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Predictor-based stabilization of finite-dimensional linear autonomous control systems with a constant input delay

Delphine Bresch-Pietri*, Christophe Prieur* Emmanuel Trélat†

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Abstract

We focus on the design of predictor-based feedbacks for finite-dimensional linear autonomous control systems with a constant input delay, with the Artstein reduction approach. This well-known method consists of reducing the delayed control system, by using an integral transform involving the state and the control, to a usual linear system without input delay, on which one can design a stabilizing feedback with usual approaches. In this paper we propose a complete analysis of this approach which is twofold. First of all we invert the Artstein transform and, as a result, we infer a Lyapunov functional for the linear control system with delay. Secondly, we show how the method can be implemented in a simple way.

1 Introduction

Many dynamical models are subject to input delay. Examples include those describing industrial process (see e.g., [4]), systems controlled through a network [15, 12], and automotive engines [17, 7]. Delay in control loops can be source of performance degradation, and even of instability if the controller has been designed by neglecting this delay, see [14, 5] for introductions of time-delay systems. To prevent these undesirable effects, predictor-based design approaches have been introduced in [1], see also [11, 13]. The basic idea of this approach is to close the loop with a system state prediction instead of the value of current state. This method has been first introduced for linear delay constant control systems (as those considered in this paper), but it has been further developed for nonlinear plants [10, 3] and for non-constant delays [2].

Focusing on (maybe unstable) linear control systems with constant input delay, using a Laplace transform approach, it is well-known that a necessary and sufficient condition ensuring exponential asymptotic stability is that all complex zeros of the transfer function Δ have negative real parts (see [6], where a Lyapunov functional is moreover provided). It can be noted that Δ has at most a finite number of zeros having a positive real part. In spite of that, the problem of designing a linear controller such that, for the closed-loop system, the analytic equation $\Delta(\lambda) = 0$ (which has an infinite number of solutions) has no zeros in the right-hand side of the complex plane is far from obvious.

In the existing literature one can find a number of sufficient conditions ensuring the exponential stability property (see [18], or see [14] for a survey). All the above conditions are sufficient but not necessary. They have the advantage to be numerically tractable since the sufficient conditions

*GIPSA-lab, 11 rue des Mathématiques, BP 46, 38402 Saint-Martin d'Hères, France (delphine.bresch-pietri@gipsa-lab.fr, Christophe.Prieur@gipsa-lab.fr).

†Université Pierre et Marie Curie (Univ. Paris 6) and Institut Universitaire de France and Team GECO Inria Saclay, CNRS UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (emmanuel.trelat@upmc.fr).

are often written in terms of Linear Matrix Inequalities (LMIs). The LMI conditions impose some restrictions on the matrices of the system or on the value of the delay which cannot be arbitrarily large, although they can be useful for many applications since checking LMIs is a numerically tractable numerical problem.

The first objective of this paper is to re-start from the so-called Artstein approach and to give a new expression of the predictor based controller. This expression is obtained by solving an implicit Volterra equation, and yields a controller that is a functional of the current and past value of the state. Therefore our first result can be seen as alternative definition of the predictor based controller, but simpler since it is given directly in terms of the state variable and is not based on the usual Artstein transform. The second objective of this approach is to emphasize that this new expression has not only a theoretical interest, but it is also useful for practical implementation. Indeed the truncation of the series defining the designed controller is studied, together with its impact in terms of the stability.

This paper is organized as follows. In Section 2, the main results are stated, namely the definition of the controller in terms of the system state directly, and the inversion of the Artstein reduction (see Theorems 1 and 2). The stability of the closed-loop system follows (see Corollary 1). The practical implementation and the impact of the truncation in the definition of the stabilizing controller are given in Section 3, together with some numerical simulations illustrating the results. Section 4 collects the proofs of all results, whereas some concluding remarks are given in Section 5.

2 Main results

2.1 Artstein's approach

In this section, we recall Artstein's approach and we provide a controller and the corresponding Lyapunov function. We consider the finite-dimensional linear control system with constant delay

$$\dot{x}(t) = Ax(t) + Bu(t - D), \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, A is a real matrix of size $n \times n$ and B is a real matrix of size $n \times m$. In order to stabilize the control system (1), let us consider the so-called Artstein model reduction (see [1], see also [11, 13, 14]), we set, for every $t \in \mathbb{R}$,

$$z(t) = x(t) + \int_{t-D}^t e^{(t-s-D)A} Bu(s) ds = x(t) + \int_t^{t+D} e^{(t-s)A} Bu(s - D) ds, \quad (2)$$

and we get immediately, from an easy computation,

$$\dot{z}(t) = Az(t) + e^{-DA} Bu(t), \quad (3)$$

which is a usual linear control system, without input delay. Therefore, assuming controllability of the pair $(A, e^{-DA}B)$, this leads to the natural control choice ¹

$$u(t) = K_D z(t) = K_D \left(x(t) + \int_{t-D}^t e^{(t-s-D)A} Bu(s) ds \right) \quad (4)$$

