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► **To cite this version:**

Federico Bribiesca Argomedo, Miroslav Krstic. Backstepping-Forwarding Control and Observation for Hyperbolic PDEs With Fredholm Integrals. IEEE Transactions on Automatic Control, 2015, 60 (8), pp.2145 - 2160. 10.1109/TAC.2015.2398882 . hal-01225219

HAL Id: hal-01225219

<https://hal.science/hal-01225219>

Submitted on 11 Apr 2019

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Backstepping-Forwarding Control and Observation for Hyperbolic PDEs with Fredholm Integrals

Federico Bribiesca-Argomedo and Miroslav Krstic

Abstract—An integral transform is introduced which allows the construction of boundary controllers and observers for a class of first-order hyperbolic PIDEs with Fredholm integrals. These systems do not have a strict-feedback structure and thus the standard backstepping approach cannot be applied. Sufficient conditions for the existence of the backstepping-forwarding transform are given in terms of spectral properties of some integral operators and, more conservatively but easily verifiable, in terms of the norms of the coefficients in the equations. An explicit transform is given for particular coefficient structures. In the case of strict-feedback systems, the procedure detailed in this paper reduces to the well-known backstepping design. The results are illustrated with numerical simulations.

I. INTRODUCTION

Backstepping, in its infinite-dimensional version, has proven to be a very effective tool for constructing boundary controllers and observers for large classes of PDEs, see for instance [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], with numerous applications such as: control of turbulent flows [11], boundary control of the Korteweg-de Vries Equation [12], output tracking on heat exchangers [13], delay compensation for finite-dimensional systems [14], and electrochemical battery models [15]. Nevertheless, the use of a Volterra transform restricts the class of systems to which it can be applied (they must have a strict-feedback structure). Recently, some results have appeared for specific classes of systems with non strict-feedback components. In particular, results are available for finite-dimensional systems with either distributed delays or some PDE in the actuation or sensing path that gives it a non strict-feedback structure, [16], [17] and certain other PDE structures, see [18].

In this article, we present an integral transform of the state of a PIDE that allows us to build a stabilizing boundary control for a class of first-order hyperbolic PIDEs with Fredholm integrals (non-strict feedback terms) that arise, for instance, when considering coupled PDE-ODE or PDE-PDE systems with boundary actuation in only one of the equations.

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Specifically, we consider systems of the form

$$\begin{aligned} \bar{u}_t(x, t) &= \bar{u}_x(x, t) + \bar{d}(x)\bar{u}(x, t) + \bar{f}(x)\bar{u}(0, t) \\ &\quad + \int_0^x \bar{g}(x, y)\bar{u}(y, t)dy \end{aligned} \quad (1)$$

$$\begin{aligned} &\quad + \int_x^1 \bar{h}(x, y)\bar{u}(y, t)dy, \\ \forall(x, t) &\in (0, 1) \times (0, T] \\ \bar{u}(1, t) &= \bar{U}(t), \quad \forall t \in (0, T] \end{aligned} \quad (2)$$

with initial condition $\bar{u}(x, 0) \doteq \bar{u}_0(x) \in L^2([0, 1]; \mathbb{R})$. Where \bar{d} , \bar{f} , \bar{g} and \bar{h} are real-valued continuous functions in their respective domains.

Using the change of variables

$$u(x, t) = e^{\int_0^x \bar{d}(\xi)d\xi} \bar{u}(x, t) \quad (3)$$

proposed in [2], we can focus without loss of generality on the stabilization of the equation (without reaction term)

$$\begin{aligned} u_t(x, t) &= u_x(x, t) + f(x)u(0, t) + \int_0^x g(x, y)u(y, t)dy \\ &\quad + \int_x^1 h(x, y)u(y, t)dy, \end{aligned} \quad (4)$$

$$\begin{aligned} \forall(x, t) &\in (0, 1) \times (0, T] \\ u(1, t) &= U(t), \quad \forall t \in (0, T] \end{aligned} \quad (5)$$

with initial condition $u(x, 0) \doteq u_0(x) \in L^2([0, 1]; \mathbb{R})$. With f , g and h real-valued continuous functions in their respective domains, and boundary control $U(t)$. For the observer design, we consider $u(0, t)$ to be the only available measure. The coefficients f , g and h can be expressed in terms of those appearing in (1) as

$$f(x) \doteq e^{\int_0^x \bar{d}(\xi)d\xi} \bar{f}(x) \quad (6)$$

$$g(x, y) \doteq e^{\int_y^x \bar{d}(\xi)d\xi} \bar{g}(x, y) \quad (7)$$

$$h(x, y) \doteq e^{-\int_x^y \bar{d}(\xi)d\xi} \bar{h}(x, y) \quad (8)$$

This class of systems is related to that presented in [2], however, the possible presence of non-strict-feedback terms (whenever h is not zero) means that it cannot, in general, be stabilized using a backstepping approach. The two integral terms appearing in equation (1) can be thought of as a Fredholm integral with a piecewise-continuous kernel, possibly having a discontinuity at $y = x$. The dependence of the kernel on x makes the problem more challenging but, at the same time, more relevant (as illustrated by the examples presented).

The control problem tackled in this article is then to find a gain kernel $\gamma \in \mathcal{C}([0, 1]; \mathbb{R})$ such that, under the control law

$$U(t) \doteq \int_0^1 \gamma(y)u(y, t)dy, \quad (9)$$

the origin of system (4)-(5) is finite-time stable in the topology of the L^2 norm.

The observation problem in turn, is formulated as a stabilization problem for the error system (the difference between the estimated and real states) and an adequate output error injection gain $\gamma_{obs,1} \in \mathcal{C}([0, 1]; \mathbb{R})$ must be found.

The results presented in the first part of Section II (up to Subsection II-E) are an extended version of those presented in [19] including complete proofs and a reworked simulation example. They concern the general form of the equation and provide different conditions for a stabilizing boundary controller to exist. In particular, concrete conditions on the magnitude of the coefficients in equation (4) will be given which are sufficient for a solution to exist and for it to be given as the limit of a given sequence. The approach presented in the second part of Section II (starting with Subsection II-F), on the other hand, restricts the class of systems under consideration by adding supplementary assumptions (on the shape of the coefficients in equation (4)) that allow the computation of an explicit controller gain for the system. Finally, Section III tackles the observer design problem.

II. BACKSTEPPING-FORWARDING CONTROL DESIGN

A. Preliminary Definitions

In order to build a stabilizing controller for system (4)-(5) we proceed by finding a bounded transform

$$w(x, t) = u(x, t) - \int_0^x p(x, y)u(y, t)dy - \int_x^1 q(x, y)u(y, t)dy \quad (10)$$

with bounded inverse

$$u(x, t) = w(x, t) + \int_0^x k(x, y)w(y, t)dy + \int_x^1 l(x, y)w(y, t)dy \quad (11)$$

and the associated control law

$$U(t) = \int_0^1 p(1, y)u(y, t)dy \quad (12)$$

such that system (4) is mapped into the (finite-time stable) target system

$$w_t(x, t) = w_x(x, t), \quad \forall (x, t) \in (0, 1) \times (0, T] \quad (13)$$

$$w(1, t) = 0, \quad \forall t \in (0, T]. \quad (14)$$

A more precise formulation of the transform will be given after the necessary spaces are defined.

It will be shown (the proof can be found in Appendix A) that the kernels of the direct transform need to satisfy the

following condition:

$$\begin{aligned} p_x(x, y) + p_y(x, y) &= -g(x, y) + q(x, 1)p(1, y) \\ &+ \int_0^y h(s, y)p(x, s)ds \\ &+ \int_y^x g(s, y)p(x, s)ds \\ &+ \int_x^1 g(s, y)q(x, s)ds, \end{aligned} \quad (15)$$

$$\forall x, y \in [0, 1] \text{ s.t. } y \leq x, y \neq 0$$

$$\begin{aligned} q_x(x, y) + q_y(x, y) &= -h(x, y) + q(x, 1)p(1, y) \\ &+ \int_x^y h(s, y)q(x, s)ds \\ &+ \int_y^1 g(s, y)q(x, s)ds \\ &+ \int_0^x h(s, y)p(x, s)ds, \end{aligned} \quad (16)$$

$$\forall x, y \in [0, 1] \text{ s.t. } x \leq y$$

with boundary condition

$$\begin{aligned} p(x, 0) &= -f(x) + \int_0^x p(x, y)f(y)dy \\ &+ \int_x^1 q(x, y)f(y)dy, \quad \forall x \in [0, 1]. \end{aligned} \quad (17)$$

In general, a second boundary condition is required for these equations to be well defined. In this section we choose to impose $q(x, 1) = 0$ which will simplify the contraction arguments required in the proofs by eliminating the nonlinear terms in (15) and (16). A somewhat different procedure is presented in Subsection II-F since the particular structure of the considered kernels reduce the system of PDEs to a first-order (nonlinear) ODE in the spatial variable, for which the condition on $p(x, 0)$ is expressed as $k_1(0) = 0$. The resulting ODE is already well defined (under some assumptions) so the boundary condition corresponding to $q(x, 1)$ is not required. Furthermore, an explicit solution can be obtained for this ODE.

The boundedness of the direct transform (as an operator mapping between adequate normed vector spaces) implies that any bounded initial condition of the original system corresponds to a bounded initial condition of the target system. The boundedness (again, as a map between adequate normed vector spaces) of the inverse transform implies that, as the norm of the target system goes to zero, so does the norm of the state of the original system. Therefore, the existence of both a bounded direct and inverse transforms imply the stability of the original system in some function space.

The natural choice of the function spaces in which to define the direct and inverse transforms (and thus the stability results) will depend on the regularity of the obtained kernels. In this article we focus only on obtaining continuous kernels. The procedure required to obtain higher regularity is analogous and more cumbersome.

Definition 1: Let us define two (closed, bounded) subsets of \mathbb{R}^2 as follows:

$$\mathcal{T}_l \doteq \{(x, y) \in [0, 1] \times [0, 1], y \leq x\} \quad (18)$$

$$\mathcal{T}_u \doteq \{(x, y) \in [0, 1] \times [0, 1], x \leq y\} \quad (19)$$

equipped with the norm

$$\|z\|_\infty \doteq \max\{|z_1|, |z_2|\}, \quad \forall z \doteq (z_1, z_2) \in \mathbb{R}^2,$$

where $|\cdot|$ denotes the absolute value of an element of \mathbb{R} (N.B. whenever necessary, we consider \mathbb{R} to be equipped with the topology induced by the absolute value metric, or the euclidean norm in \mathbb{R}).

We should note that $(\mathcal{T}_l, \|\cdot\|_\infty)$ and $(\mathcal{T}_u, \|\cdot\|_\infty)$ are compact in the topology induced by their norms. Hereafter, unless otherwise explicitly stated, we assume \mathcal{T}_l and \mathcal{T}_u to be equipped with these norms. Furthermore, the chosen $\|\cdot\|_\infty$ norm is equivalent to the usual Euclidean norm.

Definition 2: We now define the spaces $X_l \doteq \mathcal{C}(\mathcal{T}_l; \mathbb{R})$ and $X_u \doteq \mathcal{C}(\mathcal{T}_u; \mathbb{R})$ equipped with the norm $\|\cdot\|_{X_l}$ (respectively $\|\cdot\|_{X_u}$) defined as

$$\|s\|_{X_l} \doteq \sup_{z \in \mathcal{T}_l} |s(z)|, \quad \forall s \in X_l \quad (20)$$

$$\|s\|_{X_u} \doteq \sup_{z \in \mathcal{T}_u} |s(z)|, \quad \forall s \in X_u. \quad (21)$$

Note that $(X_l, \|\cdot\|_{X_l})$ and $(X_u, \|\cdot\|_{X_u})$ are Banach spaces. These are the spaces in which we will define the kernels in our integral operators.

