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# Robust observed-state feedback design for discrete-time systems rational in the uncertainties \*

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## Abstract

Design of controllers in the form of a state-feedback coupled to a state observer is studied in the context of uncertain systems. The classical approach by Luenberger is revisited. Results provide a heuristic design procedure that mimics the independent state-feedback / observer gains design by minimizing the coupling of observation error dynamics on the ideal state-feedback dynamics. The proposed design and analysis conditions apply to linear systems rationally-dependent on uncertainties defined in the cross-product of polytopes. Convex linear matrix inequality results are given thanks to the combination of a new descriptor multi-affine representations of systems and the S-variable approach. Stability and  $H_\infty$  performances are assessed by multi-affine parameter-dependent Lyapunov matrices for both cases of constant and time-varying uncertainties (or combinations of the two). Numerical complexity issues and ways to keep it as limited as possible are discussed and illustrated on an academic example.

*Keywords:* Robust control, Descriptor systems, State observers, State Feedback, Linear systems, Uncertain systems, Convex optimization

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\*This manuscript is a improved and extended version of results presented at IFAC 2014 World Congress [29, 30]. It has been submitted to Automatic in November 2015.

# 1 Introduction

The goal of the paper is to investigate, in the robust control context, the two step design strategy proposed by Luenberger [24]: *“The first phase is design of the control law assuming that the state vector is available. The second phase is the design of a system that produces an approximation to the state vector. This system is called an observer, or Luenberger observer”*. The main conclusion of our work is that, at the difference of the case of systems without uncertainties, the separation principle does not hold in the robust case. Stability of the closed-loop cannot any more be guaranteed by independent choices of stabilizing state-feedback and observer gains. A heuristic systematic procedure can nevertheless be constructed with the aim of minimizing the inevitable coupling between the state-feedback and observer dynamics. The procedure has two design steps combined to two analysis steps and we provide a detailed exposure of this procedure with up-to-date linear matrix inequality (LMI) convex optimization formulations for each step.

Controllers with observed-state feedback structure form a sub-class of dynamic output feedback controller models. Moreover, the observed-state feedback controllers build with identity Luenberger observers are special representations of dynamic controllers of the same order as the plant (full-order controllers). The issue of designing full-order controllers has convex LMI-based solutions as long as the systems are not affected by uncertainties, see [35, 3], to cite just a few. Unfortunately, as soon as the systems are affected by uncertainties, the design problem is either non-convex or highly conservative. The conservative case is when all uncertainties are embedded in one non-structured norm-bounded uncertainty. In that case, some results are available, for example [28] or also [21, 22] for results providing controllers with observed-state feedback structure. When aiming at preserving the knowledge of the uncertainty structure, results boil down to solving non-convex bilinear matrix inequality (BMI) constraints as for example in [19, 16]. As suggested in these papers, heuristics can be used to solve the BMIs. The strategy we propose in the paper can be considered as one of such.

The plants to be stabilized by the proposed approach are in discrete-time with uncertainties  $\theta$  described in state-space by  $x_{k+1} = A_r(\theta)x_k + B_r(\theta)u_k$  where  $u_k$  is the vector of inputs and  $y_k = Cx_k$  is the vector of outputs. The final goal is to design a Luenberger like identity observer with dynamics  $\hat{x}_{k+1} = A_o\hat{x}_k + B_o u_k + L(C\hat{x}_k - y_k)$  such that the system in closed-loop with the observed-state feedback  $u_k = K\hat{x}_k$  is stable for all uncertainties.

The first design step of the proposed procedure is, as in the classical Luenberger strategy, the design of a robust static state-feedback control  $K$ . At this step the state is assumed to be measured. This topic has been studied in the past by many authors, see [5, 27] for example. The result we provide for this step is a variation on results from [27], extended to systems rational in the uncertainties using properties exposed in [13]. The provided LMI-based result is in terms of  $H_\infty$  performance, but it could as well include other specifications such as  $H_2$  performance or pole location. The steps that follow can be applied whatever the methodology and whatever the specifications imposed at this state-feedback design step.

The second design step is an observer design type problem. The observer design problem in case of systems with uncertainties has diverse solutions in the literature.

Two types of results can be distinguished. A first category tackles the observer problem as part of the output filtering issue. [17] gives for that problem an LMI formulation in the case of systems with structured polytopic uncertainties. As discussed in section VI of that paper, the significant feature of robust filtering (which is usually considered as dual with respect to state-feedback) is that it needs to optimize over more decision variables than just one gain. At the difference of state-feedback where only the gain  $K$  is designed, the methodology of [17] illustrates that robust filtering involves the design of both an observer-like gain  $L$  and of the state matrix  $A_o$  of the filter. The same conclusions hold for results of [36] in the context of IQCs. The second category of results tackles directly the observer design problem. These, at the difference of filter-design results, have the main advantage not to assume open-loop stability of the plant. Only the error  $e_k = x_k - \hat{x}_k$  between the plant states and the observer states is required to be asymptotically stable. Surprisingly, robust observer results do not consider the upper formulated issue about having the state matrix  $A_o$  as a design variable. For example recent results of [1, 26] consider only the design of the  $L$  matrix while  $A_o$  is chosen a priori to be the one of the nominal system ( $A_o = A_r(0)$ ). In [22] the assumption of a fixed  $A_o$  is alleviated, but results are restricted to unstructured norm-bounded uncertainties. Our result considers both matrices  $A_o$  and  $B_o$  as free to design variables. As discussed in [34] this problem is a difficult one, and we do not claim to provide a final answer.

The closed-loop dynamics in terms of plant state  $x_k$  and observation error  $e_k$  are driven by the following closed-loop state matrix

$$A_c(\theta) = \begin{bmatrix} A_r(\theta) + B_r(\theta)K & -B_r(\theta)K \\ \Delta_A(\theta) + \Delta_B(\theta)K & A_o + LC - \Delta_B(\theta)K \end{bmatrix}$$

where  $\Delta_A(\theta) = A_r(\theta) - A_o$  and  $\Delta_B(\theta) = B_r(\theta) - B_o$ . From this formula it is trivial that if  $\Delta_A = 0$  and  $\Delta_B = 0$  then the closed loop dynamics depend only on the choice of stabilizing gains  $K$  and  $L$  which could be done independently. This is the separation that is no more achievable in the robust context. A way around this difficulty would be to minimize the norms of  $\Delta_A(\theta)$  and  $\Delta_B(\theta)$  independently of any other consideration such as stability or performance. It is not the choice we adopt. The proposed procedure is to minimize the effects of  $\epsilon_k = Ke_k$  on the state-feedback dynamics (due to the upper-right block in  $A_c(\theta)$ ) seen as outputs of the error dynamics perturbed by  $(\Delta_A(\theta) + \Delta_B(\theta)K)x_k$  (lower-left block in  $A_c(\theta)$ ). We show, that formalized in this way, the observer design problem has convex LMI formulations. Moreover, several types of input-output performances can be minimized simultaneously where  $x_k$  is the input and  $\epsilon_k$  is the output. [20] exposes in details the inevitable waterbed effects that occur in observer design. When improving time response or disturbance rejection of the observer the drawback is increased peak response due to non-zero initial conditions. To handle this tradeoff between performance in the long term and short term spikes, we propose an observer design that combines norm-to-norm and norm-to-peak performances.

Note that  $x_k$  is seen in the observer design phase as some input perturbation but it is not just any bounded signal. For this reason our proposed procedure includes an analysis step between the state-feedback design and the observer design steps. This

analysis provides information on expected trajectories of  $x_k$  for known state-feedback gain  $K$ . Not only this analysis step allows to improve the quality of the observer design but it also allows in the end to assess for closed-loop stability with small-gain theorem argument. This small-gain argument happens to be conservative. Hence if it is not positive, a closed-loop analysis LMI test is also proposed to analyse stability and performance of the global observed-state feedback system.

The overall design procedure with two design steps and two analysis steps is exposed for systems rationally-dependent on uncertainties  $\theta$ . A new type of modeling is proposed where the uncertainties lie in the cross-product of independent polytopes. A sub-case is when uncertainties are scalars in intervals. For this type of models we propose a transformation called descriptor multi-affine representation (DMAR for short) which is an alternative to linear-fractional representations [11, 18]. The DMAR is directly inspired by results from [39, 6, 25] and is shown to have smaller dimensions than the conventional linear-fractional representations. The rational dependence is converted into a multi-affine dependence at the expense of introducing exogenous fictive signals and rewriting the plant in descriptor form. As exposed in [10, 15, 13] this descriptor structure happens to be well adapted for deriving S-variable LMI results (also known as "extended LMIs" [9, 33], "dilated LMIs" [12], "enhanced LMIs" [2] or "slack-variable approach" [32]). Our results are extensions of these results for the case of multi-affine dependence. We moreover provide a methodology inspired from [7, 8, 30] to handle systems where some uncertainties are time-varying.

The outline of the paper is as follows. Section 2 is dedicated to the exposure of the new descriptor multi-affine representation of rationally dependent uncertain systems. The four LMI results for state-feedback design, state-feedback loop analysis, observer design and observed-state feedback loop analysis are given in Section 3. Based on these, the heuristic procedure with two design steps and two analysis steps is presented in section 4. Section 5 brings some important technical tools such as: ways to keep the numerical burden as small as possible and extensions to systems with time-varying uncertainties. A numerical example illustrates the results in section 6. Conclusions are drawn in the final section.

