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Christine Vespa

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EXTENSIONS BETWEEN FUNCTORS FROM GROUPS

CHRISTINE VESPA

ABSTRACT. We compute Ext-groups between tensor powers composed by the abelianization functor. More precisely, we compute the groups $Ext_{\mathcal{F}(\mathbf{gr})}^*(T^n \circ \mathbf{a}, T^m \circ \mathbf{a})$ where T^n is the n -th tensor power functor and \mathbf{a} is the abelianization functor from the category of free groups to abelian groups. These groups are shown to be non-zero if and only if $* = m - n \geq 0$ and $Ext_{\mathcal{F}(\mathbf{gr})}^{m-n}(T^n \circ \mathbf{a}, T^m \circ \mathbf{a}) = \mathbb{Z}[\Omega(m, n)]$ where $\Omega(m, n)$ is the set of surjections from the set having m elements to the one having n elements. We make explicit the action of symmetric groups on these groups and the Yoneda and external products. We use this computation in order to obtain other computations of Ext-groups between functors from groups.

Stable homology with twisted coefficients of various families of groups can be computed thanks to functor homology in a suitable category (see [5] [9] [3] [4]). In particular, stable homology of automorphism groups of free groups with coefficients given by a reduced polynomial covariant functor is trivial (see [4]). Recently, Djament proved in [1] that stable homology (resp. cohomology) of automorphism groups of free groups with coefficients given by a reduced polynomial contravariant (resp. covariant) functor is governed by Tor groups (resp. Ext groups) in the category of functors from finitely generated free groups. The aim of this paper is to give explicit computations of Ext groups between concrete functors from groups to abelian groups in order to obtain explicit computations of stable homology of automorphism groups of free groups with coefficients given by a contravariant functor or a bifunctor. Let \mathbf{gr} be the category of finitely generated free groups and $\mathcal{F}(\mathbf{gr})$ the category of functors from \mathbf{gr} to abelian groups. The principal result of this paper is:

Theorem. *Let n and m be integers, we have an isomorphism:*

$$Ext_{\mathcal{F}(\mathbf{gr})}^*(T^n \circ \mathbf{a}, T^m \circ \mathbf{a}) \simeq \begin{cases} \mathbb{Z}[\Omega(m, n)] & \text{if } * = m - n \\ 0 & \text{otherwise} \end{cases}$$

where Ω is the category of finite sets with surjections.

The actions of the symmetric groups \mathfrak{S}_m and \mathfrak{S}_n on $Ext_{\mathcal{F}(\mathbf{gr})}^*(T^n \circ \mathbf{a}, T^m \circ \mathbf{a})$ are induced by the composition in Ω via the previous isomorphism, up to a sign (see Proposition 2.3 for the precise signs).

The Yoneda product is induced by the composition in Ω (up to a sign) via the previous isomorphism and the external product is induced by the symmetric monoidal structure on Ω given by the disjoint union (up to a sign).

In particular for $n = m = 1$ we obtain that $\text{Hom}_{\mathcal{F}(\mathbf{gr})}(\mathbf{a}, \mathbf{a}) = \mathbb{Z}$ and $\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, \mathbf{a}) = 0$ for $* > 0$. These groups should be compared with the groups $\text{Ext}_{\mathcal{F}(\mathbf{ab})}^*(\text{Id}, \text{Id})$ corresponding to the MacLane homology of \mathbb{Z} . This homology is computed by Bökstedt by topological methods (unpublished) and reobtained in [6] by algebraic methods and is non trivial. So this result is an illustration of the fact that functor homology in $\mathcal{F}(\mathbf{gr})$ is easier than functor homology in $\mathcal{F}(\mathbf{ab})$ (see also [2] for another illustration of this fact).

The proof of this result can be decomposed into two steps corresponding to the first two sections of this paper. In the first step we compute $\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, T^m \circ \mathbf{a})$ using an explicit projective resolution of the abelianization functor \mathbf{a} . In the second step we deduce our result using the sum-diagonal adjunction and an exponential-type property of tensor power functor T^\bullet . The next section is devoted to the study of products on this Ext-groups. In the last section we deduce computations of other Ext-groups. In particular, we obtain the following results.

Proposition. *Let n and m be natural integers, we have isomorphisms:*

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*((\Lambda^n \circ \mathbf{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathbf{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{\rho(m,n)} & \text{if } * = m - n \\ 0 & \text{otherwise} \end{cases}$$

where $\rho(m, n)$ denotes the number of partitions of m into n parts.

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*((S^n \circ \mathbf{a}) \otimes \mathbb{Q}, (S^m \circ \mathbf{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m \text{ and } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*((\Lambda^n \circ \mathbf{a}) \otimes \mathbb{Q}, (S^m \circ \mathbf{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m = 0 \text{ and } * = 0 \\ & \text{or } n = m = 1 \text{ and } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*((S^n \circ \mathbf{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathbf{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m = 0 \text{ and } * = 0 \\ & \text{or } n = 1 \text{ and } * = m - 1 \\ 0 & \text{otherwise} \end{cases}$$

Let q_n be the left adjoint of the inclusion functor from polynomial functors of degree $\leq n$ to $\mathcal{F}(\mathbf{gr})$ and \bar{P} in $\mathcal{F}(\mathbf{gr})$ such that $\mathbb{Z}[\mathbf{gr}(\mathbb{Z}, -)] \simeq \mathbb{Z} \oplus \bar{P}$, we have:

Proposition. *Let m and n be integers such that $m \geq n > 0$. We have:*

$$\text{Ext}^{m-n}(q_n(\bar{P}), T^m \circ \mathbf{a}) \neq 0.$$

Notations: We denote by \mathbf{Gr} the category of groups, \mathbf{Ab} the category of abelian groups, \mathbf{gr} (resp. \mathbf{ab}), the full subcategory of \mathbf{Gr} (resp. \mathbf{Ab}), having as objects finitely generated free objects.

