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Finite-dimensional predictor-based feedback stabilization of a 1D linear reaction-diffusion equation with boundary input delay

Delphine Bresch-Pietri Christophe Prieur* Emmanuel Trélat[†]

Preliminary version

Abstract

We consider a one-dimensional controlled reaction-diffusion equation, where the control acts on the boundary and is subject to a constant delay. Such a model is a paradigm for more general parabolic systems coupled with a transport equation. We prove that this is possible to stabilize (in H^1 norm) this process by means of an explicit predictor-based feedback control that is designed from a finite-dimensional subsystem. The implementation is very simple and efficient and is based on standard tools of pole-shifting. Our feedback acts on the system as a finite-dimensional predictor. We compare our approach with the backstepping method.

1 Introduction and main result

Let $L > 0$ and let $c \in L^\infty(0, L)$. We consider the 1D heat equation on $(0, L)$ with a delayed boundary control

$$\begin{aligned} y_t &= y_{xx} + c(x)y, \\ y(t, 0) &= 0, \quad y(t, L) = u_D(t) = u(t - D), \end{aligned} \tag{1}$$

where the state is $y(t, \cdot) : [0, L] \rightarrow \mathbb{R}$ and the control is $u_D(t) = u(t - D)$, with $D > 0$ a constant delay.

Our objective is to design a feedback control stabilizing (1).

There have been a number of works in the literature dealing with the stabilization of processes with input delays but only few contributions do exist for processes driven by PDE's. The academic problem that we investigate here has been studied in [10] with a backstepping approach.

To be more precise with initial conditions, we assume that we are only interested in what happens for $t \geq 0$. We consider an initial condition

$$y(0, \cdot) = y_0(\cdot) \in L^2(0, L),$$

and since the boundary control is retarded with the delay D , we assume that no control is applied within the time interval $(0, D)$. In other words, we assume that $u_D(t) = 0$ for every $t \in (0, D)$.

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For every $t > D$ on, a nontrivial control $u_D(t)$ can then be applied. In what follows we are going to design a feedback control whose value $u(t - D)$ only depends on the values of $X_1(s)$ with $0 < s < t - D$.

Our strategy begins with a spectral analysis of the operator underlying the control system (1) which is split in two parts. The first part of the system is finite dimensional and contains all unstable modes, whereas the second part is infinite dimensional and contains all stable modes. The design of our feedback is realized on the finite-dimensional part of the system. We use the Artstein model reduction and then design a Kalman gain matrix in a standard way with the pole-shifting theorem. Then we invert the Artstein transform and end up with the desired feedback. We stress that the feedback that we design in such a way is very easy to implement in practice. We first show that this feedback stabilizes exponentially the finite-dimensional part of the system, and then, using an appropriate Lyapunov function, we prove that it stabilizes as well the whole system. Note that the exponential asymptotic stability result holds true for every possible value of the delay $D \geq 0$.

Theorem 1. *The equation (1) with boundary input delay is exponentially stabilizable, with a feedback that is built with a finite-dimensional linear control system with input delay. More precisely, with this feedback the function $t \mapsto \|y(t, \cdot)\|_{H^1(0,L)}$ converges exponentially to 0 as t tends to $+\infty$.*

2 Construction of the feedback and proof of Theorem 1

2.1 Spectral reduction

First of all, in order to deal rather with a homogeneous Dirichlet problem (which is more convenient), we set

$$w(t, x) = y(t, x) - \frac{x}{L}u_D(t), \quad (2)$$

and we suppose that the control u_D is derivable for all positive times (this will be true in the construction that we will carry out). This leads to

$$\begin{aligned} w_t &= w_{xx} + cw + \frac{x}{L}cu_D - \frac{x}{L}u'_D, \quad \forall t > 0, \forall x \in (0, 1), \\ w(t, 0) &= w(t, L) = 0, \quad \forall t > 0, \\ w(0, x) &= y(0, x) - \frac{x}{L}u_D(0), \quad \forall x \in (0, 1). \end{aligned} \quad (3)$$

We define the operator

$$A = \partial_{xx} + c(\cdot)\text{id}, \quad (4)$$

on the domain $D(A) = H^2(0, L) \cap H_0^1(0, L)$. Then the above control system is

$$w_t(t, \cdot) = Aw(t, \cdot) + a(\cdot)u_D(t) + b(\cdot)u'_D(t), \quad (5)$$

with $a(x) = \frac{x}{L}c(x)$ and $b(x) = -\frac{x}{L}$ for every $x \in (0, L)$.