Definition 3: Given functions $\phi \in X_l$, $\psi \in X_u$ we define the operator $\Pi^{\phi, \psi} : L^2([0, 1]; \mathbb{R}) \rightarrow L^2([0, 1]; \mathbb{R})$ as

$$\Pi^{\phi, \psi}[\xi](x) = \int_0^x \phi(x, s)\xi(s)ds + \int_x^1 \psi(x, s)\xi(s)ds, \quad (22)$$

for all $\xi \in L^2([0, 1]; \mathbb{R})$, and all $x \in [0, 1]$.

Based on this definition, we can write the transforms in (10) and (11) as

$$w(x, t) = (\mathbb{I}_{L^2} - \Pi^{p, q})[u(\cdot, t)](x) \quad (23)$$

and

$$u(x, t) = (\mathbb{I}_{L^2} + \Pi^{k, l})[w(\cdot, t)](x), \quad (24)$$

for all $(x, t) \in [0, 1] \times [0, T]$, where \mathbb{I}_{L^2} is the identity operator on $L^2([0, 1]; \mathbb{R})$.

Assumption 1: The coefficients in (4) satisfy: $f \in \mathcal{C}([0, 1]; \mathbb{R})$, $g \in X_l$ and $h \in X_u$.

Definition 4: Define now the space

$$X \doteq X_l \times X_u \quad (25)$$

equipped with the norm

$$\|\varphi\|_X \doteq \max\{\|\varphi_1\|_{X_l}, \|\varphi_2\|_{X_u}\}, \quad \forall \varphi \doteq (\varphi_1, \varphi_2) \in X. \quad (26)$$

As defined, $(X, \|\cdot\|_X)$ is a Banach space.

We now introduce an integral operator T related to the PDEs the kernels in (10) must satisfy in order to map the dynamics of (4) to those of (13).

Definition 5: Define the integral operator $T : X \rightarrow X$ (for $A_{1,1} : X_l \rightarrow X_l$, $A_{1,2} : X_u \rightarrow X_l$, $A_{2,1} : X_l \rightarrow X_u$, $A_{2,2} : X_u \rightarrow X_u$, $F_1 \in X_l$ $F_2 \in X_u$), for all $p \in X_l$, $q \in X_u$ as

$$\begin{aligned} T \begin{bmatrix} p \\ q \end{bmatrix} &\doteq A \begin{bmatrix} p \\ q \end{bmatrix} + F \\ &\doteq \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \end{aligned} \quad (27)$$

where

$$\begin{aligned} A_{1,1}[p](x, y) &\doteq \int_0^{x-y} f(s)p(x-y, s)ds \\ &+ \int_0^y \int_0^\sigma h(s, \sigma)p(\sigma+x-y, s)ds d\sigma \\ &+ \int_0^y \int_0^{x-y} g(s+\sigma, \sigma) \\ &\times p(\sigma+x-y, \sigma+s)ds d\sigma \end{aligned} \quad (28)$$

$$\begin{aligned} A_{1,2}[q](x, y) &\doteq \int_0^{1-x+y} f(x-y+s)q(x-y, x-y+s)ds \\ &+ \int_0^y \int_0^{1-\sigma-x+y} g(\sigma+x-y+s, \sigma) \\ &\times q(\sigma+x-y, \sigma+x-y+s)ds d\sigma \end{aligned} \quad (29)$$

$$\begin{aligned} A_{2,1}[p](x, y) &\doteq - \int_0^{1-y} \int_0^{\sigma+x} h(s, \sigma+y)p(\sigma+x, s)ds d\sigma \end{aligned} \quad (30)$$

$$\begin{aligned} A_{2,2}[q](x, y) &\doteq - \int_0^{1-y} \int_0^{y-x} h(s+\sigma+x, \sigma+y) \\ &\times q(\sigma+x, \sigma+x+s)ds d\sigma \\ &- \int_0^{1-y} \int_0^{1-\sigma-y} g(s+\sigma+y, \sigma+y) \\ &\times q(\sigma+x, \sigma+y+s)ds d\sigma \end{aligned} \quad (31)$$

$$F_1(x, y) \doteq -f(x-y) - \int_0^y g(\sigma+x-y, \sigma)d\sigma \quad (32)$$

$$F_2(x, y) \doteq \int_0^{1-y} h(\sigma+x, \sigma+y)d\sigma \quad (33)$$

in their respective domains.

Next, we introduce an integral operator R related to the conditions required for (11) to be a left-inverse of (10). This operator is obtained by substituting (10) into (11).

Definition 6: Given functions $\phi \in X_l$, $\psi \in X_u$, we define an operator $R^{\phi, \psi} : X \rightarrow X$ as

$$\begin{aligned} R^{\phi, \psi} \begin{bmatrix} k \\ l \end{bmatrix} &\doteq S^{\phi, \psi} \begin{bmatrix} k \\ l \end{bmatrix} + \begin{bmatrix} \phi \\ \psi \end{bmatrix} \\ &\doteq \begin{bmatrix} S_{1,1}^{\phi, \psi} & S_{1,2}^{\phi, \psi} \\ S_{2,1}^{\phi, \psi} & S_{2,2}^{\phi, \psi} \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} + \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \end{aligned} \quad (34)$$

where

$$S_{1,1}^{\phi,\psi}[k](x,y) \doteq \int_0^y \psi(s,y)k(x,s)ds + \int_y^x \phi(s,y)k(x,s)ds \quad (35)$$

$$S_{1,2}^{\phi,\psi}[l](x,y) \doteq \int_x^1 \phi(s,y)l(x,s)ds \quad (36)$$

$$S_{2,1}^{\phi,\psi}[k](x,y) \doteq \int_0^x \psi(s,y)k(x,s)ds \quad (37)$$

$$S_{2,2}^{\phi,\psi}[l](x,y) \doteq \int_x^y \psi(s,y)l(x,s)ds + \int_y^1 \phi(s,y)l(x,s)ds \quad (38)$$

in their respective domains.

Finally, we define an operator related to the PDE conditions that the kernels of (11) must verify in order to map the dynamics of (13) into those of (4).

Definition 7: Define also the integral operator $\bar{T} : X \rightarrow X$ (for $\bar{A}_{1,1} : X_l \rightarrow X_l$, $\bar{A}_{1,2} : X_u \rightarrow X_l$, $\bar{A}_{2,1} : X_l \rightarrow X_u$, $\bar{A}_{2,2} : X_u \rightarrow X_u$, $\bar{F}_1 \in X_l$, $\bar{F}_2 \in X_u$), for all $k \in X_l$, $l \in X_u$ as

$$\begin{aligned} \bar{T} \begin{bmatrix} k \\ l \end{bmatrix} &\doteq \bar{A} \begin{bmatrix} k \\ l \end{bmatrix} + \bar{F} \\ &\doteq \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ \bar{A}_{2,1} & \bar{A}_{2,2} \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} + \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix}, \end{aligned} \quad (39)$$

where

$$\begin{aligned} \bar{A}_{1,1}[k](x,y) &\doteq - \int_0^y \int_0^{x-y} g(\sigma+x-y, s+\sigma)k(s+\sigma, \sigma)ds d\sigma \\ &\quad - \int_0^y \int_0^{1-\sigma-x+y} h(\sigma+x-y, s+\sigma+x-y) \\ &\quad \times k(s+\sigma+x-y, \sigma)ds d\sigma \end{aligned} \quad (40)$$

$$\begin{aligned} \bar{A}_{1,2}[l](x,y) &\doteq - \int_0^y f(\sigma+x-y)l(0, \sigma)d\sigma \\ &\quad - \int_0^y \int_0^\sigma g(\sigma+x-y, s)l(s, \sigma)ds d\sigma \end{aligned} \quad (41)$$

$$\begin{aligned} \bar{A}_{2,1}[k](x,y) &\doteq \int_0^{1-y} \int_0^{1-y-\sigma} h(\sigma+x, s+\sigma+y) \\ &\quad \times k(s+\sigma+y, \sigma+y)ds d\sigma \end{aligned} \quad (42)$$

$$\begin{aligned} \bar{A}_{2,2}[l](x,y) &\doteq \int_0^{1-y} f(\sigma+x)l(0, \sigma+y)d\sigma \\ &\quad + \int_0^{1-y} \int_0^{\sigma+x} g(\sigma+x, s) \\ &\quad \times l(s, \sigma+y)ds d\sigma \\ &\quad + \int_0^{1-y} \int_0^{y-x} h(\sigma+x, s+\sigma+x) \\ &\quad \times l(s+\sigma+x, \sigma+y)ds d\sigma \end{aligned} \quad (43)$$

$$\bar{F}_1(x,y) \doteq -f(x-y) - \int_0^y g(\sigma+x-y, \sigma)d\sigma \quad (44)$$

$$\bar{F}_2(x,y) \doteq \int_0^{1-y} h(\sigma+x, \sigma+y)d\sigma \quad (45)$$

in their respective domains.

B. Direct transform

Proposition 1: If the operator T , as defined in (27), has a unique fixed point in X (i.e. there exists a unique $\zeta \in X$ s.t. $T\zeta = \zeta$), then transform (10) with kernels

$$\begin{bmatrix} p \\ q \end{bmatrix} \doteq \zeta \quad (46)$$

maps system (4)-(9), with

$$\gamma(y) \doteq p(1,y), \quad \forall y \in [0,1] \quad (47)$$

into (13)-(14).

The proof of this result is given in Appendix A.

An equivalent condition to that of Proposition 1 is that 1 belongs to the resolvent set of the operator A , as defined in (27). For the conditions required for a value to belong to the spectrum (or the resolvent) of a bounded operator on a Banach space the reader is directed to [20, Lemma 1.2.13].

Using Banach's contraction mapping principle, see for example [21, Theorem 3.1], we can establish sufficient conditions for the previous results to hold.

Corollary 2: If the operator T , as defined in (27), is a contraction then transform (10) with kernels

$$\begin{bmatrix} p \\ q \end{bmatrix} \doteq \lim_{n \rightarrow \infty} T^n \vartheta_0 \quad (48)$$

for any $\vartheta_0 \in X$, maps system (4)-(9), with

$$\gamma(y) \doteq p(1,y), \quad \forall y \in [0,1] \quad (49)$$

into (13)-(14).

In particular, if T is a contraction, it implies that the spectral radius of A is less than 1 (and therefore 1 does not belong to the spectrum of A). Even though this condition is conservative, it allows for a constructive result to be given (the kernels can be found using Picard iterations).

Particularly noteworthy is the fact that this corollary depends on the choice of norm used in the definition of the Banach space X . A similar result can be obtained whenever there exists a positive integer n for which T^n is a contraction. However, since the computations become extremely cumbersome after more than a couple iterations (except for very particular cases) we only give the proofs for the case where T is a contraction mapping.

Using the supremum norm, associated to our space X , we can give a sufficient condition in terms of the magnitude of the coefficients in (4) for the direct transform to exist. It should be noted that this bound is conservative since few conditions are imposed on the coefficients. For some particular cases it can be easily relaxed (for instance, if $f(x) = 0$ this bound is doubled).