¹Note that this approach is formally equivalent to ones considering a pole placement in terms of the original dynamics matrices A, B as it is done, e.g., in [9]. In details, choosing K_D such that $A + e^{-DA}BK_D$ is Hurwitz is equivalent to choosing K such that $A + BK$ is Hurwitz and the two gains can actually be explicitly related as $K = K_D e^{-DA}$.

in which the gain matrix K_D is chosen such that $A + e^{-DA}BK_D$ is Hurwitz. Then, by construction $t \mapsto z(t)$ converges exponentially to 0, and hence both $t \mapsto u(t)$ and $t \mapsto \int_{t-D}^t e^{(t-D-s)A}Bu(s) ds$ converge exponentially to 0 as well. Then the equality (2) implies that $t \mapsto x(t)$ converges exponentially to the origin.

Theoretically, the predictor-based control (4) stabilizes exponentially the retarded control system (1), whatever the value of the delay D may be, and without any restriction on the matrices of the system. This is in contrast with the sufficient conditions mentioned in the previous section, however the feedback (4) is more complicated than a classical one $u(t) = Kx(t)$ since it involves an integral term depending on the past values of u .

From the elements provided above, one can straightforwardly construct a Lyapunov functional. Indeed, assuming that the pair (A, B) satisfies the Kalman condition, the well-known pole-shifting theorem and Lyapunov theorem imply the existence of a stabilizing gain matrix [8, 16].

Lemma 1. *For every $D \geq 0$ there exists a gain matrix K_D of size $m \times n$ such that $A + Be^{-DA}K_D$ admits -1 as an eigenvalue with order n . Moreover there exists a symmetric positive definite matrix P_D of size $n \times n$ such that*

$$P_D (A + Be^{-DA}K_D) + (A + e^{-DA}BK_D)^\top P_D = -I_n. \quad (5)$$

In particular, the function

$$V(z) = \frac{1}{2} z^\top P_D z \quad (6)$$

is a Lyapunov function for the closed-loop system $\dot{z}(t) = (A + e^{-DA}BK_D)z(t)$.

Remark 1. From Lemma 1 we infer that for every $D \geq 0$ there exists $C_D > 0$ (depending smoothly on D) such that

$$\frac{d}{dt} V(z(t)) = -\|z(t)\|^2 \leq -C_D V(z(t)), \quad (7)$$

where $\|\cdot\|$ is the usual Euclidean norm in \mathbb{R}^n .

2.2 Inversion of the Artstein transform

We now invert the Artstein transform (2), and in particular we will infer from our study a Lyapunov function depending only on $x(\cdot)$ and involving integral terms. In the next section, we are going to investigate the implementation aspects of the stabilization through Artstein's approach and show that the method can be implemented in a very simple way.

By inverting the Artstein transform, we mean that we are going to solve the fixed point implicit equality (4). For every function f defined on \mathbb{R} and locally integrable, we define

$$(T_D f)(t) = K_D \int_{\max(t-D, 0)}^t e^{(t-D-s)A} B f(s) ds.$$

It follows that (4) can thus be rewritten as $u(t) = K_D x(t) + (T_D u)(t)$, for every $t \geq D$. Actually, one can obtain an explicit reformulation of this relation as given in the following theorem (proved in Section 4.1).

Theorem 1. *There holds*

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ \sum_{j=0}^{+\infty} (T_D^j K_D x)(t) & \text{if } t \geq 0, \end{cases} \quad (8)$$

and the series is convergent, whatever the value of the delay $D \geq 0$ may be.

Note that the value of the feedback u at time t ,

$$\begin{aligned} u(t) &= K_D x(t) + K_D \int_{\max(t-D,0)}^t e^{(t-D-s)A} B K_D x(s) ds \\ &\quad + K_D \int_{\max(t-D,0)}^t e^{(t-D-s)A} B K_D \int_{\max(s-D,0)}^s e^{(s-D-\tau)A} B K_D x(\tau) d\tau ds \\ &\quad + \dots \end{aligned}$$

depends on the past values of x over the time interval $(0, t)$.

We are next going to express z as a function of x , that is, to invert the equality

$$z(t) = x(t) + \int_{(t-D,t) \cap (0,+\infty)} e^{(t-s-D)A} B K_D z(s) ds \quad (9)$$

coming from (2) and (4). Although it is technical and not directly useful to derive the exponential stability of z , it will however allow us to express the Lyapunov functional V defined by (6), to be compared with those existing in the literature. Note that

$$(t-D, t) \cap (0, +\infty) = \begin{cases} \emptyset & \text{if } t < 0, \\ (0, t) & \text{if } 0 < t < D, \\ (t-D, t) & \text{if } D < t. \end{cases} \quad (10)$$

In particular if $t < 0$ then $z(t) = x(t)$. Actually we have the following precise result (proved in Section 4.2).

