$$

Denote by $ev^{\bar{\mu}} : Z^{\bar{\mu}} \rightarrow \mathbb{A}^1$ the map $ev^{\bar{\mu}} = \sum_{i \in \mathcal{J}} ev_i$. The restriction $\chi_0^\lambda|_{b_0}$ equals $ev^{\bar{\mu}} \mathfrak{q}^{\bar{\mu}}$.

Clearly, $ev^{\bar{\mu}} : Z^{\bar{\mu}} \rightarrow \mathbb{A}^1$ is dominant if and only if there is $i \in \mathcal{J}$ such that $ev_i : \mathrm{Gr}_{B(M_i)}^\lambda \cap \mathrm{Gr}_{B^-(M_i)}^{\mu_i} \rightarrow \mathbb{A}^1$ is dominant. The latter condition is equivalent to

$$\phi_i(\bar{b}) = \langle \lambda - \mu_i, \frac{\check{\alpha}_i}{2} \rangle > \langle \lambda, \check{\alpha}_i \rangle$$

Indeed, the multiplication by t^λ identifies $\mathrm{Gr}_{B(M_i)}^0 \cap \mathrm{Gr}_{B^-(M_i)}^{\mu_i-\lambda} \xrightarrow{\sim} \mathrm{Gr}_{B(M_i)}^\lambda \cap \mathrm{Gr}_{B^-(M_i)}^{\mu_i}$. Under the latter isomorphism ev_i identifies with some map $\chi_\lambda^0 : \mathrm{Gr}_{B(M_i)}^0 \cap \mathrm{Gr}_{B^-(M_i)}^{\mu_i-\lambda} \rightarrow \mathbb{A}^1$ for the group M_i . Our claim follows. \square

The local system $(\chi_0^\lambda)^* \mathcal{L}_\psi \otimes \gamma^* \mathcal{L}_\zeta$ is constant on b if and only if $\chi_0^\lambda : b \rightarrow \mathbb{A}^1$ is not dominant and the local system $\gamma^* \mathcal{L}_\zeta$ is constant on b . The map γ intertwines the natural $T(\mathcal{O})$ -action on $\mathrm{Gr}_B^\lambda \cap \mathrm{Gr}_{B^-}^\mu$ with the $T(\mathcal{O})$ -action on \mathbb{G}_m by the character $T(\mathcal{O}) \rightarrow T \xrightarrow{\bar{\kappa}(\lambda-\mu)} \mathbb{G}_m$. So, the condition $\lambda - \mu \in \Lambda^\sharp$ is necessary (but not sufficient) for $\gamma^* \mathcal{L}_\zeta$ to be trivial. Theorem 4.12.5 follows now from Lemmas 4.12.8 and 4.12.2. \square

4.12.9. *Special case.* Our purpose now is to understand the spaces V_μ^λ under the additional assumption $\lambda \in \Lambda^{\sharp,+}$.

Lemma 4.12.10. *Let $\mu \leq \lambda$ with $\mu \in \Lambda$, $\lambda \in \Lambda^{\sharp,+}$. Then over $\mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_B^\lambda \cap \mathrm{Gr}_{B^-}^\mu$ there is an isomorphism $(s_{B^-}^\mu)^* \mathcal{A}_\varepsilon^\lambda \xrightarrow{\sim} (\gamma_\nu^\mu)^* \mathcal{L}_\zeta$.*

Proof. Recall that for any $\lambda \in \Lambda^+$ we have a section $s_\lambda : \mathrm{Gr}_{G,x}^\lambda \rightarrow \widetilde{\mathrm{Gr}}_{G,x}^\lambda$ defined in ([22], Section 2.4.2) and associated to a square root $\Omega^{\frac{1}{2}}(\mathcal{O}_x)$ of $\Omega(\mathcal{O}_x)$ picked in Section 0.0.2. In turn, $s_B^\lambda : \mathrm{Gr}_B^\lambda \rightarrow \mathrm{Gr}_G^\lambda$ yields a section denoted $s_B^\lambda : \mathrm{Gr}_B^\lambda \rightarrow \widetilde{\mathrm{Gr}}_B^\lambda$ by abuse of

notation. Since $\mathrm{Gr}_B^\lambda \cap \mathrm{Gr}_G^\lambda$ is an affine space, the local system $(s_B^\lambda)^* \mathcal{A}_\varepsilon^\lambda$ is trivial on $\mathrm{Gr}_B^\lambda \cap \mathrm{Gr}_G^\lambda$. Our claim follows. \square

For $\lambda \in \Lambda^{\sharp,+}$ write $V(\lambda)$ for the irreducible representation of \check{G}_ζ of highest weight λ . For $\mu \in \Lambda^\sharp$ let $V(\lambda)_\mu \subset V(\lambda)$ denote the subspace of \check{T}_ζ -weight μ .

Theorem 4.12.11. *Let $\mu \in \Lambda^\sharp, \lambda \in \Lambda^{\sharp,+}$ with $\mu \leq \lambda$. Then the vector space V_μ^λ in the formula (42) of Theorem 4.12.5 identifies canonically with $V(\lambda)_\mu$.*

Proof. By ([22], Lemma 3.2) applied to B^- instead of B , the space $V(\lambda)_\mu$ admits a canonical base indexed by those $b \in \mathrm{Irr}(\mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_{B^-}^\mu)$ over which the shifted local system $(s_{B^-}^\mu)^* \mathcal{A}_\varepsilon^\lambda$ is trivial. The space V_μ^λ has a canonical base of $b \in \mathrm{Irr}(\mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_{B^-}^\mu)$ such that $(\gamma_\lambda^\mu)^* \mathcal{L}_\zeta$ is trivial at the generic point of b . Our claim follows now from Lemma 4.12.10. \square

5. HECKE FUNCTORS

5.1. **Action on $D_\zeta(\widetilde{\mathrm{Bun}}_G)$.** In the case of G simple simply-connected the Hecke functors on $D_\zeta(\widetilde{\mathrm{Bun}}_G)$ are defined in ([23], Section 3.2). Let us first define their analogs in our setting.

Write \mathcal{H}_G for the Hecke stack classifying $\mathcal{F}, \mathcal{F}' \in \mathrm{Bun}_G, x \in X$ and an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}'|_{X-x}$. We have a diagram

$$\mathrm{Bun}_G \times X \xleftarrow{h_G^\leftarrow \times \pi} \mathcal{H}_G \xrightarrow{h_G^\rightarrow} \mathrm{Bun}_G,$$

where h_G^\leftarrow (resp., h_G^\rightarrow) sends the above point to \mathcal{F} (resp., to \mathcal{F}'). Here $\pi(\mathcal{F}, \mathcal{F}', x) = x$.

Let $\mathrm{Gr}_{G,X}$ be the ind-scheme classifying $x \in X$ and a G -torsor \mathcal{F} on X with a trivialization $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_G^0|_{X-x}$. Let G_X be the group scheme over X classifying $x \in X$ and an automorphism of \mathcal{F}_G^0 over D_x . The restriction of $\mathcal{L}^{\bar{\kappa}}$ under $\mathrm{Gr}_{G,X} \rightarrow \mathrm{Bun}_G$ is also denoted $\mathcal{L}^{\bar{\kappa}}$. Let $\widetilde{\mathrm{Gr}}_{G,X}$ denote the gerb of N -th roots of $\mathcal{L}^{\bar{\kappa}}$ over $\mathrm{Gr}_{G,X}$.

Write $\mathrm{Bun}_{G,X}$ for the stack classifying $(\mathcal{F} \in \mathrm{Bun}_G, x \in X, \nu)$, where $\nu : \mathcal{F} \xrightarrow{\sim} \mathcal{F}_G^0|_{D_x}$ is a trivialization over D_x . Let $\widetilde{\mathrm{Bun}}_{G,X} = \mathrm{Bun}_{G,X} \times_{\mathrm{Bun}_G} \widetilde{\mathrm{Bun}}_G$. Denote by γ^\leftarrow (resp., γ^\rightarrow) the isomorphism

$$\mathrm{Bun}_{G,X} \times_{G_X} \mathrm{Gr}_{G,X} \xrightarrow{\sim} \mathcal{H}_G$$

such that the projection to the first term corresponds to h_G^\leftarrow (resp., h_G^\rightarrow). The line bundle ${}^\omega \mathcal{L}^{\bar{\kappa}} \boxtimes \mathcal{L}^{\bar{\kappa}}$ on $\mathrm{Bun}_{G,X} \times \mathrm{Gr}_{G,X}$ is G_X -equivariant, we denote by ${}^\omega \mathcal{L}^{\bar{\kappa}} \widetilde{\boxtimes} \mathcal{L}^{\bar{\kappa}}$ its descent to $\mathrm{Bun}_{G,X} \times_{G_X} \mathrm{Gr}_{G,X}$. We have canonically

$$(44) \quad (\gamma^\rightarrow)^*(h_G^\leftarrow)^*({}^\omega \mathcal{L}^{\bar{\kappa}}) \xrightarrow{\sim} {}^\omega \mathcal{L}^{\bar{\kappa}} \widetilde{\boxtimes} \mathcal{L}^{\bar{\kappa}}$$

Let $\mathcal{H}_{\check{G}}$ be the stack obtained from $\widetilde{\mathrm{Bun}}_G \times \widetilde{\mathrm{Bun}}_G$ by the base change $h_G^\leftarrow \times h_G^\rightarrow : \mathcal{H}_G \rightarrow \mathrm{Bun}_G \times \mathrm{Bun}_G$. A point of $\mathcal{H}_{\check{G}}$ is given by $(\mathcal{F}, \mathcal{F}', x) \in \mathcal{H}_G$ and lines $\mathcal{U}, \mathcal{U}'$ equipped with

$$(45) \quad \mathcal{U}^N \xrightarrow{\sim} ({}^\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}}, \quad \mathcal{U}'^N \xrightarrow{\sim} ({}^\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}'}$$

We get the diagram of projections

$$\widetilde{\mathrm{Bun}}_G \xleftarrow{\check{h}_G^\leftarrow} \mathcal{H}_{\check{G}} \xrightarrow{\check{h}_G^\rightarrow} \widetilde{\mathrm{Bun}}_G$$

As in ([23], Section 3.2), the isomorphism (44) yields a G_X -torsor

$$\tilde{\gamma}^\rightarrow : \widetilde{\text{Bun}}_{G,X} \times_X \widetilde{\text{Gr}}_{G,X} \rightarrow \mathcal{H}_{\tilde{G}}$$

extending the G_X -torsor $\text{Bun}_{G,X} \times_X \text{Gr}_{G,X} \rightarrow \text{Bun}_{G,X} \times_{G_X} \text{Gr}_{G,X} \xrightarrow{\gamma^\rightarrow} \mathcal{H}_G$. Namely, it sends

$$(x, \nu' : \mathcal{F}' \xrightarrow{\sim} \mathcal{F}_G^0 |_{D_x}, \nu_1 : \mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}_G^0 |_{X-x}, \mathcal{U}'^N \xrightarrow{\sim} (\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}'}, \mathcal{U}_1^N \xrightarrow{\sim} \mathcal{L}_{(\mathcal{F}_1, \nu_1, x)}^{\bar{\kappa}})$$

to

$$(\mathcal{F}, \mathcal{F}', \nu : \mathcal{F} \xrightarrow{\sim} \mathcal{F}' |_{X-x}, \mathcal{U}, \mathcal{U}'),$$

where \mathcal{F} is obtained as the gluing of $\mathcal{F}' |_{X-x}$ with $\mathcal{F}_1 |_{D_x}$ via $\nu_1^{-1} \circ \nu' : \mathcal{F}' \xrightarrow{\sim} \mathcal{F}_1 |_{D_x}$. We have canonically $(\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}'} \otimes \mathcal{L}_{(\mathcal{F}_1, \nu_1, x)}^{\bar{\kappa}} \xrightarrow{\sim} (\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}}$, and $\mathcal{U} = \mathcal{U}' \otimes \mathcal{U}_1$ is equipped with the induced isomorphism $\mathcal{U}^N \xrightarrow{\sim} (\omega \mathcal{L}^{\bar{\kappa}})_{\mathcal{F}}$.

