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The postulations *à la D'Alembert* and *à la Cauchy* for higher gradient continuum theories are equivalent: a review of existing results

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In order to found continuum mechanics, two different postulations have been used. The first, introduced by Lagrange and Piola, starts by postulating how the work expended by internal interactions in a body depends on the virtual velocity field and its gradients. Then, by using the divergence theorem, a representation theorem is found for the volume and contact interactions which can be exerted at the boundary of the considered body. This method assumes an *a priori* notion of internal work, regards stress tensors as dual of virtual displacements and their gradients, deduces the concept of contact interactions and produces their representation in terms of stresses using integration by parts. The second method, conceived by Cauchy and based on the celebrated tetrahedron argument, starts by postulating the type of contact interactions which can be exerted on the boundary of every (suitably) regular part of a body. Then it proceeds by proving the existence of stress tensors from a balance-type postulate. In this paper, we review some relevant literature on the subject, discussing how the two postulations can be reconciled in the case of higher gradient theories. Finally, we underline the importance of the concept of contact surface, edge and wedge s -order forces.

1. Introduction

Continuum mechanics always supplies approximate models for physical systems, in which a more fundamental (possibly discrete or inhomogeneous) microstructure may be somehow neglected. Actually, the founders of continuum mechanics, Piola, Poisson, Navier and many others did try to justify continuum theories by means of an average procedure based on atomistic models.

Cauchy continuum theory (or Cauchy–Navier theory as described in its historical development by Benvenuto [1]) describes efficiently, at a macroscopic level, the behaviour of a mechanical system only when the inhomogeneities which the model does not take into account do have a characteristic length scale much smaller than the macro-scale where phenomena are observed. The aforementioned condition of scale separation is not by itself a sufficient criterion for ensuring that Cauchy theory supplies a suitable model: the best-known example is the case of deformable porous media for which both stress tensor for matrix and pressure for fluid are needed to describe its mechanical state [2–8]. Another example is given by the case of a periodic fibre-reinforced elastic medium with high contrast of mechanical properties. The mechanical description of these systems needs in addition to the standard stress tensor a higher order hyper-stress tensor [9,10].

Such examples of media for which the Cauchy theory is not sufficient are not so exotic: in the literature, there are nowadays many particular physical phenomena described in the framework of generalized continuum theory (e.g. [11–25]). Some seminal contributions on the foundation of generalized continuum theories have been given in the papers [26–29]. Actually, immediately after the development of the Cauchy format of continuum mechanics, Gabrio Piola [30–32] already considered systems whose structure at micro-level requires a more sophisticated macroscopic model.

(a) Higher gradient and microstructured continua

Also it is now widely accepted that in some circumstances, it is necessary to add to the placement field some extra kinematical fields, to take into account, at a macroscopic level, some aspects of the mechanical behaviour of materials having complex microscopic structures. In the aforementioned direction, a first relevant generalization of Cauchy continuum models was conceived by Eugène and François Cosserat: their efforts were not continued until late in the twentieth century. The Cosserat brothers described continuum bodies in which a complete kinematical description of considered continua can be obtained by adding suitable micro-rotation fields. In Cosserat models, contact interactions were to be modelled not only by means of surface forces, but also by means of surface couples. The conceptual differences between Cauchy-type continuum mechanics and Piola or Cosserat-type continuum mechanics were relevant, and the second one cannot be obtained by means of simple modifications of the first one. The remarkable mathematical difficulties confronted by Piola and Cosserat rendered their work difficult to be understood and accepted, and, for a long period, their results were almost completely ignored. This circumstance can be easily understood: the mathematical structure of Piola and Cosserat contact interactions is really complex. For instance, as shown in [33], in Piola’s continua, one needs a N -tuple of stress tensors whose order is increasing from the second to the $N + 1$ th and contact interactions do not reduce to forces per unit area, but include k -forces¹ which may be concentrated on areas, on lines or even in wedges. On the other hand in Cosserat continua, one needs a couple stress tensor together with Cauchy stress tensor in order to represent contact couples.

As clearly stated already in his works by Germain [28,29], the Principle of Virtual Work supplies a suitable tool for extending the Cauchy–Navier format of continuum mechanics when it has to be generalized to include the so-called Generalized or Micro-Structured Continua. This principle has been successfully used for instance in [11,12,34–42] or in [43–53].

¹As defined, for example, in [33] and in what follows.

(b) Applicability range of generalized continuum theories

It has been widely recognized that higher gradient or microstructured models are needed for describing systems in which strong inhomogeneities and high contrast of physical properties are present at (possibly) different length scales (e.g. [9,10,54–68]). Therefore, many efforts have been directed towards more or less mathematically rigorous homogenization procedures leading to this class of continua (e.g. [69–75]). In particular, it has been noted that the introduction of N th-order models is suitable for describing non-local effects [76–80], some bio-mechanical phenomena [81–88], damage phenomena occurring in crack formation and growth (see those described in, for example, [89–96]) and internal friction in solids [97]. Theoretical prediction of band gaps has been recently provided in the case of granular media [98].

Bifurcation analysis of higher gradient continua has been performed, and the bifurcation condition for such models, when the ordinary first gradient contribution and the second gradient one are decoupled, only adds a size effect to classical conditions [99]. Further investigation on bifurcation results in generalized continua will require the employment of recent refined tools such as those developed in [100–102]. Finally, from the point of view of numerical investigation, generalized continua present numerous specific challenges; the development of powerful FE tools allowing high regularity between the elements, such as isogeometric analysis [103–105], is particularly useful for the numerical study of higher gradient continua.

(c) Generalized contact interactions

Higher gradient or microstructured theories are sometimes developed and used taking into consideration in a too simplistic way the boundary conditions. Indeed, specifying these boundary conditions needs a precise understanding of the very special nature of mechanical contact interactions in these continua. Actually, the delicate but needed extension of Cauchy–Navier concepts of contact forces to more complex contact interactions have repelled mechanicians for a long time. Many results are available by now (e.g. [9,63,71,106–111]) indicating that it is physically needed or mathematically consistent to consider macroscopic continuum models where contact interactions expend work on high order virtual displacement gradients on dividing surfaces. These interactions are exactly those which are called s -forces, following Green & Rivlin [43,44,46] or [28,33]. This seems to be an essential common property of all systems that show highly contrasted physical properties at micro-level (see also [10,65]). On the purely macroscopic point of view, the necessity of considering such interactions has been proved in the two very elegant papers [28,29] by Germain when one wants to consistently consider continuum models in which deformation energy depend on second gradient of displacement (for higher gradients, see [33]). The conceptual framework introduced by Truesdell & Noll [112] is not general enough for encompassing such models (see, for example, the difficulties arising in [113,114] and clarified in [115]). The reader should be aware that the misunderstood range of validity of Noll's theorem persuaded many authors that the dependence of the deformation energy on higher gradient were forbidden by the second principle of thermodynamics (e.g. [116,117]) or that the second principle of thermodynamics needed to be modified [118,119]. In fact, this is not true as clearly proved, for example, in [12,28,50,120,121].

Generalized contact interactions are not usually considered in the literature. One can find two different reasons for this circumstance. First, this is due to the fact that the concept of virtual work is not always the preferred tool for mechanicians while, on the other hand, it gives the conceptual framework in which generalized contact interactions arise naturally [122]. Secondly, it is a fact that many usual materials are properly modelled by the classical Cauchy stress theory.

Assuming that contact interactions can be modelled by surface contact forces is indeed a *constitutive* assumption so deeply rooted in the mind of many authors that it has been very often accepted unconsciously and we emphasize that Noll's theorem [123,124] cannot be proved without starting from this assumption.

2. Interactions are to be modelled as work distributions

In his fundamental textbook [125], Lagrange introduces the concept of *moment* and discusses its roots in the works of Galileo. In modern terms, the word *moment*, as used by Lagrange, means *work*. It is evident that describing a force (respectively, a force field) F is equivalent to describing the linear form which, to any test vector V (respectively, test field), associates the expended work $F \cdot V$ (respectively, $\int F \cdot V$). In this dual view, forces are regarded as distributions in the sense of Laurent Schwartz. If the use of one of these two points of view is indifferent when dealing with the simplest mechanical interactions (i.e. forces), the second one is clearly more suitable for describing higher order interactions. Is there indeed a better way for defining for instance the mechanical meaning of a couple Γ (respectively, of a field of couples γ or of a stress field σ) than specifying that the work expended² is $\varepsilon_{ijk}\Gamma_i(\nabla V)_{jk}$ (respectively, $\int \varepsilon_{ijk}\gamma_i(\nabla V)_{jk}$ or $\int \sigma_{jk}(\nabla V)_{jk}$)?

In this paper, as our aim is to review results about complex interactions, the description in terms of distributions is mandatory. This is true even when the Principle of Virtual Work is not invoked.

(a) Description of mechanical interactions in terms of distributions

It is natural to admit that the set of all admissible infinitesimal displacement fields for a continuous body B contains the set \mathcal{D} of all test functions (i.e. infinitely differentiable functions having compact support).

In accordance to what we have discussed in the previous section (as also done, for example, in [126] or [28,29,127]), we recognize that the mechanical interactions applied to an open subbody $D \subset B$ are distributions (in the sense of Schwartz) concentrated on \bar{D} , where \bar{D} denotes the topological closure of the set D .

Therefore, theorems and definitions of the theory of distributions are really relevant also in continuum mechanics. In particular, we have to remind that [128, pp. 82–103]: (i) every distribution having regular³ compact support \bar{D} can be represented as the sum of a finite number of derivatives of measures all having their support included in \bar{D} ; (ii) a distribution is said to have order smaller than or equal to N if one can represent it as the sum of derivatives with order smaller than or equal to N of measures; and (iii) every distribution having support included in a regular embedded submanifold M can be uniquely decomposed as a finite sum of transverse derivatives of extensions of distributions defined on M .

In consequence, any mechanical interaction applied to D has the following structure:

$$V \in \mathcal{D} \mapsto \sum_{s=0}^{N_D} \int (\nabla^s V) | dT_D^s, \quad (2.1)$$

where dT_D^i are tensor valued measures having support in \bar{D} and the symbol $|$ stands for the inner product between tensors.

The kinematics of considered continua may here be very general (e.g. the one specified in [129]). The configuration field may take values in a manifold and the velocity field in its tangent bundle, which can be of any tensorial nature. This tensorial nature is irrelevant for the validity of the presented results. For the sake of efficiency, we operate in this paper as if the kinematics were described by a real-valued function. Therefore, the tensor $\nabla^s V$ is considered to be of order s , as well as its dual quantities. It is straightforward, by applying the presented results componentwise, to extend them to the case where V is a tensor and, in particular, in the classical case where V is a vector.

In order to ensure uniqueness in the representation formulae, it is natural to ask that the measures dT_D^i respect the same symmetry as $\nabla^s V$, that is to be invariant with respect to any

²Here, we introduce Levi-Civita indicator ε_{ijk} and use Einstein summation convention on repeated indices.

³Here, ‘regular’ must be understood in the sense of Whitney (cf. [128, p. 98]). This condition is weak enough and all sets considered in this paper verify it.

permutation of tensorial arguments. We call *complete symmetry* this property and denote $\text{Sym}(X)$ the completely symmetric part of any tensor X .

(b) Frontier and inside-the-body interactions

One of the greatest challenges of any continuum mechanics theory is to describe the way in which the measures dT_D^i depend on the shape of D . The class of subbodies which are to be considered cannot be limited to domains with smooth boundaries. Indeed, tetrahedrons have to belong to this class if we want to follow the trail of Cauchy. Therefore, we admit subbodies D with boundaries (or Cauchy dividing surface) which are piecewise regular. The topological boundary ∂D is constituted by regular surfaces called faces (their union being denoted $\partial_2 D$), the boundary of which is constituted by regular curves called edges (their union being denoted $\partial_1 D$), concurring at wedges (their union being denoted $\partial_0 D$). We denote n_k the external normal to D on the face \mathcal{F}_k . On an edge \mathcal{L}_j , two faces $\mathcal{F}_k : k \in [\mathcal{L}_j]$ concur. Hence, $[\mathcal{L}_j]$ denotes the pair of subscripts of the faces concurring there. We denote e_j a unit vector tangent to the edge \mathcal{L}_j and v_k^j the unit vector orthogonal to the line \mathcal{L}_j , tangent to the face \mathcal{F}_k and external to it. On a wedge $\{x_\ell\}$, a finite number of edges $\mathcal{L}_j : j \in [x_\ell]$ concur, where $[x_\ell]$ denotes the set of subscripts of the edges concurring in the wedge $\{x_\ell\}$.

