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Computable bounds of ℓ^2 -spectral gap for discrete Markov chains with band transition matrices

Loïc HERVÉ, and James LEDOUX *

Abstract

We analyse the $\ell^2(\pi)$ -convergence rate of irreducible and aperiodic Markov chains with N -band transition probability matrix P and with invariant distribution π . This analysis is heavily based on: first the study of the essential spectral radius $r_{ess}(P|_{\ell^2(\pi)})$ of $P|_{\ell^2(\pi)}$ derived from Hennion's quasi-compactness criteria; second the connection between the Spectral Gap property (SG₂) of P on $\ell^2(\pi)$ and the V -geometric ergodicity of P . Specifically, (SG₂) is shown to hold under the condition

$$\alpha_0 := \sum_{m=-N}^N \limsup_{i \rightarrow +\infty} \sqrt{P(i, i+m)P^*(i+m, i)} < 1.$$

Moreover $r_{ess}(P|_{\ell^2(\pi)}) \leq \alpha_0$. Effective bounds on the convergence rate can be provided from a truncation procedure.

AMS subject classification : 60J10; 47B07

Keywords : V -geometric ergodicity, Essential spectral radius.

1 Introduction

Let $P := (P(i, j))_{(i, j) \in \mathbb{N}^2}$ be a Markov kernel on the countable state space \mathbb{N} . Throughout the paper we assume that P is irreducible and aperiodic, that P has a unique invariant probability measure denoted by $\pi := (\pi(i))_{i \in \mathbb{N}}$, and finally that

$$\exists i_0 \in \mathbb{N}, \exists N \in \mathbb{N}^*, \forall i \geq i_0 : |i - j| > N \implies P(i, j) = 0. \quad (\mathbf{AS1})$$

We denote by $(\ell^2(\pi), \|\cdot\|_2)$ the Hilbert space of sequences $(f(i))_{i \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\|f\|_2 := [\sum_{i \geq 0} |f(i)|^2 \pi(i)]^{1/2} < \infty$. Then P defines a linear contraction on $\ell^2(\pi)$, and its adjoint operator P^* on $\ell^2(\pi)$ is defined by $P^*(i, j) := \pi(j)P(j, i)/\pi(i)$. If $\pi(f) := \sum_{i \geq 0} f(i)\pi(i)$, then the kernel P is said to have the spectral gap property on $\ell^2(\pi)$ if there exists $\rho \in (0, 1)$ and $C \in (0, +\infty)$ such that

$$\forall n \geq 1, \forall f \in \ell^2(\pi), \quad \|P^n f - \Pi f\|_2 \leq C \rho^n \|f\|_2 \quad \text{with} \quad \Pi f := \pi(f)1_{\mathbb{N}}. \quad (\mathbf{SG}_2)$$

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A standard issue is to compute the value (or to find an upper bound) of

$$\varrho_2 := \inf\{\rho \in (0, 1) : (\mathbf{SG}_2) \text{ holds true}\}. \quad (1)$$

In this work the quasi-compactness criteria of [Hen93] is used to study (\mathbf{SG}_2) and to estimate ϱ_2 . In Section 2 it is proved that (\mathbf{SG}_2) holds when

$$\alpha_0 := \sum_{m=-N}^N \limsup_{i \rightarrow +\infty} \sqrt{P(i, i+m) P^*(i+m, i)} < 1. \quad (\mathbf{AS2})$$

