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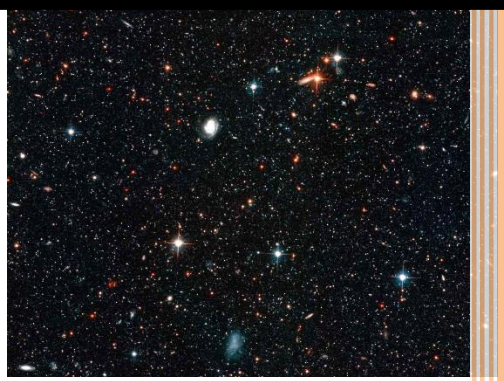
An energy and determinist approach of quantum mechanics



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I

Introduction

This manuscript describes a set of reflections on the formalism describing the quantum mechanics in its status at the beginning of 21st century. The first chapters are devoted to the resumption of the physical concepts that accompany the theory through the prism of the DIRAC equation. Abstraction is made of some strong mathematical concepts useful in the synthesis of the theory, but not necessary for the physical understanding of phenomena. The following chapters are entirely devoted to the implementation of extensive work on solutions in the form of stationary modes of the DIRAC equation.

A century of research, both theoretical and experimental helped to significantly refine the knowledge of particle physics. If the experimental tools regularly lead to significant advances in the observation and measurement of phenomena, the theoretical framework seems to be frozen since tens of years, and without a compelling perspective on a next evolution.

This theoretical framework is facing a problem that appears insurmountable: the particles behave both in the manner of a wave, and in the manner of a body of material. Unable to account for this phenomenon, the theory is reduced to treat experimental observations on a statistical and probabilistic point of view. It succeeds in a remarkable way, but at the price of a mathematical complexity that is made necessary to overcome the fact that the physics underlying the observed phenomena is not known to us with sufficient accuracy to lighten the mathematical formalism.

To advance in a reflection that allows physically account for the wave-particle duality, we must develop elements which, while being in perfect coherence with all of the existing formalisms, are breaking with the probabilistic vision of this part of physics. This can be achieved only by a deterministic approach to clarify how the material moves between its wavelike and its corpuscular aspects.

The work presented in this document is based on a set of exact, but new solutions to this day, of the DIRAC equation. This approach ensures therefore, intrinsically, consistency with all the theoretical properties built around this equation. It led, ultimately, to a description deterministic and no more probabilistic, of the wave-particle duality.

I - An energy approach

In the classical wave equations, the dimension of the quantity that spreads under the form of a wave is usually set by the second member. One can illustrate this remark by one example chosen in electromagnetism, concerning the vector and scalar potentials:

$$\vec{\nabla}^2.\vec{A} - \varepsilon_0.\mu_0.\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0.\vec{J} \qquad \text{(I-1)}$$

$$\vec{\nabla}^2.\varphi - \varepsilon_0.\mu_0.\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \qquad \text{(I-2)}$$

In the wave equation of Schrödinger or Klein-Gordon, the solution ψ function is a quantity without information on its dimensions, because it is present in the second part of the equation:

$$j\hbar\frac{\partial}{\partial t}(\psi) = -\frac{\hbar^2}{2m}\nabla^2(\psi) + E_p\psi \qquad \text{(I-3)}$$

$$\nabla^2(\psi) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\psi) = \frac{m^2c^2}{\hbar^2}\psi \qquad \text{(I-4)}$$

This means that the ψ solution reflects the function of propagation of the wave. The spreading quantity can therefore be chosen on the basis of physical considerations.

The fact that the function ψ is complex, coupled with a probabilistic view of quantum mechanics, have led to give to the quantity $\psi\psi^*$ the meaning of a volumetric density of probability, involving a normalization on any volume where the particle is located with certainty:

$$\iiint_V \psi\psi^* dv = 1 \qquad \text{(I-5)}$$

If we adopt a deterministic vision of the phenomena, there is no more probability density. The physical quantity that spreads can be considered to be an energy, and consistency with the probabilistic vision suggests to give to the quantity $\psi\psi^*$ the meaning of a volumetric energy density. The condition of standardization is obtained by expressing that integration on a volume V where the particle extends must give the total energy E of the particle:

$$\iiint_V \psi\psi^* dv = E \qquad \text{(I-6)}$$

Wave functions issued from the probabilistic theory and from the deterministic theory are therefore proportional in a ratio of square root of E.

The wave function ψ becomes homogeneous with the square root of a volume energy density. And since there are negative energies, the complex number $j = \sqrt{-1}$ appears in a natural way and with a clear physical meaning in the solutions of the Dirac equation describing both positive energy particles and particles of negative energies.

I - 1 The different kinds of energy

In special relativity, four-vectors are identified as invariant physical quantities by change of frame. It follows that their pseudo-norm is constant and does not depends on the frame in which it is evaluated.

For a mass m moving at speed v, four-vector momentum-energy is expressed as follows:

$$\tilde{P} = \begin{pmatrix} m\dfrac{dx}{dt} \\ m\dfrac{dy}{dt} \\ m\dfrac{dz}{dt} \\ \dfrac{E}{c} \end{pmatrix} = \begin{pmatrix} mv_x \\ mv_y \\ mv_z \\ \dfrac{E}{c} \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ \dfrac{E}{c} \end{pmatrix} \qquad \text{(I-7)}$$

In this expression, E means the whole energy of the particle of mass m, moving with a speed v, and therefore linked to rest mass by the relationship:

$$m = \frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad \text{(I-8)}$$

The pseudo-norm raised to the square of the four-vector pulse energy is written:

$$\left(m\frac{dx}{dt}\right)^2 + \left(m\frac{dy}{dt}\right)^2 + \left(m\frac{dz}{dt}\right)^2 - \left(\frac{E}{c}\right)^2 = cte \qquad \text{(I-9)}$$

or again:

$$E^2 - p^2c^2 = cte \qquad \text{(I-10)}$$

where p is the pulse module:

$$p = \sqrt{\left(m\frac{dx}{dt}\right)^2 + \left(m\frac{dy}{dt}\right)^2 + \left(m\frac{dz}{dt}\right)^2} = m\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = mv \qquad \text{(I-11)}$$

One determines the constant by writing that, according to the relation which links energy and mass, the total energy of a particle of mass m is also equal to:

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad \text{(I-12)}$$

where m_0 is the rest mass.

We can deduce the value of the constant:

$$E^2 - p^2c^2 = \left(\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - \left(\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 c^2 = \left(m_0c^2\right)^2 \qquad \text{(I-13)}$$

and finally the expression of the conservation of energy in special relativity:

$$E^2 = (pc)^2 + \left(m_0c^2\right)^2 \qquad \text{(I-14)}$$

We accept as a postulate from the expression above, that within a particle of mass m and moving at speed v, there may be only three specific forms of energy:

- A form of energy representative of the whole energy E, which we will subsequently justify the designation of wave energy
- A form of energy representing pulse energy: pc
- A form of energy representing the mass energy at rest: m_0c^2

Let us also assume that, in the world of the infinitely small, these three forms of energy are distinct, in the same way as electrical and magnetic energy in electromagnetism, or potential and kinetic energy in mechanics.

I - 2 Stationary energy

Consider a particle at rest, and therefore which has no pulse energy. One is led to the assumption that its mass energy is located in a finite spatial extent. This energy is constant in time, and we therefore call it stationary energy. Since there is no pulse energy, the total energy in this particular case, that is to say the wave energy, is equal to the mass energy.

We then ask about what we know on the stationarity of the energy in the physics of waves in general. This stationarity is a property of systems that retain the trapped wave in a limited portion of the space. If we assume the system without loss, and if we introduce the wave energy in this system, it is preserved in time.

The fact of importance is that this energy never stays at rest in the portion of the space where it is confined, but settles in stationary modes that include at least two types of energies that are exchanged in general in time quadrature.

In mechanics, the waves can propagate in a medium with certain elasticity, and exchange occurs between the elastic potential energy and kinetic energy of the moving elements.

In electromagnetism, the exchange is between the electric and magnetic energy.

If we now return to the situation of a particle at rest that contains both of the wave energy and the mass energy, we conjecture that these two forms of energy exchange in the portion of the space where they are confined. This assumption stems directly from the particle behavior under the form of a wave or a particle.

If the particle is moving, it appears a third form of energy which is the impulse energy. We conjecture that this energy will participate in exchanges with the two previous ones in a

form that remains to be determined, but which verifies the equation of conservation of energy (I-14).

Later in this document, we will not wonder on the way in which the different types of energy are confined in a region of space: the answer to this question is not known to us. But we will show that if we make the assumption that the different kinds of energy exchange in stationary modes, this led to a quantum physics deterministic and consistent with the actual probabilistic theory.

II

DIRAC equation

The DIRAC equation is part of some fundamental equations of contemporary physics. It was obtained in the wake of two other very important equations of quantum mechanics: the SCHRÖDINGER equation and the KLEIN-GORDON equation. It has supplanted the latter two because it describes with more precision the reality of observed phenomena.

The objective of this short chapter of presentation is not to discuss the contribution of this equation on quantum physics that the reader will find in books and websites related to this subject and to which we will return later. It's just, in a first time, to retrace the approach leading to the DIRAC equation, and underline some difficulties on the physical interpretation of its solution.

I - The SCHRÖDINGER equation

Any linear physical phenomenon $\psi(x, y, z, t)$ which propagates at speed v in a three dimensions space as time flows can be described by a wave equation:

$$\nabla^2\psi(x,y,z,t) - \frac{1}{v^2}\frac{\partial^2\psi(x,y,z,t)}{\partial t^2} = 0 \qquad \text{(II-1)}$$

When dealing with a signal with a sinusoidal temporal variation, we can put:

$$\psi(x,y,z,t) = \varphi(x,y,z)\exp(\pm j\omega t) \qquad \text{(II-2)}$$

and the wave equation for the quantity φ independent of time becomes:

$$\nabla^2\varphi(x,y,z) + \frac{\omega^2}{v^2}\varphi(x,y,z) = 0 \qquad \text{(II-3)}$$

Following Louis De BROGLIE assuming that one can associate to any particle of mass m moving at a velocity v a wave of wavelength $\lambda = h/mv$, where h is the PLANCK constant, the time-independent wave equation attached to the particle becomes:

$$\nabla^2\varphi(x,y,z) + \frac{\omega^2}{v^2}\varphi(x,y,z) = \nabla^2\varphi(x,y,z) + \left(\frac{2\pi}{\lambda}\right)^2\varphi(x,y,z) = 0 \qquad \text{(II-4)}$$

On the other hand, in the context of classical mechanics, energy total E of this particle is the sum of its kinetic energy E_c and its potential energy E_p, which allows to write:

$$E_c = \frac{1}{2}mv^2 = E - E_p \qquad \text{(II-5)}$$

from which is deducting successively:

$$mv = \sqrt{2m(E - E_p)} \qquad \text{(II-6)}$$

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2m(E - E_p)}} \qquad \text{(II-7)}$$

This last relationship put in the wave equation (I-4) provides the time-independent SCHRÖDINGER equation:

$$\nabla^2\varphi(x,y,z) + \frac{2m}{\hbar^2}(E - E_p)\varphi(x,y,z) = 0 \qquad \text{with } \hbar = \frac{h}{2\pi} \qquad \text{(II-8)}$$

The heuristic approach that has been proposed for this relationship can be completed to bring up the time dependence.

For a locally flat, monochromatic wave going away towards infinity, we can write the wave function in a general manner under the form:

$$\psi(\vec{r},t) = \psi_0 \exp\left[j(\vec{k}\vec{r} - \omega t)\right] \text{ with } j^2 = -1 \qquad \text{(II-9)}$$

In the quantum world where $\lambda = h/mv$ and where the whole energy E is related to the frequency of the wave by the relation:

$$E = h\nu = \frac{h}{2\pi}\omega = \hbar\omega \qquad \text{(II-10)}$$

we obtain by substitution:

$$\psi(\vec{r},t) = \psi_0 \exp\left[\frac{j}{\hbar}(\vec{p}\vec{r} - Et)\right] \text{ with } \vec{p} = m\vec{v} \qquad \text{(II-11)}$$

In deriving this expression over time, it follows:

$$\frac{\partial}{\partial t}(\psi(\vec{r},t)) = -\frac{j}{\hbar}E\psi_0 \exp\left[\frac{j}{\hbar}(\vec{p}\vec{r} - Et)\right] = -\frac{j}{\hbar}E\psi(\vec{r},t) \qquad \text{(II-12)}$$

and taking the gradient of this same expression (II-11):

$$\vec{\nabla}(\psi(\vec{r},t)) = \frac{j}{\hbar}\vec{p}\psi(\vec{r},t) \qquad \text{(II-13)}$$

From the relations (II-12) and (II-13), are deduced the quantization rules of classical or relativistic mechanics equations which allow to obtain similar equations in the quantum field:

$$j\hbar\frac{\partial}{\partial t}(\psi) = E\psi \qquad \text{(II-14)}$$

$$-j\hbar\vec{\nabla}(\psi) = \vec{p}\psi \qquad \text{(II-15)}$$

For a particle whose total energy is given in classical mechanics by the sum of kinetic energy and potential energy:

$$E = E_c + E_p = \frac{1}{2}mv^2 + E_p = \frac{p^2}{2m} + E_p \qquad \text{(II-16)}$$

It is deduced after multiplication by the wave function ψ:

$$E\psi = \frac{p^2}{2m}\psi + E_p\psi \qquad \text{(II-17)}$$

and finally by using the rules of quantification (II-14) (II-15):

$$j\hbar\frac{\partial}{\partial t}(\psi) = \frac{\left(-j\hbar\vec{\nabla}\right)^2}{2m}(\psi) + E_p\psi = -\frac{\hbar^2}{2m}\nabla^2(\psi) + E_p\psi \qquad \text{(II-18)}$$

which is the time dependent SCHRÖDINGER equation.

This equation allows to find the main series describing the emission lines of the hydrogen atom as well as other parameters like the diameters of the BOHR orbit, but it is in default when there is interest in more subtle phenomena such as the fine structure levels for a hydrogen atom or taking into account of the spin of the electron.

II - The KLEIN-GORDON equation

The inadequacies of the SCHRÖDINGER equation are attributed to the fact that the quantization rules have been applied to an energy balance using classical mechanics.

Relativity introduces a relationship whose scope is much broader because it is invariant under the LORENTZ transformation:

$$E^2 = p^2c^2 + m_0^2c^4 \qquad \text{(II-19)}$$

and we obtain immediately by applying the rules of quantification (II-14) (II-15):

$$-\hbar^2\frac{\partial^2}{\partial t^2}(\psi) = -\hbar^2c^2\nabla^2(\psi) + m_0^2c^4\psi \qquad \text{(II-20)}$$

which is the KLEIN-GORDON equation. We can present it in a form that make the wave equation appearing with a second member:

$$\nabla^2(\psi) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\psi) = \frac{m_0^2c^2}{\hbar^2}\psi \qquad \text{(II-21)}$$

A detailed review of its solutions allows to show some inadequacies, particularly in his inability to describe the spin of the electron which is of order 1/2 as well as some fine levels of energy of the hydrogen atom.

III - The DIRAC equation

Beginning with the relativistic energy conservation equation:

$$E^2 = p^2c^2 + m_0^2c^4 \qquad \text{(II-22)}$$

and applying the rules of quantification of quantum mechanics:

$$E \to j\hbar\frac{\partial}{\partial t} \qquad p_x \to -j\hbar\frac{\partial}{\partial x} \qquad p_y \to -j\hbar\frac{\partial}{\partial y} \qquad p_z \to -j\hbar\frac{\partial}{\partial z} \qquad \text{(II-23)}$$

It transforms the KLEIN-GORDON equation in the formula:

$$\left(j\hbar\frac{\partial}{\partial t}\right)^2(\psi) = \left(-j\hbar c\frac{\partial}{\partial x}\right)^2(\psi) + \left(-j\hbar c\frac{\partial}{\partial y}\right)^2(\psi) + \left(-j\hbar c\frac{\partial}{\partial z}\right)^2(\psi) + m_0^2c^4(\psi) \qquad \text{(II-24)}$$

According to the formalism proposed by DIRAC, we have to find coefficients α_i such that the above equation is verified in the form of a first order partial differential equation raised to square:

$$\left(j\hbar\frac{\partial}{\partial t}\right)^2(\psi) = \left[\alpha_1\left(-j\hbar c\frac{\partial}{\partial x}\right) + \alpha_2\left(-j\hbar c\frac{\partial}{\partial y}\right) + \alpha_3\left(-j\hbar c\frac{\partial}{\partial z}\right) + \alpha_0\left(m_0c^2\right)\right]^2(\psi) \qquad \text{(II-25)}$$

If the relationship above is checked through the presence of coefficients α_i, the solution can be obtained by solving the equation obtained by removing the squares:

$$\left(j\hbar\frac{\partial}{\partial t}\right)(\psi) = \left[\alpha_1\left(-j\hbar c\frac{\partial}{\partial x}\right) + \alpha_2\left(-j\hbar c\frac{\partial}{\partial y}\right) + \alpha_3\left(-j\hbar c\frac{\partial}{\partial z}\right) + \alpha_0\left(m_0c^2\right)\right](\psi) \qquad \text{(II-26)}$$

It is impossible to find a α_i real or complex in response to conditions (II-25) and (II-26), but you can find matrices α_i who meet the following, necessary and sufficient conditions to transform equation (II-24) into equation (II-25):

$$\alpha_i^{\,2} = 1 \qquad \text{(II-27)}$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \text{(II-28)}$$

These matrices are not unique, but a simple choice is that proposed by Dirac:

$$\alpha_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{II-29}$$

The following three are defined from the called PAULI matrices σ_i:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{II-30}$$

or still explicitly:

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -j \\ 0 & 0 & j & 0 \\ 0 & -j & 0 & 0 \\ j & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{II-31}$$

The structure of the matrixes α_i requires the search for the solution ψ in the form of a column vector:

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{II-32}$$

In order to put the system of differential equations in a presentation such as the mass energy appears in factor with the identity matrix, we multiply the equation (II-26) by the matrix α_0. It is known that its square is equal to the identity matrix:

$$\alpha_0\left(j\hbar\frac{\partial}{\partial t}\right)(\psi) = \left[\alpha_0\alpha_1\left(-j\hbar c\frac{\partial}{\partial x}\right) + \alpha_0\alpha_2\left(-j\hbar c\frac{\partial}{\partial y}\right) + \alpha_0\alpha_3\left(-j\hbar c\frac{\partial}{\partial z}\right) + \left(m_0c^2\right)\right](\psi) \tag{II-33}$$

We put then:

$$\gamma_0 = \alpha_0 \qquad \gamma_1 = \alpha_0\alpha_1 \qquad \gamma_2 = \alpha_0\alpha_2 \qquad \gamma_3 = \alpha_0\alpha_3 \tag{II-34}$$

and we gather all terms containing partial derivatives:

$$\left[\gamma_0\left(\frac{\partial}{\partial t}\right) + \gamma_1\left(c\frac{\partial}{\partial x}\right) + \gamma_2\left(c\frac{\partial}{\partial y}\right) + \gamma_3\left(c\frac{\partial}{\partial z}\right)\right](\psi) = \frac{m_0c^2}{j\hbar}(\psi) \tag{II-35}$$

The matrix identity, implicit in the right term will be omitted in the rest of the document.

The matrices γ_i are obtained explicitly as follows:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{II-36}$$

$$\gamma_1 = \alpha_0\alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{II-37}$$

$$\gamma_2 = \alpha_0\alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -j \\ 0 & 0 & j & 0 \\ 0 & -j & 0 & 0 \\ j & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -j \\ 0 & 0 & j & 0 \\ 0 & j & 0 & 0 \\ -j & 0 & 0 & 0 \end{pmatrix} \tag{II-38}$$

$$\gamma_3 = \alpha_0\alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{II-39}$$

The arrangement of these matrices can be synthesized on the basis of the PAULI matrices:

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{II-40}$$

and we can check the following property:

$$(\gamma_0)^2 = 1, (\gamma_1)^2 = (\gamma_2)^2 = (\gamma_3)^2 = -1 \tag{II-41}$$

Considering now that the quantity ψ of the initial wave equation arises in the form of a vector with 4 column, complete writing of the matrix system is given by:

$$\frac{m_0 c}{\hbar}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=\frac{j}{c}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial t}\\ \frac{\partial\psi_1}{\partial t}\\ \frac{\partial\psi_2}{\partial t}\\ \frac{\partial\psi_3}{\partial t}\end{pmatrix}+j\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial x}\\ \frac{\partial\psi_1}{\partial x}\\ \frac{\partial\psi_2}{\partial x}\\ \frac{\partial\psi_3}{\partial x}\end{pmatrix}$$
$$+j\begin{pmatrix}0&0&0&-j\\0&0&j&0\\0&j&0&0\\-j&0&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial y}\\ \frac{\partial\psi_1}{\partial y}\\ \frac{\partial\psi_2}{\partial y}\\ \frac{\partial\psi_3}{\partial y}\end{pmatrix}+j\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial z}\\ \frac{\partial\psi_1}{\partial z}\\ \frac{\partial\psi_2}{\partial z}\\ \frac{\partial\psi_3}{\partial z}\end{pmatrix} \qquad \text{(II-42)}$$

or again, in a more condensed manner, under the form of a partial derivatives equations system of 4 equations, after putting $\eta=\frac{m_0 c}{\hbar}$:

$$\begin{aligned}\eta\psi_0&=j\frac{\partial\psi_0}{\partial(ct)}+j\frac{\partial\psi_3}{\partial x}+\frac{\partial\psi_3}{\partial y}+j\frac{\partial\psi_2}{\partial z}\\ \eta\psi_1&=j\frac{\partial\psi_1}{\partial(ct)}+j\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_2}{\partial y}-j\frac{\partial\psi_3}{\partial z}\\ \eta\psi_2&=-j\frac{\partial\psi_2}{\partial(ct)}-j\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_1}{\partial y}-j\frac{\partial\psi_0}{\partial z}\\ \eta\psi_3&=-j\frac{\partial\psi_3}{\partial(ct)}-j\frac{\partial\psi_0}{\partial x}+\frac{\partial\psi_0}{\partial y}+j\frac{\partial\psi_1}{\partial z}\end{aligned} \qquad \text{(II-43)}$$

These equations show that the 4 quantity associated with the wave of material ψ interact, without it is possible to specify in a clear and detailed manner the physical nature of their interaction. The representation in terms of spinors provides a rigorous framework on which are based of multiple developments, but the notion of spinor remains abstract, despite the important efforts of the scientific community to give concrete illustrations.