The description of the mechanical behaviour of a body needs the partition of the mechanical interactions applied to any subbody D into two subclasses: those which are applied inside the body and those which are applied on its frontier:

$$\mathfrak{S}^{\text{ins}}(D, V) = - \sum_{s=0}^{N_D} \int (\nabla^s V) | d\tau_{s,D}, \quad \mathfrak{S}^{\text{fro}}(D, V) = \sum_{s=0}^{N_D} \int (\nabla^s V) | dF_{s,D},$$

where $\tau_{s,D}$ are tensor-valued measures concentrated in D , while $dF_{s,D}$ are tensor-valued measures having support in the topological boundary of D . At this point, the distinction between these two kinds of interactions is completely arbitrary. We emphasize that it has been shown in [28,33] how an expression of type $\mathfrak{S}^{\text{ins}}(D, V)$ can be transformed in an expression of type $\mathfrak{S}^{\text{fro}}(D, V)$, while in [120,130] the converse is shown.

Actually, the necessity for mechanicians to divide the mechanical interactions into these subclasses comes from their desire to find constitutive laws for the tensors τ and F which only involve local quantities. This distinction will be now on assumed as granted.

When accepting the point of view by Cauchy, it is the functional $\mathfrak{S}^{\text{fro}}$ which characterizes the stress state of the body. When there exists an integer $N = N_D$ such that the previous representation holds for all subbodies of the considered body B then it is said that the body B has a *stress state of order N in the sense of Cauchy*. We deal with measures $dF_{s,D}$ constituted by three parts concentrated on $\partial_i D, i = 0, 1, 2$, each one being, respectively, absolutely continuous with respect to the corresponding natural Hausdorff measures:

$$dF_{s,D} = F_s^2 d\mathcal{H}_{|\partial_2 D}^2 + F_s^1 d\mathcal{H}_{|\partial_1 D}^1 + F_s^0 d\mathcal{H}_{|\partial_0 D}^0. \quad (2.2)$$

At this point, one should avoid a frequent and misleading confusion: indeed, when establishing the balance of forces on the boundary of a domain D containing a surface S carrying energy, one has to take into account a concentration of external forces along the line $S \cap \partial D$ which in general is not an edge of ∂D . This situation should not be confused with the concentration of external forces represented by $F_s^1 d\mathcal{H}_{|\partial_1 D}^1$ on an edge included in the topological boundary of D . The first case corresponds to physical concentration of energy (like surface tension or deformation energy of shells), while the second one is related to the geometrical singularity of a Cauchy Cut. The representation (2.2) allows only for the concentration of interactions on geometrical singularities of the frontier of D : it is a limiting circumstance. To our knowledge, in the literature, there is no unified theory encompassing lower dimensional concentration of energy and the only way which has been followed up to now for studying, for instance, a continuum containing surfaces

endowed with surface tension was to use together a three-dimensional theory for the continuum and a two-dimensional theory for the contained surfaces: an approach which dates up to Laplace.

Moreover, we deal with fields F_s^i which are smooth tensor fields orthogonal to the manifold where they are applied:

$$F_s^i \perp \partial_i D.$$

This is a further limitation as in the Schwartz decomposition of distributions concentrated on manifolds dual quantities to tangential components of test functions may appear. Note, however, that, if these tangential dual quantities are smooth enough to be integrated by parts, they reduce to functions plus dual quantities concentrated on lower order manifolds. Therefore, frontier interactions have the form:⁴

$$\mathfrak{S}^{fr}o(D, V) = \sum_{s=0}^{N-1} \int_{\partial_2 D} F_s^2 | (\nabla^s V)_{\perp} d\mathcal{H}^2 + \sum_{s=0}^{N-2} \int_{\partial_1 D} F_s^1 | (\nabla^s V)_{\perp} d\mathcal{H}^1 + \sum_{s=0}^{N-3} \int_{\partial_0 D} F_s^0 | \nabla^s V d\mathcal{H}^0. \quad (2.3)$$

The tensor fields (F_s^2, F_s^1, F_s^0) , which depend on D and on the material particle, are naturally completely symmetric and normal to the manifolds where they are applied. They are called *the contact* $(s + 1)$ -*forces*.

One of the essential points of Cauchy approach (see [123] or [131]) is the determination of the dependence of the fields F_s^i on the (shape of the) subbody D . The densities F_s^i are assumed to depend in a sufficiently regular way on the position and to depend on the considered subbody only in a local way through its *shape*: a notion which contains all local geometrical characteristics of the frontier (including its direction). This notion is precisely defined in [120] where two domains are said to have the same shape if they coincide locally up to a translation.

When accepting the point of view by D'Alembert, it is the functional \mathfrak{S}^{ins} that characterizes the stress state of the body. When there exists an integer N such that the previous representation holds for all subbodies of considered body B , it is said that the body B has a *stress state of order N in the sense of D'Alembert*. The tensor measures $d\tau_{s,D}$ are naturally completely symmetric. They are called *the sth-order (hyper)-stress tensors*. In the literature, the only tensor measures which were considered are absolutely continuous with respect to the volume measure $d\mathcal{H}^3$, $d\tau_{k,D} = \tau_{k,D} d\mathcal{H}_{|D}^3$, with completely symmetric tensor densities. Moreover, the densities are supposed to be smooth enough to be repeatedly integrated by parts.

It has also to be remarked that the only possible way for the densities $\tau_{s,D}$ to depend on the local shape of D is to be independent of D . Finally, one deals with representations of the type:

$$\mathfrak{S}^{ins}(D, V) = - \sum_{k=0}^N \int_D (\nabla^k V) | \tau_k d\mathcal{H}^3. \quad (2.4)$$

To our knowledge, this type of representation has first been considered by Green & Rivlin [43–46] who called the tensors τ_s the *sth-order stresses*.

(c) Alternative: D'Alembert versus Cauchy

The mechanical postulation *à la D'Alembert* consists in assuming given a stress state \mathfrak{S}^{ins} . Then, the procedure is to rewrite it as the sum of a term of type $\int_D V | f d\mathcal{H}^3$ plus an expression similar to $\mathfrak{S}^{fr}o$. This deduction is simply obtained by a repeated application of the divergence theorem.

The mechanical postulation *à la Cauchy* uses a reverse procedure. It consists in assuming an expression for $\mathfrak{S}^{fr}o$ and rewriting it in a form similar to \mathfrak{S}^{ins} . This is a more difficult procedure and, to be completed, it needs the following (Quasi-)Balance Postulate: for every test field V , there

⁴The chosen summation bounds may seem restrictive. This is not the case, as one can easily add some extra terms with vanishing densities. We will see later on the reason for preferring to write the distribution in this form.

exists a constant K_V such that, for every subbody D

$$|\mathfrak{S}^{\text{f}^{\text{to}}}(D, V)| \leq K_V \mathcal{H}^3(D). \quad (2.5)$$

The reader should note that, when considering Cauchy continua and rigid virtual velocity fields V , the inequality (2.5) reduces to the quasi-balances of forces and moments put forward by Noll & Virga [132], but, as remarked in [120], these quasi-balances are not sufficient for obtaining a complete description of a stress state of order two or higher. While inequality (2.5) could seem a very weak assumption, it has been emphasized in [130] that it rules out some possible stress states, as for instance those occurring in continua including material surfaces or continua including interfaces with Laplace surface tension.

Even if often not explicitly stated, both procedures (Cauchy type as well as in D'Alembert type) are always completed by using the Postulate of Work Balance (or Postulate of Virtual Work) and the aforementioned uniqueness result by L. Schwartz. This postulate states that the total mechanical interactions vanish. In formula:

$$\mathfrak{S}^{\text{ins}} + \mathfrak{S}^{\text{f}^{\text{to}}} = 0.$$

For a presentation of the ideas inspiring this postulate, we refer to [28,29,126] or to the works [30,31] (translated in [32]), [133,134]. This equality, which holds for every admissible subbody and test field, dates back to the pioneering works of D'Alembert, Lagrange and Piola [30–32,78] where it is shown that this principle is a generalization of Newton second law which is more suitable when dealing with more general systems than finite systems of material points (see also [135–137]).

Note that this postulate is usually written in a slightly different way: indeed, mechanical interactions are usually distinguished into internal and external ones. Here, $\mathfrak{S}^{\text{f}^{\text{to}}}$ includes only contact interactions, while the external long range forces are included in $\mathfrak{S}^{\text{ins}}$. Note also that since the works by D'Alembert, inertial forces are treated like external interactions. Obviously, one should keep in mind that the distinction between interactions included in $\mathfrak{S}^{\text{f}^{\text{to}}}$ or in $\mathfrak{S}^{\text{ins}}$, as well as the distinction between the internal and external interactions, are relative to the considered subbody.

(d) The case of first gradient continua

The two methods we just described and their relationship are well known since the works by Piola [31,32] in the case of first gradient continua ($N = 1$).

In that case, following D'Alembert, one assumes that the stress state is given by

$$\mathfrak{S}^{\text{ins}}(D, V) = - \int_D V | \tau_0 \, d\mathcal{H}^3 - \int_D \nabla V | \tau_1 \, d\mathcal{H}^3.$$

This can be rewritten under the desired form by using divergence theorem as (n being the normal to $\partial_2 D$):

$$\mathfrak{S}^{\text{ins}}(D, V) = \int_D V | (-\tau_0 + \text{div } \tau_1) \, d\mathcal{H}^3 - \int_{\partial_2 D} V | (\tau_1 \cdot n) \, d\mathcal{H}^3.$$

Using the Postulate of Virtual Work $\mathfrak{S}^{\text{ins}} + \mathfrak{S}^{\text{f}^{\text{to}}} = 0$, we get $(\tau_0 - \text{div } \tau_1) = 0$ in D and $\mathfrak{S}^{\text{f}^{\text{to}}}(D, V) = \int_{\partial_2 D} V | F_0^2 \, d\mathcal{H}^2$ with $F_0^2 = \tau_1 \cdot n$. Here, we recognize the classical force balances inside and at the frontier of the body.

Following Cauchy, one instead assumes that the stress state is given by $\mathfrak{S}^{\text{f}^{\text{to}}}(D, V) = \int_{\partial_2 D} V | F_0^2 \, d\mathcal{H}^2$. To proceed, one needs to confront two difficulties: establishing that F_0^2 depends only on the normal n to $\partial_2 D$ and establishing that F_0^2 depends linearly on it. The second result, due to Cauchy, is universally known and is based on the quasi-balance of force applied to a tetrahedron. Quasi-balance of force $|\int_{\partial_2 D} F_0^2 \, d\mathcal{H}^2| \leq K\mathcal{H}^3(D)$ is simply deduced from the Quasi-Balance Postulate (2.5) by considering constant fields V . Only in the case of stress states of order one, this consequence is sufficient for proceeding. The same consequence is used to prove Noll's Theorem (e.g. [112, 120,123,126,138,139]) which states that: *the contact surface 1-force F_0^2 depends on the shape of D only*

through the normal n of $\partial_2 D$. Based on Quasi-Balance and on Noll's result (which at the time of Cauchy was assumed as a Postulate), Cauchy tetrahedron theorem (see same references as above) states that there exists a tensor τ_1 such that $f_0^2 = \tau_1 \cdot n$. Using the Postulate of Virtual Work, we get

$$\mathfrak{S}^{\text{ins}}(D, V) = -\mathfrak{S}^{\text{ro}}(D, V) = - \int_{\partial_2 D} V | (\tau_1 \cdot n) d\mathcal{H}^3 = - \int_D V | (\text{div } \tau_1) d\mathcal{H}^3 - \int_D \nabla V | \tau_1 d\mathcal{H}^3.$$

One can then observe that, in the case $N = 1$, both procedures lead to the same theory.

3. The foundation of the mechanics of continuous bodies *à la D'Alembert*

Following D'Alembert, Lagrange and Piola, one can found continuum mechanics by postulating a form for the work functional expressing internal interactions. Starting from this postulate, one can deduce, by means of a successive application of the theorem of divergence, the structure of the functionals expressing the contact interactions which can be exerted at the boundary of the considered body. Hence, this method starts from the notion of stress tensors (as dual of virtual displacements and their gradients) and deduces from it the concept and the structure of contact interactions by using the D'Alembert Principle of Virtual Work. This principle is undoubtedly a great tool in mechanics. It has not been improved since its original first (and standard) formulation (differently to what stated, for example, in [140]). This is a position generally maintained in the literature (see for instance in [141]).

In the approach *à la D'Alembert*, one assumes the Principle of Virtual Work to be valid for every subbody of considered continuous body. This is done in all the literature directly based on Lagrange's and Piola's works (e.g. [26–29,43–53,64,121,127,133,134,142–146]). An unduly restricted version of the principle has been formulated in [147, pp. 595–600]. For this reason, many authors, at different times, rediscovered its correct and complete formulation.