*Notation:*

$I$  stands for the identity matrix.  $A^T$  is the transpose of the matrix  $A$ .  $\{A\}^S$  stands for the symmetric matrix  $\{A\}^S = A + A^T$ .  $\text{diag}(F_1, \dots, F_i, \dots)$  stands for a block-diagonal matrix whose diagonal blocks are the  $F_i$  matrices.  $A \succ B$  is the matrix inequality stating that  $A - B$  is symmetric positive definite. The terminology "congruence operation of  $A$  on  $B$ " is used to denote  $A^T B A$ . If  $A$  is full rank, and  $B \succ 0$ , the congruence operation of  $A$  on  $B$  gives a positive definite matrix:  $A^T B A \succ 0$ . A matrix inequality of the type  $N(X) \succ 0$  is said to be a linear matrix inequality (LMI for short), if  $N(X)$  is affine in the decision variables  $X$ . In the following, decision variables are highlighted using the **blue color**.  $\Xi_{\bar{v}} = \{\xi_{v=1 \dots \bar{v}} \geq 0, \sum_{v=1}^{\bar{v}} \xi_v = 1\}$  is the unitary simplex in  $\mathbb{R}^{\bar{v}}$ . The elements  $\xi$  of unitary simplexes are used to describe polytopic type uncertainties. In the following, uncertainties are highlighted using the **red color**. For a discrete-time signal  $v_{k \geq 0}$ ,  $\|v\|_2^2 = \sum_{k=0}^{\infty} v_k^T v_k$  is the squared  $l_2$  norm and  $\|v\|_{2, \bar{k}}^2 = \sum_{k=0}^{\bar{k}} v_k^T v_k$  stands for the truncated squared norm.  $\|v\|_p = \max_{k \geq 0} (v_k^T v_k)^{1/2}$  denotes the peak of the euclidian norm over time.

## 2 Descriptor multi-affine modeling of rationally dependent uncertain systems

We shall consider in this paper parameter-dependent systems such as

$$\begin{aligned} x_{k+1} &= A_r(\theta)x_k + B_r(\theta)u_k + B_{rw}(\theta)w_k \\ z_k &= C_{rz}(\theta)x_k + D_{rzu}(\theta)u_k + D_{rzw}(\theta)w_k \\ y_k &= Cx_k \end{aligned} \quad (1)$$

where all the matrices except  $C$  are rational in uncertain parameters gathered in the notation  $\theta$ . The parameters  $\theta$  are assumed to lie in a set  $\Theta$  defined as the cross product of  $\bar{p}$  sets:

$$\theta \in \Theta = \{(\theta_1, \dots, \theta_{\bar{p}}) \in \Theta_1 \times \dots \times \Theta_{\bar{p}}\}. \quad (2)$$

The  $\bar{p}$  components of  $\theta$  are independent vectors of  $\mathbb{R}^{m_p}$ . Each set  $\Theta_p$  is assumed to be a polytope with  $\bar{v}_p$  vertices from the set  $\mathcal{V}_p = \{\theta_p^{[1]}, \dots, \theta_p^{[\bar{v}_p]}\}$ .  $\Theta_p$  is the convex hull of the vertices, or equivalently, each  $\theta_p$  writes as the weighted sum of vertices with weight from unitary simplexes:

$$\Theta_p = Co(\mathcal{V}_p) = \left\{ \theta_p = \sum_{v=1}^{\bar{v}_p} \xi_{p,v} \theta_p^{[v]} : \xi_p \in \Xi_{\bar{v}_p} \right\}. \quad (3)$$

In the following,  $\mathcal{V} = \mathcal{V}_1 \times \dots \times \mathcal{V}_{\bar{p}}$  is the set of all extremal values of the parameters gathered in  $\theta$ . A generic element of  $\mathcal{V}$  will be denoted  $\theta^{[v]}$  with  $v = (v_1, \dots, v_{\bar{p}})$  the vector of indices of vertices for each component.  $\mathcal{I}$  is the set of all vectors of indices  $v$ .  $\theta^{[v]}$  is the one to one map from  $\mathcal{I}$  to  $\mathcal{V}$ . The cardinality of  $\mathcal{V}$  is  $\bar{v} = \prod_{p=1}^{\bar{p}} \bar{v}_p$ .

Uncertainties  $\theta$  are assumed to be constant (except for subsection 5.3 that extends the results to the time-varying case).

A matrix  $M(\theta)$  is said to be multi-affine in the parameters if it is affine in each  $\theta_p$  taken independently. For example  $M(\theta) = 1 + \theta_1 + \theta_1\theta_2$  is multi-affine in the two uncertainties  $\theta_1, \theta_2$ . In the considered case of scalar uncertainties, they belong to polytopes with two vertices  $\theta_1 \in [\theta_1^{[1]}, \theta_1^{[2]}]$ ,  $\theta_2 \in [\theta_2^{[1]}, \theta_2^{[2]}]$ . It is easy to see that the following holds

$$\begin{aligned} M(\theta) &= \xi_{1,1}\xi_{2,1}(1 + \theta_1^{[1]} + \theta_1^{[1]}\theta_2^{[1]}) \\ &\quad + \xi_{1,1}\xi_{2,2}(1 + \theta_1^{[1]} + \theta_1^{[1]}\theta_2^{[2]}) \\ &\quad + \xi_{1,2}\xi_{2,1}(1 + \theta_1^{[2]} + \theta_1^{[2]}\theta_2^{[1]}) \\ &\quad + \xi_{1,2}\xi_{2,2}(1 + \theta_1^{[2]} + \theta_1^{[2]}\theta_2^{[2]}). \end{aligned}$$

This holds for all cases. A matrix, multi-affine with respect to uncertainties defined in the cross-product of polytopes, can be written as the multi-sum of weighted vertices from  $\mathcal{V}$  denoted as

$$M(\theta) = \sum_{v \in \mathcal{I}} \xi_{1,v_1} \dots \xi_{\bar{p},v_{\bar{p}}} M(\theta^{[v]}).$$

The simplest case is when the  $\theta_p$  elements are scalars ( $m_p = 1$ ) defined in intervals  $\theta_p \in [\theta_p^{[1]}, \theta_p^{[2]}]$ , which are polytopes of  $\bar{v}_p = 2$  vertices. For the case when all

elements are scalar, the cardinality of  $\mathcal{V}$  is  $\bar{v} = 2^{\bar{p}}$ . In case of two scalar parameters, the multi-sum reads as

$$\begin{aligned} M(\theta) &= \sum_{v \in \{(1,1), (1,2), (2,1), (2,2)\}} \xi_{1,v_1} \xi_{2,v_2} M(\theta^{[v]}) \\ &= \xi_{1,1} \xi_{2,1} M(\theta^{[(1,1)]}) + \xi_{1,1} \xi_{2,2} M(\theta^{[(1,2)]}) \\ &\quad + \xi_{1,2} \xi_{2,1} M(\theta^{[(2,1)]}) + \xi_{1,2} \xi_{2,2} M(\theta^{[(2,2)]}). \end{aligned}$$

It is trivial to notice that the multi-affine matrix  $M(\theta)$  can also be described as included in the polytope of four vertices  $M(\theta^{[(1,1)]})$ ,  $M(\theta^{[(1,2)]})$ ,  $M(\theta^{[(2,1)]})$ ,  $M(\theta^{[(2,2)]})$ . The converse is not true in general. There are potentially elements in the convex hull of these four vertices that are not in the multi-affine model. An example to this fact is the matrix  $M(\theta) = \begin{bmatrix} \theta_1 & \theta_1 \theta_2 & \theta_2 \end{bmatrix}$  with  $\theta_1 \in [1, 2]$  and  $\theta_2 \in [1, 2]$ . The middle of vertices  $M(\theta^{[(1,1)]}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$  and  $M(\theta^{[(2,2)]}) = \begin{bmatrix} 2 & 4 & 2 \end{bmatrix}$  is equal to  $\begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{3}{2} \end{bmatrix}$ . It is in the polytope, but is not a realization of  $M(\theta)$  since  $\frac{5}{2} \neq (\frac{3}{2})^2$ .

**Lemma 1** Any rationally dependent matrix  $R(\theta)$  admits a descriptor multi-affine representation (DMAR) of the form  $R(\theta) = E_1(\theta)E_2^{-1}(\theta)E_3(\theta)$  where the  $E_1(\theta)$ ,  $E_2(\theta)$  and  $E_3(\theta)$  matrices are multi-affine in  $\theta$ .

*Proof:* Recall [11, 18] that any rational matrix admits a linear-rational representation (LFR) of the type  $R(\theta) = R_1 + R_2 \Delta(\theta) (I - R_4 \Delta(\theta))^{-1} R_3$  where  $\Delta(\theta)$  is linear in the elements of  $\theta$ . Taking  $E_1(\theta) = \begin{bmatrix} R_1 & R_2 \Delta(\theta) \end{bmatrix}$ ,  $E_2(\theta) = \begin{bmatrix} I & 0 \\ 0 & I - R_4 \Delta(\theta) \end{bmatrix}$ ,  $E_3 = \begin{bmatrix} I \\ R_3 \end{bmatrix}$  concludes the proof. ■

The LFR used in the proof usually generates a DMAR of larger dimensions than needed, and imposes many more manipulations. This is not only because the LFR generates an affine DMAR (not multi-affine). For example, take the following rational system in two scalar uncertainties

$$x_{k+1} = \frac{\theta_1}{1+\theta_2} x_k + \theta_1^2 \theta_2 u_k + w_k, \quad z_k = \frac{1}{\theta_1} x_k, \quad y_k = x_k.$$

Its model depends on the following parameter-dependent matrix with DMAR

$$\begin{aligned} &\begin{bmatrix} \frac{\theta_1}{1+\theta_2} & \theta_1^2 \theta_2 \\ \frac{1}{\theta_1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \theta_1 & \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 + \theta_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta_1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \theta_1 \theta_2 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Obtaining this DMAR is very simple. Meanwhile, an equivalent LFR needs additional manipulations and leads to a  $\Delta(\theta)$  matrix with at least 4 rows. The DMAR build using the formulas in the proof of Lemma 1 would have an  $E_2(\theta)$  matrix with 6 rows.