We denote by $\mathbf{a} : \mathbf{gr} \rightarrow \mathbf{ab}$ the abelianization functor.

Let \mathcal{C} be a small pointed category having finite coproduct. We denote by $\mathcal{F}(\mathcal{C})$ the category of functors from \mathcal{C} to \mathbf{Ab} . This category is abelian and has enough projective and injective objects. We denote by P_n the representable functor $P_n := \mathbb{Z}[\mathbf{gr}(\mathbb{Z}^{*n}, -)]$ in $\mathcal{F}(\mathbf{gr})$. A *reduced* functor $F \in \mathcal{F}(\mathcal{C})$ satisfies $F(0) = 0$. We denote by \bar{P} the reduced part of P_1 (i.e. $P_1 \simeq \mathbb{Z} \oplus \bar{P}$).

For $d \in \mathbb{N}$ we denote by $\mathcal{F}_d(\mathbf{gr})$ the full subcategory of $\mathcal{F}(\mathbf{gr})$ of polynomial functors of degree $\leq d$ and $i_d : \mathcal{F}_d(\mathbf{gr}) \rightarrow \mathcal{F}(\mathbf{gr})$ the inclusion functor. This functor has a left adjoint denoted by q_d . The functors $q_i(\bar{P})$ are called Passi functors (see [7]).

1. COMPUTATION OF $Ext_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, T^m \circ \mathbf{a})$

This section is based on the existence of an explicit projective resolution of \mathbf{a} in $\mathcal{F}(\mathbf{gr})$. This resolution is already mentioned in [8, Proposition 5.1] and used in [4].

Consider the following simplicial object in $\mathcal{F}(\mathbf{gr})$:

$$(1.0.1) \quad \dots \rightrightarrows P_{n+1} \rightrightarrows P_n \rightrightarrows \dots \rightrightarrows P_2 \rightrightarrows P_1 \rightrightarrows P_0$$

where $\delta_i : P_{n+1} \rightarrow P_n$ for $0 \leq i \leq n+1$ are defined by:

$$\delta_0[g_1, g_2, \dots, g_n, g_{n+1}] = [g_2, \dots, g_n, g_{n+1}]$$

$$\delta_i[g_1, g_2, \dots, g_n, g_{n+1}] = [g_1, \dots, g_i g_{i+1}, \dots, g_n, g_{n+1}] \quad \text{for } 1 \leq i \leq n$$

$$\delta_{n+1}[g_1, g_2, \dots, g_n, g_{n+1}] = [g_1, g_2, \dots, g_n]$$

and $\epsilon_i : P_n \rightarrow P_{n+1}$ for $1 \leq i \leq n+1$ are defined by:

$$\epsilon_i[g_1, \dots, g_n] = [g_1, \dots, g_{i-1}, 1, g_i, \dots, g_n].$$

We denote by C_\bullet the unnormalized chain complex associated to this simplicial object and $D_\bullet = (C_\bullet)_{>1}$ the complex obtained by reindexing the truncation above 1 of C_\bullet .

Recall that the homology of a free group is naturally isomorphic to its abelianization in degree 1 and is zero in degree > 1 . So D_\bullet is the bar resolution of \mathbf{a} and we obtain:

Proposition 1.1. (Cf. [8, Proposition 5.1]) *The exact sequence in $\mathcal{F}(\mathbf{gr})$:*

$$\dots P_{n+1} \xrightarrow{d_n} P_n \rightarrow \dots \rightarrow P_2 \xrightarrow{d_1} P_1$$

is a projective resolution of the abelianization functor $\mathbf{a} : \mathbf{gr} \rightarrow \mathbf{Ab}$. The natural transformation $d_n : P_n \rightarrow P_{n-1}$ is given on a group G by the linear map $\mathbb{Z}[G^{n+1}] \rightarrow \mathbb{Z}[G^n]$ such that:

$$d_n([g_1, \dots, g_{n+1}]) = [g_2, \dots, g_{n+1}]$$

$$+ \sum_{i=1}^n (-1)^i [g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}] + (-1)^{n+1} [g_1, \dots, g_n]$$

for all $(g_1, \dots, g_{n+1}) \in G^{n+1}$.

This explicit projective resolution play a crucial rôle in [4].

For $F : \mathbf{gr} \rightarrow \mathbf{Ab}$ we have

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, F) = \pi_{*+1}(F(\mathbb{Z}^\bullet))$$

where $F(\mathbb{Z}^\bullet)$ is the simplicial abelian group obtained when we apply $\text{Hom}_{\mathcal{F}(\mathbf{gr})}(-, F)$ to the simplicial object 1.0.1 and using the natural isomorphism $\text{Hom}_{\mathcal{F}(\mathbf{gr})}(P_n, F) \simeq F(\mathbb{Z}^{*n})$.