The D'Alembert spirit has been resumed by Casal [144,145], Toupin or Mindlin. Subsequently, Germain, in his enlightening papers [28,29], framed D'Alembert postulation by using the modern concepts of functional analysis. The works of Germain have been taken up again and again (e.g. in [140,148,149]), sometimes rephrasing them without introducing any notable amelioration. The Principle of Virtual Work is now being revived by many authors (e.g. [131,140,148–156]) who recognize that it is really a suitable conceptual basis for continuum mechanics. More detailed historical studies would be required to describe how and why the importance of the Principle of Virtual Work has been underestimated for long periods in the literature.

In order to construct in a more general case contact interactions as a derived concept from stresses, following the procedure *à la D'Alembert* which we already illustrated in the case of first gradient continua, we start by assuming that the representation (2.4) for $\mathfrak{S}^{\text{ins}}$ holds for all the stress states of the considered material. This is fundamentally a constitutive assumption which specifies the order and the smoothness of internal interactions which are considered to be admissible inside the body. The constitutive theory will be completed only once the dependence of the tensors τ_s on suitably introduced measures of deformation is specified. In the case of standard continuum models, for instance, the constitutive assumptions which specifies the way in which the stress tensor τ_1 depends on Green–Saint–Venant deformation tensor and possibly on its time rate, are made after a more fundamental *constitutive* assumption: indeed, it is, usually implicitly, accepted that the stress state is of order one. It is these two constitutive assumptions which determine the set of external contact interactions which a material is able to sustain (this point is carefully discussed, for example, in [27–29,43,53,127,142,156–158] and in many other papers).

When the fundamental constitutive assumption is that the stress state is of order N , it has been determined which contact interactions are compatible with the general representation (2.4).

Indeed, in [33], the following identity is proved:

$$\begin{aligned} \mathfrak{S}^{\text{ins}}(D, V) = & - \int_D \tilde{F}_0^3 | V \, d\mathcal{H}^3 - \sum_{s=0}^{N-1} \int_{\partial_2 D} \tilde{F}_s^2 | (\nabla^k V)_\perp \, d\mathcal{H}^2 \\ & - \sum_{s=0}^{N-2} \int_{\partial_1 D} \tilde{F}_s^1 | (\nabla^s V)_\perp \, d\mathcal{H}^1 - \sum_{s=0}^{N-3} \int_{\partial_0 D} \tilde{F}_s^0 | \nabla^s V \, d\mathcal{H}^0, \end{aligned} \quad (3.1)$$

where the tensors \tilde{F}_s^p are explicitly given in terms of the stress tensors τ_s . The proof consists in integrating by parts the highest order term $\int_D \tau_N | \nabla^N V \, d\mathcal{H}^3$. One obtains the boundary term $\int_{\partial_2 D} (\tau_N \cdot n) | (\nabla^{N-1} V) \, d\mathcal{H}^2$ plus a volume term of order $N - 1$. At this point, a difficulty arises as it is necessary to write $\nabla^{N-1} V$ as the sum of a purely transverse term plus a tangent derivative.

This imposes the introduction of some geometrical and tensorial operators: for any smooth submanifold M with boundary, one introduces the operators defined by setting, for any tensors X, Y and T of order $q, p - q$ and p , respectively, and any vector v

$$\mathbb{P}_{M,q}^p(X \otimes Y) := X_{\perp M} \otimes Y_{//M} \quad (3.2)$$

and

$$\mathbb{P}_M^p(T) \cdot v := \text{Sym} \left(\sum_{q=0}^{p-1} \binom{p}{q} \mathbb{P}_{M,q}^{p-1}(T \cdot (\Pi_M \cdot v)) \right), \quad (3.3)$$

where $\binom{q}{p}$ denote the binomial coefficients and the subscripts $\perp M$ and $//M$ stand for the parts of tensors totally orthogonal or parallel to M . One also denotes div_M the tangential divergence operator on M and the composed operator $\text{div}_{\nu M}^\alpha$ by setting in a recursive way:

$$\text{div}_{\nu M}^0 T := T, \quad \text{div}_{\nu M}^1(T) := \text{div}_M(\mathbb{P}(T)), \quad \text{div}_{\nu M}^\alpha(T) := \text{div}_{\nu M}^1(\text{div}_{\nu M}^{\alpha-1}(T)).$$

One can thus use the following integration by parts formula:

$$\int_M X | \nabla^p V = \int_M X_\perp | (\nabla^p V)_\perp - \int_M \text{div}_{\nu M}(X) | \nabla^{p-1} V + \int_{\partial M} \mathbb{P}(X) \cdot v | \nabla^{p-1} V,$$

which holds for any C^1 completely symmetric tensor field X of order p defined on M and any C^p vector field V defined in some neighbourhood of M . Using this formula, the totally orthogonal part of the boundary term produces an addend in (3.1), a term of lesser order which will be dealt with later and a new term on the curves $\partial_1 D$. The procedure is repeated along the edges up to the wedges. Highest order terms are thus dealt with. At this point, the reader understands why the summation bounds decrease in formula (3.1). The lower terms are then treated in a similar way without forgetting that some quantities resulting from the higher order integration by parts need to be accounted for.

The expressions for all the tensors \tilde{F}_s^i which result from this procedure are the following:

$$\tilde{F}_0^3 = \sum_{q=0}^N (-1)^q \text{div}^q(\tau_q) \quad (3.4)$$

and

$$\tilde{F}_s^i = \left(\sum_{q=s}^{N-3+i} (-1)^{q-s} (\text{div}_{\nu \partial_1 D})^{q-s} T_q^i \right)_{\perp \partial_1 D}, \quad (3.5)$$

where the quantities T_q^i are defined on each face \mathcal{F}_k , on each edge \mathcal{L}_j and on each wedge \hat{x} (and, respectively, denoted there $T_{\mathcal{F}_k,q}^2$, $T_{\mathcal{L}_j,q}^1$ and $T_{\hat{x},q}^0$) by:

$$T_{\mathcal{F}_k,q}^2 := \sum_{r=q+1}^N (-1)^{r-1-q} (\operatorname{div}^{r-1-q}(\tau_r)) \cdot n_k, \quad (3.6)$$

$$T_{\mathcal{L}_j,q}^1 = \sum_{k \in [\mathcal{L}_j]} \left(\sum_{r=q+1}^{N-1} (-1)^{r-1-q} \mathbb{P}_{\mathcal{F}_k}((\operatorname{div}_{\mathcal{F}_k})^{r-1-q}(T_{\mathcal{F}_k,r}^2)) \cdot v_k^j \right) \quad (3.7)$$

and

$$T_{\hat{x},q}^0 = \sum_{j \in [\hat{x}]} \sum_{k \in [\mathcal{L}_j]} \left(\sum_{r=q+1}^{N-2} (-1)^{r-1-q} \mathbb{P}_{\mathcal{L}_j}((\operatorname{div}_{\mathcal{L}_j})^{r-1-q}(T_{\mathcal{L}_j,r}^1)) \cdot e_j \right). \quad (3.8)$$

The expressions thus obtained for the tensors \tilde{F}_s^i are complex. For the highest order terms, they reduce to the simpler form:

$$\tilde{F}_{N-1}^2 = (\tau_N \cdot n)_{\perp \mathcal{F}_k}, \quad (3.9)$$

$$\tilde{F}_{N-2}^1 = \sum_{k \in [\mathcal{L}_j]} (\mathbb{P}_{\mathcal{F}_k}^{N-1}(\tau_N \cdot n_k) \cdot v_k^j)_{\perp \mathcal{L}_j} \quad (3.10)$$

and

$$\tilde{F}_{N-3}^0 = \sum_{j \in [\hat{x}]} \sum_{k \in [\mathcal{L}_j]} (\mathbb{P}_{\mathcal{L}_j}^{N-2}(\mathbb{P}_{\mathcal{F}_k}^{N-1}(\tau_N \cdot n_k) \cdot v_k^j) \cdot e_j). \quad (3.11)$$

The use of the virtual work principle together with the uniqueness result for the representation of distributions in terms of transverse derivatives, allows to identify the tensors \tilde{F}_s^i ($i > 0$) with the actual contact interactions F_s^i . These results show clearly the strict relationship between surface, edge and wedge contact interactions. First of all, one cannot assume in general (as done in [123, 124,159]) that contact interactions can be represented in terms of surface integrals only. Moreover, differently from what was done in [132], one cannot take into account, for instance, 1–forces on edges without taking into account also 2–forces on faces (this fact was already understood by Rivlin *et al.* [26,43,44,46,50]).

An important consequence of identity (3.1) and representation formula (3.11) is the uniqueness of the representation (2.4) for inside-the-body interactions. Indeed, if the quantity

$$\mathfrak{S}^{\text{ins}}(D, V) = - \sum_{k=0}^N \int_D (\nabla^s V) \cdot \tau_s \, d\mathcal{H}^3 \quad (3.12)$$

vanishes for all fields V and all subdomains D of B , then all tensors τ_s are identically vanishing. To prove this, it is enough to remark that (3.1) provides for any D with smooth boundary, the representation of $\mathfrak{S}^{\text{ins}}(D, V)$ in term of transverse derivatives. As the uniqueness of this representation is ensured by Schwartz result, we can deduce, in particular, that $\tilde{F}_{N-1}^2 = 0$ and thus $\tau_N \cdot n^{\otimes N} = 0$. Varying arbitrarily D , we know that this equality is true at any point in B and for any unit vector n . Recalling that the polarization formula gives the expression of any completely symmetric N -linear form in terms of diagonal terms, we get $\tau_N = 0$. A simple induction argument proves that all $\tau_s = 0$ have to vanish. This consequence is non-trivial as, in general, a distribution can be written in infinitely many ways under the form (2.4). Here, it is the particular dependence of $\mathfrak{S}^{\text{ins}}(D, V)$ with respect to D and the fact that the tensors τ_s do not depend on it which provide this uniqueness result.

4. Postulation of the mechanics of continuous bodies *à la Cauchy*

At the beginning of the nineteenth century, Cauchy founded continuum mechanics by assuming that the surrounding material exerts on a part of a continuum, a mechanical interaction limited to a surface density of contact forces concentrated on the dividing surface. Then, by assuming that

these contact forces depend only on the normal of dividing surface and are balanced by some volume density of force (including inertia), he played with tetrahedrons and proved the existence of the so-called Cauchy stress tensor.

As noted in [130], many authors consider tetrahedron argument as the untouchable basis of continuum mechanics (see [116,138] and the criticism raised in [160] and in [78]). In 1959, Noll [123] crystallized this faith by proving that the so-called Cauchy Postulate that is the dependence of contact forces only on the normal of dividing surfaces, is indeed equivalent to the seemingly weaker assumption of uniform boundedness of contact forces for all dividing surfaces. We underline that Cauchy Postulate, despite its designation, is not a fundamental Principle of Mechanics as sometimes believed but simply a constitutive assumption: nothing comparable, for what concerns generality, for instance to the balance of force, energy or to the Principle of Virtual Work. The merit of Noll's result consists in pointing out the relationship between tetrahedron argument and measure theory (e.g. [161]); the drawback is in camouflaging behind a technical hypothesis the physical assumption that the contact forces depend only on the normal. Actually, the contact force per unit surface at any regular point of a Cauchy cut (in what is called face here) does not depend, in general, only on the orientation of such surface (i.e. only on its normal n). Although many authors (among which Richard Toupin [26,121]) were aware of this fact, no effort has been attempted to generalize the tetrahedron construction in order to encompass theories of higher gradient continua until the works [115,120,130] (see also Maugin G. MathReview MR1437786 (98d:73003) 73A05 (73B18 73S10) on the paper [120]). The reason is probably due to the mathematical difficulties, as explicitly remarked in [115,120,141,162], which are encountered when dealing with the double dependence of power functional $\mathfrak{S}^{\text{tr}\circ}(D, V)$ on velocity fields and on subbodies of the considered continuum. The efforts of Banfi *et al.* [141], Marzocchi & Musesti [162,163] and Degiovanni *et al.* [164] are directed, with remarkable results, to the search of a generalized Schwartz representation theorem adapted to this context.

In *De la pression ou tension dans un corps solide*, Cauchy wrote [165, pp. 61–64] that ‘a small element experiences on its different faces and at each point of them a determined pressure or tension [...] which can depend on the orientation of the surface. This being set, [...] and that ‘equilibrium should hold between inertial force and the forces to which are reduced all pressures and tensions exerted on the surface’. In his proof, Cauchy applied the balance of forces to domains with a ‘volume very small, so that every dimension is an infinitesimal quantity of first order’ the mass being ‘an infinitesimal quantity of the third order’ and finally he stated that pressure and tension on a small face ‘experience, by moving from one point to another one on a face, infinitesimal variations of the first order’. Clearly, Cauchy accepted the following hypotheses: (i) *contact interactions reduce to surface forces on the boundary and depend on its normal*; (ii) *contact interactions are balanced by volume forces*; and (iii) *contact interactions depend at least continuously on the position*.

