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EXTENSIONS BETWEEN FUNCTORS FROM GROUPS

CHRISTINE VESPA

Abstract. We compute Ext-groups between tensor powers composed by the abelianization functor. More precisely, we compute the groups $Ext^*_{\mathcal{F}(\mathbf{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a})$ where T^n is the n-th tensor power functor and $\mathfrak{a}$ is the abelianization functor from the category of free groups to abelian groups. These groups are shown to be non-zero if and only if $* = m - n \geq 0$ and $Ext^{m-n}_{\mathcal{F}(\mathbf{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) = \mathbb{Z}[Surj(m,n)]$ where $Surj(m,n)$ is the set of surjections from the set having m elements to the one having n elements. We make explicit the action of symmetric groups on these groups and the Yoneda and external products. We deduce from these computations those of other Ext-groups between functors from groups such as symmetric and exterior powers. Combining these computations with a recent result of Djament we obtain explicit computations of stable homology of automorphism groups of free groups with coefficients given by particular contravariant functors.

Stable homology with twisted coefficients of various families of groups can be computed thanks to functor homology in a suitable category (see [6] [15] [4] [5]). In particular, stable homology of automorphism groups of free groups with coefficients given by a reduced polynomial covariant functor is trivial (see [5]). Recently, Djament proved in [2] that stable homology (resp. cohomology) of automorphism groups of free groups with coefficients given by a reduced polynomial contravariant (resp. covariant) functor is governed by Tor groups (resp. Ext groups) in the category of functors from finitely generated free groups. The aim of this paper is to give explicit computations of Ext and Tor groups between concrete functors from groups to abelian groups in order to obtain explicit computations of stable homology of automorphism groups of free groups with coefficients given by a contravariant functor. Let $\mathbf{gr}$ be a small skeleton of the category of finitely generated free groups, $\mathcal{F}(\mathbf{gr})$ the category of functors from $\mathbf{gr}$ to abelian groups and $\mathfrak{a}$ the abelianization functor in $\mathcal{F}(\mathbf{gr})$. The principal result of this paper is:

Theorem 0.1. *Let n and m be integers, we have an isomorphism:*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \begin{cases} \mathbb{Z}[Surj(m,n)] & \text{if } * = m - n \\ 0 & \text{otherwise} \end{cases}$$

where $Surj(m,n)$ is the set of surjections from the set having m elements to the one having n elements.

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The actions of the symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$ on $Ext^*_{\mathcal{F}(\boldsymbol{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a})$ are induced by the composition of surjections via the previous isomorphism, up to a sign (see Proposition 2.5 for the precise signs).

The Yoneda product is induced by the composition of surjections, up to a sign, (see Proposition 3.1 for the precise signs) via the previous isomorphism and the external product is induced by the disjoint union of sets.

In particular for $n = m = 1$ we obtain that $Hom_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}) = \mathbb{Z}$ and $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}) = 0$ for $* > 0$. Let $\mathbf{ab}$ be a small skeleton of the category of finitely generated free abelian groups and $\mathcal{F}(\mathbf{ab})$ be the categoy of functors from $\mathbf{ab}$ to abelian groups. The previous groups should be compared with the groups $Ext^*_{\mathcal{F}(\mathbf{ab})}(Id, Id)$ corresponding to the MacLane homology of $\mathbb{Z}$. This homology is computed by Bökstedt by topological methods (unpublished) and reobtained in [7] by algebraic methods and is non trivial. So this result is an illustration of the fact that functor homology in $\mathcal{F}(\mathbf{gr})$ is easier than functor homology in $\mathcal{F}(\mathbf{ab})$ (see also [3] for another illustration of this fact).

Theorem 0.1 is used in [3, Proposition 4.1 and 4.6] in order to compute homological dimension in a category of polynomial functors.

The proof of the first part of Theorem 0.1 can be decomposed into two steps corresponding to the first two sections of this paper. In the first step we compute $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^m \circ \mathfrak{a})$ using an explicit projective resolution of the abelianization functor $\mathfrak{a}$. In the second step we deduce our result using the sum-diagonal adjunction and an exponential-type property of tensor power functor $T^\bullet$. The next section is devoted to the study of products on these Ext-groups and the description of the PROP governing these Ext-groups. More precisely,

Let $g\mathbf{Ab}^-$ be the category of graded abelian groups where the commutativity is given by: $v \otimes w \mapsto (-1)^{|v|.|w|} w \otimes v$.

We deduce from Theorem 0.1 the following results:

Proposition 0.2. *(Propositions 3.2 and 3.4)*

1. *The graded symmetric sequence $\Lambda = \{\Lambda(n)\}_{n\geq 0}$ given by $\Lambda(n) = Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a}, T^n \circ \mathfrak{a})$ is an operad in $g\boldsymbol{Ab}^-$.*
2. *The graded category $\Omega(\Lambda)$ having as objects the finite sets n and such that*

$$Hom_{\Omega(\Lambda)}(m, n) = Ext^*_{\mathcal{F}(\boldsymbol{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a})$$

is the graded PROP associated to the operad Λ.

In the last section we deduce from Theorem 0.1 other computations of Ext-groups between symmetric power functor S^k, exterior power functor Λ^k and tensor power functor. In particular, we obtain the following results.

Proposition 0.3. *Let n and m be natural integers, we have isomorphisms:*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{\rho(m,n)} & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

where $\rho(m,n)$ denotes the number of partitions of m into n parts.

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (S^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m = 0 \text{ and } * = 0 \\ & \text{or } n = m = 1 \text{ and } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (T^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{S(m,n)} & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

where $S(m,n)$ denotes the Stirling partition number (i.e. the number of ways to partition a set of m elements into n non-empty subsets).

By duality arguments we can obtain similar results for Tor groups.

In [2] Djament gives the description of stable homology of automorphisms of free groups with coefficients twisted by contravariant functors in terms of Tor-groups between functors in $\mathcal{F}(\mathbf{gr})$. More precisely, for $N : \mathbf{gr}^{op} \to \mathbb{Q}$-Mod, F_n the free group in n generators and all integer i, Djament obtains in [2, Théorème 1.10] the existence of an isomorphism:

$$\operatorname*{colim}_{n\in\mathbb{N}} H_i(Aut(F_n), N(F_n)) \simeq \bigoplus_{k+l=i} Tor_k^{\mathbf{gr}}(N, \Lambda^l \circ (\mathfrak{a} \otimes \mathbb{Q}))$$

where Λ^l is the l-th exterior power functor. This statement is equivalent to the following: for $M : \mathbf{gr} \to \mathbb{Q}$-Mod, there exists an isomorphism:

$$\operatorname*{colim}_{n\in\mathbb{N}} H^i(Aut(F_n), M(F_n)) \simeq \bigoplus_{k+l=i} Ext^k_{\mathbf{gr}}(\Lambda^l \circ (\mathfrak{a} \otimes \mathbb{Q}), M).$$

To a covariant functor $M : \mathbf{gr} \to \mathbb{Q}$-Mod wa can associate a contravariant functor $N : \mathbf{gr}^{op} \to \mathbb{Q}$-Mod given by $N(G) = M(Hom_{\mathbf{Gr}}(F_n, \mathbb{Q}))$. As an application of Proposition 0.3, using the previous result of Djament, we obtain the following computations of stable homology:

Proposition 0.4. *Let $H_n = Hom_{\boldsymbol{Gr}}(F_n, \mathbb{Q})$. We have*

$$\operatorname*{colim}_{n\in\mathbb{N}} H_*(Aut(F_n), T^d \circ (\mathfrak{a} \otimes \mathbb{Q})(H_n)) \simeq \begin{cases} \mathbb{Q}^{B(d)} & \text{if } * = d \\ 0 & \text{otherwise} \end{cases}$$

where $B(d)$ denotes the d-th Bell number (i.e. the number of partitions of a set of d elements),

$$\operatorname*{colim}_{n\in\mathbb{N}} H_*(Aut(F_n), \Lambda^d \circ (\mathfrak{a} \otimes \mathbb{Q})(H_n)) \simeq \begin{cases} \mathbb{Q}^{\rho(d)} & \text{if } * = d \\ 0 & \text{otherwise} \end{cases}$$

where $\rho(d)$ denotes the number of partitions of d, and

$$\underset{n\in\mathbb{N}}{colim}\ H_*(Aut(F_n), S^d\circ(\mathfrak{a}\otimes\mathbb{Q})(H_n)) \simeq \begin{cases} \mathbb{Q} & if * = d = 0 \\ & or * = d = 1 \\ 0 & otherwise \end{cases}$$

where S^d is the d-th symmetric power.