Moreover $r_{ess}(P|_{\ell^2(\pi)}) \leq \alpha_0$. We refer to [Hen93] for the definition of the essential spectral radius $r_{ess}(T)$ and for quasi-compactness of a bounded linear operator T on a Banach space. Under the assumptions

$$\forall m = -N, \dots, N, \quad P(i, i+m) \xrightarrow{i \rightarrow +\infty} a_m \in [0, 1] \quad (\mathbf{AS3})$$

$$\frac{\pi(i+1)}{\pi(i)} \xrightarrow{i \rightarrow +\infty} \tau \in [0, 1] \quad (\mathbf{AS4})$$

$$\sum_{k=-N}^N k a_k < 0, \quad (\mathbf{NERI})$$

Property $(\mathbf{AS2})$ holds (hence (\mathbf{SG}_2)) and α_0 can be explicitly computed in function of τ and the a_m 's. Moreover, using the inequality $r_{ess}(P|_{\ell^2(\pi)}) \leq \alpha_0$, Property (\mathbf{SG}_2) is proved to be connected to the V -geometric ergodicity of P for $V := (\pi(n)^{-1/2})_{n \in \mathbb{N}}$. In particular, denoting the minimal V -geometrical ergodic rate by ϱ_V , it is proved that, either ϱ_2 and ϱ_V are both less than α_0 , or $\varrho_2 = \varrho_V$. As a result, an accurate bound of ϱ_2 can be obtained for random walks (RW) with i.d. bounded increments using the results of [HL14b]. Actually, any estimation of ϱ_V , for instance that derived in Section 3 from the truncation procedure of [HL14a], provides an estimation of ϱ_2 . We point out that all the previous results hold without any reversibility properties.

The spectral gap property for Markov processes has been widely investigated in the discrete and continuous-time cases (e.g. see [Ros71, Che04]). There exist different definitions of the spectral gap property according that we are concerned with discrete or continuous-time case (e.g. see [Yue00, MS13]). The focus of our paper is on the discrete time case. In the reversible case, the equivalence between the geometrical ergodicity and (\mathbf{SG}_2) is proved in [RR97] and Inequality $\varrho_2 \leq \varrho_V$ is obtained in [Bax05, Th.6.1.]. This equivalence fails in the non-reversible case (see [KM12]). The link between ϱ_2 and ϱ_V stated in our Proposition 1 is obtained with no reversibility condition. Formulae for ϱ_2 are provided in [SW11, Wüb12] in terms of isoperimetric constants which are related to P in reversible case and to P and P^* in non-reversible case. However, to the best of our knowledge, no explicit value (or upper bounds) of ϱ_2 can be derived from these formulae for discrete Markov chains with band transition matrices. Our explicit bound $r_{ess}(P|_{\ell^2(\pi)}) \leq \alpha_0$ in Theorem 1 is the preliminary key results in this work. Recall that $r_{ess}(P|_{\ell^2(\pi)})$ is a natural lower bound of ϱ_2 (apply [HL14b, Prop. 2.1] with the Banach space $\ell^2(\pi)$). The essential spectral radius of Markov operators on a \mathbb{L}^2 -type space is investigated for Markov chains with general state space in [Wu04], but no explicit bound for $r_{ess}(P|_{\ell^2(\pi)})$ can be derived a priori from these theoretical results for Markov chains with band transition matrices, except in the reversible case [Wu04, Th. 5.5.].

2 Property (SG₂) and V-geometrical ergodicity

Theorem 1 *If Condition (AS2) holds, then P satisfies (SG₂). Moreover $r_{ess}(P|_{\ell^2(\pi)}) \leq \alpha_0$.*

Proof. Consider the Banach space $\ell^1(\pi) := \{(f(i))_{i \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|f\|_1 := \sum_{i \geq 0} |f(i)| \pi(i) < \infty\}$.