When accepting the form (3.1), one weakens the assumption (i) and when accepting the Quasi-Balance Postulate (2.5), one adapts assumption (ii) to the new context. In order to generalize Cauchy procedure, one still needs assumptions similar to (iii) with some extra assumptions relative to the way in which contact interactions depend on the shape of the subdomains. We do not recall here these rather technical assumptions which are used in [120,130] to prove the existence of a field C_N of completely symmetric tensors of order N such that, at any point of a face \mathcal{F}_k ,

$$\mathbb{F}_{N-1}^2 = (C_N | n_k^{\otimes N}) \otimes n_k^{\otimes N-1}, \quad (4.1)$$

at any point of an edge \mathcal{L}_j

$$\mathbb{F}_{N-2}^1 = \text{Sym} \left(\sum_{k \in [\mathcal{L}_j]} (\mathcal{P}_{\mathcal{F}_k}^{N-1}(C_N \cdot n_k) \cdot v_k^j) \right)_{\perp \mathcal{L}_j}, \quad (4.2)$$

and, at any wedge point \hat{x} ,

$$F_{N-3}^0 = \text{Sym} \left(\sum_{j \in [\hat{x}]} \mathcal{P}_{\mathcal{L}_j}^{N-2} \left(\sum_{k \in [\mathcal{L}_j]} (\mathcal{P}_{\mathcal{F}_k}^{N-1}(C_N \cdot n_k) \cdot v_k^j) \right) \cdot e_j \right), \quad (4.3)$$

where the operator \mathcal{P}_M , for any submanifold M of the physical space, is defined by setting for any tensor X , v and Y of order $q - 1$, 1 and $p - q$, respectively,

$$\mathcal{P}_{M,q}^p(X \otimes v \otimes Y) := X \otimes Y_{\perp M} \otimes v_{\parallel M}, \quad \mathcal{P}_M^p := \sum_{q=1}^p \mathcal{P}_{M,q}^p. \quad (4.4)$$

The proof given in [130] is inspired by the Cauchy tetrahedron construction. A family of tetrahedra with height tending to zero is considered and tested with a polynomial velocity field of order N . In the Quasi-Balance inequality, the terms involving F_{N-1}^2 , F_{N-2}^1 and F_{N-3}^0 are preponderant and thus must balance each other. One chooses one face \mathcal{F}_k of the tetrahedron and one defines a tensor \tilde{C}_N in terms of all quantities F_{N-1}^2 , F_{N-2}^1 and F_{N-3}^0 calculated on the other faces and on the edges and wedges which are not part of the boundary of \mathcal{F}_k . Therefore, the expression (4.1) is proved to be valid with the same \tilde{C}_N for all n_k in the unit sphere. The proof is constructive but intricate: it is first obtained for all n_k inside a trihedron and then extended in the whole unit sphere via a topological argument. Straightforward calculations allow to check that equations (4.2) and (4.3) are identities as soon as applied to the contact forces involved in the definition of \tilde{C}_N . Cumbersome tensorial computations are needed to prove that \tilde{C}_N can be replaced by its completely symmetric part C_N which make all equations actual representation formulae. This first result concerns only highest order forces and tetrahedral shapes.

In order to extend it to more general shapes, a theorem (analogous to Noll theorem [123]) is needed which states that the highest order terms in $\mathfrak{S}^{\text{fto}}$ depend on the shape of the domain only through the tangent tetrahedral shape (see [120,130] for precise definitions). The idea of the proof (which can be found in [166,167] or [130]) is again to apply the Quasi-Balance inequality to a shrinking family of domains made by the intersection of D and suitable polyhedra.

Finally, in order to obtain representation formulae for the lower order contact interactions, one observes that the highest order terms balance each other up to lower order ones. Indeed, it is proved in [130] that the quantity

$$\begin{aligned} \tilde{\mathfrak{S}}^{\text{fto}}(D, V) := & \int_D (C_N | \nabla^N V + \text{div}(C_N) | \nabla^{N-1} V) - \int_{\partial_2 D} F_{N-1}^2 | \nabla^{N-1} V \\ & - \int_{\partial_1 D} F_{N-2}^1 | \nabla^{N-2} V - \int_{\partial_0 D} F_{N-3}^0 | \nabla^{N-3} V \end{aligned}$$

is a stress state in the sense of Cauchy of order $N - 1$. Let us introduce the truncated stress state (of order $N - 1$)

$$\tilde{\mathfrak{S}}^{\text{fto}}(D, V) := \mathfrak{S}^{\text{fto}}(D, V) - \int_{\partial_2 D} F_{N-1}^2 | \nabla^{N-1} V - \int_{\partial_1 D} F_{N-2}^1 | \nabla^{N-2} V - \int_{\partial_0 D} F_{N-3}^0 | \nabla^{N-3} V.$$

The difference $\tilde{\mathfrak{S}}^{\text{fto}} - \mathfrak{S}^{\text{fto}}$ is the sum of $\mathfrak{S}^{\text{fto}}$ plus a volume term and thus is quasi-balanced. As it is also the difference of two stress states of order $N - 1$, it is a stress state of order $N - 1$. The result concerning highest order interactions can be applied to this new stress state obtaining further representation formulae. Iterating this procedure, one has constructed a sequence of stress tensors C_1, \dots, C_N representing all terms in $\mathfrak{S}^{\text{fto}}(D, V)$ but this iterative construction does not easily lead to explicit formulae.

To be more precise, we are using here an inductive definition: we say that *the sequence* (C_1, \dots, C_N) *represents* $\mathfrak{S}^{\text{fto}}$ if (i) C_N represents the highest order terms and (ii) the sequence (C_1, \dots, C_{N-1}) represents $\tilde{\mathfrak{S}}^{\text{fto}} - \mathfrak{S}^{\text{fto}}$.

In [130], this construction has been made explicit up to third gradient theories.

The representing sequence (C_1, \dots, C_N) enable us to write $\mathfrak{S}^{\text{fr}_0}$ in a form similar to $\mathfrak{S}^{\text{ins}}$. Indeed, we have

Lemma 4.1. *For any $N > 0$ and any quasi-balanced $\mathfrak{S}^{\text{fr}_0}$ having expression (3.1), let (C_1, \dots, C_N) be the associated stress tensors obtained via the Cauchy type procedure described in §4. Then the following identity holds:*

$$\mathfrak{S}^{\text{fr}_0}(D, V) = \sum_{s=0}^N \int_D \tilde{\tau}_s | \nabla^s V \, d\mathcal{H}^3, \quad (4.5)$$

with $\tilde{\tau}_0 := \text{div}(C_1)$, $\tilde{\tau}_s := C_s + \text{div}(C_{s+1})$ for $0 < s < N$ and $\tilde{\tau}_N := C_N$.

Proof. We use an induction argument. In the case $N = 1$, the identity (4.5) reads

$$\int_{\partial_2 D} F_0^2 | V \, d\mathcal{H}^2 = \int_D (\tilde{\tau}_0 | V + \tilde{\tau}_1 | \nabla V) \, d\mathcal{H}^3$$

that is

$$\int_{\partial_2 D} (C_1 \cdot n) | V \, d\mathcal{H}^2 = \int_D (\text{div}(C_1) | V + C_1 | \nabla V) \, d\mathcal{H}^3$$

which results directly from the divergence theorem.

Assume now that the Lemma holds for any quasi-balanced Cauchy stress state of order $N - 1$. By construction, (C_1, \dots, C_{N-1}) represents $\tilde{\mathfrak{S}}^{\text{fr}_0} - \tilde{\mathfrak{S}}^{\text{fr}_0}$. Therefore, owing to the induction assumption, we have

$$\tilde{\mathfrak{S}}^{\text{fr}_0}(D, V) - \tilde{\mathfrak{S}}^{\text{fr}_0}(D, V) = \sum_{s=0}^{N-2} \int_D \tilde{\tau}_s | \nabla^s V \, d\mathcal{H}^3 + \int_D C_{N-1} | \nabla^{N-1} V \, d\mathcal{H}^3.$$

Hence,

$$\begin{aligned} \mathfrak{S}^{\text{fr}_0}(D, V) &= \tilde{\mathfrak{S}}^{\text{fr}_0}(D, V) - \tilde{\mathfrak{S}}^{\text{fr}_0}(D, V) + \int_D (C_N | \nabla^N V + \text{div}(C_N) | \nabla^{N-1} V), \\ &= \sum_{s=0}^N \int_D \tilde{\tau}_s | \nabla^s V \, d\mathcal{H}^3. \end{aligned}$$

■

By assuming the Principle of Virtual Work, the last expression coincides with $\mathfrak{S}^{\text{ins}}(D, V)$ and the unicity result we stated in §3 implies that the tensors τ_s appearing in $\mathfrak{S}^{\text{ins}}(D, V)$ coincide with the tensors $\tilde{\tau}_s$.

5. Cauchy versus D'Alembert postulations: the two methods can be reconciled

In fact the two methods can be reconciled. Their equivalence has already been explicitly established by Gabrio Piola [32] for stress states of order one. Much later, the same equivalence has been proved for stress states of order two: this results has been obtained in [115,120] where the relationship between the concept of contact line force and surface double force was clearly established by obtaining a representation formula relating the two concepts (on line forces see also [168,169]).

The results we have recalled or established in the previous sections show that the operators \mathcal{O}_D which associate the s -forces \tilde{F}_s^i to the tensors τ_s as specified by the formulae (3.5)–(3.8), resulting from the D'Alembert type procedure, are identical to the operators \mathcal{O}_C which associate the s -forces F_s^i to the tensors $\tilde{\tau}_s$ following the Cauchy type procedure described in §4.

Indeed, let us consider some family (τ_s) of stress tensors representing a D'Alembert stress state

$$\mathfrak{S}^{\text{ins}}(D, V) = \sum_{s=0}^N \int_D \tau_s | \nabla^s V \, d\mathcal{H}^3.$$

The d'Alembert procedure provides a family of s -forces $(\tilde{f}_s^i) = \mathcal{O}_D((\tau_s))$ such that

$$\sum_{s=0}^N \int_D \tau_s | \nabla^s V \, d\mathcal{H}^3 = \sum_{i=0}^2 \left(\sum_{s=0}^{N-3+i} \int_{\partial_i D} \tilde{f}_s^i | (\nabla^s V)_\perp \, d\mathcal{H}^i \right).$$

On the other hand, starting from the \tilde{f}_s^i , the Cauchy procedure provides a family (C_s) and an associated family $(\tilde{\tau}_s)$ such that $(\tilde{f}_s^i) = \mathcal{O}_C((\tilde{\tau}_s))$ and

$$\sum_{s=0}^N \int_D \tilde{\tau}_s | \nabla^s V \, d\mathcal{H}^3 = \sum_{i=0}^2 \left(\sum_{s=0}^{N-3+i} \int_{\partial_i D} \tilde{f}_s^i | (\nabla^s V)_\perp \, d\mathcal{H}^i \right).$$

We have proved at the end of §3 the uniqueness of the representation of $\mathfrak{S}^{\text{ins}}$ in terms of stress tensors. Hence, $\tilde{\tau}_s = \tau_s$ and $\tilde{f}_s^i = \mathcal{O}_D((\tau_s)) = \mathcal{O}_C((\tau_s))$.

The previous proof is indirect and it has sometimes been objected that the explicit formulae giving the highest order forces while presenting some similarities, were different following Cauchy or D'Alembert procedures. We show now, using only algebraic arguments, that they are equivalent.

Lemma 5.1. *The operators which associate the highest order forces (that is the surface N -force \tilde{f}_{N-1}^2 on any face \mathcal{F}_k , the line $N-1$ -force \tilde{f}_{N-2}^1 on any edge \mathcal{L}_j and the $N-2$ -force \tilde{f}_{N-3}^0 on wedge \hat{x}) to the tensor τ_N as specified by the formulae (3.9)–(3.11) resulting from the D'Alembert type procedure, are identical to the operators which associate F_{N-1}^2 , F_{N-2}^1 and F_{N-3}^0 to the tensor C_N as specified by formulae (4.1)–(4.3) resulting from the Cauchy-type procedure.*

Proof. The proof needs some rather technical steps which, for the sake of clarity, we postpone to appendix A.

The fact that (3.9) is equivalent to (4.1) is obvious. The fact that (3.10) is equivalent to (4.2) is a simple consequence of the fact that, for any submanifold M and any completely symmetric tensor X of order p ,

$$\text{Sym}(\mathcal{P}_M^p(Y) \cdot v) = \mathbb{P}_M^p(Y) \cdot v,$$

the proof of which is postponed to appendix A (lemma A.1). Indeed, it is enough to apply this identity for every $k \in [\mathcal{L}_j]$ with $M = \mathcal{F}_k$, $X = C_N \cdot n_k$ and $v = v_k^j$.