DMARs are used in the following to convert rational systems (1) into equivalent descriptor form representations where all matrices are multi-affine in the uncertain parameters. Three important examples of such representations follow. They are based on

the DMAR

$$\begin{aligned} & \begin{bmatrix} A_r(\theta) & B_r(\theta) & B_{rw}(\theta) \\ C_{rz}(\theta) & D_{rzu}(\theta) & D_{rzw}(\theta) \end{bmatrix} \\ &= \begin{bmatrix} E_x(\theta) \\ E_z(\theta) \end{bmatrix} E_\pi^{-1}(\theta) [ A(\theta) \quad B(\theta) \quad B_w(\theta) ] \end{aligned} \quad (4)$$

or its transposed version

$$\begin{aligned} & \begin{bmatrix} A_r^T(\theta) & C_{rz}^T(\theta) \\ B_r^T(\theta) & D_{rzu}^T(\theta) \\ B_{rw}^T(\theta) & D_{rzw}^T(\theta) \end{bmatrix} \\ &= \begin{bmatrix} E_{dx}(\theta) \\ E_{dy}(\theta) \\ E_{dz}(\theta) \end{bmatrix} E_{d\pi}^{-1}(\theta) [ A_d(\theta) \quad B_{dw}(\theta) ]. \end{aligned} \quad (5)$$

Trivially the DMAR (4) can be obtained from (5), and conversely, with the choice  $E_x(\theta) = A_d^T(\theta)$ ,  $E_z(\theta) = B_{dw}^T(\theta)$ ,  $E_\pi(\theta) = E_{d\pi}^T(\theta)$ ,  $A(\theta) = E_{dx}^T(\theta)$ ,  $B(\theta) = E_{dy}^T(\theta)$  and  $B_w(\theta) = E_{dz}^T(\theta)$ . The subscripts  $d$  in the DMAR (5) are used to highlight that the matrices describe a dual system. This property is used in the proof of the following lemma.

**Lemma 2** *Assume the DMAR (5), the  $H_\infty$  performance of the rationally dependent system (1) in closed-loop with a state-feedback  $u_k = Kx_k$ , is equal to the  $H_\infty$  performance of the descriptor multi-affine system  $E_d(\theta)\eta_{d,k} = 0$  defined by*

$$\begin{aligned} E_d(\theta) &= \begin{bmatrix} I & 0 & E_{dx}(\theta) + K^T E_{dy}(\theta) & 0 & 0 \\ 0 & I & E_{dz}(\theta) & 0 & 0 \\ 0 & 0 & E_{d\pi}(\theta) & A_d(\theta) & B_{dw}(\theta) \end{bmatrix} \\ \eta_{d,k}^T &= ( x_{d,k+1}^T \quad z_{d,k}^T \quad \pi_{d,k}^T \quad x_{d,k}^T \quad w_{d,k}^T ). \end{aligned} \quad (6)$$

*Proof:* System (1) with state-feedback  $u_k = Kx_k$  reads as

$$\begin{aligned} x_{k+1} &= A_{r,K}(\theta)x_k + B_{rw}(\theta)w_k \\ z_k &= C_{rz,K}(\theta)x_k + D_{rzw}(\theta)w_k \end{aligned} \quad (7)$$

where  $A_{r,K}(\theta) = (A_r(\theta) + B_r(\theta)K)$  and  $C_{rz,K}(\theta) = (C_{rz}(\theta) + D_{rzu}(\theta)K)$ . Its  $H_\infty$  performance is equal to the performance of the dual system defined by

$$\begin{aligned} x_{d,k+1} &= A_{r,K}^T(\theta)x_{d,k} + C_{rz,K}^T(\theta)w_{d,k} \\ z_{d,k} &= B_{rw}^T(\theta)x_{d,k} + D_{rzw}^T(\theta)w_{d,k} \end{aligned} \quad (8)$$

Using the DMAR (5) these equations read exactly as

$$\begin{aligned} x_{d,k+1} + (E_{dx}(\theta) + K^T E_{dy}(\theta))\pi_{d,k} &= 0 \\ z_{d,k} + E_{dz}(\theta)\pi_{d,k} &= 0 \end{aligned}$$

which correspond to the two first rows of (6), with  $\pi_{d,k}$  as

$$\pi_{d,k} = -E_{d\pi}^{-1}(\theta) (A_d(\theta)x_{d,k} + B_{dw}(\theta)w_{d,k}).$$



Pre-multiplying the last equation by  $E_{d\pi}(\theta)$  gives the last row of (6). ■

Lemma 2 will be used in the next section for building state-feedback design conditions. As exposed in the following, the conditions are finite dimensional LMI. Conditions need only to be tested on vertices of the polytopes, thanks to convexity of matrix inequalities, and to the multi-affine nature of the descriptor representation.

Since states of a plant are in general not measured, state-feedback control needs in practice to be replaced by a feedback  $u_k = K\hat{x}_k$  based on estimations of the state  $\hat{x}$ . This supposes the design of some observer. The following Lemma presents the descriptor multi-affine modeling related to this problem.

**Lemma 3** Assume the DMAR (4) and consider the descriptor multi-affine system  $E_o(\theta)\eta_{o,k} = 0$  defined by

$$\begin{aligned} \hat{A}_o &= A_o + B_o K + LC \\ E_o(\theta) &= \begin{bmatrix} I & E_x(\theta) & -\hat{A}_o & A_o + B_o K \\ 0 & E_\pi(\theta) & -B(\theta)K & A(\theta) + B(\theta)K \end{bmatrix} \\ \eta_{o,k}^T &= ( e_{k+1}^T \quad \pi_{o,k}^T \quad e_k^T \quad x_k^T ) \end{aligned} \quad (9)$$

where  $e_k$  plays the role of the state and  $x_k$  is an input. The stability and input/output performances of  $E_o(\theta)\eta_{o,k} = 0$  are equivalent to stability of the error  $e_k = x_k - \hat{x}_k$  and performance of the transfert from  $x_k$  to  $\epsilon_k = Ke_k$ , for the Luenberger type observer defined by

$$\hat{x}_{k+1} = A_o \hat{x}_k + B_o u_k + L(C\hat{x}_k - y_k), \quad (10)$$

when assuming observed-state feedback  $u_k = K\hat{x}_k$ .

*Proof:* Simple manipulations with equations (1), (10) and  $u_k = K\hat{x}_k$  provide that, for zero perturbations  $w_k = 0$ , the error between the plant state  $x_k$  and the state of the observer  $\hat{x}_k$  is driven by the following equation

$$\begin{aligned} e_{k+1} &= (A_o + (B_o - B_r(\theta))K + LC)e_k \\ &\quad + (A_r(\theta) - A_o + (B_r(\theta) - B_o)K)x_k. \end{aligned} \quad (11)$$

Using the DMAR (4), equation (11) also reads as

$$\begin{bmatrix} I & E_x(\theta) & -\hat{A}_o & A_o + B_o K \end{bmatrix} \eta_{o,k} = 0$$

where  $\pi_{o,k} = -E_\pi^{-1}(\theta)(B(\theta)K e_k - (A(\theta) + B(\theta)K)x_k)$ . This definition of  $\pi_{o,k}$  is exactly the last row in (9). ■

The two last lemmas are used in the next section for designing separately the state-feedback gain  $K$  and then the observer matrices  $A_o, B_o, L$ . Since separation principle does not hold for uncertain systems, a last step is then needed for robust stability analysis of the overall observer-state feedback control. The third descriptor multi-affine model that follows is used for that purpose.

**Lemma 4** Assume the DMAR (4) and (5) and consider the descriptor multi-affine systems  $E_c(\theta)\eta_{c,k} = 0$  and  $E_{dc}(\theta)\eta_{dc,k} = 0$  defined respectively by

$$E_c(\theta) = \begin{bmatrix} I & 0 & 0 & 0 & -\hat{A}_o & LC & 0 \\ 0 & I & 0 & E_x(\theta) & 0 & 0 & 0 \\ 0 & 0 & I & E_z(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & E_\pi(\theta) & B(\theta)K & A(\theta) & B_w(\theta) \end{bmatrix}, \quad (12)$$

$$\eta_{c,k}^T = \begin{pmatrix} \hat{x}_{k+1}^T & x_{k+1}^T & z_k^T & \pi_{c,k}^T & \hat{x}_k^T & x_k^T & w_k^T \end{pmatrix}$$

and

$$E_{dc}(\theta) = \begin{bmatrix} I & 0 & 0 & K^T E_{dy}(\theta) & -\hat{A}_o^T & 0 & 0 \\ 0 & I & 0 & E_{dx}(\theta) & C^T L^T & 0 & 0 \\ 0 & 0 & I & E_{dz}(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{d\pi}(\theta) & 0 & A_d(\theta) & B_{dw}(\theta) \end{bmatrix}, \quad (13)$$

$$\eta_{dc,k}^T = \begin{pmatrix} \hat{x}_{d,k+1}^T & x_{d,k+1}^T & z_{d,k}^T & \pi_{dc,k}^T & \hat{x}_{d,k}^T & x_{d,k}^T & w_{d,k}^T \end{pmatrix}.$$

The stability and input/output performances of  $E_c(\theta)\eta_{c,k} = 0$  and  $E_{dc}(\theta)\eta_{dc,k} = 0$  are equivalent to stability and input/output performances of the closed-loop system composed of the plant (1), the observer (10) and the feedback  $u_k = K\hat{x}_k$ .

*Proof:* The proof follows exactly the same lines as the two previous ones and involve respectively the DMAR (4) applied to the system equations and the DMAR (5) applied to the equations of the dual system. The detailed proof is not provided for conciseness purpose. ■

**Remark 1** Lemma 2, 3 and 4 assume implicitly that the rationally dependent system (1) is well-posed in the sense that all matrices are bounded when  $\theta \in \Theta$ . This assumption is equivalent to stating that the  $E_\pi(\theta)$  and  $E_{d\pi}(\theta)$  are invertible for all uncertainties  $\theta \in \Theta$ . It also implies that the descriptor representations have no impulsive modes. All vectors are uniquely defined and finite for given finite states and perturbations. Stability of the descriptor systems is hence in the usual sense of the boundedness and convergence of the state.