III

DIRAC bi-spinors

In the previous chapter, we concluded that solutions of the Dirac equation are presented in the form of a quantity with 4 components:

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \qquad \text{(III-1)}$$

A careful review of the system of Dirac:

$$\begin{aligned} \eta\psi_0 &= \mathrm{j}\frac{\partial\psi_0}{\partial(\mathrm{ct})} + \mathrm{j}\frac{\partial\psi_3}{\partial\mathrm{x}} + \frac{\partial\psi_3}{\partial\mathrm{y}} + \mathrm{j}\frac{\partial\psi_2}{\partial\mathrm{z}} \\ \eta\psi_1 &= \mathrm{j}\frac{\partial\psi_1}{\partial(\mathrm{ct})} + \mathrm{j}\frac{\partial\psi_2}{\partial\mathrm{x}} - \frac{\partial\psi_2}{\partial\mathrm{y}} - \mathrm{j}\frac{\partial\psi_3}{\partial\mathrm{z}} \\ \eta\psi_2 &= -\mathrm{j}\frac{\partial\psi_2}{\partial(\mathrm{ct})} - \mathrm{j}\frac{\partial\psi_1}{\partial\mathrm{x}} - \frac{\partial\psi_1}{\partial\mathrm{y}} - \mathrm{j}\frac{\partial\psi_0}{\partial\mathrm{z}} \\ \eta\psi_3 &= -\mathrm{j}\frac{\partial\psi_3}{\partial(\mathrm{ct})} - \mathrm{j}\frac{\partial\psi_0}{\partial\mathrm{x}} + \frac{\partial\psi_0}{\partial\mathrm{y}} + \mathrm{j}\frac{\partial\psi_1}{\partial\mathrm{z}} \end{aligned} \qquad \text{(III-2)}$$

shows that these solutions can be grouped by 2 in a behavior with analogies. To display these analogies, it is opportune to introduce the system in a slightly different arrangement:

$$\begin{aligned} \mathrm{j}\frac{\partial\psi_0}{\partial(\mathrm{ct})} &= \eta\psi_0 - \mathrm{j}\frac{\partial\psi_3}{\partial\mathrm{x}} - \frac{\partial\psi_3}{\partial\mathrm{y}} - \mathrm{j}\frac{\partial\psi_2}{\partial\mathrm{z}} \\ \mathrm{j}\frac{\partial\psi_1}{\partial(\mathrm{ct})} &= \eta\psi_1 - \mathrm{j}\frac{\partial\psi_2}{\partial\mathrm{x}} + \frac{\partial\psi_2}{\partial\mathrm{y}} + \mathrm{j}\frac{\partial\psi_3}{\partial\mathrm{z}} \\ \mathrm{j}\frac{\partial\psi_2}{\partial(\mathrm{ct})} &= -\eta\psi_2 - \mathrm{j}\frac{\partial\psi_1}{\partial\mathrm{x}} - \frac{\partial\psi_1}{\partial\mathrm{y}} - \mathrm{j}\frac{\partial\psi_0}{\partial\mathrm{z}} \\ \mathrm{j}\frac{\partial\psi_3}{\partial(\mathrm{ct})} &= -\eta\psi_3 - \mathrm{j}\frac{\partial\psi_0}{\partial\mathrm{x}} + \frac{\partial\psi_0}{\partial\mathrm{y}} + \mathrm{j}\frac{\partial\psi_1}{\partial\mathrm{z}} \end{aligned} \qquad \text{(III-3)}$$

Therefore, if we put:

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \qquad \text{with } \varphi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \qquad \text{and} \qquad \chi = \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} \tag{III-4}$$

Dirac system breaks down into two coupled systems:

$$j\frac{\partial}{\partial(ct)}\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \eta\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} - j\begin{pmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - j\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + j\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} \tag{III-5}$$

$$j\frac{\partial}{\partial(ct)}\begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} = -\eta\begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} - j\begin{pmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - j\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + j\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \tag{III-6}$$

in which it can be shown that the quantities to both φ and χ components behave as mathematical objects known as spinors.

A relief of writing is proposed by calling M the matrix:

$$M = -j\begin{pmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - j\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + j\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{pmatrix} \tag{III-7}$$

The coupled system (III-5, 6) in terms of spinors then takes the form:

$$\begin{aligned} j\frac{\partial}{\partial(ct)}(\varphi) &= \eta(\varphi) + M(\chi) \\ j\frac{\partial}{\partial(ct)}(\chi) &= -\eta(\chi) + M(\varphi) \end{aligned} \tag{III-8}$$

or still:

$$j\frac{\partial}{\partial(ct)}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \eta & M \\ M & -\eta \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{III-9}$$

In the absence of additional assumptions, it is impossible to go forward in the writing of the cross-relationships between two spinors.

On the other hand, we can seek special solutions with a temporal dependence in exp(-jωt) with a total energy $E = \hbar\omega$, hence a time in dependence $\exp\left(-j\frac{E}{\hbar}t\right)$.

It is important to note that the j which appears in the exponential has nothing to do with the complex representation of a physical wave in cos(ωt). Its physical meaning already discussed in chapter I, will be detailed in the following chapters.

After taking account of the time derivation in the relationship (III-9), the coupled spinors system has the form:

$$\frac{E}{\hbar c}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=\begin{pmatrix}\eta & M\\ M & -\eta\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \qquad \text{(III-10)}$$

or again taking account of $\eta=\frac{mc}{\hbar}$:

$$E\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=\begin{pmatrix}mc^2 & \hbar cM\\ \hbar cM & -mc^2\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \qquad \text{(III-11)}$$

A rigorous writing should reveal the 2 dimension identity matrix multiplied by η for the relationship (III-10) terms and the terms in mc² for relationship (III-11). The use wants when this matrix is implied, as necessary for the coherence of the dimensions, it is not necessarily indicated for relief of writing.

I - Spinors for a not moving particle

It is possible to show solutions with positive energy that describe the particles of mass m, and negative energy solutions that describe the same anti-particles mass. To show this property, we treat the case of immobile particles: if x, y, z are fixed and constant, then the derivatives with respect to x, y, and z are zero, and the matrix M is zero. We derive from (III-11):

$$E\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=\begin{pmatrix}mc^2 & 0\\ 0 & -mc^2\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \qquad \text{(III-12)}$$

or still:

$$\begin{aligned}E(\varphi)&=mc^2(\varphi)\\ E(\chi)&=-mc^2(\chi)\end{aligned} \qquad \text{(III-13)}$$

We identify that the energy associated with the spinor φ is equal to its mass energy, while the one associated with the spinor χ is equal to its opposite, and thus represents the energy associated with the anti-particle. We can deduce that the spinor φ is associated with the particle, while the spinor χ is associated with the antiparticle. The solution to the Dirac equation, which includes two spinors simultaneously, allows to describe both the behavior of the particle and its antiparticle.

In a very general manner, any spinor φ can be decomposed on the canonical basis in the following manner:

$$\varphi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \psi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(III-14)}$$

where the two independent spinors of the canonical basis are related to two possible spin for an electron States. The practice is that the spinor $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is associated with the state of spin said «up» (↑), and the spinor $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is associated with the state of spin said «down» (↓).

As for the particle with negative energy, we adopt the following decomposition:

$$\chi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(III-15)}$$

and identify the two spin states described above.

Gathering these partial results, we can represent four distinct states of the DIRAC bi-spinors in the frame where the particle is at rest:

$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ Particle at positive energy in spin "up".

$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ "Particle at positive energy in spin "down".

$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ Particle with negative energy in spin "up".

$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ Particle with negative energy in spin "down".

II - Spinors for a particle in motion

When the particle is moving, the matrix M is not null, and we use the relationship (III-11) called for memory:

$$E\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=\begin{pmatrix}mc^2 & \hbar cM\\ \hbar cM & -mc^2\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \qquad \text{(III-16)}$$

This relationship shows that two spinors interact and that they are not independent of one another. As a necessary consequence, we can think that particle and antiparticle form a whole, and that one cannot move without the other is associated with this movement.

It comes, developing the matrix relationship (III-16):

$$\begin{aligned}E(\varphi)&=mc^2(\varphi)+\hbar cM(\chi)\\ E(\chi)&=\hbar cM(\varphi)-mc^2(\chi)\end{aligned} \qquad \text{(III-17)}$$

or still

$$\begin{aligned}(E-mc^2)(\varphi)&=\hbar cM(\chi)\\ (E+mc^2)(\chi)&=\hbar cM(\varphi)\end{aligned} \qquad \text{(III-18)}$$

or still:

$$\begin{aligned}(\varphi)&=\frac{\hbar c}{E-mc^2}M(\chi)\\ (\chi)&=\frac{\hbar c}{E+mc^2}M(\varphi)\end{aligned} \qquad \text{(III-19)}$$

These relationships indicate that in the special case where we fit, that is to say in the case where the 4 elements of the Dirac bispinors have a time in dependence $\exp\left(-j\frac{E}{\hbar}t\right)$, from the knowledge of one of the spinors, we can deduce the other.

It is possible to develop a more advanced formalism of relations (III-19) above. We must take again the matrix M and describe it on the basis of the PAULI matrices called for memory:

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_2=\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix} \qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \text{(III-20)}$$

It gives:

$$M=-j\begin{pmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}-j\frac{\partial}{\partial y}\\ \frac{\partial}{\partial x}+j\frac{\partial}{\partial y} & -\frac{\partial}{\partial z}\end{pmatrix}=-j\left(\frac{\partial}{\partial x}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix}+\frac{\partial}{\partial z}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) \qquad \text{(III-21)}$$

We introduce by substituting the partial derivatives (III-21), the operator pulse of quantum mechanics, whose three components are explicitly:

$$\hat{p}_x = -j\hbar\frac{\partial}{\partial x} \qquad \hat{p}_y = -j\hbar\frac{\partial}{\partial y} \qquad \hat{p}_z = -j\hbar\frac{\partial}{\partial z} \tag{III-22}$$

This allows to present the M matrix in the form:

$$M = \frac{1}{\hbar}\left\{\hat{p}_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \hat{p}_y\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix} + \hat{p}_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\} \tag{III-23}$$

It follows an abuse of writing that would give cold sweats to more than a teacher of mathematics:

We call:

$$\vec{\sigma} = \begin{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ \begin{pmatrix}0 & -j\\ j & 0\end{pmatrix}\\ \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{pmatrix} \tag{III-24}$$

a "vector" containing the three PAULI matrices, and:

$$\vec{p} = \begin{pmatrix}-j\hbar\frac{\partial}{\partial x}\\ -j\hbar\frac{\partial}{\partial y}\\ -j\hbar\frac{\partial}{\partial z}\end{pmatrix} \tag{III-25}$$

a "vector" which represents the pulse operator of the quantum mechanics.

Therefore the relation expressing the matrix M (III-23) may come under the following condensed form, using the usual formalism of the scalar product between two vectors:

$$M = \frac{\vec{\sigma}.\vec{p}}{\hbar} \tag{III-26}$$

We can deduce the condensed writing which expresses the relationship between the two spinors which constitute the solution of DIRAC, in the event of a time in dependence $\exp\left(-j\frac{E}{\hbar}t\right)$:

$$\begin{aligned}(\varphi) &= \frac{\vec{\sigma}.\vec{p}c}{E - mc^2}(\chi)\\ (\chi) &= \frac{\vec{\sigma}.\vec{p}c}{E + mc^2}(\varphi)\end{aligned} \tag{III-27}$$

By substitution of one relationship in the other, for example the second in the first, we obtain:

$$(\varphi)=\frac{\vec{\sigma}.\vec{p}c}{E-mc^2}\frac{\vec{\sigma}.\vec{p}c}{E+mc^2}(\varphi)=\frac{(\vec{\sigma}.\vec{p}c)^2}{E^2-(mc^2)^2}(\varphi) \qquad \text{(III-28)}$$

Using the property of the PAULI matrices whose square gives the identity matrix, yields finally:

$$(\varphi)=\frac{(\vec{p}c)^2}{E^2-(mc^2)^2}(\varphi) \qquad \text{(III-29)}$$

This relationship will be checked only if:

$$(\vec{p}c)^2=E^2-(mc^2)^2 \qquad \text{(III-30)}$$

or again:

$$(\vec{p}c)^2+(mc^2)^2=E^2 \qquad \text{(III-31)}$$

This relationship confirms the consistency of solutions expressed with relativistic energy conservation, but also with the possibility of solutions with negative energies of the form:

$$E=\pm\sqrt{(\vec{p}c)^2+(mc^2)^2} \qquad \text{(III-32)}$$

IV

Spin ½ of the electron

The solution in terms of spinors mentioned in the previous chapter suggests that there is within the electron something that spins, without knowing precisely what. We will therefore focus in this chapter on the kinematics of rotation, and try to put in relation with the solution to the system of DIRAC.

I - The concept of angular momentum

For a material point of mass m, located in M, and moving around an origin O, the kinetic momentum that will be designated by L is expressed as the pulse momentum:

$$\vec{L} = \overrightarrow{OM} \wedge \vec{p} = \overrightarrow{OM} \wedge m\vec{v} \quad \text{(IV-1)}$$

It is a vector quantity which is brought by the axis of rotation.

If one denotes by x, y, z, the components of the position vector, and p_x, $p_y p_z$, the components of the momentum vector:

$$\overrightarrow{OM} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \qquad \text{(IV-2)}$$

we get the components of the angular momentum in the form:

$$\begin{aligned} L_x &= y p_z - z p_y \\ L_y &= z p_x - x p_z \\ L_z &= x p_y - y p_x \end{aligned} \qquad \text{(IV-3)}$$

The transition to quantum mechanics imposes one overrides the "position"' quantities and "quantity of motion" with operators "position" and "quantity of motion", which allows to define the components of the angular momentum operator in the form:

$$\begin{aligned} \hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{aligned} \qquad \text{(IV-4)}$$

where the definition of the position and pulse momentum, applied to a wave function ψ operators is recalled below:

$$\hat{p}_x(\psi) = -j\hbar\frac{\partial\psi}{\partial x} \qquad \hat{p}_y(\psi) = -j\hbar\frac{\partial\psi}{\partial y} \qquad \hat{p}_z(\psi) = -j\hbar\frac{\partial\psi}{\partial z} \qquad \text{(IV-5)}$$
$$\hat{x}(\psi) = x\psi \qquad \hat{y}(\psi) = y\psi \qquad \hat{z}(\psi) = z\psi$$

The description of the above angular momentum operator allows to treat without problem the orbital angular momentum of a particle of mass m which revolves around an origin, but she is distressed to describe the angular momentum of spin. This is an internal movement to the particle which is poorly known and we don't know how to express it in terms of the position and momentum operators.

To overcome this difficulty, it was necessary to look for a property of the kinetic moment which do not involve the position and momentum operators, and that contains sufficient information to describe the rotation.

This property is constructed from a function called switch whose we can give the following definition in quantum mechanics:

If A and B are two operators, the switch of A and B is defined by:

$$[A,B] = AB - BA \qquad \text{(IV-6)}$$

For example, we can show that the operators position and speed applied to a wave function do not switch, which is to say that these two operators switch is non-zero:

$$\left[x, \frac{\partial}{\partial x}\right](\psi) = x\left(\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial x}(x\psi) = x\left(\frac{\partial\psi}{\partial x}\right) - \psi - x\left(\frac{\partial\psi}{\partial x}\right) = -\psi \qquad \text{(IV-7)}$$

If we built switches of angular momentum operators (IV-4), they have the following property:

$$\begin{aligned}
\left[\hat{L}_x, \hat{L}_y\right](\psi) &= \left(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x\right)(\psi) = j\hbar\hat{L}_z(\psi) \\
\left[\hat{L}_z, \hat{L}_x\right](\psi) &= \left(\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z\right)(\psi) = j\hbar\hat{L}_y(\psi) \\
\left[\hat{L}_y, \hat{L}_z\right](\psi) &= \left(\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y\right)(\psi) = j\hbar\hat{L}_x(\psi)
\end{aligned} \qquad \text{(IV-8)}$$

The demonstration is obtained directly from the definition of angular momentum operators (IV-4). On the example of the first line of (IV - 8) yields successively:

$$\hat{L}_x(\psi) = -j\hbar\left(y\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial y}\right) \qquad \hat{L}_y(\psi) = -j\hbar\left(z\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial z}\right) \qquad \text{(IV-9)}$$

$$\hat{L}_y\left(\hat{L}_x(\psi)\right) = -\hbar^2\left\{z\frac{\partial}{\partial x}\left(y\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial y}\right) - x\frac{\partial}{\partial z}\left(y\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial y}\right)\right\} \qquad \text{(IV-10)}$$

$$\hat{L}_y\left(\hat{L}_x(\psi)\right)=-\hbar^2\left\{zy\frac{\partial^2\psi}{\partial x\partial z}-z^2\frac{\partial^2\psi}{\partial x\partial y}-xy\frac{\partial^2\psi}{\partial z^2}+x\frac{\partial f}{\partial y}+xz\frac{\partial^2\psi}{\partial z\partial y}\right\} \qquad \text{(IV-11)}$$

and by reversing the order of operators:

$$\hat{L}_x\left(\hat{L}_y(\psi)\right)=-\hbar^2\left\{y\frac{\partial}{\partial z}\left(z\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial z}\right)-z\frac{\partial}{\partial y}\left(z\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial z}\right)\right\} \qquad \text{(IV-12)}$$

$$\hat{L}_x\left(\hat{L}_y(\psi)\right)=-\hbar^2\left\{y\frac{\partial\psi}{\partial x}+yz\frac{\partial^2\psi}{\partial x\partial z}-yx\frac{\partial^2\psi}{\partial z^2}-z^2\frac{\partial^2\psi}{\partial y\partial x}+xz\frac{\partial^2\psi}{\partial z\partial y}\right\} \qquad \text{(IV-13)}$$

By subtraction of (IV-13) from (IV-11), yields the result presented in (IV-8):

$$\hat{L}_x\left(\hat{L}_y(\psi)\right)-\hat{L}_y\left(\hat{L}_x(\psi)\right)=-\hbar^2\left\{y\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial y}\right\}=j\hbar\hat{L}_z \qquad \text{(IV-14)}$$

The property (IV-8) no longer depends on the position and momentum operators. Even if the image should be taken with care, one has somehow built a system of three equations and three unknowns which presents for these unknown similar properties to those that are generated by relations (IV-4) that define the angular momentum in quantum mechanics.

II - The spin angular momentum operator

The reasoning of the previous paragraph has enabled us to clarify the relationships that define an operator of angular momentum for a wave function ψ. This operator is defined from the relationships of commutation recalled to memory:

$$\begin{aligned}\left[\hat{L}_x,\hat{L}_y\right](\psi)&=\left(\hat{L}_x\hat{L}_y-\hat{L}_y\hat{L}_x\right)(\psi)=j\hbar\hat{L}_z(\psi)\\ \left[\hat{L}_z,\hat{L}_x\right](\psi)&=\left(\hat{L}_z\hat{L}_x-\hat{L}_x\hat{L}_z\right)(\psi)=j\hbar\hat{L}_y(\psi)\\ \left[\hat{L}_y,\hat{L}_z\right](\psi)&=\left(\hat{L}_y\hat{L}_z-\hat{L}_z\hat{L}_y\right)(\psi)=j\hbar\hat{L}_x(\psi)\end{aligned} \qquad \text{(IV-15)}$$

Solutions issue from the formulation of DIRAC appear in the form of spinors, that is to say in the form of a mathematical being containing two wave functions. In the previous chapter, we showed for example that the solution to positive energy was represented by the spinor:

$$\varphi=\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} \qquad \text{(IV-16)}$$

The question that arises is: how can we generalize the (IV-15) relationships that define the operator of angular momentum for a wave function, in order to define an operator of angular momentum for a spinor that contains not one but two wave functions?