In order to prove that (3.11) is equivalent to (4.3), we remark, by applying twice lemma A.1 that, for any $j \in [\hat{x}]$ and $k \in [\mathcal{L}_j]$,

$$\begin{aligned} \mathbb{P}_{\mathcal{L}_j}^{N-2}(\mathbb{P}_{\mathcal{F}_k}^{N-1}(C_N \cdot n_k) \cdot v_k^j) \cdot e_j &= \mathbb{P}_{\mathcal{L}_j}^{N-2}(\text{Sym}(\mathcal{P}_{\mathcal{F}_k}^{N-1}(C_N \cdot n_k) \cdot v_k^j)) \cdot e_j \\ &= \text{Sym}(\mathcal{P}_{\mathcal{L}_j}^{N-2}(\text{Sym}(\mathcal{P}_{\mathcal{F}_k}^{N-1}(C_N \cdot n_k) \cdot v_k^j)) \cdot e_j) \\ &= \text{Sym}(\mathcal{P}_{\mathcal{L}_j}^{N-2}(\mathcal{P}_{\mathcal{F}_k}^{N-1}(C_N \cdot n_k) \cdot v_k^j) \cdot e_j). \end{aligned}$$

The last equality being due to the fact that, for any line \mathcal{L} , any completely symmetric tensor X of order p and any vector e tangent to \mathcal{L} ,

$$\text{Sym}(\mathcal{P}_{\mathcal{L}}^p(Y) \cdot e) = \text{Sym}(\mathcal{P}_{\mathcal{L}}^p(\text{Sym}(Y)) \cdot e),$$

the proof of which is postponed to appendix A (lemma A.2). ■

6. Some perspectives for future researches

Even if the modelling of N th gradient continua is well founded by the two methods which are now reconciled, many questions about stress states remain open. It has to be remarked that the available results are far to include all possible shapes for bodies. It is not clear if it is possible to determine the set of domains to which a theory can apply independently of the considered constitutive equations. On the other hand, all the works we have described accept a bound for the order of the stress state while it would also be interesting, at least from a theoretical point of view, to understand what happens in a body where the order of the stress state varies from point to point, being unbounded.

We have already emphasized the fact that the presented results cannot encompass stress states for which there are stress concentrations along lower dimensional manifolds, models which are needed if one wants to model in a unifying way for instance a two-dimensional plate included in a three-dimensional elastic body. To our knowledge, the theoretical tools for attacking this important problem remain to be developed.

Based on the original ideas of Lagrange himself, the principles of power balance have received attention also in dynamics, namely in vibrations and acoustics. In this field, some authors (e.g. [170,171]) attempted to write a self-contained set of equations to describe power migration through a continuum medium: this situation resembles the one in which Dunn & Serrin [113,114] found themselves in the context of incomplete second gradient theories. In our opinion (generalizing what is done in [120]), higher gradient theories may complete the cited attempt or, in general, supply a regularized model when non-convex energy functions need to be introduced (as in Cahn–Hilliard and Korteweg fluids [172] or in many other physical situations, see e.g. [40,173–176]). Finally, the power balance equations can be also approached in the context of uncertainties in the constitutive relationships, where some randomness affects the physical parameters of the equivalent continuum (e.g. [177]). In this case, higher order gradients would be related to the introduction of some statistical average and ergodic assumption.

Higher order gradient theories are needed when boundary layer phenomena must be described: when considering impact phenomena (e.g. [89,178–180]) in general some *ad hoc* assumptions are imposed, especially when choosing boundary conditions. More detailed models for impact between solids or between solids and fluids, involving some space–time length scales, may cure some of the singularities which are present in many models presented in the literature: in particular, one could conceive to describe the phenomena of water spray formation or turbulence (see, respectively, [179] with references there cited or [181]) by means of suitable contact edge forces.

Finally, the constant technological progress allows now to conceive and built metamaterials with designed mechanical properties (a general review on the subject is [182], while interesting developments are in [183]). Making metamaterials in which the higher order effects are preponderant is a real challenge. The search for possible applications of such materials is a free field for future research.

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Appendix A. Two technical lemmas

We use the operator defined by setting, for any tensors X and Y of order q and $p - q$, respectively,

$$\mathbb{R}_{M,q}^p(X \otimes Y) := X \otimes Y_{\perp M} \quad (\text{A } 1)$$

and we note that, for any completely symmetric tensor X of order p and any vector v tangent to M , we have $\mathcal{P}_{M,q}^p(X) \cdot v = \mathbb{R}_{M,q-1}^{p-1}(X \cdot v)$ and consequently

$$\mathcal{P}_M^p(X) \cdot v = \sum_{q=1}^p \mathcal{P}_{M,q}^p(X) \cdot v = \sum_{q=0}^{p-1} \mathbb{R}_{M,q}^{p-1}(X \cdot v). \quad (\text{A } 2)$$

Lemma A.1. *For any completely symmetric tensor Y of order p , the following identity holds:*

$$\text{Sym} \left(\sum_{q=0}^p \mathbb{R}_{M,q}^p(Y) \right) = \text{Sym} \left(\sum_{q=0}^p \binom{q}{p+1} \mathbb{P}_{M,q}^p(Y) \right). \quad (\text{A } 3)$$

As a consequence, for any vector v tangent to M , we have

$$\text{Sym}(\mathcal{P}_M^p(Y) \cdot v) = \mathbb{P}_M^p(Y) \cdot v. \quad (\text{A } 4)$$

Proof. Let us first remark that a simple induction argument leads to the formula

$$\sum_{q=r}^N \binom{q}{r} = \binom{N+1}{r+1}. \quad (\text{A } 5)$$

To prove (A 3), it is enough to check the identity with tensors of the type

$$Y = \text{Sym}(t_1 \otimes t_2 \otimes \cdots \otimes t_\alpha \otimes n_1 \otimes n_2 \otimes \cdots \otimes n_\beta),$$

where the vectors t_i are tangent to M , the vectors n_i are normal to it and $\alpha + \beta = p$. Computing the number of permutations in the symmetrization of $t_1 \otimes t_2 \otimes \cdots \otimes t_\alpha \otimes n_1 \otimes n_2 \otimes \cdots \otimes n_\beta$ which give non-vanishing results we obtain

$$\begin{aligned} \text{Sym}(\mathbb{P}_{M,q}^p(Y)) &= \begin{cases} \frac{\alpha! \beta!}{p!} Y & \text{if } q = \beta, \\ 0 & \text{otherwise.} \end{cases} \\ \text{Sym}(\mathbb{R}_{M,q}^p(Y)) &= \begin{cases} \frac{\beta! q!}{p!(q-\alpha)!} Y & \text{if } q \geq \alpha, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

From the first equation, we deduce

$$\text{Sym} \left(\sum_{q=0}^p \binom{p+1}{q} \mathbb{P}_{M,q}^p(Y) \right) = \binom{p+1}{\beta} \frac{\alpha! \beta!}{p!} Y = \frac{p+1}{\alpha+1} Y,$$

and from the second one,

$$\text{Sym} \left(\sum_{q=0}^p \mathbb{R}_{M,q}^p(Y) \right) = \left(\sum_{q=\alpha}^p \binom{q}{\alpha} \right) \frac{\alpha! \beta!}{p!} Y = \frac{p+1}{\alpha+1} Y.$$

The first identity is thus proved. The second one is an obvious consequence. Indeed.

$$\text{Sym}(\mathcal{P}_M^p(X) \cdot v) = \text{Sym} \left(\sum_{q=0}^{p-1} \mathbb{R}_{M,q}^{p-1}(X \cdot v) \right) = \text{Sym} \left(\sum_{q=0}^{p-1} \binom{q}{p} \mathbb{P}_{M,q}^{p-1}(X \cdot v) \right) = \mathbb{P}_M^p(X) \cdot v. \quad \blacksquare$$

Lemma A.2. *For any line \mathcal{L} , any completely symmetric tensor X of order p and any vector e tangent to \mathcal{L} ,*

$$\text{Sym}(\mathcal{P}_{\mathcal{L}}^p(X) \cdot e) = \text{Sym}(\mathcal{P}_{\mathcal{L}}^p(\text{Sym}(X)) \cdot e).$$

Proof. To that aim, let us introduce $\mathcal{P}_{\mathcal{L}}^{p*}$ the adjoint operator of $\mathcal{P}_{\mathcal{L}}^p$ and check this last equality by checking, for any completely symmetric tensor Y of order $p - 1$,

$$(\mathcal{P}_{\mathcal{L}}^p(X) \cdot e) | Y = (\mathcal{P}_{\mathcal{L}}^p(\text{Sym}(X)) \cdot e) | Y$$

or

$$X | \mathcal{P}_{\mathcal{L}}^{p*}(Y \otimes e) = \text{Sym}(X) | \mathcal{P}_{\mathcal{L}}^{p*}(Y \otimes e).$$

We are thus reduced to proving that, when Y is a completely symmetric tensor of order $p - 1$ and e a vector tangent to a line \mathcal{L} , $\mathcal{P}_{\mathcal{L}}^{p*}(Y \otimes e)$ is completely symmetric or, equivalently, invariant with respect to any permutation of indices $(\ell, \ell + 1)$ for $\ell \in \{1, \dots, p - 1\}$. Noticing that

$$\mathcal{P}_{\mathcal{L}}^{p*}(Y \otimes e) = \sum_{r=1}^p \mathcal{P}_{\mathcal{L},r}^{p*}(Y \otimes e) \quad (\text{A } 6)$$

and that

$$(\mathcal{P}_{\mathcal{L},r}^{p*}(Y \otimes e))_{i_1 \dots i_p} = Y_{i_1 \dots i_{r-1} j_{r+1} \dots j_p} \Lambda_{i_{r+1}}^{j_{r+1}} \dots \Lambda_{i_p}^{j_p} e_i$$

(where Λ denotes the projector onto the orthogonal space to the line \mathcal{L}), it is immediately clear that the symmetry of Y implies the invariance with respect to the permutation of indices $(\ell, \ell + 1)$ of all terms in the sum (A 6) for which $r < \ell - 1$ or $r > \ell$. Noticing that

$$Y_{i_1 \dots i_{\ell-1} j_{\ell+1} \dots j_p} = Y_{i_1 \dots i_{\ell-2} j_{\ell} \dots j_p} (\Lambda_{i_{\ell-1}}^{j_{\ell}} + e_{i_{\ell-1}} e_{j_{\ell}})$$

the sum of the two terms corresponding to $r = \ell - 1$ and $r = \ell$ reads

$$\left(\sum_{r=\ell-1}^{\ell} \mathcal{P}_{\mathcal{L},r}^{p*}(Y \otimes e) \right)_{i_1 \dots i_p} = Y_{i_1 \dots i_{\ell-2} j_{\ell} \dots j_p} (\Lambda_{i_{\ell}}^{j_{\ell}} e_{i_{\ell-1}} + \Lambda_{i_{\ell-1}}^{j_{\ell}} e_{i_{\ell}} + e_{i_{\ell-1}} e_{j_{\ell}}) \Lambda_{i_{\ell+1}}^{j_{\ell+1}} \dots \Lambda_{i_p}^{j_p}.$$

It becomes now clear that the sum of these two terms, and thus the whole sum, are invariant with respect to the permutation of indices $(\ell, \ell + 1)$. ■

References

1. Benvenuto E. 1981 *La scienza delle costruzioni e il suo sviluppo storico*. Firenze, Italy: Sansoni.
2. Collin F, Chambon R, Charlier R. 2006 A finite element method for poro mechanical modelling of geotechnical problems using local second gradient models. *Int. J. Num. Method Eng.* **65**, 1749–1772. (doi:10.1002/nme.1515)
3. dell'Isola F, Guarascio M, Hutter K. 2000 A variational approach for the deformation of a saturated porous solid. A second gradient theory extending Terzaghi's effective stress principle. *Arch. Appl. Mech.* **70**, 323–337. (doi:10.1007/s004199900020)
4. dell'Isola F, Sciarra G, Batra RC. 2003 Static deformations of a linear elastic porous body filled with an inviscid fluid. Essays and papers dedicated to the memory of Clifford Ambrose Truesdell III: volume III. *J. Elasticity* **72**, 99–120. (doi:10.1023/B:ELAS.0000018765.68432.bb)
5. Sciarra G, dell'Isola F, Hutter K. 2001 A solid-fluid mixture model allowing for solid dilatation under external pressure. *Contin. Mech. Thermodyn.* **13**, 287–306. (doi:10.1007/s001610100053)
6. Sciarra G, dell'Isola F, Coussy O. 2007 Second gradient poromechanics. *Int. J. Solids Struct.* **44**, 6607–6629. (doi:10.1016/j.ijsolstr.2007.03.003)
7. Sciarra G, dell'Isola F, Ianiro N, Madeo A. 2008 A variational deduction of second gradient poroelasticity. Part I: general theory. *J. Mech. Mater. Struct.* **3**, 507–526. (doi:10.2140/jomms.2008.3.507)
8. Salençon J. 2001 *Mechanics*. Berlin, Germany: Springer. (Mécanique des milieux continus (éd. I Tome). École polytechnique, Palaiseau; Ellipses, Paris, (2002)–(2005)).
9. Pideri C, Seppelcher P. 1997 A second gradient material resulting from the homogenization of an heterogeneous linear elastic medium. *Contin. Mech. Thermodyn.* **9**, 241–257. (doi:10.1007/s001610050069)