The two last results of the previous proposition were conjectured by Randal-Williams in [13] which motivated the present work. More precisely, combining this proposition with the stable range of such homology groups obtained by Randal-Williams and Wahl in [14] we obtain the conjecture of Randal-Williams in [13, Corollary 6.4]. The previous proposition is also obtained in a recent work of Randal-Williams [12] using independent topological methods. Proposition 0.4 answers partially to Problem 17 asked by Morita in [11].

Last section contains also the following proposition used in the application section of [3]. Let q_n be the left adjoint of the inclusion functor from polynomial functors of degree $\leq n$ to $\mathcal{F}(\mathbf{gr})$ and $\bar{P}$ in $\mathcal{F}(\mathbf{gr})$ such that $\mathbb{Z}[\mathbf{gr}(\mathbb{Z},-)]\simeq\mathbb{Z}\oplus\bar{P}$, we have:

Proposition 0.5. *Let m and n be integers such that $m\geq n>0$. We have:*

$$Ext^{m-n}(q_n(\bar{P}), T^m\circ\mathfrak{a})\neq 0.$$

Notation: We denote by **Gr** the category of groups, **Ab** the category of abelian groups , **gr** (resp. **ab**), a small skeleton of the full subcategory of **Gr** (resp. **Ab**), having as objects finitely generated free objects. For example, the objects of **gr** are the groups $\mathbb{Z}^{*n}$ for $n\in\mathbb{N}$ where $*$ is the free product.

We denote by $\mathfrak{a}:\mathbf{gr}\to\mathbf{ab}$ the abelianization functor.

Let $\mathcal{C}$ be a small pointed category having finite coproduct. We denote by $\mathcal{F}(\mathcal{C})$ the category of functors from $\mathcal{C}$ to **Ab**. This category is abelian and has enough projective and injective objects. We denote by P_n the representable functor $P_n:=\mathbb{Z}[\mathbf{gr}(\mathbb{Z}^{*n},-)]$ in $\mathcal{F}(\mathbf{gr})$. A *reduced* functor $F\in\mathcal{F}(\mathcal{C})$ satisfies $F(0)=0$. We denote by $\bar{P}$ the reduced part of P_1 (i.e. $P_1\simeq\mathbb{Z}\oplus\bar{P}$).

For $d\in\mathbb{N}$ we denote by $\mathcal{F}_d(\mathbf{gr})$ the full subcategory of $\mathcal{F}(\mathbf{gr})$ of polynomial functors of degree $\leq d$ and $i_d:\mathcal{F}_d(\mathbf{gr})\to\mathcal{F}(\mathbf{gr})$ the inclusion functor. This functor has a left adjoint denoted by q_d. The functors $q_i(\bar{P})$ are called Passi functors (see [9]). For $G\in\mathbf{gr}$ we have a natural isomorphism $q_i(\bar{P})(G)\simeq IG/I^{i+1}G$ where IG is the augmentation ideal (see, for example [3, Proposition 3.7]).

In the following, we sometimes write $Ext^*(F,G)$ instead of $Ext^*_{\mathcal{F}(\mathbf{gr})}(F,G)$ to shorten formulas.

We denote by $g\mathbf{Ab}^-$ the symmetric monoidal category whose objects are graded abelian groups $V^\bullet=\bigoplus_{i\in\mathbb{Z}}V^i$ and morphisms are linear maps

preserving the grading. The symmetric monoidal structure is given by the tensor product. The commutativity isomorphism $V^\bullet \otimes W^\bullet \to W^\bullet \otimes V^\bullet$ is defined by:

$$v \otimes w \mapsto (-1)^{|v|.|w|} w \otimes v.$$

1. Computation of $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^m \circ \mathfrak{a})$

This section is based on the existence of an explicit projective resolution of $\mathfrak{a}$ in $\mathcal{F}(\mathbf{gr})$. This resolution is already mentioned in [10, Proposition 5.1] and used in [5].

Consider the following simplicial object in $\mathcal{F}(\mathbf{gr})$:

$$\ldots \overrightarrow{\overrightarrow{\cdots}} P_{n+1} \overrightarrow{\overrightarrow{\cdots}} P_n \overrightarrow{\overrightarrow{\cdots}} \ldots \overrightarrow{\overrightarrow{\overrightarrow{\overrightarrow{}}}} P_2 \overrightarrow{\overrightarrow{\overrightarrow{}}} P_1 \overrightarrow{\overrightarrow{}} P_0 \tag{1.0.1}$$

where $\delta_i : P_{n+1} \to P_n$ for $0 \leq i \leq n+1$ are defined by:

$$\delta_0[g_1, g_2, \ldots, g_n, g_{n+1}] = [g_2, \ldots, g_n, g_{n+1}]$$

$$\delta_i[g_1, g_2, \ldots, g_n, g_{n+1}] = [g_1, \ldots, g_i g_{i+1}, \ldots, g_n, g_{n+1}] \quad \text{for} \quad 1 \leq i \leq n$$

$$\delta_{n+1}[g_1, g_2, \ldots, g_n, g_{n+1}] = [g_1, g_2, \ldots, g_n]$$

and $\epsilon_i : P_n \to P_{n+1}$ for $1 \leq i \leq n+1$ are defined by:

$$\epsilon_i[g_1, \ldots, g_n] = [g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n].$$

We denote by $C_\bullet$ the unnormalized chain complex associated to this simplicial object and $D_\bullet$ the complex defined by $D_i = C_{i+1}$ for $i \geq 0$ and $D_i = 0$ for $i < 0$.

Recall that the homology of a free group is naturally isomorphic to its abelianization in degree 1 and is zero in degree > 1. So $D_\bullet$ is a resolution of $\mathfrak{a}$ and we obtain:

Proposition 1.1. *(Cf.* [10, Proposition 5.1]*) The exact sequence in* $\mathcal{F}(\boldsymbol{gr})$*:*

$$\ldots P_{n+1} \xrightarrow{d_n} P_n \to \ldots \to P_2 \xrightarrow{d_1} P_1$$

is a projective resolution of the abelianization functor $\mathfrak{a} : \boldsymbol{gr} \to \boldsymbol{Ab}$*. The natural transformation* $d_n : P_n \to P_{n-1}$ *is given on a group* G *by the linear map* $\mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ *such that:*

$$d_n([g_1, \ldots, g_{n+1}]) = [g_2, \ldots, g_{n+1}]$$

$$+\sum_{i=1}^{n} (-1)^i [g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}] + (-1)^{n+1} [g_1, \ldots, g_n]$$

for all $(g_1, \ldots, g_{n+1}) \in G^{n+1}$*.*

This explicit projective resolution play a crucial rôle in [5].

For $F : \mathbf{gr} \to \mathbf{Ab}$ and n an integer, we have

$$Ext^n_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, F) = \pi^{n+1}(F(\mathbb{Z}^{*\bullet})) = H^{n+1}(NF(\mathbb{Z}^{*\bullet}))$$

where $F(\mathbb{Z}^{*\bullet})$ is the cosimplicial abelian group obtained when we apply $Hom_{\mathcal{F}(\mathbf{gr})}(-,F)$ to the simplicial object 1.0.1 and using the natural isomorphism $Hom_{\mathcal{F}(\mathbf{gr})}(P_n,F)\simeq F(\mathbb{Z}^{*n})$ and $NF(\mathbb{Z}^{*\bullet})$ is the normalized cochain complex.

Proposition 1.2. *Let $m\geq 1$ be a natural integer, we have an isomorphism:*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})\simeq\begin{cases}\mathbb{Z} & \text{if } *=m-1\\ 0 & \text{otherwise}\end{cases}$$

The proof of this proposition is based on the following lemma.

Lemma 1.3. *If $F:\boldsymbol{gr}\to\boldsymbol{Ab}$ and $G:\boldsymbol{gr}\to\boldsymbol{Ab}$ are functors, then there is a graded morphism*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},F)\otimes Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},G)\to Ext^{*+1}_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},F\otimes G)$$

which is an isomorphism if F and G take values in finitely generated free abelian groups and $Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},F)$ is torsion free.