Given an object \mathcal{S} of the G_X -equivariant derived category on $\widetilde{\text{Gr}}_{G,X}$ and $\mathcal{T} \in \text{D}(\widetilde{\text{Bun}}_G)$ we can form their twisted external product $(\mathcal{T} \boxtimes \mathcal{S})^r$, which is the descent of $\mathcal{T} \boxtimes \mathcal{S}$ via $\tilde{\gamma}^\rightarrow$. Similarly, one may define $\tilde{\gamma}^\leftarrow$ and the complex $(\mathcal{T} \boxtimes \mathcal{S})^l$ on $\mathcal{H}_{\tilde{G}}$. If $\mu_N(k)$ acts on \mathcal{S} by ζ , and $\mathcal{T} \in \text{D}_\zeta(\widetilde{\text{Bun}}_G)$ then $(\tilde{h}_G^\leftarrow \times \pi)_!(\mathcal{T} \boxtimes \mathcal{S})^r \in \text{D}_\zeta(\widetilde{\text{Bun}}_G \times X)$.

In ([22], Remark 2.2) we introduced a covariant functor $\text{Perv}_{G,\zeta} \rightarrow \text{Perv}_{G,\zeta^{-1}}, K \mapsto *K$. It is induced by the map $\mathbb{E} \rightarrow \mathbb{E}, z \mapsto z^{-1}$.

Our choice of $\Omega^{\frac{1}{2}}$ gives rise to the fully faithful functor $\tau^0 : \text{Perv}_{G,\zeta} \rightarrow \text{Perv}_{G,\zeta,X}$ defined in ([22], Section 2.6). The abelian category $\text{Perv}_{G,\zeta,X}$, defined in *loc.cit.*, is the category of G_X -equivariant perverse sheaves (cohomologically shifted by 1 to the right) on $\widetilde{\text{Gr}}_{G,X}$ on which $\mu_N(k)$ acts by ζ . Now for $\mathcal{S} \in \text{Perv}_{G,\zeta}$ we define following [18]

$$\text{H}_G^\leftarrow : \text{Perv}_{G,\zeta^{-1}} \times \text{D}_\zeta(\widetilde{\text{Bun}}_G) \rightarrow \text{D}_\zeta(\widetilde{\text{Bun}}_G \times X)$$

$$\text{H}_G^\rightarrow : \text{Perv}_{G,\zeta} \times \text{D}_\zeta(\widetilde{\text{Bun}}_G) \rightarrow \text{D}_\zeta(\widetilde{\text{Bun}}_G \times X)$$

by

$$\text{H}_G^\rightarrow(\mathcal{S}, K) = (\tilde{h}_G^\leftarrow \times \pi)_!(K \boxtimes \tau_0(\mathcal{S}))^r \quad \text{and} \quad \text{H}_G^\leftarrow(\mathcal{S}, K) = (\tilde{h}_G^\rightarrow \times \pi)_!(K \boxtimes \tau_0(*\mathcal{S}))^l$$

Set $\Lambda^{\sharp,+} = \Lambda^\sharp \cap \Lambda^+$. For $\nu \in \Lambda^{\sharp,+}$ we have the associated irreducible object $\mathcal{A}_\xi^\nu \in \text{Perv}_{G,\zeta}$ defined in ([22], Section 2.4.2). Note that $*\mathcal{A}_\xi^\nu \xrightarrow{\sim} \mathcal{A}_\xi^{-w_0(\nu)}$.

5.2. Action on $\text{D}_\zeta(\mathfrak{M}_x)$. Pick $x \in X$. Let ${}_x\mathcal{H}_{\tilde{G}}$ denote the fibre of $\mathcal{H}_{\tilde{G}}$ over $x \in X$. Set $Z = {}_x\mathcal{H}_{\tilde{G}} \times_{\widetilde{\text{Bun}}_G} \widetilde{\mathfrak{M}}_x$, where we used the map $\tilde{h}_G^\rightarrow : {}_x\mathcal{H}_{\tilde{G}} \rightarrow \widetilde{\text{Bun}}_G$ in the fibred product.

Lemma 5.2.1. *There is a map $'h^\leftarrow : Z \rightarrow \widetilde{\mathfrak{M}}_x$ that renders the diagram*

$$\begin{array}{ccccc} \widetilde{\mathfrak{M}}_x & \xleftarrow{'h^\leftarrow} & Z & \xrightarrow{'h^\rightarrow} & \widetilde{\mathfrak{M}}_x \\ \downarrow \tilde{\mathfrak{p}} & & \downarrow \mathfrak{p}_Z & & \downarrow \tilde{\mathfrak{p}} \\ \widetilde{\text{Bun}}_G & \xleftarrow{\tilde{h}_G^\leftarrow} & {}_x\mathcal{H}_{\tilde{G}} & \xrightarrow{\tilde{h}_G^\rightarrow} & \widetilde{\text{Bun}}_G \end{array}$$

commutative. The left square in the above diagram is also cartesian.

Proof. The stack Z classifies $(\mathcal{F}, \mathcal{F}', \nu : \mathcal{F} \xrightarrow{\sim} \mathcal{F}' |_{X-x}, \mathcal{U}, \mathcal{U}')$ with isomorphisms (45), and inclusions for $\check{\lambda} \in \check{\Lambda}^+$

$$\kappa^{\check{\lambda}} : \Omega^{\langle \rho, \check{\lambda} \rangle} \rightarrow \mathcal{V}_{\mathcal{F}'}^{\check{\lambda}}(\infty x)$$

subject to the Plücker relations. From κ and ν we get a system of maps

$$\kappa'^{\check{\lambda}} : \Omega^{\langle \rho, \check{\lambda} \rangle} \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}}(\infty x)$$

satisfying the Plücker relations ([18], Proposition 5.3.4). Let the map $'h^{\leftarrow}$ send the above point to $(\mathcal{F}, \kappa', \mathcal{U})$. \square

As in Section 5.1, given $\mathcal{S} \in \mathbb{P}erv_{G, \zeta}$ and $K \in D_{\zeta}(\mathfrak{M}_x)$, we may form their twisted external product $(K \boxtimes \mathcal{S})^r \in D(Z)$ using the fibration $'h^{\rightarrow} : Z \rightarrow \widetilde{\mathfrak{M}}_x$ with fibre $\widetilde{\text{Gr}}_{G, x}$. Analogously, the map $'h^{\leftarrow}$ gives rise to $(K \boxtimes \mathcal{S})^l \in D(Z)$. We define

$$\mathbf{H}_G^{\leftarrow} : \mathbb{P}erv_{G, \zeta^{-1}} \times D_{\zeta}(\mathfrak{M}_x) \rightarrow D_{\zeta}(\mathfrak{M}_x) \quad \text{and} \quad \mathbf{H}_G^{\rightarrow} : \mathbb{P}erv_{G, \zeta} \times D_{\zeta}(\mathfrak{M}_x) \rightarrow D_{\zeta}(\mathfrak{M}_x)$$

by

$$\mathbf{H}_G^{\rightarrow}(\mathcal{S}, K) = ('h^{\leftarrow})_!(K \boxtimes \mathcal{S})^r \quad \text{and} \quad \mathbf{H}_G^{\leftarrow}(\mathcal{S}, K) = ('h^{\rightarrow})_!(K \boxtimes \mathcal{S})^l$$

We have functorial isomorphisms

$$\mathbf{H}_G^{\leftarrow}(\mathcal{S}_1, \mathbf{H}_G^{\leftarrow}(\mathcal{S}_2, K)) \xrightarrow{\sim} \mathbf{H}_G^{\leftarrow}(\mathcal{S}_1 * \mathcal{S}_2, K) \quad \text{and} \quad \mathbf{H}_G^{\rightarrow}(\mathcal{S}_1, \mathbf{H}_G^{\rightarrow}(\mathcal{S}_2, K)) \xrightarrow{\sim} \mathbf{H}_G^{\rightarrow}(\mathcal{S}_2 * \mathcal{S}_1, K)$$

Lemma 5.2.2. *The functors $\mathbf{H}_G^{\leftarrow}, \mathbf{H}_G^{\rightarrow}$ preserve the subcategory $D \text{Whit}_x^{\kappa} \subset D_{\zeta}(\mathfrak{M}_x)$.*

Proof. This is analogous to ([19], Proposition 7.3). For a collection of points \bar{y} the action of the Hecke groupoid on $\widetilde{\mathfrak{M}}_x$ yields an action on $(\widetilde{\mathfrak{M}}_x)_{\text{good at } \bar{y}}$, which in turn lifts to an action on the torsor ${}_{\bar{y}}\widetilde{\mathfrak{M}}_x$. \square

5.3. Write $\text{Whit}_x^{\kappa, ss} \subset \text{Whit}_x^{\kappa}$ for the full subcategory consisting of objects, which are finite direct sums of irreducible ones.

Theorem 5.3.1. *i) The functor $\mathbf{H}_G^{\rightarrow} : \mathbb{P}erv_{G, \zeta} \times D \text{Whit}_x^{\kappa} \rightarrow D \text{Whit}_x^{\kappa}$ is exact for the perverse t -structures, so induces a functor*

$$\mathbf{H}_G^{\rightarrow} : \mathbb{P}erv_{G, \zeta} \times \text{Whit}_x^{\kappa} \rightarrow \text{Whit}_x^{\kappa}$$

ii) For $\gamma \in \Lambda^{\sharp, +}$ we have $\mathbf{H}_G^{\rightarrow}(\mathcal{A}_{\mathcal{E}}^{\gamma}, \mathcal{F}_{\emptyset}) \xrightarrow{\sim} \mathcal{F}_{x, \gamma}$.

iii) The functor $\mathbf{H}_G^{\rightarrow}$ preserves the subcategory $\text{Whit}_x^{\kappa, ss}$.

The point ii) of the above theorem is an analog of ([18], Theorem 4) in our setting.

5.4. Proof of Theorem 5.3.1.

5.4.1. Pick $\lambda \in \Lambda^+$, $\gamma \in \Lambda^{\sharp, +}$. First, we show that

$$(46) \quad \mathbf{H}_G^{\rightarrow}(\mathcal{A}_{\mathcal{E}}^{-w_0(\gamma)}, \mathcal{F}_{x, \lambda}) \xrightarrow{\sim} ('h^{\leftarrow})_!(\mathcal{F}_{x, \lambda} \boxtimes \mathcal{A}_{\mathcal{E}}^{-w_0(\gamma)})^r$$

is perverse. To simplify the notation, from now on we suppress the upper index r in the latter formula.