10. Chesnais C, Boutin C, Hans S. 2012 Effects of the local resonance on the wave propagation in periodic frame structures: generalized Newtonian mechanics. *J. Acoust. Soc. Am.* **132**, 2873–2886. (doi:10.1121/1.4744975)
11. Forest S. 2006 *Milieux continus généralisés et matériaux hétérogènes*. Paris, France: Les Presses de l'École des Mines de Paris.
12. Forest S. 2005 Generalized continua. In *Encyclopedia of materials: science and technology updates* (eds KHJ Buschow, RW Cahn, MC Flemings, B Ilshner, EJ Kramer, S Mahajan), pp. 1–7. Oxford, UK: Elsevier.
13. Forest S, Amestoy M. 2004, 2005, 2006 Mécanique des milieux continus. Cours de l'École des Mines de Paris n 3121.
14. Forest S. 2004 Milieux continus généralisés et matériaux hétérogènes. Mémoire d'habilitation à diriger des recherches.
15. Kirchner N, Steinmann P. 2006 On the material setting of gradient hyperelasticity [English summary.] *Math. Mech. Solids* **12**, 559–580. (doi:10.1177/1081286506067073)
16. Maugin GA, Metrikine AV. 2010 *Mechanics of generalized continua, one hundred years after the cosserats*. New York, NY: Springer.
17. Maugin GA. 2013 The principle of virtual power: from eliminating metaphysical forces to providing an efficient modelling tool in memory of Paul Germain (1920–2009). *Cont. Mech. Thermodyn.* **25**, 1–20. (doi:10.1007/s00161-011-0196-7)
18. Daher N, Maugin GA. 1986 Virtual power and thermodynamics for electromagnetic continua with interfaces. *J. Math. Phys.* **27**, 3022–3035. (doi:10.1063/1.527231)
19. Daher N, Maugin GA. 1986 The method of virtual power in continuum mechanics. Application to media presenting singular surfaces and interfaces. *Acta Mech.* **60**, 217–240. (doi:10.1007/BF01176354)
20. Gavriluyk S, Gouin H. 1999 A new form of governing equations of fluids arising from Hamilton's principle. *Int. J. Eng. Sci.* **37**, 1495–1520. (doi:10.1016/S0020-7225(98)00131-1)
21. Li J, Ostoja-Starzewski M. 2011 Micropolar continuum mechanics of fractal media. *Int. J. Eng. Sci.* **49**, 1302–1310. (doi:10.1016/j.ijengsci.2011.03.010)
22. Suiker ASJ, Chang CS. 2000 Application of higher-order tensor theory for formulating enhanced continuum models. *Acta Mech.* **142**, 223–234. (doi:10.1007/BF01190020)
23. Sokolowski M. 1970 *Theory of couple-stresses in bodies with constrained rotations*. CISM Courses and Lectures, vol. 26. Berlin, Germany: Springer.
24. Steinmann P. 2008 On boundary potential energies in deformational and configurational mechanics. *J. Mech. Phys. Solids* **56**, 772–800. (doi:10.1016/j.jmps.2007.07.001)
25. Sunyk R, Steinmann P. 2003 On higher gradients in continuum-atomistic modelling. *Int. J. Solids Struct.* **40**, 6877–6896. (doi:10.1016/j.ijsolstr.2003.07.001)
26. Toupin RA. 1962 Elastic materials with couple-stresses. *Arch. Rat. Mech. Anal.* **11**, 385–414. (doi:10.1007/BF00253945)
27. Mindlin RD. 1964 Micro-structure in linear elasticity. *Arch. Rat. Mech. Anal.* **16**, 51–78. (doi:10.1007/BF00248490)
28. Germain P. 1973 La méthode des puissances virtuelles en mécanique des milieux continus. Première partie. Théorie du second gradient. *J. Mécanique* **12**, 235–274.
29. Germain P. 1973 The method of virtual power in continuum mechanics. Part 2: microstructure. *SIAM J. Appl. Math.* **25**, 556–575. (doi:10.1137/0125053)
30. Piola G. 1825 Sull'applicazione de' principj della meccanica analitica del Lagrange ai principali problemi. Memoria di Gabrio Piola presentata al concorso del premio e coronata dall'I.R. Istituto di Scienze, ecc. nella solennità del giorno 4 ottobre 1824, Milano, Imp. Regia stamperia.
31. Piola G. 1846 Memoria intorno alle equazioni fondamentali del movimento di corpi qualsivogliono considerati secondo la naturale loro forma e costituzione, Modena, Tipi del R.D. Camera.
32. dell'Isola F, Maier G, Perego U, Andreaus U, Esposito R, Forest S. 2014 The complete works of Gabrio Piola: volume I. *Adv. Struct. Mater.* **38**. (doi:10.1007/978-3-319-00263-7)
33. dell'Isola F, Seppecher P, Madeo A. 2012 How contact interactions may depend on the shape of Cauchy cuts in N th gradient continua: approach a' la D'Alembert. *Z. Angew. Math. Phys.* **63**, 1119–1141. (doi:10.1007/s00033-012-0197-9)
34. dell'Isola F, Seppecher P. 2011 Commentary on the paper 'Hypertractions and hyperstresses convey the same mechanical information Continuum Mech. Thermodyn. (2010) 22: 163–176'

- by Prof. Podio Guidugli and Prof. Vianello and some related papers on higher gradient theories. *Contin. Mech. Thermodyn.* **23**, 473–478. (doi:10.1007/s00161-010-0176-3)
35. Forest S, Amestoy M, Cantournet S, Damamme G, Kruch S. 2005–2006 Mécanique des Milieux Continus, ECOLE DES MINES DE PARIS Année.
 36. Forest S. 1998 Mechanics of generalized continua: construction by homogenization. *J. Phys. IV France* **8**, Pr4-39–Pr4-48.
 37. Placidi L. 2014 A variational approach for a nonlinear one-dimensional damage-elasto-plastic second-gradient continuum model. *Contin. Mech. Thermodyn.*, 1–19. (doi:10.1007/s00161-014-0405-2)
 38. Placidi L. 2015 A variational approach for a nonlinear 1-dimensional second gradient continuum damage model. *Contin. Mech. Thermodyn.* **27**, 623–638. (doi:10.1007/s00161-014-0338-9)
 39. Pietraszkiewicz W, Eremeyev VA. 2009 On natural strain measures of the non-linear micropolar continuum. *Int. J. Solids Struct.* **46**, 774–787. (doi:10.1016/j.ijsolstr.2008.09.027)
 40. Eremeyev VA, Pietraszkiewicz W. 2004 The nonlinear theory of elastic shells with phase transitions. *J. Elast.* **74**, 67–86. (doi:10.1023/B:ELAS.0000026106.09385.8c)
 41. Eremeyev VA, Lebedev LP. 2013 Existence of weak solutions in elasticity. *Math. Mech. Solids* **18**, 204–217. (doi:10.1177/1081286512462187)
 42. Eremeyev VA, Altenbach H. 2014 Equilibrium of a second-gradient fluid and an elastic solid with surface stresses. *Meccanica* **49**, 2635–2643. (doi:10.1007/s11012-013-9851-3)
 43. Green AE, Rivlin RS. 1964 Simple force and stress multipoles. *Arch. Ration. Mech. Anal.* **16**, 325–353. (doi:10.1007/BF00281725)
 44. Green AE, Rivlin RS. 1964 Multipolar continuum mechanics. *Arch. Ration. Mech. Anal.* **17**, 113–147. (doi:10.1007/BF00253051)
 45. Green AE, Rivlin RS. 1964 On Cauchy's equations of motion. *Z. Angew. Math. Phys.* **15**, 290–292. (doi:10.1007/BF01607019)
 46. Green AE, Rivlin RS. 1965 Multipolar continuum mechanics: functional theory I. *Proc. R. Soc. Lond. A* **284**, 303–324. (doi:10.1098/rspa.1965.0065)
 47. Mindlin RD. 1965 Stress functions for a Cosserat continuum. *Int. J. Solids Struct.* **1**, 265–271. (doi:10.1016/0020-7683(65)90033-8)
 48. Mindlin RD. 1965 Complex representation of displacements and stresses in plane strain with couple-stresses. In *Applications of the Theory of Functions in Continuum Mechanics (Proc. Int. Symp., Tbilisi)*, vol. I, pp. 256–259. [*Mechanics of Solids* (Russian) Moscow, Russia: Izdat Nauka, Moscow (1963).]
 49. Mindlin RD. 1965 On the equations of elastic materials with micro-structure. *Int. J. Solids Struct.* **1**, 73–78. (doi:10.1016/0020-7683(65)90016-8)
 50. Mindlin RD. 1965 Second gradient of strain and surface tension in linear elasticity. *Int. J. Solids Struct.* **1**, 417–438. (doi:10.1016/0020-7683(65)90006-5)
 51. Mindlin RD, Tiersten HF. 1962 Effects of couple-stresses in linear elasticity. *Arch. Ration. Mech. Anal.* **11**, 415–448. (doi:10.1007/BF00253946)
 52. Mindlin RD. 1962 Influence of couple-stresses on stress concentrations Main features of Cosserat theory are reviewed by lecturer and some recent solutions of the equations, for cases of stress concentration around small holes in elastic solids, are described. *Exp. Mech.* **3**, 1–7. (doi:10.1007/BF02327219)
 53. Mindlin RD, Eshel NN. 1968 On first strain-gradient theories in linear elasticity. *Int. J. Solids Struct.* **4**, 109–124. (doi:10.1016/0020-7683(68)90036-X)
 54. Steigmann DJ, Pipkin AC. 1991 Equilibrium of elastic nets. *Phil. Trans. R. Soc. Lond. A* **335**, 419–454. (doi:10.1098/rsta.1991.0056)
 55. Nadler B, Papadopoulos P, Steigmann DJ. 2006 Multiscale constitutive modeling and numerical simulation of fabric material. *Int. J. Solids Struct.* **43**, 206–221. (doi:10.1016/j.ijsolstr.2005.05.020)
 56. Steigmann DJ. 1996 The variational structure of a nonlinear theory for spatial lattices. *Meccanica* **31**, 441–455. (doi:10.1007/BF00429932)
 57. Altenbach H, Eremeyev VA, Lebedev LP. 2010 On the existence of solution in the linear elasticity with surface stresses. *J. Appl. Math. Mech.* **90**, 231–240. (doi:10.1002/zamm.20090311)
 58. Altenbach H, Eremeyev VA, Lebedev LP. 2011 On the spectrum and stiffness of an elastic body with surface stresses. *J. Appl. Math. Mech.* **91**, 699–710. (doi:10.1002/zamm.201000214)