## 3 LMIs for robust synthesis and analysis

### 3.1 Robust $H_\infty$ state-feedback design

**Theorem 1** If there exist  $\bar{v}$  symmetric positive definite matrices  $P_d^{[v]} \succ 0$  and matrices  $S_{dx}$ ,  $S_{dy}$ ,  $S_{d\pi}$  of appropriate dimensions such that the following LMIs hold simultaneously for all  $v \in \mathcal{I}$

$$\begin{aligned} & \text{diag} \left( P_d^{[v]}, I, 0, -P_d^{[v]}, -\mu_d^2 I \right) \\ & \prec \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} S_{dx} & 0 & S_{dx}E_{dx}(\theta^{[v]}) + S_{dy}E_{dy}(\theta^{[v]}) & 0 & 0 \end{bmatrix} \right\}^S \\ & + \left\{ S_{d\pi} \begin{bmatrix} 0 & I & E_{dz}(\theta^{[v]}) & 0 & 0 \\ 0 & 0 & E_{d\pi}(\theta^{[v]}) & A_d(\theta^{[v]}) & B_{dw}(\theta^{[v]}) \end{bmatrix} \right\}^S, \end{aligned} \quad (14)$$

then  $K = S_{dy}^T (S_{dx}^T)^{-1}$  is a robustly stabilizing state-feedback gain that guarantees that the closed-loop with  $u_k = Kx_k$  has an  $H_\infty$  performance smaller than  $\mu_d$  whatever  $\theta \in \Theta$ .

*Proof:* First we shall use the fact that the matrices are multi-affine in the parameters. Take the multi-sum of all inequalities (14) with positive weights  $\xi_{p,v_p} \geq 0$ . For all  $\theta^{[v]}$ -dependent terms the result gives any generic uncertain matrix as

$$\sum_{v \in \mathcal{I}} \xi_{1,v_1} \cdots \xi_{\bar{p},v_{\bar{p}}} A_d(\theta^{[v]}) = A_d(\theta).$$

The weighted multi-sum of constant parameters has no effect on the constant parameters when the  $\xi_p$  vectors are in the unitary simplexes

$$\sum_{v \in \mathcal{I}} \xi_{1,v_1} \cdots \xi_{\bar{p},v_{\bar{p}}} S_x = S_x.$$

Finally we shall denote  $P_d(\theta)$  the multi-affine matrix uniquely defined for a given unitary simplex definition of the parameters:

$$\sum_{v \in \mathcal{I}} \xi_{1,v_1} \cdots \xi_{\bar{p},v_{\bar{p}}} P_d^{[v]} = P_d(\theta).$$

Thanks to convexity of matrix inequalities and multi-affine dependence of matrices, the  $\bar{v}$  LMIs (14) hold if and only if for all uncertainties  $\theta \in \Theta$  one has

$$\begin{aligned} & \text{diag} ( P_d(\theta), I, 0, -P_d(\theta), -\mu_d^2 I ) \\ & \prec \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} \left[ \begin{array}{cccccc} S_{dx} & 0 & S_{dx} E_{dx}(\theta) + S_{dy} E_{dy}(\theta) & 0 & 0 & \end{array} \right] \right\}^S \\ & + \left\{ S_{d\pi} \begin{bmatrix} 0 & I & E_{dz}(\theta) & 0 & 0 \\ 0 & 0 & E_{d\pi}(\theta) & A_d(\theta) & B_{dw}(\theta) \end{bmatrix} \right\}^S. \end{aligned}$$

With the change of variable  $S_{dy} = S_{dx} K^T$  these inequalities also read as

$$\text{diag} ( P_d(\theta), I, 0, -P_d(\theta), -\mu_d^2 I ) \prec \{ S_d E_d(\theta) \}^S$$

where

$$S_d = \left[ \begin{array}{c|c} S_{dx} & \\ \hline 0 & S_{d\pi} \end{array} \right] \quad (15)$$

and  $E_d(\theta)$  is the matrix defining the descriptor multi-affine system of Lemma 2. After congruence operation of  $\eta_{d,k} \neq 0$  on this last matrix inequality, one gets along the trajectories ( $E_d(\theta)\eta_{d,k} = 0$ ):

$$\begin{aligned} & x_{d,k+1}^T P_d(\theta) x_{d,k+1} - x_{d,k}^T P_d(\theta) x_{d,k} \\ & + z_{d,k}^T z_{d,k} - \mu_d^2 w_{d,k}^T w_{d,k} < 0. \end{aligned}$$

In case of zero disturbances ( $w_{d,k} = 0$ ) this inequality implies that the quadratic parameter-dependent Lyapunov function  $x_d^T P_d(\theta) x_d$  is decreasing. The system is

asymptotically stable.

In case of zero initial conditions ( $x_{d,0} = 0$ ), and taking the sum from  $k = 0$  to  $k = \bar{k}$ , one gets

$$\|z_d\|_{2,\bar{k}}^2 \leq \mu_d^2 \|w_d\|_{2,\bar{k}}^2 - x_{d,\bar{k}+1}^T P_d(\theta) x_{d,\bar{k}+1}$$

which, since  $P_d(\theta)$  is positive definite, implies

$$\|z_d\|_{2,\bar{k}}^2 \leq \mu_d^2 \|w_d\|_{2,\bar{k}}^2.$$

For  $\bar{k}$  growing to infinity one concludes that  $\|z_d\|_2 \leq \mu_d \|w_d\|_2$ . The  $L_2$  induced norm of the system is smaller than  $\mu_d$ . Since we are dealing with LTI systems it is equivalent to stating that the  $H_\infty$  performance is smaller than  $\mu_d$ . Both stability and performance are ensured for all uncertainties  $\theta \in \Theta$  thus proving both robust stability and robust performance. ■

The proof indicates that the LMI condition implies stability and performance of the state-feedback control. The result is conservative. The sources of conservatism are twofold. The first one comes from restricting the S-variable  $S_d$  to being parameter-independent. The second comes from its structuring in an upper-triangular form (15). To understand further these conservatism sources, the reader is suggested to see [13].

### 3.2 Analysis of a given state-feedback gain

The second source of conservatism mentioned above is needed for rendering convex the state-feedback design, but it can be removed for analysis purpose. This feature is used in the next analysis result. To emphasize that the state-feedback gain  $K$  is from now on considered as fixed, it is no more written in blue color.

**Theorem 2** *If there exist  $\bar{v}$  symmetric positive definite matrices  $P^{[v]} \succ 0$  and matrices  $Q$  and  $S$  of appropriate dimensions such that the following LMIs hold simultaneously for all  $v \in \mathcal{I}$*

$$\text{diag} ( P^{[v]}, 0, Q - P^{[v]}, -I ) \prec \left\{ S \begin{bmatrix} I & E_x(\theta^{[v]}) & 0 & 0 \\ 0 & E_\pi(\theta^{[v]}) & A(\theta^{[v]}) + B(\theta^{[v]})K & B(\theta^{[v]}) \end{bmatrix} \right\}^S, \quad (16)$$

*then the system with state-feedback  $u_k = Kx_k + \epsilon_k$  is robustly stable and the state of the plant  $x_k$  is bounded for bounded errors on the control signal  $\epsilon_k$  by  $\|Wx\|_2 \leq \|\epsilon\|_2$  where  $W = Q^{1/2}$ .*

*Proof:* For a start note that the plant in closed-loop with  $u_k = Kx_k + \epsilon_k$  and for zero input perturbations  $w_k = 0$  reads as

$$x_{k+1} = (A_r(\theta) + B_r(\theta)K) x_k + B_r(\theta)\epsilon_k.$$

Using the DMAR (4) and introducing a  $\pi_k$  exogenous vector as in proof of Lemma 3, the system equivalently reads as

$$\begin{bmatrix} I & E_x(\theta) & 0 & 0 \\ 0 & E_\pi(\theta) & A(\theta) + B(\theta)K & B(\theta) \end{bmatrix} \eta_k = 0$$

$$\eta_k^T = ( x_{k+1}^T \quad \pi_k^T \quad x_k^T \quad \epsilon_k^T ).$$

Now, thanks to convexity of matrix inequalities and multi-affine dependence of the matrices, the  $\bar{v}$  LMIs (16) hold if and only if for all uncertainties  $\theta \in \Theta$  one has

$$\text{diag} ( P(\theta), 0, Q - P(\theta), -I ) \prec \left\{ S \begin{bmatrix} I & E_x(\theta) & 0 & 0 \\ 0 & E_\pi(\theta) & A(\theta) + B(\theta)K & B(\theta) \end{bmatrix} \right\}^S,$$

where  $P(\theta)$  is defined the same way as  $P_d(\theta)$  in the proof of Theorem 1. After congruence operation of  $\eta_k \neq 0$  on this last inequality, one gets

$$x_{k+1}^T P(\theta) x_{k+1} - x_k^T P(\theta) x_k + x_k^T Q x_k - \epsilon_k^T \epsilon_k < 0.$$

In case of zero error on the control signal  $\epsilon_k = 0$  the inequality proves stability thanks to the decreasing parameter-dependent Lyapunov function  $x^T P(\theta)x$ . In case of zero initial conditions  $x_0 = 0$ , and taking the sum over all positive  $k$ , one gets  $\|Wx\|_2 \leq \|\epsilon\|_2$  where  $W = Q^{1/2}$ . ■

In the conference version of this manuscript [29] that addressed the sub-case of polytopic systems, it is shown that feasibility of LMIs from the theorem corresponding to Theorem 1, implies feasibility of LMIs from the theorem corresponding to Theorem 2. We are unfortunately not able to prove such implication in the present more general case with rationally dependent uncertain systems.