Noting that A is self-adjoint and of compact inverse, we consider a Hilbert basis $(e_j)_{j \geq 1}$ of $L^2(0, L)$ consisting of eigenfunctions of A , associated with the sequence of eigenvalues $(\lambda_j)_{j \geq 1}$. Note that

$$-\infty < \dots < \lambda_j < \dots < \lambda_1 \quad \text{and} \quad \lambda_j \xrightarrow{j \rightarrow +\infty} -\infty,$$

and that $e_j(\cdot) \in H_0^1(0, L) \cap C^2([0, L])$ for every $j \geq 1$. Every solution $w(t, \cdot) \in H^2(0, L) \cap H_0^1(0, L)$ of (5) can be expanded as a series in the eigenfunctions $e_j(\cdot)$, convergent in $H_0^1(0, L)$,

$$w(t, \cdot) = \sum_{j=1}^{\infty} w_j(t)e_j(\cdot),$$

and one gets the infinite-dimensional control system

$$w'_j(t) = \lambda_j w_j(t) + a_j u_D(t) + b_j u'_D(t), \quad (6)$$

with

$$\begin{aligned} a_j &= \langle a(\cdot), e_j(\cdot) \rangle_{L^2(0,L)} = \frac{1}{L} \int_0^L x c(x) e_j(x) dx, \\ b_j &= \langle b(\cdot), e_j(\cdot) \rangle_{L^2(0,L)} = -\frac{1}{L} \int_0^L x e_j(x) dx, \end{aligned} \quad (7)$$

for every $j \in \mathbb{N}^*$. We define

$$\alpha_D(t) = u'_D(t), \quad (8)$$

and we consider from now on $u_D(t)$ as a state and $\alpha_D(t)$ as a control (destinated to be a delayed feedback, with constant delay D), so that equations (6) and (8) form an infinite-dimensional control system controlled by α_D , written as

$$\begin{aligned} u'_D(t) &= \alpha_D(t), \\ w'_1(t) &= \lambda_1 w_1(t) + a_1 u_D(t) + b_1 \alpha_D(t), \\ &\vdots \\ w'_j(t) &= \lambda_j w_j(t) + a_j u_D(t) + b_j \alpha_D(t), \\ &\vdots \end{aligned} \quad (9)$$

Let n be the number of nonnegative eigenvalues, and let $\eta > 0$ be such that

$$\forall k > n \quad \lambda_k < -\eta < 0. \quad (10)$$

Let π_1 be the orthogonal projection onto the subspace of $L^2(0, L)$ spanned by $e_1(\cdot), \dots, e_n(\cdot)$, and let

$$w^1(t) = \pi_1 w(t, \cdot) = \sum_{j=1}^n w_j(t) e_j(\cdot). \quad (11)$$

With the matrix notations

$$X_1(t) = \begin{pmatrix} u_D(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & \lambda_n \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad (12)$$

the n first equations of (9) form the finite-dimensional control system with input delay

$$X'_1(t) = A_1 X_1(t) + B_1 \alpha_D(t) = A_1 X_1(t) + B_1 \alpha(t - D). \quad (13)$$

Note that the state $X_1(t) \in \mathbb{R}^{n+1}$ involves the term $u_D(t)$ which is destinated to be delayed.

Our objective is to design a feedback control α exponentially stabilizing the infinite-dimensional system (9). We follow an idea used in [3, 4] in order to stabilize nonlinear heat and wave equations around a steady-state. The idea consists of first designing a feedback control exponentially stabilizing the finite-dimensional system (13), and then of proving that this feedback actually stabilizes the whole system (9). The idea underneath is that the finite-dimensional system (13) contains the

unstable modes of the whole system (9), and thus has to be stabilized. It is however not obvious that this feedback stabilizing the unstable finite-dimensional part actually stabilizes as well the whole system (9), and this is proved using an appropriate Lyapunov functional.

Before going into details, we stress that this stabilization procedure is carried out with a very simple approach, easy to implement, and using very classical and well-known tools from the finite-dimensional linear setting.

2.2 Stabilization of the unstable finite-dimensional part

Let us design a feedback control stabilizing the control system with input delay (13), as well as a Lyapunov functional. First of all, following the so-called Artstein model reduction (see [1, 15]), we set, for every $t \in \mathbb{R}$,

$$Z_1(t) = X_1(t) + \int_{t-D}^t e^{(t-s-D)A_1} B_1 \alpha(s) ds = X_1(t) + \int_0^D e^{-\tau A_1} B_1 \alpha(t-D+\tau) d\tau, \quad (14)$$

and we get immediately

$$\dot{Z}_1(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \alpha(t), \quad (15)$$

which is a usual linear control system, without input delay, in \mathbb{R}^{n+1} .

Lemma 1. *For every $D \geq 0$, the pair $(A_1, e^{-DA_1} B_1)$ satisfies the Kalman condition, that is,*

$$\text{rank}(e^{-DA_1} B_1, A_1 e^{-DA_1} B_1, \dots, A_1^n e^{-DA_1} B_1) = n + 1. \quad (16)$$

Proof. Since A_1 and e^{-DA_1} commute, and since e^{-DA_1} is invertible, we have

$$\begin{aligned} & \text{rank}(e^{-DA_1} B_1, A_1 e^{-DA_1} B_1, \dots, A_1^n e^{-DA_1} B_1) \\ &= \text{rank}(e^{-DA_1} B_1, e^{-DA_1} A_1 B_1, \dots, e^{-DA_1} A_1^n B_1) \\ &= \text{rank}(B_1, A_1 B_1, \dots, A_1^n B_1), \end{aligned}$$

and hence it suffices to prove that the pair (A_1, B_1) satisfies the Kalman condition. A simple computation leads to

$$\det(B_1, A_1 B_1, \dots, A_1^n B_1) = \prod_{j=1}^n (a_j + \lambda_j b_j) \text{VdM}(\lambda_1, \dots, \lambda_n), \quad (17)$$

where $\text{VdM}(\lambda_1, \dots, \lambda_n)$ is a Van der Monde determinant, and thus is never equal to zero since the real numbers λ_j , $j = 1 \dots n$, are all distinct. On the other part, using the fact that every $e_j(\cdot)$ is an eigenfunction of A and belongs to $H_0^1(0, L)$, we have, for every integer j ,

$$a_j + \lambda_j b_j = \frac{1}{L} \int_0^L x (c(x) e_j(x) - \lambda_j e_j(x)) dx = -\frac{1}{L} \int_0^L x e_j''(x) dx = -e_j'(L),$$

which is not equal to zero since $e_j(L) = 0$ and $e_j(\cdot)$ is a nontrivial solution of a linear second-order scalar differential equation. The lemma is proved. \square