Proof. Let $(F\otimes G)(\mathbb{Z}^{*\bullet})$ be the cosimplicial abelian group obtained when we apply $Hom_{\mathcal{F}(\mathbf{gr})}(-,F\otimes G)$ to the cosimplicial object 1.0.1. We have

$$Hom_{\mathcal{F}(\mathbf{gr})}(P_n,F\otimes G)\simeq(F\otimes G)(n)=F(n)\otimes G(n).$$

So $(F\otimes G)(\mathbb{Z}^{*\bullet})=F(\mathbb{Z}^{*\bullet})\times G(\mathbb{Z}^{*\bullet})$ where $\times$ denote the product of cosimplicial abelian groups. By the Eilenberg-Zilber theorem, the map

$$N(F(\mathbb{Z}^{*\bullet})\times G(\mathbb{Z}^{*\bullet}))\to NF(\mathbb{Z}^{*\bullet})\otimes NG(\mathbb{Z}^{*\bullet})$$

is a quasi-isomorphism. So, we have isomorphisms

$$Ext^{n+1}(\mathfrak{a},F\otimes G)\simeq H^{n+1}\big(N(F(\mathbb{Z}^{*\bullet})\times G(\mathbb{Z}^{*\bullet}))\big)\simeq H^{n+1}(NF(\mathbb{Z}^{*\bullet})\otimes NG(\mathbb{Z}^{*\bullet})).$$

Since F and G take values in finitely generated free abelian groups, the complexes $NF(\mathbb{Z}^{*\bullet})$ and $NG(\mathbb{Z}^{*\bullet})$ are finitely generated free so, by the Künneth formula (see [16] Theorem 5.5.11), there is a natural short exact sequence

$$0\to Ext^*(\mathfrak{a},F)\otimes Ext^*(\mathfrak{a},G)\to H^*(NF(\mathbb{Z}^{*\bullet})\otimes NG(\mathbb{Z}^{*\bullet}))\to Tor^1_{\mathbb{Z}}(Ext^*(\mathfrak{a},F),Ext^*(\mathfrak{a},G))\to 0$$

We deduce the isomorphism of the statement from the torsion free hypothesis. □

Proof of Proposition 1.2. We prove the result by induction on m. To start the induction, $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$ is the homology of the complex:

$$\ldots\longleftarrow\mathbb{Z}^{n+1}\xleftarrow{\delta_n}\mathbb{Z}^n\longleftarrow\ldots\longleftarrow\mathbb{Z}^2\xleftarrow{\delta_1}\mathbb{Z}$$

which is trivial for $*\geq 1$ and is isomorphic to $\mathbb{Z}$ for $*=0$. In particular $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$ is torsion free. Assume that the statement is true for m. Applying Lemma 1.3 to $\mathfrak{a}$ and $T^m\circ\mathfrak{a}$ which take finitely generated free values, we have a graded isomorphism:

$$Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},T^{m+1}\circ\mathfrak{a})\simeq Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})\otimes Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$$

and we obtain the result by the inductive step and the computation of $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$.

□

Remark 1.4. Proposition 1.2 is used in [3] to prove that the global dimension of the category of polynomial functors of degree $\leq m$ from $\mathbf{gr}$ to the category of $\mathbb{Q}$-vector spaces has global dimension $m-1$ (see [3, Proposition 4.6]).

Remark 1.5. Note that

$$Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},T^0\circ\mathfrak{a})\simeq Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathbb{Z})=0$$

The symmetric group $\mathfrak{S}_m$ acts on T^m by permuting the factors of the tensor product. This action induces an action of $\mathfrak{S}_m$ on the extension groups $Ext^{m-1}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})$ that we make explicit in the following proposition.

Proposition 1.6. *The symmetric group $\mathfrak{S}_m$ acts on $Ext^{m-1}_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})$ by the following way: for $\sigma\in\mathfrak{S}_m$ and $E\in Ext^{m-1}_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})$*

$$E.\sigma=\epsilon(\sigma)E$$

where ϵ is the signature.

The proof of this proposition relies on the following wellknown lemma (see [1, X 62]).

Lemma 1.7. *Let A be a commutative ring, (C,d) and (C',d') be complexes of A-modules. The map $\sigma(C,C'):C\underset{A}{\otimes}C'\to C'\underset{A}{\otimes}C$ given by $\sigma(C,C')(x\otimes x')=(-1)^{|x|\ |x'|}x'\otimes x$ is an isomorphism of complexes.*

Proof of Proposition 1.6. Since $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$ is torsion free by Proposition 1.2 and $\mathfrak{a}$ takes finitely generated free values, by applying iteratively Lemma 1.3 for $F=\mathfrak{a}$ and $G=\mathfrak{a}$ we obtain that the morphism

$$\psi:\bigotimes_{i=1}^{m}Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})\to Ext^{*+m-1}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})$$

is an isomorphism.

Let $\sigma\in\mathfrak{S}_m$, the action of σ on $Ext^{m-1}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})$ is given by the following composition where $\phi:\bigotimes_{i=1}^{m}Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})\to\bigotimes_{i=1}^{m}Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$ is given by the action of σ on $\bigotimes_{i=1}^{m}Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$

$$Ext^{*+m-1}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})\xrightarrow[\psi^{-1}]{\simeq}\bigotimes_{i=1}^{m}Ext^*(\mathfrak{a},\mathfrak{a})\xrightarrow{\phi}\bigotimes_{i=1}^{m}Ext^*(\mathfrak{a},\mathfrak{a})\xrightarrow[\psi]{\simeq}Ext^{*+m-1}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},T^m\circ\mathfrak{a})$$

By Lemma 1.7, we have

$$\phi(x_1\otimes\ldots\otimes x_n)=\epsilon(\sigma)x_{\sigma(1)}\otimes\ldots\otimes x_{\sigma(m)}$$

where x_k is a generator of the k-th copy of $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$ in $\bigotimes_{i=1}^{m} Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a},\mathfrak{a})$.

□

We denote by $\Lambda(m)[m-1]$ the symmetric sequence of objects of $g\mathbf{Ab}^-$ which is equal to the sign representation of $\mathfrak{S}_m$ placed in degree $m-1$ and 0 in other degrees. It follows from Propositions 1.2 and 1.6 that $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^m\circ\mathfrak{a}) = \Lambda(m)[m-1]$.

2. Computation of $Ext^*_{\mathcal{F}(\mathbf{gr})}(T^n\circ\mathfrak{a}, T^m\circ\mathfrak{a})$

This section is based on the sum-diagonal adjunction, the exponential type property of tensor powers and the Künneth formula.

We begin this section with some recollections on these ingredients. We refer the reader to [4, Appendice B] for more details.

For F and G in $\mathcal{F}(\mathbf{gr})$, their external tensor product $F\boxtimes G$ is the functor sending (X,Y) to $F(X)\otimes G(Y)$. This yields a functor:

$$-\boxtimes- : \mathcal{F}(\mathbf{gr})\times\mathcal{F}(\mathbf{gr})\to\mathcal{F}(\mathbf{gr}\times\mathbf{gr}).$$

We denote by $\pi_d : \mathbf{gr}^{\times d}\to\mathbf{gr}$ the functor obtained by iteration of the free product (which is the categorical sum in $\mathbf{gr}$) and $\delta_d : \mathbf{gr}\to\mathbf{gr}^{\times d}$ the diagonal functor. The functor δ_d is right adjoint of the functor π_d. We deduce that the functor given by precomposition $\delta_d^* : \mathcal{F}(\mathbf{gr}^{\times d})\to\mathcal{F}(\mathbf{gr})$ is left adjoint of the functor given by precomposition $\pi_d^* : \mathcal{F}(\mathbf{gr})\to\mathcal{F}(\mathbf{gr}^{\times d})$.

The tensor power (graded) functor $T^\bullet$ is not exponential but we have a similar property which is useful in the following computations:

$$\pi_n^*(T^m\circ\mathfrak{a}) \simeq \bigoplus_{i_1+\ldots+i_n=m} (T^{i_1}\circ\mathfrak{a}\boxtimes\ldots\boxtimes T^{i_n}\circ\mathfrak{a}) \underset{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m] \tag{2.0.1}$$

where $T^0\circ\mathfrak{a}\simeq\mathbb{Z}$.

The following lemmas will be useful below.

