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# Semi-active suspension control problem: some new results using an $LPV/H_\infty$ state feedback input constrained control

M.Q.Nguyen<sup>1\*</sup>, J.M. Gomes da Silva Jr<sup>2</sup>, O.Sename<sup>1</sup>, L.Dugard<sup>1</sup>

**Abstract**—The semi-active suspension control problem faces the challenge of the dissipativity constraints of the semi-active dampers. This induces some compromises (actuator saturation, comfort, road holding...) which need to be taken into account in the control design step. In this paper, a state feedback input constrained control problem for LPV systems is considered with  $H_\infty$  performance objective. Stabilization conditions based on the Finsler's Lemma are derived in order to ensure the stability in the presence of the input saturation, and to attenuate the disturbance effects. To this aim, two different Lyapunov functions are used. For the stability analysis, a generalized sector condition for LPV systems is applied to treat the nonlinearity caused by the actuator saturation. The considered performance objective regards the reduction of  $\mathcal{L}_2$  gain from the disturbance to the controlled output. The LPV controller is computed from the solution of LMIs considering a polytopic representation for the LPV closed-loop system. These theoretical results are applied to a semi-active suspension system where the dissipativity conditions of the semi-active dampers are recast as saturation conditions on the control inputs. The comfort criteria is used as a performance objective in this study. Some simulation results are presented in order to illustrate the effectiveness of the proposed approach.

**Keywords:** Semi-active suspension, State Feedback,  $LPV/H_\infty$  control, Input saturation.

## I. INTRODUCTION

The suspension system plays a key role in vehicle dynamic system. Semi-active suspensions are nowadays widely used in automotive industry thanks to their low price and low energy consumption. However, the control design for semi-active suspension systems must face the challenge induced by the dissipativity constraints. Several control design problems for semi-active suspension systems have then been tackled with many different approaches during the last decades. In the works of [1], [2], the authors presented several control strategies for semi-active suspensions (Skyhook, Groundhook, ADD,...). Moreover, to cope with the dissipativity constraints of semi-active dampers, some control approaches using the LPV techniques have been presented. In [3], a kind of LPV gain scheduling anti-windup strategy was proposed by using a scheduling parameter which represents in some sense the excess of active control. More specifically, in some recent works, the dissipativity constraint of the semi-active damper has been recast as an input saturation problem. In

([4], [5]), the nonlinearities of the semi-active suspension (including the saturation of the control input) are taken into account and written in an LPV form but are not explicitly considered in the design step. In [6], an output feedback LPV control with input saturation and state constraints was designed (taking explicitly the constraints into account).

On the other hand, in many practical control applications, the actuator saturation is a challenge for the control system designer because it induces a nonlinear behavior for the closed-loop system even if the plant is linear. Actually, the input saturation is source of instability in control and loss in performance. In more recent years, researchers have focused on the problem of input saturation control. First, several models of the saturation nonlinearity were proposed. In [7] and [8], a full discussion about the saturation modeling based on the use of the polytopic differential inclusions is given. In [8], a generalized sector condition approach, where the saturation term is replaced by a dead-zone nonlinearity function, is also presented. Then, these models for the saturation are used to treat the stability and stabilization problems for the class of LTI system. In [9] and [10], anti-windup design has been addressed for polytopic LPV systems. Another anti-windup synthesis for LPV systems under the Linear Fractional Representation form is presented in [11]. Moreover, regarding the works that cope not only with the input saturation but also with the disturbance attenuation problems, we can cite for instance [12], which uses a polytopic approach and [13] that uses a sector condition and Finsler's lemma to give a solution to the  $\mathcal{L}_2$  stabilization problem. The interest of using such a Finsler approach is to decouple the Lyapunov matrix from the join variables. The controller derived from the synthesis thus does not depend on the Lyapunov matrix. Hence, in the multiple objective case, we can use the different candidate Lyapunov functions which potentially allows to reduce the conservatism

In this work, a LPV state feedback is designed for the semi-active suspension control problem based on the Finsler's lemma. Firstly, the semi active suspension system is rewritten in LPV form. The dissipativity constraints are recast as an input saturation, which is tackled by a generalized sector condition modified for LPV systems. Then, two objectives are considered: the stability when the input control is saturated and the disturbance attenuation. Thanks to the Finsler's lemma, two different Lyapunov functions are used: one for stability analysis and another for the disturbance rejection. Then, considering a polytopic system framework, conditions in the form of quasi-Linear Matrix Inequality

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(quasi-LMI: coupling between a matrix variable and a scalar variable) conditions are derived. They are LMIs provided some scalars are fixed. The resolution of these LMIs allows to compute a LPV state feedback input constrained control that ensures the semi-activeness while improving the comfort passenger.

The paper is organised as follows. Section 2 describes the semi-active control problem. Section 3 presents the general control problem of LPV system subject to input saturation. Section 4 gives the different steps to design the controller. Some simulation results are given in the section 5. Finally, some conclusions are drawn in the section 5.