### 3.3 Robust observer design

All elements are now available for the design of a robust observer for the system. The goal of this observer is to make possible the usage of the already designed state-feedback in case the states are not measured. The objective is hence not to reconstruct exactly the states (which is an ill posed problem for systems with uncertainties) but to build some dynamic filter that provides an estimate  $\hat{x}_k$  that minimizes the error  $\epsilon_k$  between the actual control  $K\hat{x}_k$  and the ideal state-feedback control  $Kx_k$ . The observer design is hence state-feedback gain  $K$  dependent. It also uses the results from the analysis phase which provides an indication on the possible trajectories of the state  $x_k$ . At this stage the state-feedback gain  $K$  obtained from Theorem 1 and the  $Q$  matrix obtained from the analysis phase (Theorem 2) are assumed fixed. To emphasize this feature, these matrices are no more written in blue color.

Let the following notation

$$\begin{aligned} \hat{S}_a &= S_a + S_b K + S_l C, \\ N_x(\theta, S_x, S_a, S_b, S_l) &= \begin{bmatrix} S_x & S_x E_x(\theta) & -\hat{S}_a & S_a + S_b K \end{bmatrix}, \\ N_\pi(\theta) &= \begin{bmatrix} 0 & E_\pi(\theta) & -B(\theta)K & A(\theta) + B(\theta)K \end{bmatrix} \end{aligned}$$

which are affine in the blue decision variables and multi-affine in the uncertainties  $\theta$ .

**Theorem 3** *If there exist  $2\bar{v}$  symmetric positive definite matrices  $P_\infty^{[v]} \succ 0$ ,  $P_p^{[v]} \succeq K^T K$  and matrices  $S_x, S_a, S_b, S_l, S_{2\pi}, S_{p\pi}$  of appropriate dimensions such that the*

following LMIs hold simultaneously for all  $v \in \mathcal{I}$

$$\begin{aligned} & \text{diag} \left( P_2^{[v]}, 0, K^T K - P_2^{[v]}, -\gamma_2^2 Q \right) \\ & \prec \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} N_x(\theta^{[v]}, S_x, S_a, S_b, S_l) \right\}^S + \{S_{2\pi} N_\pi(\theta^{[v]})\}^S, \end{aligned} \quad (17)$$

$$\begin{aligned} & \text{diag} \left( P_p^{[v]}, 0, -P_p^{[v]}, -\gamma_p^2 Q \right) \\ & \prec \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} N_x(\theta^{[v]}, S_x, S_a, S_b, S_l) \right\}^S + \{S_{p\pi} N_\pi(\theta^{[v]})\}^S, \end{aligned} \quad (18)$$

then  $A_o = S_x^{-1} S_a$ ,  $B_o = S_x^{-1} S_b$ ,  $L = S_x^{-1} S_l$  define an observer (10) that guarantees the following two norm-to-norm and norm-to-peak properties:

$$\|\epsilon\|_2 \leq \gamma_2 \|Wx\|_2, \quad \|\epsilon\|_p \leq \gamma_p \|Wx\|_2$$

where  $\epsilon_k = Ke_k$ . The properties hold whatever bounded  $x$  and whatever uncertainty  $\theta \in \Theta$ .

*Proof:* The first step, as in the two previous proofs, is based on convexity of the matrix inequalities and multi-affine nature of the parameter-dependent matrices. The LMIs (17) and (18) are feasible if and only if the following inequalities hold for all uncertainties  $\theta \in \Theta$ :

$$\begin{aligned} & \text{diag} \left( P_2(\theta), 0, K^T K - P_2(\theta), -\gamma_2^2 Q \right) \\ & \prec \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} N_x(\theta, S_x, S_a, S_b, S_l) \right\}^S + \{S_{2\pi} N_\pi(\theta)\}^S, \\ & \text{diag} \left( P_p(\theta), 0, -P_p(\theta), -\gamma_p^2 Q \right) \\ & \prec \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} N_x(\theta, S_x, S_a, S_b, S_l) \right\}^S + \{S_{p\pi} N_\pi(\theta)\}^S \end{aligned}$$

where the  $P_2(\theta)$  and  $P_p(\theta)$  matrices have the same multi-affine definition as  $P_d(\theta)$  in the proof of Theorem 1. With the change of variable  $S_a = S_x A_o$ ,  $S_b = S_x B_o$ ,  $S_l = S_x L$  these inequalities also read as

$$\begin{aligned} & \text{diag} \left( P_2(\theta), 0, K^T K - P_2(\theta), -\gamma_2^2 Q \right) \prec \{S_2 E_o(\theta)\}^S \\ & \text{diag} \left( P_p(\theta), 0, -P_p(\theta), -\gamma_p^2 Q \right) \prec \{S_p E_o(\theta)\}^S \end{aligned}$$

where  $S_2 = \begin{bmatrix} S_x & | & S_{2\pi} \\ 0 & & \end{bmatrix}$ ,  $S_p = \begin{bmatrix} S_x & | & S_{p\pi} \\ 0 & & \end{bmatrix}$  and  $E_o(\theta)$  is the matrix defined in (9). After congruence operation of  $\eta_{o,k} \neq 0$  on these two inequalities one gets that the observation error satisfies both following properties

$$\begin{aligned} & e_{k+1}^T P_2(\theta) e_{k+1} - e_k^T P_2(\theta) e_k + \epsilon_k^T \epsilon_k - \gamma_2^2 x_k^T Q x_k < 0, \\ & e_{k+1}^T P_p(\theta) e_{k+1} - e_k^T P_p(\theta) e_k - \gamma_p^2 x_k^T Q x_k < 0. \end{aligned}$$

From the first inequality, one directly conclude that the observation error  $e_k$  is internally asymptotically stable (proved by the Lyapunov function  $x^T P_2(\theta)x$ ). Moreover, taking the sum over all positive  $k$ , the first inequality gives  $\|\epsilon\|_2 \leq \gamma_2 \|Wx\|_2$  for all zero initial conditions. For zero initial conditions as well, taking the sum from  $k = 0$  to  $k = \bar{k} - 1$  of the second inequality gives

$$e_{\bar{k}}^T P_p(\theta) e_{\bar{k}} \leq \gamma_p^2 \|Wx\|_{2, \bar{k}-1}^2 \leq \gamma_p^2 \|Wx\|_2^2.$$

Recall that the  $P_p^{[v]}$  matrices are constrained as  $P_p^{[v]} \succeq K^T K$ . By convexity and multi-affine property of  $P_p(\theta)$ , the condition implies that for all  $\theta \in \Theta$  one has  $P_p(\theta) \succeq K^T K$  and hence  $\epsilon_{\bar{k}}^T \epsilon_{\bar{k}} \leq e_{\bar{k}}^T P_p(\theta) e_{\bar{k}} \leq \gamma_p^2 \|Wx\|_2^2$ . Since the property hold for all  $\bar{k}$  one concludes that it also holds for the peak value, hence  $\|\epsilon\|_p \leq \gamma_p \|Wx\|_2$ . ■

### 3.4 Analysis of the observed-state feedback loop

At this stage we have proposed LMI results that allow to design separately some state-feedback gain and some Luemberger like observer. Since separation principle does not hold for uncertain systems, there is not guarantee yet that the closed-loop system combining the two is stable. The goal of this section is to analyse the closed-loop. From this point we assume the state-feedback gain  $K$  provided by Theorem 1, the  $Q$  matrix obtained from Theorem 2, and the observer matrices  $A_o, B_o, L$  obtained from Theorem 3, are all fixed. To emphasize this feature, these matrices are no more written in blue color.

Based on the upper computed value  $\gamma_2$ , one can directly conclude about stability using the small-gain theorem:

**Theorem 4** *If  $\gamma_2 < 1$ , the closed-loop composed of (1) and*

$$\hat{x}_{k+1} = (A_o + B_o K + LC)\hat{x}_k - Ly_k, \quad u_k = K\hat{x}_k \quad (19)$$

*is robustly stable for all  $\theta \in \Theta$ .*

*Proof:* Introducing again the error signal  $e_k = x_k - \hat{x}_k$  the closed-loop system writes as the feedback interconnection of

$$x_{k+1} = (A_r(\theta) + B_r(\theta)K)x_k - B_r(\theta)\epsilon_k, \quad \tilde{x}_k = Wx_k$$

with the error dynamics

$$e_{k+1} = (A_o + (B_o - B_r(\theta))K + LC)e_k + (A_r(\theta) - A_o + (B_r(\theta) - B_o)K)W^{-1}\tilde{x}_k, \quad \epsilon_k = Ke_k.$$

Theorem 2 guarantees that the  $l_2$ -induced norm of the first system with  $\epsilon_k$  as inputs and  $\tilde{x}_k$  as outputs, is less than 1. Meanwhile, Theorem 3 guarantees the  $l_2$ -induced norm of the second with  $\tilde{x}_k$  as inputs and  $\epsilon_k$  as outputs, is less than  $\gamma_2$ . By small-gain theorem the closed-loop is hence stable if  $\gamma_2 < 1$ . Since the upper bounds are valid for all uncertainties  $\theta \in \Theta$ , stability is robust. ■

An other test, is to perform a closed-loop LMI-based analysis of the system with observed-state feedback control.