In other words, if S is one such operator, we want to be able to write:

$$\begin{aligned}
\left[\hat{S}_x,\hat{S}_y\right](\varphi) &= \left(\hat{S}_x\hat{S}_y-\hat{S}_y\hat{S}_x\right)(\varphi) = j\hbar\hat{S}_z(\varphi)\\
\left[\hat{S}_z,\hat{S}_x\right](\varphi) &= \left(\hat{S}_z\hat{S}_x-\hat{S}_x\hat{S}_z\right)(\varphi) = j\hbar\hat{S}_y(\varphi)\\
\left[\hat{S}_y,\hat{S}_z\right](\varphi) &= \left(\hat{S}_y\hat{S}_z-\hat{S}_z\hat{S}_y\right)(\varphi) = j\hbar\hat{S}_x(\varphi)
\end{aligned} \qquad \text{(IV-17)}$$

or still:

$$\begin{aligned}
\left[\hat{S}_x,\hat{S}_y\right]\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} &= \left(\hat{S}_x\hat{S}_y-\hat{S}_y\hat{S}_x\right)\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = j\hbar\hat{S}_z\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}\\
\left[\hat{S}_z,\hat{S}_x\right]\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} &= \left(\hat{S}_z\hat{S}_x-\hat{S}_x\hat{S}_z\right)\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = j\hbar\hat{S}_y\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}\\
\left[\hat{S}_y,\hat{S}_z\right]\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} &= \left(\hat{S}_y\hat{S}_z-\hat{S}_z\hat{S}_y\right)\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = j\hbar\hat{S}_x\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}
\end{aligned} \qquad \text{(IV-18)}$$

It appears that operators $\hat{S}_x,\hat{S}_y,\hat{S}_z$ can no longer be defined as simple operators used for a single wave function: they must be made by a matrix 2 X 2 of operators and these matrices must check the characteristic relations of angular momentum operators:

$$\begin{aligned}
\left[\hat{S}_x,\hat{S}_y\right] &= \left(\hat{S}_x\hat{S}_y-\hat{S}_y\hat{S}_x\right) = j\hbar\hat{S}_z\\
\left[\hat{S}_z,\hat{S}_x\right] &= \left(\hat{S}_z\hat{S}_x-\hat{S}_x\hat{S}_z\right) = j\hbar\hat{S}_y\\
\left[\hat{S}_y,\hat{S}_z\right] &= \left(\hat{S}_y\hat{S}_z-\hat{S}_z\hat{S}_y\right) = j\hbar\hat{S}_x
\end{aligned} \qquad \text{(IV-19)}$$

The PAULI matrices, mentioned below for memory, are good candidates for this role:

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_2=\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix} \qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \text{(IV-20)}$$

By multiplying by the quantity ħ, the first switch (IV-19) is written:

$$\begin{aligned}
\left[\hbar\sigma_1,\hbar\sigma_2\right] &= \hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}j\hbar\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix}-\hbar\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix}j\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\
\left[\hbar\sigma_1,\hbar\sigma_2\right] &= \hbar^2\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix}-\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}\\
\left[\hbar\sigma_1,\hbar\sigma_2\right] &= \hbar^2\left\{\begin{pmatrix}j & 0\\ 0 & -j\end{pmatrix}-\begin{pmatrix}-j & 0\\ 0 & j\end{pmatrix}\right\} = 2j\hbar^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}
\end{aligned} \qquad \text{(IV-21)}$$

or still:

$$\left[\hbar\sigma_1,\hbar\sigma_2\right] = 2j\hbar(\hbar\sigma_3) \qquad \text{(IV-22)}$$

We don't find exactly the relationship of switching expressed in (IV - 19), since there is a factor of 2 which takes place. If we want to find exactly the relationship that defines the components of an angular momentum, we must necessarily introduce a factor ½ in the PAULI matrices. It is this factor that will induce the spin ½ of the electron.

In summary, the components $\hat{S}_x, \hat{S}_y, \hat{S}_z$ sought, and which therefore verifies (IV-19), are as follows:

$$\hat{S}_x = \frac{\hbar}{2}\sigma_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\hat{S}_y = \frac{\hbar}{2}\sigma_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} \qquad \text{(IV-23)}$$
$$\hat{S}_z = \frac{\hbar}{2}\sigma_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If we have well advanced in the formalism for describing the spin angular momentum, we are away considerably from its physical meaning. The passage, which seems obvious, from relations (IV-3) to (IV-4), is a passage where relations (IV-3) have a real physical meaning, while relations (IV-4) have already no more because they are supported by operators. Their generalization to spinors even increases the level of abstraction, and distance with the physical representation of the angular momentum that is used in classical mechanics.

III - The interpretation of the angular momentum in terms of rotation operator

The description of rotation takes meaning only if it allows to specify the angle of rotation. We will therefore focus in this part to show how one can express this angle on the basis of the elements that were used to characterize this rotation in the preceding paragraphs, and in particular the components of the angular momentum.

The problem is not simple, because it is necessary to characterize not a classic rotation in a three-dimensional space, but a rotation operator that acts on a wave function ψ in a first time, and on a spinor φ with two components in a second time.

We must first establish the matrix of a rotation of angle θ in an Euclidean space, whose axis is chosen arbitrarily toward Oz. This rotation belong to the xOy plane, and we represent below the rotation between two points M and M' with coordinates M(x,y) and M'(x',y').

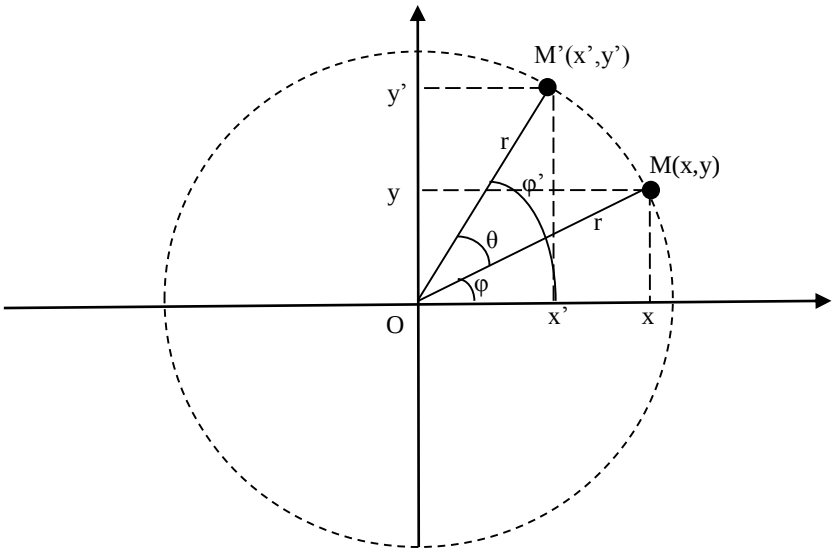


Figure (IV-1): *Rotation in the xOy plane*

The relationship between the coordinates (x,y) and (x',y') may be established geometrically as follows:

$$x = r.\cos(\varphi)$$
$$y = r.\sin(\varphi) \qquad \text{(IV - 24)}$$

$$x' = r.\cos(\varphi') = r.\cos(\theta + \varphi) = r(\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)) = x.\cos(\theta) - y.\sin(\theta)$$
$$y' = r.\sin(\varphi') = r.\sin(\theta + \varphi) = r.(\sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi)) = x.\sin(\theta) + y.\cos(\theta) \qquad \text{(IV-25)}$$

or still with a matrix written in three dimensions:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{(IV-26)}$$

For an opposite angle rotation - θ, the matrix is obtained by changing the sign of sinus:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{(IV-27)}$$

In the description of the rotation operator applied to a given function, if we assimilate the function to an object to make a concrete representation, a rotation of the axes to the right can be considered a rotation equivalent of the object to the left.

For an operator of angle rotation θ, everything happens as if the coordinates used by the function were under a rotation of angle - θ.

Taking into account these elements, we can specify what is the rotation operator applied to a function ψ, and we will adopt the following definition for an Oz axis rotation of angle θ and applied to a wave function ψ:

$$\hat{R}_{z,\theta}\psi(x,y,z)=\psi(x',y',z') \qquad \text{(IV-28)}$$

In this relationship, in accordance with the previous comments, we use the transformation of coordinates (IV-21).

The next step is to establish a link between this rotation operator, and the angular momentum operator $\hat{L}=(\hat{L}_x,\hat{L}_y,\hat{L}_z)$ which was used to characterize the movement of rotation in (IV-4), inspired from relations (IV-3) of classical mechanics. This link will be formalized in a first time, for an infinitesimal rotation.

Starting from the definition of the operator rotation (IV-28) given above:

$$\hat{R}_{z,\theta}\psi(x,y,z)=\psi(x.\cos\theta+y.\sin\theta,-x.\sin\theta+y.\cos\theta,z) \qquad \text{(IV-29)}$$

the introduction of the developments limited to first order of sin (θ) and cos (θ) leads to the infinitesimal rotation operator of angle δθ:

$$\hat{R}_{z,\delta\theta}\psi(x,y,z)=\psi(x.+y.\delta\theta,-x.\delta\theta+y,z) \qquad \text{(IV-30)}$$

Making use of the development limited to first order of a function of two variables x and y which vary from an infinitesimal amount respectively equal to ε and η:

$$f(x+\varepsilon,y+\eta)=f(x,y)+\varepsilon\frac{\partial f(x,y)}{\partial x}+\eta\frac{\partial f(x,y)}{\partial y} \qquad \text{(IV-31)}$$

the infinitesimal rotation operator (IV-30) goes in the form:

$$\hat{R}_{z,\delta\theta}\psi(x,y,z)=\psi(x,y,z)+y(\delta\theta)\frac{\partial\psi}{\partial x}-x(\delta\theta)\frac{\partial\psi}{\partial y} \qquad \text{(IV-32)}$$

It is interpreted using the operators "position" and "pulse" recalled to memory below:

$$\hat{p}_x(\psi)=-j\hbar\frac{\partial\psi}{\partial x} \qquad \hat{p}_y(\psi)=-j\hbar\frac{\partial\psi}{\partial y} \qquad \hat{p}_z(\psi)=-j\hbar\frac{\partial\psi}{\partial z} \qquad \text{(IV-33)}$$

$$\hat{x}(\psi)=x\psi \qquad \hat{y}(\psi)=y\psi \qquad \hat{z}(\psi)=z\psi$$

to give the following expression:

$$\hat{R}_{z,\delta\theta}\psi(x,y,z)=\psi(x,y,z)-j\frac{\delta\theta}{\hbar}(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)\psi(x,y,z) \qquad \text{(IV-34)}$$

We recognize the operator of angular momentum (IV-4):

$$\hat{L}_z=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x \qquad \text{(IV-35)}$$

We deduce the expression of the infinitesimal rotation operator around Oz axis and angle θ, applied to a wave function ψ, depending on the angular momentum operator:

$$\hat{R}_{z,\delta\theta}\psi(x,y,z)=\psi(x,y,z)-j\frac{\delta\theta}{\hbar}\hat{L}_z\psi(x,y,z)=\left\{\hat{I}-j\frac{\delta\theta}{\hbar}\hat{L}_z\right\}\psi(x,y,z) \qquad \text{(IV-36)}$$

where $\hat{I}$ denotes the identity operator.

We can now deduce the intrinsic expression of the infinitesimal rotation operator:

$$\hat{R}_{z,\delta\theta}=\hat{I}-j\frac{\delta\theta}{\hbar}\hat{L}_z \qquad \text{(IV-37)}$$

This relationship can be generalized to a rotation of angle θ, in any of several ways. One of the simplest is to divide this angle by an integer N, which tends towards infinity: we can assimilate the infinitely small angle δθ in the relationship (IV-37) with the angle θ/N.

$$\hat{R}_{z,\frac{\theta}{N}}=\hat{I}-\frac{1}{N}\left(j\frac{\theta}{\hbar}\hat{L}_z\right) \qquad \text{(IV-38)}$$

One then writes that for a rotation of angle θ, we must applied N times the infinitesimal rotation of angle θ/N:

$$\hat{R}_{z,\theta}=\left(\hat{R}_{z,\frac{\theta}{N}}\right)^N=\left(\hat{I}-\frac{1}{N}\left(j\frac{\theta}{\hbar}\hat{L}_z\right)\right)^N \qquad \text{(IV-39)}$$

And we can now write a transition to the limit:

$$\hat{R}_{z,\theta}=\lim_{N\to\infty}\left(\hat{R}_{z,\frac{\theta}{N}}\right)^N=\lim_{N\to\infty}\left(\hat{I}-\frac{1}{N}\left(j\frac{\theta}{\hbar}\hat{L}_z\right)\right)^N \qquad \text{(IV-40)}$$

Using the known result:

$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N=\exp(x) \qquad \text{(IV-41)}$$

we finally get the expression of the rotation of axis Oz and angle θ operator, according to the angular momentum operator $\hat{L}_z$:

$$\hat{R}_{z,\theta}=\exp\left(j\frac{\theta}{\hbar}\hat{L}_z\right) \qquad \text{(IV-42)}$$

It remains to conclude this chapter, to generalize this operator to rotation of spinors which are functions of waves in two dimensions.

It has been shown above that the operator $\hat{L}=\left(\hat{L}_x,\hat{L}_y,\hat{L}_z\right)$ should be replaced by a matrix operator $\hat{S}=\left(\hat{S}_x,\hat{S}_y,\hat{S}_z\right)$ able to act on mathematical two-dimensional beings, and that this operator is inferred from the PAULI matrices following relationship (IV-23) called for memory:

$$\begin{aligned}\hat{S}_x&=\frac{\hbar}{2}\sigma_1=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ \hat{S}_y&=\frac{\hbar}{2}\sigma_2=\frac{\hbar}{2}\begin{pmatrix}0&-j\\j&0\end{pmatrix}\\ \hat{S}_z&=\frac{\hbar}{2}\sigma_3=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{aligned}\qquad\text{(IV-43)}$$

The operator of rotation of axis Oz and angle θ, able to act on a spinor, therefore has the following form:

$$\hat{R}_{z,\theta}=\exp\left(j\frac{\theta}{\hbar}\hat{S}_z\right)\qquad\text{(IV-44)}$$

or still:

$$\hat{R}_{z,\theta}=\exp\left(j\frac{\theta}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)\qquad\text{(IV-45)}$$

From a practical point of view, we can rarely put any exponential matrix in the form of a 2 X 2 matrix whose each term is exactly known. We may nevertheless obtained an approximate solution using the development in series of the exponential function. For any matrix M, this gives:

$$\exp(M)=\sum_{k=0}^{\infty}\frac{M^k}{k!}=1+M+\frac{M^2}{2!}+\frac{M^3}{3!}+....\qquad\text{(IV-46)}$$

However, it is possible to obtain exactly the operator matrix of the rotation around the main axis Ox, Oy, and Oz.

<u>Operator of rotation around the Oz axis:</u>

The operator matrix is diagonal, which allows to give the explicit form of the exponential matrix:

$$\hat{R}_{z,\theta}=\exp\left(j\frac{\theta}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)=\begin{pmatrix}\exp\left(j\frac{\theta}{2}\right)&0\\0&\exp\left(-j\frac{\theta}{2}\right)\end{pmatrix}=\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)+j\sin\left(\frac{\theta}{2}\right)&0\\0&\cos\left(\frac{\theta}{2}\right)-j\sin\left(\frac{\theta}{2}\right)\end{pmatrix}\qquad\text{(IV-47)}$$

<u>Operator of rotation around the Oy axis:</u>

$$\hat{R}_{y,\theta} = \exp\left(j\frac{\theta}{\hbar}\hat{S}_y\right) = \exp\left(j\frac{\theta}{2}\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix}\right) = \exp\left(\frac{\theta}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right) \qquad \text{(IV-48)}$$

The particular shape of the 2 X 2 matrix allows to show that series of the exponential expansion gives the development in series of cos and sin functions in terms of the Matrix result following the relationship:

$$A = \begin{pmatrix}0 & x\\ -x & 0\end{pmatrix} \qquad \exp(A) = \begin{pmatrix}\cos(x) & \sin(x)\\ -\sin(x) & \cos(x)\end{pmatrix} \qquad \text{(IV-49)}$$

Yields the expression of the operator of rotation about the Oy axis:

$$\hat{R}_{y,\theta} = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right)\\ -\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{pmatrix} \qquad \text{(IV-50)}$$

Operator of rotation around Ox axis:

$$\hat{R}_{x,\theta} = \exp\left(j\frac{\theta}{\hbar}\hat{S}_x\right) = \exp\left(j\frac{\theta}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) \qquad \text{(IV-51)}$$

As in the previous example, the particular shape of the 2 X 2 matrix allows to show that the series of the exponential expansion give the hyperbolic functions cosh and sinh series expansion in the terms of the Matrix result following the relationship:

$$A = \begin{pmatrix}0 & x\\ x & 0\end{pmatrix} \qquad \exp(A) = \begin{pmatrix}\cosh(x) & \sinh(x)\\ \sinh(x) & \cosh(x)\end{pmatrix} \qquad \text{(IV-52)}$$

Yields the expression of the operator of rotation around Ox axis:

$$\hat{R}_{x,\theta} = \begin{pmatrix}\mathrm{ch}\left(j\frac{\theta}{2}\right) & \mathrm{sh}\left(j\frac{\theta}{2}\right)\\ \mathrm{sh}\left(j\frac{\theta}{2}\right) & \mathrm{ch}\left(j\frac{\theta}{2}\right)\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & j\sin\left(\frac{\theta}{2}\right)\\ j\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{pmatrix} \qquad \text{(IV-53)}$$

V

Covariance of the DIRAC equation

The DIRAC equation on which we relied to establish the system of differential equations has the form:

$$j\left[\gamma_0\left(\frac{\partial}{\partial(ct)}\right)+\gamma_1\left(\frac{\partial}{\partial x}\right)+\gamma_2\left(\frac{\partial}{\partial y}\right)+\gamma_3\left(\frac{\partial}{\partial z}\right)\right](\psi)=\frac{m_0 c}{\hbar}(\psi) \qquad \text{(V-1)}$$

By adopting the notation:

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z \qquad \text{(V-2)}$$

we can condense the writing of (V-1) using the summation over indices rule:

$$\left(j\gamma^\mu \frac{\partial}{\partial x^\mu}-\frac{m_0 c}{\hbar}\right)\psi\left(x^\mu\right)=0 \qquad \mu = 0, 1, 2, 3 \qquad \text{(V-3)}$$

It turns out that this equation is covariant, it means that she keeps the same shape when changing frame as it is given by special relativity. The demonstration, a bit laborious, may be found in references.

The invariance of the laws of physics by change of frame is probably the most important criterion on the general validity of a physical law. It can be summarized as follows:

Let us consider a frame in which the DIRAC equation is written in the form (V-3) above. If we change the frame, it will induce a change of spatial coordinates and time x^μ in each of the spacetime points that will become x'^μ; and this will induce a change in the wave function ψ which will become ψ'.

The invariance of the laws of physics by frame change requires that the DIRAC equation is written in the frame (R'), under the form:

$$\left(j\gamma^\mu \frac{\partial}{\partial x'^\mu}-\frac{m_0 c}{\hbar}\right)\psi'\left(x'^\mu\right)=0 \qquad \text{(V-4)}$$

Another very general law of physics is the law of local conservation. It expresses the fact that when a physical quantity evolves in time and space, the conservation of this quantity is expressed using a four divergence equal to zero. One of the best-known examples is the conservation of the electric load that is obtained from the four-vector current density (ρc, j_x, j_y, j_z) under the form:

$$\frac{\partial(\rho c)}{\partial(ct)}+\frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}+\frac{\partial j_z}{\partial z}=0 \qquad \text{(V-5)}$$

It is possible to show that there is a quantity kept by the solutions of the DIRAC equation, and this quantity is called current of DIRAC in reference to the relationship (V-5) above.

If we call J this quantity, the condensed writing of a four-divergence equal to zero is as follows:

$$\frac{\partial}{\partial x^\mu}J^\mu=\partial_\mu J^\mu=0 \qquad \text{(V-6)}$$

A few non-trivial manipulations detailed in many courses allow to switch from the DIRAC equation into the following relationship:

$$\frac{\partial}{\partial x^\mu}\left(\overline{\psi}\gamma^\mu\psi\right)=0 \qquad \text{(V-7)}$$

in which:

$$\overline{\psi}=\left(\psi^*\right)^T\gamma^0=\left(\psi_0^*,\psi_1^*,\psi_2^*,\psi_3^*\right)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=\left(\psi_0^*,\psi_1^*,-\psi_2^*,-\psi_3^*\right) \qquad \text{(V-8)}$$

From (V-6) and (V-7), we can deduce the four-vector current of DIRAC:

$$J^\mu=\overline{\psi}\gamma^\mu\psi \qquad \text{(V-9)}$$

where γ^μ represent matrices of DIRAC introduced in chapter II.