Lemma 2.1. *Let $Surj(m,n)$ be the set of surjections from the set having m elements to the one having n elements. For $f\in Surj(m,n)$ such that for all $k\in\{1,\ldots,n\}$, $|f^{-1}(k)| = i_k$ there are unique maps $\alpha\in Surj(n,n)$ and $s\in Surj(m,n)$ such that $f = s\circ\alpha$ where α is the inverse of a $(i_1,\ldots,i_n)$-shuffle and s is an order preserving surjection.*

Lemma 2.2. *If $F : \boldsymbol{gr}\to\boldsymbol{Ab}$ and $G : \boldsymbol{gr}\to\boldsymbol{Ab}$ are functors, then there is a graded morphism*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},F)\otimes Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a},G)\to Ext^*_{\mathcal{F}(\boldsymbol{gr}\times\boldsymbol{gr})}(\mathfrak{a}\boxtimes\mathfrak{a}, F\boxtimes G)$$

which is an isomorphism if F and G take values in finitely generated free abelian groups and $Ext^*(\mathfrak{a},F)$ is torsion free.

Proof. There is a canonical isomorphism $P^{\mathbf{gr}}_A\boxtimes P^{\mathbf{gr}}_B\simeq P^{\mathbf{gr}\times\mathbf{gr}}_{(A,B)}$ so

$$Hom_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr})}(P^{\mathbf{gr}}_A\boxtimes P^{\mathbf{gr}}_B, F\boxtimes G)\simeq F(A)\otimes G(B).$$

Then the statement is a consequence of the Künneth formula. □

Our first main result is the following theorem.

Theorem 2.3. *Let n and m be integers, we have an isomorphism:*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \begin{cases} \mathbb{Z}[Surj(m,n)] & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

where $Surj(m,n)$ is the set of surjections from the set having m elements to the one having n elements.

Proof.

$$\begin{aligned}
& Ext^*_{\mathcal{F}(\mathbf{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \\
\simeq\ & Ext^*_{\mathcal{F}(\mathbf{gr}\times\ldots\times\mathbf{gr})}(\mathfrak{a}^{\boxtimes n}, T^m \circ \mathfrak{a} \circ \pi_n) \text{ by the sum-diagonal adjunction} \\
\simeq\ & Ext^*_{\mathcal{F}(\mathbf{gr}\times\ldots\times\mathbf{gr})}\big(\mathfrak{a}^{\boxtimes n}, \bigoplus_{i_1+\ldots+i_n=m} (T^{i_1}\circ\mathfrak{a}\boxtimes\ldots\boxtimes T^{i_n}\circ\mathfrak{a}) \underset{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m]\big) \text{ by 2.0.1} \\
\simeq\ & \bigoplus_{i_1+\ldots+i_n=m} Ext^*_{\mathcal{F}(\mathbf{gr}\times\ldots\times\mathbf{gr})}\big(\mathfrak{a}^{\boxtimes n}, (T^{i_1}\circ\mathfrak{a}\boxtimes\ldots\boxtimes T^{i_n}\circ\mathfrak{a}) \underset{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m]\big) \\
\simeq\ & \bigoplus_{i_1+\ldots+i_n=m} \big(\bigotimes_{k=1}^{n} Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^{i_k}\circ\mathfrak{a})\big) \underset{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m] \text{ by Lemma 2.2} \\
\simeq\ & \begin{cases} \bigoplus\limits_{\substack{i_1+\ldots+i_n=m \\ i_k>0,\forall k}} \big(\mathbb{Z} \underset{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m]\big) & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

where the last isomorphism follows from Proposition 1.2 and Remark 1.5.

Let $f \in Surj(m,n)$, by Lemma 2.1 f admits a unique decomposition of the form $f = s\circ\alpha$ where α is the inverse of a $(|f^{-1}(1)|,\ldots,|f^{-1}(n)|)$-shuffle and s is an order preserving surjection. So we deduce that the map:

$$\kappa : \mathbb{Z}[Surj(m,n)] \to \bigoplus_{\substack{i_1+\ldots+i_n=m \\ i_k>0,\forall k}} \big(\mathbb{Z} \underset{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m]\big)$$

given by $\kappa([f]) = 1\otimes\alpha \in \mathbb{Z} \underset{\mathfrak{S}_{|f^{-1}(1)|}\times\ldots\times\mathfrak{S}_{|f^{-1}(n)|}}{\otimes} \mathbb{Z}[\mathfrak{S}_m]$ is an isomorphism.

□

Remark 2.4. The previous theorem is used in [3] to prove that the homological dimension of the functor $T^n\circ\mathfrak{a}$ in the category of polynomial functors of degree $\leq m$ from **gr** to **Ab** is $m-n$ (see [3, Proposition 4.1]).

For A_n and B_n ordered sets of cardinality n and $f : A_n \to B_n$ a bijection. We denote by $\bar{f} : B_n \to B_n$ the unique permutation of B_n such that $\bar{f}\circ u = f$ where $u : A_n \to B_n$ is the unique order preserving map.

In the following proposition we make explicit the actions of symmetric groups on $Ext^*_{\mathcal{F}(\mathbf{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a})$.

Proposition 2.5. *The symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$ act on $Ext^{m-n}_{\mathcal{F}(\boldsymbol{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \mathbb{Z}[Surj(m,n)]$ by the following way: for $\sigma \in \mathfrak{S}_m$, $\tau_{k,l} \in \mathfrak{S}_n$ the transposition of k and l where $k, l \in \{1, \dots, n\}$ and $f \in Surj(m,n)$*

$$[f].\sigma = \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f\circ\sigma)^{-1}(i)}})[f \circ \sigma]$$

where $\sigma_{|(f\circ\sigma)^{-1}(i)} : (f \circ \sigma)^{-1}(i) \to \sigma((f \circ \sigma)^{-1}(i))$

$$\tau_{k,l}.[f] = (-1)^{(|f^{-1}(k)|-1)(|f^{-1}(l)|-1)}[\tau_{k,l} \circ f].$$

Proof. Let f be an element of $Surj(m,n)$ such that $f = s \circ \sigma$ is the unique decomposition of f of the form described in Lemma 2.1.

For $\sigma \in \mathfrak{S}_m$, the action of σ on $\mathbb{Z}[Surj(m,n)]$ is given by the following composition, where ϕ is the map given by the action of σ on $\bigoplus_{i_1+\dots+i_n=m} (\overset{n}{\underset{k=1}{\otimes}} Ext^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k})) \underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m])$

$$\begin{array}{ccc}
\mathbb{Z}[Surj(m,n)] & \xrightarrow[\kappa]{\simeq} & \bigoplus_{i_1+\dots+i_n=m} \mathbb{Z} \underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m] \\
 & & \downarrow \simeq \\
 & & \bigoplus_{i_1+\dots+i_n=m} (\overset{n}{\underset{k=1}{\otimes}} Ext^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k})) \underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m]) \\
 & & \downarrow \phi \\
 & & \bigoplus_{i_1+\dots+i_n=m} (\overset{n}{\underset{k=1}{\otimes}} Ext^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k})) \underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m]) \\
 & & \downarrow \simeq \\
\mathbb{Z}[Surj(m,n)] & \xleftarrow[\kappa^{-1}]{} & \bigoplus_{i_1+\dots+i_n=m} \mathbb{Z} \underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes} \mathbb{Z}[\mathfrak{S}_m]
\end{array}$$

we have

$$\kappa^{-1} \circ \phi \circ \kappa([f]) = \kappa^{-1} \circ \phi \circ \kappa([s \circ \alpha]) = \kappa^{-1} \circ \phi \left((1 \otimes [\alpha])_{|f^{-1}(1)|,\dots,|f^{-1}(n)|}\right)$$

$$= \kappa^{-1}\Big(\prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f\circ\sigma)^{-1}(i)}})\ (1 \otimes [\alpha \circ \sigma])_{|f^{-1}(1)|,\dots,|f^{-1}(n)|}\Big) = \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f\circ\sigma)^{-1}(i)}})[s \circ \alpha \circ \sigma]$$

$$= \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f\circ\sigma)^{-1}(i)}})[f \circ \sigma]$$

where we deduce the third equality from Proposition 1.6 and where $-_{i_1,\dots,i_n}$ corresponds to the summand $i_1, \dots, i_n$ in the sum $\bigoplus_{i_1+\dots+i_n=m}$.

For $\tau_{k,l} \in \mathfrak{S}_n$, we have $\tau_{k,l} \circ f = \tau_{k,l} \circ s \circ \alpha = s' \circ T_{k,l} \circ \alpha$ where s' is the order preserving surjection from m to n such that $|s'^{-1}(l)| = |s^{-1}(k)|$, $|s'^{-1}(k)| = |s^{-1}(l)|$ and $|s'^{-1}(q)| = |s'^{-1}(q)|$ for $q \in \{1, \dots, m\} \setminus \{k, l\}$ and $T_{k,l}$ is the permutation obtained by the bloc transposition of $s^{-1}(k)$

and $s^{-1}(l)$. Note that $T_{k,l}\circ\alpha$ is the inverse of a $(|(\tau_{k,l}\circ f)^{-1}(1)|,\dots,|(\tau_{k,l}\circ f)^{-1}(n)|)$-shuffle.