Lemma 1 *For any $\alpha > \alpha_0$, there exists a positive constant $L \equiv L(\alpha)$ such that*

$$\forall f \in \ell^2(\pi), \quad \|Pf\|_2 \leq \alpha \|f\|_2 + L \|f\|_1.$$

Since the identity map is compact from $\ell^2(\pi)$ into $\ell^1(\pi)$ (from the Cantor diagonal procedure), it follows from Lemma 1 and from [Hen93] that P is quasi-compact on $\ell^2(\pi)$ with $r_{ess}(P|_{\ell^2(\pi)}) \leq \alpha$. Since α can be chosen arbitrarily close to α_0 , this gives $r_{ess}(P|_{\ell^2(\pi)}) \leq \alpha_0$. Then (SG₂) is deduced from aperiodicity and irreducibility assumptions. \square

Proof. Under Assumption (AS1), define

$$\forall i \geq i_0, \forall m = -N, \dots, N, \quad \beta_m(i) := \sqrt{P(i, i+m) P^*(i+m, i)}. \quad (2)$$

Let $\alpha > \alpha_0$, with α_0 given in (AS2). Fix $\ell \equiv \ell(\alpha) \geq i_0$ such that $\sum_{m=-N}^N \sup_{i \geq \ell} \beta_m(i) \leq \alpha$. For $f \in \ell^2(\pi)$ we have from Minkowski's inequality and the band structure of P for $i \geq \ell$

$$\begin{aligned} \|Pf\|_2 &\leq \left[\sum_{i < \ell} |(Pf)(i)|^2 \pi(i) \right]^{1/2} + \left[\sum_{i \geq \ell} \left| \sum_{m=-N}^N P(i, i+m) f(i+m) \right|^2 \pi(i) \right]^{1/2} \\ &\leq L \sum_{i < \ell} |(Pf)(i)| \pi(i) + \left[\sum_{i \geq \ell} \left| \sum_{m=-N}^N P(i, i+m) f(i+m) \right|^2 \pi(i) \right]^{1/2} \end{aligned} \quad (3)$$

where $L \equiv L_\ell > 0$ comes from equivalence of norms on \mathbb{C}^ℓ . Moreover we have $\sum_{i < \ell} |(Pf)(i)| \pi(i) \leq \|Pf\|_1 \leq \|f\|_1$. To control the second term in (3), define $F_m = (F_m(i))_{i \in \mathbb{N}} \in \ell^2(\pi)$ by $F_m(i) := P(i, i+m) f(i+m) (1 - 1_{\{0, \dots, \ell-1\}}(i))$ for $-N \leq m \leq N$. Then

$$\left[\sum_{i \geq \ell} \left| \sum_{m=-N}^N P(i, i+m) f(i+m) \right|^2 \pi(i) \right]^{1/2} = \left\| \sum_{m=-N}^N F_m \right\|_2 \leq \sum_{m=-N}^N \|F_m\|_2.$$

$$\begin{aligned} \text{and } \|F_m\|_2^2 &= \sum_{i \geq \ell} P(i, i+m)^2 |f(i+m)|^2 \pi(i) \\ &= \sum_{i \geq \ell} P(i, i+m) \frac{\pi(i) P(i, i+m)}{\pi(i+m)} |f(i+m)|^2 \pi(i+m) \\ &\leq \sup_{i \geq \ell} \beta_m(i)^2 \|f\|_2^2 \quad (\text{from the definition of } P^* \text{ and from (2)}). \end{aligned}$$

The statement in Lemma 1 can be deduced from the previous inequality and from (3). \square

The core of our approach to estimate ϱ_2 is the relationship between Property **(SG₂)** and the V -geometric ergodicity. Indeed, specify Theorem 1 in terms of the V -geometric ergodicity with $V := (\pi(n)^{-1/2})_{n \in \mathbb{N}}$. Let $(\mathcal{B}_V, \|\cdot\|_V)$ denote the space of sequences $(g(n))_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\|g\|_V := \sup_{n \in \mathbb{N}} V(n)^{-1} |g(n)| < \infty$. Recall that P is said to be V -geometrically ergodic if P satisfies the spectral gap property on \mathcal{B}_V , namely: there exist $C \in (0, +\infty)$ and $\rho \in (0, 1)$ such that

$$\forall n \geq 1, \forall f \in \mathcal{B}_V, \quad \|P^n f - \Pi f\|_V \leq C \rho^n \|f\|_V. \quad (\mathbf{SG}_V)$$

