# The postulations á la D'Alembert and á la Cauchy for higher gradient continuum theories are equivalent: a review of existing results

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In order to found continuum mechanics, two different postulations have been used. The first, introduced by Lagrange and Piola, starts by postulating how the work expended by internal interactions in a body depends on the virtual velocity field and its gradients. Then, by using the divergence theorem, a representation theorem is found for the volume and contact interactions which can be exerted at the boundary of the considered body. This method assumes an *a priori* notion of internal work, regards stress tensors as dual of virtual displacements and their gradients, deduces the concept of contact interactions and produces their representation in terms of stresses using integration by parts. The second method, conceived by Cauchy and based on the celebrated tetrahedron argument, starts by postulating the type of contact interactions which can be exerted on the boundary of every (suitably) regular part of a body. Then it proceeds by proving the existence of stress tensors from a balance-type postulate. In this paper, we review some relevant literature on the subject, discussing how the two postulations can be reconciled in the case of higher gradient theories. Finally, we underline the importance of the concept of contact surface, edge and wedge $s$-order forces.

# 1. Introduction

Continuum mechanics always supplies approximate models for physical systems, in which a more fundamental (possibly discrete or inhomogeneous) microstructure may be somehow neglected. Actually, the founders of continuum mechanics, Piola, Poisson, Navier and many others did try to justify continuum theories by means of an average procedure based on atomistic models.

Cauchy continuum theory (or Cauchy–Navier theory as described in its historical development by Benvenuto [1]) describes efficiently, at a macroscopic level, the behaviour of a mechanical system only when the inhomogeneities which the model does not take into account do have a characteristic length scale much smaller than the macro-scale where phenomena are observed. The aforementioned condition of scale separation is not by itself a sufficient criterion for ensuring that Cauchy theory supplies a suitable model: the best-known example is the case of deformable porous media for which both stress tensor for matrix and pressure for fluid are needed to describe its mechanical state [2–8]. Another example is given by the case of a periodic fibre-reinforced elastic medium with high contrast of mechanical properties. The mechanical description of these systems needs in addition to the standard stress tensor a higher order hyper-stress tensor [9,10].

Such examples of media for which the Cauchy theory is not sufficient are not so exotic: in the literature, there are nowadays many particular physical phenomena described in the framework of generalized continuum theory (e.g. [11–25]). Some seminal contributions on the foundation of generalized continuum theories have been given in the papers [26–29]. Actually, immediately after the development of the Cauchy format of continuum mechanics, Gabrio Piola [30–32] already considered systems whose structure at micro-level requires a more sophisticated macroscopic model.

## (a) Higher gradient and microstructured continua

Also it is now widely accepted that in some circumstances, it is necessary to add to the placement field some extra kinematical fields, to take into account, at a macroscopic level, some aspects of the mechanical behaviour of materials having complex microscopic structures. In the aforementioned direction, a first relevant generalization of Cauchy continuum models was conceived by Eugène and François Cosserat: their efforts were not continued until late in the twentieth century. The Cosserat brothers described continuum bodies in which a complete kinematical description of considered continua can be obtained by adding suitable micro-rotation fields. In Cosserat models, contact interactions were to be modelled not only by means of surface forces, but also by means of surface couples. The conceptual differences between Cauchy-type continuum mechanics and Piola or Cosserat-type continuum mechanics were relevant, and the second one cannot be obtained by means of simple modifications of the first one. The remarkable mathematical difficulties confronted by Piola and Cosserat rendered their work difficult to be understood and accepted, and, for a long period, their results were almost completely ignored. This circumstance can be easily understood: the mathematical structure of Piola and Cosserat contact interactions is really complex. For instance, as shown in [33], in Piola's continua, one needs a *N-tuple* of stress tensors whose order is increasing from the second to the $N+1$th and contact interactions do not reduce to forces per unit area, but include $k$-forces[1] which may be concentrated on areas, on lines or even in wedges. On the other hand in Cosserat continua, one needs a couple stress tensor together with Cauchy stress tensor in order to represent contact couples.

As clearly stated already in his works by Germain [28,29], the Principle of Virtual Work supplies a suitable tool for extending the Cauchy–Navier format of continuum mechanics when it has to be generalized to include the so-called Generalized or Micro-Structured Continua. This principle has been successfully used for instance in [11,12,34–42] or in [43–53].

[1]As defined, for example, in [33] and in what follows.

## (b) Applicability range of generalized continuum theories

It has been widely recognized that higher gradient or microstructured models are needed for describing systems in which strong inhomogeneities and high contrast of physical properties are present at (possibly) different length scales (e.g. [9,10,54–68]). Therefore, many efforts have been directed towards more or less mathematically rigorous homogenization procedures leading to this class of continua (e.g. [69–75]). In particular, it has been noted that the introduction of $N$th-order models is suitable for describing non-local effects [76–80], some bio-mechanical phenomena [81–88], damage phenomena occurring in crack formation and growth (see those described in, for example, [89–96]) and internal friction in solids [97]. Theoretical prediction of band gaps has been recently provided in the case of granular media [98].

Bifurcation analysis of higher gradient continua has been performed, and the bifurcation condition for such models, when the ordinary first gradient contribution and the second gradient one are decoupled, only adds a size effect to classical conditions [99]. Further investigation on bifurcation results in generalized continua will require the employment of recent refined tools such as those developed in [100–102]. Finally, from the point of view of numerical investigation, generalized continua present numerous specific challenges; the development of powerful FE tools allowing high regularity between the elements, such as isogeometric analysis [103–105], is particularly useful for the numerical study of higher gradient continua.

## (c) Generalized contact interactions

Higher gradient or microstructured theories are sometimes developed and used taking into consideration in a too simplistic way the boundary conditions. Indeed, specifying these boundary conditions needs a precise understanding of the very special nature of mechanical contact interactions in these continua. Actually, the delicate but needed extension of Cauchy–Navier concepts of contact forces to more complex contact interactions have repelled mechanicians for a long time. Many results are available by now (e.g. [9,63,71,106–111]) indicating that it is physically needed or mathematically consistent to consider macroscopic continuum models where contact interactions expend work on high order virtual displacement gradients on dividing surfaces. These interactions are exactly those which are called $s$-forces, following Green & Rivlin [43,44,46] or [28,33]. This seems to be an essential common property of all systems that show highly contrasted physical properties at micro-level (see also [10,65]). On the purely macroscopic point of view, the necessity of considering such interactions has been proved in the two very elegant papers [28,29] by Germain when one wants to consistently consider continuum models in which deformation energy depend on second gradient of displacement (for higher gradients, see [33]). The conceptual framework introduced by Truesdell & Noll [112] is not general enough for encompassing such models (see, for example, the difficulties arising in [113,114] and clarified in [115]). The reader should be aware that the misunderstood range of validity of Noll's theorem persuaded many authors that the dependence of the deformation energy on higher gradient were forbidden by the second principle of thermodynamics (e.g. [116,117]) or that the second principle of thermodynamics needed to be modified [118,119]. In fact, this is not true as clearly proved, for example, in [12,28,50,120,121].

Generalized contact interactions are not usually considered in the literature. One can find two different reasons for this circumstance. First, this is due to the fact that the concept of virtual work is not always the preferred tool for mechanicians while, on the other hand, it gives the conceptual framework in which generalized contact interactions arise naturally [122]. Secondly, it is a fact that many usual materials are properly modelled by the classical Cauchy stress theory.

Assuming that contact interactions can be modelled by surface contact forces is indeed a *constitutive* assumption so deeply rooted in the mind of many authors that it has been very often accepted unconsciously and we emphasize that Noll's theorem [123,124] cannot be proved without starting from this assumption.

# 2. Interactions are to be modelled as work distributions

In his fundamental textbook [125], Lagrange introduces the concept of *moment* and discusses its roots in the works of Galileo. In modern terms, the word *moment*, as used by Lagrange, means *work*. It is evident that describing a force (respectively, a force field) $F$ is equivalent to describing the linear form which, to any test vector $V$ (respectively, test field), associates the expended work $F \cdot V$ (respectively, $\int F \cdot V$). In this dual view, forces are regarded as distributions in the sense of Laurent Schwartz. If the use of one of these two points of view is indifferent when dealing with the simplest mechanical interactions (i.e. forces), the second one is clearly more suitable for describing higher order interactions. Is there indeed a better way for defining for instance the mechanical meaning of a couple $\Gamma$ (respectively, of a field of couples $\gamma$ or of a stress field $\sigma$) than specifying that the work expended[2] is $\varepsilon_{ijk}\Gamma_i(\nabla V)_{jk}$ (respectively, $\int \varepsilon_{ijk}\gamma_i(\nabla V)_{jk}$ or $\int \sigma_{jk}(\nabla V)_{jk}$)?