## II. PROBLEM FORMULATION

### A. Quarter-car suspension model

Let us consider a quarter vehicle model, with  $m_s$  and  $m_{us}$  standing for the sprung and unsprung masses, respectively;  $k_s$  is the suspension stiffness. The tire model is given by a passive damper with coefficient  $c_t$  and a spring with stiffness coefficient  $k_t$ ;  $z_r(t)$  is the vertical road displacement;  $z_s(t)$  and  $z_{us}(t)$  represent the vertical displacements of the sprung and unsprung masses, respectively. Then the dynamic equations of the quarter vehicle around the equilibrium are governed by [1]:

$$\begin{cases} m_s \ddot{z}_s = -k_s(z_s - z_{us}) - F_{damper} \\ m_{us} \ddot{z}_{us} = k_s(z_s - z_{us}) + F_{damper} - k_t(z_{us} - z_r) - c_t(\dot{z}_{us} - \dot{z}_r) \end{cases} \quad (1)$$

where  $z_{def} = z_s - z_{us}$ : the damper deflection is assumed to be measured or estimated, and  $F_{damper}$  is the semi-active controlled damper force:

$$F_{damper} = c(\cdot)\dot{z}_{def} \quad (2)$$

with  $\dot{z}_{def} = \dot{z}_s - \dot{z}_{us}$  being the deflection velocity and  $c(\cdot)$  is the damping coefficient assumed to be varying for control purpose. To ensure the dissipativity constraint of the semi-active damper, the following constraint must be satisfied:

$$0 \leq c_{min} \leq c(\cdot) \leq c_{max} \quad (3)$$

Rewriting now (2) as follows:

$$F_{damper} = c(\cdot)\dot{z}_{def} = (c_0 + u^{H_\infty})\dot{z}_{def} = c_0\dot{z}_{def} + u^{H_\infty}\rho \quad (4)$$

with  $c_0 = (c_{max} + c_{min})/2$  and  $\rho = \dot{z}_{def}$  being a time-varying scheduling parameter. Replacing  $F_{damper}$  into (1), one obtains the following state space representation:

$$\begin{cases} \dot{x}_s = A_s x_s + B_{s1} w + B_{s2}(\rho) u \\ z = C_z x_s + D_z(\rho) u \end{cases} \quad (5)$$

where  $x_s = (z_s - z_{us}, \dot{z}_s, z_{us} - z_r, \dot{z}_{us})^T$ ,  $w = \dot{z}_r$ ,  $z = \ddot{z}_s$ , and  $u = u^{H_\infty}$ .

$$A_s = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-k_s}{m_s} & \frac{-c_0}{m_s} & 0 & \frac{c_0}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_{us}} & \frac{c_0}{m_{us}} & \frac{-k_t}{m_{us}} & \frac{-c_0 - c_t}{m_{us}} \end{bmatrix}, D_z = \begin{bmatrix} \frac{-\rho}{m_s} \end{bmatrix}$$

$$B_{s1} = \begin{bmatrix} 0 & 0 & -1 & \frac{c_t}{m_{us}} \end{bmatrix}^T, B_{s2} = \begin{bmatrix} 0 & \frac{-\rho}{m_s} & 0 & \frac{\rho}{m_{us}} \end{bmatrix}^T$$

$$C_z = \begin{bmatrix} \frac{-k_s}{m_s} & \frac{-c_0}{m_s} & 0 & \frac{c_0}{m_s} \end{bmatrix}.$$

Note that in order to allow for application of the polytopic approach,  $B_{s2}$  and  $D_z$  must be constant. As the scheduling parameter enters on the input matrices  $B_{s2}$  and  $D_z$ , the

requirement on LPV polytopic is not satisfied. A simple way to make it possible is to introduce a strict low-pass filter as in [14].

### B. Input and State constraints

Denoting  $c(\cdot) = c_0 + u^{H_\infty}$ , the dissipativity constraint (3) is now recast into the following input constraint:

$$|u^{H_\infty}| < (c_{max} - c_{min})/2 \quad (6)$$

It is worth noting that (5) is actually a quasi-LPV system since the scheduling parameter  $\rho = \dot{z}_{def}$  depends on the system state. It is supposed that  $|\dot{z}_{def} = \dot{z}_s - \dot{z}_{us}| < 1$ . Moreover, in this study, the following constraint on the suspension stroke limit is considered:  $|z_{def} = z_s - z_{us}| < 0.125$ . These two constraints can be rewritten into following one:

$$\begin{bmatrix} \dot{z}_{def} \\ z_{def} \end{bmatrix} = |Hx| \leq \begin{bmatrix} 1 \\ 0.125 \end{bmatrix} \quad (7)$$

where  $x$  is the system state and  $H = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$  is the state constraint matrix.

### C. Performance objective and control problem

In this work, we aim at enhancing the comfort evaluated using the vertical body acceleration. Therefore, the minimization of the  $\mathcal{L}_2$  gain  $\gamma$  of the closed-loop transfer function from the disturbance  $w$  to the controlled output  $\ddot{z}_s$  (while taking into account the constraints on the control input and states) is considered as performance criterion.

The control problem to be addressed can be therefore be stated as follows:

*Control problem:* Design a suspension control that improves the passenger comfort and satisfies the input saturation constraints (6) and the state constraint (7).

To tackle this problem, we consider an LPV approach detailed in the sequel.

## III. GENERAL INPUT CONSTRAINED CONTROL PROBLEM

Consider a generalized quasi-LPV system  $\mathfrak{S}_\rho$  as follows:

$$\begin{cases} \dot{x} = A(\rho)x + B_1(\rho)w + B_2u \\ z = C_1(\rho)x + D_{11}(\rho)w + D_{12}u \end{cases} \quad (8)$$

where  $x \in \mathcal{R}^n$  is the state vector,  $u \in \mathcal{R}^m$  is the control vector,  $z \in \mathcal{R}^p$  is the controlled output vector and  $w \in \mathcal{R}^q$  is the input disturbance signal vector.  $\rho(t) = (\rho_1, \dots, \rho_k)$  is the time-varying parameter vector. Assume that  $\rho(t)$  is measurable during the operation of the system, and that it is bounded as follows:

$$\rho(t) \in \Omega_\rho = \left\{ \rho_i(t) \mid \underline{\rho}_i \leq \rho_i(t) \leq \bar{\rho}_i, i = 1, \dots, k \right\}$$