For $\tau_{k,l}\in\mathfrak{S}_n$, the action of $\tau_{k,l}$ on $\mathbb{Z}[Surj(m,n)]$ is given by the following composition, where ϕ' is the map given by the action of $\tau_{k,l}$ on $\bigoplus_{i_1+\dots+i_n=m}(\overset{n}{\underset{k=1}{\otimes}} Ext^{i_k-1}(\mathfrak{a},\mathfrak{a}^{\otimes i_k}))\underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes}\mathbb{Z}[\mathfrak{S}_m])$

$$\begin{array}{ccc}
\mathbb{Z}[Surj(m,n)] & \xrightarrow[\kappa]{\simeq} & \bigoplus_{i_1+\dots+i_n=m}\mathbb{Z}\underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes}\mathbb{Z}[\mathfrak{S}_m]\\
 & & \downarrow\simeq\\
 & & \bigoplus_{i_1+\dots+i_n=m}(\overset{n}{\underset{k=1}{\otimes}} Ext^{i_k-1}(\mathfrak{a},\mathfrak{a}^{\otimes i_k}))\underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes}\mathbb{Z}[\mathfrak{S}_m])\\
 & & \downarrow\phi'\\
 & & \bigoplus_{i_1+\dots+i_n=m}(\overset{n}{\underset{k=1}{\otimes}} Ext^{i_k-1}(\mathfrak{a},\mathfrak{a}^{\otimes i_k}))\underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes}\mathbb{Z}[\mathfrak{S}_m])\\
 & & \downarrow\simeq\\
\mathbb{Z}[Surj(m,n)] & \xleftarrow[\kappa^{-1}]{} & \bigoplus_{i_1+\dots+i_n=m}\mathbb{Z}\underset{\mathfrak{S}_{i_1}\times\dots\times\mathfrak{S}_{i_n}}{\otimes}\mathbb{Z}[\mathfrak{S}_m]
\end{array}$$

we have

$$\kappa^{-1}\circ\phi'\circ\kappa([f])=\kappa^{-1}\circ\phi'\circ\kappa([s\circ\alpha])=\kappa^{-1}\circ\phi'\left((1\otimes[\alpha])_{|f^{-1}(1)|,\dots,|f^{-1}(n)|}\right)$$
$$=\kappa^{-1}((-1)^{(i_k-1)(i_l-1)}1\otimes[T_{k,l}\circ\alpha]_{|(\tau_{k,l}\circ f)^{-1}(1)|,\dots,|(\tau_{k,l}\circ f)^{-1}(n)|})$$
$$=(-1)^{(i_k-1)(i_l-1)}[s'\circ\alpha']=(-1)^{(i_k-1)(i_l-1)}[\tau_{k,l}\circ f]$$

where the third equality is a consequence of Lemma 1.7. □

3. Products and the structure of PROP

The aim of this section is to explicit the Yoneda product and the external product on $Ext^*_{\mathcal{F}(\mathbf{gr})}(T^n\circ\mathfrak{a},T^m\circ\mathfrak{a})$ and to deduce the PROP governing the graded groups $Ext^*_{\mathcal{F}(\mathbf{gr})}(T^n\circ\mathfrak{a},T^m\circ\mathfrak{a})$.

Proposition 3.1. (1) *The Yoneda product:*

$$\mathcal{Y}: Ext^{m-l}_{\mathcal{F}(\boldsymbol{gr})}(T^l\circ\mathfrak{a},T^m\circ\mathfrak{a})\otimes Ext^{n-m}_{\mathcal{F}(\boldsymbol{gr})}(T^m\circ\mathfrak{a},T^n\circ\mathfrak{a})\to Ext^{n-l}_{\mathcal{F}(\boldsymbol{gr})}(T^l\circ\mathfrak{a},T^n\circ\mathfrak{a})$$

is induced, via the isomorphism obtained in Theorem 2.3, by the map

$$Y:\mathbb{Z}[Surj(m,l)]\otimes\mathbb{Z}[Surj(n,m)]\to\mathbb{Z}[Surj(n,l)]$$

given by

$$Y([g]\otimes[f])=\prod_{k=1}^{l}\epsilon(\overline{\tau_{i,j\,|g^{-1}(k)}})(-1)^{(|f^{-1}(i)|-1)(|f^{-1}(j)|-1)}[g\circ f]$$

where $g \in \mathbb{Z}[Surj(m,l)]$, $f \in \mathbb{Z}[Surj(n,m)]$ *and* $g = s \circ \tau_{i,j}$ *where* s *is an order preserving surjection and* $\tau_{i,j} \in \mathfrak{S}_m$ *denote the transposition of* i *and* j.

(2) *The external product*

$$Ext^{m-l}_{\mathcal{F}(\boldsymbol{gr})}(T^l\circ\mathfrak{a}, T^m\circ\mathfrak{a})\otimes Ext^{n-p}_{\mathcal{F}(\boldsymbol{gr})}(T^p\circ\mathfrak{a}, T^n\circ\mathfrak{a}) \to Ext^{m+n-l-p}_{\mathcal{F}(\boldsymbol{gr})}(T^{l+p}\circ\mathfrak{a}, T^{m+n}\circ\mathfrak{a})$$

is induced by the disjoint union of sets via the isomorphism obtained in Theorem 2.3.

Proof. (1) Considering the action of $Hom_{\mathcal{F}(\mathbf{gr})}(T^m\circ\mathfrak{a}, T^m\circ\mathfrak{a}) \simeq \mathbb{Z}[\mathfrak{S}_m]$ on $Ext^{m-l}_{\mathcal{F}(\mathbf{gr})}(T^l\circ\mathfrak{a}, T^m\circ\mathfrak{a})$ and on $Ext^{n-m}_{\mathcal{F}(\mathbf{gr})}(T^m\circ\mathfrak{a}, T^n\circ\mathfrak{a})$ we obtain that the map

$$Ext^{m-l}_{\mathcal{F}(\mathbf{gr})}(T^l\circ\mathfrak{a}, T^m\circ\mathfrak{a})\otimes Ext^{n-m}_{\mathcal{F}(\mathbf{gr})}(T^m\circ\mathfrak{a}, T^n\circ\mathfrak{a}) \to Ext^{n-l}_{\mathcal{F}(\mathbf{gr})}(T^l\circ\mathfrak{a}, T^n\circ\mathfrak{a})$$

induces a map:

$$Ext^{m-l}(T^l\circ\mathfrak{a}, T^m\circ\mathfrak{a}) \underset{\mathfrak{S}_m}{\otimes} Ext^{n-m}(T^m\circ\mathfrak{a}, T^n\circ\mathfrak{a}) \to Ext^{n-l}(T^l\circ\mathfrak{a}, T^n\circ\mathfrak{a}).$$

So the signs can easily be deduced from Proposition 2.5.

(2) We have

$$Ext^{m-l}(T^l\circ\mathfrak{a}, T^m\circ\mathfrak{a}) \simeq \bigoplus_{i_1+\ldots+i_l=m} (\overset{l}{\underset{k=1}{\otimes}} Ext^{i_l-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_l}) \bigotimes_{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_l}} \mathbb{Z}[\mathfrak{S}_m])$$

$$Ext^{n-p}(T^p\circ\mathfrak{a}, T^n\circ\mathfrak{a}) \simeq \bigoplus_{j_1+\ldots+j_p=n} (\overset{p}{\underset{k=1}{\otimes}} Ext^{j_p-1}(\mathfrak{a}, \mathfrak{a}^{\otimes j_p}) \bigotimes_{\mathfrak{S}_{j_1}\times\ldots\times\mathfrak{S}_{j_p}} \mathbb{Z}[\mathfrak{S}_n])$$

and

$$Ext^{m+n-l-p}(T^{l+p}\circ\mathfrak{a}, T^{m+n}\circ\mathfrak{a}) \simeq \bigoplus_{i_1+\ldots+i_{l+p}=m+n} (\overset{l+p}{\underset{k=1}{\otimes}} Ext^{i_{l+p}-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_{l+p}}) \bigotimes_{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_{l+p}}} \mathbb{Z}[\mathfrak{S}_{m+n}])$$