In this paper, as our aim is to review results about complex interactions, the description in terms of distributions is mandatory. This is true even when the Principle of Virtual Work is not invoked.

## (a) Description of mechanical interactions in terms of distributions

It is natural to admit that the set of all admissible infinitesimal displacement fields for a continuous body $B$ contains the set $\mathcal{D}$ of all test functions (i.e. infinitely differentiable functions having compact support).

In accordance to what we have discussed in the previous section (as also done, for example, in [126] or [28,29,127]), we recognize that the mechanical interactions applied to an open subbody $D \subset B$ are distributions (in the sense of Schwartz) concentrated on $\bar{D}$, where $\bar{D}$ denotes the topological closure of the set $D$.

Therefore, theorems and definitions of the theory of distributions are really relevant also in continuum mechanics. In particular, we have to remind that [128, pp. 82–103]: (i) every distribution having regular[3] compact support $\bar{D}$ can be represented as the sum of a finite number of derivatives of measures all having their support included in $\bar{D}$; (ii) a distribution is said to have order smaller than or equal to $N$ if one can represent it as the sum of derivatives with order smaller than or equal to $N$ of measures; and (iii) every distribution having support included in a regular embedded submanifold $M$ can be uniquely decomposed as a finite sum of transverse derivatives of extensions of distributions defined on $M$.

In consequence, any mechanical interaction applied to $D$ has the following structure:

$$V \in \mathcal{D} \mapsto \sum_{s=0}^{N_D} \int (\nabla^s V) \mid \mathrm{d}T_D^s, \tag{2.1}$$

where $\mathrm{d}T_D^i$ are tensor valued measures having support in $\bar{D}$ and the symbol $\mid$ stands for the inner product between tensors.

The kinematics of considered continua may here be very general (e.g. the one specified in [129]). The configuration field may take values in a manifold and the velocity field in its tangent bundle, which can be of any tensorial nature. This tensorial nature is irrelevant for the validity of the presented results. For the sake of efficiency, we operate in this paper as if the kinematics were described by a real-valued function. Therefore, the tensor $\nabla^s V$ is considered to be of order $s$, as well as its dual quantities. It is straightforward, by applying the presented results componentwise, to extend them to the case where $V$ is a tensor and, in particular, in the classical case where $V$ is a vector.

In order to ensure uniqueness in the representation formulae, it is natural to ask that the measures $\mathrm{d}T_D^i$ respect the same symmetry as $\nabla^s V$, that is to be invariant with respect to any

[2]Here, we introduce Levi–Civita indicator $\varepsilon_{ijk}$ and use Einstein summation convention on repeated indices.

[3]Here, 'regular' must be understood in the sense of Whitney (cf. [128, p. 98]). This condition is weak enough and all sets considered in this paper verify it.

permutation of tensorial arguments. We call *complete symmetry* this property and denote Sym($X$) the completely symmetric part of any tensor $X$.

## (b) Frontier and inside-the-body interactions

One of the greatest challenges of any continuum mechanics theory is to describe the way in which the measures $\mathrm{d}T^i_D$ depend on the shape of $D$. The class of subbodies which are to be considered cannot be limited to domains with smooth boundaries. Indeed, tetrahedrons have to belong to this class if we want to follow the trail of Cauchy. Therefore, we admit subbodies $D$ with boundaries (or Cauchy dividing surface) which are piecewise regular. The topological boundary $\partial D$ is constituted by regular surfaces called faces (their union being denoted $\partial_2 D$), the boundary of which is constituted by regular curves called edges (their union being denoted $\partial_1 D$), concurring at wedges (their union being denoted $\partial_0 D$). We denote $n_k$ the external normal to $D$ on the face $\mathcal{F}_k$. On an edge $\mathcal{L}_j$, two faces $\mathcal{F}_k : k \in [\mathcal{L}_j]$ concur. Hence, $[\mathcal{L}_j]$ denotes the pair of subscripts of the faces concurring there. We denote $e_j$ a unit vector tangent to the edge $\mathcal{L}_j$ and $\nu^j_k$ the unit vector orthogonal to the line $\mathcal{L}_j$, tangent to the face $\mathcal{F}_k$ and external to it. On a wedge $\{x_\ell\}$, a finite number of edges $\mathcal{L}_j : j \in [x_\ell]$ concur, where $[x_\ell]$ denotes the set of subscripts of the edges concurring in the wedge $\{x_\ell\}$.

The description of the mechanical behaviour of a body needs the partition of the mechanical interactions applied to any subbody $D$ into two subclasses: those which are applied inside the body and those which are applied on its frontier:

$$\mathfrak{S}^{\mathrm{ins}}(D,V) = -\sum_{s=0}^{N_D} \int (\nabla^s V) \mid \mathrm{d}\tau_{s,D}, \mathfrak{S}^{\mathfrak{fro}}(D,V) = \sum_{s=0}^{N_D} \int (\nabla^s V) \mid \mathrm{d}F_{s,D},$$

where $\tau_{s,D}$ are tensor-valued measures concentrated in $D$, while $\mathrm{d}F_{s,D}$ are tensor-valued measures having support in the topological boundary of $D$. At this point, the distinction between these two kinds of interactions is completely arbitrary. We emphasize that it has been shown in [28,33] how an expression of type $\mathfrak{S}^{\mathfrak{ins}}(D,V)$ can be transformed in an expression of type $\mathfrak{S}^{\mathfrak{fro}}(D,V)$, while in [120,130] the converse is shown.

Actually, the necessity for mechanicians to divide the mechanical interactions into these subclasses comes from their desire to find constitutive laws for the tensors $\tau$ and $F$ which only involve local quantities. This distinction will be now on assumed as granted.

When accepting the point of view by Cauchy, it is the functional $\mathfrak{S}^{\mathfrak{fro}}$ which characterizes the stress state of the body. When there exists an integer $N = N_D$ such that the previous representation holds for all subbodies of the considered body $B$ then it is said that the body $B$ has a *stress state of order N in the sense of Cauchy*. We deal with measures $\mathrm{d}F_{s,D}$ constituted by three parts concentrated on $\partial_i D, i = 0, 1, 2$, each one being, respectively, absolutely continuous with respect to the corresponding natural Hausdorff measures:

$$\mathrm{d}F_{s,D} = \mathsf{F}^2_s \,\mathrm{d}\mathcal{H}^2_{|\partial_2 D} + \mathsf{F}^1_s \,\mathrm{d}\mathcal{H}^1_{|\partial_1 D} + \mathsf{F}^0_s \,\mathrm{d}\mathcal{H}^0_{|\partial_0 D}. \tag{2.2}$$

At this point, one should avoid a frequent and misleading confusion: indeed, when establishing the balance of forces on the boundary of a domain $D$ containing a surface $S$ carrying energy, one has to take into account a concentration of external forces along the line $S \cap \partial D$ which in general is not an edge of $\partial D$. This situation should not be confused with the concentration of external forces represented by $\mathsf{F}^1_s d\mathcal{H}^1_{|\partial_1 D}$ on an edge included in the topological boundary of $D$. The first case corresponds to physical concentration of energy (like surface tension or deformation energy of shells), while the second one is related to the geometrical singularity of a Cauchy Cut. The representation (2.2) allows only for the concentration of interactions on geometrical singularities of the frontier of $D$: it is a limiting circumstance. To our knowledge, in the literature, there is no unified theory encompassing lower dimensional concentration of energy and the only way which has been followed up to now for studying, for instance, a continuum containing surfaces endowed with surface tension was to use together a three-dimensional theory for the continuum and a two-dimensional theory for the contained surfaces: an approach which dates up to Laplace.