Lemma 3: If the coefficients in equation (4) verify $c \doteq \max\{\sup_{s \in [0,1]} |f(s)|, \|g\|_{X_l}, \|h\|_{X_u}\} < \frac{1}{2}$, then transform (10) with kernels

$$\begin{bmatrix} p \\ q \end{bmatrix} \doteq \zeta \quad (50)$$

$$\doteq \lim_{n \rightarrow \infty} T^n \vartheta_0$$

for any $\vartheta_0 \in X$, maps system (4)-(5), with

$$\gamma(y) \doteq p(1,y), \quad \forall y \in [0,1], \quad (51)$$

into (13)-(14). Furthermore,

$$\|\zeta\|_X \leq \frac{\|F\|_X}{1-2c}. \quad (52)$$

Proof: If we can show that there exists $C \in [0, 1)$ such that:

$$\|T\varphi - T\tilde{\varphi}\|_X \leq C\|\varphi - \tilde{\varphi}\|_X, \quad \forall \varphi, \tilde{\varphi} \in X \quad (53)$$

then the operator T is a contraction.

We start by noting that

$$\|T\varphi - T\tilde{\varphi}\|_X = \|A\varphi - A\tilde{\varphi}\|_X, \quad \forall \varphi, \tilde{\varphi} \in X \quad (54)$$

and

$$\|A\varphi - A\tilde{\varphi}\|_X = \|A(\varphi - \tilde{\varphi})\|_X. \quad (55)$$

Let us denote $K \doteq \|\varphi - \tilde{\varphi}\|_X$, and c defined as in the theorem statement, then after some computations we obtain the norm estimate

$$\|A(\varphi - \tilde{\varphi})\|_X \leq \max\{cK \sup_{y \in [0,1]} (1+y), cK \sup_{y \in [0,1]} (1-y)\}, \quad (56)$$

which in turn implies

$$\|A(\varphi - \tilde{\varphi})\|_X \leq 2cK. \quad (57)$$

If $2c < 1$, T defines a contraction mapping. The application of Banach's contraction mapping principle [21, Theorem 3.1] completes the first part of the proof. The norm estimate comes from rewriting

$$\zeta = \sum_{n=0}^{\infty} A^n F \quad (58)$$

and noting that it implies, using (57),

$$\|\zeta\|_X \leq \|F\|_X \sum_{n=0}^{\infty} (2c)^n. \quad (59)$$

This expression and the condition $c < \frac{1}{2}$ complete the proof. ■

C. Inverse Transform

In this section we focus on the computation of the inverse transform (assuming the direct transform has already been obtained). The first results use the definition of the operator $R^{p,q}$ to give conditions for the left-inverse of the direct transform to exist. Similar conditions can be found for its right-inverse and it can be shown, using the associativity of linear operators from a space to itself, that if the left- and right-inverse exist they are equal. Where necessary, this condition is given in terms of the spectrum of the operator $\Pi^{p,q}$.

Proposition 4: Given kernels $p \in X_l$ and $q \in X_u$, if the operator $R^{p,q}$, as defined in (34) has a unique fixed point $\tilde{\varphi} \in X$, then transform (11) with kernels

$$\begin{bmatrix} k \\ l \end{bmatrix} \doteq \tilde{\varphi} \quad (60)$$

is the left-inverse of transform (10).

The proof of this Proposition follows by applying first the direct and then the inverse transform to an arbitrary function in $L^2([0, 1]; \mathbb{R})$ and requiring the result to be the original function. A condition equivalent to that in the Lemma is that

1 belongs to the resolvent set of the operator $S^{p,q}$, as defined in (34).

After applying Banach's contraction mapping principle, the following corollary is obtained:

Corollary 5: Given kernels $p \in X_l$ and $q \in X_u$, if the operator $R^{p,q}$ as defined in (34) is a contraction, then transform (11) with kernels

$$\begin{bmatrix} k \\ l \end{bmatrix} \doteq \lim_{n \rightarrow \infty} (R^{p,q})^n \varphi_0, \quad (61)$$

for any $\varphi_0 \in X$, is the left-inverse of transform (10).

Using the norm estimate obtained in Lemma 3 we obtain the following sufficient condition for the existence of an inverse transform (left- and right-inverse):

Lemma 6: If the coefficients in equation (4) verify $\max\{\sup_{s \in [0,1]} |f(s)|, \|g\|_{X_l}, \|h\|_{X_u}\} < \frac{1}{4}$, then for kernels $p \in X_l$ and $q \in X_u$ as defined in Lemma 3, transform (11) with kernels

$$\begin{bmatrix} k \\ l \end{bmatrix} \doteq \lim_{n \rightarrow \infty} (R^{p,q})^n \varphi_0, \quad (62)$$

for any $\varphi_0 \in X$, is the inverse of transform (10). Furthermore, the operator $\Pi^{p,q}$ defined in (22) has a spectral radius less than 1.

Proof: Applying Lemma 3, the condition in this result implies that the direct transform exists and that the operator T has a unique fixed point (since the norm of the coefficients is less than 1/2). The stronger 1/4 bound on the coefficients required here, together with the norm estimate at the end of Lemma 3, implies that $R^{p,q}$ is a contraction and that $\Pi^{p,q}$ has an operator norm less than one, which implies that $(\mathbb{I}_{L^2} - \Pi^{p,q})$ is boundedly invertible (and thus its left- and right-inverse is the same). Finally, using Corollary 5 we obtain that (11) is the left-inverse of (10) and must therefore be its inverse. This completes the proof. ■

Repeating the procedure in Proposition 1 but mapping from the target system to the original one, we obtain an operator \bar{T} analogous to the previously considered operator T . In practice, the Picard iterations for this operator converge more easily than those of $R^{p,q}$ and therefore the following conditions may be easier to test:

Lemma 7: If the coefficients in equation (4) verify $\max\{\sup_{s \in [0,1]} |f(s)|, \|g\|_{X_l}, \|h\|_{X_u}\} < \frac{1}{2}$, then for kernels $p \in X_l$ and $q \in X_u$ as defined in Lemma 3, if the unique fixed point of \bar{T} is also the fixed point of $R^{p,q}$ and 1 belongs to the resolvent set of $\Pi^{p,q}$, then transform (11) with kernels

$$\begin{bmatrix} k \\ l \end{bmatrix} \doteq \lim_{n \rightarrow \infty} (\bar{T})^n \varphi_0, \quad (63)$$

for any $\varphi_0 \in X$, is the inverse of transform (10).

Proof: Following a procedure analogous to that used in the proof of Lemma 3, the condition on the coefficients implies that \bar{T} is a contraction and therefore has a unique fixed point. Furthermore, by a similar procedure to the one used in the proof of Proposition 1, we obtain that the transform (11), with the kernels given by the fixed point of \bar{T} maps system (13)-(14) into (4)-(9). The condition that \bar{T} is also the fixed point of $R^{p,q}$ guarantees that (11) is the left-inverse of (10) and, since

1 belongs to the resolvent set of $\Pi^{p,q}$ it is also the right-inverse thus completing the proof. ■

This formulation ensures that \bar{T} is a contraction and therefore does not require the spectral radius of $S^{p,q}$ to be less than 1. The resulting conditions on the coefficients are weaker than those needed for $R^{p,q}$ to be a contraction and the result can, therefore, be more easily applied. We must stress that requiring one of these operators to be a contraction is not necessary for the backstepping-forwarding technique to work but guarantees that Picard iterations can be used to find the necessary fixed points of the operators.

D. Closed-loop L^2 Stability

The previous sections gave conditions for the direct and inverse transforms to exist. In this section we present the first main result in this paper.

Proposition 8: If 1 belongs to the resolvent set of the operators A (defined in (27)) and $\Pi^{p,q}$ (defined in (22)), with kernels

$$\begin{bmatrix} p \\ q \end{bmatrix} \doteq (\mathbb{I}_X - A)^{-1}F \quad (64)$$

then the origin of system (4)-(9), with

$$\gamma(y) \doteq p(1, y), \quad \forall y \in [0, 1] \quad (65)$$

is finite-time stable in the topology of the $L^2([0, 1]; \mathbb{R})$ norm.

Proof: The first condition in the Theorem guarantees, by Proposition 1, that transform (10) is bounded and maps system (4)-(9), with

$$\gamma(y) \doteq p(1, y), \quad \forall y \in [0, 1], \quad (66)$$

into (13)-(14). The second condition guarantees that the inverse transform exists and is bounded [20, Lemma 1.2.13]. Finally, the finite-time convergence to zero of the state of the target system (13)-(14) completes the proof. ■

A conservative (but easy to verify) sufficient condition for the above result to hold is:

Theorem 9: If the coefficients in (4) verify that $\max \left\{ \sup_{s \in [0, 1]} |f(s)|, \|g\|_{X_t}, \|h\|_{X_u} \right\} < \frac{1}{4}$ then the origin of system (4)-(9) is finite-time stable in the topology of the $L^2([0, 1]; \mathbb{R})$ norm, with

$$\gamma(y) \doteq p(1, y), \quad \forall y \in [0, 1] \quad (67)$$

where

$$\begin{aligned} \begin{bmatrix} p \\ q \end{bmatrix} &\doteq \zeta \\ &\doteq \lim_{n \rightarrow \infty} T^n \vartheta_0 \end{aligned} \quad (68)$$

for any $\vartheta_0 \in X$.

Proof: The conditions in this result imply, by Lemma 3, that the direct transform exists and maps (4)-(9), with

$$\gamma(y) \doteq p(1, y), \quad \forall y \in [0, 1] \quad (69)$$

into (13)-(14). Lemma 6 completes the proof. ■

As was the case in the inverse transform, a more practical condition to verify may be:

Proposition 10: If the following conditions are verified:

- (i) the operator T defined in (27) is a contraction in some norm equivalent to $\|\cdot\|_X$ and therefore has a unique fixed point $\zeta \in X$,
- (ii) the operator \bar{T} defined in (39) is a contraction in some norm equivalent to $\|\cdot\|_X$ and therefore has a unique fixed point $\vartheta \in X$, and
- (iii) setting

$$\begin{bmatrix} p \\ q \end{bmatrix} \doteq \zeta \quad (70)$$

1 belongs to the resolvent set of $\Pi^{p,q}$ and ϑ is the fixed point of $R^{p,q}$

then the origin of system (4)-(9), with

$$\gamma(y) \doteq p(1, y), \quad \forall y \in [0, 1] \quad (71)$$

is finite-time stable in the topology of the $L^2([0, 1]; \mathbb{R})$ norm.

Proof: Conditions (i) and (ii) are set in order to find the fixed points of T and \bar{T} using Picard iterations (they give directly a constructive solution method for the resulting kernel integral equations). As a direct consequence, Since 1 belongs to the resolvent set of $\Pi^{p,q}$, the transform (10) is invertible and, ϑ being the fixed point of $R^{p,q}$, by Proposition 4, its inverse is given by (11) with

$$\begin{bmatrix} k \\ l \end{bmatrix} \doteq \vartheta. \quad (72)$$

■

E. Application to a PDE-ODE interconnected system

Consider the following first-order PDE coupled with a second order ODE:

$$u_t(x, t) = u_x(x, t) + au(0, x) - bv(x, t) \quad (73)$$

$$0 = v_{xx}(x, t) - cv(x, t) + du_x(x, t) \quad (74)$$

with $a, b > 0$ and boundary conditions

$$u(1, t) = U(t) \quad (75)$$

$$v_x(0, t) = 0 \quad (76)$$

$$v(1, t) = 0. \quad (77)$$

This system closely resembles the Korteweg-de Vries-like equation presented in [2]. The only two differences (other than notation) are the addition of a (destabilizing) term $au(0, t)$ and the use of only one boundary to control the full interconnected system (instead of using one boundary of each subsystem).

