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FROM HARD SPHERE DYNAMICS TO THE STOKES-FOURIER EQUATIONS: AN L^2 ANALYSIS OF THE BOLTZMANN-GRAD LIMIT

THIERRY BODINEAU, ISABELLE GALLAGHER AND LAURE SAINT-RAYMOND

Abstract. We derive the linear acoustic and Stokes-Fourier equations as the limiting dynamics of a system of N hard spheres of diameter ε in two space dimensions, when $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon = \alpha \to \infty$, using the linearized Boltzmann equation as an intermediate step. Our proof is based on Lanford's strategy [15], and on the pruning procedure developed in [4] to improve the convergence time. The main novelty here is that uniform L^2 a priori estimates combined with a subtle symmetry argument provide a useful cumulant expansion describing the asymptotic decorrelation between the particles. A refined geometric analysis of recollisions is also required in order to discard the possibility of multiple recollisions.

1. Introduction to the Boltzmann-Grad limit and statement of the result

The sixth problem raised by Hilbert in 1900 on the occasion of the International Congress of Mathematicians addresses the question of the axiomatization of mechanics, and more precisely of describing the transition between atomistic and continuous models for gas dynamics by rigorous mathematical convergence results. Even though it is quite restrictive (since only perfect gases can be considered by this process), Hilbert further suggested using Boltzmann's kinetic equation as an intermediate step to understand the appearance of irreversibility and dissipative mechanisms [13].

A huge amount of literature has been devoted to these asymptotic problems, but up to now they remain still largely open. Important breakthroughs [6, 2] have allowed for a complete study of some hydrodynamic limits of the Boltzmann equation, especially in incompressible viscous regimes leading to the Navier-Stokes equations (see [9] for instance). Note that other regimes such as the compressible Euler limit (which is the most immediate from a formal point of view) are still far from being understood.

But, at this stage, the main obstacle seems actually to come from the other step, namely the derivation of the Boltzmann equation from a system of interacting particles: the best result to this day concerning this low density limit which is due to Lanford in the case of hard-spheres [15] (see also [5, 22, 7] for a complete proof) is indeed valid only for short times, i.e. breaks down before any relaxation can be observed.

Theorem 1.1. *Consider a system of N hard-spheres of diameter ε on $\mathbb{T}^d = [0,1]^d$ (with $d \geq 2$), initially "independent" and identically distributed with density f_0 such that*

$$\left\| f_0 \exp(\mu + \frac{\beta}{2}|v|^2) \right\|_{L^\infty(\mathbb{T}^d_x \times \mathbb{R}^d_v)} \leq 1 \, ,$$

for some $\beta > 0, \mu \in \mathbb{R}$.

Fix $\alpha > 0$, then, in the Boltzmann-Grad limit $N \to \infty$ with $N\varepsilon^{d-1} = \alpha$, the density of a typical particle converges almost everywhere to the solution of the Boltzmann equation

$$\begin{aligned}
&\partial_t f + v \cdot \nabla_x f = \alpha \, Q(f,f), \\
(1.1) \qquad &Q(f,f)(v) := \iint_{\mathbb{S}^{d-1} \times \mathbb{R}^d} [f(v')f(v'_1) - f(v)f(v_1)] \left((v - v_1) \cdot \nu\right)_+ dv_1 d\nu \, , \\
&v' = v + \nu \cdot (v_1 - v)\, \nu \, , \quad v'_1 = v_1 - \nu \cdot (v_1 - v)\, \nu \, ,
\end{aligned}$$

on a time interval $[0, C(\beta,\mu)/\alpha]$.

By independent we mean here that the correlations, which are due to the non overlapping condition, vanish asymptotically as $\varepsilon \to 0$.

The main reason why the convergence is not known to hold for longer time intervals is that the nonlinearity in the Boltzmann equation (1.1) is treated as if the equation was of the type $\partial_t f = \alpha f^2$: the cancellations between gain and loss terms in $Q(f,f)$ are yet to be understood. The only information we are able to get,about these compensations is the existence of invariant measures; in this work we improve the time of convergence by considering very small fluctuations around such equilibria.

1.1. **Setting of the problem.**

1.1.1. *The model.* In the following, we consider only the case of dimension $d = 2$. We are interested in describing the macroscopic behavior of a gas consisting in N hard spheres of diameter ε in a periodic domain $\mathbb{T}^2 = [0,1]^2$ of $\mathbb{R}^2$, with positions and velocities $(x_i, v_i)_{1\leq i\leq N}$ in $(\mathbb{T}^2 \times \mathbb{R}^2)^N$, the dynamics of which is given by

$$\frac{dx_i}{dt} = v_i\,, \quad \frac{dv_i}{dt} = 0 \quad \text{as long as } \; |x_i(t) - x_j(t)| > \varepsilon \quad \text{for } \; 1 \leq i \neq j \leq N\,, \tag{1.2}$$

with specular reflection at a collision

$$\left.\begin{aligned} v_i' &:= v_i - \frac{1}{\varepsilon^2}(v_i - v_j)\cdot(x_i - x_j)\,(x_i - x_j) \\ v_j' &:= v_j + \frac{1}{\varepsilon^2}(v_i - v_j)\cdot(x_i - x_j)\,(x_i - x_j) \end{aligned}\right\} \quad \text{if } |x_i(t) - x_j(t)| = \varepsilon\,. \tag{1.3}$$

By macroscopic behavior, we mean that we look for a statistical description averaging both on the number of particles $N \to \infty$, and on the initial configurations.

Denote $X_N := (x_1, \dots, x_N) \in \mathbb{T}^{2N}$, $V_N := (v_1, \dots, v_N) \in \mathbb{R}^{2N}$ and $Z_N := (X_N, V_N) \in \mathbb{D}^N$ where $\mathbb{D}^N := \mathbb{T}^{2N} \times \mathbb{R}^{2N}$. Defining the Hamiltonian

$$H_N(V_N) := \frac{1}{2}\sum_{i=1}^N |v_i|^2\,,$$

we consider the Liouville equation in the $4N$-dimensional phase space

$$\mathcal{D}^N_\varepsilon := \big\{ Z_N \in \mathbb{D}^N \,/\, \forall i \neq j\,, \quad |x_i - x_j| > \varepsilon \big\}\,. \tag{1.4}$$

The Liouville equation is the following

$$\partial_t f_N + \{H_N, f_N\} = 0\,,$$

or in other words

$$\partial_t f_N + V_N \cdot \nabla_{X_N} f_N = 0\,, \tag{1.5}$$

with specular reflection on the boundary, meaning that if Z_N belongs to $\partial\mathcal{D}^{N+}_\varepsilon(i,j)$ then we impose that

$$f_N(t, Z_N) = f_N(t, Z_N')\,, \tag{1.6}$$

where $X_N' = X_N$ and $v_k' = v_k$ if $k \neq i,j$ while (v_i', v_j') are given by (1.3). We have also defined

$$\begin{aligned} \partial\mathcal{D}^{N\pm}_\varepsilon(i,j) := \Big\{ & Z_N \in \mathbb{D}^N \,/\, |x_i - x_j| = \varepsilon\,, \quad \pm(v_i - v_j)\cdot(x_i - x_j) > 0 \\ & \text{and} \quad \forall (k,\ell) \in [1,N]^2 \setminus \{(i,j)\}\,, k \neq \ell\,, |x_k - x_\ell| > \varepsilon \Big\}\,. \end{aligned} \tag{1.7}$$

In the following we assume that f_N is symmetric under permutations of the N particles, meaning that the particles are exchangeable, and we define f_N on the whole phase space $\mathbb{D}^N$ by setting $f_N \equiv 0$ on $\mathbb{D}^N \setminus \mathcal{D}_\varepsilon^N$.

We recall, as shown in [1] for instance, that the set of initial configurations leading to ill-defined characteristics (due to clustering of collision times, or collisions involving more than two particles) is of measure zero in $\mathcal{D}_\varepsilon^N$.

In the following we shall denote by $\mathbf{\Psi}_N$ the solution operator to the ODE (1.2-1.3) and by $\mathbf{S}_N$ the group associated to free transport in $\mathcal{D}_\varepsilon^N$ with specular reflection on the boundary. In other words, for a function φ_N defined on $\mathcal{D}_\varepsilon^N$, we write

$$\mathbf{S}_N(\tau)\varphi_N(Z_N) = \varphi_N\big(\mathbf{\Psi}_N(-\tau)Z_N\big) \, .$$

1.1.2. *The BBGKY and Boltzmann hierarchies.* We are interested in the limiting behaviour of the previous system when $N \to \infty$ and $\varepsilon \to 0$ under the Boltzmann-Grad scaling $N\varepsilon = \alpha$, with $\alpha = O(1)$ or diverging slowly to infinity. The quantities which are expected to have finite limits in the Boltzmann-Grad limit are the marginals

$$f_N^{(s)}(t, Z_s) := \int_{\mathbb{D}^{N-s}} f_N(t, Z_N) dz_{s+1} \dots dz_N$$

for every $s < N$.

A formal computation based on Green's formula (see [5, 7] for instance) leads to the following BBGKY hierarchy for $s < N$

$$(\partial_t + \sum_{i=1}^s v_i \cdot \nabla_{x_i}) f_N^{(s)}(t, Z_s) = \alpha\big(C_{s,s+1} f_N^{(s+1)}\big)(t, Z_s) \tag{1.8}$$

on $\mathcal{D}_\varepsilon^s$, with the boundary condition as in (1.6)

$$f_N^{(s)}(t, Z_s) = f_N^{(s)}(t, Z_s') \text{ on } \partial D_\varepsilon^{s+}(i,j) \, .$$

The collision term is defined by

$$\begin{aligned}
&\big(C_{s,s+1} f_N^{(s+1)}\big)(Z_s) := (N-s)\varepsilon\alpha^{-1} \\
&\times \Big(\sum_{i=1}^s \int_{\mathbb{S}\times\mathbb{R}^2} f_N^{(s+1)}(\dots, x_i, v_i', \dots, x_i + \varepsilon\nu, v_{s+1}') \big((v_{s+1} - v_i)\cdot\nu\big)_+ d\nu dv_{s+1} \\
&\quad - \sum_{i=1}^s \int_{\mathbb{S}\times\mathbb{R}^2} f_N^{(s+1)}(\dots, x_i, v_i, \dots, x_i + \varepsilon\nu, v_{s+1}) \big((v_{s+1} - v_i)\cdot\nu\big)_- d\nu dv_{s+1} \Big) \\
&\text{with} \quad v_i' := v_i - (v_i - v_{s+1})\cdot\nu\,\nu \, , \quad v_{s+1}' := v_{s+1} + (v_i - v_{s+1})\cdot\nu\,\nu \, ,
\end{aligned} \tag{1.9}$$

where $\mathbb{S}$ denotes the unit sphere in $\mathbb{R}^2$. Note that the collision integral is split into two terms according to the sign of $(v_i - v_{s+1})\cdot\nu$ and we used the trace condition on $\partial\mathcal{D}_\varepsilon^{N+}(i, s+1)$ to express all quantities in terms of pre-collisional configurations: in the following we shall also use the notation

$$C_{s,s+1}^{i,+} f_{s+1}(Z_s) := (N-s)\varepsilon\alpha^{-1} \int f_{s+1}(\dots, x_i, v_i', \dots, x_i + \varepsilon\nu, v_{s+1}') \Big((v_{s+1} - v_i)\cdot\nu\Big)_+ d\nu dv_{s+1}$$

$$C_{s,s+1}^{i,-} f_{s+1}(Z_s) := (N-s)\varepsilon\alpha^{-1} \int f_{s+1}(\dots, x_i, v_i, \dots, x_i - \varepsilon\nu, v_{s+1}) \Big((v_{s+1} - v_i)\cdot\nu\Big)_+ d\nu dv_{s+1} \, ,$$

so that

$$C_{s,s+1} = \sum_{i=1}^s (C_{s,s+1}^{i,+} - C_{s,s+1}^{i,-}) \, . \tag{1.10}$$

The closure for $s = N$ is given by the Liouville equation (1.5).

To obtain the Boltzmann hierarchy, we compute the formal limit of the transport and collision operators when ε goes to 0. Recalling that $(N-s)\varepsilon \sim \alpha$, the limit hierarchy is given by

$$(\partial_t + \sum_{i=1}^s v_i \cdot \nabla_{x_i}) f^{(s)}(t, Z_s) = \alpha \big(\bar C_{s,s+1} f^{(s+1)} \big)(t, Z_s) \tag{1.11}$$

in $(\mathbb{T}^2 \times \mathbb{R}^2)^s$, where $\bar C_{s,s+1}$ are the limit collision operators defined by

$$\begin{aligned}\big(\bar C_{s,s+1} f^{(s+1)}\big)(Z_s) := \sum_{i=1}^s \int f^{(s+1)}(\dots, x_i, v_i', \dots, x_i, v_{s+1}') \Big((v_{s+1} - v_i) \cdot \nu \Big)_+ d\nu dv_{s+1} \\ - \sum_{i=1}^s \int f^{(s+1)}(\dots, x_i, v_i, \dots, x_i, v_{s+1}) \Big((v_{s+1} - v_i) \cdot \nu \Big)_+ d\nu dv_{s+1} \, .\end{aligned}$$

1.1.3. *Initial data and closures for the Boltzmann hierarchy.* Consider chaotic initial data of the form $(f_0^{\otimes s})_{s \in \mathbb{N}^*}$, with

$$f_0^{\otimes s}(Z_s) := \prod_{i=1}^s f_0(z_i) \quad \text{with} \quad \int_{\mathbb{D}} f_0(z) dz = 1 \, ,$$

and denote by $f(t)$ the solution of the nonlinear Boltzmann equation (1.1) which can be rewritten as

$$(\partial_t + v \cdot \nabla_x) f = \alpha \bar C_{1,2} f^{\otimes 2} \, , \qquad f_{|t=0} = f_0 \, .$$

Then an easy computation shows that $(f(t)^{\otimes s})_{s \in \mathbb{N}^*}$ is a chaotic solution to the Boltzmann hierarchy, whose first marginal is nothing else than $f(t)$. Note that, even though it may look like a very particular case, it is somehow generic as any symmetric initial data may in fact be decomposed as a superposition of chaotic distributions (this is known as the Hewitt-Savage theorem, see [12]). This means that the Boltzmann hierarchy, even though consisting of linear equations, encodes nonlinear phenomena. In the absence of suitable uniform a priori estimates, we therefore may expect the solution to blow up after a finite time. This is actually the main obstacle to get a rigorous derivation of the Boltzmann equation over time intervals larger than the mean free time $O(1/\alpha)$.

A different structure of initial data can lead to other types of equations. Recall that the Maxwellian

$$M_\beta(v) := \frac{\beta}{2\pi} \exp\left(-\beta \frac{|v|^2}{2} \right)$$

is an equilibrium for the Boltzmann dynamics, so that $(M_\beta^{\otimes s})_{s \geq 1}$ is a stationary solution to the Boltzmann hierarchy. Consider an initial data which is a perturbation of this stationary solution

$$f_0^{(s)}(Z_s) = M_\beta^{\otimes s}(V_s) \sum_{i=1}^s g_{\alpha,0}(z_i) \, , \tag{1.12}$$

where we added a dependency of $g_{\alpha,0}$ on α for later purposes. This form is stable under the dynamics [3] so that a solution to the Boltzmann hierarchy (1.11) is

$$f^{(s)}(t, Z_s) = M_\beta^{\otimes s}(V_s) \sum_{i=1}^s g_\alpha(t, z_i) \tag{1.13}$$

where g_α is a solution of the linearized Boltzmann equation

$$
\begin{aligned}
&(\partial_t + v\cdot\nabla_x) g_\alpha = -\alpha \mathcal{L}_\beta g_\alpha \, , \\
&\mathcal{L}_\beta \, g_\alpha(v) := -\frac{1}{M_\beta} \bar C_{1,2}(M_\beta\otimes M_\beta g_\alpha + M_\beta g_\alpha \otimes M_\beta)(v) \\
&\qquad = \int M_\beta(v_1)\Big(g_\alpha(v)+g_\alpha(v_1)-g_\alpha(v')-g_\alpha(v'_1)\Big)\Big((v_1-v)\cdot\nu\Big)_+ d\nu dv_1\, ,
\end{aligned} \tag{1.14}
$$

with initial data $g_{\alpha,0}$, because the associate norm is a Lyapunov functional for (1.14) (see Appendix A). The functional space $L^2(dxM_\beta dv)$ is natural to study the linearized Boltzmann equation. As we will heavily use it later on, we introduce the following notation, for $p=1,2$: for any function g_s defined on $\mathbb{D}^s$,

$$
\|g_s\|_{L^p_\beta(\mathbb{D}^s)} := \Big(\int M_\beta^{\otimes s}(V_s)|g_s|^p(Z_s)\, dZ_s\Big)^{\frac{1}{p}} . \tag{1.15}
$$

We now turn to the particle dynamics and discuss the counterpart of the initial data (1.12). The Gibbs measure

$$
M_{N,\beta}(Z_N) := \frac{1}{\mathcal{Z}_N}\mathbf{1}_{\mathcal{D}^N_\varepsilon}(X_N)\, M_\beta^{\otimes N}(V_N)\, , \qquad \mathcal{Z}_N := \int_{\mathbb{T}^{2N}} \prod_{1\le i\neq j\le N} \mathbf{1}_{|x_i-x_j|>\varepsilon}\, dX_N
$$

is invariant for the dynamics. An idea to get such linear asymptotics is to consider small fluctuations around an equilibrium of the form

$$
f_{N,0}(Z_N) = M_{N,\beta}(Z_N)\prod_{i=1}^N \big(1+\delta g_{\alpha,0}(z_i)\big)\, .
$$

However whatever the smallness of δ, such a sequence of initial data is never a small correction to $M_{N,\beta}$. Thus, we shall tune the size of the perturbation with N

$$
\begin{aligned}
f_{N,0}(Z_N) &= M_{N,\beta}(Z_N)\prod_{i=1}^N \big(1+\frac{\delta}{N} g_{\alpha,0}(z_i)\big) \\
&= M_{N,\beta}(Z_N) + \frac{\delta}{N} M_{N,\beta}(Z_N) \sum_{i=1}^N g_{\alpha,0}(z_i) + O(\delta^2)\, .
\end{aligned} \tag{1.16}
$$

At the first order in δ, we recover an initial data for the BBGKY hierarchy of the form (1.12)

$$
f_{N,0}(Z_N) = M_{N,\beta}(Z_N)\sum_{i=1}^N g_{\alpha,0}(z_i) \quad \text{with} \quad \int M_\beta g_{\alpha,0}(z)dz = 0\, . \tag{1.17}
$$

This initial data records only the perturbation and it is no longer a probability measure. In particular

$$
\int f_{N,0}(Z_N) dZ_N = 0\, ,
$$

and this property is preserved by the Liouville equation (1.5). The question is then to know if the solution of the BBGKY hierarchy obeys a form similar to (1.13), at least approximately, and if one can obtain good enough bounds in L^2 spaces to prove long-time convergence to $f^{(s)}$ defined in (1.13).

Remark 1.1. *Note that another type of (non symmetric) perturbation was dealt with in* [4], *namely an initial data of the form*

$$
f_{N,0}(Z_N) = M_{N,\beta}(Z_N) g_0(z_1)\, . \tag{1.18}
$$

This describes the motion of a tagged particle in a background close to equilibrium, and we have shown that it satisfies asymptotically the linear Boltzmann equation, and the tagged particle dynamics converges to the Brownian motion in the diffusive limit. However the proof is less complicated since all quantities of interest are uniformly controlled in L^∞, which will not be the case with the initial data (1.17)*.*

1.2. **Statement of the main result.** Our main result is the following.

Theorem 1.2. *Consider N hard spheres on the space $\mathbb{D} = \mathbb{T}^2 \times \mathbb{R}^2$, initially distributed according to $f_{N,0}$ defined as in* (1.17) *where $g_{\alpha,0}$ is a bounded, Lipschitz function on $\mathbb{D}$ with zero average, and satisfying the following bound for some constant C_1*

$$\|g_{\alpha,0}\|_{W^{1,\infty}} \leq C_1 \exp(C_1\alpha^2)\,. \tag{1.19}$$

Then the one-particle distribution $f_N^{(1)}(t,z)$ is close to $M_\beta(v)g_\alpha(t,z)$, where $g_\alpha(t,z)$ is the solution of the linearized Boltzmann equation (1.14) *with initial data $g_{\alpha,0}(z)$.*

More precisely, there exists a non negative constant C such that for all $T > 1$ and all $\alpha > 1$, in the limit $N \to \infty$, $N\varepsilon\alpha^{-1} = 1$,

$$\sup_{t\in[0,T]} \left\| f_N^{(1)}(t) - M_\beta g_\alpha(t) \right\|_{L^2(\mathbb{D})} \leq \frac{T^2 e^{C\alpha^2}}{\sqrt{\log\log N}}\,\cdot \tag{1.20}$$

Note that some L^∞ convergence was established in [3] following Lanford's strategy, but only for short times. Equilibrium fluctuations (for arbitrary time) have been derived by F. Rezakhanlou [18] for microscopic dynamics with random collisions.

Once Theorem 1.2 is known, it is possible to take the limit $\alpha \to \infty$ while conserving a small error on the right-hand side of (1.20). Using the classical convergence of the linearized Boltzmann equation to the acoustic equation (see Appendix A), one infers the following result.

Corollary 1.2. *Consider N hard spheres on the space $\mathbb{D} = \mathbb{T}^2 \times \mathbb{R}^2$, initially distributed according to $f_{N,0}$ defined as in* (1.17) *with a sequence $(g_{\alpha,0})$ of functions satisfying the assumptions of Theorem* 1.2 *and converging in $L^2_\beta(\mathbb{D})$ as α diverges to*

$$g_0(x,v) := \rho_0(x) + \sqrt{\beta}\, u_0(x)\cdot v + \frac{\beta|v|^2 - 4}{2}\,\theta_0(x) \quad \textit{with} \quad \int_{\mathbb{T}^2} \rho_0(x)dx = 0\,.$$

Then as $N \to \infty$, $N\varepsilon = \alpha \to \infty$ much slower than $\sqrt{\log\log\log N}$, the distribution $f_N^{(1)}(t)$ converges in $L^2(\mathbb{D})$-norm to $M_\beta g(t)$ with

$$g(t,x,v) := \rho(t,x) + \sqrt{\beta}\, u(t,x)\cdot v + \frac{\beta|v|^2 - 2}{2}\theta(t,x)\,,$$

where (ρ, u, θ) satisfies the acoustic equations

$$\begin{cases} \partial_t\rho + \dfrac{1}{\sqrt{\beta}}\,\nabla_x\cdot u = 0 \\ \partial_t u + \dfrac{1}{\sqrt{\beta}}\,\nabla_x(\rho+\theta) = 0 \\ \partial_t\theta + \dfrac{1}{\sqrt{\beta}}\,\nabla_x\cdot u = 0 \end{cases}$$

with initial data (ρ_0, u_0, θ_0).

It is even possible to rescale time as $t = \alpha\tau$ and to take the limit $\alpha \to \infty$. For well-prepared initial data, we then obtain the following diffusive approximation by the Stokes-Fourier dynamics.

Corollary 1.3. *Consider N hard spheres on the space $\mathbb{D} = \mathbb{T}^2 \times \mathbb{R}^2$, initially distributed according to $f_{N,0}$ defined in* (1.17) *with a sequence $(g_{\alpha,0})$ of functions satisfying the assumptions of Theorem* 1.2 *and converging in L^2_β as α diverges to*

$$g_0(x,v) := \sqrt{\beta}\, u_0(x) \cdot v + \frac{\beta|v|^2 - 2}{2}\, \theta_0(x)\,, \quad \nabla_x \cdot u_0 = 0\,.$$

Then as $N \to \infty$, $N\varepsilon = \alpha \to \infty$ much slower than $\sqrt{\log\log\log N}$, the distribution $f_N^{(1)}(\alpha\tau)$ converges in $L^2(\mathbb{D})$ norm to $M_\beta g(\tau)$ with

$$g(\tau,x,v) := \sqrt{\beta}\, u(\tau,x) \cdot v + \frac{\beta|v|^2 - 2}{2} \theta(\tau,x)\,,$$

where (u,θ) satisfies the Stokes-Fourier equations

$$\begin{cases} \partial_\tau u - \dfrac{1}{\sqrt{\beta}}\, \mu_\beta \Delta_x u = 0 \\ \nabla_x \cdot u = 0 \\ \partial_\tau \theta - \dfrac{1}{\sqrt{\beta}}\, \kappa_\beta \Delta_x \theta = 0 \end{cases} \tag{1.21}$$

with initial data (u_0, θ_0), and

$$\mu_\beta := \frac{1}{4} \int \Phi_\beta \mathcal{L}_\beta^{-1} \Phi_\beta M_\beta(v) dv \quad \textit{with} \quad \Phi_\beta(v) := \beta^2 (v \otimes v - \frac{|v|^2}{2} \mathrm{Id})\,,$$
$$\kappa_\beta := \frac{1}{4} \int \Psi_\beta \mathcal{L}_\beta^{-1} \Psi_\beta M_\beta(v) dv \quad \textit{with} \quad \Psi_\beta(v) := \sqrt{\beta}\, v \left(\beta \frac{|v|^2}{2} - 2 \right)\,,$$

where the operator $\mathcal{L}_\beta$ was introduced in (1.14).

Remark 1.4. *In the case of general, ill-prepared initial data, the asymptotics is also well known* [8]*. Details are provided in Appendix A.*

2. Strategy of the proof

In the sequel, we focus on the proof of Theorem 1.2, as it is the new contribution of this work. Even though it follows some ideas introduced in [4], it represents a real improvement of what has been done up to now:

- First of all, we are able to capture a fluctuation of order $O(1/N)$ around an equilibrium (1.16), and in particular there is no more positivity.
- Second, we deal with a much weaker functional setting than the L^∞ framework of Lanford's strategy [15].

Let us recall that, up to now, all the results regarding the low density limit of systems of particles have been established following Lanford's strategy [15]. In this section, we describe the main objects involved in the proof, and the pruning procedure introduced in [4]. We then show the main differences between our setting and that of [4] and finally explain how to adapt the pruning procedure to our setting.

2.1. **The series expansion.** The starting point is the series expansion obtained by iterating Duhamel's formula for the BBGKY hierarchy (1.8) :

$$
\begin{aligned}
f_N^{(s)}(t) = \sum_{n=0}^{N-s} \alpha^n \int_0^t \int_0^{t_{s+1}} \dots \int_0^{t_{s+n-1}} \mathbf{S}_s(t-t_{s+1}) C_{s,s+1} \mathbf{S}_{s+1}(t_{s+1}-t_{s+2}) C_{s+1,s+2} \\
\dots \mathbf{S}_{s+n}(t_{s+n}) f_{N,0}^{(s+n)}\, dt_{s+n} \dots dt_{s+1} \,,
\end{aligned}
\tag{2.1}
$$

where recall that $\mathbf{S}_s$ denotes the group associated to free transport in $\mathcal{D}^s_\varepsilon$ with specular reflection on the boundary. By abuse of notation, the term $n=0$ in (2.1) should be interpreted as $\mathbf{S}_s(t) f^{(s)}_{N,0}$ as n records the number of collision operators up to time 0. Denoting by $\mathbf{S}^0_s$ the free flow, one can derive formally the limiting Boltzmann hierarchy

$$
\begin{aligned}
f^{(s)}(t) = \sum_{n\geq 0} \alpha^n \int_0^t \int_0^{t_{s+1}} \dots \int_0^{t_{s+n-1}} \mathbf{S}^0_s(t-t_{s+1}) \bar C_{s,s+1} \mathbf{S}^0_{s+1}(t_{s+1}-t_{s+2}) \bar C_{s+1,s+2} \\
\dots \mathbf{S}^0_{s+n}(t_{s+n}) f_0^{(s+n)}\, dt_{s+n} \dots dt_{s+1} \,,
\end{aligned}
\tag{2.2}
$$

and one aims at proving the convergence of one hierarchy to the other (actually one is only interested in the convergence of the first marginal, but as it involves the second marginal, and so on, one is led to proving the convergence of each marginal).

These series expansions have graphical representations which play a key role in the analysis as explained in [15, 5, 7]. This interpretation in terms of collision trees is described below.

Let us extract combinatorial information from the iterated Duhamel formula (2.1). We describe the adjunction of new particles (in the backward dynamics) by ordered trees.

Definition 2.1 (Collision trees)**.** *Let $s > 1$ be fixed. An (ordered) collision tree $a \in \mathcal{A}_s$ is defined by a family $(a(i))_{2\leq i\leq s}$ with $a(i) \in \{1,\dots,i-1\}$.*

Note that $|\mathcal{A}_s| \leq (s-1)!$.

Once we have fixed a collision tree $a \in \mathcal{A}_s$, we can reconstruct pseudo-dynamics starting from any point in the one-particle phase space $z_1 = (x_1, v_1) \in \mathbb{T}^2 \times \mathbb{R}^2$ at time t.

Definition 2.2 (Pseudo-trajectory)**.** *Given $z_1 \in \mathbb{T}^2 \times \mathbb{R}^2$, consider a collection of times, angles and velocities $(T_{2,s}, \Omega_{2,s}, V_{2,s}) = (t_i, \nu_i, v_i)_{2\leq i\leq s}$ with $0 \leq t_s \leq \dots \leq t_2 \leq t$. We then define recursively the pseudo-trajectories in terms of the backward BBGKY dynamics as follows*

- *in between the collision times t_i and t_{i+1} the particles follow the i-particle backward flow with specular reflection;*
- *at time t_i^+, particle i is adjoined to particle $a(i)$ at position $x_{a(i)} + \varepsilon\nu_i$ and with velocity v_i. If $(v_i - v_{a(i)}(t_i^+))\cdot \nu_i > 0$, velocities at time t_i^- are given by the scattering laws*

$$
\begin{aligned}
v_{a(i)}(t_i^-) &= v_{a(i)}(t_i^+) - (v_{a(i)}(t_i^+) - v_i)\cdot \nu_i\, \nu_i \,,\\
v_i(t_i^-) &= v_i + (v_{a(i)}(t_i^+) - v_i)\cdot \nu_i\, \nu_i \,.
\end{aligned}
\tag{2.3}
$$

We denote by $z_i(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, \tau)$ the position and velocity of the particle labeled i, at time τ (provided $\tau < t_i$). The configuration obtained at the end of the tree, i.e. at time 0, is $Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)$.

Similarly, we define the pseudo-trajectories associated with the Boltzmann hierarchy. These pseudo-trajectories evolve according to the backward Boltzmann dynamics as follows

- *in between the collision times t_i and t_{i+1} the particles follow the i-particle backward free flow;*

- *at time t_i^+, particle i is adjoined to particle $a(i)$ at exactly the same position $x_{a(i)}$. Velocities are given by the laws* (2.3)*.*

We denote $\bar Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)$ the initial configuration.

The following semantic distinction will be important later on.

Definition 2.3 (Collisions/Recollisions)**.** *In the BBGKY hierarchy, the term collision will be used only for the creation of a new particle, i.e. for a branching in the collision trees. A shock between two particles in the backward BBGKY dynamics will be called a recollision.*

Note that no recollision occurs in the Boltzmann hierarchy as the particles have zero diameter.

With these notations the iterated Duhamel formula (2.1) for the first marginal ($s = 1$) can be rewritten

$$
\begin{aligned}
f_N^{(1)}(t) = \sum_{s=1}^{N} (N-1)\dots\big(N-(s-1)\big)\varepsilon^{s-1} \sum_{a\in\mathcal{A}_s} \int_{\mathcal{T}_{2,s}} dT_{2,s} \int_{\mathbb{S}^{s-1}} d\Omega_{2,s} \int_{\mathbb{R}^{2(s-1)}} dV_{2,s} \\
\times \Big(\prod_{i=2}^{s} \big((v_i - v_{a(i)}(t_i)) \cdot \nu_i \Big) f_{N,0}^{(s)}\big(Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)\big)\,,
\end{aligned}
\tag{2.4}
$$

denoting

$$
\mathcal{T}_{2,s} := \big\{ (t_i)_{2\le i\le s} \in [0,t]^{s-1} \,/\, 0 \le t_s \le \dots \le t_2 \le t \big\}\,,
$$

while in the limit

$$
\begin{aligned}
f^{(1)}(t) = \sum_{s=1}^{\infty} \alpha^{s-1} \sum_{a\in\mathcal{A}_s} \int_{\mathcal{T}_{2,s}} dT_{2,s} \int_{\mathbb{S}^{s-1}} d\Omega_{2,s} \int_{\mathbb{R}^{2(s-1)}} dV_{2,s} \\
\times \Big(\prod_{i=2}^{s} (v_i - v_{a(i)}(t_i)) \cdot \nu_i \Big) f_0^{(s)}\big(\bar Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)\big)\,.
\end{aligned}
\tag{2.5}
$$

2.2. **Lanford's strategy.** The proof of Lanford relies then on two steps :

(i) proving a short time bound for the series (2.4) expressing the correlations of the system of N particles and a similar bound for the corresponding quantities associated with the Boltzmann hierarchy;

(ii) proving the termwise convergence of each term of the series, which actually consists in proving that the BBGKY and Boltzmann pseudo-trajectories $Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)$ and $\bar Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)$ stay close to each other, except for a set of parameters $(t_i, \nu_i, v_i)_{2\le i\le s}$ of vanishing measure.

Note that step (i) alone is responsible for the fact that the low density limit is only known to hold for short times (of the order of $1/\alpha$). This is due to the fact that the uniform bound is essentially obtained by replacing the hierarchy by equations of the type $\partial_t F = \alpha F^2$, neglecting all cancellations present in the nonlinear term.

More precisely, defining the operator associated to the series (2.1)

$$
\begin{aligned}
Q_{s,s+n}(t) := \alpha^n \int_0^t \int_0^{t_{s+1}} \dots \int_0^{t_{s+n-1}} \mathbf{S}_s(t - t_{s+1}) C_{s,s+1} \mathbf{S}_{s+1}(t_{s+1} - t_{s+2}) C_{s+1,s+2} \dots \\
\dots \mathbf{S}_{s+n}(t_{s+n})\, dt_{s+n} \dots dt_{s+1}
\end{aligned}
\tag{2.6}
$$

we overestimate all contributions by considering rather the operators $|Q_{s,s+n}|$ defined by

$$
\begin{aligned}
|Q_{s,s+n}|(t) := \alpha^n \int_0^t \int_0^{t_{s+1}} \dots \int_0^{t_{s+n-1}} \mathbf{S}_s(t - t_{s+1}) |C_{s,s+1}| \mathbf{S}_{s+1}(t_{s+1} - t_{s+2}) |C_{s+1,s+2}| \dots \\
\dots \mathbf{S}_{s+n}(t_{s+n})\, dt_{s+n} \dots dt_{s+1}
\end{aligned}
\tag{2.7}
$$

where $C_{s,s+1}$ in (1.10) is replaced by

$$|C_{s,s+1}|f_{s+1} := \sum_{i=1}^{s}(C^{i,+}_{s,s+1} + C^{i,-}_{s,s+1})|f_{s+1}|\,.$$

In the same way for the Boltzmann hierarchy, the iterated collision operator is denoted by

$$\bar Q_{s,s+n}(t) := \alpha^n \int_0^t \int_0^{t_{s+1}} \dots \int_0^{t_{s+n-1}} \mathbf{S}^0_s(t-t_{s+1})\bar C_{s,s+1}\mathbf{S}^0_{s+1}(t_{s+1}-t_{s+2})\bar C_{s+1,s+2}\dots \\ \dots \mathbf{S}^0_{s+n}(t_{s+n})\,dt_{s+n}\dots dt_{s+1}$$

which is bounded from above by

$$|\bar Q_{s,s+n}|(t) := \alpha^n \int_0^t \int_0^{t_{s+1}} \dots \int_0^{t_{s+n-1}} \mathbf{S}^0_s(t-t_{s+1})|\bar C_{s,s+1}|\mathbf{S}^0_{s+1}(t_{s+1}-t_{s+2})|\bar C_{s+1,s+2}|\dots \\ \dots \mathbf{S}_{s+n}(t_{s+n})\,dt_{s+n}\dots dt_{s+1}\,,$$

where $|\bar C_{s,s+1}|$ is defined as $|C_{s,s+1}|$ above.

Notation. From now on we shall denote by C a constant which may change from line to line, and which may depend on β, but not on N and α. We also write $A \lesssim B$ for $A \leq CB$ for some constant C, and $A \ll B$ for $A \leq CB$ if we further require that C is small enough. Finally we write B^s_R for the ball of $\mathbb{R}^{2s}$ of radius R, and $B_R = B^1_R$.

We have the following continuity estimates (see [7, 4]).

Proposition 2.4. *There is a constant C such that for all $s, n \in \mathbb{N}^*$ and all $h, t \geq 0$, the operator $|Q|$ satisfies the following continuity estimates: if g_s, g_{s+n} belong to $L^\infty(\mathbb{D}^s)$ and $L^\infty(\mathbb{D}^{s+n})$ respectively, then*

$$\forall z_1 \in \mathbb{D}\,,\quad \begin{aligned} &\big(|Q_{1,s}|(t)M_{s,\beta}g_s\big)(z_1) \leq (C\alpha t)^{s-1}M_{3\beta/4}(z_1)\|g_s\|_{L^\infty(\mathbb{D}^s)} \\ &\big(|Q_{1,s}|(t)\,|Q_{s,s+n}|(h)M_{s+n,\beta}g_{s+n}\big)(z_1) \leq (C\alpha)^{s+n-1}t^{s-1}h^n M_{3\beta/4}(z_1)\|g_{s+n}\|_{L^\infty(\mathbb{D}^{s+n})}\,. \end{aligned}$$

Similar estimates hold for $|\bar Q|$.

Sketch of proof. The estimate is simply obtained from the fact that the transport operators preserve the Gibbs measures, along with the continuity of the elementary collision operators :

- the transport operators satisfy the identities

$$\mathbf{S}_k(t)M_{k,\beta} = M_{k,\beta}$$

- the collision operators satisfy the following bounds in the Boltzmann-Grad scaling $N\varepsilon = \alpha$ (see [7])

$$|C_{k,k+1}|M_{k+1,\beta}(Z_k) \leq C\beta^{-1}\Big(k\beta^{-\frac12} + \sum_{1\leq i\leq k}|v_i|\Big)M_{k,\beta}(Z_k)$$

  almost everywhere on $\mathbb{R}_t \times \mathcal{D}^k_\varepsilon$.

Estimating the operator $|Q_{s,s+n}|(h)$ follows from piling together those inequalities (distributing the exponential weight evenly on each occurence of a collision term). We notice indeed that by the Cauchy-Schwarz inequality

$$\sum_{1\leq i\leq k}|v_i|\exp\Big(-\frac{\beta}{8n}|V_k|^2\Big) \leq \Big(k\frac{4n}{\beta}\Big)^{\frac12}\Bigg(\sum_{1\leq i\leq k}\frac{\beta}{4n}|v_i|^2\exp\Big(-\frac{\beta}{4n}|V_k|^2\Big)\Bigg)^{1/2} \\ \leq \Big(\frac{4nk}{e\beta}\Big)^{1/2} \leq \frac{2}{\sqrt{e\beta}}(s+n)\,, \tag{2.8}$$

where the last inequality comes from the fact that $k \leq s+n$. Each collision operator gives therefore a loss of $C\beta^{-3/2}(s+n)$ together with a loss on the exponential weight, while the integration with respect to time provides a factor $h^n/n!$. By Stirling's formula, we have

$$\frac{(s+n)^n}{n!} \leq \exp\left(n \log \frac{n+s}{n} + n\right) \leq \exp(s+n)\,.$$

As a consequence

$$|Q_{s,s+n}|(h) M_{s+n,\beta}(Z_s) \leq C^{s+n}\,(\alpha h)^n M_{s,3\beta/4}(Z_s)\,.$$

The proof of Proposition 2.4 follows from this upper bound. □

The iteration of the first estimate in Proposition 2.4 is the key to the local wellposedness of the hierarchy (see [7]) : we indeed prove that, if the initial data satisfies

$$f_{N,0}^{(s)} \leq \exp(\mu s) M_{s,\beta}$$

the series expansion (2.1) converges (uniformly in N) on a time such that $t\alpha \ll 1$.