Moreover, we deal with fields $\mathsf{F}_s^i$ which are smooth tensor fields orthogonal to the manifold where they are applied:

$$\mathsf{F}_s^i \perp \partial_i D.$$

This is a further limitation as in the Schwartz decomposition of distributions concentrated on manifolds dual quantities to tangential components of test functions may appear. Note, however, that, if these tangent dual quantities are smooth enough to be integrated by parts, they reduce to functions plus dual quantities concentrated on lower order manifolds. Therefore, frontier interactions have the form:[4]

$$\mathfrak{S}^{\mathfrak{fro}}(D,V) = \sum_{s=0}^{N-1} \int_{\partial_2 D} \mathsf{F}_s^2 \mid (\nabla^s V)_\perp \, \mathrm{d}\mathcal{H}^2 + \sum_{s=0}^{N-2} \int_{\partial_1 D} \mathsf{F}_s^1 \mid (\nabla^s V)_\perp \, \mathrm{d}\mathcal{H}^1 + \sum_{s=0}^{N-3} \int_{\partial_0 D} \mathsf{F}_s^0 \mid \nabla^s V \, \mathrm{d}\mathcal{H}^0. \quad (2.3)$$

The tensor fields $(\mathsf{F}_s^2, \mathsf{F}_s^1, \mathsf{F}_s^0)$, which depend on $D$ and on the material particle, are naturally completely symmetric and normal to the manifolds where they are applied. They are called *the contact* $(s+1)$*-forces*.

One of the essential points of Cauchy approach (see [123] or [131]) is the determination of the dependence of the fields $\mathsf{F}_s^i$ on the (shape of the) subbody $D$. The densities $\mathsf{F}_s^i$ are assumed to depend in a sufficiently regular way on the position and to depend on the considered subbody only in a local way through its *shape*: a notion which contains all local geometrical characteristics of the frontier (including its direction). This notion is precisely defined in [120] where two domains are said to have the same shape if they coincide locally up to a translation.

When accepting the point of view by D'Alembert, it is the functional $\mathfrak{S}^{\mathfrak{ins}}$ that characterizes the stress state of the body. When there exists an integer $N$ such that the previous representation holds for all subbodies of considered body $B$, it is said that the body $B$ has a *stress state of order* $N$ *in the sense of D'Alembert*. The tensor measures $\mathrm{d}\tau_{s,D}$ are naturally completely symmetric. They are called *the sth-order (hyper)-stress tensors*. In the literature, the only tensor measures which were considered are absolutely continuous with respect to the volume measure $\mathrm{d}\mathcal{H}^3$, $\mathrm{d}\tau_{k,D} = \tau_{k,D}\,\mathrm{d}\mathcal{H}^3_{|D}$, with completely symmetric tensor densities. Moreover, the densities are supposed to be smooth enough to be repeatedly integrated by parts.

It has also to be remarked that the only possible way for the densities $\tau_{s,D}$ to depend on the local shape of $D$ is to be independent of $D$. Finally, one deals with representations of the type:

$$\mathfrak{S}^{\mathfrak{ins}}(D,V) = -\sum_{k=0}^{N} \int_D (\nabla^s V) \mid \tau_s \, \mathrm{d}\mathcal{H}^3. \quad (2.4)$$

To our knowledge, this type of representation has first been considered by Green & Rivlin [43–46] who called the tensors $\tau_s$ the $s$th-order stresses.

## (c) Alternative: D'Alembert versus Cauchy

The mechanical postulation *à la D'Alembert* consists in assuming given a stress state $\mathfrak{S}^{\mathfrak{ins}}$. Then, the procedure is to rewrite it as the sum of a term of type $\int_D V \mid f \, \mathrm{d}\mathcal{H}^3$ plus an expression similar to $\mathfrak{S}^{\mathfrak{fro}}$. This deduction is simply obtained by a repeated application of the divergence theorem.

The mechanical postulation *à la Cauchy* uses a reverse procedure. It consists in assuming an expression for $\mathfrak{S}^{\mathfrak{fro}}$ and rewriting it in a form similar to $\mathfrak{S}^{\mathfrak{ins}}$. This is a more difficult procedure and, to be completed, it needs the following (Quasi-)Balance Postulate: for every test field $V$, there

[4]The chosen summation bounds may seem restrictive. This is not the case, as one can easily add some extra terms with vanishing densities. We will see later on the reason for preferring to write the distribution in this form.

exists a constant $K_V$ such that, for every subbody $D$

$$|\mathfrak{S}^{\mathfrak{fro}}(D,V)| \leq K_V \mathcal{H}^3(D). \tag{2.5}$$

The reader should note that, when considering Cauchy continua and rigid virtual velocity fields $V$, the inequality (2.5) reduces to the quasi-balances of forces and moments put forward by Noll & Virga [132], but, as remarked in [120], these quasi-balances are not sufficient for obtaining a complete description of a stress state of order two or higher. While inequality (2.5) could seem a very weak assumption, it has been emphasized in [130] that it rules out some possible stress states, as for instance those occurring in continua including material surfaces or continua including interfaces with Laplace surface tension.

Even if often not explicitly stated, both procedures (Cauchy type as well as in D'Alembert type) are always completed by using the Postulate of Work Balance (or Postulate of Virtual Work) and the aforementioned uniqueness result by L. Schwartz. This postulate states that the total mechanical interactions vanish. In formula:

$$\mathfrak{S}^{\mathfrak{ins}} + \mathfrak{S}^{\mathfrak{fro}} = 0.$$

For a presentation of the ideas inspiring this postulate, we refer to [28,29,126] or to the works [30,31] (translated in [32]), [133,134]. This equality, which holds for every admissible subbody and test field, dates back to the pioneering works of D'Alembert, Lagrange and Piola [30–32,78] where it is shown that this principle is a generalization of Newton second law which is more suitable when dealing with more general systems than finite systems of material points (see also [135–137]).

Note that this postulate is usually written in a slightly different way: indeed, mechanical interactions are usually distinguished into internal and external ones. Here, $\mathfrak{S}^{\mathfrak{fro}}$ includes only contact interactions, while the external long range forces are included in $\mathfrak{S}^{\mathfrak{ins}}$. Note also that since the works by D'Alembert, inertial forces are treated like external interactions. Obviously, one should keep in mind that the distinction between interactions included in $\mathfrak{S}^{\mathfrak{fro}}$ or in $\mathfrak{S}^{\mathfrak{ins}}$, as well as the distinction between the internal and external interactions, are relative to the considered subbody.

## (d) The case of first gradient continua

The two methods we just described and their relationship are well known since the works by Piola [31,32] in the case of first gradient continua ($N=1$).

In that case, following D'Alembert, one assumes that the stress state is given by

$$\mathfrak{S}^{\mathfrak{ins}}(D,V) = -\int_D V \mid \tau_0 \, \mathrm{d}\mathcal{H}^3 - \int_D \nabla V \mid \tau_1 \, \mathrm{d}\mathcal{H}^3.$$

This can be rewritten under the desired form by using divergence theorem as ($n$ being the normal to $\partial_2 D$):

$$\mathfrak{S}^{\mathfrak{ins}}(D,V) = \int_D V \mid (-\tau_0 + \operatorname{div} \tau_1) \, \mathrm{d}\mathcal{H}^3 - \int_{\partial_2 D} V \mid (\tau_1 \cdot n) \, \mathrm{d}\mathcal{H}^3.$$

Using the Postulate of Virtual Work $\mathfrak{S}^{\mathfrak{ins}} + \mathfrak{S}^{\mathfrak{fro}} = 0$, we get $(\tau_0 - \operatorname{div} \tau_1) = 0$ in $D$ and $\mathfrak{S}^{\mathfrak{fro}}(D,V) = \int_{\partial_2 D} V \mid \mathsf{F}_0^2 \, \mathrm{d}\mathcal{H}^2$ with $\mathsf{F}_0^2 = \tau_1 \cdot n$. Here, we recognize the classical force balances inside and at the frontier of the body.