We can give the explicit expressions of each of the components of the four-vector J^μ representing s the DIRAC's currents:

<u>index component 0:</u>

$$J^0=\overline{\psi}\gamma^0\psi=\left(\psi_0^*,\psi_1^*,-\psi_2^*,-\psi_3^*\right)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\psi_0\\\psi_1\\\psi_2\\\psi_3\end{pmatrix}=\left(\psi_0^*,\psi_1^*,\psi_2^*,\psi_3^*\right)\begin{pmatrix}\psi_0\\\psi_1\\\psi_2\\\psi_3\end{pmatrix} \qquad \text{(V-10)}$$

$$J^0=\psi_0^*\psi_0+\psi_1^*\psi_1+\psi_2^*\psi_2+\psi_3^*\psi_3 \qquad \text{(V-11)}$$

<u>index component 1:</u>

$$J^1 = \overline{\psi}\gamma^1\psi = \left(\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*\right)\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \left(\psi_3^*, \psi_2^*, \psi_1^*, \psi_0^*\right)\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \qquad \text{(V-12)}$$

$$J^1 = \psi_3^*\psi_0 + \psi_2^*\psi_1 + \psi_1^*\psi_2 + \psi_0^*\psi_3 \qquad \text{(V-13)}$$

index component 2:

$$J^2 = \overline{\psi}\gamma^2\psi = \left(\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*\right)\begin{pmatrix} 0 & 0 & 0 & -j \\ 0 & 0 & j & 0 \\ 0 & j & 0 & 0 \\ -j & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \left(j\psi_3^*, -j\psi_2^*, j\psi_1^*, -j\psi_0^*\right)\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \qquad \text{(V-14)}$$

$$J^2 = j\psi_3^*\psi_0 - j\psi_2^*\psi_1 + j\psi_1^*\psi_2 - j\psi_0^*\psi_3 \qquad \text{(V-15)}$$

index component 3:

$$J^3 = \overline{\psi}\gamma^3\psi = \left(\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*\right)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \left(\psi_2^*, -\psi_3^*, \psi_0^*, -\psi_1^*\right)\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \qquad \text{(V-16)}$$

$$J^3 = \psi_2^*\psi_0 - \psi_3^*\psi_1 + \psi_0^*\psi_2 - \psi_1^*\psi_3 \qquad \text{(V-17)}$$

Hence in summary:

$$\begin{aligned} J^0 &= \psi_0^*\psi_0 + \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 \\ J^1 &= \psi_3^*\psi_0 + \psi_2^*\psi_1 + \psi_1^*\psi_2 + \psi_0^*\psi_3 \\ J^2 &= j\psi_3^*\psi_0 - j\psi_2^*\psi_1 + j\psi_1^*\psi_2 - j\psi_0^*\psi_3 \\ J^3 &= \psi_2^*\psi_0 - \psi_3^*\psi_1 + \psi_0^*\psi_2 - \psi_1^*\psi_3 \end{aligned} \qquad \text{(V-18)}$$

The J^μ represent the density currents of probability of presence of the particle.

Second part

Energy approach of Dirac equation and its exact solutions in Cartesian coordinates

VI

Stationary modes

In a general manner, one presents the possible solutions to the DIRAC equations under the form of plane waves of the type:

$$\psi(x,y,z,t) = A\exp\left[j\left(\omega t - \vec{k}\vec{r}\right)\right] \qquad \text{(VI-1)}$$

This relationship is characteristic of a wave which propagates to the angular frequency ω and along the wave vector $\vec{k}$. It does not describe exchanges of energy such as those that can arise in an electromagnetic cavity for example. It is also subject to question about the number $j = \sqrt{-1}$ which is present and has nothing to do with the complex formalism usual for the description of waves.

We may think to put into evidence stationary modes by summing two or more travelling solutions, but it seems extremely complex to get a general formulation of the stationary solutions from DIRAC system solutions (VI-1).

We have therefore to formalize a method of searching for stationary solutions to the DIRAC system recalled to memory:

$$\begin{aligned}
\eta\psi_0 &= j\frac{\partial\psi_0}{\partial(ct)} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z}\\
\eta\psi_1 &= j\frac{\partial\psi_1}{\partial(ct)} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z}\\
\eta\psi_2 &= -j\frac{\partial\psi_2}{\partial(ct)} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z}\\
\eta\psi_3 &= -j\frac{\partial\psi_3}{\partial(ct)} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z}
\end{aligned} \qquad \text{(VI-2)}$$

and we hypothesize that any wave function can be written as a linear combination of stationary modes which may be present in a three-dimensional cavity:

$$\psi_0 = \{a_0 S_x S_y S_z + b_0 S_x C_y S_z + c_0 C_x S_y S_z + d_0 C_x C_y S_z + e_0 S_x S_y C_z + f_0 S_x C_y C_z + g_0 C_x S_y C_z + h_0 C_x C_y C_z\} C_t$$
$$+ \{i_0 S_x S_y S_z + j_0 S_x C_y S_z + k_0 C_x S_y S_z + l_0 C_x C_y S_z + m_0 S_x S_y C_z + n_0 S_x C_y C_z + o_0 C_x S_y C_z + p_0 C_x C_y C_z\} S_t$$
$$\psi_1 = \{a_1 S_x S_y S_z + b_1 S_x C_y S_z + c_1 C_x S_y S_z + d_1 C_x C_y S_z + e_1 S_x S_y C_z + f_1 S_x C_y C_z + g_1 C_x S_y C_z + h_1 C_x C_y C_z\} C_t$$
$$+ \{i_1 S_x S_y S_z + j_1 S_x C_y S_z + k_1 C_x S_y S_z + l_1 C_x C_y S_z + m_1 S_x S_y C_z + n_1 S_x C_y C_z + o_1 C_x S_y C_z + p_1 C_x C_y C_z\} S_t$$
$$\psi_2 = \{a_2 S_x S_y S_z + b_2 S_x C_y S_z + c_2 C_x S_y S_z + d_2 C_x C_y S_z + e_2 S_x S_y C_z + f_2 S_x C_y C_z + g_2 C_x S_y C_z + h_2 C_x C_y C_z\} C_t$$
$$+ \{i_2 S_x S_y S_z + j_2 S_x C_y S_z + k_2 C_x S_y S_z + l_2 C_x C_y S_z + m_2 S_x S_y C_z + n_2 S_x C_y C_z + o_2 C_x S_y C_z + p_2 C_x C_y C_z\} S_t$$
$$\psi_3 = \{a_3 S_x S_y S_z + b_3 S_x C_y S_z + c_3 C_x S_y S_z + d_3 C_x C_y S_z + e_3 S_x S_y C_z + f_3 S_x C_y C_z + g_3 C_x S_y C_z + h_3 C_x C_y C_z\} C_t$$
$$+ \{i_3 S_x S_y S_z + j_3 S_x C_y S_z + k_3 C_x S_y S_z + l_3 C_x C_y S_z + m_3 S_x S_y C_z + n_3 S_x C_y C_z + o_3 C_x S_y C_z + p_3 C_x C_y C_z\} S_t \qquad \text{(VI-3)}$$

In this definition, the following abbreviated notation has been used:

$$S_x = \sin(k_x x)\ S_y = \sin(k_y y)\ S_z = \sin(k_z z)\ S_t = \sin(k_t ct) \qquad \text{(VI-4)}$$

$$C_x = \cos(k_x x)\ C_y = \cos(k_y y)\ C_z = \cos(k_z z)\ C_t = \cos(k_t ct) \qquad \text{(VI-5)}$$

The wave vector is represented by its k_x, k_y, k_z components, while for a homogeneous notation and consistent with relativity, the product ωt has been replaced by the expression $k_t.ct$, which allows to highlight the two four-vectors:

$$\text{4-vector position: } \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \qquad \text{4-vector wave: } \begin{pmatrix} k_x \\ k_y \\ k_z \\ k_t = \dfrac{\omega}{c} \end{pmatrix} \qquad \text{(VI-6)}$$

Coefficients a_i, b_i, c_i, d_i, e_i, f_i, g_i h_i, i_i, j_i, k_i, l_i, m_i, n_i, o_i p_i , for i = 0, 1, 2, 3, are real or complex constants that weigh each of the modes and will serve as an unknown in the search for the wave functions ψ_0 , ψ_1 , ψ_2 , ψ_3 solutions of the DIRAC system.

This leads, for each equation of the DIRAC system, to express the partial derivatives of the wave functions ψ_0 , ψ_1 , ψ_2 , ψ_3 and to formulate a homogeneous system of 16 equations for the coefficients a_i, b_i, c_i, d_i, e_i, f_i, g_i h_i, i_i, j_i, k_i, l_i, m_i, n_i, o_i p_i . The obtained global system will therefore be a homogeneous system of 64 equations with 64 unknowns.

Calculations are a bit laborious but without difficulties. They are presented in their entirety in the following lines where we remember having put $k_t = \omega/c$.

First equation of the DIRAC system: $j\frac{\partial\psi_0}{\partial(ct)}+j\frac{\partial\psi_3}{\partial x}+\frac{\partial\psi_3}{\partial y}+j\frac{\partial\psi_2}{\partial z}-\eta\psi_0=0$

$$
\begin{aligned}
&j\frac{\partial\psi_0}{\partial(ct)}=-jk_t\left\{a_0S_xS_yS_z+b_0S_xC_yS_z+c_0C_xS_yS_z+d_0C_xC_yS_z+e_0S_xS_yC_z+f_0S_xC_yC_z+g_0C_xS_yC_z+h_0C_xC_yC_z\right\}S_t\\
&+jk_t\left\{i_0S_xS_yS_z+j_0S_xC_yS_z+k_0C_xS_yS_z+l_0C_xC_yS_z+m_0S_xS_yC_z+n_0S_xC_yC_z+o_0C_xS_yC_z+p_0C_xC_yC_z\right\}C_t\\
&j\frac{\partial\psi_3}{\partial x}=jk_x\left\{a_3C_xS_yS_z+b_3C_xC_yS_z-c_3S_xS_yS_z-d_3S_xC_yS_z+e_3C_xS_yC_z+f_3C_xC_yC_z-g_3S_xS_yC_z-h_3S_xC_yC_z\right\}C_t\\
&+jk_x\left\{i_3C_xS_yS_z+j_3C_xC_yS_z-k_3S_xS_yS_z-l_3S_xC_yS_z+m_3C_xS_yC_z+n_3C_xC_yC_z-o_3S_xS_yC_z-p_3S_xC_yC_z\right\}S_t\\
&\frac{\partial\psi_3}{\partial y}=k_y\left\{a_3S_xC_yS_z-b_3S_xS_yS_z+c_3C_xC_yS_z-d_3C_xS_yS_z+e_3S_xC_yC_z-f_3S_xS_yC_z+g_3C_xC_yC_z-h_3C_xS_yC_z\right\}C_t\\
&k_y\left\{i_3S_xC_yS_z-j_3S_xS_yS_z+k_3C_xC_yS_z-l_3C_xS_yS_z+m_3S_xC_yC_z-n_3S_xS_yC_z+o_3C_xC_yC_z-p_3C_xS_yC_z\right\}S_t\\
&j\frac{\partial\psi_2}{\partial z}=jk_z\left\{a_2S_xS_yC_z+b_2S_xC_yC_z+c_2C_xS_yC_z+d_2C_xC_yC_z-e_2S_xS_yS_z-f_2S_xC_yS_z-g_2C_xS_yS_z-h_2C_xC_yS_z\right\}C_t\\
&+jk_z\left\{i_2S_xS_yC_z+j_2S_xC_yC_z+k_2C_xS_yC_z+l_2C_xC_yC_z-m_2S_xS_yS_z-n_2S_xC_yS_z-o_2C_xS_yS_z-p_2C_xC_yS_z\right\}S_t\\
&\eta\psi_0=\eta\left\{a_0S_xS_yS_z+b_0S_xC_yS_z+c_0C_xS_yS_z+d_0C_xC_yS_z+e_0S_xS_yC_z+f_0S_xC_yC_z+g_0C_xS_yC_z+h_0C_xC_yC_z\right\}C_t\\
&+\eta\left\{i_0S_xS_yS_z+j_0S_xC_yS_z+k_0C_xS_yS_z+l_0C_xC_yS_z+m_0S_xS_yC_z+n_0S_xC_yC_z+o_0C_xS_yC_z+p_0C_xC_yC_z\right\}S_t
\end{aligned}
\qquad \text{(VI-7)}
$$

We can deduce the homogeneous system associated with the first equation of DIRAC system:

$$
\begin{aligned}
&-jk_ta_0-jk_xk_3-k_yj_3-jk_zm_2-\eta i_0=0\\
&-jk_tb_0-jk_xl_3+k_yi_3-jk_zn_2-\eta j_0=0\\
&-jk_tc_0+jk_xi_3-k_yl_3-jk_zo_2-\eta k_0=0\\
&-jk_td_0+jk_xj_3+k_yk_3-jk_zp_2-\eta l_0=0\\
&-jk_te_0-jk_xo_3-k_yn_3+jk_zi_2-\eta m_0=0\\
&-jk_tf_0-jk_xp_3+k_ym_3+jk_zj_2-\eta n_0=0\\
&-jk_tg_0+jk_xm_3-k_yp_3+jk_zk_2-\eta o_0=0\\
&-jk_th_0+jk_xn_3+k_yo_3+jk_zl_2-\eta p_0=0\\
&jk_ti_0-jk_xc_3-k_yb_3-jk_ze_2-\eta a_0=0\\
&jk_tj_0-jk_xd_3+k_ya_3-jk_zf_2-\eta b_0=0\\
&jk_tk_0+jk_xa_3-k_yd_3-jk_zg_2-\eta c_0=0\\
&jk_tl_0+jk_xb_3+k_yc_3-jk_zh_2-\eta d_0=0\\
&jk_tm_0-jk_xg_3-k_yf_3+jk_za_2-\eta e_0=0\\
&jk_tn_0-jk_xh_3+k_ye_3+jk_zb_2-\eta f_0=0\\
&jk_to_0+jk_xe_3-k_yh_3+jk_zc_2-\eta g_0=0\\
&jk_tp_0+jk_xf_3+k_yg_3+jk_zd_2-\eta h_0=0
\end{aligned}
\qquad \text{(VI-8)}
$$

Second equation of DIRAC system: $j\frac{\partial\psi_1}{\partial(ct)}+j\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_2}{\partial y}-j\frac{\partial\psi_3}{\partial z}-\eta\psi_1=0$

$$
\begin{aligned}
&j\frac{\partial\psi_1}{\partial(ct)}=-jk_t\{a_1S_xS_yS_z+b_1S_xC_yS_z+c_1C_xS_yS_z+d_1C_xC_yS_z+e_1S_xS_yC_z+f_1S_xC_yC_z+g_1C_xS_yC_z+h_1C_xC_yC_z\}S_t\\
&+jk_t\{i_1S_xS_yS_z+j_1S_xC_yS_z+k_1C_xS_yS_z+l_1C_xC_yS_z+m_1S_xS_yC_z+n_1S_xC_yC_z+o_1C_xS_yC_z+p_1C_xC_yC_z\}C_t\\
&j\frac{\partial\psi_2}{\partial x}=jk_x\{a_2C_xS_yS_z+b_2C_xC_yS_z-c_2S_xS_yS_z-d_2S_xC_yS_z+e_2C_xS_yC_z+f_2C_xC_yC_z-g_2S_xS_yC_z-h_2S_xC_yC_z\}C_t\\
&+jk_x\{i_2C_xS_yS_z+j_2C_xC_yS_z-k_2S_xS_yS_z-l_2S_xC_yS_z+m_2C_xS_yC_z+n_2C_xC_yC_z-o_2S_xS_yC_z-p_2S_xC_yC_z\}S_t\\
&-\frac{\psi_2}{\partial y}=-k_y\{a_2S_xC_yS_z-b_2S_xS_yS_z+c_2C_xC_yS_z-d_2C_xS_yS_z+e_2S_xC_yC_z-f_2S_xS_yC_z+g_2C_xC_yC_z-h_2C_xS_yC_z\}C_t\\
&-k_y\{i_2S_xC_yS_z-j_2S_xS_yS_z+k_2C_xC_yS_z-l_2C_xS_yS_z+m_2S_xC_yC_z-n_2S_xS_yC_z+o_2C_xC_yC_z-p_2C_xS_yC_z\}S_t\\
&-j\frac{\partial\psi_3}{\partial z}=-jk_z\{a_3S_xS_yC_z+b_3S_xC_yC_z+c_3C_xS_yC_z+d_3C_xC_yC_z-e_3S_xS_yS_z-f_3S_xC_yS_z-g_3C_xS_yS_z-h_3C_xC_yS_z\}C_t\\
&-jk_z\{i_3S_xS_yC_z+j_3S_xC_yC_z+k_3C_xS_yC_z+l_3C_xC_yC_z-m_3S_xS_yS_z-n_3S_xC_yS_z-o_3C_xS_yS_z-p_3C_xC_yS_z\}S_t\\
&\eta\psi_1=\eta\{a_1S_xS_yS_z+b_1S_xC_yS_z+c_1C_xS_yS_z+d_1C_xC_yS_z+e_1S_xS_yC_z+f_1S_xC_yC_z+g_1C_xS_yC_z+h_1C_xC_yC_z\}C_t\\
&+\eta\{i_1S_xS_yS_z+j_1S_xC_yS_z+k_1C_xS_yS_z+l_1C_xC_yS_z+m_1S_xS_yC_z+n_1S_xC_yC_z+o_1C_xS_yC_z+p_1C_xC_yC_z\}S_t
\end{aligned}
\qquad \text{(VI-9)}
$$

We can deduce the homogeneous system associated with the second equation of DIRAC system:

$$
\begin{aligned}
&-jk_ta_1-jk_xk_2+k_yj_2+jk_zm_3-\eta i_1=0\\
&-jk_tb_1-jk_xl_2-k_yi_2+jk_zn_3-\eta j_1=0\\
&-jk_tc_1+jk_xi_2+k_yl_2+jk_zo_3-\eta k_1=0\\
&-jk_td_1+jk_xj_2-k_yk_2+jk_zp_3-\eta l_1=0\\
&-jk_te_1-jk_xo_2+k_yn_2-jk_zi_3-\eta m_1=0\\
&-jk_tf_1-jk_xp_2-k_ym_2-jk_zj_3-\eta n_1=0\\
&-jk_tg_1+jk_xm_2+k_yp_2-jk_zk_3-\eta o_1=0\\
&-jk_th_1+jk_xn_2-k_yo_2-jk_zl_3-\eta p_1=0\\
&jk_ti_1-jk_xc_2+k_yb_2+jk_ze_3-\eta a_1=0\\
&jk_tj_1-jk_xd_2-k_ya_2+jk_zf_3-\eta b_1=0\\
&jk_tk_1+jk_xa_2+k_yd_2+jk_zg_3-\eta c_1=0\\
&jk_tl_1+jk_xb_2-k_yc_2+jk_zh_3-\eta d_1=0\\
&jk_tm_1-jk_xg_2+k_yf_2-jk_za_3-\eta e_1=0\\
&jk_tn_1-jk_xh_2-k_ye_2-jk_zb_3-\eta f_1=0\\
&jk_to_1+jk_xe_2+k_yh_2-jk_zc_3-\eta g_1=0\\
&jk_tp_1+jk_xf_2-k_yg_2-jk_zd_3-\eta h_1=0
\end{aligned}
\qquad \text{(VI-10)}
$$

Third equation of DIRAC system: $-j\frac{\partial\psi_2}{\partial(ct)}-j\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_1}{\partial y}-j\frac{\partial\psi_0}{\partial z}-\eta\psi_2=0$

$$
\begin{aligned}
&-j\frac{\partial\psi_2}{\partial(ct)}=jk_t\{a_2S_xS_yS_z+b_2S_xC_yS_z+c_2C_xS_yS_z+d_2C_xC_yS_z+e_2S_xS_yC_z+f_2S_xC_yC_z+g_2C_xS_yC_z+h_2C_xC_yC_z\}S_t\\
&-jk_t\{i_2S_xS_yS_z+j_2S_xC_yS_z+k_2C_xS_yS_z+l_2C_xC_yS_z+m_2S_xS_yC_z+n_2S_xC_yC_z+o_2C_xS_yC_z+p_2C_xC_yC_z\}C_t\\
&-j\frac{\partial\psi_1}{\partial x}=-jk_x\{a_1C_xS_yS_z+b_1C_xC_yS_z-c_1S_xS_yS_z-d_1S_xC_yS_z+e_1C_xS_yC_z+f_1C_xC_yC_z-g_1S_xS_yC_z-h_1S_xC_yC_z\}C_t\\
&-jk_x\{i_1C_xS_yS_z+j_1C_xC_yS_z-k_1S_xS_yS_z-l_1S_xC_yS_z+m_1C_xS_yC_z+n_1C_xC_yC_z-o_1S_xS_yC_z-p_1S_xC_yC_z\}S_t\\
&-\frac{\partial\psi_1}{\partial y}=-k_y\{a_1S_xC_yS_z-b_1S_xS_yS_z+c_1C_xC_yS_z-d_1C_xS_yS_z+e_1S_xC_yC_z-f_1S_xS_yC_z+g_1C_xC_yC_z-h_1C_xS_yC_z\}C_t\\
&-k_y\{i_1S_xC_yS_z-j_1S_xS_yS_z+k_1C_xC_yS_z-l_1C_xS_yS_z+m_1S_xC_yC_z-n_1S_xS_yC_z+o_1C_xC_yC_z-p_1C_xS_yC_z\}S_t\\
&-j\frac{\partial\psi_0}{\partial z}=-jk_z\{a_0S_xS_yC_z+b_0S_xC_yC_z+c_0C_xS_yC_z+d_0C_xC_yC_z-e_0S_xS_yS_z-f_0S_xC_yS_z-g_0C_xS_yS_z-h_0C_xC_yS_z\}C_t\\
&-jk_z\{i_0S_xS_yC_z+j_0S_xC_yC_z+k_0C_xS_yC_z+l_0C_xC_yC_z-m_0S_xS_yS_z-n_0S_xC_yS_z-o_0C_xS_yS_z-p_0C_xC_yS_z\}S_t\\
&\eta\psi_2=\eta\{a_2S_xS_yS_z+b_2S_xC_yS_z+c_2C_xS_yS_z+d_2C_xC_yS_z+e_2S_xS_yC_z+f_2S_xC_yC_z+g_2C_xS_yC_z+h_2C_xC_yC_z\}C_t\\
&+\eta\{i_2S_xS_yS_z+j_2S_xC_yS_z+k_2C_xS_yS_z+l_2C_xC_yS_z+m_2S_xS_yC_z+n_2S_xC_yC_z+o_2C_xS_yC_z+p_2C_xC_yC_z\}S_t
\end{aligned}
\qquad \text{(VI-11)}
$$