Theorem 2. *For every $t \in \mathbb{R}$, there holds*

$$x(t) = z(t) - \int_0^t \Phi_D(t, s) x(s) ds \quad (11)$$

where Φ_D is defined as, for $(t, s) \in \mathbb{R}^2$,

$$\Phi_D(t, s) = f_{\lfloor \frac{t-s}{D} \rfloor} \left(t - s - \lfloor \frac{t-s}{D} \rfloor D \right), \quad (12)$$

denoting $\lfloor \cdot \rfloor$ the integer part of a real number, and the sequence of functions $f_i : [0, D] \rightarrow \mathcal{M}_n(\mathbb{R})$ is defined as follows:

- f_0 is the solution of the fixed-point equation

$$f_0(r) = \tilde{f}(r) + (\tilde{T}_0 f_0)(r), r > 0 \quad (13)$$

with, for $r > 0$,

$$\begin{aligned} \tilde{f}(r) &= e^{(r-D)A} B K_D \\ (\tilde{T}_0 f_0)(r) &= \int_0^r e^{(r-\tau-D)A} B K_D f_0(\tau) d\tau \end{aligned}$$

- for $i \in \mathbb{N}$, f^{i+1} is the solution of the fixed-point equation

$$f_{i+1}(r) = (\psi f_i)(r) + (\tilde{T}_D f_{i+1})(r), r > 0 \quad (14)$$

with, for $r > 0$,

$$\begin{aligned}(\psi f_i)(r) &= \int_r^D e^{(r-\tau)A} BK_D f_i(\tau) d\tau \\ (\tilde{T}_D f_{i+1})(r) &= \int_0^r e^{(r-\tau-D)A} BK_D f_{i+1}(\tau) d\tau\end{aligned}$$

With this expression in mind, using (9), it follows that the feedback u defined by (4) can be as well written as

$$\begin{aligned}u(t) &= \chi_{(0,+\infty)}(t) K_D z(t) \\ &= \chi_{(0,+\infty)}(t) K_D x(t) + K_D \int_0^t \Phi_D(t, s) x(s) ds\end{aligned}$$

and we recover an expression which is consistent with the one provided in (8) and which was derived in Theorem 1.

Plugging this feedback into the control system (1) yields, for $t > 0$, the closed-loop system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t - D) \\ &= Ax(t) + BK_D x(t - D) + BK_D \int_0^{t-D} \Phi_D(t - D, s) x(s) ds\end{aligned}\tag{15}$$

which is, as said above, exponentially stable. This can be shown independently from the arguments previously used, using only the definitions and properties introduced in Theorem 2.

Corollary 1. *The variable z defined through (11) as*

$$z(t) = x(t) + \int_0^t \Phi_D(t, s) x(s)$$

satisfies

$$\dot{z} = (A + e^{-AD} BK_D) z$$

Further, z and x converge exponentially to the origin.

Proof. We start by observing that, due to the definition of Φ_D , one can write

$$\begin{aligned}z(t) &= x(t) + \int_0^t \Phi_D(t, s) x(s) ds \\ &= x(t) + \sum_{i=1}^{\lfloor \frac{t}{D} \rfloor} \int_{\max\{0, t-(i+1)D\}}^{t-iD} f_i(t-s-iD) x(s) ds + \int_{t-D}^t f_0(t-s) x(s) ds\end{aligned}$$

Taking a time-derivative of this last expression, one gets

$$\begin{aligned}\dot{z}(t) &= Ax(t) + Bu(t - D) + f_0(0)x(t) + \sum_{i=0}^{\lfloor \frac{t}{D} \rfloor - 1} [f_{i+1}(0) - f_i(D)] x(t - iD) \\ &\quad + \sum_{i=1}^{\lfloor \frac{t}{D} \rfloor} \int_{\max\{0, t-(i+1)D\}}^{t-iD} \dot{f}_i(t-s-iD) x(s) ds + \int_{t-D}^t \dot{f}_0(t-s) x(s) ds\end{aligned}$$