59. Altenbach H, Eremeyev VA. 2011 On the shell theory on the nanoscale with surface stresses. *Int. J. Eng. Sci.* **49**, 1294–1301. (doi:10.1016/j.jengsci.2011.03.011)
60. Altenbach H, Eremeyev VA, Morozov NF. 2012 Surface viscoelasticity and effective properties of thin-walled structures at the nanoscale. *Int. J. Eng. Sci.* **59**, 83–89. (doi:10.1016/j.jengsci.2012.03.004)
61. Exadaktylos GE, Vardoulakis I. 2001 Microstructure in linear elasticity and scale effects: a reconsideration of basic rock mechanics and rock fracture mechanics. *Tectonophysics* **335**, 81–109. (doi:10.1016/S0040-1951(01)00047-6)
62. Eremeyev VA, Lebedev LP. 2015 Mathematical study of boundary-value problems within the framework of Steigmann-Ogden model of surface elasticity. *Contin. Mech. Thermodyn.* 1–16. (doi:10.1007/s00161-015-0439-0)
63. Alibert J-J, Seppecher P, dell'Isola F. 2003 Truss modular beams with deformation energy depending on higher displacement gradients. *Math. Mech. Solids* **8**, 51–73. (doi:10.1177/1081286503008001658)
64. Polizzotto C. 2007 Strain-gradient elastic-plastic material models and assessment of the higher order boundary conditions. *Eur. J. Mech. A Solids* **26**, 189–211. (doi:10.1016/j.euromechsol.2006.07.005)
65. Chesnais C, Boutin C, Hans S. 2013 Wave propagation and non-local effects in periodic frame materials: generalized continuum mechanics. *Math. Mech. Solids* **20**, 929–958. (doi:10.1177/1081286513511092)
66. Aifantis EC. 1992 On the role of gradients in the localization of deformation and fracture. *Int. J. Eng. Sci.* **30**, 1279–1299. (doi:10.1016/0020-7225(92)90141-3)
67. Aifantis EC. 1999 Strain gradient interpretation of size effects. *Int. J. Fract.* **95**, 299–314. (doi:10.1023/A:1018625006804)
68. Placidi L, Rosi G, Giorgio I, Madeo A. 2013 Reflection and transmission of plane waves at surfaces carrying material properties and embedded in second-gradient materials. *Math. Mech. Solids* **19**, 1–25. (doi:10.1177/1081286512474016)
69. Bardenhagen S, Triantafyllidis N. 1994 Derivation of higher order gradient continuum theories in 2,3-D nonlinear elasticity from periodic lattice models. *J. Mech. Phys. Solids* **42**, 111–139. (doi:10.1016/0022-5096(94)90051-5)
70. Triantafyllidis N, Bardenhagen S. 1993 On higher order gradient continuum theories in 1-D nonlinear elasticity. Derivation from and comparison to the corresponding discrete models. *J. Elast.* **33**, 259–293. (doi:10.1007/BF00043251)
71. Triantafyllidis N, Aifantis EC. 1986 A gradient approach to localization of deformation. I. Hyperelastic materials. *J. Elast.* **16**, 225–237. (doi:10.1007/BF00040814)
72. Fannjiang AC, Chan YS, Paulino GH. 2001 Strain gradient elasticity for antiplane shear cracks: a hypersingular integrodifferential equation approach. *SIAM J. Appl. Math.* **62**, 1066–1091. (doi:10.1137/S0036139900380487)
73. Larsson R, Diebels SA. 2007 A second-order homogenization procedure for multi-scale analysis based on micropolar kinematics. *Int. J. Numer. Methods Eng.* **69**, 2485–2512. (doi:10.1002/nme.1854)
74. Forest S. 2002 Homogenization methods and the mechanics of generalized continua: part 2. *Theor. Appl. Mech.* **28–29**, 113–143. (doi:10.2298/TAM0229113F)
75. Carcaterra A, dell'Isola F, Esposito R, Pulvirenti M. 2015 Macroscopic description of microscopically strongly inhomogeneous systems: a mathematical basis for the synthesis of higher gradients metamaterials. *Arch. Ration. Mech. Anal.* **218**, 1239–1262. (doi:10.1007/s00205-015-0879-5)
76. Ganghoffer JF, Sluys LJ, De Borst R. 1999 A reappraisal of nonlocal mechanics. *Eur. J. Mech. A Solids* **18**, 17–46. (doi:10.1016/S0997-7538(99)80002-3)
77. Ganghoffer JF, De Borst R. 2000 A new framework in nonlocal mechanics. *Int. J. Eng. Sci.* **38**, 453–486. (doi:10.1016/S0020-7225(99)00030-0)
78. dell'Isola F, Andreaus U, Placidi L. 2015 At the origins and in the vanguard of peridynamics, non-local and higher gradient continuum mechanics. An underestimated and still topical contribution of Gabrio Piola. *Math. Mech. Solids* **20**, 887–928. (doi:10.1177/1081286513509811)
79. D'Agostino MV, Giorgio I, Greco L, Madeo A, Boisse P. 2015 Continuum and discrete models for structures including (quasi-)inextensible elasticae with a view to the design and modeling of composite reinforcements. *Int. J. Solids Struct.* **59**, 1–17. (doi:10.1016/j.ijsolstr.2014.12.014)

80. Ferretti M, Madeo A, dell'Isola F, Boisse P. 2014 Modeling the onset of shear boundary layers in fibrous composite reinforcements by second-gradient theory. *Z. Angew. Math. Phys.* **65**, 587–612. (doi:10.1007/s00033-013-0347-8)
81. Madeo A, George D, Remond Y. 2013 Second-gradient models accounting for some effects of microstructure on remodelling of bones reconstructed with bioresorbable materials. *Comput. Methods Biomech. Biomed. Eng.* **16**, 260–261. (doi:10.1080/10255842.2013.815856)
82. Andreaus U, Giorgio I, Lekszycki T. 2014 A 2-D continuum model of a mixture of bone tissue and bio-resorbable material for simulating mass density redistribution under load slowly variable in time. *J. Appl. Math. Mech.* **94**, 978–1000. (doi:10.1002/zamm.201200182)
83. Andreaus U, Giorgio I, Madeo A. 2015 Modeling of the interaction between bone tissue and resorbable biomaterial as linear elastic materials with voids. *J. Appl. Math. Phys.* **66**, 209–237. (doi:10.1007/s00033-014-0403-z)
84. Giorgio I, Andreaus U, Madeo A. 2014 The influence of different loads on the remodeling process of a bone and bio-resorbable material mixture with voids. *Contin. Mech. Thermodyn.*, 1–20. (doi:10.1007/s00161-014-0397-y)
85. Federico S, Grillo A. 2012 Elasticity and permeability of porous fibre-reinforced materials under large deformations. *Mech. Mater.* **44**, 58–71. (doi:10.1016/j.mechmat.2011.07.010)
86. Lekszycki T, dell'Isola F. 2012 A mixture model with evolving mass densities for describing synthesis and resorption phenomena in bones reconstructed with bio-resorbable materials. *J. Appl. Math. Mech.* **6**, 426–444. (doi:10.1002/zamm.201100082)
87. Madeo A, Lekszycki T, dell'Isola F. 2011 A continuum model for the bio-mechanical interactions between living tissue and bio-resorbable graft after bone reconstructive surgery. *CRAS Mécanique* **339**, 625–682. (doi:10.1016/j.crme.2011.07.004)
88. Hamed E, Lee Y, Jasiuk I. 2010 Multiscale modeling of elastic properties of cortical bone. *Acta Mech.* **213**, 131–154. (doi:10.1007/s00707-010-0326-5)
89. Andreaus U, Baragatti P. 2009 Fatigue crack growth, free vibrations, and breathing crack detection of aluminium alloy and steel beams. *J. Strain Anal. Eng. Des.* **44**, 595–608. (doi:10.1243/03093247JSA527)
90. Rinaldi A, Placidi L. 2014 A microscale second gradient approximation of the damage parameter of quasi-brittle heterogeneous lattices. *J. Appl. Math. Mech.* **94**, 862–877. (doi:10.1002/zamm.201300028)
91. Chambolle A, Francfort GA, Marigo J-J. 2010 Revisiting energy release rates in brittle fracture. *J. Nonlinear Sci.* **20**, 395–424. (doi:10.1007/s00332-010-9061-2)
92. Chambolle A, Francfort GA, Marigo J-J. 2009 When and how do cracks propagate. *J. Mech. Phys. Solids* **56**, 1614–1622. (doi:10.1016/j.jmps.2009.05.009)
93. Francfort GA, Marigo J-J. 1998 Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids* **46**, 1319–1342. (doi:10.1016/S0022-5096(98)00034-9)
94. Yang Y, Misra A. 2010 Higher-order stress-strain theory for damage modeling implemented in an element-free Galerkin formulation. *Comput. Model. Eng. Sci.* **64**, 1–36.
95. Yang Y, Ching WY, Misra A. 2011 Higher-order continuum theory applied to fracture simulation of nano-scale intergranular glassy film. *J. Nanomech. Micromech.* **1**, 60–71. (doi:10.1061/(ASCE)NM.2153-5477.0000030)
96. Yang Y, Misra A. 2012 Micromechanics based second gradient continuum theory for shear band modeling in cohesive granular materials following damage elasticity. *Int. J. Solids Struct.* **49**, 2500–2514. (doi:10.1016/j.ijsolstr.2012.05.024)
97. Scerrato D, Giorgio I, Madeo A, Limam A, Darve F. 2014 A simple non-linear model for internal friction in modified concrete. *Int. J. Eng. Sci.* **80**, 136–152. (doi:10.1016/j.ijengsci.2014.02.021)
98. Misra A, Poorsolhjoui P. 2015 Granular micromechanics based micromorphic model predicts frequency band gaps. *Contin. Mech. Thermodyn.*, 1–20. (doi:10.1007/s00161-015-0420-y)
99. Bésuelle P, Chambon R. 2006 *Modeling the post-localization regime with local second gradient models: non-uniqueness of solutions and nonpersistent shear bands*, pp. 209–221. Berlin, Germany: Springer.
100. Zulli D, Luongo A. 2012 Bifurcation and stability of a two-tower system under wind-induced parametric, external and self-excitation. *J. Sound Vib.* **331**, 365–383. (doi:10.1016/j.jsv.2011.09.008)

101. Luongo A, D'Annibale F. 2013 Double zero bifurcation of non-linear viscoelastic beams under conservative and non-conservative loads. *Int. J. Non-Linear Mech.* **55**, 128–139. (doi:10.1016/j.ijnonlinmec.2013.05.007)
102. Piccardo G, D'Annibale F, Zulli D. 2014 On the contribution of Angelo Luongo to mechanics: in honor of his 60th birthday. *Contin. Mech. Thermodyn.*, **27**, 507–529. (doi:10.1007/s00161-014-0388-z)
103. Cazzani A, Malagù M, Turco E. 2014 Isogeometric analysis: a powerful numerical tool for the elastic analysis of historical masonry arches. *Contin. Mech. Thermodyn.*, 1–18. (doi:10.1007/s00161-014-0409-y)
104. Greco L, Cuomo M. 2013 B-Spline interpolation of Kirchhoff-Love space rods. *Comput. Methods Appl. Mech. Eng.* **256**, 251–269. (doi:10.1016/j.cma.2012.11.017)
105. Greco L, Cuomo M. 2014 An implicit G1 multi patch B-spline interpolation for Kirchhoff-Love space rod. *Comput. Methods Appl. Mech. Eng.* **269**, 173–197. (doi:10.1016/j.cma.2013.09.018)
106. Seppecher P, Alibert JJ, dell'Isola F. 2011 Linear elastic trusses leading to continua with exotic mechanical. *J. Phys. Conf. Ser.* **319**, 012018. (doi:10.1088/1742-6596/319/1/012018)
107. Ghiba ID, Neff P, Madeo A, Placidi L, Rosi G. 2014 The relaxed linear micromorphic continuum: existence, uniqueness and continuous dependence in dynamics. *Math. Mech. Solids.* (doi:10.1177/1081286513516972)
108. Madeo A, Neff P, Ghiba ID, Placidi L, Rosi G. 2014 Band gaps in the relaxed linear micromorphic continuum. *J. Appl. Math. Mech.* **95**, 880–887. (doi:10.1002/zamm.201400036)
109. Neff P, Ghiba I-D, Madeo A, Placidi L, Rosi G. 2014 A unifying perspective: the relaxed linear micromorphic continuum. *Contin. Mech. Thermodyn.* **26**, 639–681. (doi:10.1007/s00161-013-0322-9)
110. dell'Isola F, Sciarra G, Vidoli S. 2009 Generalized Hooke's law for isotropic second gradient materials. *Proc. R. Soc. A* **465**, 2177–2196. (doi:10.1098/rspa.2008.0530)
111. Schuricht F. 2007 A new mathematical foundation for contact interactions in continuum physics. *Arch. Ration. Mech. Anal.* **184**, 495–551. (doi:10.1007/s00205-006-0032-6)
112. Truesdell C, Noll W. 1965 The non-linear field theories of mechanics. In *Fliigg'e's encyclopedia of physics* (ed. SS Antman), vol. III/3, pp. 1–662. Berlin, Germany: Springer.
113. Dunn JE, Serrin J. 1985 On the thermomechanics of interstitial working. *Arch. Ration. Mech. Anal.* **88**, 95–133. (doi:10.1007/BF00250907)
114. Dunn JE. 1986 Interstitial working and a non classical continuum thermodynamics. In *New perspectives in Thermodynamics* (ed. J Serrin), pp. 187–222. Berlin, Germany: Springer.
115. dell'Isola F, Seppecher P. 1995 The relationship between edge contact forces, double force and interstitial working allowed by the principle of virtual power. *C.R. Acad. Sci. IIb Mec. Phys. Chim. Astron.* **321**, 303–308.
116. Gurtin ME. 1965 Thermodynamics and the possibility of spatial interaction in elastic materials. *Arch. Ration. Mech. Anal.* **19**, 339–352. (doi:10.1007/BF00253483)
117. Coleman BD, Noll W. 1963 The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Ration. Mech. Anal.* **13**, 167–178. (doi:10.1007/BF01262690)
118. Mueller I, Ruggeri T. 1993 *Extended thermodynamics*, 2nd edn. New York, NY: Springer.
119. Mueller I, Ruggeri T. 1998 *Rational extended thermodynamics*. New York, NY: Springer.
120. dell'Isola F, Seppecher P. 1997 Edge contact forces and quasi-balanced power. *Meccanica* **32**, 33–52. (doi:10.1023/A:1004214032721)
121. Toupin RA. 1964 Theories of elasticity with couple-stress. *Arch. Ration. Mech. Anal.* **17**, 85–112. (doi:10.1007/BF00253050)
122. dell'Isola F, Placidi L. 2012 Variational principles are a powerful tool also for formulating field theories. In *Variational models and methods in solid and fluid mechanics*. CISM Courses and Lectures, vol. 535, pp. 1–15. Vienna, Austria: Springer.
123. Noll W. 1959 The foundations of classical mechanics in the light of recent advances in continuum mechanics. In *Proc. of the Berkeley Symp. on the Axiomatic Method, Berkeley, CA, 26 December 1957–4 January 1958*, pp. 226–281. Berlin, Heidelberg: Springer.
124. Noll W. 1973 Lectures on the foundations of continuum mechanics and thermodynamics. *Arch. Ration. Mech. Anal.* **52**, 62–92. (doi:10.1007/BF00249093)
125. Lagrange. J-L. 1989 *Mécanique analytique*. Sceaux, France: Éditions Jaques Gabay.
126. Salençon J. 1988–1995 *Mécanique des milieux continus. Handbook of Continuum*. Paris, France: Ellipses.