Since the control system (15) satisfies the Kalman condition, the well-known pole-shifting theorem and Lyapunov theorem imply the existence of a stabilizing gain matrix and of a Lyapunov functional (see, e.g., [9, 17]). This yields the following corollary.

Corollary 1. For every $D \geq 0$ there exists a $1 \times (n+1)$ matrix $K_1(D) = (k_0(D), k_1(D), \dots, k_n(D))$ such that $A_1 + B_1 e^{-DA_1} K_1(D)$ admits -1 as an eigenvalue with order $n+1$. Moreover there exists a $(n+1) \times (n+1)$ symmetric positive definite matrix $P(D)$ such that

$$P(D) (A_1 + B_1 e^{-DA_1} K_1(D)) + (A_1 + e^{-DA_1} B_1 K_1(D))^\top P(D) = -I_{n+1}. \quad (18)$$

In particular, the function

$$V_1(Z_1) = \frac{1}{2} Z_1^\top P(D) Z_1 \quad (19)$$

is a Lyapunov function for the closed-loop system $\dot{Z}_1(t) = (A_1 + e^{-DA_1} B_1 K_1(D)) Z_1(t)$.

Remark 1. It can even be proved that $K_1(D)$ and $P(D)$ are smooth (i.e., of class C^∞) with respect to D , but we do not need this property in this paper.

Remark 2. In the statement above we chose -1 as an eigenvalue of $A_1 + B_1 e^{-DA_1} K_1(D)$, but actually the pole-shifting theorem implies that, for every $(n+1)$ -tuple (μ_0, \dots, μ_n) of eigenvalues there exists a $1 \times (n+1)$ matrix $K_1(D)$ such that the eigenvalues $A_1 + B_1 e^{-DA_1} K_1(D)$ are exactly (μ_0, \dots, μ_n) . The eigenvalue -1 was chosen for simplicity. What is important is to ensure that $A_1 + B_1 e^{-DA_1} K_1(D)$ is a Hurwitz matrix (that is, whose eigenvalues have a negative real part).

In practice other choices can be done, which can be more efficient according to such or such criterion (see [17]). For instance, instead of using the pole-shifting theorem, one could design a stabilizing gain matrix K_1 by using a standard Riccati procedure.

Remark 3. From Corollary 1 we infer that for every $D \geq 0$ there exists $C_1(D) > 0$ (depending smoothly on D) such that

$$\frac{d}{dt} V_1(Z_1(t)) = -\|Z_1(t)\|_{\mathbb{R}^{n+1}}^2 \leq -C_1(D) V_1(Z_1(t)), \quad (20)$$

where $\|\cdot\|_{\mathbb{R}^{n+1}}$ is the usual Euclidean norm in \mathbb{R}^{n+1} .

From Corollary 1, the feedback $\alpha(t) = K_1(D) Z_1(t)$ stabilizes exponentially the control system (15). Since $\alpha(t-D)$ is used in the control system (13), and since in general we are only concerned with prescribing the future of a system, starting at time 0, we assume that the control system (13) is uncontrolled for $t < 0$, and from the starting time $t = 0$ on we let the feedback act on the system. In other words, we set

$$\alpha(t) = \begin{cases} 0 & \text{if } t < D, \\ K_1(D) Z_1(t) & \text{if } t \geq D, \end{cases} \quad (21)$$

so that, with this control, the control system (13) with input delay is written as

$$X_1'(t) = A_1 X_1(t) + \chi_{(D, +\infty)}(t) B_1 K_1(D) Z_1(t-D),$$

with Z_1 given by (14). Here the notation χ_E stands the characteristic function of E , that is the function defined by $\chi_E(t) = 1$ whenever $t \in E$ and $\chi_E(t) = 0$ otherwise. Using (14), the feedback α defined by (21) is such that

$$\alpha(t) = \begin{cases} 0 & \text{if } t < D, \\ K_1(D) X_1(t) + K_1(D) \int_{\max(t-D, D)}^t e^{(t-D-s)A_1} B_1 \alpha(s) ds & \text{if } t \geq D. \end{cases} \quad (22)$$

In other words, the value of the feedback control α at time t depends on $X_1(t)$ and of the controls applied in the past – more precisely, of the values of α over the time interval $(\max(t-D, D), t)$.