Note that  $\rho$  will be used instead of  $\rho(t)$  for simplicity. The matrices  $A(\rho), B_1(\rho), C_1(\rho), D_{11}(\rho)$  are assumed to depend affinely on the parameter  $\rho = (\rho_1, \dots, \rho_k)$ , that is:

$$\mathcal{A}(\rho) = \mathcal{A}_0 + \rho_1 \mathcal{A}_1 + \dots + \rho_k \mathcal{A}_k$$

where  $\mathcal{A}$  stands for matrices  $A, B_1, C_1, D_{11}$ . Then, provided that  $\rho$  is bounded in a polytope, the system  $\mathfrak{S}_\rho$  can be written

as a convex combination of the vertices  $\mathfrak{S}_\rho^j$  of the polytope as follows:  $\mathfrak{S}_\rho = \sum_{j=1}^{2^k} \alpha_j(\rho) \mathfrak{S}_\rho^j$  where  $\sum_{j=1}^{2^k} \alpha_j(\rho) = 1$  and  $\mathfrak{S}_\rho^j = [A_j, B_{1j}, B_2, C_{1j}, D_{11j}, D_{12}]$ ,  $j = 1, \dots, 2^k$ . Let us now consider the following assumptions:

- The applied control signal  $u$  takes value in the compact set:

$$\mathcal{U} = \{u \in \mathcal{R}^m / -u_{0i} \leq u_i \leq u_{0i}, i = 1, \dots, m\} \quad (9)$$

- The input disturbances  $w$  are supposed to be bounded in amplitude i.e  $w$  belongs to a set  $\mathcal{W}$ :

$$\mathcal{W} = \{w \in \mathcal{R}^q / w^T w < \delta\} \quad (10)$$

- The state vector is assumed to be known (measured or estimated). Moreover, from (7), the trajectories of system must belong to a region  $\mathcal{X}$  defined as follows:

$$\mathcal{X} = \{x \in \mathcal{R}^n / |H_i x| \leq h_{0i}, i = 1, \dots, k\} \quad (11)$$

In this work, a state feedback control law is considered (Fig.1) and the control signal  $v(t)$  is given by:

$$v(t) = K(\rho(t))x(t)$$

where  $K(\rho) \in \mathcal{R}^{m \times n}$  is a parameter dependent state feedback matrix gain, under the form:

$$K(\rho) = \sum_{j=1}^{2^k} \alpha_j(\rho) K_j$$

where  $K_j$  is the state feedback gain which is computed at each vertice  $\mathfrak{S}_\rho^j$  of the polytope.

Then, by virtue of the input constraints (9), the applied control  $u$  to system (8) is a saturated one, i.e:

$$u(t) = \text{sat}(v(t)) = \text{sat}(K(\rho(t))x(t)) \quad (12)$$

where the saturated function  $\text{sat}(\cdot)$  is defined by:

$$\text{sat}(v_i(t)) = \begin{cases} u_{0i} & \text{if } v_i(t) > u_{0i} \\ v_i(t) & \text{if } -u_{0i} \leq v_i(t) \leq u_{0i} \\ -u_{0i} & \text{if } v_i(t) < -u_{0i} \end{cases} \quad (13)$$

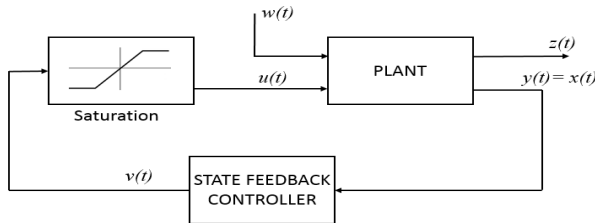


Fig. 1. State feedback control with input saturation

The closed-loop system obtained from the application of (12) in (8) reads as follows:

$$\begin{cases} \dot{x} = A(\rho)x + B_1(\rho)w + B_2 \text{sat}(K(\rho)x) \\ z = C_1(\rho)x + D_{11}(\rho)w + D_{12} \text{sat}(K(\rho)x) \end{cases} \quad (14)$$

Let us define now the vector-valued dead-zone function  $\phi(K(\rho)x)$ :

$$\phi(K(\rho)x) = \text{sat}(K(\rho)x) - K(\rho)x \quad (15)$$

From (15), the closed-loop system can therefore be re-written as follows:

$$\begin{cases} \dot{x} = (A(\rho) + B_2 K(\rho))x + B_2 \phi(K(\rho)x) + B_1(\rho)w \\ z = (C_1(\rho) + D_{12} K(\rho))x + D_{12} \phi(K(\rho)x) + D_{11}(\rho)w \end{cases} \quad (16)$$

#### A. Problem definition

It should be noticed that under the input saturation, the state may become unbounded for large disturbances ([8]). Hence, in this work, we propose the design of a state feedback  $K(\rho)$  for the LPV system (14) in order to satisfy the following conditions:

- When the control input signal is saturated, the nonlinear behavior of the closed-loop system must be considered and the stability has to be guaranteed both internally as well as in the input to state context, that is:
  - for  $w \in \mathcal{W}$ , the trajectories of the closed-loop system must be bounded.
  - if  $w(t) = 0$  for  $t > t_1 > 0$  then the trajectory of the system converge asymptotically to the origin.
- The control performance objective consists in minimizing the upper bound for the  $\mathcal{L}_2$  gain from the disturbance  $w$  to the controlled output  $z$ . In the detail, the following optimization problem is considered:

$$\min \gamma, \quad \text{such that: } \sup_{w \in \mathcal{W}} \frac{\|z\|_2}{\|w\|_2} < \gamma \quad (17)$$

In order to reduce the conservatism, it is worth noting that in this work, the  $\mathcal{L}_2$  performance problem is solved only in the case that the input saturation is not activated. Actually, this is appropriated in reality because in the presence of actuator saturation, the main concern is to guarantee that the trajectories are bounded and the state constraints are not violated.