For $\nu \in \Lambda$ write $\mathfrak{M}_{x, \leq \nu} \subset \mathfrak{M}_x$ for the substack given by the property that for any $\check{\lambda}$ the map

$$(47) \quad \Omega^{\langle \rho, \check{\lambda} \rangle} \rightarrow \mathcal{V}_{\mathcal{F}}^{\check{\lambda}}(\langle \nu, \check{\lambda} \rangle x)$$

is regular over X . Let $\mathfrak{M}_{\tilde{x}, \leq \nu} \subset \mathfrak{M}_{x, \leq \nu}$ be the open substack given by the property that (47) has no zeros in a neighbourhood of x . Let $\mathfrak{M}_{x, \nu} \subset \mathfrak{M}_{\tilde{x}, \leq \nu}$ be the open substack given by requiring that (47) has no zeros over X . Write $\widetilde{\mathfrak{M}}_{x, \nu}$, $\widetilde{\mathfrak{M}}_{\tilde{x}, \nu}$ and so on for the restriction of the gerb $\widetilde{\mathfrak{M}}_x$ to the corresponding stack.

Denote by $K_{\tilde{x}}^\nu$ (resp., K^ν) the $*$ -restriction of (46) to $\widetilde{\mathfrak{M}}_{\tilde{x}, \nu}$ (resp., to $\widetilde{\mathfrak{M}}_{x, \nu}$). Since (46) is Verdier self-dual (up to replacing ψ by ψ^{-1} and ζ by ζ^{-1}), it suffices to prove the following.

Lemma 5.4.2. *If $\nu \in \Lambda$ then $K_{\tilde{x}}^\nu$ is placed in perverse degrees ≤ 0 .*

5.4.3. For $\nu, \nu' \in \Lambda$ define the locally closed substacks of Z

$$\begin{aligned} Z_{\tilde{x}}^{\nu, ?} &= ('h^\leftarrow)^{-1}(\widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu}), & Z^{\nu, ?} &= ('h^\leftarrow)^{-1}(\widetilde{\mathfrak{M}}_{x, \nu}) \\ Z_{\tilde{x}}^{?, \nu'} &= ('h^\rightarrow)^{-1}(\widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu'}), & Z^{?, \nu'} &= ('h^\rightarrow)^{-1}(\widetilde{\mathfrak{M}}_{x, \nu'}) \\ Z_{\tilde{x}}^{\nu, \nu'} &= Z_{\tilde{x}}^{\nu, ?} \cap Z_{\tilde{x}}^{?, \nu'}, & Z^{\nu, \nu'} &= Z^{\nu, ?} \cap Z^{?, \nu'} \end{aligned}$$

For $\mu \in \Lambda^+$ let ${}_x\mathcal{H}^\mu$ be the locally closed substack $\gamma^{\leftarrow}(\mathrm{Bun}_{G, x} \times_{G(\mathcal{O}_x)} \mathrm{Gr}_{G, x}^\mu) \subset {}_x\mathcal{H}_G$. Let ${}_x\mathcal{H}_{\tilde{G}}^\mu$ be its preimage in ${}_x\mathcal{H}_{\tilde{G}}$. Set

$$\begin{aligned} Z_{\tilde{x}}^{\nu, ?, \mu} &= Z_{\tilde{x}}^{\nu, ?} \cap \mathfrak{p}_Z^{-1}({}_x\mathcal{H}_{\tilde{G}}^\mu), & Z_{\tilde{x}}^{?, \nu', \mu} &= Z_{\tilde{x}}^{?, \nu'} \cap \mathfrak{p}_Z^{-1}({}_x\mathcal{H}_{\tilde{G}}^\mu) \\ Z_{\tilde{x}}^{\nu, \nu', \mu} &= Z_{\tilde{x}}^{\nu, \nu'} \cap \mathfrak{p}_Z^{-1}({}_x\mathcal{H}_{\tilde{G}}^\mu), & Z^{\nu, \nu', \mu} &= Z^{\nu, \nu'} \cap \mathfrak{p}_Z^{-1}({}_x\mathcal{H}_{\tilde{G}}^\mu) \end{aligned}$$

Denote by $K_{\tilde{x}}^{\nu, \nu', \mu}$ the $!$ -direct image under $'h^\leftarrow : Z_{\tilde{x}}^{\nu, \nu', \mu} \rightarrow \widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu}$ of the $*$ -restriction of $\mathcal{F}_{x, \lambda} \boxtimes \mathcal{A}_{\mathcal{E}}^{-w_0(\gamma)}$ to $Z_{\tilde{x}}^{\nu, \nu', \mu}$. Denote by $K^{\nu, \nu', \mu}$ the restriction of $K_{\tilde{x}}^{\nu, \nu', \mu}$ to the open substack $\widetilde{\mathfrak{M}}_{x, \nu}$. Lemma 5.4.2 is reduced to the following.

Lemma 5.4.4. (1) *The complex $K_{\tilde{x}}^{\nu, \nu', \mu}$ is placed in perverse degrees ≤ 0 , and the inequality is strict unless $\mu = \gamma$ and $\nu' = \lambda$.*

(2) *The $*$ -restriction of $K_{\tilde{x}}^{\nu, \lambda, \gamma}$ to the closed substack $\widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu} - \widetilde{\mathfrak{M}}_{x, \nu}$ vanishes.*

Choose for each $\nu \in \Lambda$ a trivialization $\epsilon_\nu : \Omega^\rho(-\nu x) \xrightarrow{\sim} \mathcal{F}_T^0|_{D_x}$. They yield a $U(\mathcal{O}_x)$ -torsor $\mathcal{U}_{\tilde{x}}^{\epsilon_\nu}$ (resp., $\mathcal{U}^{\epsilon_\nu}$) over $\widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu}$ (resp., over $\widetilde{\mathfrak{M}}_{x, \nu}$) classifying a point of the latter stack together with a trivialization of the corresponding U -torsor over D_x . The projection $'h^\leftarrow$ identifies $Z_{\tilde{x}}^{\nu, ?}$ (resp., $'h^\rightarrow$ identifies $Z_{\tilde{x}}^{?, \nu'}$) with the fibration

$$\mathcal{U}_{\tilde{x}}^{\epsilon_\nu} \times_{U(\mathcal{O}_x)} \widetilde{\mathrm{Gr}}_{G, x} \rightarrow \widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu}$$

(resp., with the fibration $\mathcal{U}_{\tilde{x}}^{\epsilon_\nu} \times_{U(\mathcal{O}_x)} \widetilde{\mathrm{Gr}}_{G, x} \rightarrow \widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu'}$). As in ([18], Lemma 7.2.4), one has the following.

Lemma 5.4.5. (1) *The stacks $Z_{\tilde{x}}^{\nu, \nu'}$ and $Z_{\tilde{x}}^{\nu, ?}, \mu$, when viewed as substack of $Z_{\tilde{x}}^{\nu, ?}$, are identified with*

$$\mathcal{U}_{\tilde{x}}^{\epsilon_\nu} \times_{U(\mathcal{O}_x)} \widetilde{\mathrm{Gr}}_{B, x}^{\nu' - \nu} \xrightarrow{'h^\leftarrow} \widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu} \quad \text{and} \quad \mathcal{U}_{\tilde{x}}^{\epsilon_\nu} \times_{U(\mathcal{O}_x)} \widetilde{\mathrm{Gr}}_{G, x}^\mu \xrightarrow{'h^\leftarrow} \widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu}$$

respectively.

(2) The stacks $Z_{\tilde{x}}^{\nu, \nu'}$ and $Z_{\tilde{x}}^{?, \nu', \mu}$, when viewed as substacks of $Z_{\tilde{x}}^{?, \nu'}$, are identified with

$$\mathcal{U}_{\tilde{x}}^{\epsilon_{\nu'}} \times_{U(\mathcal{O}_x)} \widetilde{\text{Gr}}_{B,x}^{\nu-\nu'} \xrightarrow{'h'} \widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu'} \quad \text{and} \quad \mathcal{U}_{\tilde{x}}^{\epsilon_{\nu'}} \times_{U(\mathcal{O}_x)} \widetilde{\text{Gr}}_{G,x}^{-w_0(\mu)} \xrightarrow{'h'} \widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu'}$$

respectively. \square

Proof of Lemma 5.4.4. (1) By Lemma 5.4.5, the $*$ -restriction of $\mathcal{F}_{x,\lambda} \widetilde{\mathcal{A}}_{\mathcal{E}}^{-w_0(\gamma)}$ to $Z_{\tilde{x}}^{?, \nu', \mu}$ is the twisted external product of complexes

$$(\mathcal{F}_{x,\lambda} |_{\widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu'}}) \widetilde{\boxtimes} (\mathcal{A}_{\mathcal{E}}^{-w_0(\gamma)} |_{\widetilde{\text{Gr}}_{G,x}^{-w_0(\mu)}}).$$

It lives in perverse degrees ≤ 0 , and the inequality is strict unless $\mu = \gamma$ and $\nu' = \lambda$. Recall also that the $*$ -restriction of $\mathcal{A}_{\mathcal{E}}^{-w_0(\gamma)}$ to $\widetilde{\text{Gr}}_{G,x}^{-w_0(\mu)}$ vanishes unless $\mu \in \Lambda^{\sharp,+}$.

Since $\mathcal{A}_{\mathcal{E}}^{-w_0(\gamma)} |_{\widetilde{\text{Gr}}_{G,x}^{-w_0(\mu)}}$ has locally constant cohomology sheaves, its $*$ -restriction to $Z_{\tilde{x}}^{\nu, \nu', \mu}$ by Lemma 5.4.5 is placed in perverse degrees

$$\leq -\text{codim}(\text{Gr}_B^{\nu-\nu'} \cap \text{Gr}_G^{-w_0(\mu)}, \text{Gr}_G^{-w_0(\mu)}) \leq -\langle \mu - \nu + \nu', \check{\rho} \rangle,$$

we have used here ([18], Proposition 7.1.3). From Lemma 5.4.5(1) we now learn that the fibres of $'h^{\leftarrow} : Z_{\tilde{x}}^{\nu, \nu', \mu} \rightarrow \widetilde{\mathfrak{M}}_{\tilde{x}, \leq \nu}$ are of dimension $\leq \dim(\text{Gr}_B^{\nu-\nu'} \cap \text{Gr}_G^{\mu}) \leq \langle \nu' - \nu + \mu, \check{\rho} \rangle$. If $f : Y \rightarrow W$ is a morphism of schemes of finite type, each fibre of f is of dimension $\leq d$, K is a perverse sheaf on Y then $f_! K$ is placed in perverse degrees $\leq d$. We are done.

(2) the $*$ -restriction of $\mathcal{F}_{x,\lambda}$ to $\widetilde{\mathfrak{M}}_{\tilde{x}, \leq \lambda} - \widetilde{\mathfrak{M}}_{x,\lambda}$ vanishes, because there are no dominant coweights < 0 . \square

Theorem 5.3.1 i) is proved. Theorem 5.3.1 iii) follows from the decomposition theorem of [4].

To establish Theorem 5.3.1 ii), keep the above notation taking $\lambda = 0$. We want to show that (46) identifies with $\mathcal{F}_{x, -w_0(\gamma)}$. It remains to analyse the complex $K^{\nu, 0, \gamma}$ on $\widetilde{\mathfrak{M}}_{x, \nu}$ placed in perverse degrees ≤ 0 . We are reduced to the following.

Lemma 5.4.6. *i) The 0-th perverse cohomology sheaf of $K^{\nu, 0, \gamma}$ vanishes unless $\nu = -w_0(\gamma)$.*

ii) The 0-th perverse cohomology sheaf of $K^{-w_0(\gamma), 0, \gamma}$ identifies with the restriction of $\mathcal{F}_{x, -w_0(\gamma)}$ to $\widetilde{\mathfrak{M}}_{x, -w_0(\gamma)}$.