**Theorem 5** *If there exist  $\bar{v}$  symmetric positive definite matrices  $P_c^{[v]} \succ 0$  and a matrix  $S_c$  of appropriate dimensions such that the following LMIs hold simultaneously for all  $v \in \mathcal{I}$*

$$\text{diag} \left( P_c^{[v]}, I, 0, -P_c^{[v]}, -\mu_c^2 I \right) \prec \{S_c E_c(\theta^{[v]})\}^S, \quad (20)$$

*then the system (1) in feedback loop with (19) is robustly stable and its  $H_\infty$  performance is smaller than  $\mu_c$  whatever  $\theta \in \Theta$ .*

*Proof:* The proof follows the same lines as the previous ones, this time based on the descriptor multi-affine model (12). It is not provided for conciseness purpose. ■

An alternative version of Theorem 5 is also possible based on the dual representation (13) given in Lemma 4.

**Theorem 6** *If there exist  $\bar{v}$  symmetric positive definite matrices  $P_{dc}^{[v]} \succ 0$  and a matrix  $S_{dc}$  of appropriate dimensions such that the following LMIs hold simultaneously for all  $v \in \mathcal{I}$*

$$\text{diag} \left( P_{dc}^{[v]}, I, 0, -P_{dc}^{[v]}, -\mu_{dc}^2 I \right) \prec \{S_{dc} E_{dc}(\theta^{[v]})\}^S, \quad (21)$$

*then the system (1) in feedback loop with (19) is robustly stable and its  $H_\infty$  performance is smaller than  $\mu_{dc}$  whatever  $\theta \in \Theta$ .*

The two analysis conditions are not equivalent (see [13]). Both give upper bounds, but there is no a priori order between  $\mu_c$  and  $\mu_{dc}$ .

## 4 Heuristic design of observed-state feedback

The previous section exposes results that allow to elaborate a heuristic for robust observed-state feedback design. Most of this heuristic algorithm is implicitly described along the exposure of the LMI results. It is now summarized.

### *Step 1 - State-feedback gain design*

Assume the goal is the design of an output feedback dynamic controller that ensures robust stability and some robust input/output  $H_\infty$  performance level  $\mu_{obj}$ . The first step is to design a state-feedback gain  $K$  using Theorem 1 for some fixed  $\mu_d < \mu_{obj}$ . Choosing  $\mu_d$  smaller than  $\mu_{obj}$  lies in the fact that the observed-state feedback that will follow will inevitably degrade the performance. Hence the ideal state-feedback should do better than the goal  $\mu_{obj}$ . One can also solve the LMIs of Theorem 1 while minimizing  $\mu_d^2$ . Note that it is also possible to do multi-objective state-feedback design combining pole location constraints as well as  $H_\infty$ ,  $H_2$  and impulse-to-peak performances. See for example [13] for methods to build LMI conditions for these other performances.

### *Step 2 - Analysis of trajectories in case of state-feedback*

Whatever performances obtained for the ideal state-feedback, the goal is now to build a robust observer that keeps the dynamics as similar as possible. To realize this,



the proposed methodology consists in first having an estimate of what type of state-trajectories are expected when applying state-feedback. This is done by searching for  $W$  matrices using LMIs of Theorem 2.

It should be noticed here that the LMIs of Theorem 2 may not be feasible, even if the state-feedback is proved to be a stabilizing one since issued from Theorem 1. This is because all these conditions are conservative and there is at our knowledge no way to prove that in general feasibility of LMIs in Theorem 1 implies feasibility of the LMIs in Theorem 2. Yet, in the special case of polytopic systems (see [29]) this property holds. But we were not able to prove this fact for the more general case of rationally dependent systems treated here. If LMIs of Theorem 2 are found unfeasible, then to proceed one needs to build less conservative analysis results. Such results can be obtained using the augmentation technique described in [13].

Assume two matrices  $W_1$  and  $W_2$  are solutions of the LMIs (16) and satisfy  $Q_1 = W_1^2 \prec Q_2 = W_2^2$ . Then for the same  $\epsilon$  the following inequalities hold  $\|W_1 x\|_2 \leq \|W_2 x\|_2 \leq \|\epsilon\|_2$ . It is clear from these that the matrix  $W_2$  provides a tighter information in terms of the effect of  $\epsilon$  on the state  $x$ . To characterize worst case effects of control errors  $\epsilon$  on the state trajectories it is hence natural to “maximize”  $W$ . One way to do so is to maximize the trace of  $Q$  or to maximize its smallest eigenvalue when solving the LMIs. In the examples we shall maximize a linear combination of the trace and the smallest eigenvalue.

#### *Step 3 - Design of an observer for state-feedback fitting*

In case of systems without uncertainties a natural (and optimal) choice of  $A_o$  and  $B_o$  is such that the last term in (11) is zero. That is, duplicating the system  $A_r$  and  $B_r$  matrices of the system, to the observer. In such case, the error is asymptotically stable if  $A_o + LC$  has all Schur stable eigenvalues. This property is part of the separation principle that makes possible the separate design of state-feedback and observer gains. In case of systems with uncertainties, this choice is no more possible. The separation principle does not hold. Therefore, the dynamics of the error are perturbed by some (hopefully small) perturbation dependent on the trajectories of the observed plant. Because of that, the actual control signal  $K\hat{x}_k$  differs from the ideal state-feedback  $Kx_k$ . The error between the two is  $\epsilon_k = Ke_k$  and should be kept small. The heuristic design of observer matrices  $A_o$ ,  $B_o$  and  $L$  we propose is hence to ensure that the error  $e_k$  is stable and the transfer from  $x_k$  to  $\epsilon_k$  is as small as possible.

The issue is to adopt some norm to measure  $\epsilon$ . One classical norm would be the  $l_2$  norm  $\|\epsilon\|_2$ . Unfortunately, such norm that measures the total energy over all time samples could be small but with high peak values. Typically it can give very fast converging observers, but that generate large, irrelevant, spikes on  $\epsilon$  at the first time instants  $k = 1, 2, 3$  etc. See [20] for discussions about this inevitable waterbed effect. To handle such phenomena the observer design of Theorem 3 allows to minimize a compromise between the  $l_2$  norm  $\|\epsilon\|_2$  and the peak  $\|\epsilon\|_p$ . In practice it can be done by minimizing the weighted sum  $\beta_2 \gamma_2^2 + \beta_p \gamma_p^2$  for a priori chosen values of  $\beta_2$  and  $\beta_p$ .

#### *Step 4 - Analysis of obtained observed-state feedback*

If  $\gamma_2 < 1$  Theorem 4 ensures that the closed loop with observed-state feedback robustly stabilizes the plant. To know more about the closed-loop one can also apply Theorems 5 and 6. The LMIs can be solved while minimizing  $\mu_c^2$  and  $\mu_{dc}^2$  respectively. The optimal values are upper bounds on the robust  $H_\infty$  performance of the observed-

state feedback loops. They can be different one from the other (see [13] for discussions about this fact). Both are expected to be greater than  $\mu_d$  in the sense that observed-state feedback will have worse performances than the pure state-feedback based on which the observer was designed. If either one of the guaranteed costs satisfies  $\mu_c \leq \mu_{obj}$  or  $\mu_{dc} \leq \mu_{obj}$ , the design goal is achieved.

The 4 steps define a heuristic algorithm. The methodology is not guaranteed to attain the design goal, even if such solution exists. To our knowledge there is no convex design result for robust output feedback design. The proposed methodology is one possible heuristic. It has the advantage to be based on classical state-feedback with observer. Moreover, each of the steps is useful by itself. Step 1 gives a new methodology for state-feedback design in presence of rationally uncertainty-dependent systems. Step 2 gives a method to evaluate bounds on state trajectories in presence of control errors. Step 3 solves the robust observer design problem for which we could not find any comparable result in the literature. Step 4 can be seen as a natural extension of analysis results from [13] for the descriptor multi-affine representations introduced by the present paper.

## 5 System dependent variations on the results

### 5.1 Reduced size LMIs, removing rows

When comparing results in the conference version of this work (see [29]) and the results exposed here, one can notice that the former results involve no  $S_\pi$  like variable, while here this S-variable is most necessary to derive the results in the case of rationally dependent uncertain systems. A question that could arise is whether this additional S-variable brings improvements to the polytopic case as well. The answer is no, and this is demonstrated by Lemma 3.1 in [13] (see also [30]).

Let us illustrate this statement on the LMIs (16). In case of systems that are multi-affine in the uncertainties (such as the polytopic systems), then the DMAR can be chosen such that  $E_x(\theta) = E_\pi(\theta) = I$ . For this choice, the data-dependent matrix that multiplies the S-variable in (16) has all its first rows that are parameter-independent. Lemma 3.1 from [13] can be applied and gives the following equivalent LMIs

$$\text{diag} ( P^{[v]}, \quad Q - P^{[v]}, \quad -I ) \\ \prec \left\{ \tilde{S} \begin{bmatrix} I & -(A(\theta^{[v]}) + B(\theta^{[v]})K) & B(\theta^{[v]}) \end{bmatrix} \right\}^S.$$

It is exactly the one that can be found in [29]. This LMI is trivially preferable since it is of reduced size and contains less decision variables (the  $\tilde{S}$  matrix has smaller dimensions than  $S$ ).

Note that the procedure described here applies in a similar way whatever parameter-independent rows in the matrix

$$\begin{bmatrix} I & E_x(\theta^{[v]}) & 0 & 0 \\ 0 & E_\pi(\theta^{[v]}) & A(\theta^{[v]}) + B(\theta^{[v]})K & B(\theta^{[v]}) \end{bmatrix}.$$

All the LMI conditions of the present manuscript admit reduced size equivalent formulations as soon as some rows in the matrices that multiply the S-variables are independent of uncertainties and of decision variables. This is an easy to check property and can be implemented readily in software tools.

## 5.2 Reduced size LMIs, removing columns

In [13] and [30] one can also find a procedure to build LMIs of reduced size when some properties hold in terms of parameter-independent columns of that same matrices (Lemma 3.2 in [13]). An example of application of this other procedure is given below. The procedure can unfortunately be conservative. The solution to the reduced size LMI result of the following Theorem 7 is such that if the LMIs are feasible then the LMIs of Theorem 1 are feasible as well for  $\mu_d = \tilde{\mu}_d$ . The converse is not guaranteed.