Proposition 1.2. *Let $m \geq 1$ be a natural integer, we have an isomorphism:*

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, T^m \circ \mathbf{a}) \simeq \begin{cases} \mathbb{Z} & \text{if } * = m - 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We prove the result by induction on m . To start the induction, $\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, \mathbf{a})$ is the homology of the complex:

$$\dots \longleftarrow \mathbb{Z}^{n+1} \xleftarrow{\delta_n} \mathbb{Z}^n \longleftarrow \dots \longleftarrow \mathbb{Z}^2 \xleftarrow{\delta_1} \mathbb{Z}$$

which is trivial for $* \geq 1$ and is isomorphic to \mathbb{Z} for $* = 0$. Assume that the statement is true for m . We have:

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^n(\mathbf{a}, T^{m+1} \circ \mathbf{a}) \simeq \pi_{n+1}((T^{m+1} \circ \mathbf{a})(\mathbb{Z}^{*\bullet})) \simeq \bigoplus_{i+j=n+1} \pi_i((T^m \circ \mathbf{a})(\mathbb{Z}^{*\bullet})) \otimes \pi_j(\mathbf{a}(\mathbb{Z}^{*\bullet}))$$

by the Künneth formula ($\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, \mathbf{a})$ is torsion free by the previous computation). So we have:

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^n(\mathbf{a}, T^{m+1} \circ \mathbf{a}) \simeq \bigoplus_{i+j=n+1} \text{Ext}_{\mathcal{F}(\mathbf{gr})}^{i-1}(\mathbf{a}, T^m \circ \mathbf{a}) \otimes \text{Ext}_{\mathcal{F}(\mathbf{gr})}^{j-1}(\mathbf{a}, \mathbf{a})$$

and we obtain the result by the inductive step and the computation of $\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, \mathbf{a})$. □

Remark 1.3. Note that

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, T^0 \circ \mathbf{a}) \simeq \text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(\mathbf{a}, \mathbb{Z}) = 0$$

The symmetric group \mathfrak{S}_m acts on T^m by permuting the factors of the tensor product. This action induces an action of \mathfrak{S}_m on the extension groups $\text{Ext}_{\mathcal{F}(\mathbf{gr})}^{m-1}(\mathbf{a}, T^m \circ \mathbf{a})$ that we make explicit in the following proposition.

Proposition 1.4. *The symmetric group \mathfrak{S}_m acts on $\text{Ext}_{\mathcal{F}(\mathbf{gr})}^{m-1}(\mathbf{a}, T^m \circ \mathbf{a})$ by the following way: for $\sigma \in \mathfrak{S}_m$ and $E \in \text{Ext}_{\mathcal{F}(\mathbf{gr})}^{m-1}(\mathbf{a}, T^m \circ \mathbf{a})$*

$$E \cdot \sigma = \epsilon(\sigma) E$$

where ϵ is the signature.

Proof. Let $\sigma \in \mathfrak{S}_m$, the action of σ on $Ext_{\mathcal{F}(\mathbf{gr})}^{m-1}(\mathbf{a}, T^m \circ \mathbf{a})$ is given by the following composition where $\phi : \pi_1(\bigotimes_{i=1}^m \mathbf{a}(\mathbb{Z}^{*\bullet})) \rightarrow \pi_1(\bigotimes_{i=1}^m \mathbf{a}(\mathbb{Z}^{*\bullet}))$ is given by the action of σ on $\pi_1(\bigotimes_{i=1}^m \mathbf{a}(\mathbb{Z}^{*\bullet}))$ and the isomorphism $\bigotimes_{i=1}^m \pi_1(\mathbf{a}(\mathbb{Z}^{*\bullet})) \simeq \pi_1(\bigotimes_{i=1}^m \mathbf{a}(\mathbb{Z}^{*\bullet}))$ is a consequence of the Künneth formula (since $\pi_1(\mathbf{a}(\mathbb{Z}^{*\bullet}))$ is flat by Proposition 1.2):

$$\begin{array}{ccc} Ext_{\mathcal{F}(\mathbf{gr})}^{m-1}(\mathbf{a}, T^m \circ \mathbf{a}) & \xrightarrow[\simeq]{\alpha} & \bigotimes_{i=1}^m \pi_1(\mathbf{a}(\mathbb{Z}^{*\bullet})) \xrightarrow{\simeq} \pi_1(\bigotimes_{i=1}^m \mathbf{a}(\mathbb{Z}^{*\bullet})) \\ & & \downarrow \phi \\ Ext_{\mathcal{F}(\mathbf{gr})}^{m-1}(\mathbf{a}, T^m \circ \mathbf{a}) & \xleftarrow[\alpha^{-1}]{} & \bigotimes_{i=1}^m \pi_1(\mathbf{a}(\mathbb{Z}^{*\bullet})) \xleftarrow[\simeq]{} \pi_1(\bigotimes_{i=1}^m \mathbf{a}(\mathbb{Z}^{*\bullet})) \end{array}$$

We have

$$\alpha^{-1}\phi\alpha(E) = \alpha^{-1}\phi(x_1 \otimes \dots \otimes x_n) = \alpha^{-1}(\epsilon(\sigma)x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(m)}) = \epsilon(\sigma)E$$

where x_k is a generator of the k -th copy of $\pi_1(\mathbf{a}(\mathbb{Z}^{*\bullet}))$ in $\bigotimes_{i=1}^m \pi_1(\mathbf{a}(\mathbb{Z}^{*\bullet}))$. \square

2. COMPUTATION OF $Ext_{\mathcal{F}(\mathbf{gr})}^*(T^m \circ \mathbf{a}, T^m \circ \mathbf{a})$

This section is based on the sum-diagonal adjunction, the exponential type property of tensor powers and the Künneth formula.