Following Cauchy, one instead assumes that the stress state is given by $\mathfrak{S}^{\mathfrak{fro}}(D,V) = \int_{\partial_2 D} V \mid \mathsf{F}_0^2 \, \mathrm{d}\mathcal{H}^2$. To proceed, one needs to confront two difficulties: establishing that $\mathsf{F}_0^2$ depends only on the normal $n$ to $\partial_2 D$ and establishing that $\mathsf{F}_0^2$ depends linearly on it. The second result, due to Cauchy, is universally known and is based on the quasi-balance of force applied to a tetrahedron. Quasi-balance of force $|\int_{\partial_2 D} \mathsf{F}_0^2 \, \mathrm{d}\mathcal{H}^2| \leq K \mathcal{H}^3(D)$ is simply deduced from the Quasi-Balance Postulate (2.5) by considering constant fields $V$. Only in the case of stress states of order one, this consequence is sufficient for proceeding. The same consequence is used to prove Noll's Theorem (e.g. [112, 120,123,126,138,139]) which states that: *the contact surface* $1-$*force* $\mathsf{F}_0^2$ *depends on the shape of* $D$ *only* *through the normal $n$ of $\partial_2 D$.* Based on Quasi-Balance and on Noll's result (which at the time of Cauchy was assumed as a Postulate), Cauchy tetrahedron theorem (see same references as above) states that there exists a tensor $\tau_1$ such that $\mathsf{F}_0^2 = \tau_1 \cdot n$. Using the Postulate of Virtual Work, we get

$$\mathfrak{S}^{\mathfrak{ins}}(D,V) = -\mathfrak{S}^{\mathfrak{fro}}(D,V) = -\int_{\partial_2 D} V \mid (\tau_1 \cdot n)\, \mathrm{d}\mathcal{H}^3 = -\int_D V \mid (\operatorname{div} \tau_1)\, \mathrm{d}\mathcal{H}^3 - \int_D \nabla V \mid \tau_1\, \mathrm{d}\mathcal{H}^3.$$

One can then observe that, in the case $N = 1$, both procedures lead to the same theory.

## 3. The foundation of the mechanics of continuous bodies *á la D'Alembert*

Following D'Alembert, Lagrange and Piola, one can found continuum mechanics by postulating a form for the work functional expressing internal interactions. Starting from this postulate, one can deduce, by means of a successive application of the theorem of divergence, the structure of the functionals expressing the contact interactions which can be exerted at the boundary of the considered body. Hence, this method starts from the notion of stress tensors (as dual of virtual displacements and their gradients) and deduces from it the concept and the structure of contact interactions by using the D'Alembert Principle of Virtual Work. This principle is undoubtedly a great tool in mechanics. It has not been improved since its original first (and standard) formulation (differently to what stated, for example, in [140]). This is a position generally maintained in the literature (see for instance in [141]).

In the approach *à la D'Alembert*, one assumes the Principle of Virtual Work to be valid for every subbody of considered continuous body. This is done in all the literature directly based on Lagrange's and Piola's works (e.g. [26–29,43–53,64,121,127,133,134,142–146]). An unduly restricted version of the principle has been formulated in [147, pp. 595–600]. For this reason, many authors, at different times, rediscovered its correct and complete formulation.

The D'Alembert spirit has been resumed by Casal [144,145], Toupin or Mindlin. Subsequently, Germain, in his enlightening papers [28,29], framed D'Alembert postulation by using the modern concepts of functional analysis. The works of Germain have been taken up again and again (e.g. in [140,148,149]), sometimes rephrasing them without introducing any notable amelioration. The Principle of Virtual Work is now being revived by many authors (e.g. [131,140,148–156]) who recognize that it is really a suitable conceptual basis for continuum mechanics. More detailed historical studies would be required to describe how and why the importance of the Principle of Virtual Work has been underestimated for long periods in the literature.

In order to construct in a more general case contact interactions as a derived concept from stresses, following the procedure *à la D'Alembert* which we already illustrated in the case of first gradient continua, we start by assuming that the representation (2.4) for $\mathfrak{S}^{\mathfrak{ins}}$ holds for all the stress states of the considered material. This is fundamentally a constitutive assumption which specifies the order and the smoothness of internal interactions which are considered to be admissible inside the body. The constitutive theory will be completed only once the dependence of the tensors $\tau_s$ on suitably introduced measures of deformation is specified. In the case of standard continuum models, for instance, the constitutive assumptions which specifies the way in which the stress tensor $\tau_1$ depends on Green–Saint–Venant deformation tensor and possibly on its time rate, are made after a more fundamental *constitutive* assumption: indeed, it is, usually implicitly, accepted that the stress state is of order one. It is these two constitutive assumptions which determine the set of external contact interactions which a material is able to sustain (this point is carefully discussed, for example, in [27–29,43,53,127,142,156–158] and in many other papers).

When the fundamental constitutive assumption is that the stress state is of order $N$, it has been determined which contact interactions are compatible with the general representation (2.4).

Indeed, in [33], the following identity is proved:

$$\mathfrak{G}^{\mathrm{ins}}(D,V) = -\int_D \tilde{\mathsf{F}}_0^3 \mid V\,\mathrm{d}\mathcal{H}^3 - \sum_{s=0}^{N-1}\int_{\partial_2 D} \tilde{\mathsf{F}}_s^2 \mid (\nabla^k V)_\perp\,\mathrm{d}\mathcal{H}^2$$
$$- \sum_{s=0}^{N-2}\int_{\partial_1 D} \tilde{\mathsf{F}}_s^1 \mid (\nabla^s V)_\perp\,\mathrm{d}\mathcal{H}^1 - \sum_{s=0}^{N-3}\int_{\partial_0 D} \tilde{\mathsf{F}}_s^0 \mid \nabla^s V\,\mathrm{d}\mathcal{H}^0, \tag{3.1}$$

where the tensors $\tilde{\mathsf{F}}_s^p$ are explicitly given in terms of the stress tensors $\tau_s$. The proof consists in integrating by parts the highest order term $\int_D \tau_N \mid \nabla^N V\,\mathrm{d}\mathcal{H}^3$. One obtains the boundary term $\int_{\partial_2 D}(\tau_N.n) \mid (\nabla^{N-1}V)\,\mathrm{d}\mathcal{H}^2$ plus a volume term of order $N-1$. At this point, a difficulty arises as it is necessary to write $\nabla^{N-1}V$ as the sum of a purely transverse term plus a tangent derivative.

This imposes the introduction of some geometrical and tensorial operators: for any smooth submanifold $M$ with boundary, one introduces the operators defined by setting, for any tensors $X$, $Y$ and $T$ of order $q$, $p-q$ and $p$, respectively, and any vector $v$

$$\mathbb{P}^p_{M,q}(X \otimes Y) := X_{\perp M} \otimes Y_{//M} \tag{3.2}$$

and

$$\mathbb{P}^p_M(T)\cdot v := \mathrm{Sym}\left(\sum_{q=0}^{p-1}\binom{p}{q}\mathbb{P}^{p-1}_{M,q}(T\cdot(\Pi_M\cdot v))\right), \tag{3.3}$$

where $\binom{q}{p}$ denote the binomial coefficients and the subscripts $\perp M$ and $//M$ stand for the parts of tensors totally orthogonal or parallel to $M$. One also denotes $\mathrm{div}_M$ the tangential divergence operator on $M$ and the composed operator $\mathrm{div}^\alpha_{\nvdash M}$ by setting in a recursive way:

$$\mathrm{div}^0_{\nvdash M} T := T, \quad \mathrm{div}^1_{\nvdash M}(T) := \mathrm{div}_M(\mathbb{P}(T)), \quad \mathrm{div}^\alpha_{\nvdash M}(T) := \mathrm{div}^1_{\nvdash M}(\mathrm{div}^{\alpha-1}_{\nvdash M}(T)).$$

One can thus use the following integration by parts formula:

$$\int_M X \mid \nabla^p V = \int_M X_\perp \mid (\nabla^p V)_\perp - \int_M \mathrm{div}_{\nvdash M}(X) \mid \nabla^{p-1} V + \int_{\partial M} \mathbb{P}(X)\cdot\nu \mid \nabla^{p-1} V,$$

which holds for any $C^1$ completely symmetric tensor field $X$ of order $p$ defined on $M$ and any $C^p$ vector field $V$ defined in some neighbourhood of $M$. Using this formula, the totally orthogonal part of the boundary term produces an addend in (3.1), a term of lesser order which will be dealt with later and a new term on the curves $\partial_1 D$. The procedure is repeated along the edges up to the wedges. Highest order terms are thus dealt with. At this point, the reader understands why the summation bounds decrease in formula (3.1). The lower terms are then treated in a similar way without forgetting that some quantities resulting from the higher order integration by parts need to be accounted for.