Lemma 2. *The feedback (22) stabilizes exponentially the control system (13).*

Proof. By construction $t \mapsto Z_1(t)$ converges exponentially to 0, and hence $t \mapsto \alpha(t)$ and thus $t \mapsto \int_{\max(t-D, D)}^t e^{(t-D-s)A_1} B_1 \alpha(s) ds$ converge exponentially to 0 as well. Then the equality (14) implies that $t \mapsto X_1(t)$ converges exponentially to 0. \square

Inversion of the Artstein transform. We show here how to invert the Artstein transform, with two motivations in mind:

- First of all, it is interesting to express the stabilizing control α (defined by (21)) directly as a feedback of X_1 .
- Secondly, it is interesting to express the Lyapunov functional V_1 (defined by (19)) as a function of X_1 .

To reach this objective, it suffices to solve the fixed point implicit equality (22). For every function f defined on \mathbb{R} and locally integrable, we define

$$(T_D f)(t) = K_1(D) \int_{\max(t-D, D)}^t e^{(t-D-s)A_1} B_1 f(s) ds.$$

It follows from (22) that $\alpha(t) = K_1(D)X_1(t) + (T_D \alpha)(t)$, for every $t \geq D$. We have the following lemma, proved in [14].

Lemma 3. *There holds*

$$\alpha(t) = \begin{cases} 0 & \text{if } t < D, \\ \sum_{j=0}^{+\infty} (T_D^j K_1(D)X_1)(t) & \text{if } t \geq D, \end{cases} \quad (23)$$

and the series is convergent, whatever the value of the delay $D \geq 0$ may be.

Note that the value of the feedback α at time t ,

$$\begin{aligned} \alpha(t) &= K_1(D)X_1(t) + K_1(D) \int_{\max(t-D, D)}^t e^{(t-D-s)A_1} B_1 K_1(D)X_1(s) ds \\ &\quad + K_1(D) \int_{\max(t-D, D)}^t e^{(t-D-s)A_1} B_1 K_1(D) \int_{\max(s-D, D)}^s e^{(s-D-\tau)A_1} B_1 K_1(D)X_1(\tau) d\tau ds \\ &\quad + \dots \end{aligned}$$

depends on the past values of X_1 over the time interval (D, t) . Since the feedback is retarded with the delay D , the term $\alpha(t-D)$ appearing at the right-hand side of (13) only depends on the values of $X_1(s)$ with $0 < s < t-D$, as desired.

We stress that in the above result the convergence of the series is the nontrivial fact. Otherwise the formula can be obtained from an immediate formal computation.

Remark 4. It is also interesting to express Z_1 in function of X_1 , that is, to invert the equality

$$Z_1(t) = X_1(t) + \int_{(t-D, t) \cap (D, +\infty)} e^{(t-s-D)A_1} B_1 K_1(D)Z_1(s) ds \quad (24)$$

coming from (14) and (21). Although it is technical and not directly useful to derive the exponential stability of Z_1 , it will however allow us to express the Lyapunov functional V_1 defined by (19). Note that

$$(t-D, t) \cap (D, +\infty) = \begin{cases} \emptyset & \text{if } t < D, \\ (D, t) & \text{if } D < t < 2D, \\ (t-D, t) & \text{if } 2D < t. \end{cases} \quad (25)$$

In particular if $t < D$ then $Z_1(t) = X_1(t)$. Actually we have the following precise result (see [14]).

Lemma 4. *For every $t \in \mathbb{R}$, there holds*

$$X_1(t) = Z_1(t) - \int_{(t-D, t) \cap (D, +\infty)} f(t-s)X_1(s) ds, \quad (26)$$

where f is defined as the unique solution of the fixed point equation

$$f(r) = f_0(r) + (\tilde{T}_D f)(r),$$

with $f_0(r) = e^{(r-D)A_1} B_1 K_1(D)$ and

$$(\tilde{T}_D f)(r) = \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) f(\tau) d\tau.$$

Moreover, we have

$$\begin{aligned} f(r) &= \sum_{j=0}^{+\infty} (\tilde{T}_D^j f_0)(r) \\ &= e^{(r-D)A_1} B_1 K_1(D) \\ &\quad + \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) e^{(\tau-D)A_1} B_1 K_1(D) d\tau \\ &\quad + \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) \int_0^\tau e^{(\tau-s-D)A_1} B_1 K_1(D) e^{(s-D)A_1} B_1 K_1(D) ds d\tau \\ &\quad + \dots \end{aligned}$$

and the series is convergent, whatever the value of the delay $D \geq 0$ may be.

With this expression and using (24) in Remark 4, the feedback α can be as well written as

$$\begin{aligned} \alpha(t) &= \chi_{(D, +\infty)}(t) K_1(D) Z_1(t) \\ &= \chi_{(D, +\infty)}(t) K_1(D) X_1(t) + K_1(D) \int_{(t-D, t) \cap (D, +\infty)} f(t-s) X_1(s) ds, \end{aligned}$$

and we recover of course the expression (23) derived in Lemma 3.

Plugging this feedback into the control system (13) yields, for $t > D$, the closed-loop system

$$\begin{aligned} X_1'(t) &= A_1 X_1(t) + B_1 \alpha(t-D) \\ &= A_1 X_1(t) + B_1 K_1(D) X_1(t-D) + B_1 K_1(D) \int_{(t-D, t) \cap (D, +\infty)} f(t-s) X_1(s) ds, \end{aligned} \quad (27)$$

which is, as said above, exponentially stable. Moreover, the Lyapunov function V_1 , which is exponentially decreasing according to Remark 3, can be written as

$$V_1(t) = \frac{1}{2} \left(X_1(t) + \int_{I_t(D)} f(t-s) X_1(s) ds \right)^\top P(D) \left(X_1(t) + \int_{I_t(D)} f(t-s) X_1(s) ds \right).$$

with $I_t(D) = (t - D, t) \cap (D, +\infty)$. We stress once again that the above feedback and Lyapunov functional stabilize the system whatever the value of the delay may be.