We begin this section with some recollections on these ingredients. We refer the reader to [3, Appendice B] for more details.

For F and G in $\mathcal{F}(\mathbf{gr})$, their external tensor product $F \boxtimes G$ is the functor sending (X, Y) to $F(X) \otimes G(Y)$. This yields a functor:

$$- \boxtimes - : \mathcal{F}(\mathbf{gr}) \times \mathcal{F}(\mathbf{gr}) \rightarrow \mathcal{F}(\mathbf{gr} \times \mathbf{gr}).$$

We denote by $\pi_d : \mathbf{gr}^{\times d} \rightarrow \mathbf{gr}$ the functor obtained by iteration of the free product (which is the categorical sum in \mathbf{gr}) and $\delta_d : \mathbf{gr} \rightarrow \mathbf{gr}^{\times d}$ the diagonal functor. The functor δ_d is right adjoint of the functor π_d . We deduce that the functor given by precomposition $\delta_d^* : \mathcal{F}(\mathbf{gr}^{\times d}) \rightarrow \mathcal{F}(\mathbf{gr})$ is left adjoint of the functor given by precomposition $\pi_d^* : \mathcal{F}(\mathbf{gr}) \rightarrow \mathcal{F}(\mathbf{gr}^{\times d})$.

The tensor power (graded) functor T^\bullet is not exponential but we have a similar property which is useful in the following computations:

$$(2.0.1) \quad \pi_n^*(T^m \circ \mathbf{a}) \simeq \bigoplus_{i_1 + \dots + i_n = m} (T^{i_1} \circ \mathbf{a} \boxtimes \dots \boxtimes T^{i_n} \circ \mathbf{a}) \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m]$$

where $T^0 \circ \mathbf{a} \simeq \mathbb{Z}$.

The following easy lemma will be useful in the sequel.

Lemma 2.1. For $f \in \Omega(m, n)$ such that for all $k \in \{1, \dots, n\}$, $|f^{-1}(k)| = i_k$ there are unique maps $\alpha \in \Omega(n, n)$ and $s \in \Omega(m, n)$ such that $f = s \circ \alpha$ where α is the inverse of a (i_1, \dots, i_n) -shuffle and s is an order preserving surjection.

Our first main result is the following theorem.

Theorem 2.2. Let n and m be integers, we have an isomorphism:

$$\text{Ext}_{\mathcal{F}(\text{gr})}^*(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \begin{cases} \mathbb{Z}[\Omega(m, n)] & \text{if } * = m - n \\ 0 & \text{otherwise} \end{cases}$$

Proof.

$$\begin{aligned} & \text{Ext}_{\mathcal{F}(\text{gr})}^*(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \\ & \simeq \text{Ext}_{\mathcal{F}(\text{gr} \times \dots \times \text{gr})}^*(\mathfrak{a}^{\boxtimes n}, T^m \circ \mathfrak{a} \circ \pi_n) \text{ by the sum-diagonal adjunction} \\ & \simeq \text{Ext}_{\mathcal{F}(\text{gr} \times \dots \times \text{gr})}^*(\mathfrak{a}^{\boxtimes n}, \bigoplus_{i_1 + \dots + i_n = m} (T^{i_1} \circ \mathfrak{a} \boxtimes \dots \boxtimes T^{i_n} \circ \mathfrak{a}) \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m]) \text{ by 2.0.1} \\ & \simeq \bigoplus_{i_1 + \dots + i_n = m} \text{Ext}_{\mathcal{F}(\text{gr} \times \dots \times \text{gr})}^*(\mathfrak{a}^{\boxtimes n}, (T^{i_1} \circ \mathfrak{a} \boxtimes \dots \boxtimes T^{i_n} \circ \mathfrak{a}) \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m]) \end{aligned}$$

By Proposition 1.2 $\text{Ext}_{\mathcal{F}(\text{gr})}^*(\mathfrak{a}, T^m \circ \mathfrak{a})$ is torsion free so, by the Künneth formula we obtain:

$$\begin{aligned} & \text{Ext}_{\mathcal{F}(\text{gr})}^*(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \\ & \simeq \bigoplus_{i_1 + \dots + i_n = m} \left(\bigotimes_{k=1}^n \text{Ext}_{\mathcal{F}(\text{gr})}^*(\mathfrak{a}, T^{i_k} \circ \mathfrak{a}) \right) \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] \\ & \simeq \begin{cases} \bigoplus_{\substack{i_1 + \dots + i_n = m \\ i_k > 0, \forall k}} \left(\mathbb{Z} \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] \right) & \text{if } * = m - n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where the last isomorphism follows from Proposition 1.2 and Remark 1.3.