**Theorem 7** *If there exist  $\bar{v}$  symmetric positive definite matrices  $P_d^{[v]} \succ 0$  and matrices  $S_{dx}, S_{dy}, S_{d\pi}$  of appropriate dimensions such that the following LMIs hold simultaneously for all  $v \in \mathcal{I}$*

$$\begin{aligned} & \text{diag} \left( P_d^{[v]}, E_{dz}^T(\theta^{[v]})E_{dz}(\theta^{[v]}), -P_d^{[v]}, -\tilde{\mu}_d^2 I \right) \\ & \prec \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} S_{dx} & S_{dx}E_{dx}(\theta^{[v]}) + S_{dy}E_{dy}(\theta^{[v]}) & 0 & 0 \end{bmatrix} \right\}^{\mathcal{S}} \\ & + \left\{ S_{d\pi} \begin{bmatrix} 0 & E_{d\pi}(\theta^{[v]}) & A_d(\theta^{[v]}) & B_{dw}(\theta^{[v]}) \end{bmatrix} \right\}^{\mathcal{S}}, \end{aligned} \quad (22)$$

then  $K = S_{dy}^T (S_{dx}^T)^{-1}$  is a robustly stabilizing state-feedback gain that guarantees that the closed-loop with  $u_k = Kx_k$  has an  $H_\infty$  performance smaller than  $\tilde{\mu}_d$  whatever  $\theta \in \Theta$ .

## 5.3 The time-varying uncertainty case

The case when some parameters are time-varying can be treated in the same framework as the one exposed in the paper using a methodology inspired from [7, 8] and that is also described in [30]. To illustrate how all LMI results exposed in the present paper can extend to the case of time-varying uncertainties, we shall introduce some additional notations and concentrate on the LMI result of Theorem 2. The others follow in the same manner.

Let us assume that in the set of parameters described in (2) some are constant and others are time-varying. The ones that are constant are with indexes  $p \in \mathcal{P}_c$  and the time-varying ones are such that  $p \in \mathcal{P}_{tv}$ . The two sets are of course disjoint and their union is  $\{1 \dots \bar{p}\}$ . The number of vertices of the multi-affine representations for each constant and time-varying parameters separately are  $\bar{v}_c = \prod_{p \in \mathcal{P}_c} \bar{v}_p$  and  $\bar{v}_{tv} = \prod_{p \in \mathcal{P}_{tv}} \bar{v}_p$  respectively. For an element  $v \in \mathcal{I}$  we shall denote  $v^+ \in \mathcal{I}$  any other vector of indices of vertices such that the indices of  $v^+$  and  $v$  coincide for the constant uncertainties, i.e.  $v_p^+ = v_p, \forall p \in \mathcal{P}_c$ . The indices  $v_{p \in \mathcal{P}_{tv}}^+$  are independent of  $v_{p \in \mathcal{P}_{tv}}$  and represent the possibility of any time-varying uncertainty to evolve to any other admissible value between the current sample of time and the next one. Finally we shall

denote  $w_c = (v_{p \in \mathcal{P}_c})$  and  $w_{tv}^+ = (v_{p \in \mathcal{P}_{tv}}^+)$  the vectors of indices related only to the constant and the next step time-varying uncertainties respectively.

With these notations two extensions of Theorem 2 are possible for the time-varying case. The first one can be understood as related to the quadratic stability concept of [4] where a common Lyapunov matrix is used for all uncertainties. Here it amounts to have the  $P(\theta)$  matrix independent of time-varying uncertainties.

**Theorem 8** *If there exist  $\bar{v}_c$  symmetric positive definite matrices  $P^{[w_c]} \succ 0$ , and two matrices  $Q$  and  $S$  of appropriate dimensions such that the following LMIs hold simultaneously for all  $v \in \mathcal{I}$*

$$\begin{aligned} & \text{diag} \left( P^{[w_c]}, 0, Q - P^{[w_c]}, -I \right) \\ & \prec \left\{ S \begin{bmatrix} I & E_x(\theta^{[v]}) & 0 & 0 \\ 0 & E_\pi(\theta^{[v]}) & A(\theta^{[v]}) + B(\theta^{[v]})K & B(\theta^{[v]}) \end{bmatrix} \right\}^S, \end{aligned} \quad (23)$$

then the system with state-feedback  $u_k = Kx_k + \epsilon_k$  is robustly stable and the state of the plant  $x_k$  is bounded for bounded errors on the control signal  $\epsilon_k$  by  $\|Wx\|_2 \leq \|\epsilon\|_2$  where  $W = Q^{1/2}$ .

The other one relies on the fact that for discrete-time systems both the S-variable and the Lyapunov matrix can depend on the time-varying uncertainties as long as one looks at all possible jumps from one value of the uncertainty to any other during each time sample. The result is as follows and the proof follows exactly the ones propped in [30].

**Theorem 9** *If there exist  $\bar{v}$  symmetric positive definite matrices  $P^{[v]} \succ 0$ , a matrix  $Q$  and  $\bar{v}_{tv}$  matrices  $S^{[w_{tv}^+]}$  of appropriate dimensions such that the following LMIs hold simultaneously for all  $v \in \mathcal{I}, v^+ \in \mathcal{I}$*

$$\begin{aligned} & \text{diag} \left( P^{[v^+]}, 0, Q - P^{[v]}, -I \right) \\ & \prec \left\{ S^{[w_{tv}^+]} \begin{bmatrix} I & E_x(\theta^{[v]}) & 0 & 0 \\ 0 & E_\pi(\theta^{[v]}) & A(\theta^{[v]}) + B(\theta^{[v]})K & B(\theta^{[v]}) \end{bmatrix} \right\}^S \end{aligned} \quad (24)$$

then the system with state-feedback  $u_k = Kx_k + \epsilon_k$  is robustly stable and the state of the plant  $x_k$  is bounded for bounded errors on the control signal  $\epsilon_k$  by  $\|Wx\|_2 \leq \|\epsilon\|_2$  where  $W = Q^{1/2}$ .

In all cases Theorem 8 is more conservative than Theorem 9 but with much less decision variables and less LMI constraints ( $\bar{v}$  LMIs in the first, while there are  $\bar{v}\bar{v}_{tv}$  LMIs in the second). For both these results, the methodology of Lemma 3.2 in [13] can often be applied to reduce the size of the LMIs, see [30] for the details. When all uncertainties are time-varying the size reduction procedure can provide LMIs with no S-variables as in classical results such as [4].

## 6 Numerical example

For illustration purpose we consider the toy example with the following dynamics

$$x_{k+1} = \begin{bmatrix} -\theta_1^2/\theta_2 & -\theta_1 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix} u_k + \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix} w_k$$

and with  $z_k = [1 \ 0] x_k + \theta_2 u_k$ ,  $y_k = [0 \ 1] x_k$ . The DMAR we shall use is given by:

$$E_x = \begin{bmatrix} \theta_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_z = [0 \ 1], \quad E_\pi = \begin{bmatrix} \theta_2 & 0 \\ 0 & 1 \end{bmatrix}, \\ A = \begin{bmatrix} -\theta_1 & -\theta_2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix}, \quad B_w = \begin{bmatrix} \theta_2 \\ 0 \end{bmatrix}.$$

The uncertain parameters are in intervals around the nominal value 1 with discrepancies  $\delta_1, \delta_2$ : *i.e.*  $\theta_1 \in [1 - \delta_1, 1 + \delta_1]$  and  $\theta_2 \in [1 - \delta_2, 1 + \delta_2]$ . The case without uncertainties is when  $(\delta_1, \delta_2) = (0, 0)$ . Note that for the nominal values  $\theta_1 = \theta_2 = 1$ , the poles of the system are on the unit disc. The nominal plant is not asymptotically stable and the plant is unstable for some values of the uncertainties. The observed-state feedback design methodology is applied with  $\mu_d = 10$  at the first step for results in Table 1 and with minimization of  $\mu_d$  for results in Table 2. At the second step the following convex criterion is maximized:  $Tr(Q) + 10 \cdot \lambda_{\min}(Q)$ . Table 3 indicates the dimensions of the LMI problems for all tested cases (depends on the number of uncertain parameters). These dimensions are those after applying the parameter independent rows reduction method. All LMIs were built using the YALMIP parser [23] and solved with SDPT3 solver [38].

The case when  $\delta = 0$  allows to see that the procedure for removing decision variables based on parameter-independent rows allows to simplify significantly the numerical burden and in particular to remove all S-variables. In this special case the values  $\gamma_2$  and  $\gamma_p$  are negligible since the optimization finds a solution such that the  $x_k$  dependent term in (11) is zero. Separation principle is confirmed.

The expected difference between Theorem 1 and Theorem 7 is as expected in terms of smaller dimensions for the second. This reduction is at the expense of possibly higher conservatism. This fact occurs only once (see third row of Table 2 where  $\mu_d < \tilde{\mu}_d$ ), yet is rather negligible. Larger gap could occur for larger size systems.

Although there is no guarantee that Theorem 5 and Theorem 6 would provide the same values, it happens to be the case for almost all tests, except for the case with maximal discrepancy (last row of Table 1). Dimensions of the analysis conditions built based on the primal or dual representations are different depending on the structure of the data.

Comparing Tables 1 and 2 there is no clear answer on how should the value  $\mu_d$  be chosen at the first step. Optimizing it does not lead to better closed-loop performances in the end.