For fixed $i_1,\ldots,i_l,j_1,\ldots j_k$ such that $i_1+\ldots+i_l=m$ and $j_1+\ldots+j_p=n$, $E_k \in Ext^{i_k-1}(\mathfrak{a},\mathfrak{a}^{\otimes i_k})$ and $F_\alpha \in Ext^{j_\alpha-1}(\mathfrak{a},\mathfrak{a}^{\otimes j_\alpha})$ and $\sigma\in\mathfrak{S}_n$ and $\sigma'\in\mathfrak{S}_m$ the external product is given by the map

$$E_1\otimes\ldots E_l\otimes\sigma\otimes F_1\otimes\ldots\otimes F_p\otimes\sigma' \mapsto E_1\otimes\ldots\otimes E_l\otimes F_1\otimes\ldots\otimes F_p\otimes\sigma\vee\sigma'$$

where $\sigma\vee\sigma' : m+n \to m+n$ is the permutation such that $\sigma\vee\sigma'(i)=\sigma(i)$ for $1\le i\le n$ and $\sigma\vee\sigma'(i)=\sigma'(i)$ for $n+1\le i\le n+m$. This map corresponds to the disjoint union via the isomorphism obtained in Theorem 2.3.

□

For $i_1,\ldots,i_k$ integers, iterating the external product, we obtain a graded map

$$\mathcal{E}^{i_1,\ldots,i_k} : Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^{i_1}\circ\mathfrak{a})\otimes\ldots\otimes Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^{i_k}\circ\mathfrak{a}) \to Ext^*_{\mathcal{F}(\mathbf{gr})}(T^k\circ\mathfrak{a}, T^{i_1+\ldots+i_k}\circ\mathfrak{a}).$$

This map is $\mathfrak{S}_{i_1} \times \ldots \times \mathfrak{S}_{i_k}$-equivariant by Propositions 1.6 and 2.5. It induces a map

$$\bigoplus_{\substack{i_1+\ldots+i_k=m \\ i_p>0,\forall p}} \bigotimes_{p=1}^{k} Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^{i_p}\circ\mathfrak{a}) \underset{\mathfrak{S}_{i_1}\times\ldots\times\mathfrak{S}_{i_k}}{\otimes} \mathbb{Z}[\mathfrak{S}_m] \to Ext^*_{\mathcal{F}(\mathbf{gr})}(T^k\circ\mathfrak{a}, T^m\circ\mathfrak{a})$$

which is an isomorphism by the exponential property (see the proof of Theorem 2.3). We deduce that the elements of $Ext^*_{\mathcal{F}(\mathbf{gr})}(T^k \circ \mathfrak{a}, T^m \circ \mathfrak{a})$ are obtained from elements of $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^i \circ \mathfrak{a})$ using the external product.

By Proposition 3.1 (2), the graded category $\Omega(\Lambda)$ having as objects natural numbers and where the morphisms from m to l are given by the groups $Ext^*_{\mathcal{F}(\mathbf{gr})}(T^l \circ \mathfrak{a}, T^m \circ \mathfrak{a})$ is a graded PROP (i.e. a graded symmetric monoidal category with objects the natural numbers whose symmetric monoidal structure is given by the sum of integer). In the following, we prove that this graded PROP is the PROP associated to the following operad.

Proposition 3.2. *Let $\Lambda = \{\Lambda(n)\}_{n\geq 0}$ be the graded symmetric sequence given by $\Lambda(n) = Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a}, T^n \circ \mathfrak{a})$. The maps*

$$\gamma(i_1, \ldots, i_k) : \Lambda(k) \otimes \Lambda(i_1) \otimes \ldots \Lambda(i_k) \to \Lambda(i_1 + \ldots + i_k)$$

given by

$$\gamma(i_1, \ldots, i_k)(x, x_{i_1}, \ldots, x_{i_k}) = \mathcal{Y}(x, \mathcal{E}^{i_1,\ldots,i_k}(x_{i_1}, \ldots, x_{i_k}))$$

for $x \in \Lambda(k)$ and $x_{i_l} \in \Lambda(i_l)$, define an operad structure on Λ.

Proof. Using Proposition 1.6, we verify easily that γ is associative, unital and equivariant. □

Remark 3.3. Notice the similarity between the operad Λ and the *determinant operad* introduced by Ginzburg and Kapranov in [8, 1.3.21]. The *determinant operad* $\mathcal{P}$ is the operad in $g\mathbf{Ab}^-$ such that $\mathcal{P}(n) = \Lambda(n)[1-n]$ (the sign representation of $\mathfrak{S}_n$ placed in degree $(1-n)$).

Recall that we can associate a unique graded PROP $\mathcal{C}$ to an operad P in a graded symmetric monoidal category such that $\mathcal{C}(1, n) = P(n)$. We refer the reader to [9, Definition 1.5] for the concrete description of the composition in this PROP in the ungraded case, which can be easily extend to the graded case. However there exists PROPs which do not come from an operad.

Theorem 2.3 and Proposition 3.1 imply the following proposition:

Proposition 3.4. *The graded category $\Omega(\Lambda)$ is the graded PROP associated to the operad Λ.*

4. Applications

In this section, we deduce from Theorem 2.3 the computation of some other Ext-groups between functors from **gr** to **Ab**.

For M an abelian group, recall that $S^n(M) = (T^n(M))_{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ acts by the permutation of variables and $\Lambda^n(M) \otimes \mathbb{Z}[1/2] = (T^n(M) \otimes \mathbb{Z}[1/2])_{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ acts by the permutation of variables and the multiplication by the signature.

Proposition 4.1. *Let n and m be natural integers, we have isomorphisms:*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{\rho(m,n)} & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

where $\rho(m,n)$ denotes the number of partitions of m into n parts.

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((S^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (S^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m \text{ and } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (S^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m = 0 \text{ and } * = 0 \\ & \text{or } n = m = 1 \text{ and } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((S^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m = 0 \text{ and } * = 0 \\ & \text{or } n = 1 \text{ and } * = m-1 \\ 0 & \text{otherwise} \end{cases}$$

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (T^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{S(m,n)} & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((S^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (T^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{S(m,n)} & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

where $S(m,n)$ denotes the Stirling partition number (i.e. the number of ways to partition a set of m elements into n non-empty subsets).

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((T^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (S^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{r(m,n)} & \text{if } m = n \text{ and } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((T^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{r(m,n)} & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

where $r(m,n)$ denotes the number of ordered partition of m into n parts.

The proof of this proposition relies on the following lemma:

Lemma 4.2. *Let G be a group, E a G-set and $\mathbb{Q}[E]$ the $\mathbb{Q}$-vector space having E as a basis. Consider an action of G on $\mathbb{Q}[E]$, given by*

$$g.[e] = \alpha(g,e)[g.e]$$

where $\alpha : G \times E \to \{1,-1\}$ is a map such that, for $e \in E$, $\alpha(-,e)$ is a group morphism, then

$$\mathbb{Q}[E]_G \simeq \mathbb{Q}[\{e \in E \mid \forall g \in G \text{ s.t. } g.e = e \text{ we have } \alpha(g,e) = 1\}/G].$$

Proof of Proposition 4.1. For F and G functors of type T^k, S^k or Λ^k we determine the action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $Ext^*_{\mathcal{F}(\mathbf{gr})}((T^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (T^m \circ \mathfrak{a}) \otimes \mathbb{Q})$ such that

$$Ext^*_{\mathcal{F}(\mathbf{gr})}((F \circ \mathfrak{a}) \otimes \mathbb{Q}, (G \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq Ext^*_{\mathcal{F}(\mathbf{gr})}((T^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (T^m \circ \mathfrak{a}) \otimes \mathbb{Q})_{\mathfrak{S}_n \times \mathfrak{S}_m}$$
$$\simeq \mathbb{Q}[Surj(m,n)]_{\mathfrak{S}_n \times \mathfrak{S}_m}.$$

By Lemma 4.2, it is sufficient to determine this action for the elements $(\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m$ such that $(\sigma', \sigma'').f = f$.