Let $f \in \Omega(m, n)$, by Lemma 2.1 f admits a unique decomposition of the form $f = s \circ \alpha$ where α is the inverse of a $(|f^{-1}(1)|, \dots, |f^{-1}(n)|)$ -shuffle and s is an order preserving surjection. So we deduce that the map:

$$\kappa : \mathbb{Z}[\Omega(m, n)] \rightarrow \bigoplus_{\substack{i_1 + \dots + i_n = m \\ i_k > 0, \forall k}} \left(\mathbb{Z} \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] \right)$$

given by $\kappa([f]) = 1 \otimes \alpha \in \mathbb{Z} \otimes_{\mathfrak{S}_{|f^{-1}(1)|} \times \dots \times \mathfrak{S}_{|f^{-1}(n)|}} \mathbb{Z}[\mathfrak{S}_m]$ is an isomorphism. \square

For A_n and B_n ordered sets of cardinality n and $f : A_n \rightarrow B_n$ a bijection. We denote by $\bar{f} : B_n \rightarrow B_n$ the unique permutation of B_n such that $\bar{f} \circ u = f$ where $u : A_n \rightarrow B_n$ is the unique order preserving map.

In the following proposition we make explicit the actions of symmetric groups on $\text{Ext}_{\mathcal{F}(\text{gr})}^*(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a})$.

Proposition 2.3. *The symmetric groups \mathfrak{S}_m and \mathfrak{S}_n act on $\text{Ext}_{\mathcal{F}(\text{gr})}^{m-n}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \mathbb{Z}[\Omega(m, n)]$ by the following way: for $\sigma \in \mathfrak{S}_m$, $\tau_{k,l} \in \mathfrak{S}_n$ the transposition of k and l where $k, l \in \{1, \dots, n\}$ and $f \in \Omega(m, n)$*

$$[f] \cdot \sigma = \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f \circ \sigma)^{-1}(i)}}) [f \circ \sigma]$$

where $\sigma_{|(f \circ \sigma)^{-1}(i)} : (f \circ \sigma)^{-1}(i) \rightarrow \sigma((f \circ \sigma)^{-1}(i))$

$$\tau_{k,l} \cdot [f] = (-1)^{(|f^{-1}(k)|-1)(|f^{-1}(l)|-1)} [\tau_{k,l} \circ f].$$

Proof. Let f be an element of $\Omega(m, n)$ such that $f = s \circ \sigma$ is the unique decomposition of f of the form described in Lemma 2.1.

For $\sigma \in \mathfrak{S}_m$, the action of σ on $\mathbb{Z}[\Omega(m, n)]$ is given by the following composition, where ϕ is the map given by the action of σ on

$$\begin{array}{ccc} \bigoplus_{i_1+\dots+i_n=m} \left(\bigotimes_{k=1}^n \text{Ext}^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k}) \right) \bigotimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] & & \\ \downarrow \simeq & & \\ \mathbb{Z}[\Omega(m, n)] & \xrightarrow{\simeq \kappa} & \bigoplus_{i_1+\dots+i_n=m} \mathbb{Z} \bigotimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] \\ & & \downarrow \simeq \\ & & \bigoplus_{i_1+\dots+i_n=m} \left(\bigotimes_{k=1}^n \text{Ext}^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k}) \right) \bigotimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] \\ & & \downarrow \phi \\ & & \bigoplus_{i_1+\dots+i_n=m} \left(\bigotimes_{k=1}^n \text{Ext}^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k}) \right) \bigotimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] \\ & & \downarrow \simeq \\ & & \bigoplus_{i_1+\dots+i_n=m} \mathbb{Z} \bigotimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] \\ & \xleftarrow{\kappa^{-1}} & \mathbb{Z}[\Omega(m, n)] \end{array}$$

we have

$$\begin{aligned} \kappa^{-1} \circ \phi \circ \kappa([f]) &= \kappa^{-1} \circ \phi \circ \kappa([s \circ \alpha]) = \kappa^{-1} \circ \phi \left((1 \otimes [\alpha])_{|f^{-1}(1)|, \dots, |f^{-1}(n)|} \right) \\ &= \kappa^{-1} \left(\prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f \circ \sigma)^{-1}(i)}}) (1 \otimes [\alpha \circ \sigma])_{|f^{-1}(1)|, \dots, |f^{-1}(n)|} \right) = \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f \circ \sigma)^{-1}(i)}}) [s \circ \alpha \circ \sigma] \\ &= \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f \circ \sigma)^{-1}(i)}}) [f \circ \sigma] \end{aligned}$$

where we deduce the third equality from Proposition 1.4 and where $-_{i_1, \dots, i_n}$ corresponds to the summand i_1, \dots, i_n in the sum $\bigoplus_{i_1+\dots+i_n=m}$.