Solving (74) with boundary conditions (76)-(77) and plugging the resulting expression into (73), we obtain a representation of the form (4) with

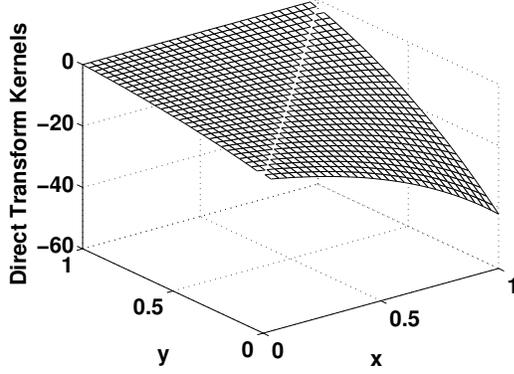
$$f(x) = a + \frac{bd \sinh(\sqrt{c}(1-x))}{\sqrt{c} \cosh(\sqrt{c})} \quad (78)$$

$$g(x, y) = -\frac{bd \cosh(\sqrt{c}x) \cosh(\sqrt{c}(1-y))}{\cosh(\sqrt{c})} + bd \cosh(\sqrt{c}(x-y)) \quad (79)$$

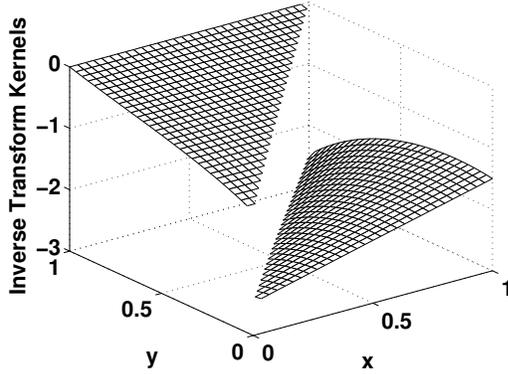
$$h(x, y) = -\frac{bd \cosh(\sqrt{c}x) \cosh(\sqrt{c}(1-y))}{\cosh(\sqrt{c})}. \quad (80)$$

We now present simulation results for $a = 1.25$, $b = 0.1$, $c = 0.1$, $d = 10$. For these coefficients, a solution can still be

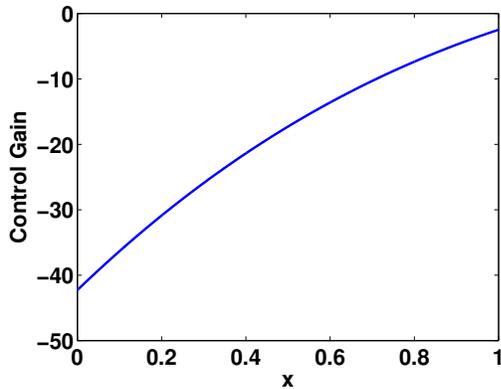
found for both systems of integral equations (even though they are larger than the sufficient condition presented in Theorem 9) and therefore the direct and inverse transforms exist and are bounded. Figures 1 (a) and 1 (b) show the obtained direct (respectively inverse) transform kernels for this system. Figure 1 (c) shows the obtained control gain. Figure 2 shows the evolution of the state in open-loop (unstable) and closed-loop (finite-time stable).



(a) Direct transform kernels $p(x, y)$ and $q(x, y)$.

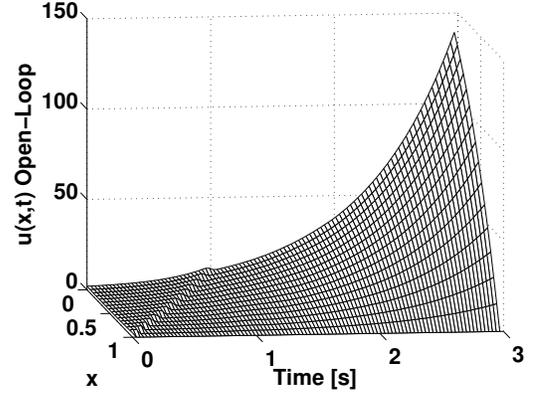


(b) Inverse transform kernels $k(x, y)$ and $l(x, y)$.

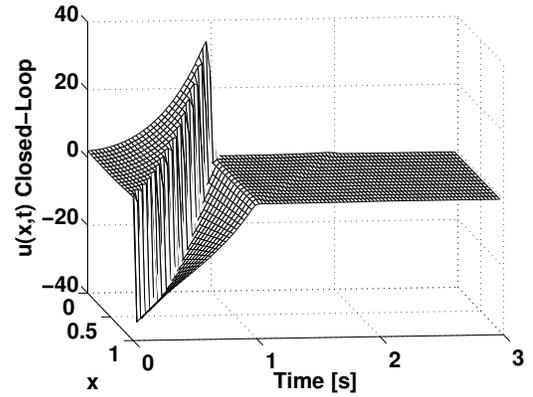


(c) Control gain $p(1, y)$.

Figure 1. Direct and inverse transform kernels obtained numerically for the interconnected PDE-ODE system and resulting control gain.



(a) Open-loop evolution of the PDE state $u(x, t)$.



(b) Closed-loop evolution of the PDE state $u(x, t)$.

Figure 2. Simulated evolution of the open-loop and closed-loop behavior of the $u(x, t)$ state of the interconnected PDE-ODE system.

F. Explicit boundary controller with shape restrictions in the coefficients

In this subsection, we impose additional conditions on the structure of the coefficients in (4) and the transform kernels in order to obtain an explicit solution to the nonlinear kernel equations (15)-(16) with boundary condition (17). With this structure (restricting the degrees of freedom for the kernels), the boundary condition on $q(x, 1)$ used in the previous section is no longer required to obtain a well-posed system under certain assumptions.

In this subsection, we will restrict the general class of systems (4)-(5) to the more particular form:

$$u_t(x, t) = u_x(x, t) + f_1 e^{\lambda x} \int_0^1 h_1(y) u(y, t) dy, \quad (81)$$

$$\forall (x, t) \in (0, 1) \times (0, T]$$

for $f_1, \lambda \in \mathbb{R}$, with boundary condition

$$u(1, t) = U_1(t) \quad (82)$$

for all $t \in (0, T]$.

This restricted form, along with the assumptions that follow (required only in this subsection) will allow us to find an explicit expression for the controller and its associated transform.

Assumption 2: $h_1(x)$ is such that

$$1 - \int_0^1 h_1(s) \int_s^1 e^{-\alpha(y-s)} f_1 e^{\lambda y} dy ds \neq 0, \quad (83)$$

where $\alpha \doteq \lambda + f_1 \int_0^1 e^{\lambda y} h_1(y) dy$.

Theorem 11: If Assumption 2 is verified, then the origin of the system (81)-(82), with control

$$U(t) = f_1 e^\lambda \int_0^1 k_1(y) u(y, t) dy, \quad (84)$$

where k_1 is given by:

$$k_1(x) = - \frac{\int_0^x e^{-\alpha(x-s)} h_1(s) ds}{1 - \int_0^1 h_1(s) \int_s^1 e^{-\alpha(y-s)} f_1 e^{\lambda y} dy ds}, \quad (85)$$

is (finite-time) stable in the topology of the L^2 norm.

The proof of this result is given in Appendix B.

G. Numerical example of explicit controller

Figure 3 shows the open-loop and closed-loop behavior under simulation of a system of the form (81)-(82) with $f_1 = 2$, $\lambda = 2$ and $h_1(x) = \cosh(x)$. The corresponding explicit controller is

$$k_1(x) = \frac{k_1^{num}(x)}{k_1^{den}} \quad (86)$$

with

$$k_1^{num}(x) = 3(\alpha - 2)e^\alpha (\alpha e^{-\alpha x} - \alpha \cosh(x) + \sinh(x)) \quad (87)$$

$$k_1^{den} = e^\alpha (3\alpha^3 - (2 + 3e + e^3)\alpha^2 + 3(-1 + e + e^3)\alpha - 2e^3 + 6e + 2) - 6e^2\alpha, \quad (88)$$

and $\alpha = \frac{1}{3}(2 + 3e + e^3)$.

III. BACKSTEPPING-FORWARDING OBSERVER DESIGN

A. Observer Structure

For any practical implementation of the controllers constructed in the previous section, the construction of an observer is required. We now turn to the observer design problem for a first-order hyperbolic system with the same structure as (4) and measured output $u(0, t)$.

We propose the following observer structure:

$$\begin{aligned} \hat{u}_t(x, t) &= \hat{u}_x(x, t) + f(x)\hat{u}(0, t) + \gamma_{obs,1}(x) [\hat{u}(0, t) - u(0, t)] \\ &\quad + \int_0^x g(x, y)\hat{u}(y, t) dy + \int_x^1 h(x, y)\hat{u}(y, t) dy, \\ \forall(x, t) &\in (0, 1) \times (0, T] \end{aligned} \quad (89)$$

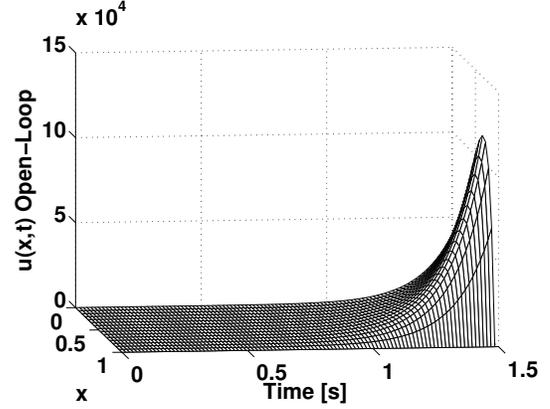
$$\hat{u}(1, t) = U(t), \quad \forall t \in (0, T] \quad (90)$$

with initial condition $\hat{u}(x, 0) \doteq \hat{u}_0(x) \in L^2([0, 1]; \mathbb{R})$. Here, $\gamma_{obs,1}(x)$ is a gain to be determined.

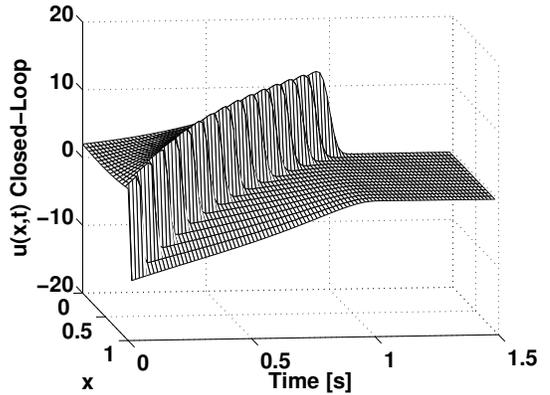
The resulting error system is given by

$$\begin{aligned} \tilde{u}_t(x, t) &= \tilde{u}_x(x, t) + f(x)\tilde{u}(0, t) + \gamma_{obs,1}(x)\tilde{u}(0, t) \\ &\quad + \int_0^x g(x, y)\tilde{u}(y, t) dy + \int_x^1 h(x, y)\tilde{u}(y, t) dy, \\ \forall(x, t) &\in (0, 1) \times (0, T] \end{aligned} \quad (91)$$

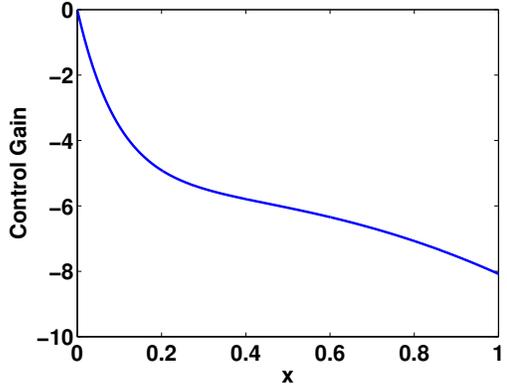
$$\tilde{u}(1, t) = 0, \quad \forall t \in (0, T] \quad (92)$$



(a) Open-loop evolution of the PDE state $u(x, t)$.