## IV. CONTROLLER DESIGN

#### A. Stability analysis

The system (14) presents an input disturbance  $w \in \mathcal{W}$  and its state variables must belong to the state region  $\mathcal{X}$ . Moreover, the saturation function induces an extra nonlinear behavior in the closed-loop system. Hence we will take into account these facts by using a regional (local) stability approach. To this aim, a modification of the generalized sector condition which uses a parameter dependant matrix  $T(\rho)$  is proposed and applied for the LPV system.

Let us first define the following polyhedral set:

$$\mathcal{S}_\rho(K, G, u_0) = \{x \in \mathcal{R}^m \mid -u_0 \preceq (K(\rho) - G(\rho))x \preceq u_0\} \quad (18)$$

where  $\preceq$  stands for componentwise inequality.

*Lemma 1:* If  $x \in \mathcal{S}_\rho(K, G, u_0)$ , then the deadzone function  $\phi$  satisfies the following inequality:

$$\phi(K(\rho)x)^T T(\rho) [\phi(K(\rho)x) + G(\rho)x] \leq 0 \quad (19)$$

for any diagonal positive definite matrix  $T(\rho) \in \mathbb{R}^{m \times m}$ .

*Proof:* The proof of the lemma can be inferred easily from ([15])

Because of the boundness of the disturbance  $w \in \mathcal{W}$ , we consider the  $\mathcal{W}$ -invariance concept ([16]):

*Definition:* The set  $\mathcal{E} \subset \mathbb{R}^n$  is said to be  $\mathcal{W}$ -invariant if  $\forall x(t_0) \in \mathcal{E}, \forall w(t) \in \mathcal{W}$  implies that the trajectory  $x(t) \in \mathcal{E}$  for all  $t \geq t_0$ .

As known, the quadratic stability can be interpreted in terms of invariant ellipsoids ([17]). In fact, considering a quadratic Lyapunov candidate function  $V = x^T P x$ , where  $P$  is an Lyapunov matrix, the level set associated to this Lyapunov function is given by the following ellipsoid:

$$\mathcal{E}(P) = \{x \in \mathbb{R}^n : x^T P x < 1\} \quad (20)$$

Then, the idea is to ensure that  $\mathcal{E}(P)$  is  $\mathcal{W}$ -invariant for the closed-loop system (16). This can be achieved if  $\dot{V}(t) < 0$  in the boundary of  $\mathcal{E}(P)$ . Thus, it suffices to ensure that  $\dot{V}(t) < 0 \forall x \notin \text{int} \mathcal{E}(P)$  (the interior of  $\mathcal{E}(P)$ ) i.e  $x^T P x \geq 1$  and for any  $w \in \mathcal{W}$  i.e  $w^T w \leq \delta$ . By using the S-procedure [17], this condition can be satisfied if there exist scalars  $\lambda_1 > 0$  and  $\lambda_2 > 0$ , such that:

$$\dot{V} + \lambda_1(x^T P x - 1) + \lambda_2(\delta - w^T w) < 0 \quad (21)$$

Then, the following theorem regards a stabilization condition for the system (14):

*Theorem 1:* If there exist a matrix  $W$  symmetric positive definite, a matrix  $N(\rho)$ -diagonal positive definite, matrices  $M, Z(\rho), Y(\rho)$  of appropriate dimensions and positive scalars  $\lambda_1, \lambda_2$ , and scalar  $\varepsilon_1$  such that the following conditions are verified:

- $LMI_1 < 0$  where  $LMI_1$  is given in (22)

•

$$\left[ \begin{array}{c|c} W & (Z_i(\rho) - Y_i(\rho))^T \\ \hline Z_i(\rho) - Y_i(\rho) & u_{0i}^2 \end{array} \right] \geq 0, i = 1, \dots, m \quad (23)$$

where  $Z_i(\rho)$  and  $Y_i(\rho)$  are  $i^{\text{th}}$  line of  $Z(\rho)$  and  $Y(\rho)$  respectively.

•

$$\left[ \begin{array}{c|c} W & M^T H_i^T \\ \hline H_i M & h_{0i}^2 \end{array} \right] \geq 0, i = 1, \dots, k \quad (24)$$

where  $H_i$  is  $i^{\text{th}}$  line the state constraint matrix  $H$ .

$$\lambda_2 \delta - \lambda_1 < 0 \quad (25)$$

Then, the system (14) with  $K(\rho) = Z(\rho)M^{-1}$  is such that:

- For any  $w \in \mathcal{W}$  and  $x(0) \in \mathcal{E}(P)$  (with  $P = M^{-T} W M^{-1}$ ) the trajectories do not leave  $\mathcal{E}(P)$ , i.e.  $\mathcal{E}(P)$  is an  $\mathcal{W}$ -invariant domain for the system (14).
- If  $x(0) \in \mathcal{E}(P)$  and  $w(t) = 0$  for  $t > t_1$ , then the corresponding trajectory converge asymptotically to the origin, i.e.  $\mathcal{E}(P)$  is included in the region of attraction of the closed-loop system (14).

In order to prove this theorem, an approach based on Finsler's lemma is used ([18]).