Remark 5. Let us make a remark on the practical implementation. Although the expression (23) has some theoretical interest, in practice we do not use it to compute $\alpha(t)$, and we use instead the gain matrix $K_1(D)$ whose computation, based on the knowledge of $(A_1, e^{-DA_1}B_1)$, is a very easy task. Moreover, instead of considering the closed-loop system (27), it is far more convenient to consider the equivalent system

$$\begin{aligned} X_1'(t) &= AX_1(t) + B_1K_1(D)Z_1(t - D), \\ Z_1'(t) &= (A_1 + e^{-DA_1}B_1K_1(D))Z_1(t), \end{aligned} \quad (28)$$

which in this form looks more like a dynamic stabilization procedure (see [16]). These implementation issues are analyzed in detail in [14].

2.3 Stabilization of the whole system

In order to prove that the feedback α designed above stabilizes the whole system (9) we have to take into account the rest of the system (consisting of modes that are naturally stable). What has to be checked is whether or not the delayed control part might destabilize this infinite-dimensional part.

Let $(u_D(\cdot), w(\cdot))$ denote a solution of (5) in which we choose the control α in the feedback form designed previously, such that $u_D(0) = 0$ and $w(0) = 0$. Here, we make a slight abuse of notation, since $w(t)$ designates the solution $w(t, \cdot) \in H^2(0, L) \cap H_0^1(0, L)$ satisfying

$$\begin{aligned} u_D' &= \alpha, \quad w' = Aw + au_D + b\alpha, \\ u_D(0) &= 0, \quad w(0, \cdot) = 0. \end{aligned} \quad (29)$$

Let $M(D)$ be a positive real number such that

$$\begin{aligned} M(D) &> \|b\|_{L^2(0,L)}^2 \|K_1(D)\|_{\mathbf{R}^{n+1}}^2 \\ &+ \max \left(2\|a\|_{L^2(0,L)}^2, \frac{\max(\lambda_1, \dots, \lambda_n)}{\lambda_{\min}(P(D))} \right) \max \left(1, De^{2D\|A_1\|} \|B_1\|_{\mathbf{R}^{n+1}}^2 \|K_1(D)\|_{\mathbf{R}^{n+1}}^2 \right), \end{aligned} \quad (30)$$

where $\|K_1(D)\|_{\mathbf{R}^{n+1}}^2 = \sum_{j=0}^n k_j(D)^2$, $\|B_1\|_{\mathbf{R}^{n+1}}^2 = 1 + \sum_{j=1}^n b_j^2$, where $\|A_1\|$ is the usual matrix norm induced from the Euclidean norm of \mathbf{R}^{n+1} , and where $\lambda_{\min}(P(D)) > 0$ is the smallest eigenvalue of the symmetric positive definite matrix $P(D)$. The precise value of $M(D)$ is not important however. What is important in what follows is that $M(D) > 0$ is large enough.

We set

$$\begin{aligned} V_D(t) &= M(D) V_1(t) + M(D) \int_{(t-D,t) \cap (D,+\infty)} V_1(s) ds - \frac{1}{2} \langle w(t), Aw(t) \rangle_{L^2(0,L)} \\ &= \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t) + \frac{M(D)}{2} \int_{(t-D,t) \cap (D,+\infty)} Z_1(s)^\top P(D) Z_1(s) ds \\ &\quad - \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2. \end{aligned} \quad (31)$$

We are going to prove that $V_D(t)$ is positive and decreases exponentially to 0. This Lyapunov functional consists of three terms. The two first terms stand for the unstable finite-dimensional

part of the system. As we will see, the integral term is instrumental in order to tackle the delayed terms. The third term stands for the infinite-dimensional part of the system. In this infinite sum actually all modes are involved, in particular those that are unstable. Then the two first terms of (31), weighted with $M(D) > 0$, can be seen as corrective terms and this weight $M(D) > 0$ is chosen large enough so that $V_D(t)$ be indeed positive. More precisely,

$$-\frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2 = -\frac{1}{2} \sum_{j=1}^n \lambda_j w_j(t)^2 - \frac{1}{2} \sum_{j=n+1}^{\infty} \lambda_j w_j(t)^2, \quad (32)$$

where $\lambda_j \geq 0$ for every $j \in \{1, \dots, n\}$ and $\lambda_j \leq -\eta < 0$ for every $j > n$ (see (10)). Therefore the second term of (32) is positive and the first term, which is nonpositive, is actually compensated by the first term of $V_D(t)$ since $M(D)$ is large enough, as proved in the following more precise lemma.

Lemma 5. *There exists $C_2(D) > 0$ such that*

$$V_D(t) \geq C_2(D) \left(u_D(t)^2 + \|w(t)\|_{H_0^1(0,L)}^2 \right), \quad (33)$$

for every $t \geq 0$.