From the definition of (14), it follows that $f_{i+1}(0) = f_i(D)$ for $i \in \mathbb{N}^*$ and, from (13), that $f_0(0) = e^{-AD}BK_D$. Besides, using again these two fixed-point equations, one gets

$$\begin{aligned} f_1(0) &= \int_0^D e^{-\tau A} BK_D f_0(\tau) d\tau \\ f_0(D) &= BK_D + \int_0^D e^{-\tau A} BK_D f(\tau) d\tau \end{aligned}$$

and therefore concludes that $f_1(0) - f_0(D) = -BK_D$. Further, differentiating the two fixed-point equations satisfied by f_i and f_0 , one straightforwardly obtains

$$\dot{f}_0(r) = (A + e^{-AD}BK_D)f_0(r) \quad (16)$$

$$\dot{f}_{i+1}(r) = (A + e^{-AD}BK_D)f_{i+1}(t, s) - BK_D f_i(r) \quad (17)$$

for $r > 0$. Hence, it follows

$$\begin{aligned} \dot{z}(t) &= (A + e^{-AD}BK_D)z(t) + Bu(t - D) - BK_D x(t - D) \\ &\quad - BK_D \sum_{i=1}^{\lfloor \frac{t}{D} \rfloor} \int_{\max\{0, t-(i+1)D\}}^{t-iD} f_{i-1}(t - s - iD)x(s) ds. \end{aligned}$$

As the choice of the control law yields

$$\begin{aligned} Bu(t - D) &= BK_D z(t - D) \\ &= BK_D x(t - D) + BK_D \sum_{i=0}^{\lfloor \frac{t-D}{D} \rfloor} \int_{\max\{0, t-(i+2)D\}}^{t-(i+1)D} f_i(t - D - s - iD)x(s) ds \\ &= BK_D x(t - D) + BK_D \sum_{i=1}^{\lfloor \frac{t}{D} \rfloor} \int_{\max\{0, t-(i+1)D\}}^{t-iD} f_{i-1}(t - s - iD)x(s) ds \end{aligned}$$

it follows that

$$\dot{z}(t) = (A + e^{-AD}BK_D)z(t)$$

In turn, the Lyapunov function (in the z variable) V can be written as

$$V(t) = \frac{1}{2} \left(x(t) + \int_0^t \Phi_D(t, s)x(s) ds \right)^\top P_D \left(x(t) + \int_0^t \Phi_D(t, s)x(s) ds \right). \quad (18)$$

which provides

$$\dot{V}(t) = - \left(x(t) + \int_0^t \Phi_D(t, s)x(s) ds \right)^\top \left(x(t) + \int_0^t \Phi_D(t, s)x(s) ds \right) \leq -\frac{1}{\lambda(P)} V(t) \leq 0$$

and implies the exponential convergence of z . Now, using the triangle inequality, one obtains

$$|x(t)| \leq |z(t)| + \left| \int_0^t \Phi_D(t, s)x(s) ds \right|$$

Consider $t \in [0, D]$. Then, using (16), one obtains

$$|\Phi_D(t, s)| \leq e^{-\frac{t-s}{2\bar{\lambda}(P)}}, \quad 0 \leq s \leq t$$

and then the previous inequality rewrites

$$|x(t)| \leq \left[2\bar{\lambda}(P) \left(1 - e^{-\frac{t}{2\bar{\lambda}(P)}} \right) + e^{-\frac{t}{2\bar{\lambda}(P)}} \right] \max_{s \in [0, t]} |x(s)|$$

One can choose freely the eigenvalue $\bar{\lambda}(P)$ by choosing K_D appropriately. In particular, one can choose $\bar{\lambda}(P) > 1$, which guarantees the existence of $\epsilon \in (0, 1)$ such that

$$|x(D)| \leq (1 - \epsilon) \max_{s \in [0, D]} |x(s)|$$

By direct iteration on time intervals of length D , one obtains the exponential decay of x . \square

3 Practical implementation and numerical simulations

3.1 Preliminary

Seeking implementation, it is worth studying the impact of truncation of the infinite sum appearing in the control law (8). With this aim in view, we consider now

$$u(t) = \sum_{j=0}^N (T_D^j K_D x)(t) \quad (19)$$

for a certain integer $N \in \mathbb{N}$. We have the following finite-time result.

Proposition 1. *Consider the closed-loop system consisting of the plant (1) and the control law (19). Define $\varepsilon > 0$ and $T > 0$. There exist $N^* \in \mathbb{N}$ and $R, \rho > 0$ such that, if $N > N^*$, the following holds*

$$\|x(t)\|^2 \leq (\varepsilon + R e^{-\rho t}) \|x(0)\|^2, \quad t \in [0, T]. \quad (20)$$

3.2 Numerical example

Let us consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t - h)$$

where A is the unstable matrix defined by $A = 0.01 \begin{pmatrix} 10 & 10 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $D = 1$ is a constant delay. The gain matrix $K_D = (-1.66 \quad -31.4)$ is chosen such that the closed-loop eigenvalues are -0.5 and -0.6 respectively.

Let us compute the feedback law from Theorem 1 with the initial condition $x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. It has been computed with $N = 100$ in (19). The time-evolutions of the state and of the control are given in Figures 1 and 2, where it can be checked that the state converges to the origin.