*Lemma 2 (Finsler's lemma):* If  $x \in \mathbb{R}^n$ ,  $\mathbf{Q}$  is a symmetric matrix,  $\mathbf{B} \in \mathbb{R}^{m \times n}$  such that  $\text{rank}(\mathbf{B}) < n$ , then the following statements are equivalent:

- $x^T \mathbf{Q} x < 0 \quad \forall \mathbf{B} x = 0, x \neq 0$
- $\exists \mathbf{X} \in \mathbb{R}^{n \times m} : \mathbf{Q} + \mathbf{X} \mathbf{B} + \mathbf{B}^T \mathbf{X}^T < 0$

*Proof of Theorem 1:* As mentioned previously,  $\mathcal{E}(P)$  is  $\mathcal{W}$ -invariant if:

$$\dot{V} + \lambda_1(x^T P x - 1) + \lambda_2(\delta - w^T w) < 0 \quad (26)$$

Now, from *Lemma 1*, provided that  $x \in \mathcal{S}_\rho(K, G, u_0)$ , (26) is satisfied if:

$$\dot{V} + \lambda_1(x^T P x - 1) + \lambda_2(\delta - w^T w) - 2\phi(K(\rho)x)^T T(\rho)[\phi(K(\rho)x) + G(\rho)x] < 0 \quad (27)$$

For the sake of simplicity, the argument  $\rho$  is omitted here, then (27) is rewritten as follows:

$$\dot{x}^T P x + x^T P \dot{x} + \lambda_1 x^T P x - \lambda_2 w^T w - 2\phi(Kx)^T T \phi(Kx) - 2\phi(Kx)^T T G x + \lambda_2 \delta - \lambda_1 < 0 \quad (28)$$

Then the condition (28) is guaranteed if both following inequalities hold:

$$\lambda_2 \delta - \lambda_1 < 0 \quad (29)$$

$$\xi^T \mathbf{P} \xi < 0 \quad (30)$$

where  $\mathbf{P} = \begin{bmatrix} \lambda_1 P & P & -G^T T & 0 \\ P & 0 & 0 & 0 \\ -T G & 0 & -2T & 0 \\ 0 & 0 & 0 & -\lambda_2 I \end{bmatrix}$  and  $\xi = [x^T \quad \dot{x}^T \quad \phi^T \quad w^T]^T$ .

Rewrite (16) in the form:  $\mathbf{B}(\rho)\xi = 0$  where  $\mathbf{B}(\rho) = [A_F(\rho) \quad -I \quad B_2 \quad B_1(\rho)]$  with  $A_F(\rho) = A(\rho) + B_2 K(\rho)$ .

Now, using the Finsler's lemma:  $\xi^T \mathbf{P} \xi < 0, \forall \mathbf{B}(\rho)\xi = 0$  if there exists a matrix  $\mathbf{X}$  such that:

$$\mathbf{P} + \mathbf{X} \mathbf{B}(\rho) + \mathbf{B}(\rho)^T \mathbf{X}^T < 0 \quad (31)$$

In particular, one chooses  $\mathbf{X} = \begin{bmatrix} F^T \\ \varepsilon_1 F^T \\ 0 \\ 0 \end{bmatrix}$ . Then (31) becomes (32) (top of the next page).

Pre and post-multiplying (32) by  $\mathbf{R}^T$  and  $\mathbf{R} = \text{diag}(F^{-1}, F^{-1}, T^{-1}, I)$ , and denoting  $F^{-1} = M, T(\rho)^{-1} = N(\rho), W = M^T P M, Z(\rho) = K(\rho)M, Y(\rho) = G(\rho)M$ , and noting that  $A_F(\rho) = A(\rho) + B_2 K(\rho)$ , condition (32) becomes (22).

Finally, to ensure that  $x(t)$  belongs effectively to  $\mathcal{S}_\rho(K, G, u_0)$  and that the state constraints are not violated, it must be proven that  $\mathcal{E}(P) \subset \mathcal{S}_\rho(K, G, u_0) \cap \mathcal{X}$ , i.e  $\mathcal{E}(P) \subset \mathcal{S}_\rho(K, G, u_0)$  and  $\mathcal{E}(P) \subset \mathcal{X}$ .

To ensure  $\mathcal{E}(P) \subset \mathcal{S}_\rho(K, G, u_0)$ , we should satisfy:

$$\left[ \begin{array}{c|c} P & (K_i(\rho) - G_i(\rho))^T \\ \hline K_i(\rho) - G_i(\rho) & u_{0i}^2 \end{array} \right] \geq 0, i = 1, \dots, m \quad (33)$$

$$LMI_1 = \left[ \begin{array}{c|c|c|c} \lambda_1 W + A(\rho)M + B_2 Z(\rho) + M^T A(\rho)^T + Z(\rho)^T B_2^T & W + \varepsilon_1 M^T A(\rho)^T + \varepsilon_1 Z(\rho)^T B_2^T - M & -Y(\rho)^T + B_2 N(\rho) & B_1(\rho) \\ \hline W + \varepsilon_1 A(\rho)M + \varepsilon_1 B_2 Z(\rho) - M^T & -\varepsilon_1 M - \varepsilon_1 M^T & \varepsilon_1 B_2 N(\rho) & \varepsilon_1 B_1(\rho) \\ \hline -Y(\rho) + N(\rho)B_2^T & \varepsilon_1 N(\rho)B_2^T & -2N(\rho) & 0 \\ \hline B_1(\rho)^T & \varepsilon_1 B_1(\rho)^T & 0 & -\lambda_2 I \end{array} \right] < 0 \quad (22)$$