Proof. First of all, by definition of $\lambda_{\min}(P(D))$, one has

$$\begin{aligned} & \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t) + \frac{M(D)}{2} \int_{t-D}^t Z_1(s)^\top P(D) Z_1(s) ds \\ & \geq M(D) \frac{\lambda_{\min}(P(D))}{2} \left(\|Z_1(t)\|_{\mathbf{R}^{n+1}}^2 + \int_{t-D}^t \|Z_1(s)\|_{\mathbf{R}^{n+1}}^2 ds \right), \end{aligned} \quad (34)$$

for every $t \geq 0$. Besides, recall that, from (24), one has

$$X_1(t) = Z_1(t) - \int_{(t-D,t) \cap (D,+\infty)} e^{(t-s-D)A_1} B_1 K_1(D) Z_1(s) ds,$$

and therefore, using the Cauchy-Schwarz inequality and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, it follows that

$$\|X_1(t)\|_{\mathbf{R}^{n+1}}^2 \leq C_3(D) \left(\|Z_1(t)\|_{\mathbf{R}^{n+1}}^2 + \int_{t-D}^t \|Z_1(s)\|_{\mathbf{R}^{n+1}}^2 ds \right), \quad (35)$$

with

$$C_3(D) = \max \left(2, 2De^{2D\|A_1\|} \|B_1\|_{\mathbf{R}^{n+1}}^2 \|K_1(D)\|_{\mathbf{R}^{n+1}}^2 \right).$$

We then infer from (34) and (35) that

$$\begin{aligned} & \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t) + \frac{M(D)}{2} \int_{t-D}^t Z_1(s)^\top P(D) Z_1(s) ds \\ & \geq M(D) \frac{\lambda_{\min}(P(D))}{2C_3(D)} \|X_1(t)\|_{\mathbf{R}^{n+1}}^2, \end{aligned} \quad (36)$$

for every $t \geq 0$.

Using (32) and the definition of X_1 in (12), we have

$$-\frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2 \geq -\frac{1}{2} \sum_{j=n+1}^{\infty} \lambda_j w_j(t)^2 - \frac{1}{2} \max_{1 \leq j \leq n} (\lambda_j) \|X_1(t)\|_{\mathbf{R}^{n+1}}^2, \quad (37)$$

and therefore, using (36), we get

$$V_D(t) \geq \left(M(D) \frac{\lambda_{\min}(P(D))}{2C_3(D)} - \frac{1}{2} \max_{1 \leq j \leq n} (\lambda_j) \right) \|X_1(t)\|_{\mathbf{R}^{n+1}}^2 - \frac{1}{2} \sum_{j=n+1}^{\infty} \lambda_j w_j(t)^2,$$

for every $t \geq 0$. By definition of $M(D)$ (see (30)), one has $M(D) \frac{\lambda_{\min}(P(D))}{2C_3(D)} - \frac{1}{2} \max_{1 \leq j \leq n} (\lambda_j) > 0$ and hence there exists $C_4(D) > 0$ such that

$$V_D(t) \geq C_4(D) \left(\|X_1(t)\|_{\mathbf{R}^{n+1}}^2 - \frac{1}{2} \sum_{j=n+1}^{\infty} \lambda_j w_j(t)^2 \right). \quad (38)$$

Using the series expansion $w(t, \cdot) = \sum_{i=1}^{+\infty} w_i(t) e_i(\cdot)$, we have

$$\|w(t)\|_{H_0^1(0,L)}^2 = \sum_{(i,j) \in (\mathbf{N}^*)^2} w_i(t) w_j(t) \int_0^L e'_i(x) e'_j(x) dx.$$

By definition, one has $e''_n + c e_n = \lambda_n e_n$ and $e_n(0) = e_n(L) = 0$, for every $n \in \mathbf{N}^*$. Integrating by parts and using the orthonormality property, we get

$$\int_0^L e'_i(x) e'_j(x) dx = \int_0^L c(x) e_i(x) e_j(x) dx - \lambda_j \delta_{ij},$$

with $\delta_{ij} = 1$ whenever $i = j$ and $\delta_{ij} = 0$ otherwise, and thus, for all $t \geq 0$,

$$\|w(t)\|_{H_0^1(0,L)}^2 = \int_0^L c(x) w(t, x)^2 dx - \sum_{j=1}^{\infty} \lambda_j w_j(t)^2. \quad (39)$$

Since $c \in L^\infty(0, L)$, it follows that

$$\begin{aligned} \|w(t)\|_{H_0^1(0,L)}^2 &\leq \|c\|_{L^\infty(0,L)} \|w(t)\|_{L^2(0,L)}^2 - \sum_{j=1}^n \lambda_j w_j(t)^2 - \sum_{j=n+1}^{\infty} \lambda_j w_j(t)^2 \\ &\leq \|c\|_{L^\infty(0,L)} \sum_{j=1}^{\infty} w_j(t)^2 - \sum_{j=n+1}^{\infty} \lambda_j w_j(t)^2 \\ &\leq \|c\|_{L^\infty(0,L)} \|X_1(t)\|_{\mathbf{R}^{n+1}}^2 - \sum_{j=n+1}^{\infty} (\lambda_j - \|c\|_{L^\infty(0,L)}) w_j(t)^2 \end{aligned}$$

and since $\lambda_j \rightarrow -\infty$ as j tends to $+\infty$, there exists $C_5 > 0$ such that

$$\|w(t)\|_{H_0^1(0,L)}^2 \leq -C_5 \left(\|X_1(t)\|_{\mathbf{R}^{n+1}}^2 - \frac{1}{2} \sum_{j=n+1}^{\infty} \lambda_j w_j(t)^2 \right).$$

Then (33) follows from (38). \square

Using (25), note that if $t < D$ then the integral term of (31) is equal to 0 and $Z_1(t) = X_1(t)$, and hence

$$V_D(t) = \frac{M(D)}{2} X_1(t)^\top P(D) X_1(t) - \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2,$$

for every $t < D$. This remark leads to the following lemma.