Proof. The situation with the additive characters is exactly the same as in ([18], Sections 7.2.6-7.2.8). Let $\overline{U(F_x)}^{\epsilon_{\nu}}$ be ind-group scheme over $\widetilde{\mathfrak{M}}_{x, \nu}$, the $\mathcal{U}^{\epsilon_{\nu}}$ -twist of $U(F_x)$ with respect to the adjoint action of $U(\mathcal{O}_x)$ on $U(F_x)$. Then $Z^{\nu, \nu'}$ carries a natural $\overline{U(F_x)}^{\epsilon_{\nu}}$ -action preseving $'h^{\leftarrow} : Z^{\nu, \nu'} \rightarrow \widetilde{\mathfrak{M}}_{x, \nu}$ and defined via the identification of Lemma 5.4.5(1).

The ind-group $\overline{U(F_x)}^{\epsilon_{\nu}}$ classifies a point $(\mathcal{F}, \kappa, \mathcal{U}) \in \widetilde{\mathfrak{M}}_{x, \nu}$ giving rise to the corresponding B -torsor \mathcal{F}_B on D_x equipped with $\mathcal{F}_B \times_B T \xrightarrow{\sim} \Omega^{\rho}(-\nu x)$, and an automorphism $g : \mathcal{F}_B \xrightarrow{\sim} \mathcal{F}_B$ over D_x^* inducing the identity on $\mathcal{F}_B \times_B T$.

The trivialization $\epsilon_\nu : \Omega^\rho(-\nu x) \xrightarrow{\sim} \mathcal{F}_T^0|_{D_x}$ gives for $i \in \mathcal{J}$ the character

$$U/[U, U](F_x) \xrightarrow{\tilde{\alpha}_i} F_x \xrightarrow{\epsilon_\nu^{-1}} \mathcal{L}_{\Omega^\rho(-\nu x)}^{\tilde{\alpha}_i}|_{D_x^*} \xrightarrow{\sim} \Omega(F_x) \xrightarrow{\text{Res}} \mathbb{A}^1$$

Their sum over $i \in \mathcal{J}$ is the character of conductor $\bar{\nu}$ denoted $\chi_\nu : U(F_x) \rightarrow \mathbb{A}^1$. Here $\bar{\nu}$ is the image of ν in the coweights lattice of G_{ad} . Twisting $U(F_x)$ by the $U(\mathcal{O}_x)$ -torsor $\mathcal{U}^{\epsilon_\nu}$, one gets the character denoted $\bar{\chi}_\nu : \overline{U(F_x)}^{\epsilon_\nu} \rightarrow \mathbb{A}^1$.

For $\nu, \nu' \in \Lambda^+$ a $(U(F_x), \chi_\nu)$ -equivariant function $\chi_{\nu'}^{\nu'-\nu} : \text{Gr}_B^{\nu'-\nu} \rightarrow \mathbb{A}^1$ gives rise to a $(\overline{U(F_x)}^{\epsilon_\nu}, \bar{\chi}_\nu)$ -equivariant function $\bar{\chi}_{\nu'}^{\nu'-\nu} : Z^{\nu, \nu'} \rightarrow \mathbb{A}^1$. For the convenience of the reader we recall the following.

Lemma 5.4.7 ([18], Lemma 7.2.7). *Assume $\nu' \in \Lambda^+$. Then*

- (1) *the map $ev_{x, \nu'} \circ 'h^\rightarrow : Z^{\nu, \nu'} \rightarrow \mathbb{A}^1$ is $(\overline{U(F_x)}^{\epsilon_\nu}, \bar{\chi}_\nu)$ -equivariant.*
- (2) *If in addition $\nu \in \Lambda^+$ then $ev_{x, \nu'} \circ 'h^\rightarrow$ coincides with the composition*

$$Z^{\nu, \nu'} \xrightarrow{\bar{\chi}_{\nu'}^{\nu'-\nu} \times 'h^\rightarrow} \mathbb{A}^1 \times \widetilde{\mathfrak{M}}_{x, \nu} \xrightarrow{\text{id} \times ev_{x, \nu}} \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{\text{sum}} \mathbb{A}^1$$

for some $\chi_{\nu'}^{\nu'-\nu}$. \square

The fibration $'h^\leftarrow : Z^{\nu, 0, \gamma} \rightarrow \widetilde{\mathfrak{M}}_{x, \nu}$ identifies with $\mathcal{U}^{\epsilon_\nu} \times_{U(\mathcal{O}_x)} (\widetilde{\text{Gr}}_{B, x}^{-\nu} \cap \widetilde{\text{Gr}}_{G, x}^\gamma) \rightarrow \widetilde{\mathfrak{M}}_{x, \nu}$. After a smooth localization $V \rightarrow \widetilde{\mathfrak{M}}_{x, \nu}$ the latter fibration becomes a direct product $V \times (\widetilde{\text{Gr}}_{B, x}^{-\nu} \cap \widetilde{\text{Gr}}_{G, x}^\gamma)$. The $*$ -restriction of $\mathcal{F}_\emptyset \boxtimes \mathcal{A}_\varepsilon^{-w_0(\gamma)}$ to $Z^{\nu, 0, \gamma}$ will descend to $V \times (\text{Gr}_{B, x}^{-\nu} \cap \text{Gr}_{G, x}^\gamma)$, and there will be of the form

$$\mathcal{E}_V \boxtimes ((\chi_\nu^{-\nu})^* \mathcal{L}_\psi \otimes \delta^* \mathcal{L}_\zeta)[\langle \gamma - \nu, 2\check{\rho} \rangle],$$

for a suitable discrepancy map $\delta : \text{Gr}_{B, x}^{-\nu} \cap \text{Gr}_{G, x}^\gamma \rightarrow \mathbb{G}_m$. Here \mathcal{E}_V is a perverse sheaf on V .

The local system $(\chi_\nu^{-\nu})^* \mathcal{L}_\psi \otimes \delta^* \mathcal{L}_\zeta$ is nonconstant on any irreducible component by ([18], Proposition 7.1.7). This proves i). Since $\text{Gr}_B^{w_0(\gamma)} \cap \text{Gr}_G^\gamma$ is the point scheme, part ii) follows from Lemma 5.4.7 and 5.4.5. \square

Theorem 5.3.1 is proved.

APPENDIX A.

A.1. In some cases we will use the following observation. Let $i \in \mathcal{J}$, $\lambda > \alpha_i$ such that $\omega_i - \lambda$ appears as a weight of \mathbb{V}^{ω_i} . Then there is $\mu \in \Lambda^+$ with $\mu \leq \omega_i$, $w \in W$ such that $\lambda = \omega_i - w\mu$. Then the property $\bar{\kappa}(\omega_i - w\mu - \alpha_i) \in N\check{\Lambda}$ is equivalent to $\bar{\kappa}(w^{-1}s_i(\omega_i) - \mu) \in N\check{\Lambda}$, where s_i is the reflection corresponding to α_i . So, one may first find the W -orbit of each ω_i . Second, find for each i all the dominant coweights satisfying $\mu \leq \omega_i$. Third, check for each $i \in \mathcal{J}$, $\mu \leq \omega_i$ dominant with $\mu \neq \nu \in W\omega_i$ the property $\bar{\kappa}(\nu - \mu) \notin N\check{\Lambda}$.

Type A_{n-1} . We may assume $G = \text{GL}_n$, $B \subset G$ is the group of upper triangular matrices, T is the group of diagonal matrices. So, $\Lambda = \mathbb{Z}^n$. We may assume $\bar{\kappa} : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ given by $\bar{\kappa} = m\kappa$, where $m \in \mathbb{Z}$ and $\kappa(a, b) = \sum_{i=1}^n a_i b_i$. Then our assumption is $m \notin N\mathbb{Z}$. Since λ is not a simple coroot, we have $n \geq 3$. We assume $\mathcal{J} =$

$\{1, \dots, n-1\}$ and $\omega_i = (1, \dots, 1, 0, \dots, 0)$, where 1 appears i times. The representation \mathbb{V}^{ω_i} is minuscule, for any $\mu \leq \omega_i$ with $\mu \in \Lambda^+$ we have $\mu = \omega_i$. Any $\nu \in W\omega_i$ is of the form $\nu = e_{j_1} + \dots + e_{j_i}$ for $1 \leq j_1 < \dots < j_i \leq n$. Let $1 \leq k \leq n$ be the smallest such that $\alpha_k = e_k - e_{k+1}$ appears in the decomposition of $\omega_i - \nu \neq 0$ into a sum of simple coroots. Then $k \leq i$ and $m = \bar{\kappa}(\lambda, e_k) \notin N\mathbb{Z}$. We are done.

Type C_n . We may assume $G = \mathrm{GSp}_{2n}$, the quotient of $\mathbb{G}_m \times \mathrm{Sp}_{2n}$ by the diagonally embedded μ_2 . Realize $G \subset \mathrm{GL}_{2n}$ as the subgroup preserving up to scalar the bilinear form given by the matrix

$$\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix},$$

where E_n is the unit matrix of GL_n . The maximal torus T of G is $\{(y_1, \dots, y_{2n}) \mid y_i y_{n+i} \text{ does not depend on } i\}$. Let $\check{\epsilon}_i \in \check{\Lambda}$ be the character that sends a point of T to y_i . The roots are

$$\check{R} = \{\pm \check{\alpha}_{ij} \ (i < j \in 1, \dots, n), \pm \check{\beta}_{ij} \ (i \leq j \in 1, \dots, n)\},$$

where $\check{\alpha}_{ij} = \check{\epsilon}_i - \check{\epsilon}_j$ and $\check{\beta}_{ij} = \check{\epsilon}_i - \check{\epsilon}_{n+j}$.

We have $\Lambda = \{(a_1, \dots, a_{2n}) \mid a_i + a_{n+i} \text{ does not depend on } i\}$. The weight lattice is

$$\check{\Lambda} = \mathbb{Z}^{2n} / \{\check{\epsilon}_i + \check{\epsilon}_{n+i} - \check{\epsilon}_j - \check{\epsilon}_{n+j}, \ i < j\}$$

Let e_i denote the standard basis of \mathbb{Z}^{2n} . The coroots are

$$R = \{\pm \alpha_{ij} \ (i < j \in 1, \dots, n), \pm \beta_{ij} \ (i \leq j \in 1, \dots, n)\},$$

where $\beta_{ij} = e_i + e_j - e_{n+i} - e_{n+j}$ for $i < j$ and $\beta_{ii} = e_i - e_{n+i}$. Besides, $\alpha_{ij} = e_i + e_{n+j} - e_j - e_{n+i}$.

Fix positive roots

$$\check{R}^+ = \{\check{\alpha}_{ij} \ (i < j \in 1, \dots, n), \check{\beta}_{ij} \ (i \leq j \in 1, \dots, n)\}$$

Then the simple roots are $\check{\alpha}_1 := \check{\alpha}_{12}, \dots, \check{\alpha}_{n-1} := \check{\alpha}_{n-1,n}$ and $\check{\alpha}_n := \check{\beta}_{n,n}$.

For $1 \leq i < n$ set $\omega_i = (1, \dots, 1, 0, \dots, 0; -1, \dots, -1, 0, \dots, 0)$, where 1 appears i times then 0 appears $n-i$ times then -1 appears i times, and 0 appears $n-i$ times. Set $\omega_n = (1, \dots, 1; 0, \dots, 0)$, where 1 appears n times, and 0 appears n times. This is our choice of the fundamental coweights corresponding to $\check{\alpha}_i$.