(b) Closed-loop evolution of the PIDE state $u(x, t)$.



(c) Control gain $f_1 e^\lambda k_1(x)$.

Figure 3. Simulated evolution of the open-loop and closed-loop behavior of the $u(x, t)$ state of the PIDE.

where $\tilde{u}(x, t) \doteq \hat{u}(x, t) - u(x, t)$.

We can define $\gamma_{obs}(x) \doteq f(x) + \gamma_{obs,1}(x)$ and focus only on the backstepping-forwarding stabilization of the error system

$$\begin{aligned} \tilde{u}_t(x, t) &= \tilde{u}_x(x, t) + \gamma_{obs}(x)\tilde{u}(0, t) + \int_0^x g(x, y)\tilde{u}(y, t) dy \\ &\quad + \int_x^1 h(x, y)\tilde{u}(y, t) dy, \quad \forall(x, t) \in (0, 1) \times (0, T] \end{aligned} \quad (93)$$

$$\tilde{u}(1, t) = 0, \quad \forall t \in (0, T]. \quad (94)$$

B. Preliminary Definitions

In order to build an observer for system (93)-(94) we proceed by finding a bounded transform

$$\begin{aligned} \tilde{u}(x, t) &= \tilde{w}(x, t) + \int_0^x k_{obs}(x, y) \tilde{w}(y, t) dy \\ &+ \int_x^1 l_{obs}(x, y) \tilde{w}(y, t) dy, \end{aligned} \quad (95)$$

with bounded inverse

$$\begin{aligned} \tilde{w}(x, t) &= \tilde{u}(x, t) - \int_0^x p_{obs}(x, y) \tilde{u}(y, t) dy \\ &- \int_x^1 q_{obs}(x, y) \tilde{u}(y, t) dy, \end{aligned} \quad (96)$$

and the associated gain

$$\gamma_{obs}(x) = -k_{obs}(x, 0) \quad (97)$$

such that the error system (93) is mapped into the (finite-time stable) target system

$$\tilde{w}_t(x, t) = \tilde{w}_x(x, t), \quad \forall (x, t) \in (0, 1) \times (0, T] \quad (98)$$

$$\tilde{w}(1, t) = 0, \quad \forall t \in (0, T]. \quad (99)$$

We remark that, for the observer design, we proceed by first finding the transform mapping from w to u and then its inverse (mapping from u to w). Assumption 1 is maintained throughout this section.

Analogously to the control case, the kernels of the inverse transform for the observer need to satisfy a set of PDEs:

$$\begin{aligned} k_{obs,x}(x, y) + k_{obs,y}(x, y) &= -g(x, y) + k_{obs}(x, 0)l_{obs}(0, y) \\ &- \int_0^y g(x, s)l_{obs}(s, y)ds \\ &- \int_y^x g(x, s)k_{obs}(s, y)ds \\ &- \int_x^1 h(x, s)k_{obs}(s, y)ds, \\ \forall x, y \in [0, 1] \text{ s.t. } y \leq x, y \neq 0 \end{aligned} \quad (100)$$

$$\begin{aligned} l_{obs,x}(x, y) + l_{obs,y}(x, y) &= -h(x, y) + k_{obs}(x, 0)l_{obs}(0, y) \\ &- \int_0^x g(x, s)l_{obs}(s, y)ds \\ &- \int_x^y h(x, s)l_{obs}(s, y)ds \\ &- \int_y^1 h(x, s)k_{obs}(s, y)ds, \\ \forall x, y \in [0, 1] \text{ s.t. } x \leq y \end{aligned} \quad (101)$$

with boundary condition

$$k_{obs}(1, y) = 0, \quad \forall y \in [0, 1]. \quad (102)$$

In this section, a second boundary condition $l_{obs}(0, y) = 0$ is chosen to cancel the nonlinearity in the kernel PDEs and simplify the contraction arguments required to solve the equations.

First, we introduce an integral operator \bar{R} related to the conditions required for (96) to be a left-inverse of (95).

Definition 8: Given functions $\phi \in X_l$, $\psi \in X_u$, define an integral operator $\bar{R}^{\phi, \psi} : X \rightarrow X$ as

$$\bar{R}^{\phi, \psi} \begin{bmatrix} p_{obs} \\ q_{obs} \end{bmatrix} \doteq \begin{bmatrix} \phi \\ \psi \end{bmatrix} - S^{\phi, \psi} \begin{bmatrix} p_{obs} \\ q_{obs} \end{bmatrix}, \quad (103)$$

with the operator $S^{\phi, \psi}$ defined as in (34).

We now introduce an integral operator T_{obs} related to the PDEs the kernels in (95) must satisfy in order to map the dynamics of (93)-(94) to those of (98)-(99).

Definition 9: Let us now define the integral operator $T_{obs} : X \rightarrow X$ (for $A_{1,1}^{obs} : X_l \rightarrow X_l$, $A_{1,2}^{obs} : X_u \rightarrow X_l$, $A_{2,1}^{obs} : X_l \rightarrow X_u$, $A_{2,2}^{obs} : X_u \rightarrow X_u$, $F_1^{obs} \in X_l$, $F_2^{obs} \in X_u$), for all $k_{obs} \in X_l$, $l_{obs} \in X_u$ as

$$\begin{aligned} T_{obs} \begin{bmatrix} k_{obs} \\ l_{obs} \end{bmatrix} &\doteq A_{obs} \begin{bmatrix} k_{obs} \\ l_{obs} \end{bmatrix} + F_{obs} \\ &\doteq \begin{bmatrix} A_{1,1}^{obs} & A_{1,2}^{obs} \\ A_{2,1}^{obs} & A_{2,2}^{obs} \end{bmatrix} \begin{bmatrix} k_{obs} \\ l_{obs} \end{bmatrix} + \begin{bmatrix} F_1^{obs} \\ F_2^{obs} \end{bmatrix}, \end{aligned} \quad (104)$$

where

$$\begin{aligned} A_{1,1}^{obs}[k_{obs}](x, y) &\doteq \int_0^{1-x} \int_0^{x-y} g(\sigma + x, s + \sigma + y) \\ &\times k_{obs}(s + \sigma + y, \sigma + y) ds d\sigma \\ &+ \int_0^{1-x} \int_0^{1-\sigma-x} h(\sigma + x, s + \sigma + x) \\ &\times k_{obs}(s + \sigma + x, \sigma + y) ds d\sigma \end{aligned} \quad (105)$$

$$\begin{aligned} A_{1,2}^{obs}[l_{obs}](x, y) &\doteq \int_0^{1-x} \int_0^{\sigma+y} g(\sigma + x, s) \\ &\times l_{obs}(s, \sigma + y) ds d\sigma \end{aligned} \quad (106)$$

$$\begin{aligned} A_{2,1}^{obs}[k_{obs}](x, y) &\doteq - \int_0^x \int_0^{1-\sigma+x-y} h(\sigma, s + \sigma - x + y) \\ &\times k_{obs}(s + \sigma - x + y, \sigma - x + y) ds d\sigma \end{aligned} \quad (107)$$

$$\begin{aligned} A_{2,2}^{obs}[l_{obs}](x, y) &\doteq - \int_0^x \int_0^\sigma g(\sigma, s)l_{obs}(s, \sigma - x + y) ds d\sigma \\ &- \int_0^x \int_0^{y-x} h(\sigma, s + \sigma) \\ &\times l_{obs}(s + \sigma, \sigma - x + y) ds d\sigma \end{aligned} \quad (108)$$

$$F_1^{obs}(x, y) \doteq \int_0^{1-x} g(\sigma + x, \sigma + y) d\sigma \quad (109)$$

$$F_2^{obs}(x, y) \doteq - \int_0^x h(\sigma, \sigma - x + y) d\sigma \quad (110)$$

in their respective domains.

Finally, we introduce an integral operator \bar{T}_{obs} related to the PDEs the kernels in (96) must satisfy in order to map the dynamics of (98)-(99) to those of (93)-(94).

Definition 10: Define the integral operator $\bar{T}_{obs} : X \rightarrow X$ (for $\bar{A}_{1,1}^{obs} : X_l \rightarrow X_l$, $\bar{A}_{1,2}^{obs} : X_u \rightarrow X_l$, $\bar{A}_{2,1}^{obs} : X_l \rightarrow X_u$,

$\bar{A}_{2,2}^{obs} : X_u \rightarrow X_u$, $\bar{F}_1^{obs} \in X_l$, $\bar{F}_2^{obs} \in X_u$), for all $p \in X_l$, $q \in X_u$ as

$$\begin{aligned} \bar{T}_{obs} \begin{bmatrix} p_{obs} \\ q_{obs} \end{bmatrix} &\doteq \bar{A}_{obs} \begin{bmatrix} p_{obs} \\ q_{obs} \end{bmatrix} + \bar{F}_{obs} \\ &\doteq \begin{bmatrix} \bar{A}_{1,1}^{obs} & \bar{A}_{1,2}^{obs} \\ \bar{A}_{2,1}^{obs} & \bar{A}_{2,2}^{obs} \end{bmatrix} \begin{bmatrix} p_{obs} \\ q_{obs} \end{bmatrix} + \begin{bmatrix} \bar{F}_1^{obs} \\ \bar{F}_2^{obs} \end{bmatrix}, \end{aligned} \quad (111)$$

where

$$\begin{aligned} \bar{A}_{1,1}^{obs}[p_{obs}](x, y) &\doteq - \int_0^{1-x} \int_0^{\sigma+y} h(s, \sigma + y) \\ &\quad \times p_{obs}(\sigma + x, s) ds d\sigma \\ &\quad - \int_0^{1-x} \int_0^{x-y} g(s + \sigma + y, \sigma + y) \\ &\quad \times p_{obs}(\sigma + x, \sigma + y + s) ds d\sigma \end{aligned} \quad (112)$$

$$\begin{aligned} \bar{A}_{1,2}^{obs}[q_{obs}](x, y) &\doteq - \int_0^{1-x} \int_0^{1-\sigma-x} g(s + \sigma + x, \sigma + y) \\ &\quad \times q_{obs}(\sigma + x, \sigma + x + s) ds d\sigma \end{aligned} \quad (113)$$

$$\begin{aligned} \bar{A}_{2,1}^{obs}[p_{obs}](x, y) &\doteq \int_0^x \int_0^\sigma h(s, \sigma - x + y) p_{obs}(\sigma, s) ds d\sigma \\ &\quad \times q_{obs}(\sigma, \sigma - x + y + s) ds d\sigma \end{aligned} \quad (114)$$

$$\begin{aligned} \bar{A}_{2,2}^{obs}[q_{obs}](x, y) &\doteq \int_0^x \int_0^{-x+y} h(s + \sigma, \sigma - x + y) q_{obs}(\sigma, \sigma + s) ds d\sigma \\ &\quad + \int_0^x \int_0^{1-\sigma+x-y} g(s + \sigma - x + y, \sigma - x + y) \\ &\quad \times q_{obs}(\sigma, \sigma - x + y + s) ds d\sigma \end{aligned} \quad (115)$$

$$\bar{F}_1^{obs}(x, y) \doteq \int_0^{1-x} g(\sigma + x, \sigma + y) d\sigma \quad (116)$$

$$\bar{F}_2^{obs}(x, y) \doteq - \int_0^x h(\sigma, \sigma - x + y) d\sigma \quad (117)$$

in their respective domains.