Lemma 6. *There exists $C_6(D) > 0$ such that*

$$V_D(t) \leq C_6(D)(u_D(t)^2 + \|w(t)\|_{H_0^1(0,L)}^2), \quad (40)$$

for every $t < D$.

Proof. Using (39), one has

$$-\sum_{j=1}^{+\infty} \lambda_j w_j(t)^2 \leq \|w(t)\|_{H_0^1(0,L)}^2 + \|c\|_{L^\infty(0,L)} \|w(t)\|_{L^2(0,L)}^2 \leq C_8(D) \|w(t)\|_{H_0^1(0,L)}^2,$$

and then the lemma follows from the Poincaré inequality $\|w(t)\|_{L^2(0,L)}^2 \leq L \|w(t)\|_{H_0^1(0,L)}^2$. \square

Lemma 7. *The functional V_D decreases exponentially to 0.*

Proof. Let us compute $V_D'(t)$ for $t > 2D$ and state a differential inequality satisfied by V_D . First of all, it follows from (18) (in Corollary 1) that

$$\frac{d}{dt} \frac{M(D)}{2} Z_1(t)^\top P(D) Z_1(t) = -M(D) \|Z_1(t)\|_{\mathbf{R}^{n+1}}^2,$$

and thus

$$\frac{d}{dt} \frac{M(D)}{2} \int_{t-D}^t Z_1(s)^\top P(D) Z_1(s) ds = -M(D) \int_{t-D}^t \|Z_1(s)\|_{\mathbf{R}^{n+1}}^2 ds.$$

Then, using (29), (31) and the fact that A is self-adjoint, we get

$$\begin{aligned} V_D'(t) &= -M(D) \|Z_1(t)\|_{\mathbf{R}^{n+1}}^2 - M(D) \int_{t-D}^t \|Z_1(s)\|_{\mathbf{R}^{n+1}}^2 ds \\ &\quad - \|Aw(t)\|_{L^2(0,L)}^2 - \langle Aw(t), a \rangle_{L^2(0,L)} u_D(t) - \langle Aw(t), b \rangle_{L^2(0,L)} K_1(D) Z_1(t), \end{aligned} \quad (41)$$

for every $t > 2D$. From Young's inequality, we derive the estimates

$$|\langle Aw(t), a \rangle_{L^2(0,L)} u_D(t)| \leq \frac{1}{4} \|Aw(t)\|_{L^2(0,L)}^2 + \|a\|_{L^2(0,L)}^2 \|X_1(t)\|_{\mathbf{R}^{n+1}}^2, \quad (42)$$

and

$$|\langle Aw(t), b \rangle_{L^2(0,L)} K_1(D) Z_1(t)| \leq \frac{1}{4} \|Aw(t)\|_{L^2(0,L)}^2 + \|b\|_{L^2(0,L)}^2 \|K_1(D)\|_{\mathbf{R}^{n+1}}^2 \|Z_1(t)\|_{\mathbf{R}^{n+1}}^2. \quad (43)$$

With the estimates (42), (43) and (35), we infer from (35) and from (41) that

$$\begin{aligned} V_D'(t) &\leq - \left(M(D) - \|b\|_{L^2(0,L)}^2 \|K_1(D)\|_{\mathbf{R}^{n+1}}^2 - \|a\|_{L^2(0,L)}^2 C_3(D) \right) \|Z_1(t)\|_{\mathbf{R}^{n+1}}^2 \\ &\quad - \left(M(D) - \|a\|_{L^2(0,L)}^2 C_3(D) \right) \int_{t-D}^t \|Z_1(s)\|_{\mathbf{R}^{n+1}}^2 ds - \frac{1}{2} \|Aw(t)\|_{L^2(0,L)}^2. \end{aligned}$$

From (30), the real number $M(D)$ has been chose large enough so that

$$M(D) - \|b\|_{L^2(0,L)}^2 \|K_1(D)\|_{\mathbf{R}^{n+1}}^2 - \|a\|_{L^2(0,L)}^2 C_3(D) > 0$$

and

$$M(D) - \|a\|_{L^2(0,L)}^2 C_3(D) > 0.$$

Therefore, there exists $C_7(D) > 0$ such that

$$V'_D(t) \leq -C_7(D) \left(\|Z_1(t)\|_{\mathbf{R}^{n+1}}^2 + \int_{t-D}^t \|Z_1(s)\|_{\mathbf{R}^{n+1}}^2 ds \right) - \frac{1}{2} \|Aw(t)\|_{L^2(0,L)}^2. \quad (44)$$