For $b \in \Lambda$ write $\bar{b} = b_i + b_{n+i}$, this is independent of i . The map $\Lambda_{ab} \xrightarrow{\sim} \mathbb{Z}$, $a \mapsto \bar{a}$ is an isomorphism. Let $\kappa : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ be given by $\kappa(a, b) = \sum_{i=1}^{2n} a_i b_i$. Then κ is W -invariant symmetric bilinear form. We have $\kappa(\alpha_{ij}, \alpha_{ij}) = \kappa(\beta_{ij}, \beta_{ij}) = 4$ for $i \neq j$, and $\kappa(\beta_{ii}, \beta_{ii}) = 2$. We may assume $\bar{\kappa} = m\kappa$ for some $m \in \mathbb{Z}$.

Note that \mathbb{V}^{ω_n} is the spinor representation of $\check{G} \xrightarrow{\sim} \mathrm{GSpin}_{2n+1}$ of dimension 2^n , \mathbb{V}^{ω_1} is the standard representation of the quotient SO_{2n+1} , and $\mathbb{V}^{\omega_i} = \wedge^i(\mathbb{V}^{\omega_1})$ for $1 \leq i < n$. We have $0 \leq \omega_1 \leq \dots \leq \omega_{n-1}$, and if $\mu \in \Lambda$ is dominant and $\mu \leq \omega_{n-1}$ then μ is in this list.

The assumption $\varrho(\alpha_i) \notin \mathbb{Z}$ for any simple coroots reads $2m \notin N\mathbb{Z}$. Assume $n = 2$. In this case it is easy to check the desired property (C).

Assume now $n \geq 3$. Then the assumption $\varrho(\alpha_i) \notin \frac{1}{2}\mathbb{Z}$ for any simple coroots reads $4m \notin N\mathbb{Z}$.

First, let $1 \leq i < n$. Suppose $\omega_i - \lambda$ appears in \mathbb{V}^{ω_i} . Then $\omega_i - \lambda$ is of the form $\sum_{k=1}^j \epsilon_k \beta_{i_k, i_k}$, where $\epsilon_k = \pm 1$, $0 \leq j \leq i$, and $1 \leq i_1 < \dots < i_j \leq n$. Let $\lambda - \alpha_i = (a_1, \dots, a_{2n})$. If $j < i$ then there is $1 \leq k \leq n$ such that $a_k = 1$, and $\kappa(\lambda - \alpha_i, \beta_{k,k}) = 2$. If $j = i$ and there is no $1 \leq k \leq n$ with this property then there is $1 \leq k \leq n$ such that $a_k = 2$, and $\kappa(\lambda - \alpha_i, \beta_{k,k}) = 4$. The case $i < n$ is done.

Let now $i = n$. The representation \mathbb{V}^{ω_n} is minuscule, its weights are the W -orbit of ω_n . The coweight λ is of the form $\lambda = \sum_{k \in S} \beta_{k,k}$, where $S \subset \{1, \dots, n\}$ is a subset, and $\lambda > \alpha_n = \beta_{n,n}$. So, there is $k \in S$ with $k < n$. We have $\kappa(\lambda - \alpha_n, \beta_{k,k}) = 2$. We are done.

Type B_n . Assume $n \geq 3$, let $G = \text{Spin}_{2n+1}$. We take $\Lambda = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum_k a_k = 0 \pmod{2}\}$, so $\mathbb{Z}^n \subset \check{\Lambda}$. The coroots are

$$R = \{\pm \alpha_{ij} (1 \leq i < j \leq n), \pm \beta_{ij} (1 \leq i \leq j \leq n)\},$$

where $\alpha_{ij} = e_i - e_j$, $\beta_{ij} = e_i + e_j$. The corresponding roots are $\check{\alpha}_{ij} = e_i - e_j$, $\check{\beta}_{ij} = e_i + e_j$ for $1 \leq i < j \leq n$, and $\check{\beta}_{ii} = e_i$. Here $\check{\alpha}_{ij}, \check{\beta}_{ij} \in \mathbb{Z}^n \subset \check{\Lambda}$. The simple roots are $\check{\alpha}_1 = \check{\alpha}_{12}, \dots, \check{\alpha}_{n-1} = \check{\alpha}_{n-1,n}$, $\check{\alpha}_n = \check{\beta}_{n,n}$.

Write \check{G}^{sc} for the simply-connected cover of \check{G} . The fundamental weights of \check{G}^{sc} , which we refer to as the fundamental coweights of G_{ad} , are $\omega_i = e_1 + \dots + e_i \in \mathbb{Z}^n$ for $1 \leq i \leq n$. We use here the canonical inclusion $\Lambda \subset \mathbb{Z}^n = \Lambda_{ad}$ as a sublattice of index 2. Here Λ_{ad} is the coweights lattice of $G_{ad} = \text{SO}_{2n+1}$. The Weyl group acts on Λ_{ad} by any permutations and any sign changes. That is, it contains the maps $\Lambda_{ad} \rightarrow \Lambda_{ad}$, $\mu = (a_1, \dots, a_n) \mapsto (\epsilon_1 a_1, \dots, \epsilon_n a_n)$ for any $\epsilon_k = \pm 1$.

Let $\kappa : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ be the unique W -invariant symmetric bilinear form such that $\kappa(\alpha, \alpha) = 2$ for a short coroot. Then κ extends uniquely to $\kappa : \Lambda_{ad} \otimes \Lambda_{ad} \rightarrow \mathbb{Z}$ as $\kappa(a, b) = \sum_{k=1}^n a_k b_k$. We get $\kappa(\beta_{ii}, \beta_{ii}) = 4$ for any $1 \leq i \leq n$, and all the other coroots are short. We may assume $\bar{\kappa} = m\kappa$, $m \in \mathbb{Z}$. Then the assumption of Conjecture 1.1.2 reads $2m \notin N\mathbb{Z}$.

Let Λ_{ad}^+ be the dominant coweights of G_{ad} then $\Lambda_{ad}^+ = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 \geq \dots \geq a_n \geq 0\}$. If $\mu \in \Lambda_{ad}^+$ and $\mu \leq \omega_i$ then $\mu = (1, \dots, 1, 0, \dots, 0)$, where 1 appears k times with $k \leq i$ and $k \equiv i \pmod{2}$. Any weight of \mathbb{V}^{ω_i} is of the form $w\mu$, $w \in W$, where $\mu \in \Lambda_{ad}^+$ and $\mu \leq \omega_i$. So, the weights of \mathbb{V}^{ω_i} are of the form $\omega_i - \lambda = \sum_{r=1}^k \epsilon_r e_{j_r}$, where $0 \leq k \leq i$, $k \equiv i \pmod{2}$, and $1 \leq j_1 < \dots < j_k \leq n$, here $\epsilon_r = \pm 1$.

If $1 \leq i < n$ then $\omega_i - \alpha_i = (1, \dots, 1, 0, 1, 0, \dots, 0)$, where 1 appears first $i - 1$ times. If $k < i$ then $\lambda - \alpha_i$ will contain an entry 1 on some m -th place and $\kappa(\lambda - \alpha_i, \beta_{m,m}) = 2$, so $\bar{\kappa}(\lambda - \alpha_i)$ is not divisible by N in this case. If $k = i$ and $\lambda - \alpha_i$ does not contain the entry 1 then $\lambda - \alpha_i$ is of the form $\sum_{j \in S} \beta_{jj}$ for some subset $S \subset \{1, \dots, n\}$ that contains at most i elements. Since $i < n$ there is a couple $j_1 \in S, j_2 \notin S$. Then $\kappa(\lambda - \alpha_i, \beta_{j_1, j_2}) = 2$, so $\bar{\kappa}(\lambda - \alpha_i)$ is not divisible by N in this case.

Let $i = n$ then $\omega_n - \alpha_n = (1, \dots, 1, -1)$. Let $\omega_i - \lambda$ be as above. If $k < n$ then $k \leq n - 2$, and $\lambda - \alpha_n$ will contain an entry 1 at some place. As above this implies that $\bar{\kappa}(\lambda - \alpha_i)$ is not divisible by N in this case. If $k = n$ then $\lambda - \alpha_n = \sum_{j \in S} \beta_{jj} + a e_n$, where $S \subset \{1, \dots, n - 1\}$ is a subset, and $a = 0$ or $a = -2$. If $\lambda - \alpha_n$ contains a entry 0 then as above one shows that $\bar{\kappa}(\lambda - \alpha_i)$ is not divisible by N . The only remaining case is $\lambda - \alpha_n = (2, \dots, 2, -2) = -\beta_{nn} + \sum_{j=1}^{n-1} \beta_{jj}$.

Recall that for any coroot α one has $\kappa(\alpha) = \frac{\kappa(\alpha, \alpha)}{2} \check{\alpha}$. We get $\kappa(\beta_{jj}) = 2\check{\beta}_{jj}$ for any j . So, $\kappa(\lambda - \alpha_n) = -2\check{\beta}_{nn} + 2\sum_{j=1}^{n-1} \check{\beta}_{jj}$. The roots lattice of G is $\mathbb{Z}^n \subset \check{\Lambda}$, and $-\check{\beta}_{nn} + \sum_{j=1}^{n-1} \check{\beta}_{jj}$ is divisible in $\check{\Lambda}$, namely $\frac{1}{2}(-\check{\beta}_{nn} + \sum_{j=1}^{n-1} \check{\beta}_{jj}) \in \check{\Lambda}$. So, we must require that $4m \notin N\mathbb{Z}$ to guarantee that $\bar{\kappa}(\lambda - \alpha_i)$ is not divisible by N . We are done.

Type G_2 . Let G be of type G_2 . Let $\Lambda = \{a \in \mathbb{Z}^3 \mid \sum_i a_i = 0\}$ with the bilinear form $\kappa : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ given by $\kappa(a, b) = \sum_i a_i b_i$ for $a, b \in \Lambda$. The coroots are the vectors $\mu \in \Lambda$ such that $\kappa(\mu, \mu) = 2$ or 6 . The coroots are

$$\pm\{e_1 - e_2, e_1 - e_3, e_2 - e_3, 2e_1 - e_2 - e_3, 2e_2 - e_1 - e_3, 2e_3 - e_1 - e_2\}$$

The form κ induces an inclusion $\kappa : \Lambda \hookrightarrow \check{\Lambda}$ such that $\check{\Lambda}/\kappa(\Lambda) \xrightarrow{\sim} \mathbb{Z}/3\mathbb{Z}$. The roots can be found from the property that for any coroot α one has $\kappa(\alpha) = \frac{\kappa(\alpha, \alpha)}{2} \check{\alpha}$. For a short coroot α one gets $\kappa(\alpha) = \check{\alpha}$, and for a long coroot α one gets $\kappa(\alpha) = 3\check{\alpha}$. We get the roots

$$\pm\{e_1 - e_2, e_1 - e_3, e_2 - e_3, e_1, e_2, e_3\} \subset \mathbb{Z}^3 / (e_1 + e_2 + e_3) = \check{\Lambda}$$

The center of G is trivial. Pick positive roots $\check{\alpha}_1 = e_1 - e_2$ and $\check{\alpha}_2 = -e_1$. They correspond to simple coroots $\alpha_1 = e_1 - e_2$, $\alpha_2 = -2e_1 + e_2 + e_3$. The dominant coweights are $\Lambda^+ = \{a \in \Lambda \mid a_2 \leq a_1 \leq 0\}$. The fundamental coweights are $\omega_1 = (0, -1, 1) = 2\alpha_1 + \alpha_2$ and $\omega_2 = (-1, -1, 2) = 3\alpha_1 + 2\alpha_2$. The positive coroots are $\{\alpha_1, \alpha_2, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \alpha_2 + 3\alpha_1, 3\alpha_1 + 2\alpha_2\}$. The representation \mathbb{V}^{ω_2} is the adjoint representation of \check{G} , $\dim \mathbb{V}^{\omega_2} = 14$ and $\dim \mathbb{V}^{\omega_1} = 7$. We have $\omega_1 \leq \omega_2$. We assume $\bar{\kappa} = m\kappa$ for some $m \in \mathbb{Z}$.