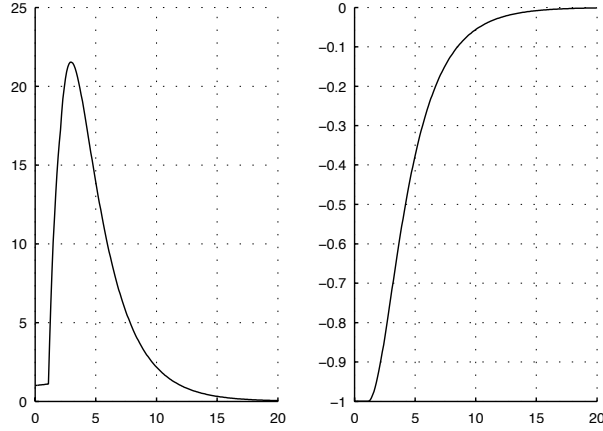


Figure 1: Time-evolution of the state component x_1 (left) and x_2 (right).

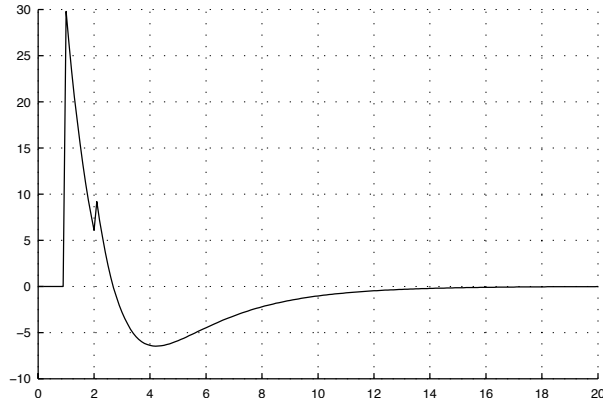


Figure 2: Time-evolution of the control u given by Theorem 1.

4 Proofs

4.1 Proof of Theorem 1

We define the functions φ_{D_j} iteratively by

$$\begin{aligned} \varphi_{D_1}(t, \tau) &= 1, \\ \varphi_{D_{j+1}}(t, \tau) &= \int_{\max(\tau, t-D)}^{\min(t, \tau+jD)} \varphi_{D_j}(s, \tau) ds, \quad j \in \mathbb{N}^*, \end{aligned} \tag{21}$$

for every $t \geq \tau$, and by $\varphi_{D_j}(t, \tau) = 0$ if $t < \tau$ and $j \in \mathbb{N}$.

Let us prove by induction that

$$\left| (T_D^j K_D x)(t) \right| \leq \|B\|^j \|K_D\|^{j+1} \int_{\max(t-jD, 0)}^t \varphi_{D_j}(t, \tau) e^{(t-jD-\tau)\|A\|} \|x(\tau)\| d\tau, \tag{22}$$

for every $j \in \mathbb{N}^*$. This is clearly true for $j = 1$, since

$$\begin{aligned} |(T_D K_D x)(t)| &= \left| K_D \int_{\max(t-D,0)}^t e^{(t-D-s)A} B K_D x(s) ds \right| \\ &\leq \|B\| \|K_D\|^2 \int_{\max(t-D,0)}^t e^{(t-D-s)\|A\|} \|x(s)\| ds. \end{aligned}$$

Assume that this is true for an integer $j \in \mathbb{N}^*$, and let us derive the estimate for $j + 1$. Since

$$(T_D^{j+1} K_D x)(t) = K_D \int_{\max(t-D,0)}^t e^{(t-D-s)A} B (T_D^j K_D x)(s) ds,$$

we get

$$\begin{aligned} |(T_D^{j+1} K_D x)(t)| &\leq \|B\|^{j+1} \|K_D\|^{j+2} \times \\ &\times \int_{\max(t-D,0)}^t e^{(t-D-s)\|A\|} \int_{\max(s-jD,0)}^s \varphi_{D_j}(s, \tau) e^{(s-jD-\tau)\|A\|} \|x(\tau)\| d\tau ds, \end{aligned}$$

and, from the Fubini theorem, noting that (τ, s) is such that

$$\max(s - jD, 0) \leq \tau \leq s, \quad \max(t - D, 0) \leq s \leq t,$$

if and only if

$$\max(t - (j + 1)D, 0) \leq \tau \leq t, \quad \max(\tau, t - D) \leq s \leq \min(t, \tau + jD),$$

we get the estimate

$$\begin{aligned} |(T_D^{j+1} K_D x)(t)| &\leq \|B\|^{j+1} \|K_D\|^{j+2} \times \\ &\int_{\max(t-(j+1)D,0)}^t \left(\int_{\max(\tau, t-D)}^{\min(t, \tau+jD)} \varphi_{D_j}(s, \tau) ds \right) e^{(t-(j+1)D-\tau)\|A\|} \|x(\tau)\| d\tau, \end{aligned}$$

and the desired estimate for $j + 1$ follows by definition of $\varphi_{D_{j+1}}$.