When this property holds, we define

$$\varrho_V := \inf\{\rho \in (0, 1) : (\mathbf{SG}_V) \text{ holds true}\}. \quad (4)$$

Remark 1 Under Assumptions **(AS3)** and **(AS4)**, we have

$$\alpha_0 := \sum_{m=-N}^N \limsup_{i \rightarrow +\infty} \sqrt{P(i, i+m) P^*(i+m, i)} = \begin{cases} \sum_{m=-N}^N a_m \tau^{-m/2} & \text{if } \tau \in (0, 1) \\ a_0 & \text{if } \tau = 0. \end{cases} \quad (5)$$

Indeed, if **(AS4)** holds with $\tau \in (0, 1)$, then the claimed formula follows from the definition of P^* . If $\tau = 0$ in **(AS4)**, then $a_m = 0$ for every $m = 1, \dots, N$ from $\sum_{m=-N}^N P(i+m, i) \pi(i+m)/\pi(i) = 1$. Thus $a_{-m} = 0$ when $m < 0$. Hence $\alpha_0 = a_0$.

Proposition 1 If P and π satisfy Assumptions **(AS3)**, **(AS4)** and **(NERI)**, then P satisfies **(AS2)** (with $\alpha_0 < 1$ given in (5)). Moreover P satisfies both **(SG₂)** and **(SG_V)** with $V := (\pi(n)^{-1/2})_{n \in \mathbb{N}}$, we have $\max(r_{ess}(P|_{\mathcal{B}_V}), r_{ess}(P|_{\ell^2(\pi)})) \leq \alpha_0$, and the next assertions hold:

(a) if $\varrho_V \leq \alpha_0$, then $\varrho_2 \leq \alpha_0$;

(b) if $\varrho_V > \alpha_0$, then $\varrho_2 = \varrho_V$.

Proof. If $\tau = 0$ in **(AS4)**, then $\alpha_0 = a_0 < 1$ from (5) and **(NERI)**. Now assume that **(AS4)** holds with $\tau \in (0, 1)$. Then $\alpha_0 = \sum_{m=-N}^N a_m \tau^{-m/2} = \psi(\sqrt{\tau})$, where: $\forall t > 0$, $\psi(t) := \sum_{k=-N}^N a_k t^{-k}$. Moreover it easily follows from the invariance of π that $\psi(\tau) = 1$. Inequality $\alpha_0 = \psi(\sqrt{\tau}) < 1$ is deduced from the following assertions: $\forall t \in (\tau, 1)$, $\psi(t) < 1$ and $\forall t \in (0, \tau) \cup (1, +\infty)$, $\psi(t) > 1$. To prove these properties, note that $\psi(\tau) = \psi(1) = 1$ and that ψ is convex on $(0, +\infty)$. Moreover we have $\lim_{t \rightarrow +\infty} \psi(t) = +\infty$ since $a_k > 0$ for some $k < 0$ (use $\psi(\tau) = \psi(1) = 1$ and $\tau \in (0, 1)$). Similarly, $\lim_{t \rightarrow 0^+} \psi(t) = +\infty$ since $a_k > 0$ for some $k > 0$. This gives the desired properties on ψ since $\psi'(1) > 0$ from **(NERI)**.

(SG₂) and $r_{ess}(P|_{\ell^2(\pi)}) \leq \alpha_0$ follow from Theorem 1. Next **(SG_V)** is deduced from the well-known link between geometric ergodicity and the following drift inequality:

$$\forall \alpha \in (\alpha_0, 1), \exists L \equiv L_\alpha > 0, \quad PV \leq \alpha V + L 1_{\mathbb{N}}. \quad (6)$$

This inequality holds from $\lim_i (PV)(i)/V(i) = \alpha_0$.