The weights of \mathbb{V}^{ω_2} are coroots and zero. So, for $i = 2$ the coweight λ is one of the following

$$\{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 2\alpha_1 + 2\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 2\alpha_2, \\ 3\alpha_1 + 3\alpha_2, 4\alpha_1 + 3\alpha_2, 5\alpha_1 + 3\alpha_2, 6\alpha_1 + 3\alpha_2, 6\alpha_1 + 4\alpha_2\}$$

Since $\kappa(\alpha_1) = \check{\alpha}_1$ and $\kappa(\alpha_2) = 3\check{\alpha}_2$, we get in this case that $\kappa(\lambda - \alpha_2)$ is an element of the set

$$\{\check{\alpha}_1, 2\check{\alpha}_1, 3\check{\alpha}_1, 2\check{\alpha}_1 + 3\check{\alpha}_2, 3\check{\alpha}_1 + 3\check{\alpha}_2, 4\check{\alpha}_1 + 3\check{\alpha}_2, 3\check{\alpha}_1 + 6\check{\alpha}_2, \\ 4\check{\alpha}_1 + 3\check{\alpha}_2, 5\check{\alpha}_1 + 6\check{\alpha}_2, 6\check{\alpha}_1 + 6\check{\alpha}_2, 6\check{\alpha}_1 + 9\check{\alpha}_2\}$$

An element of this set may be divisible in $\check{\Lambda}$ by $2, 3, 6$. So, in order to guarantee that $\frac{m}{N}\kappa(\lambda - \alpha_2) \notin \check{\Lambda} = \mathbb{Z}\check{\alpha}_1 \oplus \mathbb{Z}\check{\alpha}_2$, we must assume $6m \notin N\mathbb{Z}$. In terms of ϱ this assumption reads $\varrho(\alpha_i) \notin \frac{1}{2}\mathbb{Z}$ for any simple coroot α_i .

Let now $i = 1$. Then $\kappa(\lambda - \alpha_1)$ is an element of the set

$$\{3\check{\alpha}_2, \check{\alpha}_1 + 3\check{\alpha}_2, 2\check{\alpha}_1 + 3\check{\alpha}_2, 2\check{\alpha}_1 + 6\check{\alpha}_2, 3\check{\alpha}_1 + 6\check{\alpha}_2\}$$

An element of this set may be divisible in $\check{\Lambda}$ by $2, 3$. So, we must assume $2m, 3m \notin N\mathbb{Z}$. Finally, it suffices to assume $6m \notin N\mathbb{Z}$. We are done.

Type D_n . Let $G = \text{Spin}_{2n}$ with $n \geq 4$. We take $\Lambda = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum_j a_j = 0 \pmod{2}\}$, so $\mathbb{Z}^n \subset \check{\Lambda}$. The group $\check{\Lambda}$ is generated by \mathbb{Z}^n and the element $\frac{1}{2}(1, \dots, 1)$. The

roots are

$$\check{R} = \{\pm\check{\alpha}_{ij} = e_i - e_j(1 \leq i < j \leq n), \pm\check{\beta}_{ij} = e_i + e_j(1 \leq i < j \leq n)\}$$

The simple roots are $\check{\alpha}_1 = \check{\alpha}_{12}, \dots, \check{\alpha}_{n-1} = \check{\alpha}_{n-1,n}, \check{\alpha}_n = \check{\beta}_{n-1,n}$. The coroots are $\alpha_{ij} = e_i - e_j, \beta_{ij} = e_i + e_j$. The Weyl group acting on Λ contains all the permutations, and also all the sign changes with the even number of sign changes. Let $\kappa : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ be given by $\kappa(a, b) = \sum_{k=1}^n a_k b_k$. Then κ is the unique W -invariant symmetric bilinear form such that $\kappa(\alpha, \alpha) = 2$ for any coroot. Let $\bar{\kappa} = m\kappa, m \in \mathbb{Z}$. The assumption of Conjecture 1.1.2 reads $m \notin N\mathbb{Z}$.

The center of G is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for n even (resp., $\mathbb{Z}/4\mathbb{Z}$ for n odd). The group Λ_{ad} is generated by \mathbb{Z}^n and the vector $\frac{1}{2}(1, \dots, 1)$. The fundamental coweights of G_{ad} in Λ_{ad} are $\omega_i = (1, \dots, 1, 0, \dots, 0) \in \mathbb{Z}^n$, where 1 appears i times for $1 \leq i \leq n-2$, and

$$\omega_n = \frac{1}{2}(1, \dots, 1), \quad \omega_{n-1} = \frac{1}{2}(1, \dots, 1, -1)$$

Here $\mathbb{V}^{\omega_{n-1}}, \mathbb{V}^{\omega_n}$ are half-spin representations of $\check{G}^{sc} \xrightarrow{\sim} \text{Spin}_{2n}$. The representation \mathbb{V}^{ω_1} is the standard representation of SO_{2n} , and $\mathbb{V}^{\omega_i} \xrightarrow{\sim} \wedge^i \mathbb{V}^{\omega_1}$ for $1 \leq i \leq n-2$. Both half-spin representations are minuscule of dimension 2^{n-1} .

The weights of \mathbb{V}^{ω_n} (resp., of $\mathbb{V}^{\omega_{n-1}}$) are $\frac{1}{2}(\epsilon_1, \dots, \epsilon_n)$, where $\epsilon_k = \pm 1$, and the number of negative signs is even (resp., odd).

If $i = n$ then λ is of the form $\lambda = \sum_{k \in S} e_k$, where $S \subset \{1, \dots, n\}$ and $|S|$ is even. For n odd here one checks that for any such λ , $\kappa(\lambda - \alpha_n)$ is not divisible in $\check{\Lambda}$, so $\bar{\kappa}(\lambda - \alpha_n) \notin N\mathbb{Z}$. For n even taking $\lambda = (1, \dots, 1, 0, 0)$ we get $\lambda - \alpha_n = (1, \dots, 1, -1, -1)$. For any $\mu \in \Lambda$, $\kappa(\lambda - \alpha_n, \mu)$ is even. So, we have to assume $2m \notin N\mathbb{Z}$ for n even. Under this assumption one checks that $\bar{\kappa}(\lambda - \alpha_n) \notin N\check{\Lambda}$.

If $i = n-1$ then $\lambda - \alpha_{n-1}$ is of the form $(\epsilon_1, \dots, \epsilon_{n-2}, 0, \epsilon_n)$, where $\epsilon_k = 0$ or 1, and the number of 1's is even; or of the form $(\epsilon_1, \dots, \epsilon_{n-2}, -1, \epsilon_n)$, where $\epsilon_k = 0$ or 1, and the number of 1's is odd (and the element $\lambda = 0$ is excluded here). In the first case $\bar{\kappa}(\lambda - \alpha_n) \notin N\check{\Lambda}$, and in the second case the only difficulty comes from $\lambda - \alpha_{n-1} = (1, \dots, 1, -1, 1)$ for n even. In this case our assumption $2m \notin N\mathbb{Z}$ for n even guarantees that $\bar{\kappa}(\lambda - \alpha_n) \notin N\check{\Lambda}$.

Let now $i \leq n-2$. Note that for any $a = (a_1, \dots, a_n) \in \Lambda$, $\kappa(a) = (a_1, \dots, a_n) \in \check{\Lambda}$. If $\mu \in \Lambda^+$ is a weight of \mathbb{V}^{ω_i} then μ is of the form $(1, \dots, 1, 0, \dots, 0)$, where 1 appears $m \leq i$ times with $i - m$ even. So, any weight of \mathbb{V}^{ω_i} is of the form $\sum_{k \in S} \epsilon_k$ with $\epsilon_k = \pm 1$, where $S \subset \{1, \dots, n\}$ is a subset of order $m \leq i$ with $i - m$ even. We have $\omega_i - \alpha_i = (1, \dots, 1, 0, 1, 0, \dots, 0)$, where 1 first appears $i-1$ times. If $\lambda - \alpha_i$ contains the entry 0 then its other entries could be only 0, 1, -1, 2. So, $\kappa(\lambda - \alpha_i)$ may be divisible at most by 2 in $\check{\Lambda}$. Since $2m \notin N\mathbb{Z}$, $\bar{\kappa}(\lambda - \alpha_i) \notin N\check{\Lambda}$ in this case. If $\lambda - \alpha_i$ does not contain the entry 0 and contains the entry 2 then $\kappa(\lambda - \alpha_i)$ may be divisible at most by 2. If $\lambda - \alpha_i$ does not contain the entries 0, 2 then $i = n/2$, n is even and $\lambda - \alpha_i = (1, \dots, 1, \epsilon_i, 1, \epsilon_{i+2}, \dots, \epsilon_n)$ with $\epsilon_k = \pm 1$. Then $\kappa(\lambda - \alpha_i)$ is divisible at most by 2. We are done.

Remark A.1.1. *Our result for the type D_n could possibly be improved by replacing Spin_{2n} with the corresponding group with connected center as in Remark 1.1.3.*

Type F_4 . Let $I = \mathbb{Z}^4$, $e = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \in (\frac{1}{2}\mathbb{Z})^4$ and $\Lambda = I \cup I'$, where $I' = e + I$. So, $\Lambda \subset (\frac{1}{2}\mathbb{Z})^4$. Let $\kappa : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ be the symmetric bilinear form given by $\kappa(a, b) = 2 \sum_k a_k b_k$. Let R be the set of $\mu \in \Lambda$ with $\kappa(\mu, \mu) = 2$ or 4 . The coroots are

$$R = \{\pm e_i(1 \leq i \leq 4), \pm(e_i - e_j), \pm(e_i + e_j)(1 \leq i < j \leq 4), \frac{1}{2}(\pm 1, \dots, \pm 1)\}$$

Pick $\alpha_1 = \frac{1}{2}(1, -1, -1, -1)$, $\alpha_2 = e_4$, $\alpha_3 = e_3 - e_4$, $\alpha_4 = e_2 - e_3$. These are simple coroots (notations from [28]), and Λ is freely generated by α_i . The map $\kappa : \Lambda \hookrightarrow \check{\Lambda}$ is an inclusion. The center of G is trivial.