Now, we claim that

$$0 \leq \varphi_{D_j}(t, \tau) \leq \frac{(t - \tau)^{j-1}}{(j - 1)!}, \quad (23)$$

for every $j \in \mathbb{N}^*$. Indeed, the nonnegativity is obvious, and the right-hand side estimate easily follows from the fact that $\varphi_{D_{j+1}}(t, \tau) \leq \int_{\tau}^t \varphi_{D_j}(s, \tau) ds$ and from a simple iteration argument.

Finally, from (22) and (23), we infer that

$$\begin{aligned} |(T_D^j K_D x)(t)| &\leq \|B\|^j \|K_D\|^{j+1} \int_{\max(t-jD,0)}^t \frac{(t - \tau)^{j-1}}{(j - 1)!} e^{(t-jD-\tau)\|A\|} \|x(\tau)\| d\tau \\ &\leq \|B\|^j \|K_D\|^{j+1} \frac{t^j}{(j - 1)!} \max_{0 \leq s \leq t} \|x(s)\|, \end{aligned}$$

whence the convergence of the series in (8).

4.2 Proof of Theorem 2

Let us search the kernel Φ_D such that there holds

$$x(t) = z(t) - \int_{-\infty}^t \Phi_D(t, s)x(s) ds,$$

postulating that $\Phi_D(t, s) = 0$ whenever $s > t$. When defining Φ_D in the sequel, we do not consider sets of null Lebesgue measure, since it does not impact the integral in (11). Namely, in the following, we omit to define Φ_D for $t - s = nD$, $n \in \mathbb{N}$. Using (9) we must have, for all $t \in \mathbb{R}$,

$$\int_{-\infty}^t \Phi_D(t, s)x(s) ds = \int_{(t-D, t) \cap (0, +\infty)} e^{(t-s-D)A} BK_D \left(x(s) + \int_{-\infty}^s \Phi_D(s, \tau)x(\tau) d\tau \right) ds.$$

We have already noted, from (9), that, for $t < 0$, $z(t) = x(t)$, and hence $\Phi_D(t, s) = 0$ for $t < 0$ and $s \in \mathbb{R}$. Hence in what follows we assume that $t > 0$. Using the Fubini theorem, the previous equation rewrites

$$\begin{aligned} \int_{-\infty}^t \Phi_D(t, s)x(s) ds &= \int_{\max(t-D, 0)}^t e^{(t-s-D)A} BK_D x(s) ds \\ &\quad + \int_{-\infty}^t \int_{\max(t-D, 0, s)}^t e^{(t-\tau-D)A} BK_D \Phi_D(\tau, s) d\tau x(s) ds. \end{aligned}$$

Since we would like this equality to hold true for every x , there must hold

$$\begin{aligned} \Phi_D(t, s) &= e^{(t-s-D)A} BK_D \chi_{(\max(t-D, 0), t)}(s) \\ &\quad + \int_{\max(t-D, 0, s)}^t e^{(t-\tau-D)A} BK_D \Phi_D(\tau, s) d\tau \end{aligned} \quad (24)$$

Let us now solve the implicit equation (24), following two cases depending on the value of t .

1. First of all, if $0 < t < D$ then $\max(t - D, 0) = 0$ and (24) yields

$$\Phi_D(t, s) = e^{(t-s-D)A} BK_D \chi_{(0, t)}(s) + \int_{\max(s, 0)}^t e^{(t-\tau-D)A} BK_D \Phi_D(\tau, s) d\tau. \quad (25)$$

There are two subcases for the value of s .

- (a) If $s < 0$ or if $s > t$ then clearly $\Phi_D(t, s) = 0$ is a solution.
- (b) If $0 < s < t$ then

$$\begin{aligned} \Phi_D(t, s) &= e^{(t-s-D)A} BK_D + \int_s^t e^{(t-\tau-D)A} BK_D \Phi_D(\tau, s) d\tau \\ &= e^{(t-s-D)A} BK_D + \int_0^{t-s} e^{(t-s-\tau-D)A} BK_D \Phi_D(\tau + s, s) d\tau \end{aligned}$$

and then setting $r = t - s$ (note that $0 < r < t < D$) we search $\Phi_D(t, s) = f_0(r)$ with

$$f_0(r) = e^{(r-D)A} BK_D + \int_0^r e^{(r-\tau-D)A} BK_D f_0(\tau) d\tau,$$

that is, $f_0(r) = \tilde{f}(r) + (\tilde{T}_0 f_0)(r)$ with

$$f_0(r) = e^{(r-D)A} B K_D,$$

and

$$(\tilde{T}_0 f_0)(r) = \int_0^r e^{(r-\tau-D)A} B K_D f_0(\tau) d\tau.$$

Formally, we get

$$\begin{aligned} f_0(r) &= \sum_{j=0}^{+\infty} (\tilde{T}_0^j \tilde{f})(r) \\ &= e^{(r-D)A} B K_D + \int_0^r e^{(r-\tau-D)A} B K_D e^{(\tau-D)A} B K_D d\tau \\ &\quad + \int_0^r e^{(r-\tau-D)A} B K_D \int_0^\tau e^{(\tau-s-D)A} B K_D e^{(s-D)A} B K_D ds d\tau + \dots \end{aligned} \quad (26)$$