We can deduce the homogeneous system associated with the third equation of DIRAC system:

$$
\begin{aligned}
&jk_ta_2+jk_xk_1+k_yj_1+jk_zm_0-\eta i_2=0\\
&jk_tb_2+jk_xl_1-k_yi_1+jk_zn_0-\eta j_2=0\\
&jk_tc_2-jk_xi_1+k_yl_1+jk_zo_0-\eta k_2=0\\
&jk_td_2-jk_xj_1-k_yk_1+jk_zp_0-\eta l_2=0\\
&jk_te_2+jk_xo_1+k_yn_1-jk_zi_0-\eta m_2=0\\
&jk_tf_2+jk_xp_1-k_ym_1-jk_zj_0-\eta n_2=0\\
&jk_tg_2-jk_xm_1+k_yp_1-jk_zk_0-\eta o_2=0\\
&jk_th_2-jk_xn_1-k_yo_1-jk_zl_0-\eta p_2=0\\
&-jk_ti_2+jk_xc_1+k_yb_1+jk_ze_0-\eta a_2=0\\
&-jk_tj_2+jk_xd_1-k_ya_1+jk_zf_0-\eta b_2=0\\
&-jk_tk_2-jk_xa_1+k_yd_1+jk_zg_0-\eta c_2=0\\
&-jk_tl_2-jk_xb_1-k_yc_1+jk_zh_0-\eta d_2=0\\
&-jk_tm_2+jk_xg_1+k_yf_1-jk_za_0-\eta e_2=0\\
&-jk_tn_2+jk_xh_1-k_ye_1-jk_zb_0-\eta f_2=0\\
&-jk_to_2-jk_xe_1+k_yh_1-jk_zc_0-\eta g_2=0\\
&-jk_tp_2-jk_xf_1-k_yg_1-jk_zd_0-\eta h_2=0
\end{aligned}
\qquad \text{(VI-12)}
$$

Fourth equation of DIRAC system: $-j\frac{\partial\psi_3}{\partial(ct)}-j\frac{\partial\psi_0}{\partial x}+\frac{\partial\psi_0}{\partial y}+j\frac{\partial\psi_1}{\partial z}-\eta\psi_3=0$

$$
\begin{aligned}
&-j\frac{\partial\psi_3}{\partial(ct)}=jk_t\left\{a_3S_xS_yS_z+b_3S_xC_yS_z+c_3C_xS_yS_z+d_3C_xC_yS_z+e_3S_xS_yC_z+f_3S_xC_yC_z+g_3C_xS_yC_z+h_3C_xC_yC_z\right\}S_t\\
&-jk_t\left\{i_3S_xS_yS_z+j_3S_xC_yS_z+k_3C_xS_yS_z+l_3C_xC_yS_z+m_3S_xS_yC_z+n_3S_xC_yC_z+o_3C_xS_yC_z+p_3C_xC_yC_z\right\}C_t\\
&-j\frac{\partial\psi_0}{\partial x}=-jk_x\left\{a_0C_xS_yS_z+b_0C_xC_yS_z-c_0S_xS_yS_z-d_0S_xC_yS_z+e_0C_xS_yC_z+f_0C_xC_yC_z-g_0S_xS_yC_z-h_0S_xC_yC_z\right\}C_t\\
&-jk_x\left\{i_0C_xS_yS_z+j_0C_xC_yS_z-k_0S_xS_yS_z-l_0S_xC_yS_z+m_0C_xS_yC_z+n_0C_xC_yC_z-o_0S_xS_yC_z-p_0S_xC_yC_z\right\}S_t\\
&\frac{\partial\psi_0}{\partial y}=k_y\left\{a_0S_xC_yS_z-b_0S_xS_yS_z+c_0C_xC_yS_z-d_0C_xS_yS_z+e_0S_xC_yC_z-f_0S_xS_yC_z+g_0C_xC_yC_z-h_0C_xS_yC_z\right\}C_t\\
&+k_y\left\{i_0S_xC_yS_z-j_0S_xS_yS_z+k_0C_xC_yS_z-l_0C_xS_yS_z+m_0S_xC_yC_z-n_0S_xS_yC_z+o_0C_xC_yC_z-p_0C_xS_yC_z\right\}S_t\\
&j\frac{\partial\psi_1}{\partial z}=jk_z\left\{a_1S_xS_yC_z+b_1S_xC_yC_z+c_1C_xS_yC_z+d_1C_xC_yC_z-e_1S_xS_yS_z-f_1S_xC_yS_z-g_1C_xS_yS_z-h_1C_xC_yS_z\right\}C_t\\
&+jk_z\left\{i_1S_xS_yC_z+j_1S_xC_yC_z+k_1C_xS_yC_z+l_1C_xC_yC_z-m_1S_xS_yS_z-n_1S_xC_yS_z-o_1C_xS_yS_z-p_1C_xC_yS_z\right\}S_t\\
&\eta\psi_3=\eta\left\{a_3S_xS_yS_z+b_3S_xC_yS_z+c_3C_xS_yS_z+d_3C_xC_yS_z+e_3S_xS_yC_z+f_3S_xC_yC_z+g_3C_xS_yC_z+h_3C_xC_yC_z\right\}C_t\\
&+\eta\left\{i_3S_xS_yS_z+j_3S_xC_yS_z+k_3C_xS_yS_z+l_3C_xC_yS_z+m_3S_xS_yC_z+n_3S_xC_yC_z+o_3C_xS_yC_z+p_3C_xC_yC_z\right\}S_t
\end{aligned}
\qquad \text{(VI-13)}
$$

We can deduce the homogeneous system associated with the fourth equation of DIRAC system:

$$
\begin{aligned}
&jk_ta_3+jk_xk_0-k_yj_0-jk_zm_1-\eta i_3=0\\
&jk_tb_3+jk_xl_0+k_yi_0-jk_zn_1-\eta j_3=0\\
&jk_tc_3-jk_xi_0-k_yl_0-jk_zo_1-\eta k_3=0\\
&jk_td_3-jk_xj_0+k_yk_0-jk_zp_1-\eta l_3=0\\
&jk_te_3+jk_xo_0-k_yn_0+jk_zi_1-\eta m_3=0\\
&jk_tf_3+jk_xp_0+k_ym_0+jk_zj_1-\eta n_3=0\\
&jk_tg_3-jk_xm_0-k_yp_0+jk_zk_1-\eta o_3=0\\
&jk_th_3-jk_xn_0+k_yo_0+jk_zl_1-\eta p_3=0\\
&-jk_ti_3+jk_xc_0-k_yb_0-jk_ze_1-\eta a_3=0\\
&-jk_tj_3+jk_xd_0+k_ya_0-jk_zf_1-\eta b_3=0\\
&-jk_tk_3-jk_xa_0-k_yd_0-jk_zg_1-\eta c_3=0\\
&-jk_tl_3-jk_xb_0+k_yc_0-jk_zh_1-\eta d_3=0\\
&-jk_tm_3+jk_xg_0-k_yf_0+jk_za_1-\eta e_3=0\\
&-jk_tn_3+jk_xh_0+k_ye_0+jk_zb_1-\eta f_3=0\\
&-jk_to_3-jk_xe_0-k_yh_0+jk_zc_1-\eta g_3=0\\
&-jk_tp_3-jk_xf_0+k_yg_0+jk_zd_1-\eta h_3=0
\end{aligned}
\qquad \text{(VI-14)}
$$

The complete system of 64 equations relating to the coefficients a_i, b_i, c_i, d_i, e_i, f_i, g_i h_i, i_i, j_i, k_i, l_i, m_i, n_i, o_i p_i. for i = 0,1,2,3 can now be summarized on 2 columns:

$$
\begin{aligned}
&-jk_t a_0 - jk_x k_3 - k_y j_3 - jk_z m_2 - \eta i_0 = 0 \\
&-jk_t b_0 - jk_x l_3 + k_y i_3 - jk_z n_2 - \eta j_0 = 0 \\
&-jk_t c_0 + jk_x i_3 - k_y l_3 - jk_z o_2 - \eta k_0 = 0 \\
&-jk_t d_0 + jk_x j_3 + k_y k_3 - jk_z p_2 - \eta l_0 = 0 \\
&-jk_t e_0 - jk_x o_3 - k_y n_3 + jk_z i_2 - \eta m_0 = 0 \\
&-jk_t f_0 - jk_x p_3 + k_y m_3 + jk_z j_2 - \eta n_0 = 0 \\
&-jk_t g_0 + jk_x m_3 - k_y p_3 + jk_z k_2 - \eta o_0 = 0 \\
&-jk_t h_0 + jk_x n_3 + k_y o_3 + jk_z l_2 - \eta p_0 = 0 \\
&jk_t i_0 - jk_x c_3 - k_y b_3 - jk_z e_2 - \eta a_0 = 0 \\
&jk_t j_0 - jk_x d_3 + k_y a_3 - jk_z f_2 - \eta b_0 = 0 \\
&jk_t k_0 + jk_x a_3 - k_y d_3 - jk_z g_2 - \eta c_0 = 0 \\
&jk_t l_0 + jk_x b_3 + k_y c_3 - jk_z h_2 - \eta d_0 = 0 \\
&jk_t m_0 - jk_x g_3 - k_y f_3 + jk_z a_2 - \eta e_0 = 0 \\
&jk_t n_0 - jk_x h_3 + k_y e_3 + jk_z b_2 - \eta f_0 = 0 \\
&jk_t o_0 + jk_x e_3 - k_y h_3 + jk_z c_2 - \eta g_0 = 0 \\
&jk_t p_0 + jk_x f_3 + k_y g_3 + jk_z d_2 - \eta h_0 = 0
\end{aligned}
$$

$$
\begin{aligned}
&-jk_t a_1 - jk_x k_2 + k_y j_2 + jk_z m_3 - \eta i_1 = 0 \\
&-jk_t b_1 - jk_x l_2 - k_y i_2 + jk_z n_3 - \eta j_1 = 0 \\
&-jk_t c_1 + jk_x i_2 + k_y l_2 + jk_z o_3 - \eta k_1 = 0 \\
&-jk_t d_1 + jk_x j_2 - k_y k_2 + jk_z p_3 - \eta l_1 = 0 \\
&-jk_t e_1 - jk_x o_2 + k_y n_2 - jk_z i_3 - \eta m_1 = 0 \\
&-jk_t f_1 - jk_x p_2 - k_y m_2 - jk_z j_3 - \eta n_1 = 0 \\
&-jk_t g_1 + jk_x m_2 + k_y p_2 - jk_z k_3 - \eta o_1 = 0 \\
&-jk_t h_1 + jk_x n_2 - k_y o_2 - jk_z l_3 - \eta p_1 = 0 \\
&jk_t i_1 - jk_x c_2 + k_y b_2 + jk_z e_3 - \eta a_1 = 0 \\
&jk_t j_1 - jk_x d_2 - k_y a_2 + jk_z f_3 - \eta b_1 = 0 \\
&jk_t k_1 + jk_x a_2 + k_y d_2 + jk_z g_3 - \eta c_1 = 0 \\
&jk_t l_1 + jk_x b_2 - k_y c_2 + jk_z h_3 - \eta d_1 = 0 \\
&jk_t m_1 - jk_x g_2 + k_y f_2 - jk_z a_3 - \eta e_1 = 0 \\
&jk_t n_1 - jk_x h_2 - k_y e_2 - jk_z b_3 - \eta f_1 = 0 \\
&jk_t o_1 + jk_x e_2 + k_y h_2 - jk_z c_3 - \eta g_1 = 0 \\
&jk_t p_1 + jk_x f_2 - k_y g_2 - jk_z d_3 - \eta h_1 = 0
\end{aligned}
$$

$$
\begin{aligned}
&jk_t a_2 + jk_x k_1 + k_y j_1 + jk_z m_0 - \eta i_2 = 0 \\
&jk_t b_2 + jk_x l_1 - k_y i_1 + jk_z n_0 - \eta j_2 = 0 \\
&jk_t c_2 - jk_x i_1 + k_y l_1 + jk_z o_0 - \eta k_2 = 0 \\
&jk_t d_2 - jk_x j_1 - k_y k_1 + jk_z p_0 - \eta l_2 = 0 \\
&jk_t e_2 + jk_x o_1 + k_y n_1 - jk_z i_0 - \eta m_2 = 0 \\
&jk_t f_2 + jk_x p_1 - k_y m_1 - jk_z j_0 - \eta n_2 = 0 \\
&jk_t g_2 - jk_x m_1 + k_y p_1 - jk_z k_0 - \eta o_2 = 0 \\
&jk_t h_2 - jk_x n_1 - k_y o_1 - jk_z l_0 - \eta p_2 = 0 \\
&-jk_t i_2 + jk_x c_1 + k_y b_1 + jk_z e_0 - \eta a_2 = 0 \\
&-jk_t j_2 + jk_x d_1 - k_y a_1 + jk_z f_0 - \eta b_2 = 0 \\
&-jk_t k_2 - jk_x a_1 + k_y d_1 + jk_z g_0 - \eta c_2 = 0 \\
&-jk_t l_2 - jk_x b_1 - k_y c_1 + jk_z h_0 - \eta d_2 = 0 \\
&-jk_t m_2 + jk_x g_1 + k_y f_1 - jk_z a_0 - \eta e_2 = 0 \\
&-jk_t n_2 + jk_x h_1 - k_y e_1 - jk_z b_0 - \eta f_2 = 0 \\
&-jk_t o_2 - jk_x e_1 + k_y h_1 - jk_z c_0 - \eta g_2 = 0 \\
&-jk_t p_2 - jk_x f_1 - k_y g_1 - jk_z d_0 - \eta h_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&jk_t a_3 + jk_x k_0 - k_y j_0 - jk_z m_1 - \eta i_3 = 0 \\
&jk_t b_3 + jk_x l_0 + k_y i_0 - jk_z n_1 - \eta j_3 = 0 \\
&jk_t c_3 - jk_x i_0 - k_y l_0 - jk_z o_1 - \eta k_3 = 0 \\
&jk_t d_3 - jk_x j_0 + k_y k_0 - jk_z p_1 - \eta l_3 = 0 \\
&jk_t e_3 + jk_x o_0 - k_y n_0 + jk_z i_1 - \eta m_3 = 0 \\
&jk_t f_3 + jk_x p_0 + k_y m_0 + jk_z j_1 - \eta n_3 = 0 \\
&jk_t g_3 - jk_x m_0 - k_y p_0 + jk_z k_1 - \eta o_3 = 0 \\
&jk_t h_3 - jk_x n_0 + k_y o_0 + jk_z l_1 - \eta p_3 = 0 \\
&-jk_t i_3 + jk_x c_0 - k_y b_0 - jk_z e_1 - \eta a_3 = 0 \\
&-jk_t j_3 + jk_x d_0 + k_y a_0 - jk_z f_1 - \eta b_3 = 0 \\
&-jk_t k_3 - jk_x a_0 - k_y d_0 - jk_z g_1 - \eta c_3 = 0 \\
&-jk_t l_3 - jk_x b_0 + k_y c_0 - jk_z h_1 - \eta d_3 = 0 \\
&-jk_t m_3 + jk_x g_0 - k_y f_0 + jk_z a_1 - \eta e_3 = 0 \\
&-jk_t n_3 + jk_x h_0 + k_y e_0 + jk_z b_1 - \eta f_3 = 0 \\
&-jk_t o_3 - jk_x e_0 - k_y h_0 + jk_z c_1 - \eta g_3 = 0 \\
&-jk_t p_3 - jk_x f_0 + k_y g_0 + jk_z d_1 - \eta h_3 = 0
\end{aligned}
$$

(VI-15)

It is a homogeneous system that allows non-zero solution only if its determinant is zero. But the literal expression of the determinant of a system of 64 equations with 64 unknowns is not trivial to obtain.

However, one can try to identify it by some physical considerations. If there is a relationship between k_x, k_y, k_z, k_t and η which allows to obtain solutions to this system, this relationship must express the conservation of energy.

It was recalled in the introduction that the pseudo-norm of the pulse energy four-vector:

$$\tilde{P} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ \dfrac{E}{c} \end{pmatrix} \qquad \text{(VI-16)}$$

does not depend on the frame in which it is expressed and it has been shown that its consistency is used to establish the relationship of conservation of energy:

$$E^2 = (pc)^2 + \left(m_0 c^2\right)^2 = (p_x c)^2 + (p_y c)^2 + (p_z c)^2 + \left(m_0 c^2\right)^2 \qquad \text{(VI-17)}$$

The counterpart in (VI-16) four-vector in quantum mechanics is obtained by multiplying the four-vector of wave by the barred PLANCK's constant:

$$\tilde{P}_Q = \hbar \begin{pmatrix} k_x \\ k_y \\ k_z \\ k_t = \dfrac{\omega}{c} \end{pmatrix} \qquad \text{(VI-18)}$$

For the same reasons as before, the pseudo-norm of this four-vector is constant and this constant is necessarily the rest mass energy divided by c². We can deduce:

$$\hbar^2 \left(k_t^2 - k_x^2 - k_y^2 - k_z^2\right) = (m_0 c)^2 \qquad \text{(VI-19)}$$

or again, by making use of the notation used in the expression of the DIRAC system recalled in (V-2):

$$\eta = \frac{m_0 c}{\hbar} \qquad \text{(VI-20)}$$

$$k_t^2 = k_x^2 + k_y^2 + k_z^2 + \eta^2 \qquad \text{(VI-21)}$$

In summary, if DIRAC system has solutions, these must necessarily be consistent with the equation of conservation of energy (VI-21).

It is now possible to show that a solution in the form of a linear combination of stationary modes is solution of the KLEIN-GORDON equation recalled below:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right](\psi)=\frac{m^2c^2}{\hbar^2}(\psi) \qquad \text{(VI-22)}$$

Consider a wave function ψ representing one any cavity three-dimensional patterns expressed in (VI-3), for example to fix ideas:

$$\psi(x,y,z,t)=A\sin(k_x x)\sin(k_y y)\cos(k_z z)\cos(k_t ct) \qquad \text{(VI-23)}$$

By substituting the function ψ from (VI-23) in (VI-22), we get:

$$\left(-k_x^2-k_y^2-k_z^2+k_t^2\right)(\psi)=\eta^2(\psi) \qquad \text{(VI-24)}$$

which suggests that DIRAC system admits solutions in the form of stationary modes provided that the equation of conservation of energy (VI-21) is satisfied.

Based on these assumptions, it can be shown that the determinant of the entire system of 64 equations in 64 unknowns has the determinant:

$$\left(-k_t^2+k_x^2+k_y^2+k_z^2+\eta^2\right)^{32} \qquad \text{(VI-35)}$$

It is concluded definitively that when this determinant is zero, that is, when the energy conservation equation is verified, there are solutions to the DIRAC system in the form of standing waves.

VII

Exact solutions to the DIRAC system

The previous chapter has allowed to show that there were stationary solutions to this system. To be convincing, we must be able to clarify them.

A detailed analysis of the system shows that when the determinant is zero, that is, when the following condition occurs:

$$k_t^2 = k_x^2 + k_y^2 + k_z^2 + \eta^2 \qquad \text{(VII-1)}$$

the choice of one of spinors is arbitrary, and the other follows.