We identify $\check{\Lambda}$ with a sublattice of \mathbb{Q}^4 such that the pairing $\langle, \rangle : \Lambda \otimes \check{\Lambda} \rightarrow \mathbb{Z}$ is the map sending (a, b) to $\sum_k a_k b_k$. The fundamental weights are $\check{\omega}_1 = 2e_1$, $\check{\omega}_2 = 3e_1 + e_2 + e_3 + e_4$, $\check{\omega}_3 = 2e_1 + e_2 + e_3$, $\check{\omega}_4 = e_1 + e_2$ in $\check{\Lambda}$. Then $\check{\Lambda}$ is freely generated by $\check{\omega}_i$. So, $\check{\Lambda} = \{a \in \mathbb{Z}^4 \mid \sum_i a_i = 0 \pmod{2}\}$. The map $\kappa : \Lambda \rightarrow \check{\Lambda}$ sends any a to $2a$. We recover the roots in $\check{\Lambda}$ from the property that $\kappa(\alpha) = \frac{\kappa(\alpha, \alpha)}{2} \check{\alpha}$ for any coroot α . The roots are

$$\check{R} = \{\pm 2e_i(1 \leq i \leq 4), \pm(e_i - e_j), \pm(e_i + e_j)(1 \leq i < j \leq 4), (\pm 1, \dots, \pm 1)\}$$

The simple roots are $\check{\alpha}_1 = (1, -1, -1, -1)$, $\check{\alpha}_2 = 2e_4$, $\check{\alpha}_3 = e_3 - e_4$, $\check{\alpha}_4 = e_2 - e_3$. The fundamental coweights are $\omega_1 = e_1$, $\omega_2 = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4)$, $\omega_3 = 2e_1 + e_2 + e_3$, $\omega_4 = e_1 + e_2$. The Weyl group acting on Λ is generated by all the permutations, all the sign changes, and the element s_1 given by

$$s_1(a_1, \dots, a_4) = \frac{1}{2}(a_1 + \dots + a_4, a_1 + a_2 - a_3 - a_4, a_1 - a_2 + a_3 - a_4, a_1 - a_2 - a_3 + a_4)$$

The element $-w_0$ acts trivially on Λ . The group W acts transitively on long (resp., short) coroots. We have $0 \leq \omega_1 \leq \omega_4 \leq \omega_2 \leq \omega_3$. The representation \mathbb{V}^{ω_4} is the adjoint one, $\dim \mathbb{V}^{\omega_2} = 273$, $\dim \mathbb{V}^{\omega_3} = 1274$. The 24 positive coroots are

$$\begin{aligned} R^+ = \{ & \alpha_i(1 \leq i \leq 4), \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4, \\ & 2\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \\ & 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4, 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + \alpha_4, 2\alpha_1 + 4\alpha_2 + 2\alpha_3 + \alpha_4, \\ & 2\alpha_2 + 2\alpha_3 + \alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4, 2\alpha_2 + \alpha_3, \\ & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \\ & \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 \} \end{aligned}$$

Let $i = 1$. The weights of \mathbb{V}^{ω_1} are known from [28], they are $\pm e_j$, $\frac{1}{2}(\pm 1, \dots, \pm 1)$, 0 . We have $\omega_1 - \alpha_1 = e$. So, $\lambda - \alpha_1$ may be $\frac{1}{2}(a_1, \dots, a_4)$, where all $a_j = 1$ except one, which is -1 or 3 ; it also may be $(a_1, \dots, a_4) \neq 0$, where each a_k is 0 or 1 ; it also maybe e . We see that $\kappa(\lambda - \alpha_1)$ may be divisible at most by 2 . Assume $\bar{\kappa} = m\kappa$ with $m \in \mathbb{Z}$. The assumption of Conjecture 1.1.2 says $2m \notin N\mathbb{Z}$. So, in this case $\bar{\kappa}(\lambda - \alpha_i)$ is not divisible by N .

Let $i = 4$. The weights of \mathbb{V}^{ω_4} are the coroots and 0 . We have $\omega_4 = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$. If $\omega_4 - \lambda$ is a weight of \mathbb{V}^{ω_4} then $\lambda \leq 2\omega_4$. Under our assumptions, we get $0 < \lambda - \alpha_4 \leq 2\omega_4 - \alpha_4 = 4\alpha_1 + 8\alpha_2 + 6\alpha_3 + 3\alpha_4$. Since $\gamma := 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + \alpha_4$ is a coroot, $\lambda - \alpha_4$ may take value $\omega_4 + \gamma - \alpha_4 = 4\alpha_1 + 8\alpha_2 + 6\alpha_3 + 2\alpha_4$. For this λ

we see that $\kappa(\lambda - \alpha_4) = 4\check{\alpha}_1 + 8\check{\alpha}_2 + 12\check{\alpha}_3 + 4\check{\alpha}_4$ is divisible by 4. So, the assumption of Conjecture 1.1.2 is not sufficient for our method to work in this case. We need to assume at least that $4m \notin N\mathbb{Z}$.

Use the method from Section A.1. The dominant coweights $\mu \in \Lambda^+$ such that $\mu \leq \omega_4$ are $\{0, \omega_1, \omega_4\}$. For $\mu = 0$ we need to check that $\bar{\kappa}(\omega_4) \notin N\check{\Lambda}$. Since $\kappa(\omega_4) = 2(e_1 + e_2)$ is only divisible by 2, and $2m \notin N\mathbb{Z}$, we see that $\bar{\kappa}(\omega_4) \notin N\check{\Lambda}$. For $\mu = \omega_1$ this property is easy. The W -orbit through ω_4 is the set of long coroots. For $\mu = \omega_4$ and a long coroot α , $\kappa(\alpha - \mu)$ may be divisible at most by 4 in the case $\alpha = -e_1 - e_2$. The assumption $4m \notin N\mathbb{Z}$ guarantees in this case that $\bar{\kappa}(\lambda - \alpha_i) \notin N\check{\Lambda}$.

Let $i = 2$. The dominant coweights μ such that $\mu \leq \omega_2$ form the set $\{0, \omega_1, \omega_4, \omega_2\}$. The W -orbit through ω_2 is the set

$$X_2 = \left\{ \frac{1}{2}(\pm 3, \pm 1, \pm 1, \pm 1), \frac{1}{2}(\pm 1, \pm 3, \pm 1, \pm 1), \frac{1}{2}(\pm 1, \pm 1, \pm 3, \pm 1), \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 3), \right. \\ \left. (\pm 1, \pm 1, \pm 1, 0), (\pm 1, \pm 1, 0, \pm 1), (\pm 1, 0, \pm 1, \pm 1), (0, \pm 1, \pm 1, \pm 1) \right\},$$

these are all the coweights of length 6. The element $\kappa(\omega_2)$ is not divisible. For $\tau \in X_2$, $\kappa(\tau - \omega_1)$ is divisible at most by 2. For $\tau \in X_2$, $\kappa(\tau - \omega_4)$ is divisible at most by 2. For $\tau \in X_2$, $\kappa(\tau - \omega_2)$ may be divisible by 2 or 3. Namely, if $\tau = \frac{1}{2}(-3, 1, 1, 1)$ then $\kappa(\tau - \omega_2) = -6e_1$ is divisible in $\check{\Lambda}$ by 3. So, we must assume $3m \notin N\mathbb{Z}$.

Let $i = 3$. The set of $\mu \in \Lambda^+$ such that $\mu \leq \omega_3$ is the set $\{0, \omega_1, \omega_4, \omega_2, 2\omega_1, \omega_1 + \omega_4, \omega_3\}$. The W -orbit through ω_3 is the set X_3 of all the coweights of length 12, it consists of $(\pm 2, \pm 1, \pm 1, 0)$ and all their permutations. The element $\kappa(\omega_3)$ is divisible by 2. For $\tau \in X_3$, $\kappa(\tau - \omega_1)$ is not divisible. For $\tau \in X_3$, $\kappa(\tau - \omega_4)$ may be divisible at most by 4. In this case our condition $4m \notin N\mathbb{Z}$ guarantees that $\bar{\kappa}(\lambda - \alpha_i) \notin N\check{\Lambda}$. For $\tau \in X_3$, $\kappa(\tau - \omega_2)$ may be divisible at most by 3. For $\tau \in X_3$, $\kappa(\tau - 2\omega_1)$ is divisible at most by 2. For $\tau \in X_3$, $\kappa(\tau - \omega_1 - \omega_4)$ is not divisible. For $\tau \in X_3$, $\kappa(\tau - \omega_3)$ may be divisible by 4 and by 6 (it is not divisible by 5 or by r with $r \geq 7$). For example, if $\tau = (-1, -2, 1, 0)$ then $\kappa(\tau - \omega_3) = 6(-1, -1, 0, 0) \in 6\check{\Lambda}$. Our condition $4m, 6m \notin N\mathbb{Z}$ guarantees that $\bar{\kappa}(\lambda - \alpha_i) \notin N\check{\Lambda}$. We are done.

A.2. Assume G is of type E_8 . We follow the notations for the corresponding root system from Bourbaki ([7], chapter 6, Section 4.10). So, $\Lambda = \Lambda_1 + \mathbb{Z}(\frac{1}{2}\sum_{i=1}^8 e_i)$, where e_i is the canonical (orthonormal) base in \mathbb{Z}^8 . Here $\Lambda_1 = \{(a_1, \dots, a_8) \in \mathbb{Z}^8 \mid \sum a_i \equiv 0 \pmod{2}\}$. The bilinear form $\kappa : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ is induced from the scalar product on \mathbb{R}^8 , where e_i is the orthonormal base. Then $\kappa : \Lambda \rightarrow \check{\Lambda}$ is an isomorphism. The element w_0 acts on Λ as -1 . The structure of W is described in ([7], exercise 1, paragraph 4, p. 228). It contains all the permutations of e_i and all the even number of sign changes (of the base elements). Our notations for ω_i and α_i is as in ([7], Section 4.10, p. 213). In particular, ω_8 is the biggest coroot, so \mathbb{V}^{ω_8} is the (quasi-minuscule) adjoint representation. We may assume $\bar{\kappa} = m\kappa$. The assumption of Conjecture 1.1.2 reads $m \notin N\mathbb{Z}$. The condition $\bar{\kappa}(\lambda - \alpha_i) \in N\check{\Lambda}$ is equivalent to $m(\lambda - \alpha_i) \in N\Lambda$.

We have the following inequalities

$$0 \leq \omega_8 \leq \omega_1 \leq \omega_7 \leq \omega_2 \leq \omega_6 \leq \omega_3 \leq \omega_5 \leq \omega_4$$

For $i = 8$ we have $\omega_8 = e_7 + e_8$ and $\alpha_8 = e_7 - e_6$. So, $\omega_8 - \alpha_8 = e_6 + e_8$, and $\omega_8 - \lambda$ is either zero or a coroot. Taking $\omega_i - \lambda = -e_6 - e_8$ we get $\lambda - \alpha_i = 2(e_6 + e_8) \in 2\Lambda$. So, we have to assume $2m \notin N\mathbb{Z}$ at least. Clearly, for $\omega_i - \lambda = \pm e_k \pm e_j$ with $k \neq j$ the element $\lambda - \alpha_i$ may be divisible at most by 2 in Λ . For $\omega_i - \lambda = \frac{1}{2}(a_1 + \dots + a_8)$ with $a_k = \pm 1$, $\sum_k a_k$ even, the element $\lambda - \alpha_8$ is not divisible. So, for $i = 8$ we are done.

In the case $i = 4$ consider $\omega_4 - \alpha_4 = e_2 + e_4 + e_5 + e_6 + e_7 + 5e_8$. Its W -orbit contains the element $\omega_4 - \lambda = e_2 + e_4 + e_5 + e_6 + e_7 - 5e_8$, for such λ we get $\lambda - \alpha_4 = 10e_8$. So, we must assume $10m \notin N\mathbb{Z}$.