for every $r \in (0, D)$. The convergence of the series follows from the estimate

$$|(\tilde{T}_0^j f_0)(r)| \leq \|B\|_{\mathbf{R}^n}^{j+1} \|K_D\|_{\mathbf{R}^n}^{j+1} \frac{r^j}{j!} e^{(r-jD)\|A\|},$$

which is straightforward to establish by induction.

2. If $t > D$ then $\max(t - D, 0) = t - D$ and (24) yields

$$\Phi_D(t, s) = e^{(t-s-D)A} B K_D \chi_{(t-D, t)}(s) + \int_{\max(s, t-D)}^t e^{(t-\tau-D)A} B K_D \Phi_D(\tau, s) d\tau$$

and we have $\lfloor \frac{t}{D} \rfloor + 2$ subcases for the value of s .

- (a) If $s < 0$ or if $s > t$ then clearly $\Phi_D(t, s) = 0$ is a solution.
- (b) If $t - D < s < t$ then, following the exact same arguments as previously, one can show that $\Phi_D(t, s) = f_0(t - s)$ in which f_0 has been previously introduced as the solution of fixed-point equation $f_0 = \tilde{f} + \tilde{T}_0 f_0$ and can be defined as in (26).
- (c) If $t - 2D < s < t - D$, then

$$\Phi_D(t, s) = \int_{t-D}^t e^{(t-\tau-D)A} B K_D \Phi_D(\tau, s) d\tau$$

and, from the previous subcase, one obtains

$$\Phi_D(t, s) = \int_{t-D}^{s+D} e^{(t-\tau-D)A} B K_D f_0(\tau - s) d\tau + \int_{s+D}^t e^{(t-\tau-D)A} B K_D \Phi_D(\tau, s) d\tau \quad (27)$$

Define $\Phi_D(t, s) = f_1(t - s - D)$ and $r = t - s - D \in [0, D]$. Then, (27) rewrites

$$f_1(r) = \int_r^D e^{(r-\xi)A} B K_D f_0(\xi) d\xi + \int_0^r e^{(r-\xi-D)A} B K_D f_1(\xi) d\xi \quad (28)$$

and, thus, f_1 is the solution of the following fixed-point equations

$$f_1(r) = (\psi f_0)(r) + (\tilde{T}_D f_1)(r), r > 0 \quad (29)$$

with, for $r > 0$,

$$\begin{aligned}(\psi f_0)(r) &= \int_r^D e^{(r-\xi)A} B K_D f_0(\xi) d\xi \\(\tilde{T}_D f_1)(r) &= \int_0^r e^{(r-\xi-D)A} B K_D f_1(\xi) d\xi\end{aligned}$$

in which f_0 has been previously introduced. Solving the fixed-point equation (29) as previously, one gets that $\Phi_D(t, s) = f_1(t - s - D) = \sum_{j=0}^{\infty} (\tilde{T}_D f_0)(t - s - D)$ for $t - 2D < s < t - D$. The convergence of this last series can be obtained with arguments similar to those previously used.

- (d) There remains $\lfloor \frac{t}{D} \rfloor - 1$ subcases for the value of s . Those can be straightforwardly investigated with the same arguments and an iteration procedure, as, in particular, the implicit equation (14) which is obtained does not depend on the index $i \in \mathbb{N}$. This concludes the definition of Φ_D and the proof of Theorem 2.

4.3 Proof of Proposition 1

Following the previous considerations, the closed-loop system consisting of the plant (1) and the control law (19) is as follows

$$\dot{x}(t) = \begin{cases} Ax(t), & t < D \\ (A + B e^{-DA} K_D)x(t) - B \sum_{j=N+1}^{\infty} (T_D^j K_D x)(t - D), & t \geq D. \end{cases}$$