Taking into account this observation, two tables of solutions have been built for spinors defined in previous chapters:

$$\varphi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \quad \text{and} \quad \chi = \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} \qquad \text{(VII-2)}$$

I - Solutions with a single mode excited on one of the components of spinors

The first table (table VII-1) was developed by choosing the spineur.φ and calculating the corresponding spinor χ to be solution for the DIRAC system. In order to sweep the set of solutions, the choice of the spinor φ has been made by setting the component ψ_0 successively with all the possible modes while maintaining $\psi_1 = 0$, then by setting the component ψ_1 successively all possible modes while keeping now $\psi_0 = 0$.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
	SxSySzCt	SxCySzCt	CxSySzCt	CxCySzCt	SxSyCzCt	SxCyCzCt	CxSyCzCt	CxCyCzCt	SxSySzSt	SxCySzSt	CxSySzSt	CxCySzSt	SxSyCzSt	SxCyCzSt	CxSyCzSt	CxCyCzSt
ψ0	a0	b0	c0	d0	e0	f0	g0	h0	i0	j0	k0	L0	m0	n0	o0	p0
ψ1	a1	b1	c1	d1	e1	f1	g1	h1	i1	j1	k1	L1	m1	n1	o1	p1
ψ2(n²-k²)	j.e0.η.Kz m0.Kt.Kz b1.η.Ky j.c1.η.Kx -j. j1.Kt.Ky k1.Kt.Kx	j.f0.η.Kz n0.Kt.Kz -a1.η.Ky j.d1.η.Kx j.i1.Kt.Ky L1.Kt.Kx	j.g0.η.Kz o0.Kt.Kz -j.a1.η.Kx d1.η.Ky -i1.Kt.Kx -j.L1.Kt.Ky	j.h0.η.Kz p0.Kt.Kz -j.b1.η.Kx -c1.η.Ky -j1.Kt.Kx j.k1.Kt.Ky	-j.a0.η.Kz -i0.Kt.Kz f1.η.Ky j.g1.η.Kx -j.n1.Kt.Ky o1.Kt.Kx	-j.b0.η.Kz -j0.Kt.Kz -e1.η.Ky j.h1.η.Kx j.m1.Kt.Ky p1.Kt.Kx	-j.c0.η.Kz -k0.Kt.Kz -j.e1.η.Kx h1.η.Ky -m1.Kt.Kx -j.p1.Kt.Ky	-j.d0.η.Kz -L0.Kt.Kz -j.f1.η.Kx -g1.η.Ky -n1.Kt.Kx j.o1.Kt.Ky	-e0.Kt.Kz j.m0.η.Kz j.b1.Kt.Ky -c1.Kt.Kx j1.η.Ky j.k1.η.Kx	-f0.Kt.Kz j.n0.η.Kz -j.a1.Kt.Ky -d1.Kt.Kx -i1.η.Ky j.L1.η.Kx	-g0.Kt.Kz j.o0.η.Kz a1.Kt.Kx j.d1.Kt.Ky -j.i1.η.Kx L1.η.Ky	-h0.Kt.Kz j.p0.η.Kz b1.Kt.Kx -j.c1.Kt.Ky -j.j1.η.Kx -k1.η.Ky	a0.Kt.Kz -j.i0.η.Kz j.f1.Kt.Ky -g1.Kt.Kx n1.η.Ky j.o1.η.Kx	b0.Kt.Kz -j.j0.η.Kz -j.e1.Kt.Ky -h1.Kt.Kx -m1.η.Ky j.p1.η.Kx	c0.Kt.Kz -j.k0.η.Kz e1.Kt.Kx j.h1.Kt.Ky -j.m1.η.Kx p1.η.Ky	d0.Kt.Kz -j.L0.η.Kz f1.Kt.Kx -j.g1.Kt.Ky -j.n1.η.Kx -o1.η.Ky
ψ3(n²-k²)	-b0.η.Ky j.c0.η.Kx j.j0.Kt.Ky k0.Kt.Kx -j.e1.η.Kz -m1.Kt.Kz	a0.η.Ky j.d0.η.Kx -j.i0.Kt.Ky L0.Kt.Kx -j.f1.η.Kz -n1.Kt.Kz	-j.a0.η.Kx -d0.η.Ky -i0.Kt.Kx j.L0.Kt.Ky -j.g1.η.Kz -o1.Kt.Kz	-j.b0.η.Kx c0.η.Ky -j0.Kt.Kx -j.k0.Kt.Ky -j.h1.η.Kz -p1.Kt.Kz	-f0.η.Ky j.g0.η.Kx j.n0.Kt.Ky o0.Kt.Kx j.a1.η.Kz i1.Kt.Kz	e0.η.Ky j.h0.η.Kx -j.m0.Kt.Ky p0.Kt.Kx j.b1.η.Kz j1.Kt.Kz	-j.e0.η.Kx -h0.η.Ky -m0.Kt.Kx j.p0.Kt.Ky j.c1.η.Kz k1.Kt.Kz	-j.f0.η.Kx g0.η.Ky -n0.Kt.Kx -j.o0.Kt.Ky j.d1.η.Kz L1.Kt.Kz	-j.b0.Kt.Ky -c0.Kt.Kx -j0.η.Ky j.k0.η.Kx e1.Kt.Kz -j.m1.η.Kz	j.a0.Kt.Ky -d0.Kt.Kx i0.η.Ky j.L0.η.Kx f1.Kt.Kz -j.n1.η.Kz	a0.Kt.Kx -j.d0.Kt.Ky -j.i0.η.Kx -L0.η.Ky g1.Kt.Kz -j.o1.η.Kz	b0.Kt.Kx j.c0.Kt.Ky -j.j0.η.Kx k0.η.Ky h1.Kt.Kz -j.p1.η.Kz	-j.f0.Kt.Ky -g0.Kt.Kx -n0.η.Ky j.o0.η.Kx -a1.Kt.Kz j.i1.η.Kz	j.e0.Kt.Ky -h0.Kt.Kx m0.η.Ky j.p0.η.Kx -b1.Kt.Kz j.j1.η.Kz	e0.Kt.Kx -j.h0.Kt.Ky -j.m0.η.Kx -p0.η.Ky -c1.Kt.Kz j.k1.η.Kz	f0.Kt.Kx j.g0.Kt.Ky -j.n0.η.Kx o0.η.Ky -d1.Kt.Kz j.L1.η.Kz

Table VII-1 : *Stationary solutions of DIRAC equation for ψ_2 and ψ_3 versus the excited modes ψ_0 and ψ_1.*

The reading of this table is made in the following way: ψ_0 and ψ_1 being chosen as a amplitude for a stationary mode x_0 or x_1 (x represents any letter included between a and p), ψ_2 and ψ_3 are determined by identifying in each columns the modes related to amplitudes x_0 or x_1.

The wave functions expressed in these solutions are dimensionless. From a purely mathematical point of view, these wave functions may have a multiplicative constant, which allows to express them in different units. We will use this property later.

Examples are proposed in order to familiarize themselves with the reading of the table (VII-1). We put for homogeneity of notation, $x_t = ct$ which allows to write the term of temporal phase under the form $\omega t = k_t x_t$.

Example 1:

$$\begin{aligned}
&\psi_0 = a_0 \sin(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t) \\
&\psi_1 = 0 \\
&\psi_2 = a_0 \frac{k_t k_z}{\eta^2 - k_t^2}\sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) - j a_0 \frac{\eta k_z}{\eta^2 - k_t^2}\sin(k_x x)\sin(k_y y)\cos(k_z z)\cos(k_t x_t) \\
&\psi_3 = a_0 \frac{\eta k_y}{\eta^2 - k_t^2}\sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) + j a_0 \frac{k_t k_y}{\eta^2 - k_t^2}\sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t) \\
&+ a_0 \frac{k_t k_x}{\eta^2 - k_t^2}\cos(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t) - j a_0 \frac{\eta k_x}{\eta^2 - k_t^2}\cos(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t)
\end{aligned} \qquad \text{(VII-3)}$$

Example 2:

$$\begin{aligned}
&\psi_0 = 0 \\
&\psi_1 = a_1 \sin(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t) \\
&\psi_2 = -a_1 \frac{\eta k_y}{\eta^2 - k_t^2}\sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) - j a_1 \frac{k_t k_y}{\eta^2 - k_t^2}\sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t) \\
&+ a_1 \frac{k_t k_x}{\eta^2 - k_t^2}\cos(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t) - j a_1 \frac{\eta k_x}{\eta^2 - k_t^2}\cos(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t) \\
&\psi_3 = -a_1 \frac{k_t k_z}{\eta^2 - k_t^2}\sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + j a_1 \frac{\eta k_z}{\eta^2 - k_t^2}\sin(k_x x)\sin(k_y y)\cos(k_z z)\cos(k_t x_t)
\end{aligned} \qquad \text{(VII-4)}$$

Example 3:

$$\begin{aligned}
&\psi_0 = b_0 \sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) \\
&\psi_1 = 0 \\
&\psi_2 = b_0 \frac{k_t k_z}{\eta^2 - k_t^2}\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) - j b_0 \frac{\eta k_z}{\eta^2 - k_t^2}\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) \\
&\psi_3 = -b_0 \frac{\eta k_y}{\eta^2 - k_t^2}\sin(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t) - j b_0 \frac{k_t k_y}{\eta^2 - k_t^2}\sin(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t) \\
&+ b_0 \frac{k_t k_x}{\eta^2 - k_t^2}\cos(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t) - j b_0 \frac{\eta k_x}{\eta^2 - k_t^2}\cos(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t)
\end{aligned} \qquad \text{(VII-5)}$$

Example 4:

$$
\begin{aligned}
&\psi_0 = 0 \\
&\psi_1 = b_1 \sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) \\
&\psi_2 = b_1 \frac{\eta k_y}{\eta^2 - k_t^2}\sin(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t) + jb_1 \frac{k_t k_y}{\eta^2 - k_t^2}\sin(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t) \\
&+ b_1 \frac{k_t k_x}{\eta^2 - k_t^2}\cos(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t) - jb_1 \frac{\eta k_x}{\eta^2 - k_t^2}\cos(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) \\
&\psi_3 = -b_1 \frac{k_t k_z}{\eta^2 - k_t^2}\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) + jb_1 \frac{\eta k_z}{\eta^2 - k_t^2}\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t)
\end{aligned}
\qquad \text{(VII-6)}
$$

The second table (table VII-2) has been developed by choosing the spinor.χ and calculating the spinor φ corresponding to be solution of the DIRAC system. As in the previous table, the choice of the spinor χ was made by setting the component ψ_2 successively with all the possible modes while maintaining $\psi_3 = 0$, then by setting the component ψ_3 successively in all possible modes while keeping now $\psi_2 = 0$.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
	SxSySzCt	SxCySzCt	CxSySzCt	CxCySzCt	SxSyCzCt	SxCyCzCt	CxSyCzCt	CxCyCzCt	SxSySzSt	SxCySzSt	CxSySzSt	CxCySzSt	SxSyCzSt	SxCyCzSt	CxSyCzSt	CxCyCzSt
ψ0	-j.e2.η.Kz m2.Kt.Kz -b3.η.Ky -j.c3.η.Kx -j.j3.Kt.Ky k3.Kt.Kx	-j.f2.η.Kz n2.Kt.Kz a3.η.Ky -j.d3.η.Kx j.i3.Kt.Ky L3.Kt.Kx	-j.g2.η.Kz o2.Kt.Kz j.a3.η.Kx -d3.η.Ky -i3.Kt.Kx -j.L3.Kt.Ky	-j.h2.η.Kz p2.Kt.Kz j.b3.η.Kx c3.η.Ky -j3.Kt.Kx j.k3.Kt.Ky	j.a2.η.Kz -i2.Kt.Kz -f3.η.Ky -j.g3.η.Kx -j.n3.Kt.Ky o3.Kt.Kx	j.b2.η.Kz -j2.Kt.Kz e3.η.Ky -j.h3.η.Kx j.m3.Kt.Ky p3.Kt.Kx	j.c2.η.Kz -k2.Kt.Kz j.e3.η.Kx -h3.η.Ky -m3.Kt.Kx -j.p3.Kt.Ky	j.d2.η.Kz -L2.Kt.Kz j.f3.η.Kx g3.η.Ky -n3.Kt.Kx j.o3.Kt.Ky	-e2.Kt.Kz -j.m2.η.Kz j.b3.Kt.Ky -c3.Kt.Kx -j3.η.Ky -j.k3.η.Kx	-f2.Kt.Kz -j.n2.η.Kz -j.a3.Kt.Ky -d3.Kt.Kx i3.η.Ky -j.L3.η.Kx	-g2.Kt.Kz -j.o2.η.Kz a3.Kt.Kx j.d3.Kt.Ky j.i3.η.Kx -L3.η.Ky	-h2.Kt.Kz -j.p2.η.Kz b3.Kt.Kx -j.c3.Kt.Ky j.j3.η.Kx k3.η.Ky	a2.Kt.Kz j.i2.η.Kz j.f3.Kt.Ky -g3.Kt.Kx -n3.η.Ky -j.o3.η.Kx	b2.Kt.Kz j.j2.η.Kz -j.e3.Kt.Ky -h3.Kt.Kx m3.η.Ky -j.p3.η.Kx	c2.Kt.Kz j.k2.η.Kz e3.Kt.Kx j.h3.Kt.Ky j.m3.η.Kx -p3.η.Ky	d2.Kt.Kz j.L2.η.Kz f3.Kt.Kx -j.g3.Kt.Ky j.n3.η.Kx o3.η.Ky
ψ1	b2.η.Ky -j.c2.η.Kx j.j2.Kt.Ky k2.Kt.Kx j.e3.η.Kz -m3.Kt.Kz	-a2.η.Ky -j.d2.η.Kx -j.i2.Kt.Ky L2.Kt.Kx j.f3.η.Kz -n3.Kt.Kz	j.a2.η.Kx d2.η.Ky -i2.Kt.Kx j.L2.Kt.Ky j.g3.η.Kz -o3.Kt.Kz	j.b2.η.Kx -c2.η.Ky -j2.Kt.Kx -j.k2.Kt.Ky j.h3.η.Kz -p3.Kt.Kz	f2.η.Ky -j.g2.η.Kx j.n2.Kt.Ky o2.Kt.Kx -j.a3.η.Kz i3.Kt.Kz	-e2.η.Ky -j.h2.η.Kx -j.m2.Kt.Ky p2.Kt.Kx -j.b3.η.Kz j3.Kt.Kz	j.e2.η.Kx h2.η.Ky -m2.Kt.Kx j.p2.Kt.Ky -j.c3.η.Kz k3.Kt.Kz	j.f2.η.Kx -g2.η.Ky -n2.Kt.Kx -j.o2.Kt.Ky -j.d3.η.Kz L3.Kt.Kz	-j.b2.Kt.Ky -c2.Kt.Kx j2.η.Ky -j.k2.η.Kx e3.Kt.Kz j.m3.η.Kz	j.a2.Kt.Ky -d2.Kt.Kx -i2.η.Ky -j.L2.η.Kx f3.Kt.Kz j.n3.η.Kz	a2.Kt.Kx -j.d2.Kt.Ky j.i2.η.Kx L2.η.Ky g3.Kt.Kz j.o3.η.Kz	b2.Kt.Kx j.c2.Kt.Ky j.j2.η.Kx -k2.η.Ky h3.Kt.Kz j.p3.η.Kz	-j.f2.Kt.Ky -g2.Kt.Kx n2.η.Ky -j.o2.η.Kx -a3.Kt.Kz -j.i3.η.Kz	j.e2.Kt.Ky -h2.Kt.Kx -m2.η.Ky -j.p2.η.Kx -b3.Kt.Kz -j.j3.η.Kz	e2.Kt.Kx -j.h2.Kt.Ky j.m2.η.Kx p2.η.Ky -c3.Kt.Kz -j.k3.η.Kz	f2.Kt.Kx j.g2.Kt.Ky j.n2.η.Kx -o2.η.Ky -d3.Kt.Kz -j.L3.η.Kz
ψ2(η²-k²)	a2	b2	c2	d2	e2	f2	g2	h2	i2	j2	k2	L2	m2	n2	o2	p2
ψ3(η²-k²)	a3	b3	c3	d3	e3	f3	g3	h3	i3	j3	k3	L3	m3	n3	o3	p3

Table VII-2 *: Stationary solutions of DIRAC equation for ψ_0 and ψ_1 versus the excited modes ψ_2 and ψ_3.*

As previously, some examples are proposed in order to familiarize themselves with the reading of the table (VII-2). It is recalled that we adopted the notation $x_t = ct$ which allows to write the term of temporal phase $\omega t = k_t x_t$.

Example 5:

$$\psi_0 = a_2 \frac{k_t k_z}{\eta^2 - k_t^2} \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + ja_2 \frac{\eta k_z}{\eta^2 - k_t^2} \sin(k_x x)\sin(k_y y)\cos(k_z z)\cos(k_t x_t)$$

$$\psi_1 = -a_2 \frac{\eta k_y}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) + ja_2 \frac{k_t k_y}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

$$+ a_2 \frac{k_t k_x}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t) + ja_2 \frac{\eta k_x}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t)$$

$$\psi_2 = a_2 \sin(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t) \qquad \text{(VII-7)}$$

$$\psi_3 = 0$$

Example 6:

$$\psi_0 = a_3 \frac{\eta k_y}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) - ja_3 \frac{k_t k_y}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

$$+ a_3 \frac{k_t k_x}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t) + ja_3 \frac{\eta k_x}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t)$$

$$\psi_1 = -a_3 \frac{k_t k_z}{\eta^2 - k_t^2} \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) - ja_3 \frac{\eta k_z}{\eta^2 - k_t^2} \sin(k_x x)\sin(k_y y)\cos(k_z z)\cos(k_t x_t)$$

$$\psi_2 = 0 \qquad \text{(VII-8)}$$

$$\psi_3 = a_3 \sin(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t)$$

Example 7:

$$\psi_0 = -h_2 \frac{k_t k_z}{\eta^2 - k_t^2} \cos(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t) - jh_2 \frac{\eta k_z}{\eta^2 - k_t^2} \cos(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t)$$

$$\psi_1 = h_2 \frac{\eta k_y}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\cos(k_z z)\cos(k_t x_t) - jh_2 \frac{k_t k_y}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$- h_2 \frac{k_t k_x}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) - jh_2 \frac{\eta k_x}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t)$$

$$\psi_2 = h_2 \cos(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) \qquad \text{(VII-9)}$$

$$\psi_3 = 0$$

Example 8:

$$\psi_0 = -h_3 \frac{\eta k_y}{\eta^2 - k_t^2} \cos(k_x x) \sin(k_y y) \cos(k_z z) \cos(k_t x_t) + j h_3 \frac{k_t k_y}{\eta^2 - k_t^2} \cos(k_x x) \sin(k_y y) \cos(k_z z) \sin(k_t x_t)$$
$$- h_3 \frac{k_t k_x}{\eta^2 - k_t^2} \sin(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_t x_t) - j h_3 \frac{\eta k_x}{\eta^2 - k_t^2} \sin(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_t x_t)$$
$$\psi_1 = h_3 \frac{k_t k_z}{\eta^2 - k_t^2} \cos(k_x x) \cos(k_y y) \sin(k_z z) \sin(k_t x_t) + j h_3 \frac{\eta k_z}{\eta^2 - k_t^2} \cos(k_x x) \cos(k_y y) \sin(k_z z) \cos(k_t x_t)$$
$$\psi_2 = 0$$
$$\psi_3 = h_3 \cos(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_t x_t) \qquad \text{(VII-10)}$$

Any linear combination of solutions to the system of DIRAC is still a solution of the system. One can thus construct alternative solutions, of which some examples are given below.

II - Travelling Solutions

A wave that spreads can be seen as the sum of two standing waves, allowing travelling solutions from previous tables (table VI-1 and 2).

One can for example choose the following modes, in which we reminded the notation of the phase time $\omega t = k_t x_t$:

$$\cos(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_t x_t) + \cos(k_x x) \cos(k_y y) \sin(k_z z) \sin(k_t x_t) \qquad \text{(VII-11)}$$

what gives after factorisation:

$$\cos(k_x x) \cos(k_y y) \{\cos(k_z z) \cos(k_t x_t) + \sin(k_z z) \sin(k_t x_t)\} \qquad \text{(VII-12)}$$

a stationary wave in x, y, and propagative wave along the z axis:

$$\cos(k_x x) \cos(k_y y) \cos(k_t x_t - k_z z) \qquad \text{(VII-13)}$$

The approach is as follows: we write the solution corresponding to each of the modes we want to add in (VII-11), which gives for the first mode:

$$\psi_0 = h_0 \cos(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_t x_t)$$
$$\psi_1 = 0 \qquad \text{(VII-14)}$$
$$\psi_2 = j h_0 \frac{\eta k_z}{\eta^2 - k_t^2} \cos(k_x x) \cos(k_y y) \sin(k_z z) \cos(k_t x_t) - h_0 \frac{k_t k_z}{\eta^2 - k_t^2} \cos(k_x x) \cos(k_y y) \sin(k_z z) \sin(k_t x_t)$$
$$\psi_3 = -h_0 \frac{\eta k_y}{\eta^2 - k_t^2} \cos(k_x x) \sin(k_y y) \cos(k_z z) \cos(k_t x_t) - j h_0 \frac{k_t k_y}{\eta^2 - k_t^2} \cos(k_x x) \sin(k_y y) \cos(k_z z) \sin(k_t x_t)$$
$$- h_0 \frac{k_t k_x}{\eta^2 - k_t^2} \sin(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_t x_t) + j h_0 \frac{\eta k_x}{\eta^2 - k_t^2} \sin(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_t x_t)$$

and for the second:

$$\psi_0 = l_0 \cos(k_x x) \cos(k_y y) \sin(k_z z) \sin(k_t x_t)$$
$$\psi_1 = 0 \qquad \text{(VII-15)}$$

$$\psi_2 = -jl_0 \frac{\eta k_z}{\eta^2 - k_t^2} \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) - l_0 \frac{k_t k_z}{\eta^2 - k_t^2} \cos(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t)$$

$$\psi_3 = -l_0 \frac{\eta k_y}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t) + jl_0 \frac{k_t k_y}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t)$$

$$+ l_0 \frac{k_t k_x}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) + jl_0 \frac{\eta k_x}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

We put then: $h_0 = l_0 = A$, we sum term-to-term sum the solutions (VII-14) and (VII-15), in order to obtain the travelling solution along z after reduction:

$$\psi_0 = A\cos(k_x x)\cos(k_y y)\cos(k_t x_t - k_z z)$$

$$\psi_1 = 0$$

$$\psi_2 = jA \frac{\eta k_z}{\eta^2 - k_t^2} \cos(k_x x)\cos(k_y y)\sin(k_t x_t - k_z z) - A \frac{k_t k_z}{\eta^2 - k_t^2} \cos(k_x x)\cos(k_y y)\cos(k_t x_t - k_z z)$$

$$\psi_3 = -A \frac{\eta k_y}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\cos(k_t x_t - k_z z) + jA \frac{k_t k_y}{\eta^2 - k_t^2} \cos(k_x x)\sin(k_y y)\sin(k_t x_t - k_z z)$$

$$+ A \frac{k_t k_x}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\sin(k_t x_t - k_z z) + jA \frac{\eta k_x}{\eta^2 - k_t^2} \sin(k_x x)\cos(k_y y)\cos(k_t x_t - k_z z)$$

(VII-16)

In reiterating this process with the other two directions, we can develop travelling solutions in x, y, z whose example is given below:

$$\psi_0 = \eta\cos(k_t x_t - k_x x - k_y y - k_z z) - jk_t \sin(k_t x_t - k_x x - k_y y - k_z z)$$

$$\psi_1 = 0$$

$$\psi_2 = -jk_z \sin(k_t x_t - k_x x - k_y y - k_z z)$$

$$\psi_3 = k_y \sin(k_t x_t - k_x x - k_y y - k_z z) - jk_x \sin(k_t x_t - k_x x - k_y y - k_z z)$$

(VII-17)

III - Other solutions

Basic solutions expressed in tables 1 and 2 above may also be combined to get solutions whose shape is a little different. Two examples are proposed.