2.3. **The pruning procedure introduced in** [4]**.** We recall now a strategy devised in [4] in order to control the growth of collision trees. The idea is to introduce some sampling in time with a (small) parameter $h > 0$. Let $\{n_k\}_{k\geq 1}$ be a sequence of integers, typically $n_k = 2^k$. We then study the dynamics up to time $t := Kh$ for some large integer K, by splitting the time interval $[0,t]$ into K intervals of size h, and controlling the number of collisions on each interval. In order to discard trajectories with a large number of collisions in the iterated Duhamel formula, we define collision trees "of controlled size" by the condition that they have strictly less than n_k branch points on the interval $[t-kh, t-(k-1)h]$. Note that by construction, the trees are actually followed "backwards", from time t (large) to time 0. So we decompose the iterated Duhamel formula (2.1), in the case $s=1$, by writing

$$\begin{aligned} f_N^{(1)}(t) := & \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{K-1},J_K}(h) f_{N,0}^{(J_K)} \\ & + \sum_{k=1}^{K} \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} \sum_{j_k \geq n_k} Q_{1,J_1}(h) \dots Q_{J_{k-2},J_{k-1}}(h)\, Q_{J_{k-1},J_k}(h) f_N^{(J_k)}(t-kh)\,, \end{aligned} \tag{2.9}$$

with $J_0 := 1$, $J_k := 1 + j_1 + \dots + j_k$. The first term on the right-hand side corresponds to the smallest trees, and the second term is the remainder: it represents trees with super exponential branching, i.e. having at least n_k collisions during the last time lapse, of size h. One proceeds in a similar way for the Boltzmann hierarchy (2.2).

The main argument of [4] consists in proving that the remainder is small, even for large t (but small h). This was achieved in [4] to derive the linear Boltzmann equation with initial data of the form (1.18). In that case, the maximum principle ensures that the L^∞ norm of the marginals are bounded at all times

$$\left|f_N^{(s)}(t, Z_s)\right| \leq C^s M_{N,\beta}^{(s)}(Z_s)\,. \tag{2.10}$$

Combining this uniform bound with the L^∞ estimate on the collision operator given in Proposition 2.4, one can gain smallness thanks to the factor h^{j_k} which controls the occurence of j_k collisions in the last time interval.

The conclusion of the proof in the linear case (see [4]) then comes from a comparison of the BBGKY and the Boltzmann pseudo-trajectories, through a geometric argument showing that recollisions are events with small probability (compared to the $O(1)$ norm of the data in L^∞), once K is fixed.

2.4. **A priori estimates.** One of the main differences here with [4] is that the initial data is no longer $O(1)$ in L^∞. We summarize below the estimates at our disposal for the initial data $f_{N,0}$ defined in (1.17) and the associate solution f_N to the Liouville equation (1.5), compared with [4].

L^∞-estimates. First, one has clearly

$$\big|f_{N,0}(Z_N)\big| \leq N\|g_{\alpha,0}\|_{L^\infty(\mathbb{D})}\, M_{N,\beta}(Z_N)\,. \tag{2.11}$$

From the maximum principle, we deduce from (2.11) that for all $t\in\mathbb{R}$,

$$\big|f_N(t,Z_N)\big| \leq N\|g_{\alpha,0}\|_{L^\infty(\mathbb{D})}\, M_{N,\beta}(Z_N)\,. \tag{2.12}$$

A classical result on the exclusion (see Lemma 6.1.2 in [7]) shows that

$$\forall 1\leq s\leq N, \qquad \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s} \leq C(1-C\alpha\varepsilon)^{-s} \leq C\exp(Cs\alpha\varepsilon)\,, \tag{2.13}$$

so from (2.12), the marginals satisfy

$$\big|f_N^{(s)}(t,Z_s)\big| \leq NM_{N,\beta}^{(s)}(Z_s)\;\|g_{\alpha,0}\|_{L^\infty(\mathbb{D})} \leq NC^s\exp(Cs\alpha\varepsilon)M_\beta^{\otimes s}(Z_s)\;\|g_{\alpha,0}\|_{L^\infty(\mathbb{D})}\,. \tag{2.14}$$

This should be compared with the counterpart in the linear case, given in (2.10) : there is a factor N difference between the two estimates.

Much better estimates can be obtained at the initial time by using the explicit structure of the measure $f_{N,0}$ defined by (1.17). In particular the discrepancy between the marginals $f_{N,0}^{(s)}$ and $f_0^{(s)}$ can be evaluated.

Proposition 2.5. *There exists $C>1$ such that as $N\to\infty$ in the scaling $N\varepsilon=\alpha\ll 1/\varepsilon$*

$$\forall s\leq N, \qquad \Big|\Big(f_{N,0}^{(s)}-f_0^{(s)}\Big)(Z_s)\mathbf{1}_{\mathcal{D}_\varepsilon^s}(X_s)\Big| \leq C^s\alpha^3\varepsilon M_\beta^{\otimes s}(V_s)\|g_{\alpha,0}\|_{L^\infty}\,.$$

As a consequence, the initial data are bounded by

$$\forall s\leq N, \qquad \big|f_{N,0}^{(s)}(Z_s)\big| \leq C^s\alpha^3 M_\beta^{\otimes s}(V_s)\|g_{\alpha,0}\|_{L^\infty}\,. \tag{2.15}$$

The proof of this Proposition can be found in Appendix D. A similar statement was derived in [3]. Note that contrary to estimate (2.10) in the linear case, we are unable to propagate the initial estimate (2.15) in time and to improve (2.14).

L^2-estimates. In our setting the L^2_β-norm (defined in (1.15)) is better behaved than the L^∞ norm. One of the specificities of dimension 2 is the fact that the normalizing factor $\mathcal{Z}_N^{-1}$ is uniformly bounded in N. From (2.13), we indeed deduce that under the Boltzmann-Grad scaling $N\varepsilon=\alpha$, one has

$$\mathcal{Z}_N^{-1}\leq C\exp(C\alpha^2)\,. \tag{2.16}$$

This upper bound and the definition of $f_{N,0}$ in (1.17) lead to

$$\begin{aligned}\int\frac{f_{N,0}^2}{M_{N,\beta}}(Z_N)dZ_N &\leq C\exp(C\alpha^2)\int M_\beta^{\otimes N}(Z_N)\left(\sum_{i=1}^N g_{\alpha,0}(z_i)\right)^2 dZ_N\\ &\leq CN\exp(C\alpha^2)\|g_{\alpha,0}\|^2_{L^2_\beta(\mathbb{D})}\,,\end{aligned} \tag{2.17}$$

where we used in the last inequality that $g_{\alpha,0}$ is mean free with respect to the measure $M_\beta dz$ due to (1.17). The weighted L^2 norm is therefore $O(\sqrt{N})$. Since the Liouville equation is conservative, we obtain from (2.17) that

$$\int\frac{f_N^2}{M_{N,\beta}}(t,Z_N)dZ_N\leq CN\exp(C\alpha^2)\|g_{\alpha,0}\|^2_{L^2_\beta(\mathbb{D})}\,. \tag{2.18}$$

The L^2 bound (2.18) is in some sense more accurate than (2.12) since it comes from the orthogonality at time 0 inherited from the structure of the initial data. In particular, if the function $f_N(t, Z_N)$ was of the same form as the initial data for all times, meaning if

$$f_N(t, Z_N) = M_{N,\beta}(Z_N) \sum_{i=1}^{N} g_\alpha(t, z_i) \text{ with } \int M_\beta g_\alpha(t, z) dz = 0 \,, \tag{2.19}$$

we would deduce a similar L^2 estimate on $f_N^{(s)}(t)$. Unfortunately this structure is not preserved by the flow. However one inherits a trace of this structure, as will be shown in Proposition 4.2.

2.5. **Estimate of the collision operators in L^2.** Proving an analogue of Proposition 2.4 in an L^2 setting is not an easy task, since one cannot compute the trace of an L^2 function on a hypersurface. However (and that is actually the way to get around a similar difficulty in L^∞, see [7]) composing the collision integral with free transport and integrating over time is a way to replace the integral over the unit sphere by an integral over a volume using a change of variables of the type

$$(Z_s, \nu_{s+1}, v_{s+1}, t) \mapsto Z_{s+1} = (Z_s - V_s t, x_s + \varepsilon\nu - v_{s+1} t, v_{s+1}) \tag{2.20}$$

(with scattering if need be). Using this idea one can hope to prove some kind of continuity estimate of $Q_{s,s+n}$ in L^2, but two additional difficulties arise:

(1) the transport operators appearing in $Q_{s,s+n}$ are not free transport operators since recollisions are possible, so the change of variables (2.20) cannot be used directly. If there is a fixed number of recollisions then one can still use a similar argument but if there is no control on the number of collisions then this method fails.
(2) Computing an L^∞ bound on the collision operator $C_{s,s+1}$ gives rise to the size of the sphere, hence ε, which compensates exactly (up to a factor α) the factor $(N - s)$; but in L^2 one only can recover $\varepsilon^{\frac{1}{2}}$, so there remains a factor $N^{\frac{1}{2}}$. Typically one can expect in general an estimate of the type

$$\big\| |Q_{1,s}|(t) g_s \big\|_{L^2_\beta} \leq (C\alpha t)^{s-1} \|g_s\|_{L^2_\beta} N^{\frac{s-1}{2}}$$

so this power of N will need to be compensated (see Section 4).

2.6. **Decomposition of the BBGKY solution and organization of the paper.** We start from decomposition (2.9) but in the remainder we need to analyze differently the trajectories with more or less than 1 recollision. This is due to the fact that as explained in Paragraph 2.5 (Point (1)), the control in L^2_β of the collision operators $Q_{s,s+n}$ requires a precise control on the number of recollisions.

Our strategy consists in adapting (2.9) in two ways: first we truncate energies defining, for some constant C_0 to be specified later in Proposition 7.1,

$$\forall s \geq 1, \qquad \mathcal{V}_s := \left\{ V_s \in \mathbb{R}^{2s} \;\middle|\; |V_s|^2 \leq C_0 |\log \varepsilon| \right\} . \tag{2.21}$$

Second we decompose

$$f_N^{(1)}(t) = f_N^{(1,K)}(t) + R_N^K(t) \tag{2.22}$$

with

$$f_N^{(1,K)}(t) := \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{K-1},J_K}(h) \left(f_{N,0}^{(J_K)} \mathbf{1}_{\mathcal{V}_{J_K}} \right),$$

with $n_k = 2^k n_0$ for some n_0 to be specified, and where $J_0 := 1$, $J_k := 1 + j_1 + \cdots + j_k$. The decomposition above is reminiscent of (2.9), except that the velocities have been truncated in the dominant term $f_N^{(1,K)}$.

We then split the remainder in three parts according to the number of recollisions in the pseudo-trajectories (see Definition 2.3) and a fourth part to take into account large velocities

$$R_N^K(t) = R_N^{K,0}(t) + R_N^{K,1}(t) + R_N^{K,>}(t) + R_N^{K,vel}(t)\,. \tag{2.23}$$

• We first introduce a truncated transport operator up to the first collision. Let us rewrite Liouville's equation (1.5) for s particles with a different boundary condition

$$\partial_t \varphi_s + V_s \cdot \nabla_{X_s} \varphi_s = 0 \qquad \text{with} \quad \varphi_s(t, Z_s) = 0 \quad \text{for} \quad Z_s \in \bigcup_{i,j \leq s} \partial \mathcal{D}_\varepsilon^{s+}(i,j)\,.$$

The corresponding semi-group is denoted by $\widehat{\mathbf{S}}_s^0$ and it coincides with the free flow $\mathbf{S}_s^0$ up to the first recollision

$$\left(\widehat{\mathbf{S}}_s^0(\tau)\varphi_s\right)(Z_s) = \begin{cases} \left(\mathbf{S}_s^0(\tau)\varphi_s\right)(Z_s) & \text{if no recollision occurs in } [0,\tau]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We define the operator $Q^0_{s,s+n}(t)$ by replacing $\mathbf{S}_s$ by $\widehat{\mathbf{S}}_s^0$ in the iterated collision operator $Q_{s,s+n}(t)$ given in (2.6)

$$\begin{aligned} Q^0_{s,s+n}(t) := \alpha^n \int_0^t \int_0^{t_{s+1}} \dots \int_0^{t_{s+n-1}} \widehat{\mathbf{S}}_s^0(t - t_{s+1}) C_{s,s+1} \widehat{\mathbf{S}}_{s+1}^0(t_{s+1} - t_{s+2}) C_{s+1,s+2} \dots \\ \dots \widehat{\mathbf{S}}_{s+n}^0(t_{s+n})\, dt_{s+n} \dots dt_{s+1}\,. \end{aligned} \tag{2.24}$$

With this definition, we set

$$R_N^{K,0}(t) := \sum_{k=1}^K \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} \sum_{j_k \geq n_k} Q^0_{1,J_1}(h) \dots Q^0_{J_{k-1},J_k}(h) \big(f_N^{(J_k)}(t - kh) \mathbf{1}_{\mathcal{V}_{J_k}} \big)\,. \tag{2.25}$$

• In a similar way, we define pseudo-dynamics involving exactly one recollision.

$$\left(\widehat{\mathbf{S}}_s^1(\tau)\varphi_s\right)(Z_s) = \begin{cases} \left(\mathbf{S}_s(\tau)\varphi_s\right)(Z_s) & \text{if exactly one recollision occurs in } [0,\tau]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Note that, contrary to $\widehat{\mathbf{S}}_s^0(\tau)$, the operator $\widehat{\mathbf{S}}_s^1(\tau)$ is not a semi-group, as the dynamics keeps memory of past events. In particular, there is no infinitesimal generator.

We then define the operator $Q^1_{s,s+n}(t)$ by replacing $\mathbf{S}_s$ by $\widehat{\mathbf{S}}_s^0$ in the iterated collision operator $Q_{s,s+n}(t)$, except for one iteration

$$\begin{aligned} Q^1_{s,s+n}(t) := \alpha^n \sum_{j=0}^n \int_0^t \int_0^{t_{s+1}} \dots \int_0^{t_{s+n-1}} \widehat{\mathbf{S}}_s^0(t - t_{s+1}) C_{s,s+1} \widehat{\mathbf{S}}_{s+1}^0(t_{s+1} - t_{s+2}) C_{s+1,s+2} \dots \\ \dots C_{s+j-1,s+j} \widehat{\mathbf{S}}_{s+j}^1(t_{s+j} - t_{s+j-1}) \dots \widehat{\mathbf{S}}_{s+n}^0(t_{s+n})\, dt_{s+n} \dots dt_{s+1}\,. \end{aligned}$$

With this definition, we set

$$\begin{aligned} R_N^{K,1}(t) := \sum_{k=1}^K \sum_{\ell=1}^k \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} \sum_{j_k \geq n_k} Q^0_{1,J_1}(h) \dots Q^1_{J_{\ell-1},J_\ell}(h) \\ \dots Q^0_{J_{k-1},J_k}(h) \big(f_N^{(J_k)}(t - kh) \mathbf{1}_{\mathcal{V}_{J_k}} \big)\,. \end{aligned} \tag{2.26}$$

- The contribution of large velocities, i.e. those which are not in $\mathcal{V}_{J_K}$, is

$$
\begin{aligned}
R_N^{K,vel}(t) := &\sum_{j_1=0}^{n_1-1}\dots\sum_{j_K=0}^{n_K-1} Q_{1,J_1}(h)Q_{J_1,J_2}(h)\dots Q_{J_{K-1},J_K}(h)\,\Big(f_{N,0}^{(J_K)}\mathbf{1}_{\mathcal{V}^c_{J_K}}\Big)\\
&+\sum_{k=1}^{K}\sum_{j_1=0}^{n_1-1}\dots\sum_{j_{k-1}=0}^{n_{k-1}-1}\sum_{j_k\geq n_k} Q_{1,J_1}(h)\dots Q_{J_{k-1},J_k}(h)\,\Big(f_N^{(J_k)}(t-kh)\mathbf{1}_{\mathcal{V}^c_{J_k}}\Big)\,.
\end{aligned}
\tag{2.27}
$$

- We finally define

$$
R_N^{K,>}(t) := R_N^K(t) - R_N^{K,0}(t) - R_N^{K,1}(t) - R_N^{K,vel}(t)\,,
\tag{2.28}
$$

which by definition corresponds to pseudo-dynamics involving at least two recollisions, with truncated velocities.

Using the notation (2.8), the counterpart of $f_N^{(1,K)}(t)$ for the Boltzmann hierarchy is

$$
\bar f_N^{(1,K)}(t) := \sum_{j_1=0}^{n_1-1}\dots\sum_{j_K=0}^{n_K-1}\bar Q_{1,J_1}(h)\bar Q_{J_1,J_2}(h)\dots\bar Q_{J_{K-1},J_K}(h)\,\big(f_0^{(J_K)}\mathbf{1}_{\mathcal{V}_{J_K}}\big)\,.
$$

and we define also

$$
\bar R_N^K(t) = \sum_{k=1}^{K}\sum_{j_1=0}^{n_1-1}\dots\sum_{j_{k-1}=0}^{n_{k-1}-1}\sum_{j_k\geq n_k}\bar Q_{1,J_1}(h)\dots\bar Q_{J_{k-1},J_k}(h)\big(f^{(J_k)}(t-kh)\mathbf{1}_{\mathcal{V}_{J_k}}\big)
$$

and

$$
\begin{aligned}
\bar R_N^{K,vel}(t) := &\sum_{j_1=0}^{n_1-1}\dots\sum_{j_K=0}^{n_K-1}\bar Q_{1,J_1}(h)\,\bar Q_{J_1,J_2}(h)\dots\bar Q_{J_{K-1},J_K}(h)\,\Big(f_0^{(J_K)}\mathbf{1}_{\mathcal{V}^c_{J_K}}\Big)\\
&+\sum_{k=1}^{K}\sum_{j_1=0}^{n_1-1}\dots\sum_{j_{k-1}=0}^{n_{k-1}-1}\sum_{j_k\geq n_k}\bar Q_{1,J_1}(h)\dots\bar Q_{J_{k-1},J_k}(h)\,\Big(f^{(J_k)}(t-kh)\mathbf{1}_{\mathcal{V}^c_{J_k}}\Big)\,.
\end{aligned}
$$

Section 3 deals with the convergence of the main part $f_N^{(1,K)}(t)$ defined in (2.9). Since the initial data is well behaved (see Proposition 2.5), the derivation of this convergence essentially follows the same lines as in [4]. In the proof of Proposition 3.1, we shall however improve the estimates of [4] on the measure of trajectories having at least one recollision, as they will be the first step to control multiple recollisions.

Section 4 is the main breakthrough of this paper, as it shows how exchangeability combined with the L^2 estimate provides a very weak chaos property (see Proposition 4.2). We then explain, in Proposition 4.4, how to use this structure to compensate the expected loss explained in Paragraph 2.5 (Point (2)), and to obtain an estimate on $R_N^{K,0}$, corresponding to pseudo-trajectories with super exponential branching but without recollision. This L^2 continuity estimate uses crucially the integration with respect to time of the free transport (see Paragraph 2.5, Point (1)). Section 5 is a refinement of this argument to estimate the remainder $R_N^{K,1}$ when there is one recollision. In fact, the same argument holds with any finite number of recollisions.

Section 6 deals with $R_N^{K>}$, which corresponds to multiple recollisions (Proposition 6.1). In this case, the extra smallness coming from the geometric control of multiple recollisions compensates exactly the $O(N)$ divergence of the L^∞-bound (2.12). The proof relies on delicate geometric estimates which are detailed in Appendix B. This allows to control the remainder $R_N^{K>}$ by using L^∞ estimates from Proposition 2.4. Note that the critical number of recollisions depends on the dimension, it is 1 only in the simple case of dimension $d = 2$. The L^∞-bound (2.12) is also used in Section 7 to control $R_N^{K,vel}$, i.e. the large velocities.

Finally, we conclude the proof in Section 8 and state some open problems.

3. Convergence of the principal parts

We recall that the principal part of the iterated Duhamel formula (2.1) for the first marginal is given by (2.9)

$$f_N^{(1,K)}(t) := \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{K-1},J_K}(h) \left(f_{N,0}^{(J_K)} \mathbf{1}_{\mathcal{V}_{J_K}} \right).$$

and its counterpart of $f_N^{(1,K)}(t)$ for the Boltzmann hierarchy is

$$\bar f_N^{(1,K)}(t) := \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \bar Q_{1,J_1}(h) \bar Q_{J_1,J_2}(h) \dots \bar Q_{J_{K-1},J_K}(h) \left(f_0^{(J_K)} \mathbf{1}_{\mathcal{V}_{J_K}} \right).$$

From now on, the exponential growth of the collision trees will be controlled by the sequence

$$n_k := 2^k n_0 ,$$

for some large integer n_0 to be tuned later.

The error $f_N^{(1,K)} - \bar f_N^{(1,K)}$ can be estimated as follows.

Proposition 3.1. *Assume that $g_{\alpha,0}$ satisfies the Lipschitz bound* (1.19) *then, under the Boltzmann-Grad scaling $N\varepsilon = \alpha \gg 1$, we have for all $T > 1$ and $t \in [0,T]$,*

$$\left\| f_N^{(1,K)}(t) - \bar f_N^{(1,K)}(t) \right\|_{L^2(\mathbb{D})} \leq \exp(C\alpha^2)(C\alpha T)^{2^{K+1} n_0} \left(\varepsilon |\log \varepsilon|^{10} + \frac{\varepsilon}{\alpha} \right). \tag{3.1}$$

The key step of the proof is Proposition 3.2 where the contribution of recollisions in the pseudo-trajectories associated with $f_N^{(1,K)}$ are shown to be negligible. Once the recollisions have been neglected, the pseudo-trajectories in both hierarchies are comparable and the rest of the proof is rather straightforward (see Section 3.2).

In the rest of this section, we assume that $g_{\alpha,0}$ satisfies the Lipschitz bound (1.19).

3.1. Geometric control of recollisions.

We are going to prove that dynamics involving recollisions contribute very little to $f_N^{(1,K)}$ so that $\mathbf{S}_s$ can be replaced by the free transport $\widehat{\mathbf{S}}_s^0$, up to a small error. With the notation (2.24), $f_N^{(1,K)}$ can be decomposed as follows:

$$f_N^{(1,K)} = f_N^{(1,K),0}(t) + f_N^{(1,K),\geq}(t)$$

with

$$f_N^{(1,K),0}(t) := \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} Q^0_{1,J_1}(h) Q^0_{J_1,J_2}(h) \dots Q^0_{J_{K-1},J_K}(h) \left(f_{N,0}^{(J_K)} \mathbf{1}_{\mathcal{V}_{J_K}} \right) \tag{3.2}$$

and the remainder encodes the occurence of at least one recollision

$$f_N^{(1,K),\geq}(t) := f_N^{(1,K)}(t) - f_N^{(1,K),0}(t). \tag{3.3}$$

Proposition 3.2. *The contribution of (at least) a recollision is bounded by*

$$\forall t \in [0,T], \quad |f_N^{(1,K),\geq}(t, z_1)| \leq \exp(C\alpha^2) \big(CT\alpha \big)^{2^{K+1} n_0} \varepsilon \big| \log \varepsilon \big|^{10} M_{5\beta/8}(v_1).$$

The core of the proof is based on a careful analysis of recollisions detailed in Section 3.1.1 below. The proof of Proposition 3.2 is completed in Section 3.1.2. Thanks to the energy cut-off $\mathcal{V}_{J_K}$, we assume, in the rest of this section, that all energies are bounded by $C_0 |\log \varepsilon|$.

3.1.1. *A local condition for a recollision.* We start by writing a geometric condition for a recollision which involves only two collision integrals: this corresponds to writing a local condition, which will then be incorporated to the other collision integral estimates in Section 3.1.2. The following notions of *pseudo-particle* and *parents* will be useful. These notions are depicted in Figures 1 and 2.

Definition 3.3 (Pseudo-particles). *Given a tree $a \in \mathcal{A}_s$ and $i \leq s$, we define recursively, moving towards the root, the pseudo-particle $\bar{\imath}$ associated with the particle i to be*

- *$\bar{\imath} = i$ as long as i exists,*
- *$\bar{\imath} = a(i)$ when i disappears, and as long $a(i)$ exists,*
- *$\bar{\imath} = a\big(a(i)\big)$ when $a(i)$ disappears, and as long as this latter exists,*
- *...*

When there is no possible confusion, we shall denote abusively i the pseudo-particle.

Note that we disregard times t_k at which the pseudo-particles encounter a new particle k with no scattering. Contrary to the case of a true particle, whose trajectory stops at its creation time, the trajectory of a pseudo-particle exists for all times. At each collision time the pseudo-particle is liable to be deviated through a scattering operator, and may jump of a distance ε in space (see Figure 1).

Each collision leading to the deviation of a pseudo-particle brings a new degree of freedom which will be essential to control the trajectories later on. This degree of freedom is associated with a new particle which we call *parent*.

Definition 3.4 (Parent). *Given a collision tree $a \in \mathcal{A}_s$ and a height in this tree, we consider a subset $\mathcal{I}$ of particles at that height. We define $(n^*)_{n\in\mathbb{N}}$ the sequence of branching points in a at which one of the pseudo-particles associated with the particles in $\mathcal{I}$ is deviated. The family $1^*, 2^*, \dots$ of particles created in these collisions are the parents of the set $\mathcal{I}$. Note that the particles $1^*, 2^*, \dots$ may coincide with the pseudo-particles (see Figure 2).*

A recollision between i and j imposes some strong constraints on the history of these particles, especially on the last two collisions at times t_{1^*} and t_{2^*} with the particles 1^* and 2^* which are the first parents of i, j (see Figure 4(i)): we prove the smallness of the collision integral associated with particle 1^* (with the measure $|(v_{1^*} - v_{a(1^*)}(t_{1^*})) \cdot \nu_{1^*}|dt_{1^*}d\nu_{1^*}dv_{1^*}$), with a singularity at small relative velocities which can be integrated out using the collision integral with respect to particle 2^*. The final result is the following.

Proposition 3.5. *Given $z_1 \in \mathbb{T}^2 \times B_R$ with $1 \leq R^2 \leq C_0|\log\varepsilon|$ and a collision tree $a \in \mathcal{A}_s$ with $s \geq 2$, consider the set of parameters $(t_n, \nu_n, v_n)_{2\leq n\leq s}$ in $\mathcal{T}_{2,s} \times \mathbb{S}^{s-1} \times \mathbb{R}^{2(s-1)}$ leading to a pseudo-trajectory with total energy bounded by R^2. Fix i, j two labels and $\max(i,j) \leq \theta \leq s$.*

We consider the pseudo-trajectories with at least one recollision such that the first recollision takes place between particles i and j during the time interval $[t_\theta, t_{\theta+1}]$. Let $1^*, 2^*$ be the indices in $\{2, \dots, s\}$ of the first two parents of the set $\{i, j\}$ starting at height $\theta+1$. For $t \geq 1$, with $R^2 + t \leq C|\log\varepsilon|$, the measure associated with these pseudo-trajectories is bounded by

$$\int \mathbf{1}_{\textit{first recollision between } (i,j) \textit{ at height } \theta} \prod_{m=1}^{2} \left|\big(v_{m^*} - v_{a(m^*)}(t_{m^*})\big) \cdot \nu_{m^*}\big)\right| dt_{m^*}d\nu_{m^*}dv_{m^*} \tag{3.4}$$
$$\leq CR^7t^3\varepsilon\,|\log\varepsilon|^3\,,$$

uniformly with respect to all other parameters $(t_n, \nu_n, v_n)_{\substack{2\leq n\leq s\\ n\neq 1^*,2^*}}$ in $\mathcal{T}_{2,s} \times \mathbb{S}^{s-1} \times B_R^{s-1}$ and z_1 in $\mathbb{T}^2 \times B_R$.

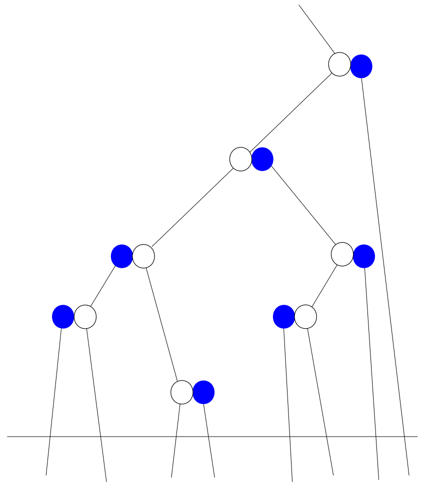

FIGURE 1. A collision tree is depicted with the trajectory of the pseudo-particle $\bar{\imath}$ in black. The pseudo-particle $\bar{\imath}$ coincides with i up to the creation time of i, it then coincides with $a(i)$ and so on. Each change of label induces a shift by ε of the pseudo-particle $\bar{\imath}$.

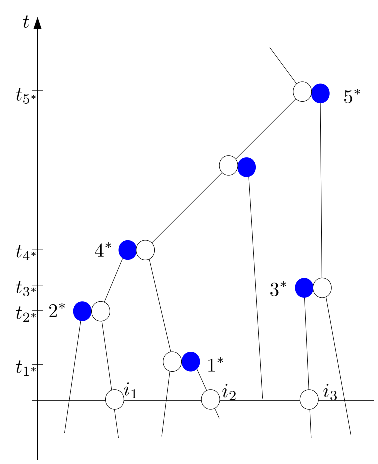

FIGURE 2. The set $\mathcal{I}$ consists in $\{i_1, i_2, i_3\}$. The parents are $1^*, \dots, 5^* \dots$ Note that between times t_{4^*} and t_{5^*} a particle has been created but with no scattering so it is not a parent.

Proof. We focus on the first recollision, which involves particles i and j by assumption.

Self-recollision. If the collision at time t_{1^*} involves i and j, a recollision may occur due to the periodicity (see Figure 3).

This has a very small cost, we indeed have for some recollision time $t_{rec} \geq 0$ and ν_{rec} in $\mathbb{S}$

$$\varepsilon \nu_{1^*} + (v'_i - v'_j)(t_{rec} - t_{1^*}) = \varepsilon \nu_{rec} + q \text{ for some } q \in \mathbb{Z}^2 \setminus \{0\} \tag{3.5}$$

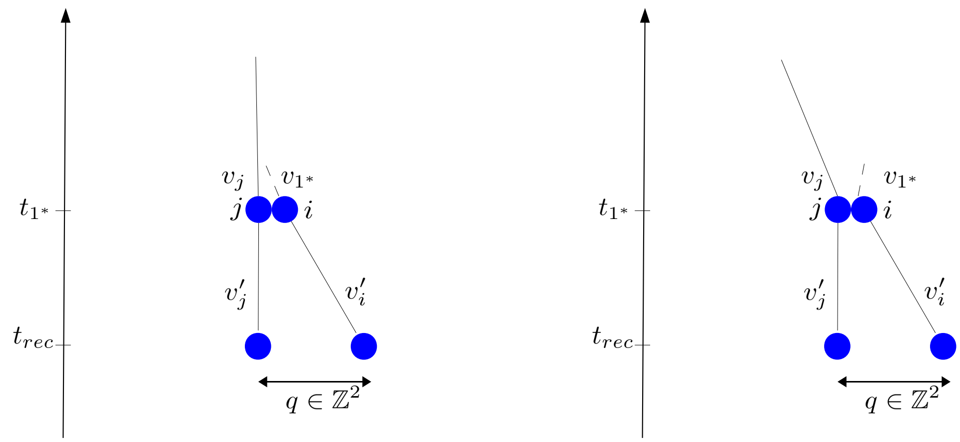

FIGURE 3. The case when a recollision is due to periodicity; on the left the collision at time t_{1^*} is without scattering, on the right it is with scattering.

assuming for instance that particle j has been created at time t_{1^*} with velocity v_{1^*}, and denoting by v'_i, v'_j the velocities after the collision.

- In the absence of scattering at time t_{1^*}, we have $v'_i = v_i$ and $v'_j = v_{1^*}$, and the equation (3.5) for self recollision implies that v_{1^*} has to belong to a cone of opening ε. Because of the assumption that the total energy is bounded by R^2,

$$\int \mathbf{1}_{(3.5)\text{ has a precollisional solution for a fixed } q} \left|\big(v_{1^*} - v_{a(1^*)}(t_{1^*})\big) \cdot \nu_{1^*}\big)\right| dt_{1^*} d\nu_{1^*} dv_{1^*} \leq C\varepsilon R^3 t\,,$$

where $a(1^*) = i$.

- In the case with scattering, recall that

$$v'_i - v'_j = (v_i - v_{1^*}) - 2(v_i - v_{1^*}) \cdot \nu_{1^*} \nu_{1^*}\,.$$

Equation (3.5) for the self recollision implies that $v'_i - v'_j$ has to belong to a cone of opening ε. For each fixed ν_{1^*}, we conclude that $v_i - v_{1^*}$ is also in a cone of opening ε. Because of the assumption that the total energy is bounded by R^2, we have as in the previous case

$$\int \mathbf{1}_{(3.5)\text{ has a postcollisional solution for a fixed } q} \left|\big(v_{1^*} - v_{a(1^*)}(t_{1^*})\big) \cdot \nu_{1^*}\big)\right| dt_{1^*} d\nu_{1^*} dv_{1^*} \leq C\varepsilon R^3 t.$$

Note that, since the total energy is assumed to be bounded by R^2 and we consider a finite time interval $[0, t]$ with $t \geq 1$, the number of q's for which the set is not empty is at most $O\big(R^2 t^2\big)$. Summing over all contributions, we end up with

$$\int \mathbf{1}_{(3.5)\text{ has a solution for some } q} \left|\big(v_{1^*} - v_{a(1^*)}(t_{1^*})\big) \cdot \nu_{1^*}\big)\right| dt_{1^*} d\nu_{1^*} dv_{1^*} \leq C\varepsilon R^5 t^3\,. \tag{3.6}$$

The collision with 2^* is not necessary to estimate the cost of a self-recollision (and the bound is better than expected). This completes (3.4) for self-recollisions.

Remark 3.6. *Notice that if the self-recollision takes place between the two first particles at play before any other collision, then there is actually no such parameter 2^*, but this is a very particular situation which we choose not to incorporate in the statement of the proposition — one could assume in this case that $1^* = 2^*$. From now on we shall always assume that there are enough degrees of freedom as needed for the computations, since if that is not the case the result will follow simply by integrating over less variables.*

Geometry of the first recollision. Without loss of generality, we may now assume that time t_{1^*} corresponds to the deviation/creation of the pseudo-particle i and that at t_{1^*} the collision does not involve both i and j. From now on, we denote by i and j the pseudo-particles, even if the actual particles may have disappeared through a collision (see Definition 3.3).

Denote by z_i and z_j the (pre-collisional) configuration of pseudo-particles i and j at time t_{2^*}.

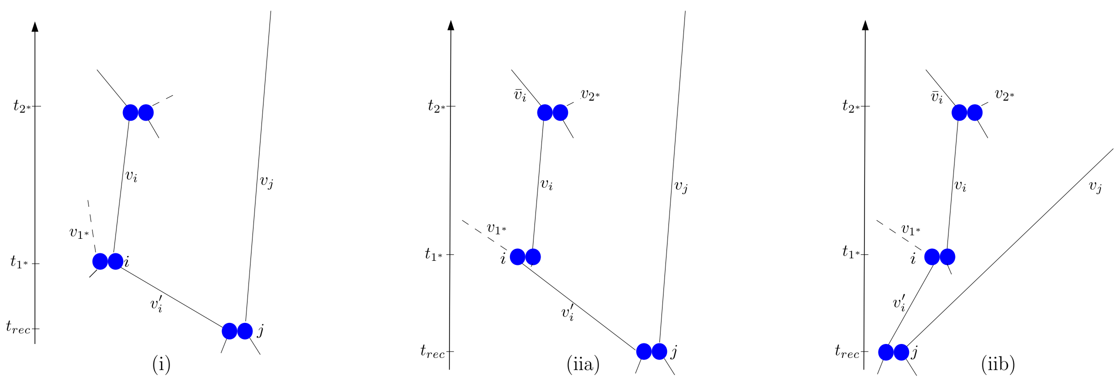

FIGURE 4. The two collisions at times t_{1^*} and t_{2^*} leading to the recollision between the pseudo particles i and j are depicted. Three different cases can occur if the first collision involves i : the particle i can be deflected (i), or created without scattering (iia) or with scattering (iib). These three cases can also occur for the recollision at 2^* but only one is depicted each time.

• In the case when the particle i already exists before t_{1^*} (as depicted in Figure 4(i)), the velocity of particle i after t_{1^*} (in the backward dynamics) is

$$v'_i = v_i - \big((v_i - v_{1^*}) \cdot \nu_{1^*}\big)\, \nu_{1^*} \,. \tag{i}$$

The condition for the recollision to hold in the backward dynamics at a time $t_{rec} \geq 0$ then states

$$(x_i - x_j) + (t_{1^*} - t_{2^*})(v_i - v_j) + (t_{rec} - t_{1^*})(v'_i - v_j) = \varepsilon \nu_{rec} + q \,, \tag{3.7}$$

for some $\nu_{rec} \in \mathbb{S}$, and $q \in \mathbb{Z}^2$.

• In the case when the particle i was created at t_{1^*}, we get

$$v'_i = v_{1^*} \tag{iia}$$

if $(v_{1^*}, \nu_{1^*}, v_i)$ is a precollisional configuration as on Figure 4(iia), and

$$v'_i = v_{1^*} + \big((v_i - v_{1^*}) \cdot \nu_{1^*}\big)\, \nu_{1^*} \,, \tag{iib}$$

if $(v_{1^*}, \nu_{1^*}, v_i)$ is a post-collisional configuration as on Figure 4(iib). The condition for the recollision states

$$(x_i - x_j + \varepsilon\nu_{1^*}) + (t_{1^*} - t_{2^*})(v_i - v_j) + (t_{rec} - t_{1^*})(v_i' - v_j) = \varepsilon\nu_{rec} + q\,,$$

for some $t_{rec} \geq 0$, $\nu_{rec} \in \mathbb{S}$ and $q \in \mathbb{Z}^2$.

As noticed previously, since the total energy is assumed to be bounded by R^2 and we consider a finite time interval $[0, t]$ with $t \geq 1$, the number of q's for which the set is not empty is at most $O\big(R^2 t^2\big)$. Let us now fix q and prove that the corresponding domain in $(t_{1^*}, v_{1^*}, \nu_{1^*})$ is small.

We denote

$$\delta x := \frac{1}{\varepsilon}(x_i - x_j - q) \quad \text{in case (i), and} \quad \delta x := \frac{1}{\varepsilon}(x_i - x_j - q) + \nu_{1^*} \quad \text{in case (ii)}\,.$$

Next we decompose δx into a component along $v_i - v_j$ and an orthogonal component, by writing

$$\delta x = \frac{\lambda}{\varepsilon}(v_i - v_j) + \delta x_\perp \quad \text{with} \quad \delta x_\perp \cdot (v_i - v_j) = 0$$

and we further rescale time as

$$\tau_1 := -\frac{1}{\varepsilon}(t_{1^*} - t_{2^*} + \lambda)\,, \qquad \tau_{rec} := -\frac{1}{\varepsilon}(t_{rec} - t_{1^*})\,. \tag{3.8}$$

Note that we have used the hyperbolic scaling invariance (by scaling the space and time variables by ε), and that only the bounds on τ_1 depend now on ε

$$|v_i - v_j|\,|\tau_1| \leq \frac{1}{\varepsilon}|v_i - v_j|t + |\delta x| \leq \frac{2Rt}{\varepsilon}\,\cdot$$

We shall gain a factor ε on the integral in time, thanks to the change of variable $t_{1^*} \mapsto \tau_1$.

In these new variables, the equation for the recollision can be restated as follows

$$v_i' - v_j = \frac{1}{\tau_{rec}}\delta x_\perp - \frac{\tau_1}{\tau_{rec}}(v_i - v_j) - \frac{1}{\tau_{rec}}\nu_{rec}\,. \tag{3.9}$$

The following lemma quantifies the size of the set of solutions to this recollision equation.