C. Direct Transform

For the observer, the direct transform (95) maps the target system to the original error system (contrary to the control case). For the existence of the direct transform we have the following results (analogous to those for the control).

Proposition 12: If the operator T_{obs} , as defined in (104), has a unique fixed point in X (i.e. there exists a unique $\zeta \in X$ s.t. $T_{obs}\zeta = \zeta$), then transform (95) with kernels

$$\begin{bmatrix} k_{obs} \\ l_{obs} \end{bmatrix} \doteq \zeta \quad (118)$$

maps system (98)-(99) into (93)-(94), with

$$\gamma_{obs}(x) \doteq -k_{obs}(x, 0), \quad \forall x \in [0, 1]. \quad (119)$$

The proof of this result is analogous to that in Appendix A and is omitted for brevity.

An equivalent condition to that in Proposition 12 is that 1 belongs to the resolvent set of the operator A_{obs} , as defined in (104).

We give a sufficient condition on the coefficients for the results to hold:

Lemma 13: If the coefficients in equation (93) verify $c_{obs} \doteq \max\{\|g\|_{X_l}, \|h\|_{X_u}\} < 1$, then transform (95) with kernels

$$\begin{bmatrix} k_{obs} \\ l_{obs} \end{bmatrix} \doteq \zeta \quad (120)$$

$$\doteq \lim_{n \rightarrow \infty} T_{obs}^n \vartheta_0$$

for any $\vartheta_0 \in X$, maps system (98)-(99) into (93)-(94), with

$$\gamma_{obs}(x) \doteq -k_{obs}(x, 0), \quad \forall x \in [0, 1] \quad (121)$$

and

$$\|\zeta\|_X \leq \frac{\|F_{obs}\|_X}{1 - c_{obs}}. \quad (122)$$

The proof is analogous to that of Lemma 3 and is therefore omitted. It should be noted that the conditions in this section are somewhat less stringent than those used for the control design. This is due to the fact that, for the observer design, $u(x, 0)$ is measured and, therefore, the coefficient $f(x)$ can be compensated perfectly.

D. Inverse Transform

In this section we focus on the computation of the inverse transform (assuming the direct transform has already been obtained). The first results use the definition of the operator $\bar{R}^{k_{obs}, l_{obs}}$ (in (103)) to give conditions for the left-inverse of the direct transform to exist. Similar conditions can be found for its right-inverse and it can be shown that if the left- and right-inverse exist they are equal. Where necessary, this condition is given in terms of the spectrum of the operator $\Pi^{k_{obs}, l_{obs}}$.

Proposition 14: Given kernels $k_{obs} \in X_l$ and $l_{obs} \in X_u$, if the operator $\bar{R}^{k_{obs}, l_{obs}}$, as defined in (103) has a unique fixed point $\bar{\varphi} \in X$, then transform (96) with kernels

$$\begin{bmatrix} p_{obs} \\ q_{obs} \end{bmatrix} \doteq \bar{\varphi} \quad (123)$$

is the left-inverse of transform (95).

A condition equivalent to that in the previous Proposition is that -1 belongs to the resolvent set of the operator $S^{k_{obs}, l_{obs}}$, as defined in (34).

Using the norm estimate obtained in Lemma 13 we obtain the following sufficient condition for the existence of an inverse transform (left- and right-inverse):

Lemma 15: If the coefficients in equation (93) verify $\max\{\|g\|_{X_l}, \|h\|_{X_u}\} < \frac{1}{2}$, then for kernels $k_{obs} \in X_l$ and $l_{obs} \in X_u$ as defined in Lemma 13, transform (96) with kernels

$$\begin{bmatrix} p_{obs} \\ q_{obs} \end{bmatrix} \doteq \lim_{n \rightarrow \infty} (\bar{R}^{k_{obs}, l_{obs}})^n \varphi_0, \quad (124)$$

for any $\varphi_0 \in X$, is the inverse of transform (95). Furthermore, the operator $\Pi^{k_{obs}, l_{obs}}$ defined in (22) has a spectral radius less than 1.

The proof is analogous to that in Lemma 6.

E. Closed-loop L^2 Stability

The previous sections gave conditions for the direct and inverse transforms to exist. In this section we present the main observation result.

Proposition 16: If 1 belongs to the resolvent set of the operators A_{obs} (defined in (104)) and $-\Pi^{k_{obs}, l_{obs}}$ (defined in (22)), with kernels

$$\begin{bmatrix} k_{obs} \\ l_{obs} \end{bmatrix} \doteq (\mathbb{I}_X - A_{obs})^{-1} F_{obs} \quad (125)$$

then the origin of system (93)-(94), with

$$\gamma_{obs}(x) \doteq -k_{obs}(x, 0), \quad \forall x \in [0, 1] \quad (126)$$

is finite-time stable in the topology of the $L^2([0, 1]; \mathbb{R})$ norm.

The proof is analogous to that of Proposition 8 and is therefore omitted.

A conservative (but easy to verify) sufficient condition for the above result to hold is:

Theorem 17: If the coefficients in (93) verify that $\max \{\|g\|_{X_l}, \|h\|_{X_u}\} < \frac{1}{2}$ then the origin of system (93)-(94) is finite-time stable in the topology of the $L^2([0, 1]; \mathbb{R})$ norm, with

$$\gamma_{obs}(x) \doteq -k_{obs}(x, 0), \quad \forall x \in [0, 1] \quad (127)$$

where

$$\begin{aligned} \begin{bmatrix} k_{obs} \\ l_{obs} \end{bmatrix} &\doteq \zeta \\ &\doteq \lim_{n \rightarrow \infty} T_{obs}^n \vartheta_0 \end{aligned} \quad (128)$$

for any $\vartheta_0 \in X$.

The proof is analogous to that of Theorem 9 and is therefore omitted.

Again, a more practical version of the results is:

Proposition 18: If the following conditions are verified:

- (i) the operator T_{obs} defined in (104) is a contraction in some norm equivalent to $\|\cdot\|_X$ and therefore has a unique fixed point $\zeta \in X$,
- (ii) the operator \bar{T}_{obs} defined in (111) is a contraction in some norm equivalent to $\|\cdot\|_X$ and therefore has a unique fixed point $\vartheta \in X$, and
- (iii) setting

$$\begin{bmatrix} k_{obs} \\ l_{obs} \end{bmatrix} \doteq \zeta \quad (129)$$

-1 belongs to the resolvent set of $\Pi^{k_{obs}, l_{obs}}$ and ϑ is the fixed point of $\bar{R}^{k_{obs}, l_{obs}}$

then the origin of system (93)-(94), with

$$\gamma_{obs}(x) \doteq -k_{obs}(x, 0), \quad \forall x \in [0, 1] \quad (130)$$

is finite-time stable in the topology of the $L^2([0, 1]; \mathbb{R})$ norm.

The proof is analogous to that of Proposition 10 and is therefore omitted.

F. Stability of Observer and Controller

In this section, we discuss the stability of the observer and controller interconnection. This means we consider systems (4)-(5) and (89)-(90) with $U(t) \doteq \int_0^1 p(1, y) \hat{u}(y, t) dy$ and initial conditions $u_0(x), \hat{u}_0(x) \in L^2([0, 1]; \mathbb{R})$. We assume that kernels $p, k_{obs} \in X_l$ and $q, l_{obs} \in X_u$ are given satisfying (15)-(17) and (100)-(102). We further assume kernels $k, p_{obs} \in X_l$ and $l, q_{obs} \in X_u$ are given such that $(\mathbb{I}_{L^2} + \Pi^{k, l})$ is the inverse of $(\mathbb{I}_{L^2} - \Pi^{p, q})$ and $(\mathbb{I}_{L^2} - \Pi^{p_{obs}, q_{obs}})$ is the inverse of $(\mathbb{I}_{L^2} + \Pi^{k_{obs}, l_{obs}})$.

Using the definition of $\tilde{u}(x, t) \doteq \hat{u}(x, t) - u(x, t)$, stability of (u, \hat{u}) is equivalent to stability of (u, \tilde{u}) . We therefore focus on equations (4)-(5) and (93)-(94) with $U(t) \doteq \int_0^1 p(1, y) u(y, t) dy + \int_0^1 p(1, y) \tilde{u}(y, t) dy$ and initial conditions $u_0(x), \tilde{u}_0(x) \doteq \hat{u}_0(x) - u_0(x) \in L^2([0, 1]; \mathbb{R})$. Applying the backstepping-forwarding transformations, we change variables to $w(x, t) = (\mathbb{I}_{L^2} - \Pi^{p, q})[u(\cdot, t)](x)$ and $\tilde{w}(x, t) = (\mathbb{I}_{L^2} - \Pi^{p_{obs}, q_{obs}})[\tilde{u}(\cdot, t)](x)$. The transformed system dynamics are given by

$$w_t(x, t) = w_x(x, t) - \int_0^1 q(x, 1) p(1, y) \times (\mathbb{I}_{L^2} + \Pi^{k_{obs}, l_{obs}})[\tilde{w}(\cdot, t)](y) dy \quad (131)$$

$$w(1, t) = \int_0^1 p(1, y) (\mathbb{I}_{L^2} + \Pi^{k_{obs}, l_{obs}})[\tilde{w}(\cdot, t)](y) dy \quad (132)$$

$$\tilde{w}_t(x, t) = \tilde{w}_x(x, t) \quad (133)$$

$$\tilde{w}(1, t) = 0 \quad (134)$$

with initial conditions $w_0(x) = (\mathbb{I}_{L^2} - \Pi^{p, q})[u_0](x)$ and $\tilde{w}_0(x) = (\mathbb{I}_{L^2} - \Pi^{p_{obs}, q_{obs}})[\tilde{u}_0](x) \in L^2([0, 1]; \mathbb{R})$. These equations can be solved as

$$w(x, t) \quad (135)$$

$$= \begin{cases} w_0(x+t) - \int_0^t \int_0^1 q(x+\sigma, 1) p(1, y) \times (\mathbb{I}_{L^2} + \Pi^{k_{obs}, l_{obs}})[\tilde{w}(\cdot, t-\sigma)](y) dy d\sigma, & \text{for } x+t \leq 1 \\ \int_0^1 p(1, y) (\mathbb{I}_{L^2} + \Pi^{k_{obs}, l_{obs}})[\tilde{w}(\cdot, x+t-1)](y) dy - \int_0^{1-x} \int_0^1 q(x+\sigma, 1) p(1, y) \times (\mathbb{I}_{L^2} + \Pi^{k_{obs}, l_{obs}})[\tilde{w}(\cdot, t-\sigma)](y) dy d\sigma, & \text{for } x+t > 1 \end{cases}$$

$$\tilde{w}(x, t) = \begin{cases} \tilde{w}_0(x+t), & \text{for } x+t \leq 1 \\ 0, & \text{for } x+t > 1 \end{cases} \quad (136)$$

for all $t \geq 0, x \in [0, 1]$.