For $\tau_{k,l} \in \mathfrak{S}_n$, we have $\tau_{k,l} \circ f = \tau_{k,l} \circ s \circ \alpha = s' \circ T_{k,l} \circ \alpha$ where s' is the order preserving surjection from m to n such that $|s'^{-1}(l)| = |s^{-1}(k)|$, $|s'^{-1}(k)| = |s^{-1}(l)|$ and $|s'^{-1}(q)| = |s^{-1}(q)|$ for $q \in \{1, \dots, m\} \setminus \{k, l\}$ and $T_{k,l}$ is the permutation obtained by the bloc transposition of $s^{-1}(k)$

and $s^{-1}(l)$. Note that $T_{k,l} \circ \alpha$ is the inverse of a $(|(\tau_{k,l} \circ f)^{-1}(1)|, \dots, |(\tau_{k,l} \circ f)^{-1}(n)|)$ -shuffle.

For $\tau_{k,l} \in \mathfrak{S}_n$, the action of $\tau_{k,l}$ on $\mathbb{Z}[\Omega(m, n)]$ is given by the following composition, where ϕ' is the map given by the action of $\tau_{k,l}$ on

$$\begin{array}{ccc}
\bigoplus_{i_1+\dots+i_n=m} \left(\bigotimes_{k=1}^n \text{Ext}^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k}) \right) \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] & & \\
\downarrow \cong & & \\
\mathbb{Z}[\Omega(m, n)] \xrightarrow{\cong \kappa} \bigoplus_{i_1+\dots+i_n=m} \mathbb{Z} \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] & & \\
\downarrow \cong & & \\
\bigoplus_{i_1+\dots+i_n=m} \left(\bigotimes_{k=1}^n \text{Ext}^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k}) \right) \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] & & \\
\downarrow \phi' & & \\
\bigoplus_{i_1+\dots+i_n=m} \left(\bigotimes_{k=1}^n \text{Ext}^{i_k-1}(\mathfrak{a}, \mathfrak{a}^{\otimes i_k}) \right) \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] & & \\
\downarrow \cong & & \\
\mathbb{Z}[\Omega(m, n)] \xleftarrow{\cong \kappa^{-1}} \bigoplus_{i_1+\dots+i_n=m} \mathbb{Z} \otimes_{\mathfrak{S}_{i_1} \times \dots \times \mathfrak{S}_{i_n}} \mathbb{Z}[\mathfrak{S}_m] & &
\end{array}$$

we have

$$\begin{aligned}
\kappa^{-1} \circ \phi' \circ \kappa([f]) &= \kappa^{-1} \circ \phi' \circ \kappa([s \circ \alpha]) = \kappa^{-1} \circ \phi' \left((1 \otimes [\alpha])_{|f^{-1}(1)|, \dots, |f^{-1}(n)|} \right) \\
&= \kappa^{-1} \left((-1)^{i_k i_l + (i_k-1)(i_l-1)} 1 \otimes [T_{k,l} \circ \alpha]_{|(\tau_{k,l} \circ f)^{-1}(1)|, \dots, |(\tau_{k,l} \circ f)^{-1}(n)|} \right) \\
&= (-1)^{i_k+i_l-1} [s' \circ \alpha'] = (-1)^{i_k+i_l-1} [\tau_{k,l} \circ f].
\end{aligned}$$

□

3. PRODUCTS

The aim of this section is to explicit the Yoneda product and the external product on $\text{Ext}_{\mathcal{F}(\mathbf{gr})}^*(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a})$.

Proposition 3.1. (1) *The Yoneda product:*

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^{m-l}(T^l \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \otimes \text{Ext}_{\mathcal{F}(\mathbf{gr})}^{n-m}(T^m \circ \mathfrak{a}, T^n \circ \mathfrak{a}) \rightarrow \text{Ext}_{\mathcal{F}(\mathbf{gr})}^{n-l}(T^l \circ \mathfrak{a}, T^n \circ \mathfrak{a})$$

is induced by the composition in the category Ω via the isomorphism obtained in Theorem 2.2 (up to a sign).

(2) *The external product*

$$\text{Ext}_{\mathcal{F}(\mathbf{gr})}^{m-l}(T^l \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \otimes \text{Ext}_{\mathcal{F}(\mathbf{gr})}^{n-p}(T^p \circ \mathfrak{a}, T^n \circ \mathfrak{a}) \rightarrow \text{Ext}_{\mathcal{F}(\mathbf{gr})}^{m+n-l-p}(T^{l+p} \circ \mathfrak{a}, T^{m+n} \circ \mathfrak{a})$$

is induced by the monoidal structure of Ω via the isomorphism obtained in Theorem 2.2 (up to a sign).

The proof of this proposition and the precise signs will be given in an update version of this preprint.

4. APPLICATIONS

In this section, we deduce from Theorem 2.2 the computation of some other Ext-groups between functors from \mathbf{gr} to \mathbf{Ab} .

For M an abelian group, recall that $S^n(M) = (T^n(M))_{\mathfrak{S}_n}$ where \mathfrak{S}_n acts by the permutation of variables and $\Lambda^n(M) \otimes \mathbb{Z}[1/2] = (T^n(M) \otimes \mathbb{Z}[1/2])_{\mathfrak{S}_n}$ where \mathfrak{S}_n acts by the permutation of variables and the multiplication by the signature.