Lemma 3.7. *Fix $t \geq 1$, $\delta x_\perp \in \mathbb{R}^2$, $v_i, v_j \in B_R$ with $R \geq 1$, and $R^2 + t \ll |\log \varepsilon|$. Then*

$$\int_{B_R\times\mathbb{S}\times[-Ct/\varepsilon, Ct/\varepsilon]} \mathbf{1}_{(3.9)\text{ has a solution}} \big|(v_{1^*} - v_i)\cdot\nu_{1^*}\big| d\tau_1 d\nu_{1^*} dv_{1^*} \leq \frac{CR^3(\log\varepsilon)^2}{|v_i - v_j|}\,\cdot$$

We postpone the proof of Lemma 3.7 and complete first the proof of Proposition 3.5. In Lemma 3.7, the measure of the set leading to a recollision is evaluated in terms of the variable $|v_i - v_j|\tau_1$. Going back to the variables $(v_{1^*}, \nu_{1^*}, t_{1^*})$ and summing over all possible q, we therefore obtain

$$\int \mathbf{1}_{\big\{(3.9)\text{ has a solution for some } q\big\}} \big|(v_{1^*} - v_i)\cdot\nu_{1^*}\big| dt_{1^*} dv_{1^*} d\nu_{1^*} \leq CR^5 t^2 \frac{\varepsilon|\log\varepsilon|^2}{|v_i - v_j|}\,\cdot \tag{3.10}$$

On the other hand, a direct computation shows that

$$\int \big|(v_{1^*} - v_i)\cdot\nu_{1^*}\big| dt_{1^*} dv_{1^*} d\nu_{1^*} \leq CR^3 t\,,$$

so using the fact that $R \geq 1$, $t \geq 1$, we find

$$\begin{aligned}\int \mathbf{1}_{\big\{(3.9)\text{ has a solution for some } q\big\}} &\big|(v_{1^*} - v_i)\cdot\nu_{1^*}\big| dt_{1^*} dv_{1^*} d\nu_{1^*} \\ &\leq CR^5 t^2 \min\Big(\frac{\varepsilon|\log\varepsilon|^2}{|v_i - v_j|}\,, 1\Big)\,\cdot\end{aligned} \tag{3.11}$$

Now we need to integrate out the singularity $1/|v_i - v_j|$, when the parameters of the preceding collision $(t_{2^*}, v_{2^*}, \nu_{2^*})$ range over $[0,t] \times B_R \times \mathbb{S}$. From (C.1), we know that the singularity $1/|v_i - v_j|$ is integrable if particles i, j are related through the same collision. Otherwise the inequality (C.4), from Lemma C.2 page 74, implies that

$$\int \min\Big(\frac{\varepsilon |\log \varepsilon|^2}{|v_j - v_i|}, 1\Big) \, \big|(v_{2^*} - v_i)\cdot \nu_{2^*}\big| dt_{2^*} dv_{2^*} d\nu_{2^*} \leq C t R^2 \varepsilon |\log \varepsilon|^3 \,,$$

and together with (3.11) this concludes the proof of Proposition 3.5.

□

Proof of Lemma 3.7. Since the total energy is bounded by R^2, the left-hand side of (3.9) is bounded by $2R$, and we get that

$$\frac{1}{|\tau_{rec}|} \leq \frac{4R}{|\tau_1||v_i - v_j|} \tag{3.12}$$

recalling that $\delta x_\perp \perp (v_i - v_j)$: the contribution $\big|\delta x_\perp\big|$ has been neglected in order to get uniform estimates with respect to the positions at time t_{2^*}.

Given $\delta x_\perp$ and $\tau_1 (v_i - v_j)$, the relation (3.9) forces $v_i' - v_j$ to belong to a rectangle $\mathcal{R}$ of size $2R \times (R \min\Big(\frac{4}{|\tau_1||v_i - v_j|}, 1\Big))$. The main axis of the rectangle $\mathcal{R}$ is $\delta x_\perp - \tau_1(v_i - v_j)$ and the length $2R$ is a consequence of the cut-off on the velocities. Applying (C.11), we deduce that

$$\begin{aligned}\int \mathbf{1}_{v_i' - v_j \in \mathcal{R}} \big|(v_{1^*} - v_i)\cdot \nu_{1^*}\big| \, dv_{1^*} d\nu_{1^*} &\leq C R^3 \min\left(\frac{4}{|\tau_1||v_i - v_j|}, 1\right) \Big(|\log(|\tau_1||v_i - v_j|)| + \log R\Big) \\ &\leq C R^3 |\log \varepsilon| \min\left(\frac{4}{|\tau_1||v_i - v_j|}, 1\right),\end{aligned}$$

recalling that $R^2 + t \ll |\log \varepsilon|$. Integrating with respect to $|v_i - v_j|\,|\tau_1|$ up to Rt/ε, we obtain that

$$\int \mathbf{1}_{\text{(3.9) has a solution}} \big|(v_{1^*} - v_i)\cdot \nu_{1^*}\big| |v_i - v_j| d\tau_1 dv_{1^*} d\nu_{1^*} \leq C R^3 (\log \varepsilon)^2.$$

Lemma 3.7 is proved. □

Instead of fixing the first two recolliding particles and the time interval for the recollision, we are going to index the recollisions in terms of the collision integrals depending on the particles $1^*, 2^*$ which are the parents leading to the first recollision (introduced in Proposition 3.5). This will be useful when estimating the norm of the iterated collision operators in order to keep a symmetric structure of the collision operators which are not involved in the first recollision. The following corollary is a consequence of the proof of Proposition 3.5.

Corollary 3.8. *Fix $z_1 \in \mathbb{T}^2 \times B_R$ with $1 \leq R^2 \leq C_0 |\log \varepsilon|$, and a collision tree $a \in \mathcal{A}_s$.*

Then there exist sets $\mathcal{P}_1(a, z_1, \sigma) \subset \mathcal{T}_{2,s} \times \mathbb{S}^{s-1} \times B_R^{s-1}$ for all $\sigma \subset \{2, \dots, s\}$ with at most two elements such that

- *the following estimate holds*

$$\int \mathbf{1}_{\mathcal{P}_1(a,z_1,\sigma)} \Big(\prod_{m \in \sigma} \big|(v_{a(m)}(t_m) - v_m)\cdot \nu_m\big|\Big) dT_\sigma dV_\sigma d\Omega_\sigma \leq C s^2 R^7 t^3 \varepsilon |\log \varepsilon|^3 \,, \tag{3.13}$$

  uniformly over z_1 and the parameters $(T_{2,s}, V_{2,s}, \Omega_{2,s})$ in $\mathcal{P}_1(a, z_1, \sigma)$ which are not indexed by σ;

- *the set $\mathcal{P}_1(a, z_1)$ of pseudo-trajectories with at least one recollision is included in* $\bigcup_\sigma \mathcal{P}_1(a, z_1, \sigma)$.

Proof. There are s choices of θ to localize the recollision time in $[t_\theta, t_{\theta+1}]$.
In the case when the recollision is a self-recollision between 1^* and $a(1^*)$ as described page 18, then (3.13) follows from Proposition 3.5 (with a factor s only) where $\sigma = 1^*$ and $\mathcal{P}_1(a, z_1, 1^*)$ is the set of parameters leading to that recollision.

If the recollision is not a self-recollision then depending on the tree a, either 2^* is the parent of 1^* or not:

- Suppose that 2^* is not the parent of 1^*, then there are exactly two particles associated with these parents and the recollision will take place among these four particles. In this case (3.13) follows from Proposition 3.5 (with a factor s only) where $\mathcal{P}_1(a, z_1, \sigma)$ is the set of parameters leading to that recollision.
- If 2^* is the parent of 1^*, then only one particle involved in the recollision is fixed (it can be either 1^* or $a(1^*)$) and we get an extra factor s for the choice of the second recolliding particle. Once the recolliding particles are prescribed, the right-hand side of (3.13) is again a consequence of Proposition 3.5. The set $\mathcal{P}_1(a, z_1, \sigma)$ is then the union of all the possible choices.

This completes the proof of Corollary 3.8.

□

3.1.2. *Global estimate.* To estimate the global error due to recollisions, we have to incorporate the local estimate provided in Corollary 3.8 (which is uniform with respect to all parameters $(t_i, \nu_i, v_i)_{i \notin \sigma}$) with all the other collision integrals. We use the fact that we have now a tree with $s-1$ or $s-2$ branching points, neglecting the constraints that $(t_j)_{j \in \sigma}$ have to be properly chosen in between other collision times, and also the constraint on the distribution of collision times on the different time intervals $[t - kh, t-(k-1)h]$. In general, a constraint on p particles leads to the following estimates.

Proposition 3.9. *We fix $z_1 \in \mathbb{T}^2 \times \mathbb{R}^2$, a set $\sigma \subset \{1, \dots, s\}$ of p indices and $\eta > 0$ such that the collection of sets $\mathcal{P}_1(a, z_1, \sigma) \subset \mathcal{T}_{2,s} \times \mathbb{S}^{s-1} \times \mathbb{R}^{2(s-1)}$ associated with the collision trees satisfies*

$$\sup_{a \in \mathcal{A}_s} \sup_{T_{2,s}^{<\sigma>}, \Omega_{2,s}^{<\sigma>}, V_{2,s}^{<\sigma>}} \int \mathbf{1}_{\mathcal{P}_1(a,z_1,\sigma)} \prod_{i \in \sigma} \big|(v_i - v_{a(i)}(t_i)) \cdot \nu_i\big| dT_\sigma d\Omega_\sigma dV_\sigma \leq \eta \,, \tag{3.14}$$

with the notation $Y^{<\sigma>} := \{y_i\}_{i \notin \sigma}$. Then for $t \geq 1$, one has

$$\begin{aligned} \sum_{a \in \mathcal{A}_s} \int \mathbf{1}_{\mathcal{T}_{2,s}} \mathbf{1}_{\mathcal{P}_1(a,z_1,\sigma)} \Big(\prod_{i=2}^s \big|(v_i - v_{a(i)}(t_i)) \cdot \nu_i\big| \Big) M_\beta^{\otimes s} \, dT_{2,s} d\Omega_{2,s} dV_{2,s} \\ \leq s^p (Ct)^{s-1-p} \eta M_{5\beta/8}(v_1) \,. \end{aligned} \tag{3.15}$$

If we further specify that the last n collision times have to be in an interval of length $h \leq 1$ (this constraint is denoted by $\mathcal{T}^h_{s-n+1,s}$)

$$\begin{aligned} \sum_{a \in \mathcal{A}_s} \int \mathbf{1}_{\mathcal{T}_{2,s}} \mathbf{1}_{\mathcal{T}^h_{s-n+1,s}} \mathbf{1}_{\mathcal{P}_1(a,z_1,\sigma)} \Big(\prod_{i=2}^s \big|(v_i - v_{a(i)}(t_i)) \cdot \nu_i\big| \Big) M_\beta^{\otimes s} \, dT_{2,s} d\Omega_{2,s} dV_{2,s} \\ \leq s^p (Ct)^{s-n-1} (Ch)^{n-p} \eta M_{5\beta/8}(v_1) \,. \end{aligned} \tag{3.16}$$

Proof. Proposition 3.9 is a consequence of the estimates on the collision operators (see Proposition 2.4) for the particles which are not in σ and the smallness assumption (3.14) for the particles in σ. Both estimates can be decoupled thanks to Fubini's theorem.

We first perform the integration with respect to the p variables in σ. Assumption (3.14) implies

$$\int \mathbf{1}_{\mathcal{P}_1(a,\sigma)} \prod_{i\in\sigma} \big|(v_i - v_{a(i)}(t_i)) \cdot \nu_i\big| dT_\sigma d\Omega_\sigma dV_\sigma \leq \eta \, .$$

Using the same estimates as in Proposition 2.4, we find that the contribution of the collision operators for the particles which are not in σ is bounded uniformly from above by

$$\sum_{(a(j))_{j\notin\sigma}} \left(\prod_{i\notin\sigma} \big|(v_i - v_{a(i)}(t_i)) \cdot \nu_i\big| \right) M^{\otimes s}_{5\beta/6}(V_s) \leq (Cs)^{s-1-p} M_{5\beta/8}(v_1) \, .$$

Then using the other part of the Gaussian weight $M^{\otimes s}_{\beta/6}$, we integrate with respect to the remaining variables. We only retain the condition for the times $(t_i)_{i\notin\sigma}$ and distinguish two cases :

- In (3.15), the time constraint $\mathcal{T}_{2,s}$ boils down to integrating over a simplex of dimension $s-1-p$, the volume of which is
$$\frac{t^{s-1-p}}{(s-1-p)!} \leq C^s \frac{t^{s-1-p}}{s^{s-1-p}}$$
by Stirling's formula.
- In (3.16), we have to add the condition that the last n times are in an interval of length $h \leq 1$. For $t \geq 1$, the worst situation is when all times $(t_i)_{i\in\sigma}$ are in this small time interval, as we loose the corresponding smallness. More precisely, we get
$$\frac{t^{s-1-n}}{(s-1-n)!} \frac{h^{n-p}}{(n-p)!} \leq C^s \frac{t^{s-1-n} h^{n-p}}{s^{s-1-p}} \, \cdot$$

The last contribution s^p comes from summing over all possible choices for $(a(j))_{j\in\sigma}$. This completes the proof of Proposition 3.9. $\square$

Proof of Proposition 3.2. Given $z_1 \in \mathbb{T}^2 \times B_R$, the set of parameters leading to a recollision is partitioned into subsets $\displaystyle\bigcup_\sigma \bigcup_a \mathcal{P}_1(a, z_1, \sigma)$ (see Corollary 3.8) with a measure bounded by the local estimate (3.13): more precisely the term $f_N^{(1,K),\geq}$ defined in (3.3) can be estimated using (3.15)

$$\begin{aligned} \Big| f_N^{(1,K),\geq}(t,z_1) \Big| \leq \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \sum_\sigma \sum_{a\in\mathcal{A}_{J_K}} \int \mathbf{1}_{\mathcal{T}^h_{2,J_K}} \mathbf{1}_{\mathcal{P}_1(a,z_1,\sigma)} \\ \times \left(\prod_{i=2}^{J_K} \big|(v_i - v_{a(i)}(t_i)) \cdot \nu_i\big| \right) \left(f_{N,0}^{(J_K)} \mathbf{1}_{\mathcal{V}_{J_K}} \right) dT_{2,J_K} d\Omega_{2,J_K} dV_{2,J_K} \, . \end{aligned} \tag{3.17}$$

We have seen in (2.15) that the marginals of the initial data are dominated by a Maxwellian

$$\big| f_{N,0}^{(J_K)}(Z_{J_K}) \big| \leq C^{J_K} M_\beta^{\otimes J_K}(V_{J_K}) \|g_{\alpha,0}\|_{L^\infty} \, .$$

Thus (3.15) (with $p \leq 2$) can be applied to estimate $f_N^{(1,K),\geq}$, recalling that $1 \leq t + R^2 \lesssim |\log\varepsilon|$

$$\Big| f_N^{(1,K),\geq}(t,z_1) \Big| \leq \|g_{\alpha,0}\|_{L^\infty} \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} C^{J_K} \alpha^{J_K-1} J_K^4 t^{J_K} \varepsilon \big|\log\varepsilon\big|^{\frac{19}{2}} M_{5\beta/8}(v_1) \, .$$

Now recalling that $n_k = 2^k n_0$ we have

$$J_K \le 2^{K+1} n_0 \quad \text{and} \quad \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \le \prod_{i=1}^{K} n_i \le n_0^K \times 2^{K^2}\,, \tag{3.18}$$

so thanks to Assumption (1.19) on the initial data $g_{\alpha,0}$, we conclude

$$\Big| f_N^{(1,K),\ge}(t,z_1) \Big| \le \exp(C\alpha^2) 2^{K^2} \big(CT\alpha\big)^{2^{K+1}n_0} \varepsilon \big|\log\varepsilon\big|^{\frac{19}{2}} M_{5\beta/8}(v_1)\,.$$

Since $2^{K^2} \ll C^{2^K}$, this completes the proof of Proposition 3.2 (bounding $\big|\log\varepsilon\big|^{\frac{19}{2}}$ by $\big|\log\varepsilon\big|^{10}$ to simplify). $\Box$

3.2. **Proof of Proposition 3.1.** Each term in the decomposition (3.3)

$$f_N^{(1,K)}(t) = f_N^{(1,K),0}(t) + f_N^{(1,K),\ge}(t)$$

can be interpreted as a restriction of the domain of integration of the times, velocities and deflection angles. For $f_N^{(1,K),\ge}$, the pseudo-trajectories associated with a tree a are integrated over $\mathcal{P}_1(a,z_1)$ as in (3.17), instead they are integrated over $\mathcal{P}_1(a,z_1)^c$ in $f_N^{(1,K),0}$ when there is no recollision.

A similar decomposition holds for the Boltzmann hierarchy

$$\bar f_N^{(1,K)}(t) := \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \bar Q_{1,J_1}(h) \bar Q_{J_1,J_2}(h) \dots \bar Q_{J_{K-1},J_K}(h) \big(f_0^{(J_K)} \mathbf{1}_{\mathcal{V}_{J_K}} \big)$$

by restricting the trajectories to the sets of parameters $\mathcal{P}_1(a,z_1)$ and $\mathcal{P}_1(a,z_1)^c$ writing

$$\bar f_N^{(1,K)}(t) = \bar f_N^{(1,K),0}(t) + \bar f_N^{(1,K),\ge}(t)\,.$$

This splitting is artificial as there are no recollisions in the Boltzmann hierarchy, however it will be useful to compare the different contributions. As a consequence of Proposition 3.2, the term $\bar f_N^{(1,K),\ge}$ is negligible

$$\Big| \bar f_N^{(1,K),\ge}(t,z_1) \Big| \le \exp(C\alpha^2) \big(CT\alpha\big)^{2^{K+1}n_0} \varepsilon \big|\log\varepsilon\big|^{10} M_{5\beta/8}(v_1)\,. \tag{3.19}$$

The last step to conclude Proposition 3.1 is to evaluate the difference $f_N^{(1,K),0}(t) - \bar f_N^{(1,K),0}(t)$. Once the recollisions have been excluded, the only discrepancies between the BBGKY and the Boltzmann pseudo-trajectories come from the micro-translations due to the diameter ε of the colliding particles (see Definition 2.2). At the initial time, the error between the two configurations is at most $O(s\varepsilon)$ after s collisions (see [7, 4])

$$\big| X_s^0(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0) - X_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0) \big| \le Cs\varepsilon\,. \tag{3.20}$$

The discrepancies are only for positions, as velocities remain equal in both hierarchies. These configurations are then evaluated either on the marginals of the initial data $f_{N,0}^{(s)}$ or of $f_0^{(s)}$ which are close to each other thanks to Proposition 2.5.

The main discrepancy between $f_N^{(1,K),0}$ and $f^{(1,K),0}$ depends on

$$\begin{aligned}
&\Big| f_0^{(s)}\Big(Z_s^0(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)\Big) - f_{N,0}^{(s)}\Big(Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)\Big) \Big| \\
&\quad \le \Big| f_0^{(s)}\Big(Z_s^0(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)\Big) - f_0^{(s)}\Big(Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)\Big) \Big| \\
&\qquad + \Big| f_0^{(s)}\Big(Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)\Big) - f_{N,0}^{(s)}\Big(Z_s(a, T_{2,s}, \Omega_{2,s}, V_{2,s}, 0)\Big) \Big|\,.
\end{aligned}$$

By the assumption (1.19), $g_{\alpha,0}$ has a Lipschitz bound $\exp(C\alpha^2)$, thus combining (3.20) and the estimate of Proposition 2.5, we get

$$\left| f_0^{(s)}\Big(Z_s^0(a,T_{2,s},\Omega_{2,s},V_{2,s},0)\Big) - f_{N,0}^{(s)}\Big(Z_s(a,T_{2,s},\Omega_{2,s},V_{2,s},0)\Big)\right| \leq C^s \exp(C\alpha^2) s\varepsilon M_\beta^{\otimes s}(V_s)\,.$$

The last source of discrepancy between the formulas defining $f_N^{(1,K),0}$ and $\bar f_N^{(1,K),0}$ comes from the prefactor $(N-1)\dots(N-s+1)\varepsilon^{s-1}$ which has been replaced by α^{s-1}. For fixed s, the corresponding error is

$$\Big(1-\frac{(N-1)\dots(N-s+1)}{N^{s-1}}\Big) \leq C\frac{s^2}{N} \leq Cs^2\frac{\varepsilon}{\alpha}$$

which, combined with the bound on the collision operators, leads to an error of the form

$$(C\alpha t)^{s-1}s^2\frac{\varepsilon}{\alpha}\,\cdot \tag{3.21}$$

Summing the previous bounds gives

$$\begin{aligned}
&\left| f_N^{(1,K),0}(t,z_1) - \bar f_N^{(1,K),0}(t,z_1)\right| \\
&\quad \leq \exp(C\alpha^2)M_\beta(v_1)\sum_{j_1=0}^{n_1-1}\dots\sum_{j_K=0}^{n_K-1}(C\alpha t)^{J_K-1}\Big(J_K^2\frac{\varepsilon}{\alpha}+J_K\varepsilon\Big) \\
&\quad \leq \exp(C\alpha^2)M_\beta(v_1)\big(CT\alpha\big)^{2^{K+1}n_0}\Big(2^{2(K+1)}\frac{\varepsilon}{\alpha}+2^{K+1}\varepsilon\Big)\,,
\end{aligned} \tag{3.22}$$

where we used the bounds (3.18) for the sequence $n_k = 2^k n_0$.

Finally Proposition 3.1 follows by combining

- Proposition 3.2 and (3.19) to control the recollisions,
- (3.22) to control the difference in the parts without recollisions.

The result is proved. □

4. Symmetry and L^2 bounds

In this section, we prove an upper bound on the contribution of super exponential collision trees without recollisions introduced in (2.25)

$$R_N^{K,0}(t) := \sum_{k=1}^{K}\sum_{\substack{j_i<n_i\\ i\leq k-1}}\sum_{j_k\geq n_k} Q^0_{1,J_1}(h)\dots Q^0_{J_{k-1},J_k}(h)\big(f_N^{(J_K)}(t-kh)\mathbf{1}_{\mathcal{V}_{J_K}}\big)\,.$$

Proposition 4.1. *Given $T>1$, $\gamma\ll 1$ and C a large enough constant (independent of γ and T), the parameters are tuned as follows*

$$h\leq \frac{\gamma^2}{\exp(C\alpha^2)T^3}\,,\qquad n_k = 2^k n_0\,. \tag{4.1}$$

Then, under the Boltzmann-Grad scaling $N\varepsilon=\alpha\gg 1$, we have for $t\in[0,T]$

$$\left\|R_N^{K,0}(t)\right\|_{L^2(\mathbb{D})}\leq \gamma\,. \tag{4.2}$$

The main step to derive Proposition 4.1 is to replace the L^∞ estimates on the collision kernel (Proposition 2.4) by L^2 estimates. To do this, we first establish an L^2_β decomposition of the marginals $f_N^{(s)}(t)$ (Proposition 4.2 in Section 4.1) and then an L^2 counterpart of Proposition 2.4 (Proposition 4.4 in Section 4.2). The proof of Proposition 4.1 is postponed to Section 4.3. Finally in Section 4.4 the counterpart of Proposition 4.1 for the Boltzmann hierarchy is stated and proved.

4.1. **Structure of symmetric functions in L^2.** We prove in Proposition 4.2 that a structure similar to (2.19) is intrinsic to symmetric functions with suitable L^2 bounds (the argument does not involve dynamics). As the density $f_N(t)$ of the particle system is symmetric and admits L^2 bounds uniform in time, we can then deduce that the higher order correlations of the marginals $f_N^{(s)}(t, Z_s)$ are small in L^2 for any time. This is a key ingredient in the proof of the main theorem.

The following proposition states a general decomposition of symmetric functions in L^2_β.

Proposition 4.2. *Let f_N be a mean free, symmetric function in $L^2_\beta(\mathbb{D}^N)$. There exist symmetric functions g_N^m on $\mathbb{D}^m$ for $1 \leq m \leq N$ such that for all $s \leq N$, the marginal of order s satisfies*

$$f_N^{(s)}(Z_s) = M_\beta^{\otimes s}(V_s) \sum_{m=1}^{s} \sum_{\sigma \in \mathfrak{S}_s^m} g_N^m(Z_\sigma) \,, \tag{4.3}$$

where $\mathfrak{S}_s^m$ denotes the set of all parts of $\{1, \dots, s\}$ with m elements. Moreover

$$\|g_N^m\|^2_{L^2_\beta(\mathbb{D}^m)} \leq \frac{1}{C_N^m} \|f_N / M_\beta^{\otimes N}\|^2_{L^2_\beta(\mathbb{D}^N)} \,.$$

Applying Proposition 4.2 to the solution $f_N(t)$ of the Liouville equation which satisfies (2.18), we deduce immediately from the control (2.16) of the exclusion that for all $s \leq N$, the marginal of order s satisfies

$$f_N^{(s)}(t, Z_s) = M_\beta^{\otimes s}(V_s) \sum_{m=1}^{s} \sum_{\sigma \in \mathfrak{S}_s^m} g_N^m(t, Z_\sigma) \,,$$

with

$$\|g_N^m(t)\|^2_{L^2_\beta(\mathbb{D}^m)} \leq \frac{CN \exp(C\alpha^2)}{C_N^m} \|g_{\alpha,0}\|^2_{L^2_\beta(\mathbb{D})} \,. \tag{4.4}$$

Proof of Proposition 4.2. Define

$$g_N^m(Z_m) := \sum_{k=1}^{m} (-1)^{m-k} \sum_{\sigma \in \mathfrak{S}_m^k} \frac{f_N^{(k)}}{M_\beta^{\otimes k}}(Z_\sigma) \,.$$

Step 1. The identity

$$\frac{f_N}{M_\beta^{\otimes N}} = \sum_{m=1}^{N} \sum_{\sigma \in \mathfrak{S}_N^m} g_N^m(Z_\sigma) \tag{4.5}$$

comes from a simple application of Fubini's theorem. We indeed have

$$\begin{aligned}
\sum_{m=1}^{N} \sum_{\sigma \in \mathfrak{S}_N^m} g_N^m(Z_\sigma) &= \sum_{m=1}^{N} \sum_{\sigma \in \mathfrak{S}_N^m} \sum_{k=1}^{m} (-1)^{m-k} \sum_{\tilde\sigma \in \mathfrak{S}_m^k} \frac{f_N^{(k)}}{M_\beta^{\otimes k}}(Z_{\tilde\sigma}) \\
&= \sum_{k=1}^{N} \sum_{\tilde\sigma \in \mathfrak{S}_N^k} \frac{f_N^{(k)}}{M_\beta^{\otimes k}}(Z_{\tilde\sigma}) \sum_{m=k}^{N} (-1)^{m-k} C_{N-k}^{m-k}
\end{aligned}$$

since the number of possible σ having $\tilde\sigma$ as a subset is C_{N-k}^{m-k}.

For $k<N$, we have

$$\sum_{m=k}^{N}(-1)^{m-k}C_{N-k}^{m-k}=\sum_{m=0}^{N-k}(-1)^{m}C_{N-k}^{m}=0^{N-k}=0\,,$$

while for $k=N$ we just obtain 1. We therefore get (4.5).

Step 2. We prove now that

$$\int g_N^m(Z_m)M_\beta(v_\ell)\,dz_\ell=0\,,\quad 1\leq \ell\leq m\,. \tag{4.6}$$

Given $1\leq\ell\leq m$, one can split the sum over $\sigma\in\mathfrak{S}_m^k$ into two pieces, depending on whether ℓ belongs to σ or not

$$\begin{aligned}
&\int g_N^m(Z_m)M_\beta(v_\ell)\,dz_\ell\\
&\quad=\sum_{k=1}^{m}(-1)^{m-k}\sum_{\substack{\sigma\in\mathfrak{S}_m^k\\ \ell\in\sigma}}\int\frac{f_N^{(k)}}{M_\beta^{\otimes k}}(Z_\sigma)M_\beta(v_\ell)dz_\ell+\sum_{k=1}^{m-1}(-1)^{m-k}\sum_{\substack{\sigma\in\mathfrak{S}_m^k\\ \ell\notin\sigma}}\int\frac{f_N^{(k)}}{M_\beta^{\otimes k}}(Z_\sigma)M_\beta(v_\ell)dz_\ell\\
&\quad=\sum_{k'=0}^{m-1}(-1)^{m-k'+1}\sum_{\substack{\sigma\in\mathfrak{S}_{m-1}^{k'}\\ \ell\notin\sigma}}\frac{f_N^{(k')}}{M_\beta^{\otimes k'}}(Z_\sigma)+\sum_{k=1}^{m-1}(-1)^{m-k}\sum_{\substack{\sigma\in\mathfrak{S}_m^k\\ \ell\notin\sigma}}\frac{f_N^{(k)}}{M_\beta^{\otimes k}}(Z_\sigma)\,.
\end{aligned}$$

The conclusion follows from the fact that the case $k'=0$ corresponds to

$$\int\frac{f_N^{(1)}}{M_\beta}(z_\ell)M_\beta(v_\ell)dz_\ell=\int f_N(Z_N)dZ_N=0\,.$$

Hence we obtain

$$\int g_N^m(Z_m)M_\beta(v_\ell)\,dz_\ell=0\,.$$

The identity (4.3) follows by integrating (4.5) with respect to $M_\beta^{\otimes(N-s)}dz_{s+1}\dots dz_N$

$$f_N^{(s)}(t,Z_s)=M_\beta^{\otimes s}\sum_{m=1}^{s}\sum_{\sigma\in\mathfrak{S}_s^m}g_N^m(t,Z_\sigma)\,.$$

Step 3. It remains to establish estimate (4.4). From (4.5) and the orthogonality condition (4.6), we also deduce that

$$\begin{aligned}
\int\frac{f_N^2}{M_\beta^{\otimes N}}dZ_N&=\int M_\beta^{\otimes N}\left(\sum_{m=1}^{N}\sum_{\sigma\in\mathfrak{S}_N^m}g_N^m(Z_\sigma)\right)^2dZ_N\\
&=\sum_{m=1}^{N}\sum_{\sigma\in\mathfrak{S}_N^m}\int M_\beta^{\otimes N}\left(g_N^m(Z_\sigma)\right)^2dZ_N\\
&=\sum_{m=1}^{N}C_N^m\|g_N^m\|^2_{L^2_\beta(\mathbb{D}^m)}\,.
\end{aligned}$$

This ends the proof of Proposition 4.2. $\square$

Remark 4.3. *The decomposition* (4.3) *shows that the higher order correlations decrease in L^2-norm according to the number of particles. This is a step towards proving local equilibrium, but these estimates are not strong enough to deduce directly that the equation on the first marginal can be closed.*

4.2. L^2 **continuity estimates for the iterated collision operators.** We will now establish an L^2 estimate for $Q^0_{1,J}(t)$ (see Proposition 4.4). As explained in the introduction (see Paragraph 2.5), it involves a loss in ε, which will be exactly compensated by the decay of the L^2_β-norm (4.4) in the expansion (4.3). This shows that the structure (2.19) is partly preserved by the collision-transport operators, as long as there is no recollision.

4.2.1. *Statement of the result and plan of the proof.* Let us first introduce some notation. As in (2.7) for $|Q_{s,s+n}|(t)$, the operator $|Q^0_{s,s+n}|(t)$ is obtained by considering the sum $C^+_{s,s+1} + C^-_{s,s+1}$ instead of the difference. Let $g_m \in L^2_\beta(\mathbb{D}^m)$, we set for $\sigma \in \mathfrak{S}^m_s$

$$g_{m,\sigma}(Z_s) = g_m(Z_\sigma)\,. \tag{4.7}$$

The key estimate is given by the following proposition. Note that the bound provided in (4.8) is not the best one can prove (in terms of the way the powers of t and h are divided) but suffices for our purposes.

Proposition 4.4. *There is a constant C (depending only on β) such that for all $J, n \in \mathbb{N}^*$ and all $t \geq 1, h \in [0,t]$, the operator $|Q^0|$ satisfies the following continuity estimate*

$$\begin{aligned}\Big\| |Q^0_{1,J}|(t)\,|Q^0_{J,J+n}|(h) \sum_{\sigma\in\mathfrak{S}^m_{J+n}} \mathbf{1}_{\mathcal{V}_{J+n}} M_\beta^{\otimes(J+n)} \big|g_{m,\sigma}\big| \Big\|_{L^2(\mathbb{D})} \\ \leq (C\alpha)^{J+n-1} t^{J+n/2-1} h^{n/2} \frac{\|g_m\|_{L^2_\beta(\mathbb{D}^m)}}{\sqrt{\varepsilon^{m-1} m!}}\,.\end{aligned} \tag{4.8}$$

Proof. To simplify the analysis, especially the treatment of large velocities, we define modified collision operators

$$\begin{aligned}\big(C^{b,\pm}_{s,s+1} h^{s+1}\big)(Z_s) &:= \frac{(N-s)\varepsilon}{\alpha} \sum_{i=1}^s \int_{\mathbb{S}\times\mathbb{R}^2} h^{s+1}(Z^{\pm,i,s+1}_{s+1}) \frac{\big((v_i - v_{s+1})\cdot\nu\big)_+}{1+|v_i - v_{s+1}|}\, d\nu dv_{s+1}\,,\\ \big(C^{q,\pm}_{s,s+1} h^{s+1}\big)(Z_s) &:= \frac{(N-s)\varepsilon}{\alpha} \sum_{i=1}^s \int_{\mathbb{S}\times\mathbb{R}^2} h^{s+1}(Z^{\pm,i,s+1}_{s+1}) \\ &\qquad\qquad \times \big(1+|v_i - v_{s+1}|\big)\big((v_i - v_{s+1})\cdot\nu\big)_+\, d\nu dv_{s+1},\end{aligned} \tag{4.9}$$

where $Z^{\pm,i,s+1}_{s+1}$ denotes the configuration after the collision between i and $s+1$ as in (1.9)

$$\begin{aligned} Z^{-,i,s+1}_{s+1} &:= (x_1, v_1, \dots, x_i, v_i, \dots, x_i - \varepsilon\nu, v_{s+1})\,,\\ Z^{+,i,s+1}_{s+1} &:= (x_1, v_1, \dots, x_i, v'_i, \dots, x_i + \varepsilon\nu, v'_{s+1})\,.\end{aligned}$$

By construction, $C^{b,\pm}_{s,s+1}$ has a bounded collision cross-section and $C^{q,\pm}_{s,s+1}$ has a collision cross-section with quadratic growth in v. Defining accordingly $|Q^{b,0}_{1,J}|$ and $|Q^{q,0}_{1,J}|$, we have by the

Cauchy-Schwarz inequality

$$\begin{aligned}\Big| |Q^0_{1,J}|(t)\, |Q^0_{J,J+n}|(h) &\sum_{\sigma\in\mathfrak{S}^m_{J+n}} M_\beta^{\otimes(J+n)} \mathbf{1}_{\mathcal{V}_{J+n}} |g_{m,\sigma}|\Big| \\ &\leq \Big(\sum_{\sigma\in\mathfrak{S}^m_{J+n}} |Q^{q,0}_{1,J}|(t)\, |Q^{q,0}_{J,J+n}|(h)\, M_\beta^{\otimes(J+n)}\Big)^{1/2} \\ &\qquad \times \Big(|Q^{b,0}_{1,J}|(t)\, |Q^{b,0}_{J,J+n}|(h) \sum_{\sigma\in\mathfrak{S}^m_{J+n}} M_\beta^{\otimes(J+n)} g^2_{m,\sigma}\Big)^{1/2},\end{aligned}$$

where the velocity cut-off $\mathcal{V}_{J+n}$ has been dropped. Thus we find directly

$$\begin{aligned}\Big| |Q^0_{1,J}|(t)\, |Q^0_{J,J+n}|(h) &\sum_{\sigma\in\mathfrak{S}^m_{J+n}} M_\beta^{\otimes(J+n)} |g_{m,\sigma}|\Big| \\ &\leq 2^{\frac{J+n}{2}} \Big(|Q^{q,0}_{1,J}|(t)\, |Q^{q,0}_{J,J+n}|(h)\, M_\beta^{\otimes(J+n)}\Big)^{1/2} \\ &\qquad \times \Big(|Q^{b,0}_{1,J}|(t)\, |Q^{b,0}_{J,J+n}|(h) \sum_{\sigma\in\mathfrak{S}^m_{J+n}} M_\beta^{\otimes(J+n)} g^2_{m,\sigma}\Big)^{1/2}.\end{aligned} \tag{4.10}$$

- The first factor can be bounded in L^∞ as in Proposition 2.4.

Proposition 4.5. *There is a constant C (depending only on β) such that for all $J, n \in \mathbb{N}^*$ and all $h, t \geq 0$, the operator $|Q^{q,0}|$ satisfies the following continuity estimates*

$$\forall z_1 \in \mathbb{D}\,, \qquad |Q^{q,0}_{1,J}|(t)\, |Q^{q,0}_{J,J+n}|(h) M_\beta^{\otimes(J+n)}(z_1) \leq (C\alpha t)^{J-1} (C\alpha h)^n M_{3\beta/4}(z_1)\,. \tag{4.11}$$

The proof is omitted as it is similar to the one of Proposition 2.4 (we just have to skip the Cauchy-Schwarz estimate in (2.8)). Note that the quadratic growth in the collision cross-section is critical in the sense that it is the highest possible power giving an admissible loss estimate.

Thus (4.10) can be bounded as follows

$$\begin{aligned}\int_{\mathbb{D}} \Big(|Q^0_{1,J}|(t)\, |Q^0_{J,J+n}|(h) &\sum_{\sigma\in\mathfrak{S}^m_{J+n}} M_\beta^{\otimes(J+n)} \mathbf{1}_{\mathcal{V}_{J+n}} |g_{m,\sigma}|\Big)^2 dz_1 \\ &\leq (C\alpha t)^{J-1} (C\alpha h)^n \int_{\mathbb{D}} |Q^{b,0}_{1,J}|(t)\, |Q^{b,0}_{J,J+n}|(h) \sum_{\sigma\in\mathfrak{S}^m_{J+n}} M_\beta^{\otimes(J+n)} g^2_{m,\sigma}\, dz_1\,.\end{aligned} \tag{4.12}$$

- The second factor can be bounded from above by relaxing the conditions on the distribution of times to retain only that the collision times have to satisfy

$$0 \leq t_{J+n-1} \leq \dots \leq t_J \leq \dots \leq t_2 \leq t+h \leq 2t\,.$$

In other words, we have

$$|Q^{b,0}_{1,J}|(t)\, |Q^{b,0}_{J,J+n}|(h) \leq |Q^{b,0}_{1,J+n}|(2t)\,.$$

This is suboptimal in the sense that it implies that powers of h will be traded for powers of t but the smallness thanks to h already present on the right-hand side of (4.12) will be enough for our purposes. To establish Proposition 4.4, it is then enough to prove the following proposition which will be applied to g^2_m.

Proposition 4.6. *Let $\varphi_m(Z_m)$ be a nonnegative symmetric function in $L^1_\beta(\mathbb{D}^m)$. For $J \geq m$, we have for any time $t \geq 1$*

$$\int_{\mathbb{D}} dz\, |Q^{b,0}_{1,J}|(t) \sum_{\sigma \in \mathfrak{S}^m_J} M^{\otimes J}_\beta \varphi_{m,\sigma} \leq \frac{(C\alpha t)^{J-1}}{m!\varepsilon^{m-1}} \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)} \,. \tag{4.13}$$

Thus this completes the derivation of Proposition 4.4. □

The idea of the proof of Proposition 4.6 is to proceed by iteration: Lemma 4.7 in Paragraph 4.2.2 shows that the structure is preserved through an integrated in time transport-collision operator, the proof of Proposition 4.6 is then completed in Paragraph 4.2.3.

4.2.2. *Stability of the structure* (4.3) *under the BBGKY dynamics.* In order to prove Proposition 4.6, we first state and prove a key lemma on the collision kernel which will be used recursively in Section 4.2.3 to prove Proposition 4.6.

Lemma 4.7. *Fix $t > 0$ and $1 \leq m \leq s+1 \leq J$, and let φ_m be a function as in Proposition* 4.6. *Then there are two symmetric functions $\Phi^{(m)}_m$ and $\Phi^{(m)}_{m-1}$ defined on $\mathbb{D}^m$ and $\mathbb{D}^{m-1}$ such that with notation* (4.7)

$$\begin{aligned}\int_0^{+\infty} d\tau\, e^{-\frac{J\tau}{t}} |C^{b,\pm}_{s,s+1}| \widehat{\mathbf{S}}^0_{s+1}(\tau) \Big(M^{\otimes(s+1)}_\beta \sum_{\sigma \in \mathfrak{S}^m_{s+1}} \varphi_{m,\sigma} \Big) \\ \leq M^{\otimes s}_\beta(V_s) \Big(\sum_{\sigma \in \mathfrak{S}^m_s} \Phi^{(m)}_{m,\sigma} + \sum_{\sigma \in \mathfrak{S}^{m-1}_s} \Phi^{(m)}_{m-1,\sigma} \Big) \,.\end{aligned}$$

Furthermore, they satisfy

$$\|\Phi^{(m)}_m\|_{L^1_\beta(\mathbb{D}^m)} \leq Ct \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)} \tag{4.14}$$

$$\|\Phi^{(m)}_{m-1}\|_{L^1_\beta(\mathbb{D}^{m-1})} \leq \frac{C}{\varepsilon(m-1)} \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)} \tag{4.15}$$

and $\Phi^{(s+1)}_{s+1} = \Phi^{(1)}_0 = 0$.