In Table 1 tests are reported with different weights on the optimization at the observer design step. As the weight  $\beta_2$  increases one naturally gets smaller values for the  $\gamma_2$  gain. Since this value (when smaller than one) is a guarantee of closed-loop

Table 1: Results for different discrepancies and optimization settings when  $\mu_d = 10$  at first step

$(\delta_1, \delta_2)$	$(\beta_2, \beta_p)$	$(\gamma_2, \gamma_p)$	$\mu_c$	$\mu_{dc}$
(0, 0)	(1, 1)	$(10^{-4}, 10^{-4})$	2.8152	2.8152
(0.1, 0)	(1, 1)	(1.0747, 1.0410)	2.6672	2.6672
(0, 0.1)	(1, 1)	(0.3947, 0.3524)	2.4546	2.4546
(0.1, 0.1)	(10, 1)	(1.2736, 1.1765)	6.5901	6.5901
(0.1, 0.1)	(1, 1)	(1.2962, 1.0985)	6.1252	6.1252
(0.1, 0.1)	(1, 10)	(1.3339, 1.0809)	5.3226	5.3226
(0.2, 0.1)	(1, 1)	(1.3006, 1.2181)	11.2285	11.2285
(0.1, 0.2)	(1, 1)	(1.3242, 1.1553)	6.8505	6.8505
(0.2, 0.2)	(1, 1)	(3.5392, 3.0228)	$\infty$	14.3142

Table 2: Results for different discrepancies when  $\mu_d$  is minimized at first step and for  $(\beta_2, \beta_p) = (1, 1)$  at third step.

$(\delta_1, \delta_2)$	$\mu_d$	$\tilde{\mu}_d$	$(\gamma_2, \gamma_p)$	$\mu_c$	$\mu_{dc}$
(0, 0)	1	1	$(10^{-4}, 10^{-4})$	3.3731	3.3731
(0.1, 0)	1.5303	1.5303	(0.2349, 0.2349)	2.8013	2.8013
(0, 0.1)	1.2497	1.2502	(0.3061, 0.2777)	2.5644	2.5644
(0.1, 0.1)	2.0284	2.0284	(0.5611, 0.5181)	4.3398	4.3398
(0.2, 0.1)	3.8506	3.8506	(1.1902, 1.0970)	10.1302	10.1302
(0.1, 0.2)	3.1161	3.1161	(1.3334, 1.1567)	7.7934	7.7934
(0.2, 0.2)	8.7776	8.7776	(3.0445, 2.5166)	$\infty$	$\infty$

Table 3: Size of the LMIs and of the S-variables in case none, one, or both parameters are uncertain:  $\{N_r, N_c\}$ .  $N_r$  gives the size of each individual LMI. This figure should be multiplied by the number of vertices to get the overall size of the LMI problem. The number of decision variables involved in the S-variables is the product  $N_r N_c$ . Th1 stand for Theorem 1, etc.

$(\delta_1, \delta_2)$	Th1	Th7	Th2	Th3	Th5	Th6
(0, 0)	{5, 0}	{5, 0}	{3, 0}	{6, 0}	{5, 0}	{5, 0}
(1, 0)	{6, 1}	{6, 1}	{5, 2}	{7, 1}	{7, 2}	{7, 2}
(0, 1)	{7, 2}	{6, 1}	{5, 2}	{8, 2}	{7, 2}	{10, 5}
(1, 1)	{7, 2}	{6, 1}	{6, 3}	{8, 2}	{8, 3}	{11, 6}

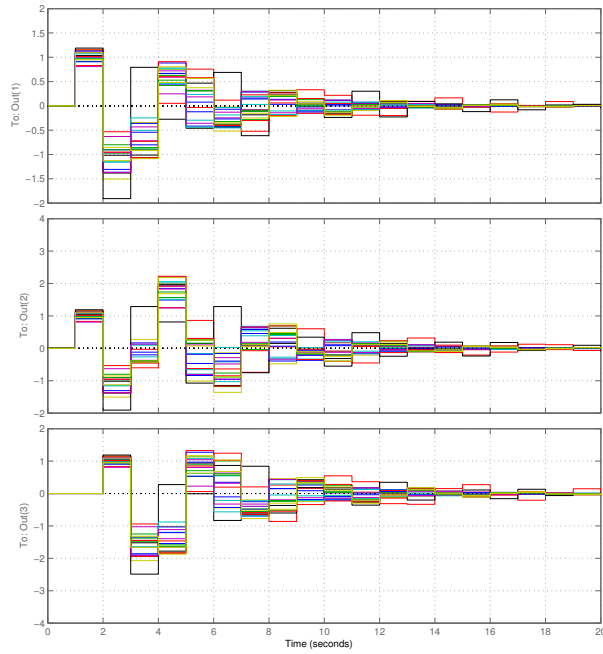


Figure 1: Impulse responses. First plot is the output  $z_k$ , the two others are the errors between the plant state and the observer state.

stability, one could expect that choosing large values for  $\beta_2$  would be preferable. This happens not to be the case as shown by the finally attained values  $\mu_c$ . To obtain a good closed-loop behavior it is important that the observer has not too large peak responses.

For the test with  $\delta_1 = \delta_2 = 0.2$  we get the following matrices

$$K = \begin{bmatrix} 0.6525 & 1.2696 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.0748 & 0.0377 \\ 0.0377 & 0.0422 \end{bmatrix}, A_o = \begin{bmatrix} -0.7832 & -1.1081 \\ 1.1373 & 0.9274 \end{bmatrix},$$

$$B_o = \begin{bmatrix} 0.1425 \\ 0.2195 \end{bmatrix}, L = \begin{bmatrix} 0.3059 \\ 0.4883 \end{bmatrix}$$

Notice that the obtained  $A_o$  matrix is far from being equal to the nominal matrix  $A_r(\theta_1 = 1, \theta_2 = 1)$ . The same comment applies to the  $B_o$  matrix.

The impulse responses ( $w_0 = 1$  and then  $w_{k>0} = 0$ ) of the plant in closed-loop with the obtained robust controller are plotted in Figure 1. The impulse responses are for 20 randomly chosen values of the uncertainties. The plots illustrate that the input-output performance is satisfactory but is not achieved by some separation with, first, a fast convergence of the observation error, and, then, a pure state-feedback control.

## 7 Comments and conclusions

### 7.1 Additional comments about the S-variable approach

All theorems and lemma exposed in the paper involve S-variables. Proofs of stability are obtained thanks to multi-affine Lyapunov matrices  $P_{\bullet}(\theta)$ . But the conditions involve additional variables  $S_{\bullet}$ . As seen in the proofs, these matrices vanish as soon as the descriptor-represented trajectories of the system are taken into account. The S-variables are related to Finsler's lemma [37] or, from a more general viewpoint, to the S-procedure [40], which is why we adopt the S-variable terminology [13].

Notice that the S-variables involved in the analysis Theorems 2, 5 and 6 are tall full matrices, while in the design Theorems 1 and 3 the corresponding S-variables have a structure as in (15). This structure is conservative, but useful because makes the design constraints linear and hence convex in the decision variables. As discussed in [31, 14, 13] other choices are possible. This one has the advantage of guaranteeing the results to be less conservative than those build with a common Lyapunov matrix for all uncertainties (see chapter 4 in [13]).

### 7.2 State-feedback / observer gain iterations

The proposed methodology suggests to first design a state-feedback and then based on it to design an observer. Having in mind that the two problems are somehow dual one of the other, it could be natural to assume possible in the same framework to design the state-feedback gain (and why not the  $A_o$  matrix) for fixed observer matrices  $L$  and  $B_o$ . This happens to be possible when structuring the S-variable in (21) as in (15) and with a mild change of variables. Nevertheless we chose not to present this result because it can be proved that such condition will never hold if the open loop system is not robustly stable. The structure (15) in this case brings a serious conservatism. Future work will be devoted to address this issue in order to propose heuristic iterative procedures that could iterate between the two state-feedback design and observer design conditions. Ideally such procedure should need to have at each step a guarantee that the next step is feasible. As discussed in section 4, this is not yet the case.

### 7.3 Conclusions

A new point-of-view on robust observer design emerges from the paper. Since exact observation is impossible for systems with uncertainties it solves the issue in an approximated way, searching for those dynamic filters that would make the observed-state feedback dynamics as close as possible to the ideal state-feedback dynamics. Although this strategy seems a natural extension of results by Luenberger, it is at our knowledge new.

Additionally to this core contribution, the paper also provides a new descriptor multi-affine representation for rationally-dependent uncertain systems. With the help of the S-variables framework, numerically tractable LMI conditions involving multi-affine parameter-dependent Lyapunov functions are produced and tested on an illustrative example. Future work will be devoted to testing the method on more realistic



examples, which could be continuous-time systems. For such systems the structure on the S-variables will need to be reconsidered.

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## **Differences between the submitted paper and its conference versions**

The manuscript entitled "*Robust observed-state feedback design for discrete-time systems rational in the uncertainties*" is an extended version of the conference paper presented at the 19th IFAC World Congress in Cape Town, year 2014, which was entitled "*LMI results for robust control design of observer-based controllers, the discrete-time case with polytopic uncertainties*". The main difference between the two papers is that the new journal version is dedicated to systems that are rational in several uncertainties while the conference paper treated the simpler case where systems are affine in one polytopic uncertainty. Related to this change, the journal version includes a new contribution in terms of modeling systems rational in the uncertainties as descriptor multi-affine systems. While the conference version concentrated on observer design since state-feedback design is already largely treated in the literature for affine polytopic systems, the journal version provides new results for both observer feedback and state-feedback in the context of rationally dependent systems. It also gives new results for the analysis of such systems.

In addition to these changes, the manuscript of the journal version includes an other contribution, also published at the 19th IFAC World Congress in Cape Town, year 2014, which was entitled "*Slack variable approach for robust stability analysis of switching discrete-time systems*". The result of that conference paper is extended here to illustrate the fact that an identical framework can be used for both time-invariant and time-varying uncertainties. In the journal version we provide a unified notation to also treat the case of mixed time-invariant / time-varying uncertainties.