The expressions for all the tensors $\tilde{\mathsf{F}}_s^i$ which result from this procedure are the following:

$$\tilde{\mathsf{F}}_0^3 = \sum_{q=0}^{N}(-1)^q\,\mathrm{div}^q(\tau_q) \tag{3.4}$$

and

$$\tilde{\mathsf{F}}_s^i = \left(\sum_{q=s}^{N-3+i}(-1)^{q-s}(\mathrm{div}_{\nvdash \partial_i D})^{q-s} T_q^i\right)_{\perp \partial_i D}, \tag{3.5}$$

where the quantities $T^i_q$ are defined on each face $\mathcal{F}_k$, on each edge $\mathcal{L}_j$ and on each wedge $\hat{x}$ (and, respectively, denoted there $T^2_{\mathcal{F}_k,q}$, $T^1_{\mathcal{L}_j,q}$ and $T^0_{\hat{x},q}$) by:

$$T^2_{\mathcal{F}_k,q} := \sum_{r=q+1}^{N} (-1)^{r-1-q} (\mathrm{div}^{r-1-q}(\tau_r)) \cdot n_k, \tag{3.6}$$

$$T^1_{\mathcal{L}_j,q} = \sum_{k\in[\mathcal{L}_j]} \left( \sum_{r=q+1}^{N-1} (-1)^{r-1-q} \mathbb{P}_{\mathcal{F}_k}((\mathrm{div}_{\not\vdash\mathcal{F}_k})^{r-1-q} (T^2_{\mathcal{F}_k,r})) \cdot \nu^j_k \right) \tag{3.7}$$

and

$$T^0_{\hat{x},q} = \sum_{j\in[\hat{x}]} \sum_{k\in[\mathcal{L}_j]} \left( \sum_{r=q+1}^{N-2} (-1)^{r-1-q} \mathbb{P}_{\mathcal{L}_j}((\mathrm{div}_{\not\vdash\mathcal{L}_j})^{r-1-q} (T^1_{\mathcal{L}_j,r})) \cdot e_j \right). \tag{3.8}$$

The expressions thus obtained for the tensors $\tilde{\mathsf{F}}^i_s$ are complex. For the highest order terms, they reduce to the simpler form:

$$\tilde{\mathsf{F}}^2_{N-1} = (\tau_N \cdot n)_{\perp\mathcal{F}_k}, \tag{3.9}$$

$$\tilde{\mathsf{F}}^1_{N-2} = \sum_{k\in[\mathcal{L}_j]} (\mathbb{P}^{N-1}_{\mathcal{F}_k}(\tau_N \cdot n_k) \cdot \nu^j_k)_{\perp\mathcal{L}_j} \tag{3.10}$$

and

$$\tilde{\mathsf{F}}^0_{N-3} = \sum_{j\in[\hat{x}]} \sum_{k\in[\mathcal{L}_j]} (\mathbb{P}^{N-2}_{\mathcal{L}_j}(\mathbb{P}^{N-1}_{\mathcal{F}_k}(\tau_N \cdot n_k) \cdot \nu^j_k) \cdot e_j). \tag{3.11}$$

The use of the virtual work principle together with the uniqueness result for the representation of distributions in terms of transverse derivatives, allows to identify the tensors $\tilde{\mathsf{F}}^i_s$ ($i > 0$) with the actual contact interactions $\mathsf{F}^i_s$. These results show clearly the strict relationship between surface, edge and wedge contact interactions. First of all, one cannot assume in general (as done in [123, 124,159]) that contact interactions can be represented in terms of surface integrals only. Moreover, differently from what was done in [132], one cannot take into account, for instance, 1−forces on edges without taking into account also 2−forces on faces (this fact was already understood by Rivlin *et al.* [26,43,44,46,50]).

An important consequence of identity (3.1) and representation formula (3.11) is the uniqueness of the representation (2.4) for inside-the-body interactions. Indeed, if the quantity

$$\mathfrak{S}^{\mathfrak{ins}}(D,V) = -\sum_{k=0}^{N} \int_D (\nabla^s V) \mid \tau_s \, \mathrm{d}\mathcal{H}^3 \tag{3.12}$$

vanishes for all fields $V$ and all subdomains $D$ of $B$, then all tensors $\tau_s$ are identically vanishing. To prove this, it is enough to remark that (3.1) provides for any $D$ with smooth boundary, the representation of $\mathfrak{S}^{\mathfrak{ins}}(D,V)$ in term of transverse derivatives. As the uniqueness of this representation is ensured by Schwartz result, we can deduce, in particular, that $\tilde{F}^2_{N-1} = 0$ and thus $\tau_N \mid n^{\otimes N} = 0$. Varying arbitrarily $D$, we know that this equality is true at any point in $B$ and for any unit vector $n$. Recalling that the polarization formula gives the expression of any completely symmetric $N$-linear form in terms of diagonal terms, we get $\tau_N = 0$. A simple induction argument proves that all $\tau_s = 0$ have to vanish. This consequence is non-trivial as, in general, a distribution can be written in infinitely many ways under the form (2.4). Here, it is the particular dependence of $\mathfrak{S}^{\mathfrak{ins}}(D,V)$ with respect to $D$ and the fact that the tensors $\tau_s$ do not depend on it which provide this uniqueness result.

## 4. Postulation of the mechanics of continuous bodies *à la Cauchy*

At the beginning of the nineteenth century, Cauchy founded continuum mechanics by assuming that the surrounding material exerts on a part of a continuum, a mechanical interaction limited to a surface density of contact forces concentrated on the dividing surface. Then, by assuming that these contact forces depend only on the normal of dividing surface and are balanced by some volume density of force (including inertia), he played with tetrahedrons and proved the existence of the so-called Cauchy stress tensor.

As noted in [130], many authors consider tetrahedron argument as the untouchable basis of continuum mechanics (see [116,138] and the criticism raised in [160] and in [78]). In 1959, Noll [123] crystallized this faith by proving that the so-called Cauchy Postulate that is the dependence of contact forces only on the normal of dividing surfaces, is indeed equivalent to the seemingly weaker assumption of uniform boundedness of contact forces for all dividing surfaces. We underline that Cauchy Postulate, despite its designation, is not a fundamental Principle of Mechanics as sometimes believed but simply a constitutive assumption: nothing comparable, for what concerns generality, for instance to the balance of force, energy or to the Principle of Virtual Work. The merit of Noll's result consists in pointing out the relationship between tetrahedron argument and measure theory (e.g. [161]); the drawback is in camouflaging behind a technical hypothesis the physical assumption that the contact forces depend only on the normal. Actually, the contact force per unit surface at any regular point of a Cauchy cut (in what is called face here) does not depend, in general, only on the orientation of such surface (i.e. only on its normal $n$). Although many authors (among which Richard Toupin [26,121]) were aware of this fact, no effort has been attempted to generalize the tetrahedron construction in order to encompass theories of higher gradient continua until the works [115,120,130] (see also Maugin G. MathReview MR1437786 (98d:73003) 73A05 (73B18 73S10) on the paper [120]). The reason is probably due to the mathematical difficulties, as explicitly remarked in [115,120,141,162], which are encountered when dealing with the double dependence of power functional $\mathfrak{S}^{\mathfrak{fro}}(D,V)$ on velocity fields and on subbodies of the considered continuum. The efforts of Banfi *et al*. [141], Marzocchi & Musesti [162,163] and Degiovanni *et al*. [164] are directed, with remarkable results, to the search of a generalized Schwartz representation theorem adapted to this context.

In *De la pression ou tension dans un corps solide*, Cauchy wrote [165, pp. 61–64] that 'a small element experiences on its different faces and at each point of them a determined pressure or tension [...] which can depend on the orientation of the surface. This being set, [...]' and that 'equilibrium should hold between inertial force and the forces to which are reduced all pressures and tensions exerted on the surface'. In his proof, Cauchy applied the balance of forces to domains with a 'volume very small, so that every dimension is an infinitesimal quantity of first order' the mass being 'an infinitesimal quantity of the third order' and finally he stated that pressure and tension on a small face 'experience, by moving from one point to another one on a face, infinitesimal variations of the first order'. Clearly, Cauchy accepted the following hypotheses: (i) *contact interactions reduce to surface forces on the boundary and depend on its normal*; (ii) *contact interactions are balanced by volume forces*; and (iii) *contact interactions depend at least continuously on the position*.