Then (\mathbf{SG}_V) is derived from (6) using aperiodicity and irreducibility. It also follows from (6) that $r_{\text{ess}}(P|_{\mathcal{B}_V}) \leq \alpha$ (see [HL14b, Prop. 3.1]). Thus $r_{\text{ess}}(P|_{\mathcal{B}_V}) \leq \alpha_0$.

Now we prove (a) and (b) using the spectral properties of [HL14b, Prop. 2.1] of both $P|_{\ell^2(\pi)}$ and $P|_{\mathcal{B}_V}$ (due to quasi-compactness, see [Hen93]). We will also use the following obvious inclusion: $\ell^2(\pi) \subset \mathcal{B}_V$. In particular every eigenvalue of $P|_{\ell^2(\pi)}$ is also an eigenvalue for $P|_{\mathcal{B}_V}$. First assume that $\varrho_V \leq \alpha_0$. Then there is no eigenvalue for $P|_{\mathcal{B}_V}$ in the annulus $\Gamma := \{\lambda \in \mathbb{C} : \alpha_0 < |\lambda| < 1\}$ since $r_{\text{ess}}(P|_{\mathcal{B}_V}) \leq \alpha_0$. From $\ell^2(\pi) \subset \mathcal{B}_V$ it follows that there is also no eigenvalue for $P|_{\ell^2(\pi)}$ in this annulus. Hence $\varrho_2 \leq \alpha_0$ since $r_{\text{ess}}(P|_{\ell^2(\pi)}) \leq \alpha_0$. Second assume that $\varrho_V > \alpha_0$. Then $P|_{\mathcal{B}_V}$ admits an eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda| = \varrho_V$. Let $f \in \mathcal{B}_V$, $f \neq 0$, such that $Pf = \lambda f$. We know from [HL14b, Prop. 2.2] that there exists some $\beta \equiv \beta_\lambda \in (0, 1)$ such that $|f(n)| = O(V(n)^\beta) = O(\pi(n)^{-\beta/2})$, so that $|f(n)|^2 \pi(n) = O(\pi(n)^{(1-\beta)})$, thus $f \in \ell^2(\pi)$ from **(AS4)**. We have proved that $\varrho_2 \geq \varrho_V$. Finally the converse inequality is true since every eigenvalue of $P|_{\ell^2(\pi)}$ is an eigenvalue for $P|_{\mathcal{B}_V}$. Thus $\varrho_2 = \varrho_V$. \square

From Proposition 1, any estimation of ϱ_V provides an estimation of ϱ_2 . This is illustrated in Example 1 and Corollary 1. Markov chains in Example 1 have been studied in details in [HL14b, Section 3]. Also mention that further technical details are reported in [HL15].

Example 1 (RWs with i.d. bounded increments) *Let P be defined as follows. There exist some positive integers $c, g, d \in \mathbb{N}^*$ such that*

$$\begin{aligned} \forall i \in \{0, \dots, g-1\}, \quad \sum_{j=0}^c P(i, j) &= 1; \\ \forall i \geq g, \forall j \in \mathbb{N}, \quad P(i, j) &= \begin{cases} a_{j-i} & \text{if } i-g \leq j \leq i+d \\ 0 & \text{otherwise.} \end{cases} \\ (a_{-g}, \dots, a_d) \in [0, 1]^{g+d+1} : a_{-g} > 0, a_d > 0, \sum_{k=-g}^d a_k &= 1. \end{aligned}$$