In the first example, two modes of the wave function ψ_1 are excited, one weighted by (-1), and the other weighted by the term $(j\eta/k_t)$:

$$\psi_0 = 0$$

$$\psi_1 = -\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j\frac{\eta}{k_t} \sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$\psi_2 = j\frac{k_y}{k_t} \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \frac{k_x}{k_t} \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$\psi_3 = \frac{k_z}{k_t} \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

(VII-18)

In the second example, these are two modes of the wave function ψ_2 who are excited, one weighted by (-1), and the other-weighted term ($j\eta/k_t$):

$$\psi_0 = -\frac{k_z}{k_t}\cos(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t)$$
$$\psi_1 = \frac{k_x}{k_t}\sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) + j\frac{k_y}{k_t}\cos(k_x x)\sin(k_y y)\sin(k_z z)\cos(k_t x_t)$$
$$\psi_2 = j\frac{\eta}{k_t}\cos(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) - \cos(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$
$$\psi_3 = 0$$

(VII-19)

VIII

The wave-particle duality

When the particle comes in the form of material, it obeys the conservation equation of energy from special relativity:

$$E^2 = p^2c^2 + m_0^2c^4 \qquad \text{(VIII-1)}$$

When the particle is in wave form, its total energy and its wave vector are such as:

$$E = \hbar\omega \qquad p_x = \hbar k_x \qquad p_y = \hbar k_y \qquad p_z = \hbar k_z \qquad \text{(VIII-2)}$$

By introducing these relations in (VIII-1), we can deduce that wave quantities must respect the following relationship for compatibility with energy conservation imposed by relativity:

$$\hbar^2\omega^2 = \hbar^2\left(k_x^2 + k_y^2 + k_z^2\right)c^2 + m_0^2c^4 \qquad \text{(VIII-3)}$$

or again:

$$\frac{\omega^2}{c^2} = \left(k_x^2 + k_y^2 + k_z^2\right) + \frac{m_0^2c^2}{\hbar^2} \qquad \text{(VIII-4)}$$

One found exactly the relationship required to get solutions to the system of DIRAC.

We can deduce that this relationship, associated with relationships (VIII-2) expresses the conservation of energy, both if the particle presents itself in the form of material or in the wave form.

The quantum relationship of conservation of energy:

$$\hbar^2\omega^2 = \hbar^2c^2\left(k_x^2 + k_y^2 + k_z^2\right) + \left(m_0c^2\right)^2 \qquad \text{(VIII-5)}$$

will play a fundamental role in the analysis of stationary solutions which will be proposed.

As for the relationship (VIII-1) issue of relativity, we distinguish three types of energy:

$\hbar\omega$: Wave energy in reference to the pulse ω that appears in this expression.
$\hbar ck_x, \hbar ck_y, \hbar ck_z$: Impulse energy following the directions x, y, z.

m_0c^2 : Mass energy

Among the different forms of solution, we choose one that allows a direct interpretation of the role of these energies in the DIRAC bispinor wave functions.

We take as starting point a solution expressed in the previous chapter:

$$\psi_0 = 0$$
$$\psi_1 = -\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j\frac{\eta}{k_t}\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_2 = j\frac{k_y}{k_t}\sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \frac{k_x}{k_t}\cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_3 = \frac{k_z}{k_t}\sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

(VIII-6)

In this solution, the wave functions are unitless. But we have, from a mathematical perspective, any freedom to multiply all of these wave functions by a constant, and one that seems indicated in this case is equal to k_t to get:

$$\psi_0 = 0$$
$$\psi_1 = -k_t \sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j\eta\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_2 = jk_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + k_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_3 = k_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

(VIII-7)

Substituting the expression $\eta = \frac{m_0 c}{\hbar}$, and multiplying again all wave functions by the constant quantity $\hbar c$,we get:

$$\psi_0 = 0$$
$$\psi_1 = -\hbar\omega\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + jm_0c^2\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_2 = j\hbar ck_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \hbar ck_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_3 = \hbar ck_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

(VIII-8)

Each wave function now has an energy dimension, and one make the observation that in this solution, each term contains an energy of different nature, considering that two pulse energies in orthogonal directions are necessarily differentiated.

If we look at a point where the mass energy is maximum, we must have at this point $|\sin(k_x.x)| = |\cos(k_y.y)| = |\cos(k_z.z)| = 1$ which returns to put $\cos(k_x.x) = \sin(k_y.y) = \sin(k_z.z) = 0$. The solution then takes the form:

$$\psi_0 = 0$$
$$\psi_1 = -\hbar\omega \sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + jm_0c^2 \sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_2 = 0$$
$$\psi_3 = 0$$

(VIII-9)

The wave function ψ_1 must retain special attention: when the mass energy is maximum, it takes the form:

$$\psi_1 = \left\{-\hbar\omega\cos(k_t x_t) + jm_0c^2 \sin(k_t x_t)\right\} \quad \text{(VIII-10)}$$

where it is recognized the wave energy $\hbar\omega$ and the mass energy m_0c^2. But the remarkable result that teaches us the relationship (VIII-9) is that these energies are evolving in time quadrature, and that when one is maximum, the other is minimum.

In other words, when the particle is in its total mass form, it has no wave energy, and when it occurs in its total wave form, it presents no mass energy. Energy present in the particle so alternates between mass and wave forms to the pulse ω defined by the equation of conservation of quantum energy, that for energy pulse equal to zero is simply written:

$$\hbar^2\omega^2 = \left(m_0c^2\right)^2 \quad \text{(VIII-11)}$$

It can be assumed that it is in this ongoing exchange of energy that lies the mystery of the wave-particle duality which appears, in the light of the relationship (VIII-10), sometimes in the form of mass, sometimes in wave form.

In the general case, it is still the wave function ψ_1 which brings these energy exchanges. The terms that carry the impulse energy are those of the second spinor:

$$\psi_2 = j\hbar ck_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \hbar ck_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_3 = \hbar ck_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

(VIII-12)

It is recalled that all these energy exchanges must agree with the law of energy conservation:

$$\hbar^2\omega^2 = \hbar^2c^2\left(k_x^2 + k_y^2 + k_z^2\right) + \left(m_0c^2\right)^2 \quad \text{(VIII-13)}$$

The fact that the pulse energy is carried by the second spinor seem to be understood by imagining that the antiparticle corresponds to the situation in which the second spinor deals with exchanges between mass energy and wave energy. On the basis of this hypothesis, one which is detected in an experiment is that whose spinor contains the energy of mass in the solution of DIRAC.

The minimum pulse ω_0 at which this exchange of energy is performed, is given by the relationship:

$$\omega_0 = \frac{m_0 c^2}{\hbar} \qquad \text{(VIII-14)}$$

The numerical application for an electron gives:

$$\omega_0 = \frac{m_0 c^2}{\hbar} = \frac{\left(9{,}11.10^{-31}\right)\left(3.10^8\right)^2}{1{,}05.10^{-34}} \approx 7{,}8.10^{20}\,\text{rd/s} \qquad \text{(VIII-15)}$$

The great value of this pulse could explain the great difficulty to see these energy exchanges from an experimental point of view.

IX

The currents of DIRAC (1)

The expression of these currents has been given in one of the previous chapters. It is recalled for memory:

$$\begin{aligned} J^0 &= \psi_0^*\psi_0 + \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 \\ J^1 &= \psi_3^*\psi_0 + \psi_2^*\psi_1 + \psi_1^*\psi_2 + \psi_0^*\psi_3 \\ J^2 &= j\psi_3^*\psi_0 - j\psi_2^*\psi_1 + j\psi_1^*\psi_2 - j\psi_0^*\psi_3 \\ J^3 &= \psi_2^*\psi_0 - \psi_3^*\psi_1 + \psi_0^*\psi_2 - \psi_1^*\psi_3 \end{aligned} \qquad \text{(IX-1)}$$

These currents check the local conservation of energy equation:

$$\frac{\partial J^0}{\partial(ct)} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z} = 0 \qquad \text{(IX-2)}$$

I - DIRAC currents for a stationary solution

A practical calculation of these currents will be performed for a stationary solution. We choose for this the solution discussed in the previous chapter:

$$\begin{aligned} \psi_0 &= 0 \\ \psi_1 &= -k_t \sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j\eta\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) \\ \psi_2 &= jk_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + k_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) \\ \psi_3 &= k_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t) \end{aligned} \qquad \text{(IX-3)}$$

Calculations are a little long, but without difficulties. Finally, we get the following expressions:

$$\begin{aligned} J^0 &= k_t^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\cos^2(k_t x_t) + \eta^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ k_y^2 \sin^2(k_x x)\sin^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) + k_x^2 \cos^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ k_z^2 \sin^2(k_x x)\cos^2(k_y y)\sin^2(k_z z)\sin^2(k_t x_t) \end{aligned} \qquad \text{(IX-4)}$$

$$J^1 = \begin{pmatrix} \eta k_y \sin^2(k_x x)\sin(2k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ -k_x k_t \sin(2k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin(k_t x_t)\cos(k_t x_t) \end{pmatrix} \quad \text{(IX-5)}$$

$$J^2 = \begin{pmatrix} k_y k_t \sin^2(k_x x)\sin(2k_y y)\cos^2(k_z z)\sin(k_t x_t)\cos(k_t x_t) \\ k_x \eta \sin(2k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \end{pmatrix} \quad \text{(IX-6)}$$

$$J^3 = k_t k_z \sin^2(k_x x)\cos^2(k_y y)\sin(2k_z z)\sin(k_t x_t)\cos(k_t x_t) \quad \text{(IX-7)}$$

We can then verify the conservation equation (IX-2). The details of the calculations is not given since it is not hard. Yields:

$$\frac{\partial J^0}{\partial x_t} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z} = k_t \sin(2k_t x_t)\sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\left(-k_t^2 + \eta^2 + k_x^2 + k_y^2 + k_z^2\right) \quad \text{(IX-8)}$$

which, taking into account the relationship of conservation of energy:

$$k_t^2 = k_x^2 + k_y^2 + k_z^2 + \eta^2 \quad \text{(IX-9)}$$

leads to the expected result:

$$\frac{\partial J^0}{\partial x_t} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z} = 0 \quad \text{(IX-10)}$$

We propose to reconsider the above results in a more physical approach. This leads to multiply the wave functions by the amount $\hbar c$ and DIRAC current by the amount $(\hbar c)^2$. In this description, the wave function has the form:

$$\begin{aligned} \psi_0 &= 0 \\ \psi_1 &= -\hbar\omega \sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j m_0 c^2 \sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) \\ \psi_2 &= j\hbar c k_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \hbar c k_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) \\ \psi_3 &= \hbar c k_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t) \end{aligned} \quad \text{(IX-11)}$$

From a mathematical point of view, we know that this wave function is defined to a multiplicative constant close, which we will call C, so that it can be put in a more general form:

$$\begin{aligned} \psi_0 &= 0 \\ \psi_1 &= C\left\{-\hbar\omega \sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j m_0 c^2 \sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)\right\} \\ \psi_2 &= C\left\{j\hbar c k_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \hbar c k_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)\right\} \\ \psi_3 &= C\left\{\hbar c k_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)\right\} \end{aligned} \quad \text{(IX-12)}$$

The constant C must allow:

1 – to give to the quantity $\psi\psi^*$ the dimension of a volumetric energy density.

2 – to ensure that the totality of the particle energy is confined in a parallelepiped with the dimension along x is between X_1 and X_2, along y between Y_1 and Y_2, and along z between Z_1 , Z_2, so its volume V is equal to:

$$V = (X_2 - X_1)\,(Y_2 - Y_1)\,(Z_2 - Z_1) \qquad \text{(IX-13)}$$

The term J^0 of the currents of DIRAC represents the total energy volume density included in the box. It has for expression:

$$\begin{aligned} J^0 &= C^2(\hbar\omega)^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\cos^2(k_t x_t) \\ &+ C^2(m_0 c^2)^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_y)^2 \sin^2(k_x x)\sin^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_x)^2 \cos^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_z)^2 \sin^2(k_x x)\cos^2(k_y y)\sin^2(k_z z)\sin^2(k_t x_t) \end{aligned} \qquad \text{(IX-14)}$$

To get the total energy E in the parallelepiped, we need to integrate on the volume of this latter, hence:

$$E = \iiint_V \left\{ \begin{aligned} &C^2(\hbar\omega)^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\cos^2(k_t x_t) \\ &+ C^2(m_0 c^2)^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_y)^2 \sin^2(k_x x)\sin^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_x)^2 \cos^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_z)^2 \sin^2(k_x x)\cos^2(k_y y)\sin^2(k_z z)\sin^2(k_t x_t) \end{aligned} \right\} dx\,dy\,dz \qquad \text{(IX-15)}$$

We will do the classical hypothesis that stationary modes settle in conditions such as the dimensions of the box are multiples of the half-wavelength along each of the x, y, z direction.

$$\begin{aligned} (X_2 - X_1) &= n\frac{\lambda_x}{2} = n\frac{\pi}{k_x} \\ (Y_2 - Y_1) &= m\frac{\lambda_y}{2} = m\frac{\pi}{k_y} \\ (Z_2 - Z_1) &= p\frac{\lambda_z}{2} = p\frac{\pi}{k_z} \end{aligned} \qquad \text{(IX-16)}$$

where m, n, p are positive or negative integers. Under these conditions, one has the following property:

$$\begin{aligned}
&\int_{X_1}^{X_2}\sin^2(k_x x)dx = \int_{X_1}^{X_2}\cos^2(k_x x)dx = \frac{X_2 - X_1}{2}\\
&\int_{Y_1}^{Y_2}\sin^2(k_y y)dy = \int_{Y_1}^{Y_2}\cos^2(k_y y)dy = \frac{Y_2 - Y_1}{2}\\
&\int_{Z_1}^{Z_2}\sin^2(k_z z)dz = \int_{Z_1}^{Z_2}\cos^2(k_z z)dz = \frac{Z_2 - Z_1}{2}
\end{aligned} \qquad \text{(IX-17)}$$

and the total energy calculated according to the relationship (IX-15) takes the form:

$$E = C^2\frac{(X_2 - X_1)(Y_2 - Y_1)(Z_2 - Z_1)}{8}\begin{Bmatrix}(\hbar\omega)^2\cos^2(k_t x_t)\\ +(m_0c^2)^2\sin^2(k_t x_t)\\ +(\hbar c k_y)^2\sin^2(k_t x_t)\\ +(\hbar c k_x)^2\sin^2(k_t x_t)\\ +(\hbar c k_z)^2\sin^2(k_t x_t)\end{Bmatrix} \qquad \text{(IX-18)}$$

or again:

$$E = C^2\frac{V}{8}\left\{(\hbar\omega)^2\cos^2(k_t x_t) + \left\{(m_0c^2)^2 + (\hbar c k_y)^2 + (\hbar c k_x)^2 + (\hbar c k_z)^2\right\}\sin^2(k_t x_t)\right\} \qquad \text{(IX-19)}$$

From relationship:

$$(\hbar\omega)^2 = (m_0c^2)^2 + (\hbar c k_y)^2 + (\hbar c k_x)^2 + (\hbar c k_z)^2 \qquad \text{(IX-20)}$$

It is deduced that the total energy in the cavity is either under the form of a wave energy, either in the form of a combination of mass and impulse energy. These energy exchanges are in time quadrature, as shown in the (IX-19) relationship: when one is maximum, the other is null and vice versa.

From relations (IX-19) and (IX-20) are deduced:

$$E = C^2\frac{V}{8}(\hbar\omega)^2 \qquad \text{(IX-21)}$$

and since the total energy E is equal to $\hbar\omega$, this imposes to the C constant the following value:

$$C = \sqrt{\frac{8}{(\hbar\omega)V}} \qquad \text{(IX-22)}$$

After taking into account of this multiplicative constant, DIRAC currents become homogeneous to a volumetric energy density, and the total energy is normalized to the energy of the particle.

The four-divergence:

$$\frac{\partial J^0}{\partial x_t}+\frac{\partial J^1}{\partial x}+\frac{\partial J^2}{\partial y}+\frac{\partial J^3}{\partial z}=0 \qquad \text{(IX-23)}$$

is interpreted in the same manner as in electromagnetism (POYNTING theorem) or in general relativity. It expresses the fact that if there is a change in energy in a volume dV = dxdydz during a time element dt, it's because this variation has crossed the border defined by the closed surface bounding the volume element.

II - DIRAC currents for a propagative solution

As an example, we choose a propagative solution in x, y, z:

$$\begin{aligned}\psi_0 &= \eta\cos(k_t x_t - k_x x - k_y y - k_z z) - jk_t \sin(k_t x_t - k_x x - k_y y - k_z z)\\ \psi_1 &= 0\\ \psi_2 &= -jk_z \sin(k_t x_t - k_x x - k_y y - k_z z)\\ \psi_3 &= k_y \sin(k_t x_t - k_x x - k_y y - k_z z) - jk_x \sin(k_t x_t - k_x x - k_y y - k_z z)\end{aligned} \qquad \text{(IX-24)}$$

Calculations give the following DIRAC currents:

$$\begin{aligned}J^0 &= \eta^2\cos^2(k_t x_t - k_x x - k_y y - k_z z) + k_t^2 \sin^2(k_t x_t - k_x x - k_y y - k_z z)\\ &+ k_x^2 \sin^2(k_t x_t - k_x x - k_y y - k_z z) + k_y^2 \sin^2(k_t x_t - k_x x - k_y y - k_z z)\\ &+ k_z^2 \sin^2(k_t x_t - k_x x - k_y y - k_z z)\end{aligned} \qquad \text{(IX-25)}$$

$$\begin{aligned}J^1 &= 2k_y \eta \sin(k_t x_t - k_x x - k_y y - k_z z)\cos(k_t x_t - k_x x - k_y y - k_z z)\\ &+ 2k_x k_t \sin^2(k_t x_t - k_x x - k_y y - k_z z)\end{aligned} \qquad \text{(IX-26)}$$

$$\begin{aligned}J^2 &= -2k_x \sin(k_t x_t - k_x x - k_y y - k_z z)\cos(k_t x_t - k_x x - k_y y - k_z z) +\\ &\left(2k_y k_t \sin^2(k_t x_t - k_x x - k_y y - k_z z)\right)\end{aligned} \qquad \text{(IX-27)}$$

$$J^3 = \left(2k_t k_z \sin^2(k_t x_t - k_x x - k_y y - k_z z)\right) \qquad \text{(IX-28)}$$

The current J^0 is of particular interest because it contains, to a multiplicative constant close, the total energy of the particle. It may come in the form:

$$J^0 = \eta^2\cos^2(k_t x_t - k_x x - k_y y - k_z z) + \left(k_t^2 + k_x^2 + k_y^2 + k_z^2\right)\sin^2(k_t x_t - k_x x - k_y y - k_z z) \qquad \text{(IX-29)}$$

After multiplication by the constant $(\hbar c)^2$ and standardization by the C^2 constant defined in the previous paragraph, it represents the total volume density attached to the particle.

Using the relation of energy conservation:

$$k_t^2 = k_x^2 + k_y^2 + k_z^2 + \eta^2 \qquad \text{(IX-30)}$$

it becomes:

$$J^0 = k_t^2 - \left(k_x^2 + k_y^2 + k_z^2\right)\left\{\cos^2(k_t x_t - k_x x - k_y y - k_z z) - \sin^2(k_t x_t - k_x x - k_y y - k_z z)\right\} \qquad \text{(IX-31)}$$

or again:

$$J^0 = k_t^2 - \left(k_x^2 + k_y^2 + k_z^2\right)\cos\left[2(k_t x_t - k_x x - k_y y - k_z z)\right] \qquad \text{(IX-32)}$$

Total energy volume density is given by the term $C^2\left(\hbar c k_t\right)^2$. It fluctuates around this value with a spatial and temporal variation in average which is zero.