When accepting the form (3.1), one weakens the assumption (i) and when accepting the Quasi-Balance Postulate (2.5), one adapts assumption (ii) to the new context. In order to generalize Cauchy procedure, one still needs assumptions similar to (iii) with some extra assumptions relative to the way in which contact interactions depend on the shape of the subdomains. We do not recall here these rather technical assumptions which are used in [120,130] to prove the existence of a field $C_N$ of completely symmetric tensors of order $N$ such that, at any point of a face $\mathcal{F}_k$,

$$\mathsf{F}^2_{N-1} = (C_N \mid n_k^{\otimes N}) \otimes n_k^{\otimes N-1}, \tag{4.1}$$

at any point of an edge $\mathcal{L}_j$

$$\mathsf{F}^1_{N-2} = \mathrm{Sym}\left(\sum_{k\in[\mathcal{L}_j]} (\mathcal{P}^{N-1}_{\mathcal{F}_k}(C_N \cdot n_k) \cdot \nu_k^j)\right)_{\perp\mathcal{L}_j}, \tag{4.2}$$

and, at any wedge point $\hat{x}$,

$$F^0_{N-3} = Sym\left(\sum_{j\in[\hat{x}]} \mathcal{P}^{N-2}_{\mathcal{L}_j}\left(\sum_{k\in[\mathcal{L}_j]} (\mathcal{P}^{N-1}_{\mathcal{F}_k}(C_N \cdot n_k) \cdot \nu^j_k)\right) \cdot e_j\right), \tag{4.3}$$

where the operator $\mathcal{P}_M$, for any submanifold $M$ of the physical space, is defined by setting for any tensor $X$, $v$ and $Y$ of order $q-1$, 1 and $p-q$, respectively,

$$\mathcal{P}^p_{M,q}(X \otimes v \otimes Y) := X \otimes Y_{\perp M} \otimes v_{//M}, \quad \mathcal{P}^p_M := \sum_{q=1}^{p} \mathcal{P}^p_{M,q}. \tag{4.4}$$

The proof given in [130] is inspired by the Cauchy tetrahedron construction. A family of tetrahedra with height tending to zero is considered and tested with a polynomial velocity field of order $N$. In the Quasi-Balance inequality, the terms involving $\mathsf{F}^2_{N-1}$, $\mathsf{F}^1_{N-2}$ and $\mathsf{F}^0_{N-3}$ are preponderant and thus must balance each other. One chooses one face $\mathcal{F}_k$ of the tetrahedron and one defines a tensor $\tilde{C}_N$ in terms of all quantities $\mathsf{F}^2_{N-1}$, $\mathsf{F}^1_{N-2}$ and $\mathsf{F}^0_{N-3}$ calculated on the other faces and on the edges and wedges which are not part of the boundary of $\mathcal{F}_k$. Therefore, the expression (4.1) is proved to be valid with the same $\tilde{C}_N$ for all $n_k$ in the unit sphere. The proof is constructive but intricate: it is first obtained for all $n_k$ inside a trihedron and then extended in the whole unit sphere via a topological argument. Straightforward calculations allow to check that equations (4.2) and (4.3) are identities as soon as applied to the contact forces involved in the definition of $\tilde{C}_N$. Cumbersome tensorial computations are needed to prove that $\tilde{C}_N$ can be replaced by its completely symmetric part $C_N$ which make all equations actual representation formulae. This first result concerns only highest order forces and tetrahedral shapes.

In order to extend it to more general shapes, a theorem (analogous to Noll theorem [123]) is needed which states that the highest order terms in $\mathfrak{S}^{\mathfrak{fro}}$ depend on the shape of the domain only through the tangent tetrahedral shape (see [120,130] for precise definitions). The idea of the proof (which can be found in [166,167] or [130]) is again to apply the Quasi-Balance inequality to a shrinking family of domains made by the intersection of $D$ and suitable polyhedra.

Finally, in order to obtain representation formulae for the lower order contact interactions, one observes that the highest order terms balance each other up to lower order ones. Indeed, it is proved in [130] that the quantity

$$\begin{aligned}\tilde{\mathfrak{S}}^{\mathfrak{fro}}(D,V) := &\int_D (C_N \mid \nabla^N V + \mathrm{div}(C_N) \mid \nabla^{N-1} V) - \int_{\partial_2 D} \mathsf{F}^2_{N-1} \mid \nabla^{N-1} V \\ &- \int_{\partial_1 D} \mathsf{F}^1_{N-2} \mid \nabla^{N-2} V - \int_{\partial_0 D} \mathsf{F}^0_{N-3} \mid \nabla^{N-3} V\end{aligned}$$

is a stress state in the sense of Cauchy of order $N-1$. Let us introduce the truncated stress state (of order $N-1$)

$$\bar{\mathfrak{S}}^{\mathfrak{fro}}(D,V) := \mathfrak{S}^{\mathfrak{fro}}(D,V) - \int_{\partial_2 D} \mathsf{F}^2_{N-1} \mid \nabla^{N-1} V - \int_{\partial_1 D} \mathsf{F}^1_{N-2} \mid \nabla^{N-2} V - \int_{\partial_0 D} \mathsf{F}^0_{N-3} \mid \nabla^{N-3} V.$$

The difference $\bar{\mathfrak{S}}^{\mathfrak{fro}} - \tilde{\mathfrak{S}}^{\mathfrak{fro}}$ is the sum of $\mathfrak{S}^{\mathfrak{fro}}$ plus a volume term and thus is quasi-balanced. As it is also the difference of two stress states of order $N-1$, it is a stress state of order $N-1$. The result concerning highest order interactions can be applied to this new stress state obtaining further representation formulae. Iterating this procedure, one has constructed a sequence of stress tensors $C_1, \ldots C_N$ representing all terms in $\mathfrak{S}^{\mathfrak{fro}}(D,V)$ but this iterative construction does not easily lead to explicit formulae.

To be more precise, we are using here an inductive definition: we say that *the sequence* $(C_1, \ldots C_N)$ *represents* $\mathfrak{S}^{\mathfrak{fro}}$ if (i) $C_N$ represents the highest order terms and (ii) the sequence $(C_1, \ldots C_{N-1})$ represents $\bar{\mathfrak{S}}^{\mathfrak{fro}} - \tilde{\mathfrak{S}}^{\mathfrak{fro}}$.

In [130], this construction has been made explicit up to third gradient theories.

The representating sequence $(C_1, \dots C_N)$ enable us to write $\mathfrak{S}^{\mathfrak{fro}}$ in a form similar to $\mathfrak{S}^{\mathfrak{ins}}$. Indeed, we have

**Lemma 4.1.** *For any $N > 0$ and any quasi-balanced $\mathfrak{S}^{\mathfrak{fro}}$ having expression (3.1), let $(C_1, \dots, C_N)$ be the associated stress tensors obtained via the Cauchy type procedure described in §4. Then the following identity holds*:

$$\mathfrak{S}^{\mathfrak{fro}}(D, V) = \sum_{s=0}^{N} \int_D \tilde{\tau}_s \mid \nabla^s V \, \mathrm{d}\mathcal{H}^3, \tag{4.5}$$

*with $\tilde{\tau}_0 := \mathrm{div}(C_1)$, $\tilde{\tau}_s := C_s + \mathrm{div}(C_{s+1})$ for $0 < s < N$ and $\tilde{\tau}_N := C_N$.*

*Proof.* We use an induction argument. In the case $N = 1$, the identity (4.5) reads

$$\int_{\partial_2 D} \mathsf{F}_0^2 \mid V \, \mathrm{d}\mathcal{H}^2 = \int_D (\tilde{\tau}_0 \mid V + \tilde{\tau}_1 \mid \nabla V) \, \mathrm{d}\mathcal{H}^3$$

that is

$$\int_{\partial_2 D} (C_1 \cdot n) \mid V \, \mathrm{d}\mathcal{H}^2 = \int_D (\mathrm{div}(C_1) \mid V + C_1 \mid \nabla V) \, \mathrm{d}\mathcal{H}^3$$

which results directly from the divergence theorem.

Assume now that the Lemma holds for any quasi-balanced Cauchy stress state of order $N - 1$. By construction, $(C_1, \dots C_{N-1})$ represents $\bar{\mathfrak{S}}^{\mathfrak{fro}} - \tilde{\mathfrak{S}}^{\mathfrak{fro}}$. Therefore, owing to the induction assumption, we have

$$\bar{\mathfrak{S}}^{\mathfrak{fro}}(D, V) - \tilde{\mathfrak{S}}^{\mathfrak{fro}}(D, V) = \sum_{s=0}^{N-2} \int_D \tilde{\tau}_s \mid \nabla^s V \, \mathrm{d}\mathcal{H}^3 + \int_D C_{N-1} \mid \nabla^{N-1} V \, \mathrm{d}\mathcal{H}^3.$$

Hence,

$$\begin{aligned} \mathfrak{S}^{\mathfrak{fro}}(D, V) &= \bar{\mathfrak{S}}^{\mathfrak{fro}}(D, V) - \tilde{\mathfrak{S}}^{\mathfrak{fro}}(D, V) + \int_D (C_N \mid \nabla^N V + \mathrm{div}(C_N) \mid \nabla^{N-1} V), \\ &= \sum_{s=0}^{N} \int_D \tilde{\tau}_s \mid \nabla^s V \, \mathrm{d}\mathcal{H}^3. \end{aligned}$$

■

By assuming the Principle of Virtual Work, the last expression coincides with $\mathfrak{S}^{\mathfrak{ins}}(D, V)$ and the unicity result we stated in §3 implies that the tensors $\tau_s$ appearing in $\mathfrak{S}^{\mathfrak{ins}}(D, V)$ coincide with the tensors $\tilde{\tau}_s$.