Let us provide an estimate of $\|Aw(t)\|_{L^2(0,L)}^2$. Since $-\lambda_j \leq \lambda_j^2$ as j tends to $+\infty$, it follows that there exists $C_8 > 0$ such that

$$\begin{aligned} -\frac{1}{2} \langle w(t), Aw(t) \rangle_{L^2(0,L)} &= -\frac{1}{2} \sum_{j=1}^n \lambda_j w_j(t)^2 - \frac{1}{2} \sum_{j=n}^{+\infty} \lambda_j w_j(t)^2 \\ &\leq -\frac{1}{2} \sum_{j=n}^{+\infty} \lambda_j w_j(t)^2 \\ &\leq \frac{1}{2C_8} \sum_{j=1}^{+\infty} \lambda_j^2 w_j(t)^2 = \frac{1}{2C_8} \|Aw\|_{L^2(0,L)}^2. \end{aligned}$$

Hence it follows from (44) that

$$V'_D(t) \leq -C_7(D) \left(\|Z_1(t)\|_{\mathbf{R}^{n+1}}^2 + \int_{t-D}^t \|Z_1(s)\|_{\mathbf{R}^{n+1}}^2 ds \right) - \frac{C_8}{2} \langle w(t), Aw(t) \rangle_{L^2(0,L)}.$$

Finally, using (34), there exists $C_9(D) > 0$ such that

$$V'_D(t) \leq -C_9(D) V_D(t),$$

for every $t > 2D$. Therefore $V_D(t)$ decreases exponentially to 0. \square

From Lemma 7, $V_D(t)$ decreases exponentially to 0. It follows from Lemmas 5 and 6 that there exists $C_{10}(D) > 0$ and $\mu > 0$ such that

$$u_D(t)^2 + \|w(t)\|_{H_0^1(0,L)}^2 \leq C_{10}(D) e^{-\mu t} (u_D(0)^2 + \|w(0)\|_{H_0^1(0,L)}^2)$$

for every $t \geq 0$. Using (2) the proof of Theorem 1 follows.

References

- [1] Z. Artstein, *Linear systems with delayed controls: A reduction*, IEEE Trans. Automat. Cont. **27** (1982), no. 4, 869–879.
- [2] R. Bellman, K.L. Cooke, *Differential difference equations*, New York: Academic Press, 1963.
- [3] J.-M. Coron, E. Trélat, *Global steady-state controllability of 1-D semilinear heat equations*, SIAM J. Control Optim. **43** (2004), no. 2, 549–569.
- [4] J.-M. Coron, E. Trélat, *Global steady-state stabilization and controllability of 1-D semilinear wave equations*, Commun. Contemp. Math. **8** (2006), no. 4, 535–567.
- [5] E. Fridman, S. Nicaise, J. Valein, *Stabilization of second order evolution equations with unbounded feedback with time-dependent delay*, SIAM J. Cont. Optim. **48** (2010), no. 8, 5028–5052.

- [6] E. Fridman, Y. Orlov, *Exponential stability of linear distributed parameter systems with time-varying delays*, *Automatica* **45** (2009), 194–201.
- [7] E. Fridman, U. Shaked, *An improved stabilization method for linear time-delay systems*, *IEEE Trans. Autom. Cont.* **47** (2002), no. 11, 1931–1937.
- [8] E.F. Infante, W.B. Castelan, *A Liapunov functional for a matrix difference-differential equation*, *J. Diff. Eq.* **29** (1978), 439–451.
- [9] H.K. Khalil, *Nonlinear systems*, Macmillan Publishing Company, New York, 1992.
- [10] M. Krstic, *Control of an unstable reaction–diffusion PDE with long input delay*, *Syst. Cont. Letters* **58** (2009), 773–782.
- [11] S. Nicaise, C. Pignotti, J. Valein, *Exponential stability of the wave equation with boundary time-varying delay*, *Discrete Cont. Dynam. Syst. Ser. S* **4** (2011), no. 3, 693–722.
- [12] S. Nicaise, J. Valein, *Stabilization of second order evolution equations with unbounded feedback with time-dependent delay*, *ESAIM Control Calc. Var.* **16** (2010), 420–456.
- [13] S. Nicaise, J. Valein, E. Fridman, *Stability of the heat and wave equations with boundary time-varying delays*, *Discrete Cont. Dynam. Syst. Ser. S* **2** (2009), no. 3, 559–581.
- [14] C. Prieur, E. Trélat, *Predictor-based stabilization of finite-dimensional linear autonomous control systems with a constant delay*, Preprint Hal (2014).
- [15] J.-P. Richard, *Time-delay systems: an overview of some recent advances and open problems*, *Automatica* **39** (2003), no. 10, 1667–1694.
- [16] E.D. Sontag, *Mathematical control theory. Deterministic finite-dimensional systems*, Second edition, *Texts in Applied Mathematics*, 6, Springer-Verlag, New York, 1998, xvi+531 pp.
- [17] E. Trélat, *Contrôle optimal (French) [Optimal control], Théorie & applications [Theory and applications]*, *Math. Concrètes [Concrete Mathematics]*, Vuibert, Paris, 2005.
- [18] J. Zhang, C.R. Knopse, P. Tsiotras, *Stability of time-delay systems: equivalence between Lyapunov and scaled small-gain conditions*, *IEEE Trans. Automat. Cont.* **46** (2001), no. 3, 482–486.