We are now interested in the last term in the right-hand side of this equation. From the elements presented in the previous subsection, one easily obtains for $t \geq D$

$$\begin{aligned}\left\| B \sum_{j=N+1}^{\infty} (T_D^j K_D x)(t - D) \right\| &\leq \sum_{j=N+1}^{\infty} \|B\|^{j+1} \|K_D\|^{j+1} \frac{(t-D)^{j-1}}{(j-1)!} \max_{s \in [0, t-D]} \|x(s)\| \\ &\leq \|B\|^2 \|K_D\|^2 \frac{\max_{s \in [0, t-D]} \|x(s)\|}{N!} \int_0^{\|B\| \|K_D\| (t-D)} e^{\|B\| \|K_D\| (t-D) - s} s^N ds\end{aligned}$$

in which the last inequality is obtained by using the Taylor expansion with integral remainder. Now, assume that $N > \|B\| \|K_D\| T$. As $t \leq T$, it follows that $N > \|B\| \|K_D\| (t - D)$ and that the function $s \geq 0 \mapsto e^{\|B\| \|K_D\| (t-D) - s} s^N$ is increasing on the interval $[0, \|B\| \|K_D\| (t - D)]$. Thus, one gets

$$\left\| B \sum_{j=N+1}^{\infty} (T_D^j K_D x)(t - D) \right\| \leq \frac{\|B\|^{N+3} \|K_D\|^{N+3} T^{N+1}}{N!} \max_{s \in [0, t-D]} \|x(s)\|.$$

One can observe that the right-hand term in this last inequality is converging to zero as N tends to infinity. Hence, for a given $\varepsilon > 0$, define $\varepsilon_0 > 0$ and $\underline{N} \in \mathbb{N}$ such that

$$\varepsilon_0 = \frac{\varepsilon e^{-2\|A\|D} \lambda(P_D)}{e^{2\|A\|D} + \varepsilon \lambda(P_D)}, \quad (30)$$

$$\frac{\|B\|^{N+3} \|K_D\|^{N+3} T^{N+1}}{N!} \leq \frac{\lambda(P_D)}{4} \varepsilon_0, \quad N \geq \underline{N}, \quad (31)$$

in which P_D satisfies (5). Now, consider $V(t) = x(t)^T P_D x(t)$. Applying Young inequality, it follows that, for $N \geq N^* = \max\{\|B\| \|K_D\| T, \underline{N}\}$ and $t \geq D$,

$$\begin{aligned} \dot{V}(t) &= -\|x(t)\|^2 - 2x(t)^T P_D B \sum_{j=N+1}^{\infty} (T_D^j K_D x)(t-D) \\ &\leq -\frac{\|x(t)\|^2}{2} + \frac{\lambda(P_D)}{2} \varepsilon_0 \max_{s \in [0, t-D]} \|x(s)\|^2 \\ &\leq -\frac{\|x(t)\|^2}{2} + \frac{\lambda(P_D)}{2} \varepsilon_0 \max_{s \in [0, t]} \|x(s)\|^2. \end{aligned}$$

Hence, one gets

$$\|x(t)\|^2 \leq \frac{1}{\lambda(P_D)} e^{-(t-D)/2} \|x(D)\|^2 + \varepsilon_0 \max_{s \in [0, t]} \|x(s)\|^2, \quad t \geq D. \quad (32)$$

with

$$\max_{s \in [0, t]} \|x(s)\|^2 \leq e^{2\|A\|D} \max_{s \in [D, t]} \|x(s)\|^2. \quad (33)$$

In particular, one obtains

$$\max_{s \in [D, t]} \|x(s)\|^2 \leq \frac{1}{\lambda(P_D)} \|x(D)\|^2 + \varepsilon_0 e^{2\|A\|D} \max_{s \in [D, t]} \|x(s)\|^2, \quad t \geq D,$$

which gives

$$\max_{s \in [D, t]} \|x(s)\|^2 \leq \frac{1}{1 - \varepsilon_0 e^{2\|A\|D}} \frac{1}{\lambda(P_D)} \|x(D)\|^2, \quad t \geq D \quad (34)$$

which is well-defined as $\varepsilon_0 e^{2\|A\|D} < 1$ by definition of ε_0 . And from (32)–(34), one obtains

$$\begin{aligned} \|x(t)\|^2 &\leq \frac{1}{\lambda(P_D)} \left(e^{-(t-D)/2} + \frac{\varepsilon_0 e^{2\|A\|D}}{1 - \varepsilon_0 e^{2\|A\|D}} \right) \|x(D)\|^2 \\ &\leq \left(\frac{e^{2\|A\|D}}{\lambda(P_D)} e^{-(t-D)/2} + \varepsilon \right) \|x(0)\|^2, \quad t \geq D. \end{aligned}$$

Thus, taking into account that $x(t) = e^{tA}x(0)$ for $t < D$, one easily obtains the existence of $R, \rho > 0$ such that (20) holds.

5 Conclusion

The contribution of this paper was to inverse the Artstein transform and to derive an explicit expression of the corresponding stabilizing controller. This derivation has been proved by using a fixed-point argument. It has been also shown how this expression is fruitful for theoretical developments on linear systems with a constant delay in the input.

This work lets many question open. The use of this Artstein transformation inversion on other classes of controlled systems is under investigation. In particular the class of systems given by partial differential equation with a delayed control (e.g., at one boundary condition) is actually studied. Generalization to delay systems is also natural.

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