# 5. Cauchy versus D'Alembert postulations: the two methods can be reconciled

In fact the two methods can be reconciled. Their equivalence has already been explicitly established by Gabrio Piola [32] for stress states of order one. Much later, the same equivalence has been proved for stress states of order two: this results has been obtained in [115,120] where the relationship between the concept of contact line force and surface double force was clearly established by obtaining a representation formula relating the two concepts (on line forces see also [168,169]).

The results we have recalled or established in the previous sections show that the operators $\mathcal{O}_D$ which associate the $s$-forces $\tilde{\mathsf{F}}_s^i$ to the tensors $\tau_s$ as specified by the formulae (3.5)–(3.8), resulting from the D'Alembert type procedure, are identical to the operators $\mathcal{O}_C$ which associate the $s$-forces $\mathsf{F}_s^i$ to the tensors $\tilde{\tau}_s$ following the Cauchy type procedure described in §4.

Indeed, let us consider some family $(\tau_s)$ of stress tensors representing a D'Alembert stress state

$$\mathfrak{S}^{\mathrm{ins}}(D,V)=\sum_{s=0}^{N}\int_D \tau_s \mid \nabla^s V \, \mathrm{d}\mathcal{H}^3.$$

The d'Alembert procedure provides a family of $s$-forces $(\tilde{\mathsf{F}}^i_s)=\mathcal{O}_D((\tau_s))$ such that

$$\sum_{s=0}^{N}\int_D \tau_s \mid \nabla^s V \, \mathrm{d}\mathcal{H}^3=\sum_{i=0}^{2}\left(\sum_{s=0}^{N-3+i}\int_{\partial_i D}\tilde{\mathsf{F}}^i_s \mid (\nabla^s V)_\perp \, \mathrm{d}\mathcal{H}^i\right).$$

On the other hand, starting from the $\tilde{\mathsf{F}}^i_s$, the Cauchy procedure provides a family $(C_s)$ and an associated family $(\tilde{\tau}_s)$ such that $(\tilde{\mathsf{F}}^i_s)=\mathcal{O}_C((\tilde{\tau}_s))$ and

$$\sum_{s=0}^{N}\int_D \tilde{\tau}_s \mid \nabla^s V \, \mathrm{d}\mathcal{H}^3=\sum_{i=0}^{2}\left(\sum_{s=0}^{N-3+i}\int_{\partial_i D}\tilde{\mathsf{F}}^i_s \mid (\nabla^s V)_\perp \, \mathrm{d}\mathcal{H}^i\right).$$

We have proved at the end of §3 the uniqueness of the representation of $\mathfrak{S}^{\mathrm{ins}}$ in terms of stress tensors. Hence, $\tilde{\tau}_s=\tau_s$ and $\tilde{\mathsf{F}}^i_s=\mathcal{O}_D((\tau_s))=\mathcal{O}_C((\tau_s))$.

The previous proof is indirect and it has sometimes been objected that the explicit formulae giving the highest order forces while presenting some similarities, were different following Cauchy or D'Alembert procedures. We show now, using only algebric arguments, that they are equivalent.

**Lemma 5.1.** *The operators which associate the highest order forces (that is the surface N-force $\tilde{\mathsf{F}}^2_{N-1}$ on any face $\mathcal{F}_k$, the line $N-1$-force $\tilde{\mathsf{F}}^1_{N-2}$ on any edge $\mathcal{L}_j$ and the $N-2$-force $\tilde{\mathsf{F}}^0_{N-3}$ on wedge $\hat{x}$) to the tensor $\tau_N$ as specified by the formulae (3.9)–(3.11) resulting from the D'Alembert type procedure, are identical to the operators which associate $\mathsf{F}^2_{N-1}$, $\mathsf{F}^1_{N-2}$ and $\mathsf{F}^0_{N-3}$ to the tensor $C_N$ as specified by formulae (4.1)–(4.3) resulting from the Cauchy-type procedure.*

*Proof.* The proof needs some rather technical steps which, for the sake of clarity, we postpone to appendix A.

The fact that (3.9) is equivalent to (4.1) is obvious. The fact that (3.10) is equivalent to (4.2) is a simple consequence of the fact that, for any submanifold $M$ and any completely symmetric tensor $X$ of order $p$,

$$\mathrm{Sym}(\mathcal{P}^p_M(Y)\cdot v)=\mathbb{P}^p_M(Y)\cdot v,$$

the proof of which is postponed to appendix A (lemma A.1). Indeed, it is enough to apply this identity for every $k\in[\mathcal{L}_j]$ with $M=\mathcal{F}_k$, $X=C_N\cdot n_k$ and $v=v^j_k$.

In order to prove that (3.11) is equivalent to (4.3), we remark, by applying twice lemma A.1 that, for any $j\in[\hat{x}]$ and $k\in[\mathcal{L}_j]$,

$$\begin{aligned}\mathbb{P}^{N-2}_{\mathcal{L}_j}(\mathbb{P}^{N-1}_{\mathcal{F}_k}(C_N\cdot n_k)\cdot v^j_k)\cdot e_j&=\mathbb{P}^{N-2}_{\mathcal{L}_j}(\mathrm{Sym}(\mathcal{P}^{N-1}_{\mathcal{F}_k}(C_N\cdot n_k)\cdot v^j_k))\cdot e_j\\&=\mathrm{Sym}(\mathcal{P}^{N-2}_{\mathcal{L}_j}(\mathrm{Sym}(\mathcal{P}^{N-1}_{\mathcal{F}_k}(C_N\cdot n_k)\cdot v^j_k))\cdot e_j)\\&=\mathrm{Sym}(\mathcal{P}^{N-2}_{\mathcal{L}_j}(\mathcal{P}^{N-1}_{\mathcal{F}_k}(C_N\cdot n_k)\cdot v^j_k)\cdot e_j).\end{aligned}$$

The last equality being due to the fact that, for any line $\mathcal{L}$, any completely symmetric tensor $X$ of order $p$ and any vector $e$ tangent to $\mathcal{L}$,

$$\mathrm{Sym}(\mathcal{P}^p_{\mathcal{L}}(Y)\cdot e)=\mathrm{Sym}(\mathcal{P}^p_{\mathcal{L}}(\mathrm{Sym}(Y))\cdot e),$$

the proof of which is postponed to appendix A (lemma A.2). ∎

# 6. Some perspectives for future researches

Even if the modelling of $N$th gradient continua is well founded by the two methods which are now reconciled, many questions about stress states remain open. It has to be remarked that the available results are far to include all possible shapes for bodies. It is not clear if it is possible to determine the set of domains to which a theory can apply independently of the considered constitutive equations. On the other hand, all the works we have described accept a bound for the order of the stress state while it would also be interesting, at least from a theoretical point of view, to understand what happens in a body where the order of the stress state varies from point to point, being unbounded.

We have already emphasized the fact that the presented results cannot encompass stress states for which there are stress concentrations along lower dimensional manifolds, models which are needed if one wants to model in a unifying way for instance a two-dimensional plate included in a three-dimensional elastic body. To our knowledge, the theoretical tools for attacking this important problem remain to be developed.

Based on the original ideas of Lagrange himself, the principles of power balance have received attention also in dynamics, namely in vibrations and acoustics. In this field, some authors (e.g. [170,171]) attempted to write a self-contained set of equations to describe power migration through a continuum medium: this situation resembles the one in which Dunn & Serrin [113,114] found themselves in the context of incomplete second gradient theories. In our opinion (generalizing what is done in [120]), higher gradient theories may complete the cited attempt or, in general, supply a regularized model when non-convex energy functions need to be introduced (as in Cahn–Hilliard and Korteweg fluids [172] or in many other physical situations, see e.g. [40,173–176]). Finally, the power balance equations can be also approached in the context of uncertainties in the constitutive relationships, where some randomness affects the physical parameters of the equivalent continuum (e.g. [177]). In this case, higher order gradients would be related to the introduction of some statistical average and ergodic assumption.