127. Germain P. 1973 *Cours de Mécanique des Milieux Continus, tome I*. Paris, France: Masson.
128. Schwartz L. 1973 *Théorie des distributions*. Paris, France: Hermann.
129. Eringen AC. 2001 *Microcontinuum field theories*. New York, NY: Springer.
130. dell'Isola F, Madeo A, Seppecher P. Submitted. Cauchy tetrahedron argument applied to higher contact interactions. *Arch. Ration. Mech. Anal.*, 1–37. (doi:10.1007/s00205-015-0922-6)
131. Fosdick R. 2011 On the principle of virtual power for arbitrary parts of a body. *Contin. Mech. Thermodyn.* **23**, 483–489. (doi:10.1007/s00161-011-0191-z)
132. Noll W, Virga EG. 1990 On edge interactions and surface tension. *Arch. Ration. Mech. Anal.* **111**, 1–31. (doi:10.1007/BF00375698)
133. Cosserat E, Cosserat F. 1908 *Note sur la théorie de l'action euclidienne*. Paris, France: Gauthier-Villars.
134. Cosserat E, Cosserat F. 1909 *Sur la Théorie des Corps Déformables*. Paris, France: Herman.
135. Boubaker BB, Haussy B, Ganghoffer JF. 2007 Discrete models of woven structures. Macroscopic approach. *Compos. B Eng.* **38**, 498–505. (doi:10.1016/j.compositesb.2006.01.007)
136. Rothenburg L, Selvadurai APS. 1981 A micromechanical definition of the Cauchy stress tensor for particulate media. In *Mechanics of structured media* (ed. APS Selvadurai). Amsterdam, The Netherlands: Elsevier.
137. Yue ZQ, Selvadurai APS. 1995 On the mechanics of a rigid disc inclusion embedded in a fluid saturated poroelastic medium. *Int. J. Eng. Sci.* **33**, 1633–1662. (doi:10.1016/0020-7225(95)00031-R)
138. Truesdell C. 1992 Cauchy and the modern mechanics of continua. *Rev. d'histoire des Sci.* **45**, 5–24. (doi:10.3406/rhs.1992.4229)
139. Truesdell CA. 1977 *A first course in rational continuum mechanics, vol. I: general concepts*. New York, NY: Academic Press.
140. Fried E, Gurtin ME. 2006 Tractions, balances, and boundary conditions for nonsimple materials with application to liquid flow at small length scales. *Arch. Ration. Mech. Anal.* **182**, 513–554. (doi:10.1007/s00205-006-0015-7)
141. Banfi C, Marzocchi A, Musesti A. 2006 On the principle of virtual powers in continuum mechanics. *Ric. Mat.* **55**, 299–310. (doi:10.1007/s11587-006-0017-7)
142. Germain P. 1972 Sur l'application de la méthode des puissances virtuelles en mécanique des milieux continus. *C. R. Acad. Sci. Paris A-B* **274**, A1051–A1055.
143. Bleustein JL. 1967 A note on the boundary conditions of Toupin's strain-gradient theory. *Int. J. Solids Struct.* **3**, 1053–1057. (doi:10.1016/0020-7683(67)90029-7)
144. Casal P. 1972 La théorie du second gradient et la capillarité. *C. R. Acad. Sci. Paris A* **274**, 1571–1574.
145. Casal P. 1961 La capillarité interne, Cahier du groupe Français de rhéologie. CNRS **VI**, 31–37.
146. Forest S, Cordero NM, Busso EP. 2011 First vs. second gradient of strain theory for capillarity effects in an elastic fluid at small length scales. *Comput. Mater. Sci.* **50**, 1299–1304. (doi:10.1016/j.commatsci.2010.03.048)
147. Truesdell C, Toupin RA. 1960 *The classical field theories*. Handbuch der Physik III/1. Berlin, Germany: Springer.
148. Podio-Guidugli P, Vianello M. 2010 Hypertractions and hyperstresses convey the same mechanical information. *Contin. Mech. Thermodyn.* **22**, 163–176. (doi:10.1007/s00161-010-0135-z)
149. Podio-Guidugli P. 2009 A virtual power format for thermomechanics. *Contin. Mech. Thermodyn.* **20**, 479–487. (doi:10.1007/s00161-009-0093-5)
150. Gurtin ME, Anand L. 2009 Thermodynamics applied to gradient theories involving the accumulated plastic strain: the theories of Aifantis and Fleck and Hutchinson and their generalization. *J. Mech. Phys. Solids* **57**, 405–421. (doi:10.1016/j.jmps.2008.12.002)
151. Gurtin ME, Fried E, Anand L. 2009 *The mechanics and thermodynamics of continua*. Cambridge, UK: Cambridge University Press.
152. Fried E, Gurtin ME. 2008 A continuum mechanical theory for turbulence: a generalized Navier–Stokes-equation with boundary conditions. *Theor. Comput. Fluid Dyn.* **182**, 513–554. (doi:10.1007/s00162-008-0083-4)
153. Gurtin ME. 2002 A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations. *Int. J. Plast.* **50**, 809–819. (doi:10.1016/s0022-5096(01)00104-1)
154. Podio-Guidugli P. 2002 Contact interactions, stress, and material symmetry, for nonsimple elastic materials. *Theor. Appl. Mech.* **28/29**, 261–276. [English, Serbo-Croatian summary.] Issue dedicated to the memory of Professor Rastko Stojanovic (Belgrade, 2002).

155. Fosdick R, Virga EG. 1989 A variational proof of the stress theorem of Cauchy. *Arch. Ration. Mech. Anal.* **105**, 95–103. (doi:10.1007/BF00250832)
156. Fosdick R. 2011 Observations concerning virtual power. *Math. Mech. Solids* **16**, 573–585. (doi:10.1177/1081286510387708)
157. Seppecher P. 1989 Etude des conditions aux limites en théorie du second gradient: cas de la capillarité. *C. R. Acad. Sci. Paris II* **309**, 497–502.
158. Seppecher P. 1987 *Etude d'une Mod Zones Capillaires Fluides: interfaces et Lignes de Contact*. Thèse de l'Université Paris VI.
159. Noll W. 1993 The geometry of contact separation and reformation of continuous bodies. *Arch. Ration. Mech. Anal.* **122**, 197–212. (doi:10.1007/BF00380254)
160. Auffray N, dell'Isola F, Eremeyev VA, Madeo A, Rosi G. 2015 Analytical continuum mechanics la Hamilton–Piola least action principle for second gradient continua and capillary fluids. *Math. Mech. Solids* **20**, 375–417. (doi:10.1177/1081286513497616)
161. DelPiero G. 2014 Non-classical continua, pseudobalance, and the law of action and reaction. *Math. Mech. Complex Syst.* **2**, 7110. (doi:10.2140/memocs.2014.2.71)
162. Marzocchi A, Musesti A. 2003 Balanced virtual powers in continuum mechanics. *Meccanica* **38**, 369–389. (doi:10.1023/A:1023301303945)
163. Marzocchi A, Musesti A. 2001 Decomposition and integral representation of Cauchy interactions associated with measures. *Cont. Mech. Thermodyn.* **13**, 149–169. (doi:10.1007/s001610100046)
164. Degiovanni M, Marzocchi A, Musesti A. 2006 Edge-force densities and second-order powers. *Ann. Mat.* **185**, 81–103. (doi:10.1007/s10231-004-0129-1)
165. Cauchy AL. 1827 De la pression ou tension dans un corps solide. *Ex. Math.* **2**, 4256. [Available at Gallica.bnf.fr.]
166. dell'Isola F, Seppecher P, Madeo A. 2011 Beyond Euler–Cauchy continua: the structure of contact actions in N th gradient generalized continua: a generalization of the Cauchy tetrahedron argument. In *CISM Lecture Notes C-1006*, ch. 2. Berlin, Germany: Springer.
167. dell'Isola F, Seppecher P, Madeo A. 2011 Fluid shock wave generation at solid-material discontinuity surfaces in porous media. In *CISM Lecture Notes C-1006*, ch. 7. Berlin, Germany: Springer.
168. Lazar M, Maugin GA. 2006 A note on line forces in gradient elasticity. *Mech. Res. Commun.* **33**, 674–680. (doi:10.1016/j.mechrescom.2005.08.004)
169. Lucchesi M, Silhavy N. 2008 Zani On the balance equation for stresses concentrated on curves. *J. Elast.* **90**, 209–223. (doi:10.1007/s10659-007-9139-8)
170. Carcaterra A. 2005 Ensemble energy average and energy flow relationships for nonstationary vibrating systems. *J. Sound Vib.* **288**, 751–790. (doi:10.1016/j.jsv.2005.07.015)
171. Carcaterra A, Sestieri A. 1995 Energy density equations and power flow in structures. *J. Sound Vib.* **188**, 269–282. (doi:10.1006/jsvi.1995.0591)
172. Seppecher P. 1996 Moving contact lines in the Cahn–Hilliard theory. *Int. J. Eng. Sci.* **34**, 977–992. (doi:10.1016/0020-7225(95)00141-7)
173. Casal P, et Gouin H. 1985 Relation entre l'équation de l'énergie et l'équation du mouvement en théorie de Korteweg de la capillarité. *C. R. Acad. Sci. Paris II N.* **7**, 231–233.
174. Forestier A, Gavriljuk S. 2011 Criterion of hyperbolicity for non-conservative quasilinear systems admitting a partially convex conservation law. *Math. Meth. Appl. Sci.* **3**, 2148–2158. (doi:10.1002/mma.1512)
175. Giusteri GG. 2013 The multiple nature of concentrated interactions in second gradient dissipative liquids. *Z. Angew. Math. Phys.* **64**, 371–380. (doi:10.1007/s00033-012-0229-5)
176. Eremeyev VA, Pietraszkiewicz W. 2011 Thermomechanics of shells undergoing phase transition. *J. Mech. Phys. Solids* **59**, 1395–1412. (doi:10.1016/j.jmps.2011.04.005)
177. Culla A, Sestieri A, Carcaterra A. 2003 Energy flow uncertainties in vibrating systems: definition of a statistical confidence factor. *Mech. Syst. Signal Process.* **17**, 635–663. (doi:10.1006/mssp.2002.1487)
178. Carcaterra A, Ciappi E, Iafrati A, Campana EF. 2000 Shock spectral analysis of elastic systems impacting on the water surface. *J. Sound Vib.* **229**, 579–605. (doi:10.1006/jsvi.1999.2517)
179. Iafrati A, Carcaterra A, Ciappi E, Campana EF. 2000 Hydroelastic analysis of a simple oscillator impacting the free surface. *J. Ship Res.* **44**, 278–289.
180. Andraeus U, Chiaia B, Placidi L. 2013 Soft-impact dynamics of deformable bodies. *Contin. Mech. Thermodyn.* **25**, 375–398. (doi:10.1007/s00161-012-0266-5)

181. Gavriluk S, Gouin H. 2010 Geometric evolution of the Reynolds stress tensor in three-dimensional turbulence. In *Waves and stability in continuous media* (eds A Greco, S Rionero, T Ruggeri), pp. 182–190. Singapore: World Scientific.
182. Del Vescovo D, Giorgio I. 2014. Dynamic problems for metamaterials: review of existing models and ideas for further research. *Int. J. Eng. Sci.* **80**, 153–172. (doi:10.1016/j.ijengsci.2014.02.022)
183. Madeo A, Placidi L, Rosi G. 2014 Towards the design of metamaterials with enhanced damage sensitivity: second gradient porous materials. *Res. Nondestruct. Evol.* **25**, 99–124. (doi:10.1080/09349847.2013.853114)