Proof. To simplify the notation, we drop the superscript (m) throughout the proof.

Let $\sigma := (i_1, \ldots, i_m)$ be a collection of ordered indices in $\{1, \ldots, s+1\}$. We first analyze the term involving $\varphi_{m,\sigma}$ and then conclude by summing over all possible σ's.

In the following, we shall use the notation $Z^{<i>}_s$ for the configuration in $\mathbb{D}^{s-1}$ defined by

$$Z^{<i>}_s := (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_s) \,.$$

When applying the collision operator $|C^{b,\pm}_{s,s+1}|$ to $\widehat{\mathbf{S}}^0_{s+1}(\tau) M^{\otimes(s+1)}_\beta \varphi_{m,\sigma}$, four different situations occur depending on whether the colliding particles $s+1$ and i belong to σ or not. Indeed recall that the collision operator consists mainly in integrating one of the variables, namely x_{s+1}, on a hypersurface $|x_i - x_{s+1}| = \varepsilon$ for some $1 \leq i \leq s$. Thus the collision may add some dependency in the arguments of $g_{m,\sigma}$.

- If z_{s+1} does not belong to σ, i.e. the variables of $\varphi_{m,\sigma}$:
  - either z_i does not belong to σ and in that case essentially nothing happens as the collision does not affect the variables in σ and the transport operator is an isometry in L^1.
  - or z_i does belong to σ and in that case v_i is modified by the scattering operator but that will be shown to be harmless thanks to the energy conservation and a change of variables by the scattering operator.
- If z_{s+1} does belong to σ:

– either z_i does not belong to σ then this is quite similar to the second case above,
– or z_i belongs to σ then by integration on the hypersurface a variable is lost (and that case alone accounts for the term $\Phi^{(m)}_{m-1}$ in the lemma).

We turn now to a detailed analysis of these cases.

Case 1. $s+1 \notin \sigma$:

This case corresponds to $\sigma \in \mathfrak{S}_s^m$ $(m \leq s)$ and will contribute partly to the function Φ_m. Recall that $\varphi_{m,\sigma}$ depends only on the coordinates Z_σ indexed by σ.

• Define the contribution $\Phi^{1,\pm}_\sigma$ corresponding to collisions between two particles of the background :

$$\Phi^{1,\pm}_\sigma(Z_s) := \int_0^{+\infty} d\tau\, e^{-\frac{J\tau}{t}} \widehat{\mathbf{S}}^0_s(\tau)\Big(\sum_{\substack{i=1\\ i\notin\sigma}}^s M_\beta^{\otimes(s-1)}\,\varphi_{m,\sigma}\Big)(V_s^{<i>}, X_\sigma)$$
$$\times \int_{\mathbb{S}\times\mathbb{R}^2} M_\beta^{\otimes 2}(v_i^{\pm,i,s+1}, v_{s+1}^{\pm,i,s+1}) \frac{\big((v_i - v_{s+1})\cdot\nu\big)_+}{1+|v_i - v_{s+1}|}\, d\nu dv_{s+1}\,.$$

Notice that by energy conservation

$$M_\beta^{\otimes 2}(v_i^{\pm,i,s+1}, v_{s+1}^{\pm,i,s+1}) = M_\beta^{\otimes 2}(v_i, v_{s+1})\,. \tag{4.16}$$

As the collision kernel is bounded, we deduce that

$$\Phi^{1,+}_\sigma(Z_s) + \Phi^{1,-}_\sigma(Z_s) \lesssim M_\beta^{\otimes s}(V_s)\Phi^1_m(Z_\sigma)\,,$$

where Φ^1_m is the first contribution to Φ_m

$$\Phi^1_m(Z_m) := 2(s-m)\int_0^{+\infty} d\tau e^{-\frac{J\tau}{t}}\ \widehat{\mathbf{S}}^0_m(\tau)\varphi_m(Z_m)\,.$$

By definition of $\widehat{\mathbf{S}}^0$, we indeed have that

$$\widehat{\mathbf{S}}^0_s(\tau) M_\beta^{\otimes s}\varphi_{m,\sigma} \leq M_\beta^{\otimes s}\widehat{\mathbf{S}}^0_m(\tau)\varphi_{m,\sigma}\,.$$

Let us compute the L^1_β norm of Φ^1_m. Note that $\widehat{\mathbf{S}}^0_m$ assigns the value 0 if a configuration has a recollision in the time interval $[0,\tau]$, so

$$\widehat{\mathbf{S}}^0_m(\tau) \leq \mathbf{S}_m(\tau). \tag{4.17}$$

Since $\varphi_m \geq 0$ and $\mathbf{S}_m$ assigns the value 0 to configurations which initially overlap, we find for $\tau \geq 0$

$$\begin{aligned}\int M_\beta^{\otimes m}(V_m)\widehat{\mathbf{S}}^0_m(\tau)\varphi_m(Z_m)dZ_m &\leq \int M_\beta^{\otimes m}(V_m)\mathbf{S}_m(\tau)\varphi_m(Z_m)dZ_m\\ &\leq \int M_\beta^{\otimes m}(V_m)\varphi_m(Z_m)dZ_m\,,\end{aligned}$$

where we used that the transport preserves the Lebesgue measure. Finally, we deduce that

$$\begin{aligned}\|\Phi^1_m\|_{L^1_\beta(\mathbb{D}^m)} &= 2(s-m)\int_0^{+\infty} d\tau e^{-\frac{J\tau}{t}}\int M_\beta^{\otimes m}(V_m)\widehat{\mathbf{S}}^0_m(\tau)\varphi_m(Z_m)dZ_m\\ &\lesssim \frac{(s-m)}{J} t\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)} \lesssim t\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,,\end{aligned} \tag{4.18}$$

where we used that $s \leq J$.

• It remains to understand what happens when the collision involves one of the particles in σ, i.e. $i \in (i_1, \dots, i_m)$. From the energy conservation (4.16) and the fact that the collision kernel is bounded, we have

$$
\begin{aligned}
(4.19)\qquad & M_\beta^{\otimes s}(V_s) \sum_{\ell=1}^m \int_0^{+\infty} d\tau e^{-\frac{J\tau}{t}} \int_{\mathbb{S}\times\mathbb{R}^2} d\nu dv_{s+1}\, M_\beta(v_{s+1}) \\
& \times \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_\sigma^{<i_\ell>}, x_{i_\ell}, v_{i_\ell}^{\pm,i_\ell,s+1}) \frac{\big((v_{i_\ell}-v_{s+1})\cdot\nu\big)_+}{1+|v_{i_\ell}-v_{s+1}|} \le M_\beta^{\otimes s}(V_s)\Phi_m^{2,\pm}(Z_\sigma)\,,
\end{aligned}
$$

where

$$
\Phi_m^{2,\pm}(Z_m) := \int_0^{+\infty} d\tau e^{-\frac{J\tau}{t}}\, \tilde{\Phi}_m^{2,\pm}(\tau, Z_m)\,,
$$

with

$$
\tilde{\Phi}_m^{2,\pm}(\tau, Z_m) := \sum_{\ell=1}^m \int_{\mathbb{S}\times\mathbb{R}^2} dv_{s+1} d\nu M_\beta(v_{s+1})\, \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_m^{<\ell>}, x_\ell, v_\ell^{\pm,\ell,s+1})\,.
$$

The function $\tilde{\Phi}_m^{2,\pm}$ is symmetric with respect to the coordinates Z_m. Using again the conservation of energy, we have

$$
\begin{aligned}
\int M_\beta^{\otimes m}(V_m)\tilde{\Phi}_m^{2,\pm}(\tau, Z_m)dZ_m &= \sum_{\ell=1}^m \int dZ_m M_\beta^{\otimes m}(V_m) \int_{\mathbb{S}\times\mathbb{R}^2} dv_{s+1} d\nu M_\beta(v_{s+1}) \\
&\qquad\qquad \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_m^{<\ell>}, x_\ell, v_\ell^{\pm,\ell,s+1}) \\
&= \sum_{\ell=1}^m \int\int_{\mathbb{S}\times\mathbb{R}^2} dZ_m dv_{s+1} d\nu M_\beta^{\otimes(m-1)}(V_m^{<\ell>}) M_\beta^{\otimes 2}(v_\ell^{\pm,\ell,s+1}, v_{s+1}^{\pm,\ell,s+1}) \\
&\qquad\qquad \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_m^{<\ell>}, x_\ell, v_\ell^{\pm,\ell,s+1})
\end{aligned}
$$

Since the change of variables

$$
(4.20)\qquad (\nu, v_\ell, v_{s+1}) \mapsto (\nu, v_\ell^{\pm,\ell,s+1}, v_{s+1}^{\pm,\ell,s+1})
$$

is an isometry and using (4.17), we deduce that for any $\tau \ge 0$,

$$
(4.21)\qquad \int M_\beta^{\otimes m}(V_m)\tilde{\Phi}_m^{2,\pm}(\tau, Z_m)dZ_m \lesssim m \int M_\beta^{\otimes m}(V_m)\,\varphi_m(Z_m)dZ_m\,.
$$

Then, integrating with respect to time and using that $m \le J$, we get

$$
\begin{aligned}
(4.22)\qquad \|\Phi_m^{2,\pm}\|_{L^1_\beta(\mathbb{D}^m)} &= \int_0^{+\infty} d\tau e^{-\frac{J\tau}{t}} \int M_\beta^{\otimes m}(V_m)\tilde{\Phi}_m^{2,\pm}(\tau, Z_m)dZ_m \\
&\lesssim \frac{mt}{J}\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)} \lesssim t\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,.
\end{aligned}
$$

From (4.19), this gives a second contribution to Φ_m for any $\sigma \in \mathfrak{S}_s^m$.

Case 2. $s+1 \in \sigma$:

Denote $I_{m-1} := \sigma \setminus \{s+1\}$. As previously, we have to distinguish if the collision with $s+1$ involves a particle $i \notin I_{m-1}$ or $i \in I_{m-1}$. The first case will lead to a third contribution to Φ_m and the second case to the term Φ_{m-1}.

• We define the contribution of the collisions with particles outside I_{m-1} as

$$
\begin{aligned}
\Psi_\sigma^{1,\pm}(Z_s) := \sum_{\substack{i=1\\ i\notin I_{m-1}}}^{s} & M_\beta^{\otimes(s-1)}(V_s^{<i>}) \int_0^{+\infty} e^{-\frac{J\tau}{t}} d\tau \int_{\mathbb{S}\times\mathbb{R}^2} M_\beta^{\otimes 2}(v_i^{\pm,i,s+1}, v_{s+1}^{\pm,i,s+1}) \\
& \times \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_\sigma^{<s+1>}, x_i \pm \varepsilon\nu, v_{s+1}^{\pm,i,s+1}) \frac{\big((v_i - v_{s+1})\cdot\nu\big)_+}{1+|v_i - v_{s+1}|}\, d\nu dv_{s+1}\,.
\end{aligned}
\tag{4.23}
$$

As the collision kernel is bounded and using the energy conservation (4.16), we get

$$
\Psi_\sigma^{1,\pm}(Z_s) \leq M_\beta^{\otimes s}(V_s) \sum_{\substack{i=1\\ i\notin I_{m-1}}}^{s} \psi_m^{\pm}(Z_\sigma^{<s+1>}, z_i)\,,
$$

with

$$
\psi_m^{\pm}(Z_{m-1}, z_i) := \int_0^{+\infty} e^{-\frac{J\tau}{t}} d\tau \int_{\mathbb{S}\times\mathbb{R}^2} \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_{m-1}, x_i \pm \varepsilon\nu, v_i^{\pm,i,s+1}) M_\beta(v_{s+1}) dv_{s+1} d\nu\,.
$$

We follow now the same arguments as in (4.21) to compute the L^1 norm of $\psi_m^{\pm}$. Using first the space translation invariance, then the isometry (4.20) and finally (4.17) and the fact that the transport preserves the Lebesgue measure, we get

$$
\begin{aligned}
\int & dZ_m\, M_\beta^{\otimes m}(V_m) \int_{\mathbb{S}\times\mathbb{R}^2} \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_{m-1}, x_m \pm \varepsilon\nu, v_m^{\pm,m,s+1}) M_\beta(v_{s+1}) dv_{s+1} d\nu \\
&= \int dZ_m\, M_\beta^{\otimes m}(V_m) \int_{\mathbb{S}\times\mathbb{R}^2} \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_{m-1}, x_m, v_m^{\pm,m,s+1}) M_\beta(v_{s+1}) dv_{s+1} d\nu \\
&= \int dZ_m\, M_\beta^{\otimes m}(V_m) \int_{\mathbb{S}\times\mathbb{R}^2} \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_{m-1}, x_m, v_{s+1}) M_\beta(v_{s+1}) dv_{s+1} d\nu \\
&\leq \int dZ_m\, M_\beta^{\otimes m}(V_m) \int \varphi_m(Z_{m-1}, x_m, v_{s+1}) M_\beta(v_{s+1}) dv_{s+1} d\nu \lesssim \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,.
\end{aligned}
$$

Finally the time integral leads to

$$
\|\psi_m^{\pm}\|_{L^1_\beta(\mathbb{D}^m)} \lesssim \frac{t}{J} \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,.
$$

Note that $\psi_m^{\pm}(Z_{m-1}, z_i)$ is only symmetric over the variables Z_{m-1} and not as a function on $\mathbb{D}^m$. However the function

$$
Z_s \to \sum_{\sigma' \in \mathfrak{S}_s^{m-1}} \sum_{i\notin\sigma'} \psi_m^{\pm}(Z_{\sigma'}, z_i)
$$

is symmetric. Thus one can check that

$$
\sum_{\sigma' \in \mathfrak{S}_s^{m-1}} \sum_{i\notin\sigma'} \psi_m^{\pm}(Z_{\sigma'}, z_i) \leq m \sum_{\sigma\in\mathfrak{S}_s^m} \widehat{\psi}_m^{\pm}(Z_\sigma)\,,
$$

where $\widehat{\psi}_m^{\pm}$ is the symmetric version of $\psi_m^{\pm}$.

Finally, the function $\Phi_m^{3,\pm}(Z_m) := m\widehat{\psi}_m^{\pm}(Z_m)$ provides an upper bound for (4.23)

$$
\sum_{\substack{\sigma\in\mathfrak{S}_{s+1}^m\\ s+1\in\sigma}} \Psi_\sigma^{1,\pm}(Z_s) \leq \sum_{\sigma\in\mathfrak{S}_s^m} \Phi_m^{3,\pm}(Z_\sigma)
$$

with

$$
\|\Phi_m^{3,\pm}\|_{L^1_\beta(\mathbb{D}^m)} \lesssim \frac{m}{J} t \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)} \lesssim t\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,.
\tag{4.24}
$$

This defines the third contribution to $\Phi_m := \Phi_m^1 + \Phi_m^{2,\pm} + \Phi_m^{3,\pm}$. Thus the upper bound (4.14) on the L^1-norm of Φ_m follows from the estimates (4.18), (4.22) and (4.24).

• It remains to understand what happens when the collision involves two particles in σ, i.e. when $i, s+1 \in \sigma$. This is a more delicate situation, as we need to take a trace on the function φ_m. The transport operator will be the key to using nevertheless an L^1 bound on φ_m. We set

$$\begin{aligned}\Psi_\sigma^{2,\pm}(Z_\sigma^{<s+1>}) &:= \sum_{i \in I_{m-1}} M_\beta^{\otimes(s-1)}(V_s^{<i>}) \int_0^{+\infty} d\tau e^{-\frac{J\tau}{t}} \int_{\mathbb{S}\times\mathbb{R}^2} M_\beta^{\otimes 2}(v_i^{\pm,i,s+1}, v_{s+1}^{\pm,i,s+1}) \\ &\qquad \times \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_\sigma^{<i,s+1>}, x_i, v_i^{\pm,i,s+1}, x_i \pm \varepsilon\nu, v_{s+1}^{\pm,i,s+1}) \frac{\big((v_i - v_{s+1})\cdot\nu\big)_+}{1+|v_i - v_{s+1}|}\, d\nu dv_{s+1} \\ &\leq M_\beta^{\otimes s}(V_s)\Phi_{m-1}(Z_\sigma^{<s+1>})\,, \end{aligned} \tag{4.25}$$

where

$$\Phi_{m-1}(Z_{m-1}) := \sum_{i=1}^{m-1} \psi_{m-1}^{i,\pm}(Z_{m-1})\,,$$

with

$$\begin{aligned}\psi_{m-1}^{i,\pm}(Z_{m-1}) := \int_0^{+\infty} d\tau \int_{\mathbb{S}\times\mathbb{R}^2} d\nu dv_m\, M_\beta(v_m)\big((v_i - v_m)\cdot\nu\big)_+ \\ \times \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_{m-1}^{<i>}, x_i, v_i^{\pm,i,m}, x_i \pm \varepsilon\nu, v_m^{\pm,i,m})\,.\end{aligned}$$

The function Φ_{m-1} is symmetric but not the functions $\psi_{m-1}^{i,\pm}$. The inequality (4.25) comes from the fact that the denominator $(1 + |v_i - v_m|)$ has been removed and the exponential factor $e^{-\frac{J\tau}{t}}$ bounded by 1. As we shall see, the time integral is still converging thanks to the cut-off on the transport operator $\widehat{\mathbf{S}}_m^0$.

We compute now the L^1_β-norm of Φ_{m-1}. Since the scattering transform

$$(v_i, v_m, \nu) \mapsto (v_i', v_m', \nu)$$

is bijective and has unit Jacobian, it is enough to study the simple case

$$\begin{aligned}\psi_{m-1}^{i,+}(Z_{m-1}) = \int_0^{+\infty} d\tau \int_{\mathbb{S}\times\mathbb{R}^2} d\nu dv_m\, M_\beta(v_m)\big((v_i - v_m)\cdot\nu\big)_+ \\ \times \widehat{\mathbf{S}}_m^0(\tau)\varphi_m(Z_{m-1}^{<i>}, x_i, v_i, x_i \pm \varepsilon\nu, v_m)\,,\end{aligned} \tag{4.26}$$

where we have used again the conservation of energy. Define the maximal subset $\mathcal{S}^{i,m}$ of the space $\mathbb{D}^{m-1} \times \mathbb{S} \times \mathbb{R}^2 \times \mathbb{R}$ such that for any initial data $(Z_{m-1}, x_i + \varepsilon\nu, v_m)$ in $\mathcal{S}^{i,m}$ no recollision takes place in the time interval $[0, \tau]$. On the domain $\mathcal{S}^{i,m}$, the map

$$\begin{aligned}\Gamma^{i,m} : \qquad \mathcal{S}^{i,m} \quad &\mapsto \quad \mathbb{D}^m \\ (Z_{m-1}, \nu, v_m, \tau) \quad &\mapsto \quad \mathbf{\Psi}(-\tau)(Z_{m-1}, x_i + \varepsilon\nu, v_m)\end{aligned} \tag{4.27}$$

is injective. This would not be true for the transport map without the restriction to $\mathcal{S}^{i,m}$ due to the periodic structure of $\mathbb{D}^m$. However, for any Z_m in the range $\mathcal{R}^{i,m}$ of the map $\Gamma^{i,m}$, the time τ is uniquely determined as the first collision time in the flow starting from Z_m. This collision will take place between i and m because the possibility of any other collision has been excluded. All the other parameters can be determined from $\mathbf{\Psi}(\tau)(Z_m)$.

Given $j \in \{1, \dots, m\} \setminus \{i\}$, we denote by $\omega^{j,m}$ the permutation which swaps the coordinates z_j, z_m of Z_m. Then $\Gamma^{i,j} = \omega^{j,m} \circ \Gamma^{i,m}$. These maps are of the same nature, however the ranges $\mathcal{R}^{i,j}$, $\mathcal{R}^{i',j'}$ are disjoint as soon as $\{i,j\} \neq \{i',j'\}$. Indeed for any configuration Z_m

in $\bigcup_{j\neq i}\mathcal{R}^{i,j}$, one can recover the associated map, as the first collision in the flow starting from Z_m will take place between i and j. Once again this is possible because we considered the truncated transport dynamics associated with the flow $\widehat{\mathbf{S}}^0$. The last important feature is that the change of variables $\Gamma^{i,m}$ maps the measure $\big((v_i-v_m)\cdot\nu\big)_+\,\varepsilon d\nu dv_m d\tau dZ_{m-1}$ to dZ_m. Thus we can rewrite (4.26) as

$$\begin{aligned}
\|\Phi_{m-1}\|_{L^1_\beta(\mathbb{D}^{m-1})} &= \sum_{i=1}^{m-1}\|\psi^{i,\pm}_{m-1}\|_{L^1_\beta(\mathbb{D}^{m-1})}\\
&= \sum_{i=1}^{m-1}\int_{\mathcal{S}^{i,m}} dZ_{m-1}d\tau d\nu dv_m\, M_\beta^{\otimes(m)}(V_m)\big((v_i-v_m)\cdot\nu\big)_+\\
&\qquad\qquad \times\varphi_m\Big(\Gamma^{i,m}(Z^{<i>}_{m-1},x_i,v_i,x_i\pm\varepsilon\nu,v_m,\tau)\Big)\\
&=\frac{1}{\varepsilon}\sum_{i=1}^{m-1}\int_{\mathcal{R}^{i,m}} dZ_m M_\beta^{\otimes m}(V_m)\varphi_m\big(Z_m\big)\\
&=\frac{1}{\varepsilon}\sum_{i=1}^{m-1}\frac{1}{m-1}\sum_{j\neq i}\int_{\mathcal{R}^{i,j}} dZ_m M_\beta^{\otimes m}(V_m)\varphi_m\big(Z_m\big)\\
&\leq \frac{1}{\varepsilon}\frac{2}{m-1}\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,,
\end{aligned}$$

where we used that the sets $\{\mathcal{R}^{i,j}\}$ cover at most twice $\mathbb{D}^m$.

Finally we notice that $\Phi^m_m=0$ because there is no loss in the number of particles only if one of the particles z_i and z_m corresponding to the collision integral is not part of the variables of Φ_m, which is impossible since it is defined on $\mathbb{D}^m$. Similarly $\Phi^1_0=0$ because there is a loss in the number of variables only if the two variables of the collision kernel are part of the variables of the function considered, which is impossible if the function only depends on one variable.

This completes the bound (4.15) and ends the proof of Lemma 4.7. □

4.2.3. *Iterated L^1 continuity estimates.* To evaluate the norm of $|Q^{b,0}_{1,J}|(t)$ and prove Proposition 4.6, we use recursively Lemma 4.7.

End of the proof of Proposition 4.6. The quantity to be controlled is of the form

$$\begin{aligned}
&\int_{\mathbb{D}} dz\,|Q^{b,0}_{1,J}|(t)M_\beta^{\otimes J}\varphi_{m,\sigma}(z)\\
&=\alpha^{J-1}\int_{\mathbb{D}} dz\int_0^t\int_0^{t_2}\dots\int_0^{t_{J-1}} dt_J\dots dt_2\widehat{\mathbf{S}}^0_1(t-t_2)|C^b_{1,2}|\widehat{\mathbf{S}}^0_2(t_2-t_3)|C^b_{2,3}|\dots\widehat{\mathbf{S}}^0_J(t_J)M_\beta^{\otimes J}\varphi_{m,\sigma}(z)\\
&=\alpha^{J-1}\int_{\mathbb{D}} dz\int_0^t\int_0^{t_2}\dots\int_0^{t_{J-1}} dt_J\dots dt_2|C^b_{1,2}|\widehat{\mathbf{S}}^0_2(t_2-t_3)|C^b_{2,3}|\dots\widehat{\mathbf{S}}^0_J(t_J)M_\beta^{\otimes J}\varphi_{m,\sigma}(z)\,.
\end{aligned}$$

Rewriting the time integrals in terms of the time increments $\tau_i=t_i-t_{i+1}$ with the constraint $\tau_2+\dots+\tau_J\leq t$, we get

$$\begin{aligned}
&\int_{\mathbb{D}} dz\,|Q^{b,0}_{1,J}|(t)M_\beta^{\otimes J}\varphi_{m,\sigma}(z)\\
&=\alpha^{J-1}\int_{\mathbb{D}} dz\int_0^\infty\int_0^\infty\dots\int_0^\infty d\tau_J\dots d\tau_2\mathbf{1}_{\{\tau_2+\dots+\tau_J\leq t\}}|C^b_{1,2}|\widehat{\mathbf{S}}^0_2(\tau_2)|C^b_{2,3}|\dots\widehat{\mathbf{S}}^0_J(\tau_J)M_\beta^{\otimes J}\varphi_{m,\sigma}(z)\,.
\end{aligned}$$

This constraint can be removed by using the inequality

$$\mathbf{1}_{\{\tau_2+\dots+\tau_J\leq t\}} \leq \exp\Big(J\big(1-\frac{\tau_2+\dots+\tau_J}{t}\big)\Big)$$

which allows to decouple the time integrals and to deal with the elementary operators

$$\int_0^{+\infty} e^{-J\frac{\tau_{s+1}}{t}}|C^b_{s,s+1}|S_{s+1}(\tau_{s+1})d\tau_{s+1}$$

separately. A factor e^J is lost in this decoupling procedure.

We proceed now by applying $J-1$ times the estimates of Lemma 4.7. One iteration transforms a symmetric sum of functions φ_ℓ depending on ℓ variables into similar sum of functions $\Phi^{(\ell)}_\ell, \Phi^{(\ell)}_{\ell-1}$ depending on ℓ or $\ell-1$ variables with the following exceptions

- $\Phi^{(\ell)}_\ell = 0$ if $\ell = s+1$,
- $\Phi^{(\ell)}_{\ell-1} = 0$ if $\ell = 1$.

We recall the bounds (4.14) and (4.15)

$$\|\Phi^{(\ell)}_\ell\|_{L^1_\beta(\mathbb{D}^\ell)} \leq Ct\|\varphi_\ell\|_{L^1_\beta(\mathbb{D}^\ell)}\,, \qquad \|\Phi^{(\ell)}_{\ell-1}\|_{L^1_\beta(\mathbb{D}^{\ell-1})} \leq \frac{C}{\varepsilon(\ell-1)}\|\varphi_\ell\|_{L^1_\beta(\mathbb{D}^\ell)}\,.$$

As the number of variables has to be dropped exactly by $m-1$, the $J-1$ iterations will lead to a sum of $C^{m-1}_{J-1} \leq 2^J$ terms. We therefore end up with

$$\int_{\mathbb{D}} dz\, |Q^{b,0}_{1,J}|(t)\Big(\sum_{\sigma\in\mathfrak{S}^m_J} M_\beta^{\otimes J}\varphi_{m,\sigma}\Big)(z) \leq (C\alpha)^{J-1}\, t^{J-m}\frac{1}{\varepsilon^{m-1}(m-1)!}\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,,$$

which is the expected estimate (bounding t^{J-m} by t^{J-1} and changing the constant C). □

4.3. **Proof of Proposition 4.1.** This Proposition is a straightforward consequence of Propositions 4.2 and 4.4. We have only to sum over all elementary contributions.

- Fix k, $j_i < n_i$ for each $i \leq k-1$ and $j_k \geq n_k$.

By relaxing the conditions on the distribution of times to retain only the constraint on the time increments

$$\tau_2 + \dots + \tau_{J_{k-1}} \leq (k-1)h \leq t\,,$$
$$\tau_{J_{k-1}+1} + \dots + \tau_{J_k} \leq h\,,$$

it is enough to consider the upper bound

$$|Q^0_{1,J_1}|(h)\dots|Q^0_{J_{k-1},J_k}|(h) \leq |Q^0_{1,J_{k-1}}|(t)\,|Q^0_{J_{k-1}+1,J_k}|(h)\,.$$

From the uniform L^2 estimates (4.4) following from Proposition 4.2 and Stirling's formula, we deduce that

$$\|g^m_N(t-kh)\|^2_{L^2_\beta(\mathbb{D}^m)} \leq \frac{CN\exp(C\alpha^2)}{C^m_N} \leq \frac{C^m\, m!\exp(C\alpha^2)}{N^{m-1}}\,.$$

Then, by Proposition 4.4, we conclude that

$$\Big(\int |Q^0_{1,J_1}|(h)\dots|Q^0_{J_{k-1},J_k}|(h)\sum_{\sigma\in\mathfrak{S}^m_{J_k}} M_\beta^{\otimes J_k}\mathbf{1}_{\mathcal{V}_{J_K}}\big|g^m_{N,\sigma}(t-kh)\big|^2dz_1\Big)^{\frac12}$$
$$\leq (C\alpha)^{J_k}\exp(C\alpha^2)t^{J_{k-1}+j_k/2}h^{j_k/2}\,,$$

with the notation $g^m_{N,\sigma}(t',Z_{J_k}) = g^m_N(t',Z_\sigma)$. We then sum over all $m\in\{1,\dots,J_k\}$ to get

$$\Big(\int\Big(|Q^0_{1,J_1}|(h)\dots|Q^0_{J_{k-1},J_k}|(h)\,|f^{(J_K)}_N(t-kh)|\mathbf{1}_{\mathcal{V}_{J_K}}\Big)^2dz_1\Big)^{\frac12} \leq (C\alpha)^{J_k}\exp(C\alpha^2)t^{J_{k-1}+\frac{j_k}{2}}h^{\frac{j_k}{2}}\,.$$

• For γ small, the scaling assumption (4.1) implies in particular that $\alpha^2 th \ll 1$, recalling that $t \geq 1$. Thus summing over all $j_k \geq n_k$ leads to

$$
\begin{aligned}
\sum_{j_k \geq n_k} \Big(\int \big(|Q^0_{1,J_1}|(h) \dots |Q^0_{J_{k-1},J_k}|(h)\, |f_N^{(J_K)}(t-kh)| \mathbf{1}_{\mathcal{V}_{J_K}}\big)^2 dz_1 \Big)^{\frac12} \\
\leq \exp(C\alpha^2)(C\alpha)^{J_{k-1}+n_k} t^{J_{k-1}+n_k/2} h^{n_k/2} \\
\leq \exp(C\alpha^2)(C\alpha)^{3n_k} t^{\frac32 n_k} h^{\frac12 n_k},
\end{aligned} \tag{4.28}
$$

where we used that $J_{k-1} \leq n_k$ as $j_\ell \leq n_\ell = 2^\ell n_0$.

Taking the sum over all possible j_i as in (3.18), we get $C^k 2^{k^2}$ such terms. From the scaling assumption (4.1) and the fact that $\alpha \geq 1$, one can choose $h \leq \gamma^2/8C\exp(C\alpha^2)\alpha^6 t^3$. This implies that

$$
\Big(\int_{\mathbb{D}} dz_1 \big| R_N^{K,0}(t,z_1) \big|^2 \Big)^{\frac12} \leq e^{C\alpha^2} \sum_{k=1}^K 2^{k^2} \big(C\alpha^6 t^3 h \big)^{\frac12 n_k} \leq \gamma\,, \tag{4.29}
$$

and Proposition 4.1 follows. □

4.4. **Super exponential branching for the Boltzmann pseudo-dynamics.** It remains then to estimate the contribution of the super-exponential branching collision trees in the Boltzmann pseudo-dynamics

$$
\bar R_N^K(t) = \sum_{k=1}^K \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} \sum_{j_k \geq n_k} \bar Q_{1,J_1}(h) \dots \bar Q_{J_{k-1},J_k}(h) \big(f^{(J_K)}(t-kh) \mathbf{1}_{\mathcal{V}_{J_K}} \big)\,.
$$

We can state a result analogous to Proposition 4.1

Proposition 4.8. *Given $T > 1$, $\gamma \ll 1$ and C a large enough constant (independent of γ and T), the parameters are tuned as follows*

$$
h \leq \frac{\gamma^2}{C\alpha^6 T^3}\,, \qquad n_k = 2^k n_0\,. \tag{4.30}
$$

Then, we have for $t \in [0,T]$

$$
\Big\| \bar R_N^K(t) \Big\|_{L^2(\mathbb{D})} \leq \gamma\,. \tag{4.31}
$$

Proof. At this stage, the constraint $\mathcal{V}_{J_K}$ is purely cosmetic and it can be removed. We use the fact that the solution (1.13) of the Boltzmann hierarchy is explicit

$$
f^{(s)}(t,Z_s) = M_\beta^{\otimes s}(V_s) \sum_{i=1}^s g_\alpha(t,z_i)\,,
$$

where g_α solves the linear Boltzmann equation (1.14) and is smooth. In particular, the weighted L^2 norm is a Lyapunov functional for the linearized Boltzmann equation, so

$$
\forall t \geq 0, \qquad \int M_\beta g_\alpha^2(t,z) dz \leq \int M_\beta g_{\alpha,0}^2(z) dz\,. \tag{4.32}
$$

The collision operators are decomposed into $\bar C^{b,\pm}_{s,s+1}$ and $\bar C^{q,\pm}_{s,s+1}$ as in (4.9). Then, following the same arguments as in the proof of Lemma 4.7 (case 1), we get for any continuous function φ in $L^1_\beta(\mathbb{D})$

$$
\bar C^{b,\pm}_{s,s+1} M_\beta^{\otimes(s+1)} \sum_{i=1}^{s+1} \varphi(z_i) = s M_\beta^{\otimes s} \sum_{i=1}^s \tilde\varphi(z_i)
$$

where

$$\int M_\beta \tilde\varphi(z)dz \leq C \int M_\beta \varphi(z) dz\,.$$

By iteration and integration with respect to time which leads to a factor $t^{J-1}/(J-1)!$, we deduce that

$$\int dz_1 |\bar Q^b_{1,J}|(t)\Big(M_\beta^{\otimes J}\sum_{i=1}^J \varphi(z_i)\Big) \leq (C\alpha t)^{J-1}\int M_\beta\varphi(z)dz\,.$$

The previous estimate can be applied to the explicit form of the Boltzmann hierarchy. Combining this upper bound with Lanford's estimate for $|\bar Q^q_{1,J}|(t)\,|\bar Q^q_{J,J+n}|(h)M_\beta^{\otimes(J+n)}$, we get by the Cauchy-Schwarz inequality as in (4.10)

$$\begin{aligned}\Big\|\bar Q_{1,J_1}(h)\dots \bar Q_{J_{k-1},J_k}(h)\mathbf{1}_{\mathcal V_{J_K}} f^{(J_K)}(t-(k-1)h)\Big\|_{L^2(\mathbb D)}\\ \leq (C\alpha t)^{J_k-1}(C\alpha h)^{j_k/2}\left(\int M_\beta g_\alpha^2(t-(k-1)h,z)dz\right)^{1/2}\\ \leq (C\alpha t)^{J_{k-1}+j_k-1}(C\alpha h)^{j_k/2}\|g_{\alpha,0}\|_{L^2_\beta(\mathbb D)}\,,\end{aligned}$$

where we used (4.32) in the last inequality.

We proceed as in (4.28), (4.29) and sum over $j_k\geq n_k$, $j_i<n_i$ for $i\leq k-1$,

$$\Big\|\bar R^K(t)\Big\|_{L^2(\mathbb D)} \leq \sum_{k=1}^K 2^{k^2}\big(C\alpha^6t^3h\big)^{\frac12 n_k}\|g_{\alpha,0}\|_{L^2_\beta(\mathbb D)}\leq\gamma\,,$$

where the last inequality follows from the condition $h\leq \gamma^2/(8C\alpha^6t^3)$. This completes the proof of Proposition 4.8. □

5. Control of super exponential trees with one recollision

In this section, we show how to modify the proof of Proposition 4.1 to take into account a finite number of recollisions (actually one here, but the argument could easily be extended to an arbitrary, finite number), and prove the following estimate for $R_N^{K,1}$.

Proposition 5.1. *Under the Boltzmann-Grad scaling $N\varepsilon=\alpha\gg1$ and with the previous notation, we have for $T>1$ and all $t\in[0,T]$, assuming $\alpha^2Th\ll1$,*

$$\Big\|R_N^{K,1}(t)\Big\|_{L^2(\mathbb D)}\leq \exp(C\alpha^2)(C\alpha T)^{2^{K+1}n_0}\,\frac{\varepsilon^{1/2}|\log\varepsilon|^6}{h}\,.$$

Given a function g_N^m, let us call distinguished the particles which are in the argument of g_N^m and the others are the background particles. Proposition 4.6 cannot be applied as a black box: indeed, the structure (4.3) is not exactly preserved by the transport operator at the time of recollision if there is scattering between one distinguished particle and one particle of the background. We have therefore to extend Lemma 4.7 to incorporate the case of one recollision. The point is to modify locally the decomposition (4.3) to ensure that the recollision will always involve either two particles of the background or two distinguished particles, in which case it is easy to adapt the proof of Proposition 4.6 .

5.1. **Extension of Lemma 4.7 to the case of one recollision.** Note that in the pseudo dynamics describing the operator

$$|C^{b,\pm}_{s,s+1}|\widehat{\mathbf{S}}^1_{s+1}(\tau)$$

there is exactly one collision occurring at the initial time and the particles evolve in straight lines with the exception of the two recolliding particles.

Lemma 5.2. *Fix $t>0$, $1\leq m\leq s+1$ and let φ_m be a nonnegative symmetric function in $L^1_\beta(\mathbb{D}^m)$. Then there are three symmetric functions $\Phi^{(m)}_m$, $\Phi^{(m)}_{m-1}$ and $\Phi^{(m)}_{m+1}$ defined respectively on $\mathbb{D}^m$, $\mathbb{D}^{m-1}$ and $\mathbb{D}^{m+1}$ such that*

$$\begin{aligned}\int_0^t d\tau\, e^{-\frac{J\tau}{t}}|C^{b,\pm}_{s,s+1}|\widehat{\mathbf{S}}^1_{s+1}(\tau)\Big(M_\beta^{\otimes(s+1)}\mathbf{1}_{\mathcal{V}_{s+1}}\sum_{\sigma\in\mathfrak{S}^m_{s+1}}\varphi_{m,\sigma}\Big)\\ \leq M_\beta^{\otimes s}(V_s)\Big(\sum_{\sigma\in\mathfrak{S}^m_s}\Phi^{(m)}_{m,\sigma}+\sum_{\sigma\in\mathfrak{S}^{m-1}_s}\Phi^{(m)}_{m-1,\sigma}+\sum_{\sigma\in\mathfrak{S}^{m+1}_s}\Phi^{(m)}_{m+1,\sigma}\Big)\,,\end{aligned}$$

where $\mathcal{V}_{s+1}$ was introduced in (2.21). *Furthermore, they satisfy*

$$\|\Phi^{(m)}_m\|_{L^1_\beta(\mathbb{D}^m)}\leq Cs^2t\,\big|\log\varepsilon\big|\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\tag{5.1}$$

$$\|\Phi^{(m)}_{m-1}\|_{L^1_\beta(\mathbb{D}^{m-1})}\leq\frac{C}{\varepsilon(m-1)}\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\tag{5.2}$$

$$\|\Phi^{(m)}_{m+1}\|_{L^1_\beta(\mathbb{D}^{m+1})}\leq Cs^3t\varepsilon\,\big|\log\varepsilon\big|\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\tag{5.3}$$

with $\Phi^{(1)}_0=\Phi^{(s)}_{s+1}=\Phi^{(s+1)}_{s+1}=\Phi^{(s+1)}_{s+2}=0$.

Unlike Lemma 4.7 which is iterated, the previous lemma will be used only once, thus there is no need to establish sharp bounds.

Proof. To simplify notation we drop the superscript (m) in the proof. We follow the main steps of the proof of Lemma 4.7.

Step 1. *Localization of the transport operators.*

Let us first fix (i,j) the pair of recolliding particles and denote by $\widehat{\mathbf{S}}^{1,(i,j)}_{s+1}(\tau)$ the corresponding transport operator. For a given $\sigma\in\mathfrak{S}^m_{s+1}$, we have to distinguish two cases.

Case 1. (i,j) belongs to σ or σ^c.

If $i,j\notin\sigma$, we have

$$\widehat{\mathbf{S}}^{1,(i,j)}_{s+1}(\tau)\,\mathbf{1}_{\mathcal{V}_{s+1}}M_\beta^{\otimes(s+1)}\varphi_m(Z_\sigma)\leq M_\beta^{\otimes(s+1)}\widehat{\mathbf{S}}^0_m(\tau)\varphi_m(Z_\sigma)\,,\tag{5.4}$$

where the transport $\widehat{\mathbf{S}}^0_m$ acts only on the m particles in σ. The distribution is therefore unchanged.