Higher order gradient theories are needed when boundary layer phenomena must be described: when considering impact phenomena (e.g. [89,178–180]) in general some *ad hoc* assumptions are imposed, especially when choosing boundary conditions. More detailed models for impact between solids or between solids and fluids, involving some space–time length scales, may cure some of the singularities which are present in many models presented in the literature: in particular, one could conceive to describe the phenomena of water spray formation or turbulence (see, respectively, [179] with references there cited or [181]) by means of suitable contact edge forces.

Finally, the constant technological progress allows now to conceive and built metamaterials with designed mechanical properties (a general review on the subject is [182], while interesting developments are in [183]). Making metamaterials in which the higher order effects are preponderant is a real challenge. The search for possible applications of such materials is a free field for future research.

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# Appendix A. Two technical lemmas

We use the operator defined by setting, for any tensors $X$ and $Y$ of order $q$ and $p-q$, respectively,

$$\mathbb{R}^{p}_{M,q}(X \otimes Y) := X \otimes Y_{\perp M} \tag{A 1}$$

and we note that, for any completely symmetric tensor $X$ of order $p$ and any vector $v$ tangent to $M$, we have $\mathcal{P}^p_{M,q}(X)\cdot v = \mathbb{R}^{p-1}_{M,q-1}(X\cdot v)$ and consequently

$$\mathcal{P}^p_M(X)\cdot v = \sum_{q=1}^{p} \mathcal{P}^p_{M,q}(X)\cdot v = \sum_{q=0}^{p-1} \mathbb{R}^{p-1}_{M,q}(X\cdot v). \tag{A 2}$$

**Lemma A.1.** *For any completely symmetric tensor $Y$ of order $p$, the following identity holds*:

$$\mathrm{Sym}\left(\sum_{q=0}^{p} \mathbb{R}^p_{M,q}(Y)\right) = \mathrm{Sym}\left(\sum_{q=0}^{p} \binom{q}{p+1} \mathbb{P}^p_{M,q}(Y)\right). \tag{A 3}$$

*As a consequence, for any vector $v$ tangent to $M$, we have*

$$\mathrm{Sym}(\mathcal{P}^p_M(Y)\cdot v) = \mathbb{P}^p_M(Y)\cdot v. \tag{A 4}$$

*Proof.* Let us first remark that a simple induction argument leads to the formula

$$\sum_{q=r}^{N} \binom{q}{r} = \binom{N+1}{r+1}. \tag{A 5}$$

To prove (A 3), it is enough to check the identity with tensors of the type

$$Y = \mathrm{Sym}(t_1 \otimes t_2 \otimes \cdots \otimes t_\alpha \otimes n_1 \otimes n_2 \otimes \cdots \otimes n_\beta),$$

where the vectors $t_i$ are tangent to $M$, the vectors $n_i$ are normal to it and $\alpha + \beta = p$. Computing the number of permutations in the symmetrization of $t_1 \otimes t_2 \otimes \cdots \otimes t_\alpha \otimes n_1 \otimes n_2 \otimes \cdots \otimes n_\beta$ which give non-vanishing results we obtain

$$\mathrm{Sym}(\mathbb{P}^p_{M,q}(Y)) = \begin{cases} \dfrac{\alpha!\beta!}{p!} Y & \text{if } q = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Sym}(\mathbb{R}^p_{M,q}(Y)) = \begin{cases} \dfrac{\beta! q!}{p!(q-\alpha)!} Y & \text{if } q \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

From the first equation, we deduce

$$\mathrm{Sym}\left(\sum_{q=0}^{p} \binom{p+1}{q} \mathbb{P}^p_{M,q}(Y)\right) = \binom{p+1}{\beta} \frac{\alpha!\beta!}{p!} Y = \frac{p+1}{\alpha+1} Y,$$

and from the second one,

$$\mathrm{Sym}\left(\sum_{q=0}^{p} \mathbb{R}^p_{M,q}(Y)\right) = \left(\sum_{q=\alpha}^{p} \binom{q}{\alpha}\right) \frac{\alpha!\beta!}{p!} Y = \frac{p+1}{\alpha+1} Y.$$

The first identity is thus proved. The second one is an obvious consequence. Indeed.

$$\mathrm{Sym}(\mathcal{P}^p_M(X)\cdot v) = \mathrm{Sym}\left(\sum_{q=0}^{p-1} \mathbb{R}^{p-1}_{M,q}(X\cdot v)\right) = \mathrm{Sym}\left(\sum_{q=0}^{p-1} \binom{q}{p} \mathbb{P}^{p-1}_{M,q}(X\cdot v)\right) = \mathbb{P}^p_M(X)\cdot v. \qquad \blacksquare$$

**Lemma A.2.** *For any line $\mathcal{L}$, any completely symmetric tensor $X$ of order $p$ and any vector $e$ tangent to $\mathcal{L}$,*

$$\mathrm{Sym}(\mathcal{P}^p_{\mathcal{L}}(X)\cdot e) = \mathrm{Sym}(\mathcal{P}^p_{\mathcal{L}}(\mathrm{Sym}(X))\cdot e).$$

*Proof.* To that aim, let us introduce $\mathcal{P}^{p*}_{\mathcal{L}}$ the adjoint operator of $\mathcal{P}^{p}_{\mathcal{L}}$ and check this last equality by checking, for any completely symmetric tensor $Y$ of order $p-1$,

$$(\mathcal{P}^{p}_{\mathcal{L}}(X)\cdot e) \mid Y = (\mathcal{P}^{p}_{\mathcal{L}}(\mathrm{Sym}(X))\cdot e) \mid Y$$

or

$$X \mid \mathcal{P}^{p*}_{\mathcal{L}}(Y\otimes e) = \mathrm{Sym}(X) \mid \mathcal{P}^{p*}_{\mathcal{L}}(Y\otimes e).$$

We are thus reduced to proving that, when $Y$ is a completely symmetric tensor of order $p-1$ and $e$ a vector tangent to a line $\mathcal{L}$, $\mathcal{P}^{p*}_{\mathcal{L}}(Y\otimes e)$ is completely symmetric or, equivalently, invariant with respect to any permutation of indices $(\ell,\ell+1)$ for $\ell\in\{1,\dots,p-1\}$. Noticing that

$$\mathcal{P}^{p*}_{\mathcal{L}}(Y\otimes e) = \sum_{r=1}^{p} \mathcal{P}^{p*}_{\mathcal{L},r}(Y\otimes e) \tag{A 6}$$

and that

$$(\mathcal{P}^{p*}_{\mathcal{L},r}(Y\otimes e))_{i_1,\dots i_p} = Y_{i_1,\dots,i_{r-1},j_{r+1}\dots j_p}\,\Lambda^{j_{r+1}}_{i_{r+1}}\dots\Lambda^{j_p}_{i_p}e_{i_r}$$

(where $\Lambda$ denotes the projector onto the orthogonal space to the line $\mathcal{L}$), it is immediately clear that the symmetry of $Y$ implies the invariance with respect to the permutation of indices $(\ell,\ell+1)$ of all terms in the sum (A 6) for which $r<\ell-1$ or $r>\ell$. Noticing that

$$Y_{i_1,\dots,i_{\ell-1},j_{\ell+1}\dots j_p} = Y_{i_1,\dots,i_{\ell-2},j_\ell\dots j_p}(\Lambda^{j_\ell}_{i_{\ell-1}} + e_{i_{\ell-1}}e_{j_\ell})$$

the sum of the two terms corresponding to $r=\ell-1$ and $r=\ell$ reads

$$\left(\sum_{r=\ell-1}^{\ell} \mathcal{P}^{p*}_{\mathcal{L},r}(Y\otimes e)\right)_{i_1,\dots i_p} = Y_{i_1,\dots,i_{\ell-2},j_\ell\dots j_p}(\Lambda^{j_\ell}_{i_\ell}e_{i_{\ell-1}} + \Lambda^{j_\ell}_{i_{\ell-1}}e_{i_\ell} + e_{i_{\ell-1}}e_{j_\ell}e_{i_\ell})\Lambda^{j_{\ell+1}}_{i_{\ell+1}}\dots\Lambda^{j_p}_{i_p}.$$

It becomes now clear that the sum of these two terms, and thus the whole sum, are invariant with respect to the permutation of indices $(\ell,\ell+1)$. ∎

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