Let f be an element of $Surj(m,n)$ such that $f = s \circ \sigma$ is the unique decomposition of f of the form described in Lemma 2.1. The group $\mathfrak{S}_n \times \mathfrak{S}_m$ acts on $Surj(m,n)$ by $(\sigma', \sigma'').f = \sigma' \circ f \circ \sigma''^{-1}$. The elements $(\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m$ such that $(\sigma', \sigma'').f = f$ are products of elements of the two following forms

- $(\tau_{i,j}, \sigma^{-1} T_{i,j} \sigma)$ for i and j such that $|f^{-1}(i)| = |f^{-1}(j)|$ and $T_{i,j}$ is the permutation obtained by the bloc transposition of $s^{-1}(i)$ and $s^{-1}(j)$;
- $(Id, \sigma_{i_1} \dots \sigma_{i_n})$ where σ_{i_k} is a permutation of $f^{-1}(k)$.

In the following, for F and G functors of type T^k, S^k or Λ^k and $(\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m$ of the previous forms, we determine the map $\alpha : (\mathfrak{S}_n \times \mathfrak{S}_m) \times Surj(m,n) \to \{1,-1\}$ such that $g.[e] = \alpha(g,e)[g.e]$.

- For $F = \Lambda^n$ and $G = \Lambda^m$, we have:

$$\alpha((\tau_{i,j}, \sigma^{-1} T_{i,j} \sigma), f) = \epsilon(\tau_{i,j}) \epsilon(\sigma^{-1} T_{i,j} \sigma) \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma^{-1} T_{i,j} \sigma_{|(f \circ (\sigma^{-1} T_{i,j} \sigma)^{-1}(i))}}) (-1)^{(|f^{-1}(i)|-1)^2} = 1$$

since

$$\epsilon(\sigma^{-1} T_{i,j} \sigma) = \epsilon(T_{i,j}) = (-1)^{|f^{-1}(i)|^2}$$

and

$$\epsilon(\overline{\sigma^{-1} T_{i,j} \sigma_{|(f \circ (\sigma^{-1} T_{i,j} \sigma)^{-1}(i))}}) = 1;$$

and

$$\alpha((Id, \sigma_{i_1} \dots \sigma_{i_n}), f) = \epsilon(\sigma_{i_1} \dots \sigma_{i_n}) \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{i_1} \dots \sigma_{i_n |(f \circ (\sigma_{i_1} \dots \sigma_{i_n})^{-1}(i))}}) = 1$$

since

$$\epsilon(\sigma_{i_1} \dots \sigma_{i_n}) = \epsilon(\sigma_{i_1}) \dots \epsilon(\sigma_{i_n})$$

and

$$\epsilon(\overline{\sigma_{i_1} \dots \sigma_{i_n |(f \circ (\sigma_{i_1} \dots \sigma_{i_n})^{-1}(k))}}) = \epsilon(\sigma_{i_k}).$$

By Lemma 4.2 we deduce that

$$Ext^*_{\mathcal{F}(\mathbf{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \mathbb{Q}[Surj(m,n)]_{\mathfrak{S}_n \times \mathfrak{S}_m} \simeq \mathbb{Q}[Surj(m,n)/\mathfrak{S}_n \times \mathfrak{S}_m]$$

$$\simeq \begin{cases} \mathbb{Q}^{\rho(m,n)} & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

- For $F = S^n$ and $G = S^m$, we have

$$\alpha((\tau_{i,j}, \sigma^{-1}T_{i,j}\sigma), f) = (-1)^{|f^{-1}(i)|^2+1},$$
$$\alpha((Id, \sigma_{i_1} \dots \sigma_{i_n}), f) = \epsilon(\sigma_{i_1}) \dots \epsilon(\sigma_{i_n}).$$

So, if $n \neq m$

$$\{f \in Surj(m,n) \mid \forall (\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m \text{ s.t. } (\sigma', \sigma'').f = f \text{ we have } \alpha((\sigma', \sigma''), f) = 1\} = \emptyset$$

and if $n = m$

$$\{f \in Surj(m,n) \mid \forall (\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m \text{ s.t. } (\sigma', \sigma'').f = f \text{ we have } \alpha((\sigma', \sigma''), f) = 1\}$$
$$= Surj(m,n).$$

- For $F = S^n$ and $G = \Lambda^m$,

$$\alpha((\tau_{i,j}, \sigma^{-1}T_{i,j}\sigma), f) = -1,$$
$$\alpha((Id, \sigma_{i_1} \dots \sigma_{i_n}), f) = 1.$$

So, if $n \neq 1$ or $n = 0$ and $m \neq 0$

$$\{f \in Surj(m,n) \mid \forall (\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m \text{ s.t. } (\sigma', \sigma'').f = f \text{ we have } \alpha((\sigma', \sigma''), f) = 1\} = \emptyset$$

and if $n = 1$ or $n = m = 0$

$$\{f \in Surj(m,n) \mid \forall (\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m \text{ s.t. } (\sigma', \sigma'').f = f \text{ we have } \alpha((\sigma', \sigma''), f) = 1\}$$

is a set of cardinality 1.

- For $F = \Lambda^n$ and $G = S^m$,

$$\alpha((\tau_{i,j}, \sigma^{-1}T_{i,j}\sigma), f) = (-1)^{|f^{-1}(i)|^2},$$
$$\alpha((Id, \sigma_{i_1} \dots \sigma_{i_n}), f) = \epsilon(\sigma_{i_1}) \dots \epsilon(\sigma_{i_n}).$$

- For $F = \Lambda^n$ and $G = T^m$, for $f \in Surj(m,n)$ and $\sigma' \in \mathfrak{S}_n$ we have $\sigma'.f \neq f$, so

$$\mathbb{Q}[Surj(m,n)]_{\mathfrak{S}_n} = \mathbb{Q}[Surj(m,n)/\mathfrak{S}_n] \simeq \mathbb{Q}^{S(m,n)}$$

where $S(m,n)$ denotes the Stirling partition number.

- For $F = S^n$ and $G = T^m$, for $f \in Surj(m,n)$ and $\sigma' \in \mathfrak{S}_n$ we have $\sigma'.f \neq f$, so

$$\mathbb{Q}[Surj(m,n)]_{\mathfrak{S}_n} = \mathbb{Q}[Surj(m,n)/\mathfrak{S}_n] \simeq \mathbb{Q}^{S(m,n)}.$$

- For $F = T^n$ and $G = S^m$. We consider the action of $\mathfrak{S}_m$ on $Ext^*_{\mathcal{F}(\mathbf{gr})}((T^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (T^m \circ \mathfrak{a}) \otimes \mathbb{Q})$. We have:

$$\alpha(\sigma_{i_1} \dots \sigma_{i_n}, f) = \epsilon(\sigma_{i_1}) \dots \epsilon(\sigma_{i_n}).$$

So, if $n \neq m$

$$\{f \in Surj(m,n) \mid \forall \sigma'' \in \mathfrak{S}_m \text{ s.t. } \sigma''.f = f \text{ we have } \alpha(\sigma'', f) = 1\} = \emptyset$$

and if $n = m$

$$\{f \in Surj(m,n) \mid \forall \sigma'' \in \mathfrak{S}_m \text{ s.t. } \sigma''.f = f \text{ we have } \alpha(\sigma'', f) = 1\} = Surj(m,n)$$

so

$$\mathbb{Q}[Surj(m,n)]_{\mathfrak{S}_m} = \mathbb{Q}[Surj(m,n)/\mathfrak{S}_m] \simeq \mathbb{Q}^{r(m,n)}$$

where $r(m,n)$ is the number of ordered partitions of m into n parts.

- For $F = T^n$ and $G = \Lambda^m$. We have:

$$\alpha(\sigma_{i_1} \dots \sigma_{i_n}, f) = (\epsilon(\sigma_{i_1}) \dots \epsilon(\sigma_{i_n}))^2 = 1.$$

So,

$$\{f \in Surj(m,n) \mid \forall \sigma'' \in \mathfrak{S}_m \text{ s.t. } \sigma''.f = f \text{ we have } \alpha(\sigma'', f) = 1\} = Surj(m,n)$$

so

$$\mathbb{Q}[Surj(m,n)]_{\mathfrak{S}_m} = \mathbb{Q}[Surj(m,n)/\mathfrak{S}_m] \simeq \mathbb{Q}^{r(m,n)}$$

where $r(m,n)$ is the number of ordered partitions of m into n parts.

□

Note that we can deduce from Proposition 4.1 computations of Tor-groups between functors. In fact, for $N : \mathbf{gr}^{op} \to \mathbb{Q} - Mod$ we denote by $N^\vee$ the postcomposition of N with the functor of duality of vector spaces $V \mapsto Hom(V, \mathbb{Q})$. The Tor-groups can be deduced from the following natural graded isomorphism:

$$Hom(Tor^{\mathbf{gr}}_\bullet(N,M), \mathbb{Q}) \simeq Ext^\bullet_{\mathcal{F}(\mathbf{gr})}(M, N^\vee) \tag{4.2.1}$$

where $M : \mathbf{gr} \to \mathbb{Q} - Mod$ (see [4, Appendice A]).