$$\left[ \begin{array}{c|c|c|c} \lambda_1 P + F^T A_F(\rho) + A_F(\rho)^T F & P + \varepsilon_1 A_F(\rho)^T F - F^T & -G(\rho)^T T(\rho) + F^T B_2 & F^T B_1(\rho) \\ \hline P + \varepsilon_1 F^T A_F(\rho) - F & -\varepsilon_1 M - \varepsilon_1 M^T & \varepsilon_1 F^T B_2 & \varepsilon_1 F^T B_1(\rho) \\ \hline B_2^T F - T(\rho)G(\rho) & \varepsilon_1 B_2^T F & -2T(\rho) & 0 \\ \hline B_1(\rho)^T F & \varepsilon_1 B_1(\rho)^T F & 0 & -\lambda_2 I \end{array} \right] < 0 \quad (32)$$

Pre and post-multiplying (33) by  $R_1^T$  and  $R_1 = \text{diag}(F^{-1}, I)$ , we obtain:

$$\left[ \begin{array}{c|c} W & (Z_i(\rho) - Y_i(\rho))^T \\ \hline (Z_i(\rho) - Y_i(\rho)) & u_{0i}^2 \end{array} \right] \succeq 0, i = 1, \dots, m \quad (34)$$

To ensure  $\mathcal{E}(P) \subset \mathcal{X}$ , the following should be verified:

$$\left[ \begin{array}{c|c} P & H_i^T \\ \hline H_i & h_{0i}^2 \end{array} \right] \succeq 0, i = 1, \dots, k \quad (35)$$

Pre and post-multiplying (35) by  $R_1^T$  and  $R_1 = \text{diag}(F^{-1}, I)$ , one obtains:

$$\left[ \begin{array}{c|c} W & M^T H_i^T \\ \hline H_i M & h_{0i}^2 \end{array} \right] \succeq 0, i = 1, \dots, k \quad (36)$$

Thus, if inequalities (22-25) are satisfied then it follows that the ellipsoid  $\mathcal{E}(P)$  is an  $\mathcal{W}$ -invariant set.

Now, let us consider the case  $w(t) = 0$ , from (26), it follows:

$\dot{V}(x(t)) \leq -\lambda_1 x^T P x$ . Thus,  $\dot{V}(x(t)) \leq -\lambda_1 V(x(t)) < 0$  i.e.  $V(x(t)) \leq e^{-\lambda_1 t} V(x(0))$ , it means that the trajectories of the system converge asymptotically to the origin.

### B. Disturbance attenuation

As mentioned before, in this work, we consider a control objective regarding the disturbance attenuation for the the unconstrained closed-loop system, i.e. when the saturation is not activated or  $\text{sat}(K(\rho)x) = K(\rho)x$ .

In order to satisfy this control objective, another candidate Lyapunov function is chosen. Let us consider another Lyapunov function  $V(x(t)) = x^T Q x$ . where  $Q$  is a Lyapunov matrix. It is well known that relation (17) is verified if the following condition holds:

$$\dot{V} + z^T z - \gamma^2 w^T w < 0 \quad (37)$$

Now, without the input saturation, the closed loop system (14) becomes:

$$\dot{x} = (A(\rho) + B_2 K(\rho))x + B_1(\rho)w \quad (38)$$

$$z = (C_1(\rho) + D_{12}K(\rho))x + D_{11}(\rho)w \quad (39)$$

Rewriting (38) as:  $\mathbf{B}_p(\rho)\Xi = 0$  where:  $\mathbf{B}_p(\rho) = \begin{bmatrix} A_F(\rho) & -I & B_1(\rho) \end{bmatrix}$  with  $A_F(\rho) = A(\rho) + B_2 K(\rho)$  and  $\Xi = \begin{bmatrix} x^T & \dot{x}^T & w^T \end{bmatrix}^T$ .

Denoting  $C_F(\rho) = C_1(\rho) + D_{12}K(\rho)$ , the condition (37) is rewritten as follows:

$$\dot{x}^T Q x + x^T Q \dot{x} + (C_F x + D_{11} w)^T (C_F x + D_{11} w) - \gamma^2 w^T w < 0$$

$$\dot{x}^T Q x + x^T Q \dot{x} + x^T C_F^T C_F x + w^T D_{11}^T C_F x + x^T C_F^T D_{11} w - w^T (D_{11}^T D_{11} - \gamma^2 I) w < 0 \quad (40)$$

Then the condition (40) is guaranteed if the following inequality holds:

$$\Xi^T Q \Xi < 0 \quad (41)$$

$$\text{where } Q = \left[ \begin{array}{c|c|c} C_F^T C_F & Q & C_F^T D_{11} \\ \hline Q & 0 & 0 \\ \hline D_{11}^T C_F & 0 & D_{11}^T D_{11} - \gamma^2 I \end{array} \right]$$

Applying the Finsler's lemma again, one has:  $\Xi^T Q \Xi < 0, \forall \mathbf{B}_p(\rho)\Xi = 0$  if there exists a matrix  $\mathbf{X}_p$  such that:

$$Q + \mathbf{X}_p \mathbf{B}_p(\rho) + \mathbf{B}_p(\rho)^T \mathbf{X}_p^T < 0 \quad (42)$$

In particular, we choose  $\mathbf{X}_p = \begin{bmatrix} F^T \\ \varepsilon_2 F^T \\ 0 \end{bmatrix}$ , then (42) becomes

(43) (top of next page).