If $i,j\in\sigma$, we have

$$\widehat{\mathbf{S}}^{1,(i,j)}_{s+1}(\tau)M_\beta^{\otimes(s+1)}\mathbf{1}_{\mathcal{V}_{s+1}}\varphi_m(Z_\sigma)\leq M_\beta^{\otimes(s+1)}\widehat{\mathbf{S}}^1_m(\tau)\varphi_m(Z_\sigma)\,.\tag{5.5}$$

In this case then the recollision involves two distinguished particles, so the distribution is modified by the scattering. However since the scattering preserves the measure $dvdv_1d\nu$, both the L^∞ and L^1 norms will be unchanged. Note that in both cases the velocity cut-off has been neglected.

Compared to the previous section, there is however one issue: if there is no recollision, then a point of the phase space cannot be in the image $\mathbf{S}^0_m(\tau)(\partial\mathcal{D}^{m,\pm}_\varepsilon)(i,j)$ for two different pairs (i,j), and that fact was the key argument to get the suitable L^1 estimate for $\Phi^{(m-1)}_m$ previously (without loosing a factor m^2). In the current situation as there is exactly one recollision, for any point in $\mathcal{D}^s_\varepsilon$ there exists a unique parametrization by one point of the boundary $\mathcal{D}^s_\varepsilon$ and one time. It is obtained by using the backward flow, going through the first collision (which is the recollision) and reaching another point of the boundary with a different (longer) time.

So in the end in both cases the analysis is exactly like the one performed in the previous section.

Case 2. i belongs to σ^c and j to σ (or the symmetric situation).
Note first that this situation can only occur when $m < s+1$.

The recollision in the transport $\widehat{\mathbf{S}}^{1,(i,j)}_{s+1}(\tau)$ induces a correlation between the particles z_i, z_j so the structure with m distinguished particles and $s+1-m$ particles at equilibrium is not stable anymore. The idea is then to add particle i to the set of distinguished particles. But in order to keep some of the structure, we then need to gain additional smallness (since $\|\varphi_m\|_{L^1_\beta}$ is expected to decay roughly as ε^{m-1}, adding a variable requires gaining a power of ε).

For any $\tau \le t$, a configuration Z_{s+1} obtained by backward transport $\widehat{\mathbf{S}}^{1,(i,j)}_{s+1}(\tau)$ will belong to the set

$$\mathcal{P}_{(i,j)} := \left\{ Z_{s+1} \in \mathbb{D}^{s+1} \;\Big|\; \exists\, u \le t,\;\; d(x_i + uv_i, x_j + uv_j) \le \varepsilon \right\}, \tag{5.6}$$

where d denotes the distance on the torus. Note that this set does not depend on $\tau \le t$. We then define a new function with $m+1$ variables which will encompass the constraint on the recollision

$$\psi^{i,j}_{m+1,\sigma^{<j>}}(Z_\sigma, z_i) := \varphi_m(Z_\sigma) \mathbf{1}_{\mathcal{P}_{(i,j)}}(z_j, z_i)\, \mathbf{1}_{\mathcal{V}_{m+1}}(Z_\sigma, z_i)\,, \tag{5.7}$$

with a velocity cut-off acting on the $m+1$ variables.

We are going to check that

$$\|\psi^{i,j}_{m+1,\sigma^{<j>}}\|_{L^1_\beta(\mathbb{D}^{m+1})} \le Ct\varepsilon\, \big|\log\varepsilon\big| \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,. \tag{5.8}$$

Thus the extra factor $\varepsilon\big|\log\varepsilon\big|$ will compensate partly the factor $1/\varepsilon$ corresponding to the shift from m to $m+1$. To prove (5.8), we first freeze the coordinates Z_σ. Integrating first over z_i, we recover the factor $Ct\varepsilon\,\big|\log\varepsilon\big|$ from the constraint $\mathcal{P}_{(i,j)}$ (as all energies are bounded by $C_0|\log\varepsilon|$), and then (5.8) after integrating over the other coordinates.

The transport operator can be localized on $m+1$ variables

$$\begin{aligned}\widehat{\mathbf{S}}^{1,(i,j)}_{s+1}(\tau)\, \mathbf{1}_{\mathcal{V}_{s+1}}\, M_\beta^{\otimes(s+1)} \varphi_m(Z_\sigma) &\le \widehat{\mathbf{S}}^{1,(i,j)}_{s+1}(\tau)\, M_\beta^{\otimes(s+1)} \varphi_m(Z_\sigma) \mathbf{1}_{\mathcal{P}_{(i,j)}}(z_j, z_i)\, \mathbf{1}_{\mathcal{V}_{m+1}}\\ &\le M_\beta^{\otimes(s+1)} \widehat{\mathbf{S}}^1_{m+1}(\tau) \psi^{i,j}_{m+1,\sigma^{<j>}}(Z_\sigma, z_i)\,,\end{aligned}$$

where we used that $\varphi_m \ge 0$.

The function (5.7) is not symmetric with respect to the i and j variables. Thus to recover the symmetry, we bound it from above by

$$\psi_{m+1}(Z_{m+1}) := \sum_{\substack{k,\ell \le m+1\\ k \ne \ell}} \varphi_m(Z^{<k>}_{m+1})\, \mathbf{1}_{\mathcal{P}_{(k,\ell)}}(z_k, z_\ell) \mathbf{1}_{\mathcal{V}_{m+1}}\,.$$

In this way, a factor s^2 has been lost compared to (5.8)

$$\|\psi_{m+1}\|_{L^1_\beta(\mathbb{D}^{m+1})} \le Cts^2\varepsilon\, \big|\log\varepsilon\big| \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,. \tag{5.9}$$

Finally, we can write

$$\widehat{\mathbf{S}}^{1,(i,j)}_{s+1}(\tau)\, \mathbf{1}_{\mathcal{V}_{s+1}}\, M_\beta^{\otimes(s+1)} \varphi_m(Z_\sigma) \le M_\beta^{\otimes(s+1)} \widehat{\mathbf{S}}^1_{m+1}(\tau)\, \psi_{m+1}(Z_\sigma, z_i). \tag{5.10}$$

Step 2. *Reduction to the estimates of Lemma* 4.7.

Using the estimates (5.4), (5.5) and (5.10), we get

$$
\begin{aligned}
|C^{b,\pm}_{s,s+1}|\widehat{\mathbf{S}}^1_{s+1}(\tau)\Big(M_\beta^{\otimes(s+1)}\,\mathbf{1}_{\mathcal{V}_{s+1}}\sum_{\sigma\in\mathfrak{S}^m_{s+1}}\varphi_{m,\sigma}\Big)
&\leq |C^{b,\pm}_{s,s+1}|\Big(M_\beta^{\otimes(s+1)}\sum_{\sigma\in\mathfrak{S}^m_{s+1}}(\widehat{\mathbf{S}}^0_m(\tau)+\widehat{\mathbf{S}}^1_m(\tau))\varphi_{m,\sigma}\Big)\\
&\quad+(m+1)|C^{b,\pm}_{s,s+1}|\Big(M_\beta^{\otimes(s+1)}\sum_{\tilde\sigma\in\mathfrak{S}^{m+1}_{s+1}}\widehat{\mathbf{S}}^1_{m+1}(\tau)\psi_{m+1,\tilde\sigma}\Big)\,,
\end{aligned}
$$

where the factor $m+1$ comes from the fact that the same function appears for each different j. The global cut-off on the velocities has been removed and the transport operator localized so that the proof of Lemma 4.7 can be applied. Note that the first term in the right-hand side will contribute to Φ_m and Φ_{m-1}, while the second term will contribute to Φ_{m+1} and Φ_m. In the latter case, an argument of the function ψ_{m+1} is dropped and the factor $1/\varepsilon$ is compensated (up to a logarithmic loss in ε) thanks to the estimate (5.9). We therefore end up with

$$
\begin{aligned}
\int_0^t d\tau\, e^{-\frac{J\tau}{t}}|C^{b,\pm}_{s,s+1}|\widehat{\mathbf{S}}^1_{s+1}(\tau)\Big(M_\beta^{\otimes(s+1)}\,\mathbf{1}_{\mathcal{V}_{s+1}}\sum_{\sigma\in\mathfrak{S}^m_{s+1}}\varphi_{m,\sigma}\Big)\\
\leq M_\beta^{\otimes s}(V_s)\Big(\sum_{\sigma\in\mathfrak{S}^m_s}\Phi_{m,\sigma}+\sum_{\sigma\in\mathfrak{S}^{m-1}_s}\Phi_{m-1,\sigma}+\sum_{\sigma\in\mathfrak{S}^{m+1}_s}\Phi_{m+1,\sigma}\Big)
\end{aligned}
$$

with

$$
\begin{aligned}
\|\Phi_{m-1}\|_{L^1_\beta(\mathbb{D}^{m-1})}&\leq\frac{C}{\varepsilon(m-1)}\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\\
\|\Phi_m\|_{L^1_\beta(\mathbb{D}^m)}&\leq Cs^2t\,\big|\log\varepsilon\big|\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\\
\|\Phi_{m+1}\|_{L^1_\beta(\mathbb{D}^{m-1})}&\leq Cs^3t\varepsilon\,\big|\log\varepsilon\big|\|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}
\end{aligned}
$$

with $\Phi_0=0$ if $m=1$ and $\Phi_s=\Phi_{s+1}=0$ if $m=s$ or $m=s+1$. This is exactly the expected estimate. □

5.2. **Estimate of $R^{K,1}_N$ (super exponential branching with exactly one recollision).** The proof of Proposition 5.1 follows the same lines as the proof of Proposition 4.1. With the notation (4.9), the iterated collision operators with quadratic and bounded collision kernels are denoted by $|Q^{q,1}_{1,J}|$, $|Q^{b,1}_{1,J}|$. The proof is split into three steps.

Step 1. *Evaluating the norm of $|Q^{b,1}_{1,J}|(t)$ in L^1_β.*

We use recursively Lemma 4.7, together with one iteration of Lemma 5.2. Using as previously the exponential to get rid of the constraint on the time increments, we have to control a quantity of the form

$$
\begin{aligned}
&\int_{\mathbb{D}}dz\,|Q^{b,1}_{1,J}|(t)M_\beta^{\otimes J}\varphi_{m,\sigma}\,\mathbf{1}_{\mathcal{V}_J}\\
&\quad\leq\alpha^{J-1}e^J\sum_{\ell=2}^J\int_{\mathbb{D}}dz\int_0^\infty\int_0^\infty\dots\int_0^\infty d\tau_J\dots d\tau_2e^{-J\frac{\tau_2}{t}}|C^b_{1,2}|\widehat{\mathbf{S}}^0_2(\tau_2)\dots\\
&\qquad\dots e^{-J\frac{\tau_\ell}{t}}\mathbf{1}_{\tau_\ell\leq t}|C^b_{\ell-1,\ell}|\widehat{\mathbf{S}}^1_\ell(\tau_\ell)\dots e^{-J\frac{\tau_J}{t}}\widehat{\mathbf{S}}^0_J(\tau_J)M_\beta^{\otimes J}\varphi_{m,\sigma}\,\mathbf{1}_{\mathcal{V}_J}\\
&\quad\leq\alpha^{J-1}e^J\sum_{\ell=2}^J\int_{\mathbb{D}}dz\int_0^\infty\int_0^\infty\dots\int_0^\infty d\tau_J\dots d\tau_2e^{-J\frac{\tau_2}{t}}|C^b_{1,2}|\widehat{\mathbf{S}}^0_2(\tau_2)\dots\\
&\qquad\dots e^{-J\frac{\tau_\ell}{t}}\mathbf{1}_{\tau_\ell\leq t}|C^b_{\ell-1,\ell}|\widehat{\mathbf{S}}^1_\ell(\tau_\ell)\,\mathbf{1}_{\mathcal{V}_\ell}\dots e^{-J\frac{\tau_J}{t}}\widehat{\mathbf{S}}^0_J(\tau_J)M_\beta^{\otimes J}\varphi_{m,\sigma},
\end{aligned}
$$

where the cut-off on the velocities in the second inequality applies only to the operator with one recollision (by using the fact that the energy is preserved by the transport operators).

We proceed now by applying $J-2$ times the estimates of Lemma 4.7, and once the estimate of Lemma 5.2. When applying Lemma 5.2, the number of variables may shift from m to $m+1$, but for all other iterations we either stay with the same number variables, or shift from m to $m-1$. As the number of variables has to be dropped to 1, the total number of possible combinations is less than 2^J. We therefore end up with

$$\int_{\mathbb{D}} dz\, |Q^{b,1}_{1,J}(t)| \sum_{\sigma\in\mathfrak{S}^m_J} M_\beta^{\otimes J} \varphi_{m,\sigma}(Z_\sigma)\, \mathbf{1}_{\mathcal{V}_J} \leq (C\alpha)^{J-1}\, t^{J-m} \frac{J^3|\log\varepsilon|}{\varepsilon^{m-1} m!} \|\varphi_m\|_{L^1_\beta(\mathbb{D}^m)}\,. \tag{5.11}$$

This estimate is similar to the one of Proposition 4.6 with an extra factor $J^3|\log\varepsilon|$. To compensate this logarithmic divergence, we are going to adapt the L^∞ estimates of Proposition 4.5 in order to gain a factor ε from the recollision.

Step 2. *Evaluating the norm of $|Q^{q,1}|$ in L^∞.*

Noticing that the recollision takes place either in the last time interval or before, we get the decomposition

$$\begin{aligned}(5.12)\quad & |Q^{q,0}_{1,J}|(t)\, |Q^{q,1}_{J,J+n}|(h)\, M_\beta^{\otimes(J+n)} \mathbf{1}_{\mathcal{V}_{J+n}} + |Q^{q,1}_{1,J}|(t)\, |Q^{q,0}_{J,J+n}|(h)\, M_\beta^{\otimes(J+n)} \mathbf{1}_{\mathcal{V}_{J+n}} \\ &\qquad \leq (C\alpha t)^{J-1} (C\alpha h)^{n-2} (J+n)^3 \varepsilon |\log\varepsilon|^{10} M_{5\beta/8}(v_1)\,,\end{aligned}$$

where we used the refined estimate (3.16) and the geometric estimates of Section 3.1 in order to recover the factor ε from the recollision. Combined with (5.11) and a Cauchy-Schwarz estimate as in (4.10), we get

$$\begin{aligned} & \Big\| |Q^1_{1,J}|(t)\, |Q^0_{J,J+n}|(h)\, M_{J+n,\beta}\, \mathbf{1}_{\mathcal{V}_{J+n}} \sum_{\sigma\in\mathfrak{S}^m_J} |g_{m,\sigma}| \Big\|_{L^2(\mathbb{D})} \\ & + \Big\| |Q^0_{1,J}|(t)\, |Q^1_{J,J+n}|(h)\, M_{J+n,\beta}\, \mathbf{1}_{\mathcal{V}_{J+n}} \sum_{\sigma\in\mathfrak{S}^m_J} |g_{m,\sigma}| \Big\|_{L^2(\mathbb{D})} \\ & \leq (C\alpha t)^{J+n/2-1} (C\alpha h)^{n/2-1} (J+n)^{\frac{3}{2}} \varepsilon^{1/2} |\log\varepsilon|^{11/2} \frac{\|g_m\|_{L^2_\beta}}{\sqrt{\varepsilon^{m-1} m!}}\,.\end{aligned} \tag{5.13}$$

The logarithmic loss in ε is compensated by the extra $\varepsilon^{1/2}$ factor from (5.12). Thus, we have obtained a counterpart of Proposition 4.4.

Step 3. *Resummation.*

The last step is then to sum over all possible contributions k, $j_i < n_i$ for $i \leq k-1$, $j_k \geq n_k$, and $m \leq J_k$. Recall from (4.4) that

$$\|g^m_N(t-kh)\|^2_{L^2_\beta} \leq \frac{CN\exp(C\alpha^2)}{C^m_N} \leq \frac{C^m\, m! \exp(C\alpha^2)}{N^{m-1}}\,.$$

Then, by (5.13), we have (rounding off the power of $\log\varepsilon$)

$$\begin{aligned} & \Big\| |Q^0_{1,J_1}|(h) \dots |Q^0_{J_{k-1},J_k}|(h) \sum_{\sigma\in\mathfrak{S}^m_{J_k}} M_\beta^{\otimes J_k}\, \mathbf{1}_{\mathcal{V}_{J_k}} |g^m_N(Z_\sigma)|\, \Big\|_{L^2(\mathbb{D})} \\ & \qquad \leq (C\alpha)^{J_k} \exp(C\alpha^2) t^{J_{k-1}+j_k/2} h^{j_k/2-1} \varepsilon^{1/2} |\log\varepsilon|^6\,.\end{aligned}$$

We then sum over all $m \in \{1,\dots,J_k\}$ to get

$$\begin{aligned} & \Big\| |Q^0_{1,J_1}|(h) \dots |Q^0_{J_{k-1},J_k}|(h)\, |f^{(J_K)}_N(t-kh)|\, \mathbf{1}_{\mathcal{V}_{J_k}} \Big\|_{L^2(\mathbb{D})} \\ & \qquad \leq (C\alpha)^{J_k} \exp(C\alpha^2) t^{J_{k-1}+j_k/2} h^{j_k/2-1} \varepsilon^{1/2} |\log\varepsilon|^6\,.\end{aligned}$$

Provided that $\alpha^2 th \ll 1$, we can first sum over all $j_k \geq n_k$, which leads to

$$\sum_{j_k \geq n_k} \int |Q^0_{1,J_1}|(h) \dots |Q^0_{J_{k-1},J_k}|(h)\, |f_N^{(J_K)}(t-kh)| \mathbf{1}_{\mathcal{V}_{J_k}} dz \leq (C\alpha t)^{J_{k-1}} \exp(C\alpha^2) \frac{\varepsilon^{1/2}}{h} |\log \varepsilon|^6 \,.$$

Taking the sum over all possible $j_i < 2^i n_0$ for $i \leq k-1$, we get $O(2^{k^2})$ such terms. We therefore end up with

$$\left\| R_N^{K,1}(t) \right\|_{L^2(\mathbb{D})} \leq \exp(C\alpha^2) 2^{K^2} (C\alpha T)^{2^{K+1} n_0} \frac{\varepsilon^{1/2} |\log \varepsilon|^6}{h} \,.$$

This concludes the proof of Proposition 5.1, since $2^{K^2} \ll C^{2^K}$. $\square$

6. Control of super-exponential trees with multiple recollisions

Recall that the remainder term R_N^K is a series expansion (2.23) with elementary terms of the form

$$\alpha^{J_k - 1} Q_{1,J_1}(h) \dots Q_{J_{k-2},J_{k-1}}(h) Q_{J_{k-1},J_k}(h) f_N^{(J_k)}(t-kh)\,,$$

which corresponds exactly to collision trees having

- $j_i < n_i$ branching points on the first $k-1$ intervals ($i < k$);
- $j_k \geq n_k$ branching points on the k-th interval;

and that $R_N^{K,>}$ is the restriction of R_N^K to pseudo-dynamics having more than one recollision, with energies bounded by $C_0 |\log \varepsilon|$.

The main result of this section is the following.

Proposition 6.1. *Let $\gamma < 1$ be given. Choose*

$$n_k = n_0 \times 2^k, \qquad h \leq \frac{\gamma}{\exp(C\alpha^2) T^3} \,.$$

Under the Boltzmann-Grad scaling $N\varepsilon = \alpha \gg 1$, there holds for all $t \in [0,T]$

$$\left\| R_N^{K,>}(t) \right\|_{L^2(\mathbb{D})} \leq \gamma \,.$$

The next two paragraphs are devoted to a quantitative estimate showing that dynamics with more than one recollision are unlikely: the statement is given in Paragraph 6.1, and its proof is in Paragraphs 6.2 and 6.3. Finally the proof of Proposition 6.1 appears in Paragraph 6.4.

6.1. Geometric control of multiple recollisions: statement of the result.

Unlike in Section 3, we need very sharp estimates to compensate the divergence of order N of the L^∞ norm (2.12). Thus, we cannot lose any power of $|\log \varepsilon|$ (which come from the bound on the energies) and this will be possible at the cost of estimating the size of trajectories having at least two recollisions rather than one.

The presence of multiple recollisions can be encoded in the domain of integration (collision times, impact parameter and velocity of the additional particles). Proposition 6.2 below is the main result of this section and provides a counterpart to Proposition 3.5 when there are at least two recollisions.

Proposition 6.2. *Given $z_1 \in \mathbb{T}^2 \times B_R$ with $1 \leq R^2 \leq C_0 |\log \varepsilon|$, $1 \leq t \leq C|\log \varepsilon|$, and a collision tree $a \in \mathcal{A}_s$ with $s \geq 2$, consider the set of parameters $(t_n, \nu_n, v_n)_{2 \leq n \leq s}$ in $\mathcal{T}_{2,s} \times \mathbb{S}^{s-1} \times \mathbb{R}^{2(s-1)}$ leading to a pseudo-trajectory with total energy bounded by R^2. Fix i,j,k,ℓ four labels (some of them may coincide) and two integers $\theta \in [\max(i,j), s], \tilde\theta \in [\max(k,\ell), s]$.*

We consider the pseudo-trajectories with at least two recollisions such that the first two recollisions involve labels (i,j) *and* (k,ℓ) *Then there exists a set* $\sigma \subset \{1,\dots,s\}$ *with at most* 6 *elements such that*

$$\int \mathbf{1}_{\textit{first two recollisions between } (i,j) \textit{ and } (k,\ell) \textit{ at } \theta,\tilde\theta}$$

$$\times \prod_{m\in\sigma} \big|(v_m - v_{a(m)}(t_m))\cdot \nu_m\big|\, dt_m d\nu_m dv_m \leq C(Rt)^r \varepsilon\,,$$

for some fixed integer r*, uniformly with respect to all other parameters* $(t_n,\nu_n,v_n)_{\substack{2\leq n\leq s\\ n\notin\sigma}}$ *in the set* $\mathcal{T}_{2,s}\times\mathbb{S}^{s-1}\times B_R^{s-1}$ *and* $z_1\in\mathbb{T}^2\times B_R$.

Remark 6.3. *The exact form of the set* σ *is not useful in the sequel, but the proof of the proposition shows that it may be constructed as follows. Define* $(1^*,2^*,3^*)$ *the first three parents of* (i,j) *at height* $\theta+1$*, and* $(\tilde 1,\tilde 2,\tilde 3)$ *the first three parents of* (k,ℓ) *at height* $\tilde\theta+1$.

- *if* $(1^*,2^*,3^*)$ *and* $(\tilde 1,\tilde 2,\tilde 3)$ *are all distinct then* $\sigma=\{1^*,2^*,3^*,\tilde 1,\tilde 2,\tilde 3\}$.
- *otherwise* σ *is made of the first 6 integers in* $\{1^*,\dots,6^*,\tilde 1,\dots,\tilde 6\}$.

The proof of Proposition 6.2 relies heavily on the computations leading to Proposition 3.5, but the two recollisions may be intertwined so more cases have to be considered. In the next paragraph we identify all possible situations that can lead to the first two recollisions in the dynamics. Paragraph 6.3 deals with each case separately, leaving the technical aspects to Appendix B.

6.2. **Classification of all possible dynamics.** We recall that a single self-recollision was analyzed in the proof of Proposition 3.5, page 18, and its cost is $O(\varepsilon R^5 t^3)$ as shown in (3.6). This is the expected power of ε given in Proposition 6.2, so we shall no longer take that possibility into account in what follows.

In the case of one recollision (recall Proposition 3.5), the key to the proof is to identify two collisions related to that recollision, i.e. two degrees of freedom, for which the constraints due to the recollision lead to a set of small measure. We proceed in the same way here and denote by (i,j) and (k,ℓ) the particles involved in the first two recollisions in the backward dynamics and by t_{rec} and $\tilde t_{rec}$ the corresponding recollision times; note that the labels are not necessarily distinct, and neither are the associate pseudo-particles, using the terminology introduced in Definition 3.3. With the notation of Proposition 6.2, we denote the first parent (starting at height θ) of the recolliding particles (i,j) by 1^*, and by $\tilde 1$ the first parent (starting at height $\tilde\theta$) of the recolliding particles (k,ℓ).

Without loss of generality we may assume that $t_{\tilde 1}\leq t_{1^*}$. To classify the dynamics, we shall consider separately the cases $t_{\tilde 1}<t_{1^*}$ and $t_{\tilde 1}=t_{1^*}$.

6.2.1. *Case 1:* $t_{\tilde 1}<t_{1^*}$. Two different types of situations may occur, depending on whether the recollisions take place "in chain" or not. Let us be more precise.

Parallel recollisions. This means that the two recollisions are not directly related in the sense that the trajectory of none of the particles k and ℓ between time $t_{\tilde 1}$ and $\tilde t_{rec}$ is affected by the recollision between particles i and j on the same time interval (see Figure 5).

Recollisions in chain. This situation is depicted in Figure 6: in this case the trajectory of k or ℓ, during the time interval $[t_{\tilde 1},\tilde t_{rec}]$, is affected by the recollision between i and j.

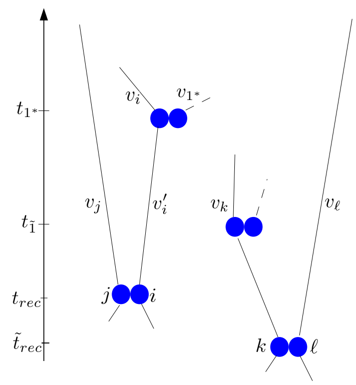

FIGURE 5. $t_{\tilde 1} < t_{1^*}$, parallel recollisions. It may happen that $a(k) = i$ as long as the collision at time $t_{\tilde 1}$ is with no scattering.

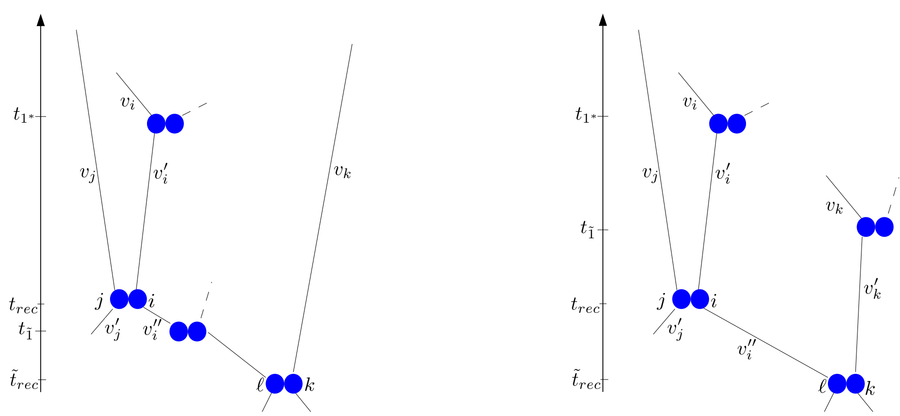

FIGURE 6. $t_{\tilde 1} < t_{1^*}$, recollisions in chain.

6.2.2. *Case 2:* $t_{\tilde 1} = t_{1^*}$. This is a very constrained case, as all the recolliding particles have the same first parent. We separate the analysis into three distinct subcases, in a similar way to Case 1.

Parallel recollisions. This case is depicted in Figure 7; the two recollisions take place with the same parent, but there is no direct link between the two couples of recolliding pseudo-particles (i, j) and (k, ℓ), meaning as previously that the trajectory of ℓ and k between time $t_{\tilde 1} = t_{1^*}$ and $\tilde t_{rec}$ is unaffected by that of i or j on the same time interval.

Recollisions in chain. In this case the two recollisions take place in chain (the trajectory of one of the recolliding particles k or ℓ is affected by i or j between time $t_{\tilde 1} = t_{1^*}$ and $\tilde t_{rec}$), but as opposed to the case of self-recollisions (see below) they do not involve the same two particles (see Figure 8).

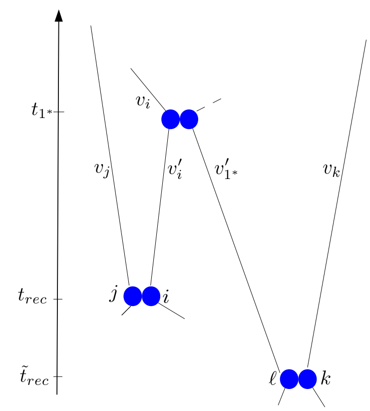

FIGURE 7. $t_{\tilde 1} = t_{1^*}$, parallel recollisions.

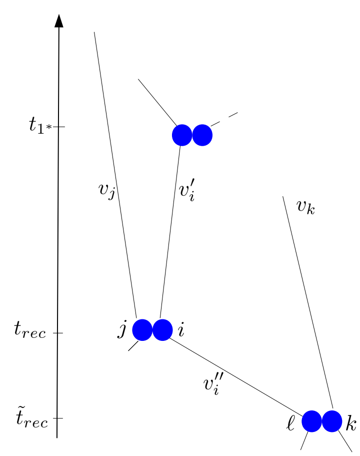

FIGURE 8. $t_{\tilde 1} = t_{1^*}$, recollisions in chain.

<u>Self-recollisions.</u> The most constrained situation is when the double recollision is due to the periodic structure of the spatial domain: it can be that the second recollision involves the same two particles as the first (see Figure 9).

6.3. **Proof of Proposition 6.2.** Let us go through the five situations described in the previous section and study the size of the set of parameters leading to those situations. Note that each case involves different degrees of freedom, i.e. different sets σ. In fact, several sets σ are often needed to cover all the situations leading to the recollisions of a given collection of particles i, j, k, ℓ.

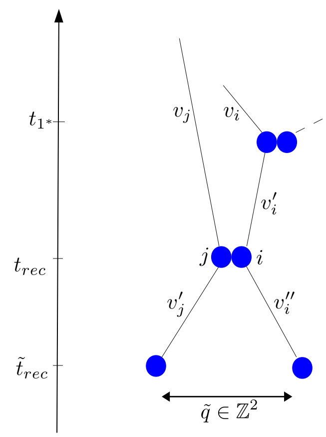

FIGURE 9. $t_{\tilde 1} = t_{1^*}$, the second recollision is due to periodicity.

Case 1, Parallel recollisions. Without loss of generality (up to exchanging the names of the particles) we assume that, with the notation of Figure 5, particles k, ℓ have three parents, called $\tilde 1$, $\tilde 2$ and $\tilde 3$.

- If $|v_k - v_\ell| \leq \varepsilon^{\frac{3}{4}}$, where v_ℓ is the velocity of particle ℓ at time $t_{\tilde 1}$ and v_k is the velocity of particle k at time $t_{\tilde 1}$ just before the collision in the backward dynamics (see Figure 5) then one can apply (C.12) of Lemma C.4 which implies that integrating over $\tilde 2$ and $\tilde 3$ gives a bound $CR^5 t^2 \varepsilon^{\frac{3}{2}} |\log \varepsilon|$ which is a stronger decay than expected. Note that this upper bound has been obtained without using the first recollision.

- If $|v_k - v_\ell| \geq \varepsilon^{\frac{3}{4}}$ then integrating over $dt_{\tilde 1} dv_{\tilde 1} d\nu_{\tilde 1}$ the constraint of having the second recollision gives, according to (3.10), the bound

$$CR^5 t^2 \varepsilon \frac{|\log \varepsilon|^2}{|v_k - v_\ell|} \leq CR^5 t^2 \varepsilon^{\frac{1}{4}} |\log \varepsilon|^2 \,.$$

Then we apply Proposition 3.5 to recollision (i, j) which gives a bound $CR^7 t^3 \varepsilon |\log \varepsilon|^3$ after integration over 1^* and 2^*. We therefore obtain again (more than) the expected result since the bound obtained is $CR^{12} t^5 \varepsilon^{\frac{5}{4}} |\log \varepsilon|^5$.

Case 1, Recollisions in chain. In this case we start by dealing with the second recollision, between (k, ℓ) (note that actually here the pseudoparticle ℓ coincides with i). Let us denote by v_i'' the velocity of particle i just after the first recollision in the backward dynamics.

- If $|v_k - v_i''| \geq \varepsilon^{\frac{3}{4}}$, then considering the second recollision and integrating over $\tilde 1$ provides, according to (3.10), the bound

$$CR^5 t^2 \varepsilon \frac{|\log \varepsilon|^2}{|v_k - v_i''|} \leq CR^5 t^2 \varepsilon^{\frac{1}{4}} |\log \varepsilon|^2 \,. \tag{6.1}$$

Note that if the collision takes place before the first recollision ($t_{\tilde 1} > t_{rec}$ is possible in the case pictured on the right of Figure 6), the result (6.1) remains valid but the proof of (3.10) has to be adapted, as follows, to take into account the deviation of particle i by the first recollision.

Rephrasing (3.7) with the notation of Figure 6 (right), the condition for the second recollision to hold is

$$\big[(x_i - x_k) + (t_{rec} - t_{\tilde 2})(v_i' - v_i'')\big] + (t_{\tilde 1} - t_{\tilde 2})(v_i'' - v_k) + (\tilde t_{rec} - t_{\tilde 1})(v_i'' - v_k') \quad = \quad \varepsilon \nu_{rec} + q,$$

where x_i, x_k are the positions at the reference time $t_{\tilde 2}$. As a consequence, $v_i'' - v_k'$ belongs to a rectangle with given axis and a width less than $\varepsilon / |\tilde\tau_1|\, |v_i'' - v_k|$, where $\tilde\tau_1$ is analogous to τ_1 defined in (3.8). We can then proceed as in (3.10) and derive an upper bound uniform with respect to the term in the brackets.

Estimate (6.1) is uniform in v_i'', thus we can apply Proposition 3.5 to the recollision (i,j) which gives a bound $CR^7 t^3 \varepsilon |\log\varepsilon|^3$ after integration over 1^* and 2^*. Again we find (more than) the expected result since the bound obtained is $CR^{12} t^5 \varepsilon^{\frac54} |\log\varepsilon|^5$.

- If $|v_k - v_i''| \leq \varepsilon^{\frac34}$ then we need a more precise geometric argument to ensure both that the first recollision occurs, and that it produces an outgoing velocity in the ball $B(v_k, \varepsilon^{3/4})$. Lemma B.1 provides the existence of κ indices $\sigma_1, \dots, \sigma_\kappa$ with $1 \leq \kappa \leq 3$ such that

$$\iint \mathbf{1}_{\text{recollision } (i,j)} \mathbf{1}_{|v_i'' - v_k| \leq \varepsilon^{3/4}} \prod_{n=1}^{\kappa} \big|(v_{\sigma_n} - v_{a(\sigma_n)}(t_{\sigma_n})) \cdot \nu_{\sigma_n}\big| dt_{\sigma_n} d\nu_{\sigma_n} dv_{\sigma_n} \leq CR^8 t^3 \varepsilon\,.$$

This is the expected decay given in Proposition 6.2.

Case 2, Parallel recollisions. The analysis is postponed to Lemma B.3 which gives the existence of κ indices $\sigma_1, \dots, \sigma_\kappa$ with $1 \leq \kappa \leq 4$ and an integer r such that

$$\iint \mathbf{1}_{\text{parallel recollisions with } t_{\tilde 1} = t_{1^*}} \prod_{n=1}^{\kappa} \big|(v_{\sigma_n} - v_{a(\sigma_n)}(t_{\sigma_n})) \cdot \nu_{\sigma_n}\big| dt_{\sigma_n} d\nu_{\sigma_n} dv_{\sigma_n} \leq C(Rt)^r \varepsilon\,.$$

This is the expected decay given in Proposition 6.2.

Case 2, Recollisions in chain. Lemma B.4 in Appendix B gives the existence of κ indices $\sigma_1, \dots, \sigma_\kappa$ with $1 \leq \kappa \leq 3$ and an integer r such that

$$\iint \mathbf{1}_{\text{recollision in chain with } t_{\tilde 1} = t_{1^*}} \prod_{n=1}^{\kappa} \big|(v_{\sigma_n} - v_{a(\sigma_n)}(t_{\sigma_n})) \cdot \nu_{\sigma_n}\big| dt_{\sigma_n} d\nu_{\sigma_n} dv_{\sigma_n} \leq C(Rt)^r \varepsilon\,.$$

This is the expected decay given in Proposition 6.2.

Case 2, Self-recollisions. This case is dealt with in Lemma B.5, which gives the existence of κ indices $\sigma_1, \dots, \sigma_\kappa$ with $1 \leq \kappa \leq 3$ such that

$$\iint \mathbf{1}_{\text{self-recollision}} \prod_{n=1}^{\kappa} \big|(v_{\sigma_n} - v_{a(\sigma_n)}(t_{\sigma_n})) \cdot \nu_{\sigma_n}\big| dt_{\sigma_n} d\nu_{\sigma_n} dv_{\sigma_n} \leq CR^9 t^2 \varepsilon\,.$$

Proposition 6.2 is proved. □

As in the case of Corollary 3.8, the following result can be deduced from the proof of Proposition 6.2. The factor s^4 on the right-hand side of (6.2) is due to the fact that there are at most two undetermined recolliding particles (one for each recollision) and two parameters $\theta, \tilde\theta$ to localize the recollision times.

Corollary 6.4. *Fix $z_1 \in \mathbb{T}^2 \times B_R$ with $1 \leq R^2 \leq C_0 |\log\varepsilon|$, and a collision tree $a \in \mathcal{A}_s$.*

Then there exist sets $\mathcal{P}_2(a, z_1, \sigma) \subset \mathcal{T}_{2,s} \times \mathbb{S}^{s-1} \times B_R^{s-1}$ for all $\sigma \subset \{2, \dots, s\}$ with at most 6 elements such that

- *the following estimate holds*

$$\int \mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)}\Big(\prod_{m\in\sigma}\big|(v_{a(m)}(t_m)-v_m)\cdot\nu_m\big|\Big)dT_\sigma dV_\sigma d\Omega_\sigma \leq Cs^4(Rt)^r\varepsilon\,, \tag{6.2}$$

  uniformly over the parameters $(T_{2,s},V_{2,s},\Omega_{2,s})$ *in* $\mathcal{P}_2(a,z_1,\sigma)$ *which are not indexed by* σ*;*
- *the set* $\mathcal{P}_2(a,z_1)$ *of pseudo-trajectories with at least two recollisions is included in* $\bigcup_\sigma \mathcal{P}_2(a,z_1,\sigma)$.

6.4. **Estimate of $R_N^{K,>}$ (super exponential trees with multiple recollisions).** Proposition 6.1 comes from a careful summation of all elementary contributions. We therefore need the following refinement of Proposition 3.9.