Using Hölder's inequality, the boundedness of the kernels (in the X_l or X_u norm, respectively), and the boundedness of $(\mathbb{I}_{L^2} + \Pi^{k_{obs}, l_{obs}})$ as an operator in $L^2([0, 1]; \mathbb{R})$ it can be shown that there exists a constant $C(p, q, k_{obs}, l_{obs}) > 0$ (i.e., depending only on p, q, k_{obs} and l_{obs}) such that the norm estimates

$$\|\tilde{w}(\cdot, t)\|_{L^2} \leq \|\tilde{w}_0\|_{L^2} \quad (137)$$

$$\|w(\cdot, t)\|_{L^2} \leq \|w_0\|_{L^2} + C(p, q, k_{obs}, l_{obs}) \|\tilde{w}_0\|_{L^2} \quad (138)$$

hold for all $t \geq 0$. Furthermore

$$\|\tilde{w}(\cdot, t)\|_{L^2} = 0, \quad \forall t \geq 1 \quad (139)$$

$$\|w(\cdot, t)\|_{L^2} = 0, \quad \forall t \geq 2. \quad (140)$$

These norm estimates guarantee the stability of the interconnected system and the finite-time convergence in 2 seconds of the transformed state (w, \tilde{w}) . Furthermore, together with the boundedness of $(\mathbb{I}_{L^2} + \Pi^{k,l})$ and $(\mathbb{I}_{L^2} + \Pi^{k_{obs}, l_{obs}})$ it implies that there exist positive constants C_1, C_2, C_3 depending only on $p, q, k, l, p_{obs}, q_{obs}, k_{obs}$ and l_{obs} such that

$$\|\tilde{u}(\cdot, t)\|_{L^2} \leq C_1 \|\tilde{u}_0\|_{L^2} \quad (141)$$

$$\|u(\cdot, t)\|_{L^2} \leq C_2 \|u_0\|_{L^2} + C_3 \|\tilde{u}_0\|_{L^2} \quad (142)$$

for all $t \geq 0$, and

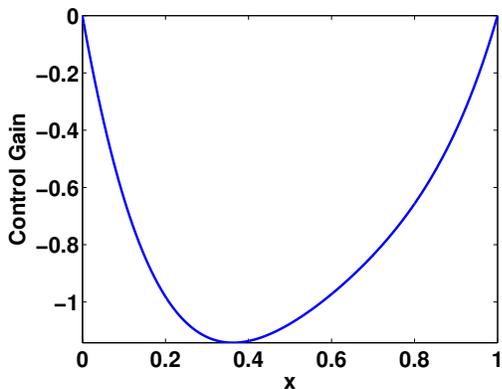
$$\|\tilde{u}(\cdot, t)\|_{L^2} = 0, \quad \forall t \geq 1 \quad (143)$$

$$\|u(\cdot, t)\|_{L^2} = 0, \quad \forall t \geq 2. \quad (144)$$

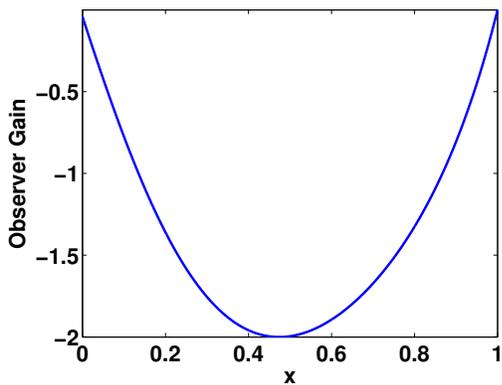
G. Application Example

In this section, we choose the following simple example to illustrate simultaneous control and observation of a first-order hyperbolic system with a Fredholm integral (with discontinuous kernel). This is, we use the observer and control design to build an output-feedback controller that drives the system to the origin in finite time (equal to the sum of the time required for the observer convergence and for closed-loop state convergence).

Consider (4) with $f(x) = 0$, $g(x, y) = 6(x - y)$ and $h(x, y) = 6(x + y)$. The control $U(t)$ is chosen as in



(a) Control gain $\gamma(y) = p(1, y)$.



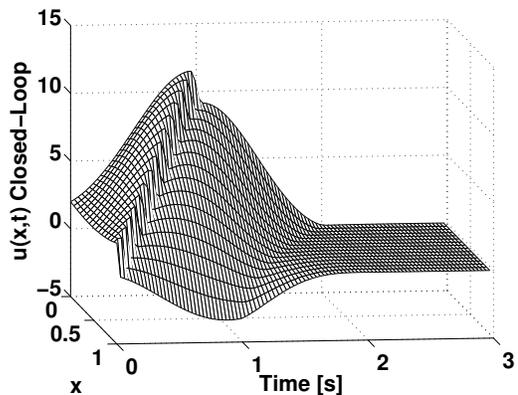
(b) Observer gain $\gamma_{obs}(x) = -k(x, 0)$.

Figure 4. Resulting control and observer gains.

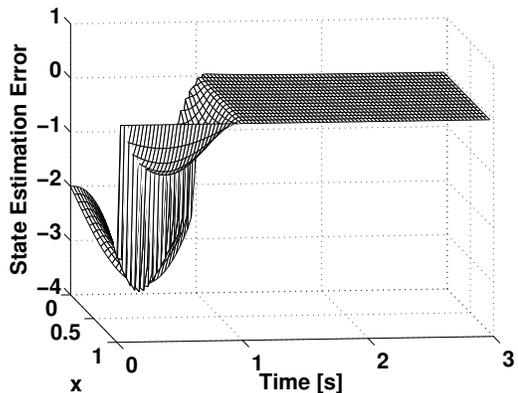
Proposition 8 and the observer gain is in turn chosen as in Proposition 16. Both the open-loop system (4)-(5) (with $U(t) = 0$) and (open-loop) error system (93)-(94) are unstable. Figures 4 (a) and 4 (b) show the obtained control and observer gains for this system. Figure 5 (a) shows the resulting state evolution (as expected, it converges in finite time). Figure 5 (b) shows the evolution of the state estimation (finite-time stable). Since the state estimation converges in 1 second and, assuming full state measurements, it takes 1 second for the controller to steer the system to the origin, using the controller and observer in the same system ensures convergence in 2 seconds.

IV. CONCLUSION

In this article, we propose an integral transform that allows the construction of stabilizing boundary controllers for a class of first-order hyperbolic PIDEs with Fredholm integrals. Sufficient conditions for this stabilizing controller and transform are given in terms of the spectrum of two integral operators on Banach spaces and (in a more conservative form) in terms of the magnitudes of the coefficients of equation (4). Also, an explicit transform and controller are given for some systems that verify additional assumptions on the shape of their coefficients. Finally, analogous conditions for the observer design are presented. This approach seems promising to deal with fully interconnected and underactuated PDE-PDE and



(a) Closed-loop evolution of the PDE state $u(x, t)$.



(b) Closed-loop evolution of the estimation error $\tilde{u}(x, t)$.

Figure 5. Simulated evolution of the closed-loop behavior of the $u(x, t)$ state and estimation error.

PDE-ODE systems, as well as systems where non-local terms appear in the evolution equation. Some research directions for future work are finding conditions that guarantee well-posedness of the kernel equations when the integral operators are not contractions (and the use of other solution methods for these cases) as well as extension of these methods to other classes of PDEs.

APPENDIX

A. Proof of Proposition 1

Proof (Proposition 1): This proof follows a similar approach to that used in standard backstepping to find sufficient conditions for the direct transform to exist.

Differentiating (10) w.r.t. x we obtain

$$\begin{aligned} w_x(x, t) &= u_x(x, t) - p(x, x)u(x, t) - \int_0^x p_x(x, y)u(y, t)dy \\ &\quad + q(x, x)u(x, t) - \int_x^1 q_x(x, y)u(y, t)dy. \end{aligned} \quad (145)$$

Differentiating (10) w.r.t. t , using (4), integrating by parts in the terms containing spatial derivatives of u and changing the order of integration in the double integrals we get

$$\begin{aligned} w_t(x, t) &= u_x(x, t) + f(x)u(0, t) + \int_0^x g(x, y)u(y, t)dy \\ &\quad + \int_x^1 h(x, y)u(y, t)dy - p(x, x)u(x, t) \\ &\quad + p(x, 0)u(0, t) + \int_0^x p_y(x, y)u(y, t)dy \\ &\quad - u(0, t) \int_0^x p(x, y)f(y)dy \\ &\quad - \int_0^x u(y, t) \int_y^x g(s, y)p(x, s)ds dy \\ &\quad - \int_0^x u(y, t) \int_0^y h(s, y)p(x, s)ds dy \\ &\quad - \int_x^1 u(y, t) \int_0^x h(s, y)p(x, s)ds dy \\ &\quad - q(x, 1)u(1, t) + q(x, x)u(x, t) \\ &\quad + \int_x^1 q_y(x, y)u(y, t)dy - u(0, t) \int_x^1 q(x, y)f(y)dy \\ &\quad - \int_0^x u(y, t) \int_x^1 g(s, y)q(x, s)ds dy \\ &\quad - \int_x^1 u(y, t) \int_y^1 g(s, y)q(x, s)ds dy \\ &\quad - \int_x^1 u(y, t) \int_x^y h(s, y)q(x, s)ds dy. \end{aligned} \quad (146)$$

Plugging (145) and (146) into (13), substituting the value of $u(1, t)$ from (12) in the term $-q(x, 1)u(1, t)$ and collecting

similar terms we obtain

$$\begin{aligned} u(0, t) &\left[f(x) - \int_0^x p(x, y)f(y)dy \right. \\ &\quad \left. - \int_x^1 q(x, y)f(y)dy + p(x, 0) \right] \\ &\quad + \int_0^x u(y, t) \left[g(x, y) + p_y(x, y) \right. \\ &\quad \left. - \int_y^x g(s, y)p(x, s)ds \right. \\ &\quad \left. - \int_x^1 g(s, y)q(x, s)ds + p_x(x, y) \right. \\ &\quad \left. - \int_0^y h(s, y)p(x, s)ds \right] dy \\ &\quad + \int_x^1 u(y, t) \left[h(x, y) + q_y(x, y) \right. \\ &\quad \left. - \int_y^1 g(s, y)q(x, s)ds \right. \\ &\quad \left. - \int_x^y h(s, y)q(x, s)ds + q_x(x, y) \right. \\ &\quad \left. - \int_0^x h(s, y)p(x, s) \right] dy \\ &\quad + \int_0^1 u(y, t) [-q(x, 1)p(1, y)] dy = 0. \end{aligned} \quad (147)$$

We therefore focus on solving the set of coupled hyperbolic PIDEs (15)-(16) with boundary conditions

$$\begin{aligned} p(x, 0) &= -f(x) + \int_0^x p(x, y)f(y)dy \\ &\quad + \int_x^1 q(x, y)f(y)dy, \quad \forall x \in [0, 1] \end{aligned} \quad (148)$$

$$q(x, 1) = 0, \quad \forall x \in [0, 1] \quad (149)$$

which cancel the nonlinear term in the domain.

Consider the (invertible) change of variables $\phi : [0, 1]^2 \rightarrow [0, 2] \times [-1, 1]$ defined as

$$\phi(x, y) \doteq (x + y, x - y), \quad \forall x, y \in [0, 1] \quad (150)$$

and

$$\begin{aligned} P(\phi(x, y)) &= P(\phi_1(x, y), \phi_2(x, y)) \doteq p(x, y), \\ \forall x, y \in [0, 1] \text{ s.t. } y &\leq x \end{aligned} \quad (151)$$

$$\begin{aligned} Q(\phi(x, y)) &= Q(\phi_1(x, y), \phi_2(x, y)) \doteq q(x, y), \\ \forall x, y \in [0, 1] \text{ s.t. } x &\leq y, \end{aligned} \quad (152)$$

where $\phi_i(x, y)$ denotes the i -th component of $\phi(x, y)$.