In the case $i = 5$ we get $\omega_5 - \alpha_5 = e_3 + e_5 + e_6 + e_7 + 4e_8$. The W -orbit of this element contains $\omega_5 - \lambda = e_3 + e_5 + e_6 + e_7 - 4e_8$. For this λ we get $\lambda - \alpha_5 = 8e_8$. So, we must assume $8m \notin N\mathbb{Z}$.

In the case $i = 6$ we get $\omega_6 - \alpha_6 = e_4 + e_6 + e_7 + 3e_8$. The W -orbit of this element contains $\omega_6 - \lambda = e_4 + e_6 + e_7 - 3e_8$. For this λ we get $\lambda - \alpha_6 = 6e_8$. So, we must assume $6m \notin N\mathbb{Z}$. The above assumptions are equivalent to the property that for a simple coroot α_i , $\varrho(\alpha_i) \notin \frac{1}{10}\mathbb{Z}, \frac{1}{8}\mathbb{Z}, \frac{1}{6}\mathbb{Z}$.

APPENDIX B. PROOF OF PROPOSITION 2.7.1

B.1. For the convenience of the reader, we first formulate the problem that could be thought of as the metaplectic Casselman-Shalika problem.

As in [18], for $\eta \in \Lambda$ we write $\chi_\eta : U(F) \rightarrow \mathbb{A}^1$ for the additive character of conductor $\bar{\eta}$, where $\bar{\eta}$ is the image of η in the coweights lattice of G_{ad} . For $\eta + \nu \in \Lambda^+$ we also write $\chi_\eta^\nu : \text{Gr}_B^\nu \rightarrow \mathbb{A}^1$ for any $(U(F), \chi_\eta)$ -equivariant function. The isomorphism $\text{Gr}_B^0 \xrightarrow{\sim} \text{Gr}_B^\eta$, $v \mapsto t^\eta v$ transforms $\chi_0^0 : \text{Gr}_B^0 \rightarrow \mathbb{A}^1$ to $\chi_{-\eta}^\eta : \text{Gr}_B^\eta \rightarrow \mathbb{A}^1$.

For $\nu \in \Lambda^{\sharp,+}$ denote by $\widetilde{\text{Gr}}_G^\nu$ the restriction of the gerb $\widetilde{\text{Gr}}_G \rightarrow \text{Gr}_G$ to $\overline{\text{Gr}}_G^\nu$. Recall the irreducible objects $\mathcal{A}_\varepsilon^\nu$ of $\mathbb{P}\text{erv}_{G,\zeta}$ defined in ([22], Section 2.4.2), we are using for their definition the choice of $\Omega^{\frac{1}{2}}$ from Section 0.0.2. The perverse sheaf $\mathcal{A}_\varepsilon^\nu$ is defined only up to a scalar automorphism (but up to a unique isomorphism for ν in the coroots lattice of G).

Any trivialization of the fibre of $\text{Gra}_G \rightarrow \text{Gr}_G$ at $t^\eta G(\mathcal{O}_x)$ yields a section $s_B^\eta : \text{Gr}_B^\eta \rightarrow \widetilde{\text{Gr}}_B^\eta$ of the gerb $\widetilde{\text{Gr}}_B^\eta \rightarrow \text{Gr}_B^\eta$.

The metaplectic Casselman-Shalika problem is the following. Given $\lambda \in \Lambda^{\sharp,+}$ and $\mu, \nu \in \Lambda$ with $\mu + \nu \in \Lambda^+$, calculate

$$(48) \quad \text{R}\Gamma_c(\text{Gr}_B^\nu \cap \overline{\text{Gr}}_G^\lambda, (\chi_\mu^\nu)^* \mathcal{L}_\psi \otimes (s_B^\nu)^* \mathcal{A}_\varepsilon^\lambda)$$

(and describe the answer in terms of the corresponding quantum group).

Pick $x \in X$. As in ([18], Section 8.2.4) for $\mu + \nu \in \Lambda^+$ we can calculate the complex $j_{x,\mu}^* \text{H}_G^{\rightarrow}(\mathcal{A}^{-w_0(\lambda)}, \mathcal{F}_{x,\mu+\nu,!})$ over $\widetilde{\mathfrak{M}}_{x,\mu}$. It will vanish unless $\mu \in \Lambda^+$, and in the latter case it will identify with

$$\mathcal{F}_{x,\mu,!} \otimes \text{R}\Gamma_c(\text{Gr}_{B,x}^\nu \cap \overline{\text{Gr}}_{G,x}^\lambda, (\chi_\mu^\nu)^* \mathcal{L}_\psi \otimes (s_B^\nu)^* \mathcal{A}_\varepsilon^\lambda)[\langle \nu, 2\check{\rho} \rangle]$$

So, the complexes (48) describe the action of the Hecke functors on the objects $\mathcal{F}_{x,\eta,!}$ for $\eta \in \Lambda^+$.

B.2. Properties ii) and iii) are clearly equivalent. For $\eta \in \Lambda$ one has

$$\mathrm{Gr}_B^0 \cap \overline{\mathrm{Gr}}_B^{-\lambda} \xrightarrow{\sim} \mathrm{Gr}_B^\eta \cap \overline{\mathrm{Gr}}_B^{\eta-\lambda}$$

By ([26], Proposition 3.5.1), if $-\eta$ is deep enough in the dominant chamber then

$$\mathrm{Gr}_B^\eta \cap \overline{\mathrm{Gr}}_B^{\eta-\lambda} = \mathrm{Gr}_B^\eta \cap \overline{\mathrm{Gr}}_G^{w_0(\eta-\lambda)}$$

Here we assume that for each $-\lambda \leq \mu \leq 0$ the coweight $\eta + \mu$ is anti-dominant, and $\eta - \lambda \in \Lambda^\sharp$. Consider the complex

$$(49) \quad \mathrm{R}\Gamma_c(\mathrm{Gr}_B^\eta \cap \overline{\mathrm{Gr}}_G^{w_0(\eta-\lambda)}, (s_B^\eta)^* \mathcal{A}_\varepsilon^{w_0(\eta-\lambda)} \otimes (\chi_{-\eta}^\eta)^* \mathcal{L}_\psi)[\langle \eta, 2\check{\rho} \rangle]$$

This complex is what should be the limiting case of the metaplectic Casselman-Shalika formula (48) as in ([26], Section 3). As in ([18], Section 8.2.4), the tensor product of $\mathcal{F}_{x,-\eta}$ by (49) is isomorphic over $\widetilde{\mathfrak{M}}_{x,-\eta}$ to $j_{x,-\eta}^* \mathrm{H}_G^\rightarrow(\mathcal{A}_\varepsilon^{\lambda-\eta}, \mathcal{F}_\emptyset)$. Recall that $\mathrm{H}_G^\rightarrow(\mathcal{A}_\varepsilon^{\lambda-\eta}, \mathcal{F}_\emptyset) \xrightarrow{\sim} \mathcal{F}_{x,\lambda-\eta}$ by Theorem 5.3.1.

The contribution of the open stratum $\mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\eta-\lambda)}$ to (49) is

$$(50) \quad \mathrm{R}\Gamma_c(\mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\eta-\lambda)}, (s_B^\eta)^* \mathcal{A}_\varepsilon^{w_0(\eta-\lambda)} \otimes (\chi_{-\eta}^\eta)^* \mathcal{L}_\psi)[\langle \eta, 2\check{\rho} \rangle]$$

Lemma B.2.1. *The complex (50) identifies with the complex (7) shifted to the left by $\langle \lambda, 2\check{\rho} \rangle$.*

Proof. Recall the local system $\mathcal{W}^{w_0(\eta-\lambda)}$ on $\widetilde{\mathrm{Gr}}_G^{w_0(\eta-\lambda)}$ defined in ([22], Section 2.4.2). The perverse sheaf $\mathcal{A}_\varepsilon^{w_0(\eta-\lambda)}$ is the intermediate extension of this (shifted) local system. The \mathbb{G}_m -torsor $\mathrm{Gra}_G \times_{\mathrm{Gr}_G} \mathrm{Gr}_B^\eta \rightarrow \mathrm{Gr}_B^\eta$ is constant with fibre $\Omega_x^{-\frac{\bar{\kappa}(\eta,\eta)}{2}} - 0$, and $T(\mathcal{O})$ acts on it by the character $T(\mathcal{O}) \rightarrow T \xrightarrow{-\bar{\kappa}(\eta)} \mathbb{G}_m$. So, the local system $(s^\eta)^* \mathcal{W}^{w_0(\eta-\lambda)}$ over $\mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\eta-\lambda)}$ changes under the action of $T(\mathcal{O})$ by the inverse image of \mathcal{L}_ζ under $T(\mathcal{O}) \rightarrow T \xrightarrow{-\bar{\kappa}(\eta)} \mathbb{G}_m$. Since $\bar{\kappa}(\eta-\lambda) \in N\check{\Lambda}$, it coincides with the inverse image of \mathcal{L}_ζ under $T(\mathcal{O}) \rightarrow T \xrightarrow{-\bar{\kappa}(\lambda)} \mathbb{G}_m$. Since the isomorphism $\mathrm{Gr}_B^0 \cap \mathrm{Gr}_B^{-\lambda} \xrightarrow{\sim} \mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\eta-\lambda)}$, $z \mapsto t^\eta z$ is $T(\mathcal{O})$ -equivariant, we are done. \square

Lemma B.2.2. *For each $-\lambda < \mu \leq 0$ the stratum $\mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\mu+\eta)}$ does not contribute to the cohomology group of (49) in degrees ≥ -1 .*

Proof. The $*$ -restriction $\mathcal{A}^{w_0(\eta-\lambda)}$ to $\widetilde{\mathrm{Gr}}_G^{w_0(\mu+\eta)}$ is placed in perverse degrees < 0 , that is, in usual degrees $\leq \langle \mu + \eta, 2\check{\rho} \rangle - 1$. Recall that $\dim \mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\mu+\eta)} = -\langle \mu, \check{\rho} \rangle$.

If $\mu \neq 0$ then, by ([18], Proposition 7.1.7), $(\chi_{-\eta}^\eta)^* \mathcal{L}_\psi$ is nonconstant on each irreducible component of $\mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\mu+\eta)}$. So, in this case

$$(51) \quad \mathrm{R}\Gamma_c(\mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\mu+\eta)}, (s_B^\eta)^* \mathcal{A}_\varepsilon^{w_0(\eta-\lambda)} \otimes (\chi_{-\eta}^\eta)^* \mathcal{L}_\psi)[\langle \eta, 2\check{\rho} \rangle]$$

lives in degrees ≤ -2 .

If $\mu = 0$ then $\mathrm{Gr}_B^\eta \cap \mathrm{Gr}_G^{w_0(\eta)}$ is a point, the $*$ -restriction of $(s_B^\eta)^* \mathcal{A}^{w_0(\eta-\lambda)}$ to this point lives in degrees $\leq \langle \eta, 2\check{\rho} \rangle - 1$. Besides, it lives only in usual degrees of the same

parity as $\langle \eta - \lambda, 2\check{\rho} \rangle$ by ([22], Lemma 2.2). Since $\langle \lambda, 2\check{\rho} \rangle \in 2\mathbb{Z}$, it is of the same parity as $\langle \eta, 2\check{\rho} \rangle$. So, it lives in degrees $\leq \langle \eta, 2\check{\rho} \rangle - 2$. \square

We conclude that the subtop cohomology property is equivalent to requiring that for any $\lambda > 0$, which is not a simple coroot, (49) is placed in degrees ≤ -2 . Proposition 2.7.1 is proved.

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