Proposition 6.5. *We fix* $z_1\in\mathbb{T}^2\times B_R$*, a set* $\sigma\subset\{1,\dots,s\}$ *of* p *indices, and* $\eta>0$ *such that the collection of sets* $\mathcal{P}_2(a,z_1,\sigma)\subset\mathcal{T}_{2,s}\times\mathbb{S}^{s-1}\times\mathbb{R}^{2(s-1)}$ *associated with the collision trees satisfies, for some integer* r*,*

$$\sup_{a\in\mathcal{A}_s}\ \sup_{T_{2,s}^{<\sigma>},\Omega_{2,s}^{<\sigma>},V_{2,s}^{<\sigma>}}\int \mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)}\mathbf{1}_{\{|V_s|\leq R\}}\prod_{i\in\sigma}\big|(v_{a(i)}(t_i)-v_i)\cdot\nu_i\big|\,dT_\sigma d\Omega_\sigma dV_\sigma\leq\eta R^r\,. \tag{6.3}$$

Then for $t\geq 1$*, one has*

$$\begin{aligned}\sum_{a\in\mathcal{A}_s}\int \mathbf{1}_{\mathcal{T}_{2,s}}\mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)}\Big(\prod_{i=2}^s\big|(v_{a(i)}(t_i)-v_i)\cdot\nu_i\big|\Big)&M_\beta^{\otimes s}(V_s)dT_{2,s}d\Omega_{2,s}dV_{2,s}\\&\leq s^p(Ct)^{s-1-p}\eta M_{\frac{5\beta}{8}}(v_1)\,.\end{aligned}$$

If we further specify that the last n *times have to be in an interval of length* $h\leq 1$ *(this constraint is denoted by* $\mathcal{T}^h_{s-n+1,s}$*) then*

$$\begin{aligned}\sum_{a\in\mathcal{A}_s}\int \mathbf{1}_{\mathcal{T}_{2,s}}\mathbf{1}_{\mathcal{T}^h_{s-n+1,s}}\mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)}\left(\prod_{i=2}^s\big|(v_{a(i)}(t_i)-v_i)\cdot\nu_i\big|\right)&M_\beta^{\otimes s}(V_s)dT_{2,s}d\Omega_{2,s}dV_{2,s}\\&\leq s^p(Ct)^{s-n-1}(Ch)^{n-p}\eta M_{\frac{5\beta}{8}}(v_1)\,.\end{aligned}$$

Proof. The proof of Proposition 6.5 follows the same lines as the proof of Proposition 3.9. The additional difficulty is to control the divergence in R^r in (6.3). To do so, we decompose the energy into blocks

$$\begin{aligned}&\sum_{a\in\mathcal{A}_s}\int \mathbf{1}_{\mathcal{T}_{2,s}}\mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)}\left(\prod_{i=2}^s\big|(v_{a(i)}(t_i)-v_i)\cdot\nu_i\big|\right)M_\beta^{\otimes s}dT_{2,s}d\Omega_{2,s}dV_{2,s}\\&\leq\sum_{m=1}^{C|\log|\log\varepsilon||}\ \sum_{a\in\mathcal{A}_s}\int \mathbf{1}_{\mathcal{T}_{2,s}}\mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)}1_{\{2^{m-1}\leq|V_s|\leq 2^m\}}\\&\qquad\qquad\times\left(\prod_{i=2}^s\big|(v_{a(i)}(t_i)-v_i)\cdot\nu_i\big|\right)M_\beta^{\otimes s}dT_{2,s}d\Omega_{2,s}dV_{2,s}\,.\end{aligned}$$

For any $R=2^m$, we first integrate with respect to the p variables indexed by σ

$$\frac{1}{R^r}\int \mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)}\mathbf{1}_{|V_s|\leq R}\prod_{i\in\sigma}\big|(v_{a(i)}(t_i)-v_i)\cdot\nu_i\big|\,dT_\sigma d\Omega_\sigma dV_\sigma\leq\eta\,.$$

The main difference here is that we use once again the Maxwellian tails to get

$$\sup_{V_s} \Big\{ \mathbf{1}_{\{R/2 \leq |V_s| \leq R\}} R^r \, M^{\otimes s}_{\beta/6}(V_s) \Big\} \leq C^s \exp(-CR^2)\,,$$

for some constant C depending only on r and β.

Then we use the proof of Proposition 2.4 to estimate uniformly the product

$$\sum_{(a_j)_{j\notin\sigma}} \Big(\prod_{i\notin\sigma} \big|(v_{a(i)}(t_i) - v_i)\cdot \nu_i\big| \Big) M^{\otimes s}_{5\beta/6}(V_s) \leq (Cs)^{s-1-p} M_{\frac{5\beta}{8}}(v_1)\,.$$

We next integrate with respect to the remaining variables. We only retain the condition for the times $(t_i)_{i\notin\sigma}$.

- In the first case, we get a simplex of dimension $s-1-p$, the volume of which is
$$\frac{t^{s-1-p}}{(s-1-p)!} \leq C^s \frac{t^{s-1-p}}{s^{s-1-p}}\,,$$
by Stirling's formula.
- In the second case, we have to add the condition that the last n times have to be in an interval of length $h \leq 1$. The worst situation is when all times $(t_i)_{i\in\sigma}$ are in this small time interval, as we loose the corresponding smallness. More precisely, we get
$$\frac{t^{s-1-n}}{(s-1-n)!}\frac{h^{n-p}}{(n-p)!} \leq C^s \frac{t^{s-1-n}h^{n-p}}{s^{s-1-p}}\,.$$

We thus conclude that for any R,

$$\begin{aligned}\sum_{a\in\mathcal{A}_s} \int \mathbf{1}_{\mathcal{T}_{2,s}} \mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)} \mathbf{1}_{\{R/2\leq|V_s|\leq R\}} \Big(\prod_{i=2}^s \big|(v_{a(i)}(t_i) - v_i)\cdot\nu_i\big| \Big) M^{\otimes s}_\beta dT_{2,s} d\Omega_{2,s} dV_{2,s} \\ \leq s^p (Ct)^{s-1-p} \eta e^{-CR^2} M_{\frac{5\beta}{8}}(v_1)\,,\end{aligned}$$

and

$$\begin{aligned}\sum_{a\in\mathcal{A}_s} \int \mathbf{1}_{\mathcal{T}_{2,s}} \mathbf{1}_{\mathcal{T}^h_{s-n+1,s}} \mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)} \mathbf{1}_{\{R/2\leq|V_s|\leq R\}} \Big(\prod_{i=2}^s \big|(v_{a(i)}(t_i) - v_i)\cdot\nu_i\big| \Big) M^{\otimes s}_\beta dT_{2,s} d\Omega_{2,s} dV_{2,s} \\ \leq s^p (Ct)^{s-n-1} (Ch)^{n-p} \eta e^{-CR^2} M_{\frac{5\beta}{8}}(v_1)\,,\end{aligned}$$

where all constants are independent of R. The factor s^p comes from the summation over the possible choices of $(a(i))_{i\in\sigma}$. Finally, the result follows by summing over $R = 2^m$. □

Proof of Proposition 6.1. To estimate the global error due to multiple recollisions, we use Corollary 6.4 together with Proposition 6.5. As a consequence, the occurrence of multiple recollisions in a collision tree of size s can be estimated by summing over all the possible σ and using Proposition 6.5 with $p \leq 6$

$$\begin{aligned}\sum_\sigma \sum_{a\in\mathcal{A}_s} \int \mathbf{1}_{\mathcal{T}_{2,s}} \mathbf{1}_{\mathcal{T}^h_{s-n+1,s}} \mathbf{1}_{\mathcal{P}_2(a,z_1,\sigma)} \Big(\prod_{i=2}^s \big|(v_{a(i)}(t_i) - v_i)\cdot\nu_i\big| \Big) \mathbf{1}_{\mathcal{V}_s} f_N^{(s)}(t-kh) dT_{2,s} d\Omega_{2,s} dV_{2,s} \\ \leq N \exp(C\alpha^2) s^{16} (Ct)^{s-n-1} (Ch)^{n-6}\, t^r \varepsilon\, M_{\frac{5\beta}{8}}(v_1)\,,\end{aligned}$$

where the a priori L^∞-bound (2.12) has been used. The factor s^{16} comes from the contribution s^4 in (6.2) and from the fact that there are at most $O(s^{12})$ choices for the elements of σ and their images by a.

We then have, since $C\alpha h \ll 1$ (and for constants C which may change from line to line), choosing $n_k = 2^k n_0$,

$$\begin{aligned}
\left|R_N^{K,>}(t,z_1)\right| &\leq M_{\frac{5\beta}{8}}(v_1) N\varepsilon \exp(C\alpha^2) \sum_{k=1}^{K} \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} \sum_{j_k=n_k}^{N-J_{k-1}} (C\alpha t)^{J_{k-1}} \alpha^6 (C\alpha h)^{j_k-6} J_k^{16} t^r \\
&\leq M_{\frac{5\beta}{8}}(v_1) \exp(C\alpha^2) \frac{\alpha}{h^6} \sum_{k=1}^{K} \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} n_k^{16} (C\alpha h)^{n_k} (C\alpha t)^{J_{k-1}} t^r \\
&\leq M_{\frac{5\beta}{8}}(v_1) \exp(C\alpha^2) \frac{\alpha}{h^6} \sum_{k=1}^{K} 2^{k^2} \left(C\alpha^2 h t\right)^{2^k n_0} t^r \\
&\leq M_{\frac{5\beta}{8}}(v_1) \alpha \exp(C\alpha^2) (C\alpha^4 h t^2)^{n_0} \, ,
\end{aligned}$$

and Proposition 6.1 follows with $h \leq \gamma / \exp(C\alpha^2) T^3$ as soon as n_0 is large enough. Note that this is the only argument in which n_0 needs to be tuned. □

7. Truncation of large velocities

In this section, we prove that collision trees with large velocities contribute very little to the iterated Duhamel series. As a consequence, the error term $R_N^{K,vel}$ introduced in (2.27) vanishes. This holds also for the analogous term in the Boltzmann hierarchy

$$\begin{aligned}
\bar R_N^{K,vel}(t) := &\sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \bar Q_{1,J_1}(h)\, \bar Q_{J_1,J_2}(h) \dots \bar Q_{J_{K-1},J_K}(h) \, \left(f_0^{(J_K)} \mathbf{1}_{|V_{J_K}|^2 > C_0 |\log \varepsilon|} \right) \\
&+ \sum_{k=1}^{K} \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} \sum_{j_k \geq n_k} \bar Q_{1,J_1}(h) \dots \bar Q_{J_{k-1},J_k}(h) \left(f^{(J_k)}(t-kh) \mathbf{1}_{|V_{J_k}|^2 > C_0 |\log\varepsilon|} \right) .
\end{aligned}$$

The contribution of the large energies can be estimated by the following result.

Proposition 7.1. *There exists a constant $C_0 \geq 0$ such that for all $t \in [0,T]$ and $\alpha h \ll 1$*

$$\left| R_N^{K,vel}(t) \right| + \left| \bar R_N^{K,vel}(t) \right| \leq \exp(C\alpha^2) (C\alpha T)^{n_0 \cdot 2^K} \varepsilon M_{5\beta/8}(z_1),$$

with the sequence $n_k = 2^k n_0$.

Proof. The remainders $R_N^{K,vel}$ (2.27) and $\bar R_N^{K,vel}$ are made of two contributions, the first one is an energy cut-off for the Duhamel series up to time 0 (with a number of collisions less than $2^K n_0$) and the second one is a truncation at an intermediate time corresponding to a large number of collisions. Both terms can be estimated with similar arguments and we shall focus on the second term which requires additional arguments as the number of collisions in the last time interval is no longer bounded. We shall also consider only the BBGKY hierarchy as $\bar R_N^{K,vel}$ can be treated similarly.

From the maximum principle (2.14), we deduce that for C_0 large enough

$$\begin{aligned}
\left| f_N^{(J_k)}(t-kh) \mathbf{1}_{|V_{J_k}|^2 \geq C_0 |\log\varepsilon|} \right| &\leq C^{J_k} N M_\beta^{\otimes J_k} \mathbf{1}_{|V_{J_k}|^2 \geq C_0 |\log\varepsilon|} \, \|g_{\alpha,0}\|_{L^\infty(\mathbb{D})} \\
&\leq \exp(C\alpha^2) C^{J_k} N M_{5\beta/6}^{\otimes J_k} \exp\left(-\frac{\beta}{12} |V_{J_k}|^2 \right) \mathbf{1}_{|V_{J_k}|^2 \geq C_0 |\log\varepsilon|} \\
&\leq \varepsilon \exp(C\alpha^2) C^{J_k} M_{5\beta/6}^{\otimes J_k} .
\end{aligned}$$

Then using the fact that

$$\left| Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{k-1},J_k}(h) \right| \leq |Q_{1,J_{k-1}}|(t) \, |Q_{1 J_{k-1},J_k}|(h) \, ,$$

together with Proposition 2.4, we get

$$\sum_{j_k \geq n_k} \Big| Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{k-1},J_k}(h) \left(f_N^{(J_k)}(t-kh) \mathbf{1}_{|V_{J_k}|^2 \geq C_0 |\log \varepsilon|} \right) \Big|$$
$$\leq \exp(C\alpha^2)(C\alpha t)^{J_{k-1}} (C\alpha h)^{n_k} \varepsilon M_{5\beta/8}(z_1) \,,$$

as soon as $\alpha h \ll 1$.

Recalling (3.18) we can sum the different contributions provided that $\alpha h \ll 1$

$$\sum_{k=1}^{K} \sum_{\substack{j_i<n_i \\ i \leq k-1}} \sum_{j_k \geq n_k} \Big| Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{k-1},J_k}(h) \left(f_N^{(J_k)}(t-kh) \mathbf{1}_{|V_{J_k}|^2 \geq C_0 |\log \varepsilon|} \right) \Big|$$
$$\leq \exp(C\alpha^2) \sum_{k=1}^{K} \sum_{\substack{j_i<n_i \\ i \leq k-1}} (C\alpha t)^{J_{k-1}} (C\alpha h)^{n_k} \varepsilon M_{5\beta/8}(z_1)$$
$$\leq \exp(C\alpha^2)\, n_0^K 2^{K^2}\, \varepsilon \sum_{k=1}^{K} (C\alpha t)^{J_{k-1}} (C\alpha h)^{n_k} M_{5\beta/8}(z_1)$$
$$\leq \exp(C\alpha^2)\, (C\alpha T)^{2^{K+1} n_0} \varepsilon M_{5\beta/8}(z_1) \,.$$

The other terms can be controlled in the same way and this concludes the proof of Proposition 7.1. □

8. End of the proof of Theorem 1.2, and open problems

8.1. **Proof of Theorem 1.2.** In this section we gather all the error estimates obtained in the previous section and conclude the proof of Theorem 1.2. Fix $T > 1$ and $t \in [0,T]$.

We recall that due to (2.22) and (2.23) we have

$$f_N^{(1)}(t) = f_N^{(1,K)}(t) + R_N^K(t)$$

and

$$R_N^K(t) = R_N^{K,0}(t) + R_N^{K,1}(t) + R_N^{K,>}(t) + R_N^{K,vel}(t) \,.$$

Similarly

$$f^{(1)}(t) = \bar f_N^{(1,K)}(t) + \bar R_N^K(t) + \bar R_N^{K,vel}(t) \,.$$

- From Proposition 3.1, we know that the difference between the dominant parts is

$$\Big\| f_N^{(1,K)}(t) - f^{(1,K)}(t) \Big\|_{L^2} \leq (C\alpha T)^{2^{K+1} n_0} \exp(C\alpha^2) \Big(\varepsilon |\log \varepsilon|^{10} + \frac{\varepsilon}{\alpha} \Big) \,.$$

This contribution will be small provided that the number of collisions is bounded by

$$K = \frac{T}{h} \ll \log|\log \varepsilon| \,. \tag{8.1}$$

Let us now gather the estimates for the remainders.

- By Propositions 4.1 and 4.8, we have

$$\Big\| R_N^{K,0}(t) \Big\|_{L^2} \leq \gamma \quad \text{and} \quad \Big\| \bar R_N^K(t) \Big\|_{L^2(\mathbb{D})} \leq \gamma$$

provided that

$$h \leq \frac{\gamma^2}{\exp(C\alpha^2) T^3} \tag{8.2}$$

for some C large enough.

- By Proposition 5.1, the remainder for 1 recollision is bounded by

$$\left\| R_N^{K,1}(t) \right\|_{L^2} \leq \exp(C\alpha^2)(C\alpha T)^{2^{K+1}n_0} \frac{\varepsilon^{1/2}|\log \varepsilon|^5}{h} \,,$$

if $\alpha^2 Th \ll 1$, which is a less stringent condition than (8.2). Again this term is small under (8.2).

- From Proposition 6.1, the remainder for multiple recollisions is bounded by

$$\left\| R_N^{K,>}(t) \right\|_{L^2} \leq \gamma$$

provided that

$$h \leq \frac{\gamma}{\exp(C\alpha^2)T^3} \,\cdot$$

Note that for $\gamma \ll 1$, $T > 1$, this condition and (8.2) put together give the condition

$$h \leq \frac{\gamma^2}{\exp(C\alpha^2)T^3} \,\cdot \tag{8.3}$$

- By Proposition 7.1 the remainders for large velocities satisfy, as soon as $\alpha h \ll 1$,

$$\left\| R_N^{K,vel}(t) \right\|_{L^2} + \left\| \bar{R}_N^{K,vel}(t) \right\|_{L^2} \leq \exp(C\alpha^2)(C\alpha T)^{2^{K+1}n_0}\varepsilon \,,$$

which is small under (8.2).

The convergence estimate (1.20) is then obtained by combining conditions (8.1) and (8.3)

$$\begin{aligned} \left\| f_N(t) - f(t) \right\|_{L^2} &\leq \left\| f_N^{(1,K)}(t) - \bar{f}_N^{(1,K)}(t) \right\|_{L^2} + \left\| R_N^{K,0}(t) \right\|_{L^2} + \left\| \bar{R}_N^{K}(t) \right\|_{L^2} \\ &\qquad + \left\| R_N^{K,1}(t) \right\|_{L^2} + \left\| R_N^{K,>}(t) \right\|_{L^2} + \left\| R_N^{K,vel}(t) \right\|_{L^2} + \left\| \bar{R}_N^{K,vel}(t) \right\|_{L^2} \\ &\leq \frac{\exp(C\alpha^2)T^2}{\sqrt{\log|\log \varepsilon|}} \,\cdot \end{aligned}$$

This concludes the proof of Theorem 1.2. □

8.2. **Open problems.** In this final section we collect some open problems related to those treated in this paper.

Finite range potentials.

We expect the same convergence results to hold if microscopic interactions are described by a repulsive compactly supported potential (instead of the singular hard-sphere interactions). The proof then involves truncated marginals and cluster estimates as in [7] (see also [17]). With the present scaling, there is however a difficulty to control triple interactions, the size of which is critical (see the computations of Appendix B).

Higher dimension.

We also expect the convergence results to extend to higher dimensions. However, there are two important simplifications in dimension 2. The first one is due to the fact that the inverse partition function associated with the exclusion is bounded uniformly in N, as shown in (2.16); in particular this makes it possible to propagate somehow the initial form of the initial data and to decompose the marginals of the solution in a quasi-orthogonal form; see Section 4. The second one is related to the control of recollisions: we have seen in this paper (namely in Section 6) that the probability of having pseudo-dynamics with multiple recollisions is $O(\varepsilon)$, which balances exactly the $O(N)$ size of the L^∞ norm of the solution, and that is not the case in higher dimension in the Boltzmann-Grad scaling since $\varepsilon \sim N^{\frac{1}{1-d}}$.

Spatial Domain.

The spatial domain we consider here is the torus $\mathbb{T}^2$, which is equivalent to a rectangular box with specular reflection on the boundary. To extend the analysis to more general domains, we would need a geometric property of the free flow on these domains, stating roughly that the probability for two trajectories to approach at a distance ε on a fixed time interval $[0,T]$ is vanishing in the limit $\varepsilon \to 0$.

Dissipation.

The control on the higher order cumulants g_N^m is the key to improve the convergence time with respect to Lanford's original argument. This estimate can be seen as playing the role of the dissipation on the limiting equation. We indeed have

$$\frac{1}{N}\int \frac{f_N^2(t)}{M_\beta^{\otimes N}} dZ_N = \|g_N^1(t)\|^2_{L^2_\beta(\mathbb{D})} + \sum_{m=2}^N \frac{C_N^m}{N}\|g_N^m(t)\|^2_{L^2_\beta(\mathbb{D}^m)} = \frac{1}{N}\int \frac{f_{N,0}^2}{M_\beta^{\otimes N}} dZ_N \,.$$

to be compared to

$$\|g(t)\|^2_{L^2_\beta(\mathbb{D})} + \alpha \int_0^t \int M_\beta g \mathcal{L}_\beta g(s,x,v)\,dvdxds = \|g_0\|^2_{L^2_\beta(\mathbb{D})}$$

for the limiting equation.

Stochastic corrections.

In [21], Spohn studied the stochastic fluctuations around the Boltzmann equation and computed the variance of the fluctuation field in a non-equilibrium state

$$\zeta^N(g,t) = \frac{1}{\sqrt{N}}\left(\chi^N(g,t) - \langle \chi^N(g,t)\rangle\right) \quad \text{with} \quad \chi^N(g,t) = \sum_{i=1}^N g(z_i(t)),$$

where g is a smooth function and $\langle\cdot\rangle$ stands for the mean. It would be of great interest to prove that the limiting field is Gaussian and to derive, even for short time, the fluctuating hydrodynamics.

Appendix A. The linearized Boltzmann equation and its fluid limits

For the sake of completeness, we recall here some by now classical results about the linearized Boltzmann equation (1.14)

$$\begin{aligned} &\frac{1}{\alpha^q}\partial_t g_\alpha + v\cdot\nabla_x g_\alpha = -\alpha\mathcal{L}_\beta g_\alpha,\\ &\mathcal{L}_\beta g(v) = \int M_\beta(v_1)\Big(g(v)+g(v_1)-g(v')-g(v_1')\Big)\big((v_1-v)\cdot\nu\big)_+ d\nu dv_1 \end{aligned} \tag{A.1}$$

and its hydrodynamic limits as $\alpha \to \infty$ (for $q = 0,1$). The results below are valid in any dimension $d \geq 2$, thus contrary to the rest of this article, we assume the space dimension to be d.

Because of the scaling invariance of the collision kernel, we shall actually restrict our attention in the sequel to the case where M_β is the reduced centered Gaussian, i.e. $\beta = 1$ (and we omit the subscript β in the following). The collision operator (A.1) will be denoted by $\mathcal{L}$.

A.1. **The functional setting.**

The linearized Boltzmann operator $\mathcal{L}$ has been studied extensively (since it governs small solutions of the nonlinear Boltzmann equation). In the case of non singular cross sections, its spectral structure was described by Hilbert [13] and Grad [10]. The main result is that it satisfies the Fredholm alternative in a weighted L^2 space. In the following we define the collision frequency

$$a(|v|) := \int M(v_1)\big((v_1-v)\cdot\nu\big)_+ d\nu dv_1$$

which satisfies, for some $C > 1$,

$$0 < a_- \leq a(|v|) \leq C(1 + |v|) .$$

Proposition A.1. *The linear collision operator $\mathcal{L}$ defined by* (A.1) *is a nonnegative unbounded self-adjoint operator on $L^2(Mdv)$ with domain*

$$\mathcal{D}(\mathcal{L}) = \{g \in L^2(Mdv) \,|\, ag \in L^2(Mdv)\} = L^2(\mathbb{R}^d; aM(v)dv)$$

and nullspace

$$\mathrm{Ker}(\mathcal{L}) = \mathrm{span}\{1, v_1, \dots, v_d, |v|^2\} .$$

Moreover the following coercivity estimate holds: there exists $C > 0$ such that, for each g in $\mathcal{D}(\mathcal{L}) \cap (\mathrm{Ker}(\mathcal{L}))^\perp$

$$\int g\mathcal{L}g(v)M(v)dv \geq C\|g\|^2_{L^2(aMdv)} .$$

Sketch of proof. • The first step consists in characterizing the nullspace of $\mathcal{L}$. It must contain the collision invariants since the integrand in $\mathcal{L}g$ vanishes identically if $g(v) = 1, v_1, v_2, \dots, v_d$ or $|v|^2$. Conversely, from the identity,

$$\int \psi\mathcal{L}gMdv = \frac{1}{4}\int (\psi+\psi_1-\psi'-\psi'_1)(g+g_1-g'-g'_1)\big((v_1 - v)\cdot\nu\big)_+ Mdvdv_1d\nu ,$$

where we have used the classical notation

$$g_1 := g(v_1) , \quad g' = g(v') , \quad g'_1 = g(v'_1) ,$$

we deduce that, if g belongs to the nullspace of $\mathcal{L}$, then

$$g + g_1 = g' + g'_1 ,$$

which entails that g is a linear combination of $1, v_1, v_2, \dots, v_d$ and $|v|^2$ (see for instance [16]).

Note that the same identity shows that $\mathcal{L}$ is self-adjoint.

• In order to establish the coercivity of the linearized collision operator $\mathcal{L}$, the key step is then to introduce Hilbert's decomposition [13], showing that $\mathcal{L}$ is a compact perturbation of a multiplication operator :

$$\mathcal{L}g(v) = a(|v|)g(v) - \mathcal{K}g(v) .$$

Proving that $\mathcal{K}$ is a compact integral operator on $L^2(Mdv)$ relies on intricate computations using Carleman's parametrization of collisions (which we also use in this paper for the study of recollisions). We shall not perform them here (see [13]).

Because n is bounded from below, $\mathcal{L}$ has a spectral gap, which provides the coercivity estimate. □

Proposition A.2. *Let $g_0 \in L^2(Mdvdx)$. Then, for any fixed α, there exists a unique solution $g_\alpha \in C(\mathbb{R}^+, L^2(Mdvdx)) \cap C^1(\mathbb{R}^+_*, L^2(Mdvdx)) \cap C(\mathbb{R}^+_*, L^2(Madvdx))$ to the linearized Boltzmann equation* (A.1).

A.2. **The acoustic and Stokes limit.** The starting point for the study of hydrodynamic limits is the scaled energy inequality

$$\|g_\alpha(t)\|^2_{L^2(Mdvdx)} + \alpha^{1+q}\int_0^t \int g_\alpha\mathcal{L}g_\alpha(t')Mdvdxdt' \leq \|g_0\|^2_{L^2(Mdv)} .$$

The uniform L^2 bound on (g_α) implies that, up to extraction of a subsequence,

$$g_\alpha \rightharpoonup g \text{ weakly in } L^2_{loc}(dt, L^2(Mdvdx)) . \tag{A.2}$$

The dissipation, together with the coercivity estimate in Proposition A.1, further provides

$$\|g_\alpha - \Pi g_\alpha\|_{L^2(Madvdxdt)} = O(\alpha^{-(q+1)/2}) ,$$

from which we deduce that

$$g(t,x,v) = \Pi g(t,x,v) \equiv \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\frac{|v|^2-d}{2}\,\cdot \tag{A.3}$$

If the Mach number α^q is of order 1, i.e. for $q=0$, one obtains asymptotically the acoustic equations. Denoting by $\langle\cdot\rangle$ the average with respect to the measure Mdv, we indeed have the following conservation laws

$$\begin{aligned}\partial_t\langle g_\alpha\rangle + \nabla_x\cdot\langle g_\alpha v\rangle &= 0\,,\\ \partial_t\langle g_\alpha v\rangle + \nabla_x\cdot\langle g_\alpha v\otimes v\rangle &= 0\,,\\ \partial_t\langle g_\alpha |v|^2\rangle + \nabla_x\cdot\langle g_\alpha v|v|^2\rangle &= 0\,.\end{aligned}$$

From (A.2) and (A.3) we then deduce that g can be written under the form

$$\begin{aligned}\partial_t\rho + \nabla_x\cdot u &= 0\,,\\ \partial_t u + \nabla_x(\rho+\theta) &= 0\,,\\ \partial_t\theta + \frac{2}{d}\nabla_x\cdot u &= 0\,.\end{aligned} \tag{A.4}$$

By uniqueness of the limiting point, we get the convergence of the whole family $(g_\alpha)_{\alpha>0}$.

Since the limiting distribution g satisfies the energy equality

$$\|g\|^2_{L^2(Mdvdx)} = \|\Pi g_0\|^2_{L^2(Mdvdx)}$$

or equivalently

$$\|g\|^2_{L^2(Mdvdx)} + \alpha\int_0^t\int g\mathcal{L}gMdvdx = \|\Pi g_0\|^2_{L^2(Mdvdx)}\,,$$

convergence is strong as soon as $g_0 = \Pi g_0$. We thus have the following result (see [8] and references therein).

Proposition A.3. *Let $g_0\in L^2(Mdvdx)$. For all α, let g_α be a solution to the scaled linearized Boltzmann equation* (A.1) *with $q=0$. Then, as $\alpha\to\infty$, g_α converges weakly in $L^2_{loc}(dt, L^2(Mdvdx))$ to the infinitesimal Maxwellian $g = \rho + u\cdot v + \frac{1}{2}\theta(|v|^2-d)$ where (ρ,u,θ) is the solution of the acoustic equations* (A.4) *with initial data $(\langle g_0\rangle, \langle g_0 v\rangle, \langle g_0\frac{1}{2}(|v|^2-d)\rangle)$.*

The convergence holds in $L^\infty_t(L^2(Mdvdx))$ provided that $g_0=\Pi g_0$.

In the diffusive regime, i.e. for $q=1$, the moment equations state

$$\begin{aligned}\frac{1}{\alpha}\partial_t\langle g_\alpha\rangle + \nabla_x\cdot\langle g_\alpha v\rangle &= 0\,,\\ \frac{1}{\alpha}\partial_t\langle g_\alpha v\rangle + \nabla_x\cdot\langle g_\alpha v\otimes v\rangle &= 0\,,\\ \frac{1}{\alpha}\partial_t\langle g_\alpha |v|^2\rangle + \nabla_x\cdot\langle g_\alpha v|v|^2\rangle &= 0\,.\end{aligned}$$

From (A.2) and (A.3) we deduce that

$$\nabla_x\cdot u = 0\,,\quad \nabla_x(\rho+\theta) = 0\,,$$

referred to as incompressibility and Boussinesq constraints.

To characterize the mean motion, we then have to filter acoustic waves, i.e. to project on the kernel of the acoustic operator

$$\partial_t P\langle g_\alpha v\rangle + \alpha P\nabla_x \cdot \langle g_\alpha (v\otimes v - \frac{1}{2}|v|^2 Id)\rangle = 0\,,$$
$$\partial_t \frac{1}{4}\langle g_\alpha (|v|^2 - d - 2)\rangle + \alpha\nabla_x \cdot \langle g_\alpha \frac{1}{4} v(|v|^2 - d - 2)\rangle = 0\,,$$

where P is the Leray projection on divergence free vector fields. Define the kinetic momentum flux $\Phi(v) = v\otimes v - \frac{1}{d}|v|^2 Id$ and the kinetic energy flux $\Psi(v) = \frac{1}{2}v(|v|^2-d-2)$. As Φ, Ψ belong to $(\mathrm{Ker}\,\mathcal{L})^\perp$, and $\mathcal{L}$ is a Fredholm operator, there exist pseudo-inverses $\tilde\Phi, \tilde\Psi \in (\mathrm{Ker}\,\mathcal{L})^\perp$ such that $\Phi = \mathcal{L}\tilde\Phi$ and $\Psi = \mathcal{L}\tilde\Psi$. Then,

$$\partial_t P\langle g_\alpha v\rangle + \alpha P\nabla_x \cdot \langle \mathcal{L} g_\alpha \tilde\Phi\rangle = 0\,,$$
$$\partial_t \frac{1}{4}\langle g_\alpha (|v|^2 - d - 2)\rangle + \alpha\nabla_x \cdot \langle \mathcal{L} g_\alpha \tilde\Psi\rangle = 0\,.$$

Using the equation

$$\alpha\mathcal{L} g_\alpha = -v\cdot\nabla_x g_\alpha - \frac{1}{\alpha}\partial_t g_\alpha$$

the Ansatz (A.3), and taking limits in the sense of distributions, we get

$$\begin{aligned}\nabla_x\cdot u = 0,\quad \nabla_x(\rho+\theta) = 0\,,\\ \partial_t u - \mu\Delta_x u = 0\,,\\ \partial_t\theta - \kappa\Delta_x\theta = 0\,.\end{aligned} \tag{A.5}$$

These are exactly the Stokes-Fourier equations with

$$\mu = \frac{1}{(d-1)(d+2)}\langle \Phi : \tilde\Phi\rangle \quad \text{and} \quad \kappa = \frac{2}{d(d+2)}\langle \Psi\cdot\tilde\Psi\rangle.$$

As previously, the limit is unique and the convergence is strong provided that the initial data is well-prepared, i.e. if

$$g_0(x,v) = \rho_0 + u_0\cdot v + \frac{1}{2}\theta_0(|v|^2 - d) \text{ with } \nabla_x\cdot u_0 = 0\,,\quad \nabla_x(\rho_0+\theta_0) = 0\,. \tag{A.6}$$

One can therefore prove the following result.

Proposition A.4. *Let $g_0 \in L^2(Mdvdx)$. For all α, let g_α be a solution to the scaled linearized Boltzmann equation* (A.1) *with $q = 1$. Then, as $\alpha\to\infty$, g_α converges weakly in $L^2_{loc}(dt, L^2(Mdvdx))$ to the infinitesimal Maxwellian $g = u\cdot v + \frac{1}{2}\theta(|v|^2-d)$ where (u,θ) is the solution of* (A.5) *with initial data $(P\langle g_0 v\rangle, \langle g_0 \frac{1}{2}(|v|^2-d)\rangle)$.*

The convergence holds in $L^\infty_t(L^2(Mdvdx))$ provided that the initial data is well-prepared in the sense of (A.6).

Remark A.5. *In both cases, the defect of strong convergence for ill-prepared initial data can be described precisely.*

If the initial profile in v is not an infinitesimal Maxwellian, i.e. if $g_0 \neq \Pi g_0$, one has a relaxation layer of size $\alpha^{-(1+q)}$ governed essentially by the homogeneous equation

$$\partial_t \Pi_\perp g_\alpha = -\alpha^{q+1}\mathcal{L} g_\alpha\,.$$

In the incompressible regime, if the initial moments do not satisfy the incompressibility and Boussinesq constraints, one has to superpose a fast oscillating component (with a time scale α^{-1}). For each eigenmode of the acoustic operator, the slow evolution is given by a diffusive equation.

A straightforward energy estimate then shows that the asymptotic behavior of g_α is well described by the sum of these three contributions (main motion, relaxation layer and acoustic waves in incompressible regime).

Appendix B. Geometrical lemmas

In this appendix, we prove several technical lemmas (namely Lemmas B.1, B.3, B.4 and B.5) which were key steps in Sections 3 and 6 in proving Propositions 3.5 and 6.2.

In the following we adopt the notation of those sections.

B.1. **A preliminary estimate.** Recall Equation (3.9) for the first recollision

$$v_i' - v_j = \frac{1}{\tau_{rec}}\delta x_\perp - \frac{\tau_1}{\tau_{rec}}(v_i - v_j) - \frac{1}{\tau_{rec}}\nu_{rec} \,. \tag{B.1}$$

The distance between particles i, j at the collision time t_{1^*} is given by

$$\big|x_i(t_{1^*}) - x_j(t_{1^*})\big| = \varepsilon\big|\delta x_\perp - \tau_1(v_i - v_j)\big| = \varepsilon\sqrt{\big|\delta x_\perp\big|^2 + \big|\tau_1(v_i - v_j)\big|^2}\,.$$

The distance between the particles varies with the collision time t_{1^*} and the closer they are, the easier it is to aim (at the collision time t_{1^*}) to create a recollision at the later time t_{rec}. The key idea is that for relative velocities $v_i - v_j \neq 0$, the particles will never remain close for a long time so that integrating over t_{1^*} allows us to recover some smallness uniformly over the initial positions at time t_{2^*}.

Suppose $|\tau_1||v_i - v_j| \leq M$. Since v_{1^*} is in a ball of size R, and ν_{1^*} belongs to $\mathbb{S}$, we have

$$\begin{aligned}\int \mathbf{1}_{\text{(B.1) has a solution}} &\mathbf{1}_{\{|\tau_1||v_i - v_j| \leq M\}}|(v_{1^*} - v_i)\cdot\nu_{1^*}| \\ &\times |v_i - v_j| d\tau_1 dv_{1^*} d\nu_{1^*} \leq CR^2 M\,.\end{aligned} \tag{B.2}$$

For later purposes, it will be useful to evaluate the integral (B.2) in terms of the integration parameter t_{1^*}: we get by the change of variable $\tau_1 = (t_{1^*} - t_{2^*} - \lambda)/\varepsilon$

$$\int \mathbf{1}_{\text{(B.1) has a solution}}\, \mathbf{1}_{\{|\tau_1||v_i - v_j| \leq M\}}\, |(v_{1^*} - v_i)\cdot\nu_{1^*}| dt_{1^*} dv_{1^*} d\nu_{1^*} \leq CR^2 M \frac{\varepsilon}{|v_i - v_j|}\cdot$$

The singularity in $|v_i - v_j|$ translates the fact that the distance between the particles may remain small during a long time if their relative velocity is small. This singularity can then be integrated out by using two extra degrees of freedom associated with the parents $2^*, 3^*$ of i or j: from (C.6) and (C.9) in Lemma C.2, we obtain the upper bound

$$\begin{aligned}\int \mathbf{1}_{\text{(B.1) has a solution}}\, &\mathbf{1}_{\{|\tau_1||v_i - v_j| \leq M\}} \\ &\times \prod_{k\in\{1^*,2^*,3^*\}} |(v_k - v_{a(k)})\cdot\nu_k| dt_k dv_k d\nu_k \leq CMR^7 t^2 \varepsilon\,.\end{aligned} \tag{B.3}$$

Note that in the case when i and j are colliding at time t_{2^*}, then it could be that there are not enough parents to carry out the previous computation (if i is particle 1 and j particle 2 for instance). As explained in Remark 3.6, this is a pathological case which can be easily handled and we shall systematically assume that the number of integration variables at our disposal is sufficient.

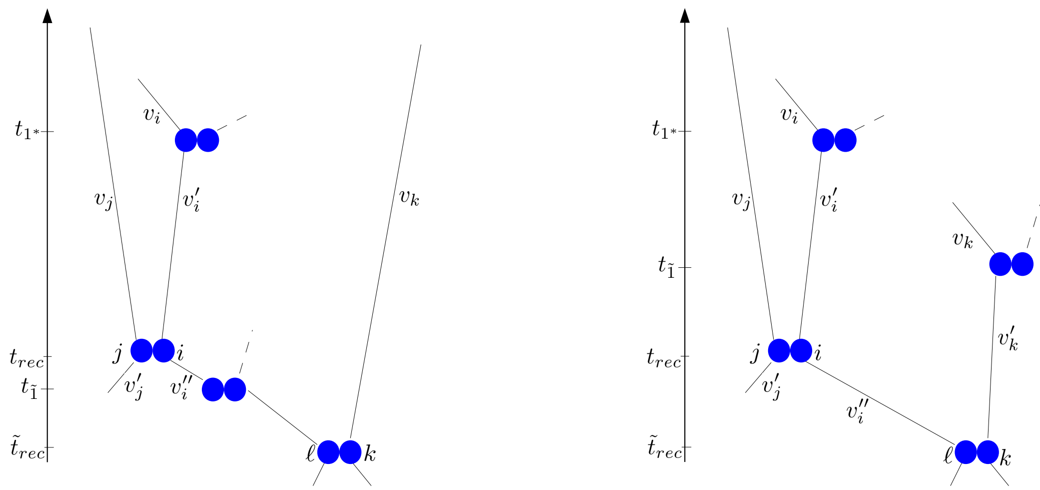

FIGURE 10.

B.2. **A recollision with a constraint on the outgoing velocity.** The following lemma deals with the cost of a recollision when one of the outgoing velocities is constrained to lie in a given ball. It was used in Section 6.3 page 49. The setting is recalled in Figure 10.

Lemma B.1. *With the notation of Figure* 10 *and Proposition* 6.2*, there are κ indices $\sigma_1, \dots, \sigma_\kappa$ with $1 \leq \kappa \leq 3$, such that assuming that the total energy $|V_s|^2$ is bounded by $1 \leq R^2 \leq |\log \varepsilon|$, and for all $t \geq 1$,*

(B.4)
$$\int \mathbf{1}_{\textit{recollision between (i,j)}} \mathbf{1}_{|v_i''-v_k| \leq \varepsilon^{3/4}} \prod_{n=1}^{\kappa} |(v_{\sigma_n} - v_{a(\sigma_n)}(t_{\sigma_n})) \cdot \nu_{\sigma_n}| dt_{\sigma_n} d\nu_{\sigma_n} dv_{\sigma_n} \leq CR^8 t^3 \varepsilon.$$

Proof. The proof follows closely the lines of the proof of Lemma 3.7.

Throughout the proof, we suppose that the parameters associated with the first recollision are such that $|\tau_1||v_i - v_j| \geq R^2$. Otherwise, the estimate (B.3) applied with $M = R^2$ leads to an upper bound of the form (B.4). Actually all the other steps of the proof lead to a better bound in terms of powers of ε, of the type $\varepsilon^\gamma |\log \varepsilon|^\delta$ with $\gamma > 1$, $\delta \geq 0$.

By definition, v_i'' is given by one of the following formulas

$$\begin{aligned} v_i'' &= v_i' - (v_i' - v_j) \cdot \nu_{rec} \nu_{rec} \,, \\ \text{or} \quad v_i'' &= v_j + (v_i' - v_j) \cdot \nu_{rec} \nu_{rec} \,. \end{aligned} \tag{B.5}$$

Note that the second choice is the value v_j' and we use this abuse of notation to describe the case when k collides with j.

We expect the condition $v_i'' \in B(v_k, \varepsilon^{3/4})$ to impose a strong constraint on the recollision angle ν_{rec}. We indeed find from (B.5) that this condition implies

$$\begin{aligned} \text{either} \qquad v_k - v_j &= (v_i' - v_j) \cdot \nu_{rec}^\perp \nu_{rec}^\perp + O(\varepsilon^{3/4}) \,, \\ \text{or} \qquad v_k - v_j &= (v_i' - v_j) \cdot \nu_{rec} \nu_{rec} + O(\varepsilon^{3/4}) \,. \end{aligned} \tag{B.6}$$

We consider now three different cases.