Defining new variables

$$\begin{aligned} \xi &\in [0, 2] \\ \eta &\in [-1, 1] \end{aligned}$$

we may rewrite (15)-(16) and the boundary conditions (148)-(149) as

$$\begin{aligned}
2P_\xi(\xi, \eta) &= -g\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \\
&+ \int_0^{\frac{\xi-\eta}{2}} h\left(s, \frac{\xi-\eta}{2}\right) P\left(\frac{\xi+\eta}{2} + s, \frac{\xi+\eta}{2} - s\right) ds \\
&+ \int_{\frac{\xi-\eta}{2}}^{\frac{\xi+\eta}{2}} g\left(s, \frac{\xi-\eta}{2}\right) P\left(\frac{\xi+\eta}{2} + s, \frac{\xi+\eta}{2} - s\right) ds \\
&+ \int_{\frac{\xi+\eta}{2}}^1 g\left(s, \frac{\xi-\eta}{2}\right) Q\left(\frac{\xi+\eta}{2} + s, \frac{\xi+\eta}{2} - s\right) ds, \\
\forall(\xi, \eta) \in [0, 2] \times [0, 1] \text{ s.t. } \eta &\leq \min\{\xi, 2 - \xi\}, \eta \neq \xi
\end{aligned} \tag{153}$$

$$\begin{aligned}
2Q_\xi(\xi, \eta) &= -h\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \\
&+ \int_{\frac{\xi-\eta}{2}}^{\frac{\xi-\eta}{2}} h\left(s, \frac{\xi-\eta}{2}\right) Q\left(\frac{\xi+\eta}{2} + s, \frac{\xi+\eta}{2} - s\right) ds \\
&+ \int_{\frac{\xi-\eta}{2}}^1 g\left(s, \frac{\xi-\eta}{2}\right) Q\left(\frac{\xi+\eta}{2} + s, \frac{\xi+\eta}{2} - s\right) ds \\
&+ \int_0^{\frac{\xi+\eta}{2}} h\left(s, \frac{\xi-\eta}{2}\right) P\left(\frac{\xi+\eta}{2} + s, \frac{\xi+\eta}{2} - s\right) ds, \\
\forall(\xi, \eta) \in [0, 2] \times [0, 1] \text{ s.t. } \eta &\geq \max\{-\xi, -2 + \xi\}, \eta \neq \xi - 2
\end{aligned} \tag{154}$$

$$\begin{aligned}
P(\eta, \eta) &= -f(\eta) + \int_0^\eta P(\eta + s, \eta - s) ds \\
&+ \int_\eta^1 Q(\eta + s, \eta - s) f(s) ds, \\
\forall \eta \in [0, 1]
\end{aligned} \tag{155}$$

$$Q(2 + \eta, \eta) = 0, \quad \forall \eta \in [-1, 0]. \tag{156}$$

Integrating (153) (w.r.t. ξ from η to ξ with boundary condition (155)) and (154) (w.r.t. ξ from ξ to $2 + \eta$ with boundary condition (156)) we obtain the following system of coupled integral equations (after inverting the change of variables and adjusting the limits of integration):

$$\begin{aligned}
p(x, y) &= \int_0^{x-y} f(s)p(x-y, s) ds \\
&+ \int_0^y \int_0^\sigma h(s, \sigma)p(\sigma+x-y, s) ds d\sigma \\
&+ \int_0^y \int_0^{x-y} g(s+\sigma, \sigma)p(\sigma+x-y, \sigma+s) ds d\sigma \\
&+ \int_0^{1-x+y} f(x-y+s)q(x-y, x-y+s) ds \\
&+ \int_0^y \int_0^{1-\sigma-x+y} g(\sigma+x-y+s, \sigma) \\
&\times q(\sigma+x-y, \sigma+x-y+s) ds d\sigma - f(x-y) \\
&- \int_0^y g(\sigma+x-y, \sigma) d\sigma, \quad \forall x, y \in [0, 1] \text{ s.t. } y \leq x
\end{aligned} \tag{157}$$

$$\begin{aligned}
q(x, y) &= - \int_0^{1-y} \int_0^{\sigma+x} h(s, \sigma+y)p(\sigma+x, s) ds d\sigma \\
&- \int_0^{1-y} \int_0^{y-x} h(s+\sigma+x, \sigma+y) \\
&\times q(\sigma+x, \sigma+x+s) ds d\sigma \\
&- \int_0^{1-y} \int_0^{1-\sigma-y} g(s+\sigma+y, \sigma+y) \\
&\times q(\sigma+x, \sigma+y+s) ds d\sigma \\
&+ \int_0^{1-y} h(\sigma+x, \sigma+y) d\sigma, \\
\forall x, y \in [0, 1] \text{ s.t. } x &\leq y.
\end{aligned} \tag{158}$$

The condition of the Proposition guarantees a unique solution to the direct transform kernel integral equations and therefore, a suitable direct transform exists. This ends the proof of Proposition 1. \blacksquare

We should note that (153)-(154) imply that the derivative of the direct transform kernels along the level curves of $x - y$ (i.e. in the ξ direction) is continuous.

B. Proof of Theorem 11

Proof (Theorem 11): We will proceed by finding a change of variables

$$w(x, t) \doteq u(x, t) - f_1 e^{\lambda x} \int_0^1 k_1(y) u(y, t) dy \tag{159}$$

that transforms system (81)-(82) into the (finite-time stable) target system

$$w_t(x, t) = w_x(x, t), \quad \forall (x, t) \in (0, 1) \times (0, T], \tag{160}$$

with boundary condition for all $t \in (0, T]$:

$$w(1, t) = 0. \tag{161}$$

The assumption in the Theorem can be shown to imply that

$$1 - \int_0^1 k_1(y) f_1 e^{\lambda y} dy \neq 0 \tag{162}$$

which, in turn, implies that the transformation (159) is boundedly invertible, with inverse given by

$$u(x, t) = w(x, t) + f_1 e^{\lambda x} \int_0^1 q_1(y) w(y, t) dy, \tag{163}$$

where $q_1(x)$ is defined as

$$q_1(x) \doteq \left(1 - \int_0^1 k_1(y) f_1 e^{\lambda y} dy\right)^{-1} k_1(x). \tag{164}$$

The proof then follows the classical backstepping paradigm of guaranteeing the stability of the closed-loop system by simultaneously finding a bounded (and boundedly invertible) transform and an associated control law that map the closed-loop system into a target stable system. The boundedness of both transforms guarantees, first, that a bounded initial condition in the original system is mapped to a bounded initial state for the target system and, second, that as the norm of the state of the target system goes to zero, the norm of the state in the original system also goes to zero.

Differentiating (159) with respect to x , we obtain

$$w_x(x, t) = u_x(x, t) - \lambda f_1 e^{\lambda x} \int_0^1 k_1(y) u(y, t) dy, \quad (165)$$

next, differentiating (159) with respect to t

$$w_t(x, t) = u_t(x, t) - f_1 e^{\lambda x} \int_0^1 k_1(y) u_t(y, t) dy. \quad (166)$$

Plugging equation (81) into (166) and integrating by parts the term containing the spatial derivative of u we obtain

$$\begin{aligned} w_t(x, t) &= u_x(x, t) + f_1 e^{\lambda x} \int_0^1 h_1(y) u(y, t) dy \\ &\quad - f_1 e^{\lambda x} k_1(1) u(1, t) + f_1 e^{\lambda x} k_1(0) u(0, t) \\ &\quad + f_1 e^{\lambda x} \int_0^1 k_1'(y) u(y, t) dy \\ &\quad - f_1 e^{\lambda x} \int_0^1 k_1(y) f_1 e^{\lambda y} \int_0^1 h_1(s) u(s, t) ds dy. \end{aligned} \quad (167)$$

Evaluating (159) at $x = 1$ we obtain the condition

$$u(1, t) = U(t) = f_1 e^{\lambda} \int_0^1 k_1(y) u(y, t) dy, \quad (168)$$

which in turn implies

$$\begin{aligned} w_t(x, t) &= u_x(x, t) + f_1 e^{\lambda x} \int_0^1 h_1(y) u(y, t) dy \\ &\quad - f_1 e^{\lambda x} k_1(1) f_1 e^{\lambda} \int_0^1 k_1(y) u(y, t) dy \\ &\quad + f_1 e^{\lambda x} k_1(0) u(0, t) + f_1 e^{\lambda x} \int_0^1 k_1'(y) u(y, t) dy \\ &\quad - f_1 e^{\lambda x} \int_0^1 k_1(y) f_1 e^{\lambda y} \int_0^1 h_1(s) u(s, t) ds dy. \end{aligned} \quad (169)$$

Substituting (165) and (169) into (81) and changing the order of integration in the resulting double integral we get

$$\begin{aligned} &f_1 e^{\lambda x} \left[\lambda \int_0^1 k_1(y) u(y, t) dy \right. \\ &\quad + \int_0^1 h_1(y) u(y, t) dy \\ &\quad - k_1(1) f_1 e^{\lambda} \int_0^1 k_1(y) u(y, t) dy \\ &\quad + k(0) u(0, t) + \int_0^1 k_1'(y) u(y, t) dy \\ &\quad \left. - \int_0^1 h_1(y) u(y, t) \int_0^1 k_1(s) f(s) ds dy \right] = 0. \end{aligned} \quad (170)$$

A sufficient condition for this equation to hold is that the following integro-differential equation is verified

$$\begin{aligned} &k_1'(y) + [\lambda - k_1(1) f_1 e^{\lambda}] k_1(y) \\ &= -h_1(y) \left[1 - \int_0^1 k_1(s) f_1 e^{\lambda s} ds \right], \end{aligned} \quad (171)$$

with boundary condition

$$k_1(0) = 0. \quad (172)$$

Defining

$$\alpha_2 \doteq \lambda - k_1(1) f_1 e^{\lambda} \quad (173)$$

and

$$g_1 \doteq 1 - \int_0^1 k_1(s) f_1 e^{\lambda s} ds \quad (174)$$

(171) can be solved as a nonhomogeneous first-order ODE with source term $-g_1 h_1(y)$ (since g_1 is different from zero, as stated in (162)) to obtain

$$k_1(y) = -g_1 \int_0^y e^{-\alpha_2(y-s)} h_1(s) ds. \quad (175)$$

Multiplying both sides of the equation by $f_1 e^{\lambda y}$, integrating from 0 to 1, using the definition of g_1 and Assumption 2 we obtain

$$g_1 = \frac{1}{1 - \int_0^1 h_1(s) \int_s^1 e^{-\alpha_2(y-s)} f_1 e^{\lambda y} dy ds}, \quad (176)$$

which implies

$$k_1(y) = -\frac{\int_0^y e^{-\alpha_2(y-s)} h_1(s) ds}{1 - \int_0^1 h_1(s) \int_s^1 e^{-\alpha_2(y-s)} f_1 e^{\lambda y} dy ds}. \quad (177)$$

The definition of α_2 in this proof can be shown to be equivalent to the expression for α given in Assumption 2 in terms of only the coefficients of the equation. This can be seen by multiplying (171) by $f_1 e^{\lambda y}$ on both sides and integrating from 0 to 1, integrating by parts the term containing the derivative of k_1 and using Assumption 2. This completes the proof of Theorem 11. ■

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