Assume that P is aperiodic and irreducible, and satisfies **(NERI)**. Then P has a unique invariant distribution π . It can be derived from standard results of linear difference equation that $\pi(n) \sim c\tau^n$ when $n \rightarrow +\infty$, with $\tau \in (0, 1)$ defined by $\psi(\tau) = 1$, where $\psi(t) := \sum_{k=-N}^N a_k t^{-k}$. Thus, if $\gamma := \tau^{-1/2}$, then $\mathcal{B}_V = \{(g(n))_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}, \sup_{n \in \mathbb{N}} \gamma^{-n} |g(n)| < \infty\}$. Then we know from [HL14b, Prop. 3.2] that $r_{\text{ess}}(P|_{\mathcal{B}_V}) = \alpha_0$ with α_0 given in (5), and that ϱ_V can be computed from an algebraic polynomial elimination. From this computation, Proposition 1 provides an accurate estimation of ϱ_2 . Property **(SG₂)** was proved in [Wüb12, Th. 2] under an extra weak reversibility assumption (with no explicit bound on ϱ_2). However, except in case $g = d = 1$ where reversibility is automatic, an RW with i.d. bounded increments is not reversible or even weak reversible in general. No reversibility condition is required here.

3 Bound for ϱ_2 via truncation

Let P be any Markov kernel on \mathbb{N} , and let us consider the k -th truncated (and augmented on the last column) matrix P_k associated with P as in [HL14a]. If $\sigma(P_k)$ denotes the set of

eigenvalues of P_k , define $\rho_k := \max \{|\lambda|, \lambda \in \sigma(P_k), |\lambda| < 1\}$. The weak perturbation method in [HL14a] provides the following general result where Condition **(AS1)** is not required and V is any unbounded increasing sequence.

Proposition 2 *Let P be an irreducible and aperiodic Markov kernel on \mathbb{N} satisfying the following drift inequality for some unbounded increasing sequence $(V(n))_{n \in \mathbb{N}}$:*

$$\exists \delta \in [0, 1[, \exists L > 0, \quad PV \leq \delta V + L 1_{\mathbb{N}}. \quad (8)$$

Let ϱ_V be defined in (4). Then, either $\varrho_V \leq \delta$ and $\limsup_k \rho_k \leq \delta$, or $\varrho_V > \delta$ and $\varrho_V = \lim_k \rho_k$.

Proof. Condition (8) ensures that the assumptions of [HL14a, Lem. 6.1] are satisfied, so that $r_{ess}(P|_{\mathcal{B}_V}) \leq \delta$. Then, using standard duality arguments, the spectral rank-stability property [HL14a, Lem. 7.2] applies to $P|_{\mathcal{B}_V}$ and P_k . If $\varrho_V \leq \delta$, then, for each r such that $\delta < r < 1$, $\lambda = 1$ is the unique eigenvalue of $P|_{\mathcal{B}_V}$ in $C_r := \{\lambda \in \mathbb{C} : r < |\lambda| \leq 1\}$ (see [Hen93]). From [HL14a, Lem. 7.2] this property holds for P_k when k is large enough, so that $\limsup_k \rho_k \leq r$. Thus $\limsup_k \rho_k \leq \delta$ since r is arbitrarily close to δ . Now assume that $\varrho_V > \delta$, and let r be such that $\delta < r < \varrho_V$. Then $P|_{\mathcal{B}_V}$ has a finite number of eigenvalues in C_r , say $\lambda_0, \lambda_1, \dots, \lambda_N$, with $\lambda_0 = 1$, $|\lambda_1| = \varrho_V$ and $|\lambda_k| \leq \varrho_V$ for $k = 2, \dots, N$ (see [Hen93]). For $a \in \mathbb{C}$ and $\varepsilon > 0$ we define $D(a, \varepsilon) := \{z \in \mathbb{C} : |z - a| < \varepsilon\}$. Now consider any $\varepsilon > 0$ such that the disks $D(\lambda_k, \varepsilon)$ for $k = 0, \dots, N$ are disjoint and are contained in C_r pour $k \geq 1$. From [HL14a, Lem. 7.2], for k large enough, 1 is the only eigenvalue of P_k in $D(1, \varepsilon)$, the others eigenvalues of P_k in C_r are contained in $\cup_{k=1}^N D(\lambda_k, \varepsilon)$, and finally each $D(\lambda_k, \varepsilon)$ contains at least one eigenvalue of P_k . Thus each eigenvalue $\lambda \neq 1$ of P_k in C_r has modulus less than $\varrho_V + \varepsilon$, so that $\rho_k \leq \varrho_V + \varepsilon$. Moreover the disk $D(\lambda_1, \varepsilon)$ contains at least an eigenvalue λ of P_k , so that $\rho_k \geq |\lambda| \geq \varrho_V - \varepsilon$. Thus, for k large enough, we have $\varrho_V - \varepsilon \leq \rho_k \leq \varrho_V + \varepsilon$. \square

Under the assumptions of Proposition 1 we deduce the following result from Proposition 2.