• If $k \neq j$ and $k \neq 1^*$, we distinguish two more cases.

- If $|v_j - v_k| > \varepsilon^{5/8} \gg \varepsilon^{3/4}$, we deduce from the constraint (B.6) that the recollision angle is in a small angular sector

$$\nu_{rec} = \frac{(v_j - v_k)^\perp}{|v_k - v_j|} + O(\varepsilon^{1/8}) \quad \text{or} \quad \nu_{rec} = \frac{v_k - v_j}{|v_k - v_j|} + O(\varepsilon^{1/8})\,.$$

Plugging this Ansatz in (B.1), we get

$$v_i' - v_j = \frac{1}{\tau_{rec}} \delta x_\perp - \frac{\tau_1}{\tau_{rec}}(v_i - v_j) - \frac{1}{\tau_{rec}} \frac{\mathcal{R}_{n'\pi/2}(v_k - v_j)}{|v_k - v_j|} + O\left(\frac{\varepsilon^{1/8}}{\tau_{rec}}\right),$$

denoting by $\mathcal{R}_\theta$ the rotation of angle θ and $n' = 0, 1$ depending on the identity in (B.6). We are then brought back to the computation leading to Lemma 3.7 page 22, except that the constraint on the recollision angle leads to the fact that $v_i' - v_j$ lies in a thinner rectangle of size $2R \times (4R\varepsilon^{1/8} \min(1, 1/|\tau_1||v_i - v_j|))$. We thus conclude by integrating in $(t_{1^*}, v_{1^*}, \nu_{1^*})$ and $(t_{2^*}, v_{2^*}, \nu_{2^*})$, exactly as in the proof of Proposition 3.5, that the contribution of these configurations is of size $O(R^7 t^3 \varepsilon^{9/8} |\log \varepsilon|^3)$.

- If $|v_j - v_k| \leq \varepsilon^{5/8}$, we simply need to integrate this constraint over the parents of j and k. Thanks to (C.12) and Lemma C.1, we get a contribution of size at most $O(R^5 t^2 \varepsilon^{5/4} |\log \varepsilon|)$.

• If $k = j$, then $|v_k - v_i''| = |v_j' - v_i''| = |v_i' - v_j| \leq \varepsilon^{3/4}$. Then, applying again (C.12) leads after two integrations to an error $O(R^5 \varepsilon^{3/2} t^2\, |\log \varepsilon|)$.

• If $k = 1^*$, then $|v_k - v_i''| = |v_{1^*}' - v_i''| \leq \varepsilon^{3/4}$. This is the most delicate case as v_i'' and v_k are linked through the same collision. One of the following identities then holds

$$\begin{aligned} & v_i' - (v_i' - v_j) \cdot \nu_{rec}\, \nu_{rec} = v_{1^*}' + O(\varepsilon^{3/4})\,, \\ \text{or} \quad & v_j + (v_i' - v_j) \cdot \nu_{rec}\, \nu_{rec} = v_i' - (v_i' - v_j) \cdot \nu_{rec}^\perp \nu_{rec}^\perp = v_{1^*}' + O(\varepsilon^{3/4})\,, \end{aligned}$$

and we further have that $|v_i' - v_{1^*}'| = |v_i - v_{1^*}|$.

- If $|v_i - v_{1^*}| \leq \varepsilon^{5/8}$, then v_{1^*} has to be in a ball of size $\varepsilon^{5/4}$ so we find a bound $O(R\varepsilon^{5/4} t)$ on integration over 1^*.

- If $|v_i - v_{1^*}| \geq \varepsilon^{5/8}$, then

$$\nu_{rec} = \pm \frac{v_i' - v_{1^*}'}{|v_i' - v_{1^*}'|} + O(\varepsilon^{1/8}) \qquad \text{or} \qquad \nu_{rec} = \pm \frac{(v_i' - v_{1^*}')^\perp}{|v_i' - v_{1^*}'|} + O(\varepsilon^{1/8})\,.$$

Plugging this Ansatz in (B.1), we get

$$v_i' - v_j = \frac{1}{\tau_{rec}} \delta x_\perp - \frac{\tau_1}{\tau_{rec}}(v_i - v_j) - \frac{1}{\tau_{rec}} \mathcal{R}_{n'\frac{\pi}{2}} \frac{v_i' - v_{1^*}'}{|v_i - v_{1^*}|} + O\left(\frac{\varepsilon^{1/8}}{\tau_{rec}}\right), \tag{B.7}$$

with $n' \in \{0, 1, 2, 3\}$.

Compared with the formulas of the same type encountered in Proposition 3.5, this one has the additional difficulty that the "unknown" v_i' is on both sides of the equation. However recalling that $|\tau_1||v_i - v_j| \gg 1$, the highest order term on the right-hand side is $\big(\delta x_\perp - \tau_1(v_i - v_j)\big)/\tau_{rec}$ so we can use this as a first order approximation for $v_i' - v_j$.

- If at time t_{1^*} there is no scattering, then $v_i' = v_{1^*}$ and $v_{1^*}' = v_i$. Defining

$$w := \delta x_\perp - (v_i - v_j)\tau_1, \quad \text{and} \quad u := |w|/\tau_{rec},$$

Equation (B.7) becomes

$$v_{1^*} - v_i = v_j - v_i + u\frac{w}{|w|} - \frac{u}{|w|}\mathcal{R}_{n\pi/2}\frac{v_{1^*} - v_i}{|v_i - v_{1^*}|} + O\left(\frac{\varepsilon^{1/8}u}{|w|}\right). \tag{B.8}$$

Recall that $|(v_i - v_j)\tau_1| \geq R^2$ thus $|w| \geq R^2 \gg 1$. From (3.12), $\frac{1}{|\tau_{rec}|} \leq \frac{4R}{|\tau_1||v_i - v_j|}$ so that the parameter u takes its values in a bounded set.

The difficulty to deduce useful information on v_{1^*} is that there is no a priori control on the direction of $v_{1^*} - v_i$. Indeed when $|v_{1^*} - v_i|$ is small, then a small perturbation in (B.8) may lead to a large variation of the angle $v_{1^*} - v_i$. Thus at first sight, (B.8) implies only that v_{1^*} lies in a rectangle $\mathcal{R}_1$ of axis $w/|w|$ and of width $CR/|w|$.

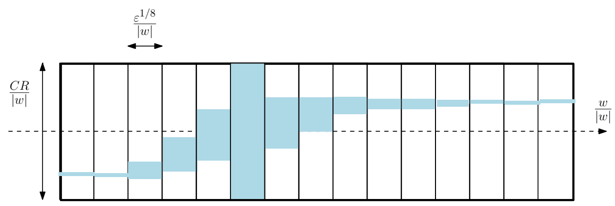

FIGURE 11. The rectangle $\mathcal{R}_1$ of axis $w/|w|$ and of width $CR/|w|$ is partitioned into bricks $\mathbb{B}_n$ of width $\frac{\varepsilon^{1/8}}{|w|}$. The solutions of (B.8) take their values in the shaded domain which size in $\mathbb{B}_n$ is of order $\frac{\varepsilon^{1/8}}{(n+1)|w|}$.

Let us assume that v_i belongs to the rectangle $\mathcal{R}_1$, since this is the most singular situation. In order to control $|v_i - v_{1^*}|$, we decompose the rectangle $\mathcal{R}_1$ into $O(R\varepsilon^{-\frac{1}{8}}|w|)$ bricks of side lenghts $\frac{\varepsilon^{1/8}}{|w|} \times \frac{CR}{|w|}$ as in Figure 11. For n ranging from 0 to $O(R\varepsilon^{-\frac{1}{8}}|w|)$, the brick $\mathbb{B}_n$ is such that

$$Cn\frac{\varepsilon^{1/8}}{|w|} \leq |v_i - v_{1^*}| \leq C(n+1)\frac{\varepsilon^{1/8}}{|w|}\cdot \tag{B.9}$$

Given $n \geq 1$, solutions of (B.8) in $\mathbb{B}_n$ belong to a restricted domain of $\mathbb{B}_n$ (see Figure 11) as the direction of $v_{1^*} - v_i$ should remain close to that of the solution of

$$\hat{v}_{1^*} - v_i = v_j - v_i + u\frac{w}{|w|} - \frac{u}{|w|}\mathcal{R}_{n\pi/2}\frac{\hat{v}_{1^*} - v_i}{|\hat{v}_{1^*} - v_i|}\,,$$

with an uncertainty of order $\frac{\varepsilon^{1/8}u}{|w|}\frac{|w|}{n\varepsilon^{1/8}} = \frac{u}{n} = O(\frac{1}{n})$, where $\frac{\varepsilon^{1/8}u}{|w|}$ is the size of the error term in (B.8). Since the width of the bricks is $\frac{\varepsilon^{1/8}}{|w|}$, the relation (B.8) implies that on each elementary brick $\mathbb{B}_n$, the solution v_{1^*} lies in a set of measure $O(\frac{\varepsilon^{1/8}}{(1+n)|w|})$. Summing over $n \leq O(R\varepsilon^{-\frac{1}{8}}|w|)$, we find finally that v_{1^*} has to be in a small domain of measure less than $O(R^2|\log\varepsilon|\varepsilon^{1/8}/(|\tau_1||v_i - v_j|))$. Thus after integration on the variables 1^* and 2^* we obtain, as in the proof of Proposition 3.5 an error of size $O(\varepsilon^{9/8}|\log\varepsilon|^3 \times R^7t^3)$.

- If at time t_{1^*}, there is scattering, then we use again Carleman's parametrization: denote

$$V'_* \equiv v_{1^*} - (v_{1^*} - v_i)\cdot\nu_{1^*}\,\nu_{1^*} \quad \text{and} \quad V' \equiv v_i + (v_{1^*} - v_i)\cdot\nu_{1^*}\,\nu_{1^*}\,.$$

As V'_* is supported by the line $v_i + \mathbb{R}(V' - v_i)^\perp$, it can be indexed in terms of $\mu \in \mathbb{R}$

$$V'_* = v_i + \mu\frac{(V' - v_i)^\perp}{|V' - v_i|}\,, \tag{B.10}$$

and we then integrate with respect to $dV'd\mu$.

- If $(v'_i, v'_{1*}) = (V', V'_*)$, we get from (B.7) that

$$V' - v_j = \frac{1}{\tau_{rec}} \delta x_\perp - \frac{\tau_1}{\tau_{rec}}(v_i - v_j) - \frac{1}{\tau_{rec}} \mathcal{R}_{n\pi/2} \frac{V' - V'_*}{|V' - V'_*|} + O\left(\frac{\varepsilon^{1/8}}{|\tau_{rec}|}\right).$$

The difficulty again is to make sure that the direction of $V' - V'_*$ does not oscillate too much so that in the end V' does not belong to a set too large. Defining as previously $u := |w|/\tau_{rec}$ with $w := \delta x_\perp - \tau_1(v_i - v_j)$ the identity above can be written

$$V' - v_i = v_j - v_i + u \frac{w}{|w|} - \frac{u}{|w|} \mathcal{R}_{n\pi/2} \frac{V' - V'_*}{|V' - V'_*|} + O\left(\varepsilon^{1/8} \frac{u}{|w|}\right).$$

For fixed μ, the size of the variations of direction of $V' - V'_*$ is the same as $V' - v_i$ by (B.10). Furthermore, we notice that thanks to (B.10),

$$\frac{V' - V'_*}{|V' - V'_*|} = \mathcal{R}_{(|V' - v_i|, \mu)} \frac{V' - v_i}{|V' - v_i|} \tag{B.11}$$

where $\mathcal{R}_{(|V' - v_i|, \mu)}$ is a rotation operator depending only on $|V' - v_i|$ and μ. Thus we can proceed as in (B.9) and decompose the rectangle $\mathcal{R}_1$ into elementary bricks of size $\varepsilon^{1/8}/|w|$ indexed by

$$Cn \frac{\varepsilon^{1/8}}{|w|} \leq |V' - v_i| \leq C(n+1) \frac{\varepsilon^{1/8}}{|w|} \cdot$$

Then following the same reasoning as with (B.8), we obtain that the error is again of size $O(\varepsilon^{9/8} |\log \varepsilon|^3 \times R^7 t^3)$.

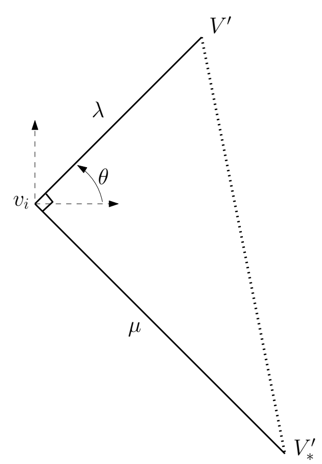

FIGURE 12. Carleman's parametrization (V', V'_*) can be evaluated in terms of the measure $dV'd\mu$ (see (B.10)) or alternatively by parametrizing $V' - v_i$ in polar coordinates by the measure $\lambda d\lambda \, d\theta d\mu$ with $\lambda = |V' - v_i|$. The direction $\frac{V' - V'_*}{|V' - V'_*|}$ can be recovered by a rotation from $V' - v_i$ (B.11) or $V'_* - v_i$ (B.12).

- If $(v'_{1*}, v'_i) = (V', V'_*)$, we write (B.7) as

$$V'_* - v_i = v_j - v_i + u \frac{w}{|w|} - \frac{u}{|w|} \mathcal{R}_{n\pi/2} \frac{V' - V'_*}{|V' - V'_*|} + O\left(\varepsilon^{1/8} \frac{u}{|w|}\right),$$

As in (B.11), it is convenient to reparametrize

$$\frac{V' - V'_*}{|V' - V'_*|} = \mathcal{R}_{(|V'_* - v_i|, \lambda)} \frac{V'_* - v_i}{|V'_* - v_i|} \tag{B.12}$$

where $\mathcal{R}_{(|V'_*-v_i|,\lambda)}$ is a rotation depending only on these two parameters and $\lambda = |V' - v_i|$ (see Figure 12). We decompose again the rectangle $\mathcal{R}_1$ into small bricks of size $\varepsilon^{1/8}/|w|$

$$Cn\frac{\varepsilon^{1/8}}{|w|} \leq |V'_* - v_i| \leq C(n+1)\frac{\varepsilon^{1/8}}{|w|} \cdot$$

Thus on each small brick $\mathbb{B}_n$, we see that V'_* has to belong to a domain of Lebesgue measure $O(\min(1,1/n)\varepsilon^{1/8}/|w|)$. This contribution can be estimated thanks to (C.12) in Lemma C.2 so the estimate is identical to the bounds found above up to the loss of an additional $|\log\varepsilon|$. Summing over n and after integration on the variables 1^* and 2^* we obtain an error of size $O(\varepsilon^{9/8}|\log\varepsilon|^4 \times R^7 t^3)$.

Lemma B.1 is proved. □

Remark B.2. *The proof of Lemma* B.1 *shows that* $\varepsilon^{3/4}$ *may be replaced by* ε^δ *for any* $\delta > 1/2$.

B.3. **Parallel recollisions.** The following result was used in Section 6.3 page 49 to deal with parallel recollisions when $t_{1^*} = t_{\bar 1}$. The setting is recalled in Figure 13.

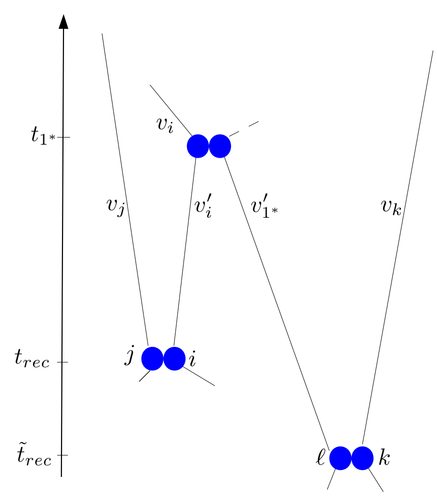

FIGURE 13

Lemma B.3. *With the notations of Figure* 13 *and Proposition 6.2, there are* κ *indices* $\sigma_1,\dots,\sigma_\kappa$ *with* $1 \leq \kappa \leq 4$*, such that*

$$\int \mathbf{1}_{recollision\ (i,j)} \mathbf{1}_{recollision\ (k,\ell)} \prod_{n=1}^{\kappa} |(v_{\sigma_n} - v_{a(\sigma_n)}(t_{\sigma_n})) \cdot \nu_{\sigma_n}| dt_{\sigma_n} d\nu_{\sigma_n} dv_{\sigma_n}$$
$$\leq C(Rt)^r \varepsilon\,.$$

Proof. As in the previous section we suppose from now on that the parameters associated with the first recollision are such that $|\tau_1||v_i - v_j| \geq R$. Otherwise, the estimate (B.3) applied with $M = R$ leads to the expected upper bound.

Denote by t_{2^*} the first time (before t_{1^*}) when one of the particles i, j or k has been deviated. Without loss of generality (up to exchanging j and k), we can assume that i and k are not colliding together at time t_{2^*}.

In the following to simplify the notation we shall denote by $O(\varepsilon)$ a quantity which is bounded by $C(Rt)^r\varepsilon$ for some integer r we do not specify. For the time being, contrary to the previous paragraphs, we do not rescale the collision constraints in ε and simply describe the recollision between (i,j) by the identity

$$v_i' - v_j = \frac{1}{t_{rec} - t_{1^*}}\big(x_i(t_{1^*}) - x_j(t_{1^*}) + q + \varepsilon\nu_{rec}\big)\,, \tag{B.13}$$

with q an element in $\mathbb{Z}^2$ which we fix from now on (in the end the estimates will be multiplied by R^2t^2 to take this fact into account). Similarly the recollision between $(k,1^*)$ can be written

$$v_{1^*}' - v_k = \frac{1}{\tilde t_{rec} - t_{1^*}}\big(x_i(t_{1^*}) + \varepsilon\nu_{1^*} - x_k(t_{1^*}) + \tilde q + \varepsilon\tilde\nu_{rec}\big) \tag{B.14}$$

with $\tilde q$ an element in $\mathbb{Z}^2$ which again we fix from now on, up to mutiplying again the estimates by R^2t^2 at the end.

Equation (B.13) implies that $v_i' - v_j$ lies in a rectangle $\mathcal{R}_1$ of main axis $x_i(t_{1^*}) - x_j(t_{1^*}) + q$, and of size $CR \times (R\varepsilon/|x_i(t_{1^*}) - x_j(t_{1^*}) + q|)$.

On the other hand, Equation (B.14) implies that $v_{1^*}' - v_k$ lies in a rectangle $\mathcal{R}_2$ of main axis $x_i(t_{1^*}) - x_k(t_{1^*}) + \tilde q$ and of size $CR \times (R\varepsilon/|x_i(t_{1^*}) - x_k(t_{1^*}) + \tilde q|)$.

We now translate these conditions with Carleman's parametrization (C.10). We introduce the notation

$$\tilde x_{i,k}(t_{1^*}) := x_i(t_{1^*}) - x_k(t_{1^*}) + \tilde q \quad \text{and} \quad x_{i,j}(t_{1^*}) := x_i(t_{1^*}) - x_j(t_{1^*}) + q\,.$$

The first condition states that V' lies in a small rectangle of size $CR \times (R\varepsilon/|x_{i,j}(t_{1^*})|)$, which is fine since we shall eventually integrate with the measure dV'. The second condition tells us that V_*' has to be in the intersection of the line orthogonal to $(V' - v_i)$ passing through v_i and the rectangle $v_k + \mathcal{R}_2$. We are going to evaluate the length of this intersection.

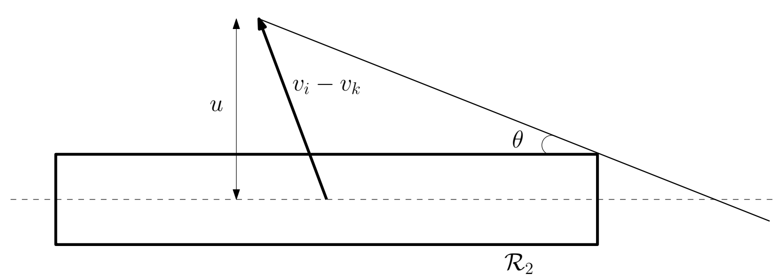

FIGURE 14. The dashed line represents the main axis of the rectangle $\mathcal{R}_2$, oriented in the direction $\tilde x_{i,k}(t_{1^*})$. The angle θ is the smallest angle between the axis of $\mathcal{R}_2$ and any line passing through v_i and intersecting the axis of $\mathcal{R}_2$.

- Suppose that $\big|(v_i - v_k) \wedge \tilde x_{i,k}(t_{1^*})/|\tilde x_{i,k}(t_{1^*})|\big| \le \varepsilon^{\frac34}$. Then we recall that

$$\tilde x_{i,k}(t_{1^*}) := x_i(t_{1^*}) - x_k(t_{1^*}) + \tilde q = x_i - x_k + \tilde q + (v_i - v_k)(t_{1^*} - t_{2^*}),$$

with x_i, x_k the positions at time t_{2^*}. Thus defining

$$\tilde x_{i,k}(t_{2^*}) := x_i - x_k + \tilde q$$

the constraint $\big|(v_i - v_k) \wedge \tilde x_{i,k}(t_{1^*})/|\tilde x_{i,k}(t_{1^*})|\big| \le \varepsilon^{\frac34}$ may be written

$$\big|(v_i - v_k) \wedge \tilde x_{i,k}(t_{2^*})\big| \le C\varepsilon^{\frac34}Rt\,. \tag{B.15}$$

Now we recall from (3.10) that the constraint (B.13) on the rectangle $\mathcal{R}_1$ produces a singularity $\varepsilon|\log\varepsilon|^2/|v_i - v_j|$, and we argue as follows:

- if $|v_i - v_j| \leq \varepsilon^{\frac{9}{16}}$, this constraint can be integrated over two parents of $\{i,j\}$ using (C.12) in Lemma C.4, and we find directly a bound $\varepsilon^{\frac{9}{8}}|\log\varepsilon|$.

- if $|\tilde x_{i,k}(t_{2^*})| \leq \varepsilon^{\frac{5}{8}}$ and $|v_i - v_j| \geq \varepsilon^{\frac{9}{16}}$, the first constraint can be seen as a kind of "recollision" between particles i and k at time t_2^*. Since $|v_i - v_j| \geq \varepsilon^{\frac{9}{16}}$, the contribution $\varepsilon|\log\varepsilon|^2/|v_i-v_j|$ of rectangle $\mathcal{R}_1$ gives a bound of the order $C\varepsilon^{\frac{7}{16}}|\log\varepsilon|$. By integrating the "recollision" (i,k) over two parents 3^* and 4^* of i,k we find a bound $C\varepsilon^{\frac{5}{8}}|\log\varepsilon|^3$ so finally this case produces, after integration over three parameters, the error $C\varepsilon^{\frac{17}{16}}|\log\varepsilon|^4$.

- if $|\tilde x_{i,k}(t_{2^*})| \geq \varepsilon^{\frac{5}{8}}$ then according to (B.15), $v_i - v_k$ must lie in a rectangle $\mathcal{R}_3$ of size $CR \times CRt\varepsilon^{\frac{1}{8}}$. This condition has to be coupled with the singularity $\varepsilon|\log\varepsilon|^2/|v_i-v_j|$ due to the constraint from the rectangle $\mathcal{R}_1$. To perform the integration, we consider $\hat 1, \hat 2$ the first two parents of $\{i,j\}$ and $\hat 3$ the first parent of $\{i,k\}$ (it might coincide with $\hat 1, \hat 2$). We split both contributions by applying the Hölder inequality

(B.16)

$$
\begin{aligned}
&\int \frac{\mathbf{1}_{v_i - v_k \in \mathcal{R}_3}}{|v_i - v_j|} \prod_{\ell = \hat 1, \dots, \hat 3} b(\nu_\ell, v_\ell)\, dv_\ell d\nu_\ell \\
&\quad \leq \left(\int \mathbf{1}_{v_i - v_k \in \mathcal{R}_3} \prod_{\ell = \hat 1, \dots, \hat 3} b(\nu_\ell, v_\ell)\, dv_\ell d\nu_\ell \right)^{1/4} \left(\int \frac{1}{|v_i - v_j|^{4/3}} \prod_{\ell = \hat 1, \dots, \hat 3} b(\nu_\ell, v_\ell)\, dv_\ell d\nu_\ell \right)^{3/4} .
\end{aligned}
$$

The term involving $\mathcal{R}_3$ can be estimated by integrating only over $\hat 3$ thanks to (C.11) (thus it does not matter if the axis of $\mathcal{R}_3$ depends on $\hat 2$). This provides a contribution of order $\varepsilon^{\frac{1}{8}}|\log\varepsilon|^2$. The singularity $1/|v_i - v_j|$ can be integrated out by (C.7), (C.8). Combining this with the contribution of the recollision between i,j, this leads to an upper bound of order $\varepsilon^{\frac{9}{8}}|\log\varepsilon|^3$.

- Suppose that $\big|(v_i - v_k)\wedge \tilde x_{i,k}(t_{1^*})/|\tilde x_{i,k}(t_{1^*})|\big| \geq \varepsilon^{\frac{3}{4}}$. The intersection of the line orthogonal to $(V' - v_i)$ passing through v_i and the rectangle $v_k + \mathcal{R}_2$ (see Figure 14) is a segment of size at most

$$
d \leq \min\Big(\frac{C\varepsilon R}{|\tilde x_{i,k}(t_{1^*})| \sin\theta}, CR \Big)
$$

where θ is the minimal angle between any line passing through v_i and intersecting $v_k + \mathcal{R}_2$. With the notation of Figure 14, we get $u \geq \varepsilon^{3/4}$ and

$$
\sin\theta \geq \frac{u}{2R} \geq \frac{C\varepsilon^{\frac{3}{4}}}{R} \cdot
$$

It follows that

$$
d \leq \frac{C\varepsilon R |v_i - v_k|}{|(v_i - v_k)\wedge \tilde x_{i,k}(t_{1^*})|} \leq \frac{C\varepsilon^{\frac{1}{4}} R^2}{|\tilde x_{i,k}(t_{1^*})|} \cdot
$$

Multiplying this estimate by the size of $\mathcal{R}_1$, we get the following upper bound for the measure in $|(v_{1^*} - v_i)\cdot \nu_{1^*}|\, dv_{1^*} d\nu_{1^*}$

$$
\frac{CR^4 \varepsilon^{\frac{5}{4}}}{\big|\tilde x_{i,k}(t_{1^*})\big|\,\big|x_{i,j}(t_{1^*})\big|} \cdot
$$

The triangle inequality

$$
\begin{aligned}
\big|(x_j(t_{1^*}) - q) - (x_k(t_{1^*}) - \tilde q)\big| &\leq \big|x_i(t_{1^*}) - (x_j(t_{1^*}) - q)\big| + \big|x_i(t_{1^*}) - (x_k(t_{1^*}) - \tilde q)\big| \\
&\leq \big|x_{i,j}(t_{1^*})\big| + \big|\tilde x_{i,k}(t_{1^*})\big|
\end{aligned}
$$

implies that

$$\frac{1}{\big|\tilde{x}_{i,k}(t_{1^*})\big|\,\big|x_{i,j}(t_{1^*})\big|} \leq \frac{1}{\big|(x_j(t_{1^*})-q)-(x_k(t_{1^*})-\tilde{q})\big|}\Big(\frac{1}{\big|\tilde{x}_{i,k}(t_{1^*})\big|}+\frac{1}{\big|x_{i,j}(t_{1^*})\big|}\Big)\,,$$

so finally the measure in $|(v_{1^*}-v_i)\cdot\nu_{1^*}|dv_{1^*}d\nu_{1^*}$ for observing 2 recollisions is bounded by

$$\frac{CR^4\varepsilon^{\frac{5}{4}}}{\big|(x_j(t_{1^*})-q)-(x_k(t_{1^*})-\tilde{q})\big|}\Big(\frac{1}{\big|\tilde{x}_{i,k}(t_{1^*})\big|}+\frac{1}{\big|x_{i,j}(t_{1^*})\big|}\Big)\,. \tag{B.17}$$

In order to integrate this in t_1^* and to get rid of the singularities, we shall distinguish according to the size of $\big|(x_j(t_{1^*})-q)-(x_k(t_{1^*})-\tilde{q})\big|$. Let $0<\delta<1/4$ be given.

- If $\big|(x_j(t_{1^*})-q)-(x_k(t_{1^*})-\tilde{q})\big|\geq\varepsilon^\delta$ then the bound becomes (neglecting as usual from now on powers of R or t)

$$C\varepsilon^{\frac{5}{4}-\delta}\Big(\frac{1}{\big|\tilde{x}_{i,k}(t_{1^*})\big|}+\frac{1}{\big|x_{i,j}(t_{1^*})\big|}\Big)$$

and we are back to usual computations, as in the proof of Proposition 3.5: we deal separately with each singularity on the right-hand side by integrating over the parents of (i,k) (resp. (i,j)) and this gives rise in the end to two integrals producing a bound of the type

$$C\varepsilon^{\frac{5}{4}-\delta}|\log\varepsilon|^3\,.$$

- If $\big|(x_j(t_{1^*})-q)-(x_k(t_{1^*})-\tilde{q})\big|\leq\varepsilon^\delta$ then we do not use the bound (B.17) but consider this condition as a "recollision" between particles j and k at time t_{1^*}. Integrating only the condition on the first rectangle at time t_{1^*} (and bounding the velocities V_*' by R, disregarding rectangle $\mathcal{R}_2$) provides the usual estimate (3.11)

$$\min\Big(\frac{\varepsilon}{|v_i-v_j|},1\Big)|\log\varepsilon|^2\,. \tag{B.18}$$

We then combine this estimate with the "recollision" at time t_{1^*}.

If the collision with 2^* involves k, we get a bound of the type

$$\min\Big(\frac{\varepsilon^\delta}{|\bar{v}_k-v_j|},1\Big)|\log\varepsilon|^2.$$

It remains to integrate the singularities in velocities using Lemma C.2. If the first parent acts on i or k, then one can integrate the singularities one after the other and get the error $\varepsilon^{1+\delta}|\log\varepsilon|^6$. If not then one first needs to reduce the singularity in order to be able to use (C.8). Indeed this singularity turns out to be too large (see Remark C.3), so we replace the above bounds by

$$\frac{C}{|v_i-v_j|^\gamma}\frac{C}{|\bar{v}_k-v_j|^\gamma}\varepsilon^{\gamma(1+\delta)}|\log\varepsilon|^4 \tag{B.19}$$

with $\gamma<1$, and the Cauchy-Schwarz inequality to deal with one or the other singularity, using again two parents. This produces finally, using four integration parameters, an error of the type

$$\varepsilon^{\gamma(1+\delta)}|\log\varepsilon|^4\,.$$

It suffices to choose γ such that $1/(1+\delta)<\gamma<1$ to have the result.

If the collision with 2^* involves j, we get as in the proof of Lemma 3.7 that v_j-v_k has to belong to a rectangle $\mathcal{R}_4$ of size $2R\times\Big(R\min\Big(\frac{4\varepsilon^{\delta-1}}{|\tau_2||\bar{v}_j-v_k|},1\Big)\Big)$, where τ_2 is a rescaled time as in (3.8). Combined with the condition (B.18), one has to integrate with respect to 2^*

$$\mathbf{1}_{\{v_j-v_k\in\mathcal{R}_4\}}\ \min\Big(\frac{\varepsilon}{|v_i-v_j|},1\Big)|\log\varepsilon|^2\,.$$

We proceed as in (C.13) and partition $\mathcal{R}_4$ into balls of radius $a = R\min\Big(\frac{4\varepsilon^{\delta-1}}{|\tau_2||\bar v_j - v_k|}, 1\Big)$ centered at points $\{w_{k'}\}_{k'\leq R/a}$

$$\begin{aligned}
&\int \mathbf{1}_{\{v_j - v_k\in\mathcal{R}_4\}}\ \min\Big(\frac{\varepsilon}{|v_i - v_j|}, 1\Big)\,\big|(\bar v_j - v_{2^*})\cdot\nu_{2^*}\big|\,dv_{2^*}d\nu_{2^*}\\
&\qquad\leq \sum_{k'=0}^{R/a}\int \mathbf{1}_{|v_j - w'_k|\leq 2a}\ \min\Big(\frac{\varepsilon}{|v_i - v_j|}, 1\Big)\,\big|(\bar v_j - v_{2^*})\cdot\nu_{2^*}\big|\,dv_{2^*}d\nu_{2^*}\,.
\end{aligned}$$

If there exists a value k'_0 such that $|v_i - w_{k'_0}|\leq a$ then one has to take care of the singularity $\frac{\varepsilon}{|v_i - v_j|}$ only for v_j in the ball indexed by k'_0. In the other balls, $\min\Big(\frac{\varepsilon}{|v_i - v_j|}, 1\Big)$ can be estimated from above by $\frac{\varepsilon}{a|k' - k'_0|}$ and decays with the distance from $w_{k'_0}$. Thus we get

$$\begin{aligned}
&\int \mathbf{1}_{\{v_j - v_k\in\mathcal{R}_4\}}\ \min\Big(\frac{\varepsilon}{|v_i - v_j|}, 1\Big)\,\big|(\bar v_j - v_{2^*})\cdot\nu_{2^*}\big|\,dv_{2^*}d\nu_{2^*}\\
&\qquad\leq \varepsilon\int \frac{\mathbf{1}_{|v_j - v_i|\leq 4a}}{|v_i - v_j|}\,\big|(\bar v_j - v_{2^*})\cdot\nu_{2^*}\big|\,dv_{2^*}d\nu_{2^*}\\
&\qquad\quad+\sum_{\substack{k'\neq k'_0\\ k'\leq R/a}}\frac{\varepsilon}{a|k' - k'_0|}\int \mathbf{1}_{|v_j - w'_k|\leq 2a}\,\big|(\bar v_j - v_{2^*})\cdot\nu_{2^*}\big|\,dv_{2^*}d\nu_{2^*}\,.
\end{aligned}$$

Using (C.6) to integrate the first term of the RHS and summing over k' (see (C.13)) in the second term, we deduce that

$$\begin{aligned}
&\int \mathbf{1}_{\{v_j - v_k\in\mathcal{R}_4\}}\ \min\Big(\frac{\varepsilon}{|v_i - v_j|}, 1\Big)\,\big|(v_j - v_{2^*})\cdot\nu_{2^*}\big|\,dv_{2^*}d\nu_{2^*}\\
&\qquad\leq C\varepsilon R^2\min\Big(1, \frac{a}{|\bar v_j - v_i|}\Big) + R^2\log(R/a)\varepsilon\min\Big(1, \frac{a}{|\bar v_j - v_i|}\Big)\,.
\end{aligned}$$

Recall that $a = R\min\Big(\frac{4\varepsilon^{\delta-1}}{|\tau_2||\bar v_j - v_k|}, 1\Big)$, we can then integrate the singularities with respect to the velocities by using further parents. As in (B.19), the double singularity $\frac{1}{|\bar v_j - v_i||\bar v_j - v_k|}$ has to be modified by a factor $\gamma<1$ before being integrated. Finally, integrating with respect to τ_2 and changing variable to t_{2^*} as in (3.10), we get an upper bound of order $O(\varepsilon^{1+\delta\gamma}|\log\varepsilon|^4)$.

The proposition is proved. □

B.4. **Recollisions in chain.** This lemma was used in Section 6.3 page 49 to deal with the case when recollisions occur in chain, with $t_{\tilde 1} = t_{1^*}$. The situation is that depicted in Figure 15.

Lemma B.4. *With the notations of Figure* 15 *and Proposition* 6.2*, there are κ indices $\sigma_1,\dots,\sigma_\kappa$ with $1\leq\kappa\leq 3$, and an integer r such that*

$$\iint \mathbf{1}_{recollision\ (i,j)}\mathbf{1}_{recollision\ (i,k)}\prod_{n=1}^{\kappa}|(v_{\sigma_n} - v_{a(\sigma_n)}(t_{\sigma_n}))\cdot\nu_{\sigma_n}|dt_{\sigma_n}d\nu_{\sigma_n}dv_{\sigma_n}\leq C(Rt)^r\varepsilon\,.$$

Proof. Recall that the condition for the first recollision states

$$v'_i - v_j = \frac{1}{\tau_{rec}}\delta x_\perp - \frac{\tau_1}{\tau_{rec}}(v_i - v_j) - \frac{1}{\tau_{rec}}\nu_{rec}\,, \tag{B.20}$$

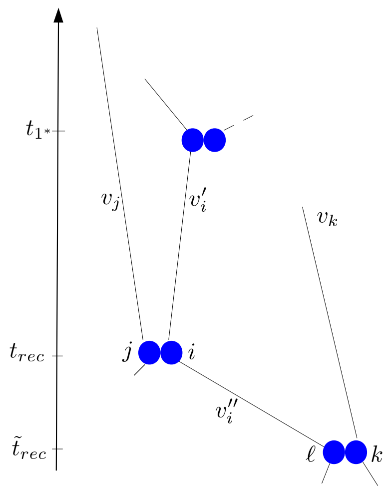

FIGURE 15

with x_i, x_j the positions at time t_{2^*}

$$\text{(B.21)}\qquad \begin{aligned} &\delta x := \frac{1}{\varepsilon}(x_i - x_j - q) = \lambda_{i,j}(v_i - v_j) + \delta x_\perp \qquad \text{with} \quad \delta x_\perp \cdot (v_i - v_j) = 0\,,\\ &\tau_1 := \frac{1}{\varepsilon}(t_{1^*} - t_{2^*} - \lambda_{i,j})\,, \qquad \tau_{rec} := \frac{1}{\varepsilon}(t_{rec} - t_{1^*})\,, \end{aligned}$$

for some q in $\mathbb{Z}^2$ of norm smaller than $O(R^2t^2)$ to take into account the periodicity.

When $|\tau_1||v_i - v_j| \leq R^2$, estimate (B.3) is enough to obtain an upper bound of order ε without taking into account the second recollision. Our goal is to prove that the constraint of having a second recollision produces an integrable function of $|\tau_1|\,|v_i - v_j| \geq R^2$: by the change of variables (B.21) that will prove the expected result. Note that unlike the previous paragraphs, the case $|\tau_1|\,|v_i - v_j| \gg 1$ does not give a better bound in terms of powers of ε.

From (B.20), we deduce as in (3.12) that

$$\text{(B.22)}\qquad \frac{1}{|\tau_{rec}|} \leq \frac{4R}{|\tau_1||v_i - v_j|} \quad \text{which implies that} \quad |\tau_{rec}| \geq R/4 \gg 1\,.$$

Two cases have to be considered: $k = 1^*$ and $k \neq 1^*$.

• If $k = 1^*$, the equation for the second recollision states

$$\tau'_{rec}(v''_i - v'_{1^*}) = \pm\nu_{1^*} - \tau_{rec}(v'_i - v'_{1^*})(+\nu_{rec}) - \tilde{\nu}_{rec}$$

where

$$\tau'_{rec} := \frac{1}{\varepsilon}(\tilde{t}_{rec} - t_{rec})\,,$$

and where the $\pm$ and the translation by ν_{rec} depend on the possible exchanges in the labels of the particles at collision times. It can be rewritten, thanks to (B.5),

$$\text{(B.23)}\qquad \begin{aligned} &\tau'_{rec}(v_j - v'_i)\cdot\nu_{rec}\,\nu_{rec} = \pm\nu_{1^*} - (\tau_{rec} + \tau'_{rec})(v'_i - v'_{1^*})(+\nu_{rec}) - \tilde{\nu}_{rec}\\ \text{or}\quad &\tau'_{rec}(v_j - v'_i)\cdot\nu^\perp_{rec}\,\nu^\perp_{rec} = \pm\nu_{1^*} - (\tau_{rec} + \tau'_{rec})(v'_i - v'_{1^*})(+\nu_{rec}) - \tilde{\nu}_{rec}\,. \end{aligned}$$

We further know that $|v_i' - v_{1^*}'| = |v_i - v_{1^*}|$.

- If $|v_i - v_{1^*}| \geq R\,|\tau_{rec}|^{-3/4}$, then the vector in the right-hand side of (B.23) has a magnitude of order

$$|\tau_{rec} + \tau_{rec}'|\,|v_i' - v_{1^*}'| \geq |\tau_{rec}|\,|v_i' - v_{1^*}'| \geq R\,|\tau_{rec}|^{1/4}\,.$$

It follows that the vector ν_{rec} has to be aligned in the direction of $v_i' - v_{1^*}'$ with a controlled error

$$\nu_{rec} = \mathcal{R}_{n\pi/2}\frac{v_i' - v_{1^*}'}{|v_i - v_{1^*}|} + O\left(\frac{1}{|\tau_{rec}|^{1/4}}\right)\,,$$

recalling that $\mathcal{R}_\theta$ is the rotation of angle θ.