Proposition 4.1. *Let n and m be natural integers, we have isomorphisms:*

$$\mathrm{Ext}_{\mathcal{F}(\mathbf{gr})}^*((\Lambda^n \circ \mathbf{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathbf{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{\rho(m,n)} & \text{if } * = m - n \\ 0 & \text{otherwise} \end{cases}$$

where $\rho(m, n)$ denotes the number of partitions of m into n parts.

$$\mathrm{Ext}_{\mathcal{F}(\mathbf{gr})}^*((S^n \circ \mathbf{a}) \otimes \mathbb{Q}, (S^m \circ \mathbf{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m \text{ and } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}_{\mathcal{F}(\mathbf{gr})}^*((\Lambda^n \circ \mathbf{a}) \otimes \mathbb{Q}, (S^m \circ \mathbf{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m = 0 \text{ and } * = 0 \\ & \text{or } n = m = 1 \text{ and } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}_{\mathcal{F}(\mathbf{gr})}^*((S^n \circ \mathbf{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathbf{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } n = m = 0 \text{ and } * = 0 \\ & \text{or } n = 1 \text{ and } * = m - 1 \\ 0 & \text{otherwise} \end{cases}$$

The proof of this proposition relies on the following lemma:

Lemma 4.2. *Let G be a group and E a G -set. Consider an action of G on the group ring $\mathbb{Q}[E]$ given by*

$$g \cdot [e] = \alpha(g, e)[g \cdot e]$$

where $\alpha : G \times E \rightarrow \{1, -1\}$ is a map such that, for $e \in E$, $\alpha(-, e)$ is a group morphism, then

$$\mathbb{Q}[E]_G \simeq \mathbb{Q}[\{e \in E \mid \forall g \in G \text{ s.t. } g \cdot e = e \text{ we have } \alpha(g, e) = 1\} / G].$$

Proof of Proposition 4.1. Let f be an element of $\Omega(m, n)$ such that $f = s \circ \sigma$ is the unique decomposition of f of the form described in Lemma 2.1. The group $\mathfrak{S}_n \times \mathfrak{S}_m$ acts on $\Omega(m, n)$ by $(\sigma', \sigma'') \cdot f = \sigma'' \circ f \circ \sigma'^{-1}$. The elements $(\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m$ such that $(\sigma', \sigma'') \cdot f = f$ are products of elements of the two following forms

- $(\tau_{i,j}, \sigma^{-1} T_{i,j} \sigma)$ for i and j such that $|f^{-1}(i)| = |f^{-1}(j)|$ and $T_{i,j}$ is the permutation obtained by the bloc transposition of $s^{-1}(i)$ and $s^{-1}(j)$;
- $(\mathrm{Id}, \sigma_{i_1} \dots \sigma_{i_n})$ where σ_{i_k} is a permutation of $f^{-1}(k)$.

We have

$$\begin{aligned} Ext_{\mathcal{F}(\mathbf{gr})}^*((\Lambda^n \circ \mathbf{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathbf{a}) \otimes \mathbb{Q}) &\simeq Ext_{\mathcal{F}(\mathbf{gr})}^*((T^n \circ \mathbf{a}) \otimes \mathbb{Q}, (T^m \circ \mathbf{a}) \otimes \mathbb{Q})_{\mathfrak{S}_n \times \mathfrak{S}_m} \\ &\simeq \mathbb{Q}[\Omega(m, n)]_{\mathfrak{S}_n \times \mathfrak{S}_m} \end{aligned}$$

where the action of $(\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m$ on $\mathbb{Q}[\Omega(m, n)]$ is given by:

$$(\sigma', \sigma'').[f] = \alpha((\sigma', \sigma''), f)[(\sigma', \sigma'').f].$$

For $(\sigma', \sigma'') = (\tau_{i,j}, \sigma^{-1}T_{i,j}\sigma)$ we have

$$\alpha((\tau_{i,j}, \sigma^{-1}T_{i,j}\sigma), f) = \epsilon(\tau_{i,j})\epsilon(\sigma^{-1}T_{i,j}\sigma) \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma^{-1}T_{i,j}\sigma|_{(f \circ (\sigma^{-1}T_{i,j}\sigma)^{-1}(i))}}) (-1)^{(|f^{-1}(i)|-1)^2} = 1$$

since

$$\epsilon(\sigma^{-1}T_{i,j}\sigma) = \epsilon(T_{i,j}) = (-1)^{|f^{-1}(i)|^2}$$

and

$$\epsilon(\overline{\sigma^{-1}T_{i,j}\sigma|_{(f \circ (\sigma^{-1}T_{i,j}\sigma)^{-1}(i))}}) = 1.$$

For $(\sigma', \sigma'') = (Id, \sigma_{i_1} \dots \sigma_{i_n})$ we have

$$\alpha((Id, \sigma_{i_1} \dots \sigma_{i_n}), f) = \epsilon(\sigma_{i_1} \dots \sigma_{i_n}) \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{i_1} \dots \sigma_{i_n} \sigma|_{(f \circ (\sigma_{i_1} \dots \sigma_{i_n})^{-1}(i))}}) = 1$$

since

$$\epsilon(\sigma_{i_1} \dots \sigma_{i_n}) = \epsilon(\sigma_{i_1}) \dots \epsilon(\sigma_{i_n})$$

and

$$\epsilon(\overline{\sigma_{i_1} \dots \sigma_{i_n} \sigma|_{(f \circ (\sigma_{i_1} \dots \sigma_{i_n})^{-1}(k))}}) = \epsilon(\sigma_{i_k}).$$

By Lemma 4.2 we deduce that

$$\begin{aligned} Ext_{\mathcal{F}(\mathbf{gr})}^*((\Lambda^n \circ \mathbf{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathbf{a}) \otimes \mathbb{Q}) &\simeq \mathbb{Q}[\Omega(m, n)]_{\mathfrak{S}_n \times \mathfrak{S}_m} \simeq \mathbb{Q}[\Omega(m, n)/\mathfrak{S}_n \times \mathfrak{S}_m] \\ &\simeq \begin{cases} \mathbb{Q}^{\rho(m,n)} & \text{if } * = m - n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Computations of the other Ext -groups are obtained by exactly the same method.