Pre and post-multiplying (43) by  $\mathbf{R}_p^T$  and  $\mathbf{R}_p = \text{diag}(F^{-1}, F^{-1}, I)$  and denoting  $F^{-1} = M, U = M^T Q M, Z(\rho) = K(\rho)M$ , one obtains the condition (44). Using the Schur's lemma and noting that  $A_F(\rho) = A(\rho) + B_2 K(\rho)$  and  $C_F(\rho) = C_1(\rho) + D_{12}K(\rho)$ , (44) becomes (45).

### C. Controller computation

The state feedback gain  $K(\rho)$  that satisfies the stability condition for the saturated system (see section IV.A) and the disturbance attenuation for the unsaturated system (see section IV.B) can be derived by solving the following optimization problem:

$$\begin{aligned} & \min_{W, U, N, M, Z, Y, \lambda_1, \lambda_2, \varepsilon_1, \varepsilon_2} \gamma^2 \\ & \text{subject to} \quad (22, 23, 24, 25, 45), \quad (46) \\ & \quad \quad \quad W, U, N > 0, \lambda_1, \lambda_2 > 0. \end{aligned}$$

Then the state feedback gain matrix  $K(\rho)$  can be computed by:

$$K(\rho) = Z(\rho)M^{-1} \quad (47)$$

It worth noting that these conditons (22, 45) are quasi-LMIs where there exist some coupling between  $W, M, Z$  and scalars  $\lambda_1, \varepsilon_1, \varepsilon_2$ . A feasible solution can be attained by fixing the scalars  $\lambda_1, \varepsilon_1, \varepsilon_2$  and solving the LMI feasibility problem. Moreover, the above optimization problem has an infinite number of LMIs to solve because the varying parameter  $\rho$  varies in the set  $\Omega$ . To relax this problem, the LMI framework

$$\left[ \begin{array}{c|c|c} C_F(\rho)^T C_F(\rho) + F^T A_F(\rho) + A_F(\rho)^T F & Q + \varepsilon_2 A_F(\rho)^T F - F^T & C_F(\rho)^T D_{11}(\rho) + F^T B_1(\rho) \\ \hline Q + \varepsilon_2 F^T A_F(\rho) - F & -\varepsilon_2 F - \varepsilon_2 F^T & \varepsilon_2 F^T B_1(\rho) \\ \hline D_{11}(\rho)^T C_F(\rho) + B_1(\rho)^T F & \varepsilon_2 B_1(\rho)^T F & D_{11}(\rho)^T D_{11}(\rho) - \gamma^2 I \end{array} \right] < 0 \quad (43)$$

$$\left[ \begin{array}{c|c|c} M^T C_F(\rho)^T C_F(\rho) M + M^T A_F(\rho)^T + A_F(\rho) M & U + \varepsilon_2 M^T A_F(\rho)^T - M & M^T C_F(\rho)^T D_{11}(\rho) + B_1(\rho) \\ \hline U + \varepsilon_2 A_F(\rho) M - M^T & -\varepsilon_2 M - \varepsilon_2 M^T & \varepsilon_2 B_1(\rho) \\ \hline D_{11}(\rho)^T C_F(\rho) M + B_1(\rho)^T & \varepsilon_2 B_1(\rho)^T & D_{11}(\rho)^T D_{11}(\rho) - \gamma^2 I \end{array} \right] < 0 \quad (44)$$

$$\left[ \begin{array}{c|c|c|c} A(\rho)M + B_2 Z(\rho) + M^T A(\rho)^T + Z(\rho)^T B_2^T & U + \varepsilon_2 M^T A(\rho)^T + \varepsilon_2 Z(\rho)^T B_2^T - M & B_1(\rho) & M^T C_1(\rho)^T + Z(\rho)^T D_{12}^T \\ \hline U + \varepsilon_2 A(\rho)M + \varepsilon_2 B_2 Z(\rho) - M^T & -\varepsilon_2 M - \varepsilon_2 M^T & \varepsilon_2 B_1(\rho) & 0 \\ \hline B_1(\rho)^T & \varepsilon_2 B_1(\rho)^T & -\gamma^2 I & D_{11}(\rho)^T \\ \hline C_1(\rho)M + D_{12} Z(\rho) & 0 & D_{11}(\rho) & -I \end{array} \right] < 0 \quad (45)$$

for the polytopic system is used, i.e we will solve the optimization problem at each vertex  $\mathfrak{S}_\rho^j$  of the polytope defined by the bounds of the varying parameters to obtain a state feedback matrix gain  $K_j$  at each vertex. Then, considering the measured value of  $\rho$ , the parameter dependent state feedback matrix  $K(\rho)$  is computed as follows:

$$K(\rho) = \sum_{j=1}^{2^k} \alpha_j(\rho) K_j, \quad \sum_{j=1}^{2^k} \alpha_j(\rho) = 1.$$

where  $\alpha_j(\rho) := \frac{\prod_{i=1}^k |\rho_i - \text{Compl}(\omega_j)_i|}{\prod_{i=1}^k (\bar{\rho}_i - \underline{\rho}_i)}$ ,  $\rho_i \in [\underline{\rho}_i \ \bar{\rho}_i]$  and

$$\sum_{j=1}^{2^k} \alpha_j(\rho) = 1, j = 1, \dots, 2^k;$$

$$\text{Compl}(\omega_j)_i := \{\bar{\rho}_i \text{ if } (\omega_j)_i = \underline{\rho}_i \text{ or } \underline{\rho}_i \text{ if } (\omega_j)_i = \bar{\rho}_i\}.$$

## V. SIMULATION RESULTS

To validate the proposed LPV state feedback input constrained control, simulations are performed on a Renault Mégane Coupé (RMC) quarter car model using the semi-active suspension. The model parameters are given in the following table:

Parameter	$m_s$ [kg]	$m_{us}$ [kg]	$k_t$ [N/m]	$k_s$ [N/m]	$c$ [Nm/s]	$c_t$ [Nm/s]
Value	315	37.5	29500	210000	{700, 5000}	100

TABLE I

QUARTER-CAR MODEL PARAMETERS

Then, the vehicle is assumed to run on a typical road profile with a bump of the following form:

$$z_r(t) = \begin{cases} \frac{A}{2} (1 - \cos(\frac{2\pi V}{L} t)) & \text{if } 0 \leq t \leq \frac{L}{V} \\ 0 & \text{otherwise} \end{cases}$$

where  $A$  and  $L$  the bump height and bump length,  $V$  the vehicle velocity. In this study, a bump is characterized by  $A = 0.1 \text{ m}$ ,  $L = 5 \text{ m}$  and the vehicle runs at  $V = 7.5 \text{ m/s}$ .

Fig. 2 shows the road profile and its velocity  $\dot{z}_r$  that is considered as the input disturbance  $w = \dot{z}_r$  of the system. Then, the input disturbance  $w$  satisfies condition (10):  $w^T w < \delta$  where  $\delta = 0.25$ .

The control input condition (6) is given by:  $|u^{H_\infty}| < 2150$ . The LPV State feedback  $K(\rho)$ , derived from the optimisation

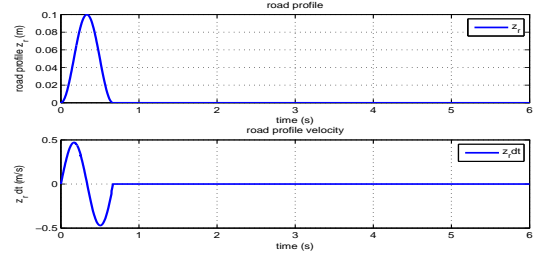


Fig. 2. Road disturbance  $z_r$  and its derivative  $z_r,dt$

problem (46), satisfies input and state constraints (6, 7), and improves the passenger comfort.

To demonstrate the efficiency of the proposed approach, we show a comparison between this approach (assumed to be called *LPV Finsler*) and another LPV controller using only one Lyapunov function for both objectives (stability and disturbance attenuation) and Bounded Real Lemma (our recently work submitted in the 8th IFAC Symposium on Robust Control Design Rocond2015, and assumed to be called *LPV Rocond*).

At first, by solving the optimisation problem (46), we obtained the value of  $\mathcal{L}_2$  gain in two cases as follows:  $\gamma_{Finsler} = 16.6685$  and  $\gamma_{Rocond} = 17.7322$ . It means that the proposed approach allows to reduce the conservatism.

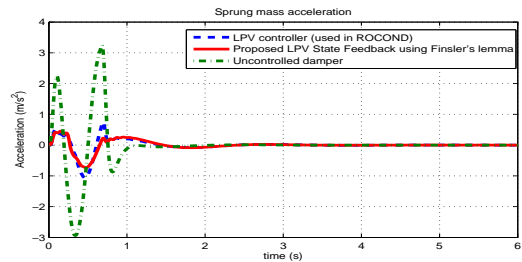


Fig. 3. Sprung mass acceleration

Fig. 4 that shows the control input applied to the system. Note that from 0–1s, the control input is saturated. In Fig. 5, the suspension deflection and its speed are depicted. This figure shows that the state constraints are not violated.

Then, Fig. (3) shows the acceleration of the sprung mass in three cases: the semi-active damper is controlled by the 'LPV

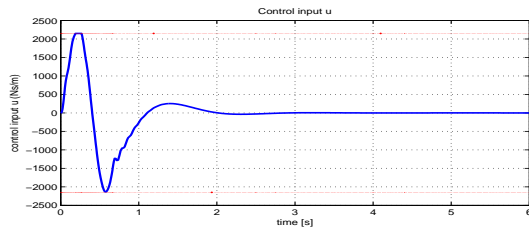


Fig. 4. Control input  $u$

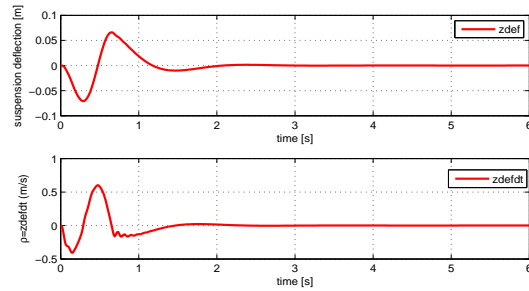


Fig. 5. Deflection ( $z_{def}$ ) and deflection speed ( $\dot{z}_{def}$ ) constraints

Finsler', by 'LPV ROCOND', and uncontrolled damper (where  $c_{damper} = c_0$  and  $u^{H_\infty} = 0$ ). It can be seen that the acceleration in the controlled case is reduced considerably and so it allows to improve the passenger comfort which is the control objective.

## VI. CONCLUSION

In this work, a LPV state feedback control is designed for the semi-active suspension control problem in order to ensure the stability in case of saturation and to improve the passenger comfort. Moreover, thanks to Finsler's lemma, the proposed approach allows to use different Lyapunov functions for multi-objective problems which allows to reduce the conservatism. The simulation results show the effectiveness of this approach: the stability is kept in case of saturated input, the state constraints are not violated and the minimization of disturbance effects. For the future works, more performance objectives could be considered (road holding,...). Moreover, the use of different parameter dependant Lyapunov function could be a next step to improve more the controller.

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