Plugging the formula for ν_{rec} into (B.20), we obtain that

$$v_i' - v_j = \frac{1}{\tau_{rec}}\delta x_\perp - \frac{\tau_1}{\tau_{rec}}(v_i - v_j) - \frac{1}{\tau_{rec}}\mathcal{R}_{n\pi/2}\frac{v_i' - v_{1^*}'}{|v_i - v_{1^*}|} + O\left(\frac{1}{|\tau_{rec}|^{5/4}}\right)\,.$$

The same arguments as in the proof of Lemma B.1 show that this provides a contribution of size $O(|\tau_{rec}|^{-5/4}|\log\tau_{rec}|)$, hence an integrable function of $\tau_1|v_i - v_j|$: from (B.22), $\dfrac{1}{|\tau_{rec}|}$ is controlled by $\dfrac{1}{|\tau_1||v_i - v_j|}$, thus integrating with respect to t_{1^*} we recover the factor ε and the singularity in $|v_i - v_j|$ is removed as usual by integration over the parents of i, j.

- If $|v_i - v_{1^*}| \leq R|\tau_{rec}|^{-3/4}$, we find that v_{1^*} has to belong to a domain of size $|\tau_{rec}|^{-3/2}$, hence again we obtain an integrable function of $|\tau_1|\,|v_i - v_j|$, with no extra gain in ε.

- If $k \neq 1^*$, the position of particle k at the time $\tilde t_{rec}$ of the second recollision is given by

$$x_k + v_k(\tilde t_{rec} - t_{2^*})\,.$$

We have written x_k for the position of particle k at time t_{2^*} and similarly x_j stands for the position of particle j at time t_{2^*}. We end up with the condition for the second recollision

$$(\tilde t_{rec} - t_{rec})(v_i'' - v_k) = x_j - x_k - (v_j - v_k)(t_{1^*} - t_{2^*} + t_{rec} - t_{1^*}) - \varepsilon\tilde\nu_{rec}(+\varepsilon\nu_{1^*} + \varepsilon\nu_{rec}) + \tilde q\,, \tag{B.24}$$

for some $\tilde q \in \mathbb{Z}^2$ not larger than $O(R^2t^2)$, and where the translations $\varepsilon\nu_{1^*}$ and $\varepsilon\nu_{rec}$ arise only if the labels of particles are exchanged at t_{1^*} and/or at t_{rec}. In the following, we fix q and $\tilde q$ and will multiply the final estimate by $(R^2t^2)^2$ to take into account the periodicity in both recollisions. Using the notation (B.21), we then rescale in ε and write

$$\delta x_{jk} := \frac{x_j - x_k + \tilde q}{\varepsilon}(+\nu_{1^*}) =: \lambda_{jk}(v_j - v_k) + \delta x_{jk}^\perp\,,\quad \delta x_{jk}^\perp\cdot(v_j - v_k) = 0\,,$$

$$\tau_1^* := \frac{1}{\varepsilon}(t_{1^*} - t_{2^*} - \lambda_{jk})\,,\quad \tau_{rec} := \frac{t_{rec} - t_{1^*}}{\varepsilon}\,,\quad \tau_{rec}' := \frac{\tilde t_{rec} - t_{rec}}{\varepsilon}\,.$$

Then Equation (B.24) for the second recollision becomes

$$\tau_{rec}'(v_i'' - v_k) = \delta x_{jk}^\perp - (v_j - v_k)(\tau_1^* + \tau_{rec}) - \tilde\nu_{rec}(+\nu_{rec})\,. \tag{B.25}$$

We consider three different cases.

- Suppose $|v_i' - v_j| \geq |\tau_{rec}|^{-5/8}$ and $|(v_j - v_k)(\tau_1^* + \tau_{rec})| \geq |\tau_{rec}|^{3/4}$. From (B.22) we know that $|\tau_{rec}| \gg 1$, so that combining the two inequalities of the assumption implies

$$|v_i' - v_j| \geq \frac{1}{|(v_j - v_k)(\tau_1^* + \tau_{rec})|^{5/6}} \gg \frac{1}{|(v_j - v_k)(\tau_1^* + \tau_{rec})|}.$$

As in the proof of Lemma 3.7, the equation (B.25) implies that v_i'' should belong to a rectangle of size $2R \times 2R/|(v_j - v_k)(\tau_1^* + \tau_{rec})|$. Furthermore v_i'' belongs as well to the circle of diameter

$[v_j, v_i']$ by definition. Since the diameter $|v_i' - v_j|$ is much larger than the width of the rectangle (see Figure 16), we deduce that ν_{rec} has to be in an angular sector of size at most

$$O\left(\frac{R}{\sqrt{|v_i' - v_j|}\sqrt{|(v_j - v_k)(\tau_1^* + \tau_{rec})|}}\right), \tag{B.26}$$

around some direction $\bar{\nu}_{rec}$.

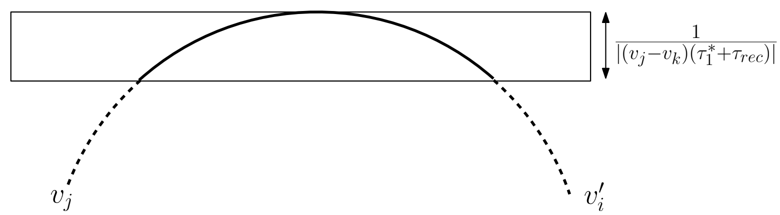

FIGURE 16. The portion of the circle with diameter $[v_j, v_i']$ intersecting the rectangle of width $\frac{1}{|(v_j-v_k)(\tau_1^*+\tau_{rec})|}$ is represented in thick line. The velocity v_i'' takes his values on this portion. Since the circle has a diameter much larger than the width, the largest overlap is obtained when the circle is tangent to the rectangle.

The assumption that $|v_i' - v_j| \geq |\tau_{rec}|^{-5/8}$ and $|(v_j - v_k)(\tau_1^* + \tau_{rec})| \geq |\tau_{rec}|^{3/4}$ combined to (B.26) implies a strong contraint on the recollision angle

$$\nu_{rec} = \bar{\nu}_{rec} + O\left(\frac{1}{|\tau_{rec}|^{1/16}}\right).$$

Plugging this formula in (B.20), we finally obtain that

$$v_i' - v_j = \frac{1}{\tau_{rec}}\delta x_\perp - \frac{\tau_1}{\tau_{rec}}(v_i - v_j) - \frac{1}{\tau_{rec}}\bar{\nu}_{rec} + O\left(\frac{1}{|\tau_{rec}|^{17/16}}\right).$$

The same arguments as in the proof of Lemma 3.7 show that this provides a contribution of size $O(\tau_{rec}^{-17/16}|\log \tau_{rec}|)$ hence an integrable function of $|\tau_1|$ with a singularity in $|v_i - v_j|$ which can be integrated by Lemma C.2. Changing variables from τ_1 to t_{1^*}, we obtain an upper bound of order ε.

- Suppose $|v_i' - v_j| \leq |\tau_{rec}|^{-5/8}$. We obtain by (C.3) that

$$\int \mathbf{1}_{|v_i'-v_j| \leq |\tau_{rec}|^{-5/8}} |(v_{1^*} - v_i) \cdot \nu_{1^*}| d\nu_{1^*} dv_{1^*} \leq C|\tau_{rec}|^{-5/4}|\log \tau_{rec}|,$$

which again produces an integrable function of $|\tau_1||v_i - v_j|$ and leads to an upper bound of order ε.

- Suppose $|(v_j - v_k)(\tau_1^* + \tau_{rec})| \leq |\tau_{rec}|^{3/4}$, this condition implies that

$$|\tau_{rec}|\Big(|v_j - v_k| + O(|\tau_{rec}|^{-1/4})\Big) = |(v_j - v_k)\tau_1^*|$$

from which we deduce that

$$\frac{1}{|\tau_{rec}|} = \frac{1}{|\tau_1^*|} + O\left(\frac{|\tau_{rec}|^{-5/4}}{|v_k - v_j|}\right).$$

This imposes a constraint on the first recollision. From (B.20), we deduce that $v_i' - v_j$ does not belong to the entire rectangle of size $2R \times (4R/|\tau_1||v_i - v_j|)$, but only to a small portion of it of size

$$2R \min\left\{\frac{C}{|\tau_{rec}|^{5/4}|v_k - v_j|}, 1\right\} \times \frac{4R}{|\tau_1||v_i - v_j|} \leq \frac{CR}{|\tau_{rec}|^{1/4}|v_k - v_j|^{1/5}} \times \frac{4R}{|\tau_1||v_i - v_j|}.$$

Note that the left-hand side has a singularity in $1/|v_k - v_j|$ which has to be integrated out later on. As previously in this section the singularity turns out to be too large (see Remark C.3), thus we replaced the singularity in the min by a power $1/5$ in order to regain some control on the relative velocities up to a loss in the power of $|\tau_{rec}|$. From the upper bound (B.22) on $|\tau_{rec}|$, we deduce that the relative velocities belong to a set of size at most

$$O\left(\frac{1}{|\tau_1|^{5/4}|v_k - v_j|^{1/5}|v_i - v_j|^{5/4}}\right).$$

The most singular case is $|v_k - v_j| = |v_i - v_j|$ and the same arguments as in the proof of Lemma B.1 show that this gives a contribution of size $O\big(\frac{\log(|\tau_1||v_i - v_j|)}{|\tau_1|^{5/4}\,|v_i - v_j|^{29/20}}\big)$. In all cases, the singularity in the velocities can be integrated by (C.7), (C.8). Finally integrating $1/|\tau_1|^{5/4}$, we recover an upper bound of order ε.

This concludes the proof of Lemma B.4. □

B.5. **Two particles recollide twice in chain due to periodicity.** The following lemma computes the cost of having a self-recollision due to periodicity (see Figure 17).

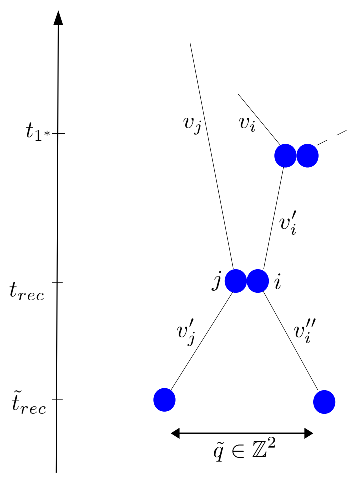

FIGURE 17

Lemma B.5. *There exist three indices $\sigma_1, \sigma_2, \sigma_3$ in $\{2, \dots, s\}^3$ such that assuming that the total energy $|V_s|^2$ is bounded by $1 \le R^2 \le |\log \varepsilon|$, and $t \ge 1$,*

$$\int \mathbf{1}_{\textit{periodic self-recollision}} \prod_{n=1}^{3} |(v_{\sigma_n} - v_{a(\sigma_n)}(t_{\sigma_n})) \cdot \nu_{\sigma_n}| dt_{\sigma_n} d\nu_{\sigma_n} dv_{\sigma_n} \le C(Rt)^r\, \varepsilon\,,$$

for some integer r.

Proof. We recall the equation (3.9) on the first recollision

$$v_i' - v_j = \frac{1}{\tau_{rec}}(\delta x_\perp - \tau_1(v_i - v_j) - \nu_{rec}) \quad \text{with} \quad \frac{1}{|\tau_{rec}|} \le \frac{4R}{|\tau_1||v_i - v_j|}\,. \tag{B.27}$$

The equation on the second recollision is

$$(v_i'' - v_j')(\tilde{t}_{rec} - t_{rec}) = \varepsilon \tilde{\nu}_{rec} + \varepsilon \nu_{rec} + \tilde{q} \tag{B.28}$$

for some $\tilde{t}_{rec} \geq 0$, $\tilde{\nu}_{rec} \in \mathbb{S}$, and $\tilde{q} \in \mathbb{Z}^2 \setminus \{0\}$. Note that $\tilde{q} \neq 0$ as the second recollision occurs from the periodicity. As usual we fix $\tilde{q}$ and multiply the estimates in the end by $O(R^2 t^2)$ to take that into account.

The condition (B.28) implies that the vector $v_i'' - v_j'$ is located in a cone of axis $\tilde{q}$ and angular sector 2ε. By definition, we have

$$v_i'' - v_j' = (v_i' - v_j) - 2(v_i' - v_j) \cdot \nu_{rec} \nu_{rec},$$

which means that $\nu_{rec}^{\perp}$ is the bisector of $v_i' - v_j$ and $v_i'' - v_j'$.

From (B.27), we deduce that the direction of $v_i' - v_j$ is

$$\frac{\delta x_\perp - \tau_1 (v_i - v_j)}{|\delta x_\perp - \tau_1 (v_i - v_j)|} + O\left(\frac{1}{|\tau_1 (v_i - v_j)|} \right).$$

From (B.28), we deduce that the direction of $v_i'' - v_j'$ is

$$\frac{\tilde{q}}{|\tilde{q}|} + O(\varepsilon).$$

Finally we get that $\nu_{rec}^{\perp}$ is known up to an error term which can be bounded by

$$\eta = \varepsilon + \frac{1}{\sqrt{|\tau_1 (v_i - v_j)|}} \cdot$$

Note that we have introduce the square root as in the proof of Lemmas B.3 and B.4 for integrability purposes.

Plugging this Ansatz in (B.27), we get that $v_i' - v_j$ has to belong to a rectangle $\mathcal{R}$ of size $R \times R\eta / |\tau_1 (v_i - v_j)|$. By Lemma C.4, we obtain

$$\int \mathbf{1}_{v_i' - v_j \in \mathcal{R}} \left| \big(v_1^* - v_j \big) \cdot \nu_1^* \big) \right| dv_1^* d\nu_1^* \leq C R^3 \frac{\varepsilon |\log \varepsilon|}{\tau_1 |v_i - v_j|} + C R^3 \frac{1}{\tau_1^{3/2} |v_i - v_j|^{3/2}} \cdot$$

By integration with respect to time, we then get

$$\int \mathbf{1}_{\text{(B.27) and (B.28)}} \left| \big(v_1^* - v_j \big) \cdot \nu_1^* \big) \right| dv_1^* d\nu_1^* dt_1^* \leq C R^3 \frac{\varepsilon^2 |\log \varepsilon|^2}{|v_i - v_j|} + C R^3 \frac{\varepsilon}{|v_i - v_j|^{3/2}} \cdot$$

We then apply twice Lemma C.2 to integrate the singularities at small relative velocities. □

Appendix C. Carleman's parametrization and scattering estimates

In Section 3 and Appendix B, we were faced with integrals containing singularities in relative velocities $v_i - v_j$ and with a multiplicative factor of the type $(v^* - \bar{v}_i) \cdot \nu^*$ where v_i is recovered from v^*, ν^* and $\bar{v}_i$ through a scattering condition. This appendix is devoted to the proof of "tool-box" lemmas for computing these singular integrals. These lemmas are used many times in this paper.

Lemma C.1. *Fix a velocity $\bar{v}_i$ and let v_i, v_j be the velocities after a collision (with or without scattering)*

$$(v_i, v_j) = (\bar{v}_i, v^*) \quad \textit{or} \quad \begin{cases} v_i = \bar{v}_i + (v^* - \bar{v}_i) \cdot \nu^* \nu^*, \\ v_j = v^* - (v^* - \bar{v}_i) \cdot \nu^* \nu^*, \end{cases}$$

with $\nu^* \in \mathbb{S}$ and $v^* \in \mathbb{R}^2$ (see Figure 18). Assume all the velocities are bounded by R then

$$\int \frac{1}{|v_i - v_j|} \left| \big(v^* - \bar{v}_i \big) \cdot \nu^* \big) \right| dv^* d\nu^* \leq C R^2. \tag{C.1}$$

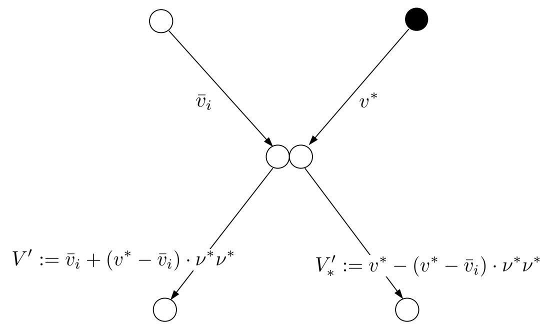

FIGURE 18. Scattering relations

Proof. In both cases, the velocities before and after the collision are related by $|v_i - v_j| = |v^* - \bar v_i|$. Inequality (C.1) follows from the fact that the singularity $1/|v^* - \bar v_i|$ is integrable. $\square$

Lemma C.2. *Fix $\bar v_i$ and v_j, and define v_i to be one of the following velocities*

$$v_i = v^* - (v^* - \bar v_i)\cdot \nu^*\nu^*, \qquad \text{(C.2)}$$
$$\text{or} \quad v_i = \bar v_i + (v^* - \bar v_i)\cdot \nu^*\nu^*,$$

with $\nu^* \in \mathbb{S}$ and $v^* \in B_R \subset \mathbb{R}^2$ (see Figure 18). Assume all the velocities are bounded by $R > 1$ and fix $\delta > 0$. Then the following estimates hold, denoting $b(\nu^*, v^*) := |(v^* - \bar v_i)\cdot \nu^*|$:

$$\int \mathbf{1}_{|v_i - v_j|\leq \delta}\, b(\nu^*, v^*)\, dv^* d\nu^* \leq CR^2\delta \min\Big(\frac{\delta}{|v_j - \bar v_i|}, 1\Big)\,, \qquad \text{(C.3)}$$

$$\int \min\Big(\frac{\delta}{|v_i - v_j|}, 1\Big)\, b(\nu^*, v^*)\, dv^* d\nu^* \leq CR^2\delta|\log\delta| + CR^3\delta\,, \qquad \text{(C.4)}$$

$$\int \frac{\mathbf{1}_{|v_i - v_j|\leq \delta}}{|v_i - v_j|}\, b(\nu^*, v^*)\, dv^* d\nu^* \leq CR^2 \min\Big(1, \frac{\delta}{|v_j - \bar v_i|}\Big) \qquad \text{(C.5)}$$

$$\int \frac{1}{|v_i - v_j|}\, b(\nu^*, v^*)\, dv^* d\nu^* \leq CR^2\Big(\big|\log|\bar v_i - v_j|\big| + R\Big)\,, \qquad \text{(C.6)}$$

$$\int \frac{1}{|v_i - v_j|^\gamma}\, b(\nu^*, v^*)\, dv^* d\nu^* \leq \frac{CR^2}{|\bar v_i - v_j|^{\gamma-1}} \quad \textit{for } \gamma \in]1,2[\,, \qquad \text{(C.7)}$$

$$\int \frac{1}{|v_i - v_j|^\gamma}\, b(\nu^*, v^*)\, dv^* d\nu^* \leq CR^3 \quad \textit{for } \gamma \in]0,1[\,, \qquad \text{(C.8)}$$

$$\int \big|\log|v_i - v_j|\big|\, b(\nu^*, v^*)\, dv^* d\nu^* \leq CR^3\,. \qquad \text{(C.9)}$$

Proof. We start by recalling Carleman's parametrization, which we shall be using many times in this Appendix: it is defined by

$$(v^*, \nu^*) \in \mathbb{R}^2 \times \mathbb{S} \mapsto \begin{cases} V'_* := v^* - (v^* - \bar v_i)\cdot \nu^*\nu^* \\ V' := \bar v_i + (v^* - \bar v_i)\cdot \nu^*\nu^* \end{cases} \qquad \text{(C.10)}$$

where (V', V'_*) belong to the set $\mathcal{C}$ defined by

$$\mathcal{C} := \Big\{(V', V'_*) \in \mathbb{R}^2 \times \mathbb{R}^2 \,/\, (V' - \bar v_i)\cdot(V'_* - \bar v_i) = 0\Big\}\,.$$

This map sends the measure $b(\nu^*, v^*)\, dv^* d\nu^*$ on the measure $dV' dS(V'_*)$, where dS is the Lebesgue measure on the line orthogonal to $(V' - \bar v_i)$ passing through $\bar v_i$.

Now let us consider the case when $|v_i - v_j| \leq \delta$ and prove (C.3). What we need here is to estimate the measure of the pre-image of the small ball of center v_j and radius δ by the scattering operator: let us study how for fixed v_j, the set $\{|v_i - v_j| \leq \delta\}$ is transformed by the inverse scattering map. Notice that the most singular case concerns the case when $v_i = V'_*$ belongs to the small ball of radius δ: indeed in the case when it is V' then the measure $b(\nu^*, v^*)\, dv^* d\nu^*$ will have support in a domain of size $O(\delta^2)$. So now assume that V'_* satisfies $|V'_* - v_j| \leq \delta$.

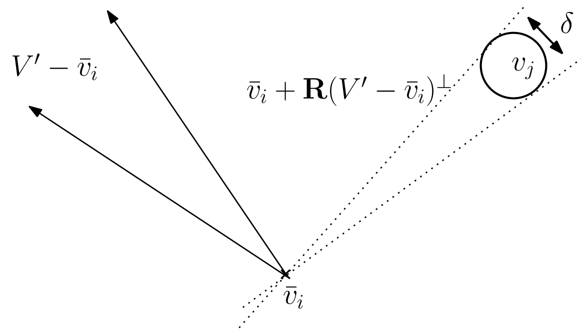

FIGURE 19. V'_* has to belong to the ball of radius δ around v_j, thus it has to be in the cone with the doted lines. By Carleman's parametrization, this imposes constraints on the angular sector of $V' - \bar{v}_i$.

- If $|v_j - \bar{v}_i| \leq \delta$, meaning that $\bar{v}_i$ is itself in the same ball, then for any $V' \in B_R$, the intersection between the small ball and the line $\bar{v}_i + \mathbb{R}(V' - \bar{v}_i)^\perp$ is a segment, the length of which is at most δ. We therefore find

$$\int \mathbf{1}_{|V'_* - v_j| \leq \delta}\, b(\nu^*, v^*)\, dv^* d\nu^* \leq CR^2 \delta\,.$$

- If $|v_j - \bar{v}_i| > \delta$, in order for the intersection between the ball and the line $\bar{v}_i + \mathbb{R}(V' - \bar{v}_i)^\perp$ to be non empty, we have the additional condition that $V' - \bar{v}_i$ has to be in an angular sector of size $\delta / |v_j - \bar{v}_i|$ (see Figure 19). We therefore have

$$\int \mathbf{1}_{|V'_* - v_j| \leq \delta}\, b(\nu^*, v^*)\, dv^* d\nu^* \leq CR^2 \frac{\delta^2}{|v_j - \bar{v}_i|} \cdot$$

We therefore conclude that (C.3) holds.

The other estimates provided in Lemma C.2 then come from Fubini's theorem: let us start with (C.4). We write

$$\begin{aligned}
\int \min\Big(\frac{\delta}{|v_i - v_j|}, 1\Big)\, b(\nu^*, v^*)\, dv^* d\nu^* &= \int \mathbf{1}_{|v_i - v_j| \leq \delta}\, b(\nu^*, v^*)\, dv^* d\nu^* \\
&\quad + \int \mathbf{1}_{|v_i - v_j| > \delta} \frac{\delta}{|v_i - v_j|}\, b(\nu^*, v^*)\, dv^* d\nu^* \\
&\leq CR^2 \delta + \int \mathbf{1}_{|v_i - v_j| > \delta} \frac{\delta}{|v_i - v_j|}\, b(\nu^*, v^*)\, dv^* d\nu^*
\end{aligned}$$

thanks to (C.3). The contribution of the velocities such that $|v_i - v_j| \geq 1$ can be bounded by $R^3 \delta$. Thus it is enough to consider

$$\begin{aligned}
\int \frac{\delta \mathbf{1}_{1 \geq |v_i - v_j| > \delta}}{|v_i - v_j|}\, b(\nu^*, v^*)\, dv^* d\nu^* &= \delta \int \Big(\int_{|v_i - v_j|}^1 \frac{dr}{r^2} + 1 \Big) \mathbf{1}_{1 \geq |v_i - v_j| > \delta}\, b(\nu^*, v^*)\, dv^* d\nu^* \\
&\leq \delta \int_\delta^1 \frac{dr}{r^2} \int \mathbf{1}_{|v_i - v_j| \leq r}\, b(\nu^*, v^*)\, dv^* d\nu^* + CR^3 \delta\,,
\end{aligned}$$

so using (C.3) again we get

$$\int \mathbf{1}_{1\geq|v_i-v_j|>\delta}\frac{\delta}{|v_i-v_j|}\,b(\nu^*,v^*)\,dv^*d\nu^* \leq CR^2\delta\int_\delta^1\frac{dr}{r}+CR^3\delta\,,$$

from which (C.4) follows.

Then we prove (C.5), by writing similarly

$$\begin{aligned}\int \frac{\mathbf{1}_{|v_i-v_j|\leq\delta}}{|v_i-v_j|}\,b(\nu^*,v^*)\,dv^*d\nu^* &= \int \Big(\int_{|v_i-v_j|}^{\delta}\frac{dr}{r^2}+\frac{1}{\delta}\Big)\mathbf{1}_{|v_i-v_j|\leq\delta}\,b(\nu^*,v^*)\,dv^*d\nu^*\\ &= \int_0^\delta\frac{1}{r^2}\Big(\int\mathbf{1}_{|v_i-v_j|\leq r}\,b(\nu^*,v^*)\,dv^*d\nu^*\Big)dr\\ &\quad+\frac{1}{\delta}\int\mathbf{1}_{|v_i-v_j|\leq\delta}\,b(\nu^*,v^*)\,dv^*d\nu^*\,.\end{aligned}$$

Applying twice (C.3), it follows that

$$\begin{aligned}\int \frac{\mathbf{1}_{|v_i-v_j|\leq\delta}}{|v_i-v_j|}\,b(\nu^*,v^*)\,dv^*d\nu^* &\leq CR^2\mathbf{1}_{|\bar v_i-v_j|\leq\delta}\Big(\int_0^{|\bar v_i-v_j|}\frac{1}{|\bar v_i-v_j|}\mathbf{1}_{r\leq\delta}\,dr+\int_{|\bar v_i-v_j|}^{\delta}\frac{dr}{r}\Big)\\ &\quad+CR^2\min\big(\frac{\delta}{|\bar v_i-v_j|},1\big)\\ &\leq CR^2\left[\min\big(\frac{\delta}{|\bar v_i-v_j|},1\big)+\mathbf{1}_{|\bar v_i-v_j|\leq\delta}\left(\delta+\Big|\log\frac{\delta}{|\bar v_i-v_j|}\Big|\right)\right]\,.\end{aligned}$$

Since the logarithmic divergence is controlled by the singularity $\delta/|\bar v_i-v_j|$, inequality (C.5) follows.

Next let us prove (C.6)-(C.8). We have as above

$$\begin{aligned}\int\frac{1}{|v_i-v_j|^\gamma}\,b(\nu^*,v^*)\,dv^*d\nu^* &= \gamma\int\Big(\int_{|v_i-v_j|}^1\frac{1}{r^{1+\gamma}}dr+1\Big)\,b(\nu^*,v^*)\,dv^*d\nu^*\\ &= \gamma\int_0^1\frac{1}{r^{\gamma+1}}\Big(\int\mathbf{1}_{|v_i-v_j|\leq r}\,b(\nu^*,v^*)\,dv^*d\nu^*\Big)dr+CR^3\\ &\leq C_\gamma R^2\Big(\int_0^{|v_j-\bar v_i|}\frac{1}{|v_j-\bar v_i|}r^{1-\gamma}dr+\int_{|v_j-\bar v_i|}^1\frac{1}{r^\gamma}dr+R\Big)\end{aligned}$$

which gives the expected estimates. Similarly

$$\begin{aligned}\int\big|\log|v_i-v_j|\big|\,b(\nu^*,v^*)\,dv^*d\nu^* &= \int\int_{|v_i-v_j|}^1\frac{1}{r}dr\,b(\nu^*,v^*)\,dv^*d\nu^*\\ &\leq CR^2\left(\int_0^{|v_j-\bar v_i|}\frac{r}{|v_j-\bar v_i|}dr+\int_{|v_j-\bar v_i|}^1dr\right)\leq CR^3\,.\end{aligned}$$

This ends the proof of Lemma C.2. □

Remark C.3. *The proof of Lemma* C.2 *shows that in order to keep control on the collision integral the power γ of the singularity must not be too large (namely smaller than* 2*).*

The singularities appearing in velocity integrals can be removed by iterating the inequalities of Lemma C.2 and integrating over the parents of a pseudo-particle.

Lemma C.4. *Consider two pseudo-particles i, j as well as their first two parents 1^* and 2^*. Denote by $\nu_{1^*}, \nu_{2^*} \in \mathbb{S}$ and $v_{1^*}, v_{2^*} \in \mathbb{R}^2$ their scattering parameters. We assume also that all the velocities are bounded by $R > 1$. Let $\mathcal{R}$ be a rectangle with sides of length a, a', then*

$$\int \mathbf{1}_{v_i - v_j \in \mathcal{R}} \left|(v_{1^*} - v_{a(1^*)}) \cdot \nu_{1^*}\right| dv_{1^*} d\nu_{1^*} \leq CR^2 \min(a, a') \, \log a \big(|\log a| + |\log a'| + 1\big), \tag{C.11}$$

$$\int \mathbf{1}_{v_i - v_j \in \mathcal{R}} \prod_{\ell = 1^*, 2^*} \left|(v_\ell - v_{a(\ell)}) \cdot \nu_\ell\right| dv_\ell d\nu_\ell \leq CR^5 a \, a' \big(|\log a| + |\log a'| + 1\big). \tag{C.12}$$

Note that in the last inequality, the direction of the axis of $\mathcal{R}$ can depend on 2^*.

Proof. Note that if i, j are generated by the same collision, then even better estimates can be obtained from Lemma C.1. Thus from now, we assume that v_i is given by (C.2).

To derive (C.11), we suppose that $a' \geq a$ and that the collision with 1^* takes place with i which had a velocity $\bar{v}_i$. We cover the rectangle $v_j + \mathcal{R}$ into $\lfloor a'/a \rfloor$ balls of radius $2a$. Let ω be the axis of the rectangle $v_j + \mathcal{R}$ and denote by $w_k = w_0 + ak\,\omega$ the centers of the balls which are indexed by the integer $k \in \{0, \dots, \lfloor a'/a \rfloor\}$. Applying (C.3) to each ball, we get

$$\begin{aligned} \int \mathbf{1}_{v_i - v_j \in \mathcal{R}} \, b(\nu_{1^*}, v_{1^*}) \, dv_{1^*} d\nu_{1^*} \quad &\leq \quad \sum_{k=0}^{a'/a} \int \mathbf{1}_{|v_i - w_k| \leq 2a} \, b(\nu_{1^*}, v_{1^*}) \, dv_{1^*} d\nu_{1^*} \\ &\leq \quad CR^2 \sum_{k=0}^{a'/a} a \min\left(\frac{a}{|w_k - \bar{v}_i|}, 1\right), \\ &\leq \quad CR^2 a \sum_{k=0}^{a'/a} \frac{a}{|w_k - \bar{v}_i| + a} \leq CR^2 a \left(\log(\frac{a'}{a}) + 1\right), \end{aligned} \tag{C.13}$$

where the log divergence in the last inequality follows by summing over k. This completes the proof of (C.11).

We turn now to the derivation of (C.12) and suppose that $a \leq a'$. Applying (C.4) in the LHS of inequality (C.13), we obtain a contribution for each ball of radius $2a$ of order $O\big(R^5 a^2 \, |\log a|\big)$ after integrating over 2^*. Summing over all these contributions, we find a bound $CR^5 a \, a' |\log a|$.

This completes the proof of Lemma C.4. □

Appendix D. Initial data estimates

This section is devoted to the proof of Proposition 2.5 stated page 12.

Using the notation $X_{k,N} := \{x_k, \dots, x_N\}$, we write

$$\begin{aligned} \Big| \left(f_N^{0(s)} - f_0^{(s)}\right) (Z_s) \mathbf{1}_{\mathcal{D}^s_\varepsilon}(X_s) \Big| \leq M_\beta^{\otimes s}(V_s) \sum_{i=1}^s \big|g_{\alpha,0}(z_i)\big| \, \Big| \mathcal{Z}_N^{-1} \int \mathbf{1}_{\mathcal{D}^N_\varepsilon}(X_N) \, dX_{s+1,N} - 1 \Big| \\ + \mathcal{Z}_N^{-1} M_\beta^{\otimes s}(V_s) \sum_{i=s+1}^N \Big| \int M_\beta(v_i) g_{\alpha,0}(z_i) \mathbf{1}_{\mathcal{D}^N_\varepsilon}(X_N) \, dv_i dX_{s+1,N} \Big|, \end{aligned}$$

where $\mathcal{D}^N_\varepsilon$ stands for the exclusion constraint on the positions (with a slight abuse of notation compared to (1.4)). The first term is estimated as in the proof of Proposition 3.3 in [7]

$$M_\beta^{\otimes s}(V_s) \sum_{i=1}^s \big|g_{\alpha,0}(z_i)\big| \, \Big| \mathcal{Z}_N^{-1} \int \mathbf{1}_{\mathcal{D}^N_\varepsilon}(X_N) \, dX_{s+1,N} - 1 \Big| \leq C^s \varepsilon \alpha M_\beta^{\otimes s}(V_s) \|g_{\alpha,0}\|_{L^\infty} \, .$$

The exchangeability of the variables allows us to rewrite the second term as

$$
\begin{aligned}
I(Z_s) &:= M_\beta^{\otimes s}(V_s) \sum_{i=s+1}^N \Big| \int M_\beta(v_i) g_{\alpha,0}(z_i) \mathbf{1}_{\mathcal{D}_\varepsilon^N}(X_N)\, dv_i dX_{s+1,N} \Big| \\
&\leq (N-s) M_\beta^{\otimes s}(V_s) \\
&\qquad \Big| \int M_\beta(v_{s+1}) g_{\alpha,0}(z_{s+1}) \Big(\prod_{k \neq s+1} \mathbf{1}_{|x_k - x_{s+1}|>\varepsilon} \Big)\, \chi_{s+2}(X_N)\, dz_{s+1} dX_{s+2,N} \Big| ,
\end{aligned}
$$

where we used the notation

$$
\chi_{s+2}(X_N) := \hat\chi^+_{s+2}(X_{s+2,N})\, \hat\chi^-_{s+2}(X_N) \tag{D.1}
$$

which distinguishes the interaction of the particles $X_{s+2,N}$ with themselves and with X_s, defining

$$
\hat\chi^+_{s+2}(X_{s+2,N}) := \frac{1}{\mathcal{Z}_N} \prod_{s+2<\ell<k\leq N} \mathbf{1}_{|x_k - x_\ell|>\varepsilon} \quad \text{and} \quad \hat\chi^-_{s+2}(X_N) := \prod_{\substack{s+2<\ell\leq N \\ 1\leq k\leq s}} \mathbf{1}_{|x_k - x_\ell|>\varepsilon} .
$$

The exclusion between $s+1$ and the rest of the system is also decomposed into a term for the interaction with X_s and another one for the interaction with $X_{s+2,N}$. Defining

$$
\chi^-_{s+1}(X_s) := \prod_{k\leq s} \mathbf{1}_{|x_k - x_{s+1}|>\varepsilon} \quad \text{and} \quad \chi^+_{s+1}(X_{s+2,N}) := \prod_{k\geq s+2} \mathbf{1}_{|x_k - x_{s+1}|>\varepsilon}
$$

we have

$$
\begin{aligned}
\prod_{k\neq s+1} \mathbf{1}_{|x_k - x_{s+1}|>\varepsilon} &= \chi^-_{s+1}(X_s)\, \chi^+_{s+1}(X_{s+2,N}) \\
&= \chi^+_{s+1}(X_{s+2,N}) - \big(1 - \chi^-_{s+1}(X_s)\big) \chi^+_{s+1}(X_{s+2,N}) .
\end{aligned}
$$

We deduce that

$$
I(Z_s) \leq M_\beta^{\otimes s}(V_s) \Big(I_1(Z_s) + I_2(Z_s) \Big)
$$

with

$$
\begin{cases}
I_1(Z_s) := N \displaystyle\int M_\beta(v_{s+1}) \big| g_{\alpha,0}(z_{s+1}) \big| \big(1 - \chi^-_{s+1}(X_s)\big) \hat\chi^+_{s+2}(X_{s+2,N})\, dz_{s+1} dX_{s+2,N} , \\
I_2(Z_s) := N \Big| \displaystyle\int M_\beta(v_{s+1}) g_{\alpha,0}(z_{s+1}) \chi^+_{s+1}(X_{s+2,N})\, \chi_{s+2}(X_N)\, dz_{s+1} dX_{s+2,N} \Big| .
\end{cases}
$$

From (2.13) and the assumption $N\varepsilon = \alpha \ll 1/\varepsilon$, we get

$$
\int \hat\chi^+_{s+2}(X_{s+2,N})\, dX_{s+2,N} = \frac{\mathcal{Z}_{N-s-2}}{\mathcal{Z}_N} \leq \exp\big(Cs\alpha\varepsilon\big) \leq \exp\big(Cs\big) .
$$

We infer that the term I_1 is bounded by the fact that x_{s+1} is close to X_s

$$
I_1(Z_s) \leq sN\varepsilon^2 \exp\big(Cs\big) \|g_{\alpha,0}\|_{L^\infty} \leq s \exp\big(Cs\big) \alpha\varepsilon \|g_{\alpha,0}\|_{L^\infty} .
$$

Using the assumption $\displaystyle\int_{\mathbb{D}} M_\beta g_{\alpha,0}(z) dz = 0$, the second term is rewritten as

$$
I_2(Z_s) = N \Big| \int M_\beta(v_{s+1}) g_{\alpha,0}(z_{s+1}) \big(1 - \chi^+_{s+1}(X_{s+2,N})\big)\, \chi_{s+2}(X_N)\, dz_{s+1} dX_{s+2,N} \Big| .
$$

Plugging the identity (D.1)

$$
\chi_{s+2}(X_N) = \hat\chi^+_{s+2}(X_{s+2,N}) - \big(1 - \hat\chi^-_{s+2}(X_N)\big) \hat\chi^+_{s+2}(X_{s+2,N})
$$

we distinguish two more contributions $I_2(Z_s) \leq I_{2,1}(Z_s) + I_{2,2}(Z_s)$ with

$$\begin{cases} I_{2,1}(Z_s) := N\|g_{\alpha,0}\|_{L^\infty} \displaystyle\int \big(1 - \chi^+_{s+1}(X_{s+2,N})\big) \, \big(1 - \hat\chi^-_{s+2}(X_N)\big)\hat\chi^+_{s+2}(X_{s+2,N}) \, dX_{s+1,N} \, , \\ I_{2,2}(Z_s) := N\Big| \displaystyle\int M_\beta(v_{s+1}) g_{\alpha,0}(z_{s+1})\big(1 - \chi^+_{s+1}(X_{s+2,N})\big) \, \hat\chi^+_{s+2}(X_{s+2,N}) \, dz_{s+1} dX_{s+2,N}\Big| \, . \end{cases}$$

The term $I_{2,1}$ takes into account two constraints : $s+1$ is close to a particle in $X_{s+2,N}$ and one particle in $X_{s+2,N}$ is close to X_s. Since $N\varepsilon = \alpha$, we deduce that

$$I_{2,1}(Z_s) \leq Ns\varepsilon^2 \, (N-s-1)^2 \varepsilon^2 \, \frac{\mathcal{Z}_{N-s-3}}{\mathcal{Z}_N} \|g_{\alpha,0}\|_{L^\infty} \leq s\alpha^3 \varepsilon \exp(Cs) \|g_{\alpha,0}\|_{L^\infty} \, .$$

The term $I_{2,2}$ does not depend on X_s, thus one can integrate over z_{s+1} and use again the assumption $\displaystyle\int_{\mathbb{D}} M_\beta g_{\alpha,0}(z) dz = 0$. To see this, it is enough to note that the function

$$x_{s+1} \mapsto \int \big(1 - \chi^+_{s+1}(X_{s+2,N})\big) \, \hat\chi^+_{s+2}(X_{s+2,N}) dX_{s+2,N}$$

is independent of x_{s+1} thanks to the periodic structure of $\mathbb{D}^{N-s-2}$. Thus $I_{2,2}(Z_s) = 0$.

Combining the previous estimates, we conclude Proposition 2.5. □

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