For $N : \mathbf{gr}^{op} \to \mathbb{Q}$-Mod, F_n the free group in n generators and all integer i, Djament obtains in [2, Théorème 1.10] the existence of an isomorphism:

$$\operatorname*{colim}_{n\in\mathbb{N}} H_i(Aut(F_n), N(F_n)) \simeq \bigoplus_{k+l=i} Tor^{\mathbf{gr}}_k(N, \Lambda^l \circ (\mathfrak{a} \otimes \mathbb{Q})) \tag{4.2.2}$$

where Λ^l is the l-th exterior power functor. As an application of Proposition 4.1, using the previous result of Djament, we obtain the following computations of stable homology:

Proposition 4.3. *Let $H_n = Hom_{\boldsymbol{Gr}}(F_n, \mathbb{Q})$. We have*

$$\operatorname*{colim}_{n\in\mathbb{N}} H_*(Aut(F_n), T^d \circ (\mathfrak{a} \otimes \mathbb{Q})(H_n)) \simeq \begin{cases} \mathbb{Q}^{B(d)} & \text{if } * = d \\ 0 & \text{otherwise} \end{cases}$$

where $B(d)$ denotes the dth Bell number (i.e. the number of partitions of a set of d elements);

$$\operatorname*{colim}_{n\in\mathbb{N}} H_*(Aut(F_n), \Lambda^d \circ (\mathfrak{a} \otimes \mathbb{Q})(H_n)) \simeq \begin{cases} \mathbb{Q}^{\rho(d)} & \text{if } * = d \\ 0 & \text{otherwise} \end{cases}$$

where $\rho(d)$ denotes the number of partitions of d, and

$$\operatorname*{colim}_{n\in\mathbb{N}} H_*(Aut(F_n), S^d \circ (\mathfrak{a} \otimes \mathbb{Q})(H_n)) \simeq \begin{cases} \mathbb{Q} & \text{if } * = d = 0 \\ & \text{or } * = d = 1 \\ 0 & \text{otherwise} \end{cases}$$

where S^d is the d-th symmetric power.

Proof. For $F : \mathbf{gr} \to \mathbb{Q}$-Mod such that for all $k \in \mathbb{N}$

$$Ext^k_{\mathcal{F}(\mathbf{gr})}(\Lambda^l \circ (\mathfrak{a} \otimes \mathbb{Q}), F \circ (\mathfrak{a} \otimes \mathbb{Q}))$$

is a finite vector space, we have the isomorphisms

$$\operatorname*{colim}_{n\in\mathbb{N}} H_i(Aut(F_n), F\circ(\mathfrak{a}\otimes\mathbb{Q})(H_n)) \simeq \bigoplus_{k+l=i} Tor^{\mathbf{gr}}_k(F\circ(\mathfrak{a}\otimes\mathbb{Q})\circ Hom_{\mathbf{Gr}}(-,\mathbb{Q}), \Lambda^l\circ(\mathfrak{a}\otimes\mathbb{Q}))$$

$$\simeq \bigoplus_{k+l=i} Hom(Ext^k_{\mathcal{F}(\mathbf{gr})}(\Lambda^l \circ (\mathfrak{a} \otimes \mathbb{Q}), F \circ (\mathfrak{a} \otimes \mathbb{Q})), \mathbb{Q})$$

where the first isomorphism if the Djament's isomorphism (4.2.2) and the second is the Tor-Ext isomorphism (4.2.1).

For $F = T^d$, by Proposition 4.1 we obtain:

$$\operatorname*{colim}_{n\in\mathbb{N}} H_i(Aut(F_n), T^d\circ(\mathfrak{a}\otimes\mathbb{Q})(H_n)) \simeq \begin{cases} 0 & \text{if } i \neq d \\ \bigoplus_{k+l=d} \mathbb{Q}^{S(d,l)} = \bigoplus_{l=0}^{d} \mathbb{Q}^{S(d,l)} = \mathbb{Q}^{B(d)} & \text{if } i = d \end{cases}$$

The other results are obtained in a similar way. □

The next result is used in the proof of corollaire 3.3 in [3].

Proposition 4.4. *Let m and n be integers such that $m \geq n > 0$. We have:*

$$Ext^{m-n}_{\mathcal{F}(\boldsymbol{gr})}(q_n(\bar{P}), T^m \circ \mathfrak{a}) \neq 0.$$

Proof. Let $g \in Hom(\mathfrak{a}^{\otimes n}, q_n\bar{P})$ be the kernel of the map $q_n\bar{P} \to q_{n-1}\bar{P}$. More concretely, for $G \in \mathbf{gr}$, $g : \mathfrak{a}^{\otimes n}(G) \to q_n\bar{P}(G) \simeq IG/I^{n+1}G$ is given by

$$g(g_1 \otimes \ldots \otimes g_n) = ([g_1] - 1) \ldots ([g_n] - 1).$$

We denote by f the canonical generator of $\mathcal{F}_n(\mathbf{gr})(q_n\bar{P}, \mathfrak{a}^{\otimes n}) \simeq \mathbb{Z}$ i.e. the natural transformation corresponding to $1 \in \mathbb{Z}$ by the natural isomorphisms

$$\mathcal{F}_n(\mathbf{gr})(q_n\bar{P}, \mathfrak{a}^{\otimes n}) \simeq \mathcal{F}(\mathbf{gr})(\bar{P}, \mathfrak{a}^{\otimes n}) \simeq cr_1(\mathfrak{a}^{\otimes n})(\mathbb{Z}) \simeq \mathbb{Z}$$

where the second isomorphism is given by the Yoneda lemma. More explicitely, for $G \in \mathbf{gr}$, the group morphism $f : IG/I^{n+1}G \simeq q_n\bar{P}(G) \to \mathfrak{a}^{\otimes n}(G)$ is given by:

$$f([g] - 1) = g^{\otimes n}.$$

Using the relation $([g_1]-1)([g_2]-1) = ([g_1+g_2]-1)-([g_1]-1)-([g_2]-1)$ in IG we obtain

$$f(([g_1] - 1)([g_2] - 1) \ldots ([g_n] - 1)) = \sum_{\sigma\in\mathfrak{S}_n} g_{\sigma(1)} \otimes \ldots \otimes g_{\sigma(n)}.$$

We deduce that the composition $\mathfrak{a}^{\otimes n} \xrightarrow{g} q_n\bar{P} \xrightarrow{f} \mathfrak{a}^{\otimes n}$ is the trace map denoted by tr. This gives rise to a commutative diagram:

$$\begin{array}{ccc} Ext^{m-n}_{\mathcal{F}(\mathbf{gr})}(T^n\circ\mathfrak{a}, T^m\circ\mathfrak{a}) & \xrightarrow{tr^*} & Ext^{m-n}_{\mathcal{F}(\mathbf{gr})}(T^n\circ\mathfrak{a}, T^m\circ\mathfrak{a}) \\ \downarrow & \nearrow & \\ Ext^{m-n}_{\mathcal{F}(\mathbf{gr})}(q_n(\bar{P}), T^m\circ\mathfrak{a}). & & \end{array}$$

We prove that the map tr^* is non zero. Let $f \in Surj(m,n)$ be the surjection defined by: $f(i) = i$ for $i \in \{1,\dots,n-1\}$ and $f(k) = n$ for $k \geq n$. By Proposition 2.5 we have:

$$\overline{tr^*}(f) = \sum_{\sigma\in\mathfrak{S}_n} \epsilon_\sigma[\sigma\circ f] = [f] + \sum_{\sigma\in\mathfrak{S}_n\setminus\{Id\}} \epsilon_\sigma[\sigma\circ f] \neq 0$$

where $\epsilon_\sigma \in \{-1,1\}$ and $\overline{tr^*}(f) : \mathbb{Z}[Surj(m,n)] \to \mathbb{Z}[Surj(m,n)]$ is the map induced by tr^* via the isomorphism in Proposition 2.3.

So, we deduce from the previous commutative diagram that $Ext^{m-n}_{\mathcal{F}(\mathbf{gr})}(q_n(\bar{P}), T^m\circ\mathfrak{a})$ is non zero. □

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