For $Ext_{\mathcal{F}(\mathbf{gr})}^*((S^n \circ \mathbf{a}) \otimes \mathbb{Q}, (S^m \circ \mathbf{a}) \otimes \mathbb{Q})$ we have

$$\alpha((\tau_{i,j}, \sigma^{-1}T_{i,j}\sigma), f) = (-1)^{|f^{-1}(i)|^2+1},$$

$$\alpha((Id, \sigma_{i_1} \dots \sigma_{i_n}), f) = \epsilon(\sigma_{i_1}) \dots \epsilon(\sigma_{i_n}).$$

So, if $n \neq m$

$$\{f \in \Omega(m, n) \mid \forall (\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m \text{ s.t. } (\sigma', \sigma'').f = f \text{ we have } \alpha((\sigma', \sigma''), f) = 1\} = \emptyset$$

and if $n = m$

$$\{f \in \Omega(m, n) \mid \forall (\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m \text{ s.t. } (\sigma', \sigma'').f = f \text{ we have } \alpha((\sigma', \sigma''), f) = 1\} = \Omega(m, n).$$

For $Ext_{\mathcal{F}(\mathbf{gr})}^*((S^n \circ \mathbf{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathbf{a}) \otimes \mathbb{Q})$,

$$\alpha((\tau_{i,j}, \sigma^{-1}T_{i,j}\sigma), f) = -1,$$

$$\alpha((Id, \sigma_{i_1} \dots \sigma_{i_n}), f) = 1.$$

So, if $n \neq 1$ or $n = 0$ and $m \neq 0$

$$\{f \in \Omega(m, n) \mid \forall (\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m \text{ s.t. } (\sigma', \sigma'').f = f \text{ we have } \alpha((\sigma', \sigma''), f) = 1\} = \emptyset$$

and if $n = 1$ or $n = m = 0$

$\{f \in \Omega(m, n) \mid \forall (\sigma', \sigma'') \in \mathfrak{S}_n \times \mathfrak{S}_m \text{ s.t. } (\sigma', \sigma'').f = f \text{ we have } \alpha((\sigma', \sigma''), f) = 1\}$

is a set of cardinality 1.

For $Ext_{\mathcal{F}(\mathbf{gr})}^*((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (S^m \circ \mathfrak{a}) \otimes \mathbb{Q})$,

$$\begin{aligned} \alpha((\tau_{i,j}, \sigma^{-1}T_{i,j}\sigma), f) &= (-1)^{|f^{-1}(i)|^2}, \\ \alpha((Id, \sigma_{i_1} \dots \sigma_{i_n}), f) &= \epsilon(\sigma_{i_1}) \dots \epsilon(\sigma_{i_n}). \end{aligned}$$

□

The next result is used in the proof of corollaire 3.3 in [2].

Proposition 4.3. *Let m and n be integers such that $m \geq n > 0$. We have:*

$$Ext_{\mathcal{F}(\mathbf{gr})}^{m-n}(q_n(\bar{P}), T^m \circ \mathfrak{a}) \neq 0.$$

Proof. For f the canonical generator of $Hom(q_n\bar{P}, \mathfrak{a}^{\otimes n}) \simeq \mathbb{Z}$ and $g \in Hom(\mathfrak{a}^{\otimes n}, q_n\bar{P})$ the kernel of the map $q_n\bar{P} \rightarrow q_{n-1}\bar{P}$, the composition $\mathfrak{a}^{\otimes n} \xrightarrow{g} q_n\bar{P} \xrightarrow{f} \mathfrak{a}^{\otimes n}$ is the norm map denoted by N . This gives rise to a commutative diagram:

$$\begin{array}{ccc} Ext_{\mathcal{F}(\mathbf{gr})}^{m-n}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) & \xrightarrow{N^*} & Ext_{\mathcal{F}(\mathbf{gr})}^{m-n}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \\ \downarrow & \nearrow & \\ Ext_{\mathcal{F}(\mathbf{gr})}^{m-n}(q_n(\bar{P}), T^m \circ \mathfrak{a}) & & \end{array}$$

where the map N^* is non zero by Lemma 2.3. So $Ext_{\mathcal{F}(\mathbf{gr})}^{m-n}(q_n(\bar{P}), T^m \circ \mathfrak{a})$ is non zero. □

Remark 4.4. In an update version of this preprint, we will compute completely $Ext_{\mathcal{F}(\mathbf{gr})}^*(q_n(\bar{P}), T^m \circ \mathfrak{a})$.

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UNIVERSITÉ DE STRASBOURG, INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE,
STRASBOURG, FRANCE.

E-mail address: vespa@math.unistra.fr