Corollary 1 *If P satisfies the assumptions of Proposition 1, then the following properties holds with α_0 given in (5):*

- (a) $\varrho_2 \leq \alpha_0 \iff \varrho_V \leq \alpha_0$, and in this case we have $\limsup_k \rho_k \leq \alpha_0$;
- (b) $\varrho_2 > \alpha_0 \iff \varrho_V > \alpha_0$, and in this case we have $\varrho_2 = \varrho_V = \lim_k \rho_k$.

As usual the reversible case is simpler. In particular we can take $C = 1$ and $\rho = \varrho_2$ in **(SG₂)**. Details and numerical illustrations for Metropolis-Hastings kernels are reported in [HL15].

References

- [Bax05] P. H. Baxendale. Renewal theory and computable convergence rates for geometrically ergodic Markov chains. *Ann. Appl. Probab.*, 15(1B):700–738, 2005.

- [Che04] M.-F. Chen. *From Markov chains to non-equilibrium particle systems*. World Scientific Publishing Co. Inc., River Edge, NJ, second edition, 2004.
- [Hen93] H. Hennion. Sur un théorème spectral et son application aux noyaux lipchitziens. *Proc. Amer. Math. Soc.*, 118:627–634, 1993.
- [HL14a] L. Hervé and J. Ledoux. Approximating Markov chains and V -geometric ergodicity via weak perturbation theory. *Stochastic Process. Appl.*, 124(1):613–638, 2014.
- [HL14b] L. Hervé and J. Ledoux. Spectral analysis of Markov kernels and application to the convergence rate of discrete random walks. *Adv. in Appl. Probab.*, 46(4):1036–1058, 2014.
- [HL15] L. Hervé and J. Ledoux. Additional material on bounds of ℓ^2 -spectral gap for discrete Markov chains with band transition matrices. *hal-01117465*, 2015.
- [KM12] I. Kontoyiannis and S. P. Meyn. Geometric ergodicity and the spectral gap of non-reversible Markov chains. *Probab. Theory Related Fields*, 154(1-2):327–339, 2012.
- [MS13] Y. H. Mao and Y. H. Song. Spectral gap and convergence rate for discrete-time Markov chains. *Acta Math. Sin. (Engl. Ser.)*, 29(10):1949–1962, 2013.
- [Ros71] M. Rosenblatt. *Markov processes. Structure and asymptotic behavior*. Springer-Verlag, New-York, 1971.
- [RR97] G. O. Roberts and J. S. Rosenthal. Geometric ergodicity and hybrid Markov chains. *Elect. Comm. in Probab.*, 2:13–25, 1997.
- [SW11] W. Stadje and A. Wübker. Three kinds of geometric convergence for Markov chains and the spectral gap property. *Electron. J. Probab.*, 16:no. 34, 1001–1019, 2011.
- [Wu04] L. Wu. Essential spectral radius for Markov semigroups. I. Discrete time case. *Probab. Theory Related Fields*, 128(2):255–321, 2004.
- [Wüb12] A. Wübker. Spectral theory for weakly reversible Markov chains. *J. Appl. Probab.*, 49(1):245–265, 2012.
- [Yue00] W. K. Yuen. Applications of geometric bounds to the convergence rate of Markov chains on \mathbf{R}^n . *Stochastic Process. Appl.*, 87(1):1–23, 2000.