The local conservation of energy equation:

$$\frac{\partial J^0}{\partial x_t} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z} = 0 \qquad \text{(IX-33)}$$

gives the following result:

$$\frac{\partial J^0}{\partial x_t} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z} = \left(k_t^2 - k_x^2 - k_y^2 - k_z^2 - \eta^2\right)\sin\left[2(k_t x_t - k_x x - k_y y - k_z z)\right] = 0 \qquad \text{(IX-34)}$$

III - Uniform DIRAC currents

We will name uniform currents, currents that are not dependent neither on time nor on space. There are exact solutions to the DIRAC equation which have this property, of which an example is provided below:

$$\begin{aligned}
\psi_0 &= k_z \exp\left\{j\left(-k_t x_t + k_x x + k_y y + k_z z\right)\right\} \\
\psi_1 &= k_x \exp\left\{j\left(-k_t x_t + k_x x + k_y y + k_z z\right)\right\} + jk_y \exp\left\{j\left(-k_t x_t + k_x x + k_y y + k_z z\right)\right\} \\
\psi_2 &= \left(k_t - \eta\right)\exp\left\{j\left(-k_t x_t + k_x x + k_y y + k_z z\right)\right\} \\
\psi_3 &= 0
\end{aligned} \qquad \text{(IX-35)}$$

Because products of wave functions by conjugated wave functions eliminate the exponential, we find:

$$\begin{aligned}
J^0 &= k_x^2 + k_y^2 + k_z^2 + \left(k_t - \eta\right)^2 \\
J^1 &= 2\left(k_t - \eta\right)k_x \\
J^2 &= 2\left(k_t - \eta\right)k_y \\
J^3 &= 2\left(k_t - \eta\right)k_z
\end{aligned} \qquad \text{(IX-36)}$$

One observe that the volume density of total energy $C^2(\hbar c)^2 J^0$ presents itself in the form of a combination of all energies working into the cavity, and it is uniform, that is to say independent of space and time.

This excitement of some modes making it uniform energy within the particle density can occur only under specific conditions. Indeed, we know that the total energy within the particle density is equal to $C^2(\hbar c k_t)^2$, hence the relationship:

$$J^0 = k_x^2 + k_y^2 + k_z^2 + (\eta - k_t)^2 = k_x^2 + k_y^2 + k_z^2 + \eta^2 + k_t^2 - 2\eta k_t = k_t^2 \qquad \text{(IX-37)}$$

By introducing the relationship of energy conservation, we can deduce:

$$2k_t^2 - 2\eta k_t = k_t^2 \qquad \text{(IX-38)}$$

still, by substituting expressions of k_t and η:

$$\eta = \frac{m_0 c}{\hbar} = \frac{\omega}{2c} \qquad \text{(IX-39}$$

The condition for obtaining a uniform density is therefore given by the relationship:

$$\hbar c k_t = 2m_0 c^2 \qquad \text{(IX-40)}$$

This relationship expresses the fact it needs a total internal energy equal to twice the mass energy particle to allow the installation of such modes.

We can connect this observation to the fact that all modes expressed in the solution (IX-35) exchange all kinds of energy in time and space quadrature between positive and negative energies identified by factor j = square root(-1). The antiparticle can appear only if the total energy is at least twice the mass energy of the particle.

X

Principle of indeterminacy

This principle, enunciated by HEISENBERG, during the early days of quantum mechanics, was popularized in the expression: "it is impossible to know both the position and momentum of a particle". From a physical point of view, it is whole contained in a relationship that connects the uncertainty on the Δx position and the uncertainty on momentum Δp_x of a particle in the quantum world:

$$\Delta x.\Delta p_x > \frac{\hbar}{2} \qquad \text{(X-1)}$$

We can deduce an alternative formulation by noting that the fundamental principle of the dynamic allows to write that the variation of the amount of movement Δp_x is done through an outdoor action F_x called force, while a duration of Δt:

$$\Delta p_x = F_x.\Delta t \qquad \text{(X-2)}$$

Yields, noting that energy can be seen as the product of a force by displacement:

$$\Delta x.\Delta p_x = \Delta x.F_x.\Delta t = \Delta E.\Delta t \geq \frac{\hbar}{2} \qquad \text{(X-3)}$$

This principle has solid theoretical foundations, based on the fact that the position and the momentum of quantum operators do not commute. Since the result of the measurement of position and pulse, made at the same place and at the same time, depends on the order in which it performs this measure, this indicates that there is necessarily an uncertainty on the result of these measures.

If we now consider an exact stationary solution of the DIRAC equation such as that which has been chosen as an example in the previous chapters:

$$\begin{aligned}
\psi_0 &= 0 \\
\psi_1 &= C\left\{-\hbar\omega\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + jm_0c^2\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)\right\} \\
\psi_2 &= C\left\{j\hbar ck_y\sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \hbar ck_x\cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)\right\} \\
\psi_3 &= C\left\{\hbar ck_z\sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)\right\}
\end{aligned} \qquad \text{(X-4)}$$

there is a question that naturally comes to mind. The solution (X-4) above is perfectly deterministic: each type of energy is known, in theory, with infinite precision for a position (x,

y, z) and an instant (t) given. This state indeed seems in contradiction with the HEISENBERG uncertainty principle.

To remove this contradiction, we must first admit in the form of postulate the following conclusion: an observer can obtain information from a physical system only if it exchanges energy with this system. A corollary of this assumption is that two systems that do not exchange energy ignore each other and do not interact: they can work simultaneously at the same time and in the same place.

On the basis of this assumption, we examine, for the above solution (X-4), the volumetric energy density present in the particle which has been calculated in the previous chapter:

$$\begin{aligned} J^0 &= C^2(\hbar\omega)^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\cos^2(k_t x_t) \\ &+ C^2(m_0c^2)^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_y)^2 \sin^2(k_x x)\sin^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_x)^2 \cos^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\sin^2(k_t x_t) \\ &+ C^2(\hbar c k_z)^2 \sin^2(k_x x)\cos^2(k_y y)\sin^2(k_z z)\sin^2(k_t x_t) \end{aligned} \qquad \text{(X-5)}$$

There are different types of energy outlined in the previous chapters:

- the total energy or wave energy: $\hbar\omega$
- the energy of mass: m_0c^2
- pulse energy: $\hbar c k_x, \hbar c k_y, \hbar c k_z$

Let us place on a point in space (x, y, z) where the volume density of mass energy of the particle is maximum. Let us assume that the position where this mass energy is maximum can be determined with any precision desired. To ensure that this condition is achieved, the coordinates x, y, z must check:

$$\sin^2(k_x x) = \cos^2(k_y y) = \cos^2(k_z z) = 1 \qquad \text{(X-6)}$$

What requires:

$$\cos^2(k_x x) = \sin^2(k_y y) = \sin^2(k_z z) = 0 \qquad \text{(X-7)}$$

It appears the following remarkable result: all impulse energy densities present in the particle are zero at this location.

In other words, if we move to a point where we can, through an exchange of energy with the energy of mass, know with precision the position of the particle, we cannot get any information on its momentum at this point because its impulse energy is zero at this place.

The reciprocal is expressed in the following way: if one moves to a place where the impulse energy following x is maximum, then mass energy and impulse energy along y and z are zero. A similar property is checked by permutation on the variables x, y, z.

These observations allow to understand how a completely deterministic theory built on exact stationary solutions to the DIRAC equation remains compatible with the HEISENBERG uncertainty principle. This principle is based on the hypothesis that measurements of position and speed are made pointwise in the same place, while the energy approach shows that the energies corresponding to these two quantities are shifted in the space. If this approach proves to be correct, it can be concluded that it is possible to know the position and velocity of a particle with arbitrary precision, provided you locate in the place where these characteristics are present in the particle

The second relation of indeterminacy (X-3) which deals with the energy and time:

$$\Delta E.\Delta t \geq \frac{\hbar}{2} \qquad \text{(X-8)}$$

gives rise to a somewhat different interpretation. It concerns total energy or wave energy, whose volume density is given by (X-5):

$$C^2(\hbar\omega)^2 \sin^2(k_x x)\cos^2(k_y y)\cos^2(k_z z)\cos^2(k_t x_t) \qquad \text{(X-9)}$$

The points of the coordinate (x, y, z) in space where this energy is maximum are the same as those where the mass energy is at a maximum, they obey therefore relations (X-6), and the volume density of the wave energy is written in these points:

$$C^2(\hbar\omega)^2 \cos^2(k_t x_t) \qquad \text{(X-10)}$$

It appears that the mesure of this energy depends on the moment in which it is carried out, in the same way as previously impulse energy or mass energy depended on the place where they were measured, and then we have similar uncertainty relations.

XI

DE BROGLIE wavelength

This chapter aims to show that the stationary solutions of the DIRAC equation are fully compatible with the conclusions of Louis DE BROGLIE on the wavelength associated with the motion of each particle. It allows to make the link between the wave description of quantum mechanics and Relativistic description of the motion of a particle.

The particle is assumed to have a straight trajectory along the Oz axis with constant velocity v. However, it is likely to have stationary modes according to Ox and Oy directions. We fall in a similar situation well known in electromagnetism, which is that of a wave in a perfectly conducting rectangular waveguide. The elements presented in this chapter have a greater analogy to those involving guided propagation.

The relationship of conservation of energy requires, to a multiplicative constant close:

$$k_t^2 = k_x^2 + k_y^2 + k_z^2 + \eta^2 = \frac{\omega^2}{c^2} \qquad \text{(XI-1)}$$

Assuming that the particle moves along Oz, it is natural to consider the space pulse k_z deduced from (XI-1):

$$k_z^2 = \frac{\omega^2}{c^2} - \left(k_x^2 + k_y^2 + \eta^2\right) \qquad \text{(XI-2)}$$

We name temporal cut pulse, the pulse ω_c such as:

$$\omega_c^2 = \left(k_x^2 + k_y^2 + \eta^2\right)c^2 \qquad \text{(XI-3)}$$

Which allows to express the space pulsation k_z in the form:

$$k_z^2 = \frac{\omega^2 - \omega_c^2}{c^2} \qquad \text{(XI-4)}$$

The curve representative $k_z = f(\omega)$ has the following look:

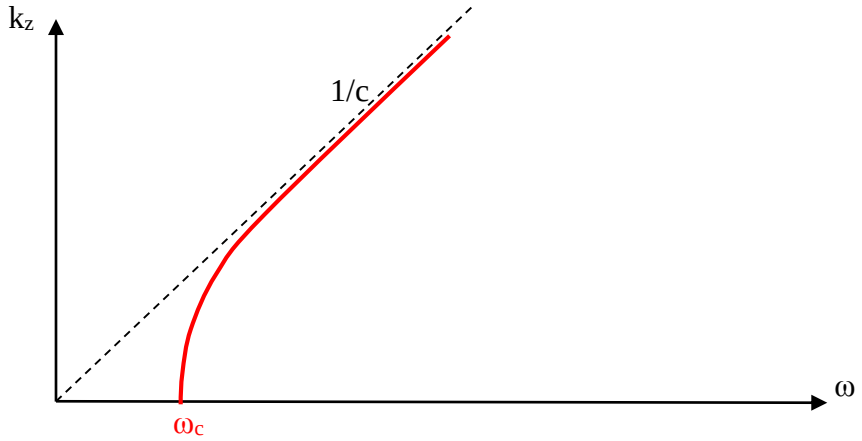


Figure (XI-1): *representation of the space pulsation versus temporal pulsation*

For a wave that will be named phase wave, and which propagates with a phase:

$$\varphi = \omega t - k_z z \qquad \text{(XI-5)}$$

we define the phase velocity as the velocity of the sliding of the phase wave:

$$v_\varphi = \frac{\omega}{k_z} \qquad \text{(XI-6)}$$

We can express this phase velocity depending on the cut pulse ω_c defined above, using the relationship (XI-4):

$$v_\varphi = \frac{c}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} \qquad \text{(XI-7)}$$

Since $\omega > \omega_c$, this speed is greater than the speed of light. It has a physical reality as it represents the sliding of the phase velocity, but it cannot represent the speed of energy which must remain below c.

We define the speed of propagation of energy, or group velocity v_g by the derivative of the temporal pulse over the space pulsation:

$$v_g = \frac{d\omega}{dk_z} \qquad \text{(XI-8)}$$

We can notice on the figure (XI-1) that the speed of propagation of energy is zero for $\omega = \omega_c$, and that it tends to c for $\omega >> \omega_c$. Energy cannot spread for $\omega < \omega_c$.

Group velocity can be expressed using the cut pulse, as has been done for the phase velocity. As a first step, one differentiates the relationship of energy conservation (XI-1) to obtain:

$$k_z.dk_z = \frac{\omega.d\omega}{c^2} \qquad \text{(XI-9)}$$

From which it takes an immediate relationship between phase velocity and group Velocity:

$$\frac{\omega}{k_z}\frac{d\omega}{dk_z} = c^2 \qquad \text{and hence: } v_\varphi v_g = c^2 \qquad \text{(XI-10)}$$

We can deduce:

$$v_g = \frac{d\omega}{dk_z} = \frac{k_z}{\omega}c^2 = \frac{c^2}{v_\varphi} = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} \qquad \text{(XI-11)}$$

To make the link with the mechanical relativistic displacement of a particle, and following Louis DE BROGLIE, we admit that a particle of mass m_0 at rest has an energy at wave pulse ω_c since there is no displacement of energy:

$$\hbar\omega_c = m_0 c^2 \qquad \text{(XI-12)}$$

When this particle is moving at speed v, it presents an increase in its total energy given by the theory of relativity, which wave representation is going to the angular frequency ω:

$$\hbar\omega = \frac{m_0 c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \qquad \text{(XI-13)}$$

On the basis of the assumptions (XI-12) and (XI-13), we can deduce that ω and ω_c pulse should check between them relations:

$$\left(\frac{\omega_c}{\omega}\right)^2 = 1-\left(\frac{v}{c}\right)^2 \qquad \text{(XI-14)}$$

From the definition of group velocity (XI-11), we derived:

$$\left(\frac{\omega_c}{\omega}\right)^2 = 1-\left(\frac{v_g}{c}\right)^2 \qquad \text{(XI-15)}$$

It follows from (XI-14) and (XI-15) that one can identify the speed of mass v to group velocity v_g of the wave's phase attached to the moveable mass. This identification ensures a representation of relativistic mass energy and quantum wave energy that is fully compatible.

The last step is to express the wavelength λ associated to phase wave of frequency ν and pulse ω that moves at the speed v_φ. From the previous paragraph, we deduced that it is associated, through its group velocity, to a mass m moving at speed v. By making use of the relationship $v_\varphi v_g = c^2$ that establishes a relationship between phase velocity and group velocity or speed of the particle, one obtains:

$$\lambda = \frac{v_\varphi}{\nu} = v_\varphi \frac{2\pi}{\omega} = \frac{c^2}{v}\frac{2\pi}{\omega} = \frac{c^2}{v}\frac{2\pi}{\frac{m_0 c^2}{\hbar\sqrt{1-\beta^2}}} = \frac{h}{mv} \qquad \text{(XI-16)}$$

XII

Generalized DIRAC equation

This part deals with the DIRAC equation for a charged particle in an electromagnetic field characterized by a scalar potential ϕ and a vector potential (A_x, A_y, A_z). These potentials are considered constant and uniform, i.e. independent of x, y, z and t. We are looking for, as previously, a solution in the form of linear combinations of stationary modes which would settle into a rectangular cavity.

Following an approach similar to chapter II, the formalism of DIRAC leads to find the solutions of the new equation in which the electromagnetic potential four-vector is introduced:

$$\left\{\gamma_0\left(j\frac{\partial}{\partial x_t}-\frac{q\phi}{c\hbar}\right)+\gamma_1\left(j\frac{\partial}{\partial x}-\frac{qA_x}{\hbar}\right)+\gamma_2\left(j\frac{\partial}{\partial y}-\frac{qA_y}{\hbar}\right)+\gamma_3\left(j\frac{\partial}{\partial z}-\frac{qA_z}{\hbar}\right)\right\}(\psi)=\frac{m_0 c}{\hbar}(\psi) \qquad \text{(XII-1)}$$

The wave function ψ is a bi-spinor with four components:

$$\psi=\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix} \qquad \text{(XII-2)}$$

The matrices γ_0, γ_1, γ_2, γ_3, are given in chapter II, and in order to simplify expressions, we put:

$$\eta=\frac{m_0 c}{\hbar} \qquad \text{(XII-3)}$$

Injecting matrices γ_i in the above equation (XII-3), we obtain the system of differential equations:

$$\eta\psi_0 = j\left(\frac{\partial\psi_0}{\partial x_t} + j\frac{q\phi}{\hbar}\psi_0\right) + j\left(\frac{\partial\psi_3}{\partial x} + j\frac{qA_x}{\hbar}\psi_3\right) + \left(\frac{\partial\psi_3}{\partial y} + j\frac{qA_y}{\hbar}\psi_3\right) + j\left(\frac{\partial\psi_2}{\partial z} + j\frac{qA_z}{\hbar}\psi_2\right)$$

$$\eta\psi_1 = j\left(\frac{\partial\psi_1}{\partial x_t} + j\frac{q\phi}{\hbar}\psi_1\right) + j\left(\frac{\partial\psi_2}{\partial x} + j\frac{qA_x}{\hbar}\psi_2\right) - \left(\frac{\partial\psi_2}{\partial y} + j\frac{qA_y}{\hbar}\psi_2\right) - j\left(\frac{\partial\psi_3}{\partial z} + j\frac{qA_z}{\hbar}\psi_3\right)$$

$$\eta\psi_2 = -j\left(\frac{\partial\psi_2}{\partial x_t} + j\frac{q\phi}{\hbar}\psi_2\right) - j\left(\frac{\partial\psi_1}{\partial x} + j\frac{qA_x}{\hbar}\psi_1\right) - \left(\frac{\partial\psi_1}{\partial y} + j\frac{qA_y}{\hbar}\psi_1\right) - j\left(\frac{\partial\psi_0}{\partial z} + j\frac{qA_z}{\hbar}\psi_0\right)$$

$$\eta\psi_3 = -j\left(\frac{\partial\psi_3}{\partial x_t} + j\frac{q\phi}{\hbar}\psi_3\right) - j\left(\frac{\partial\psi_0}{\partial x} + j\frac{qA_x}{\hbar}\psi_0\right) + \left(\frac{\partial\psi_0}{\partial y} + j\frac{qA_y}{\hbar}\psi_0\right) + j\left(\frac{\partial\psi_1}{\partial z} + j\frac{qA_z}{\hbar}\psi_1\right) \qquad \text{(XII-4)}$$

or again, by isolating the differential system of the free particle:

$$\begin{aligned}
&\left(\eta + \frac{q\phi}{\hbar c}\right)\psi_0 + \frac{qA_x}{\hbar}\psi_3 - j\frac{qA_y}{\hbar}\psi_3 + \frac{qA_z}{\hbar}\psi_2 = j\frac{\partial\psi_0}{\partial x_t} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z}\\
&\left(\eta + \frac{q\phi}{\hbar c}\right)\psi_1 + \frac{qA_x}{\hbar}\psi_2 + j\frac{qA_y}{\hbar}\psi_2 - \frac{qA_z}{\hbar}\psi_3 = j\frac{\partial\psi_1}{\partial x_t} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z}\\
&\left(\eta - \frac{q\phi}{\hbar c}\right)\psi_2 - \frac{qA_x}{\hbar}\psi_1 + j\frac{qA_y}{\hbar}\psi_1 - \frac{qA_z}{\hbar}\psi_0 = -j\frac{\partial\psi_2}{\partial x_t} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z}\\
&\left(\eta - \frac{q\phi}{\hbar c}\right)\psi_3 - \frac{qA_x}{\hbar}\psi_0 - j\frac{qA_y}{\hbar}\psi_0 + \frac{qA_z}{\hbar}\psi_1 = -j\frac{\partial\psi_3}{\partial x_t} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z}
\end{aligned} \qquad \text{(XII-5)}$$

Again for a relief of notation, we put:

$$\eta_t = \frac{q\phi}{\hbar c} \qquad \eta_x = \frac{qA_x}{\hbar} \qquad \eta_y = \frac{qA_y}{\hbar} \qquad \eta_z = \frac{qA_z}{\hbar} \qquad \text{(XII-6)}$$

which leads to the system representing the DIRAC equation:

$$\begin{aligned}
&(\eta + \eta_t)\psi_0 + \eta_x\psi_3 - j\eta_y\psi_3 + \eta_z\psi_2 = j\frac{\partial\psi_0}{\partial x_t} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z}\\
&(\eta + \eta_t)\psi_1 + \eta_x\psi_2 + j\eta_y\psi_2 - \eta_z\psi_3 = j\frac{\partial\psi_1}{\partial x_t} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z}\\
&(\eta - \eta_t)\psi_2 - \eta_x\psi_1 + j\eta_y\psi_1 - \eta_z\psi_0 = -j\frac{\partial\psi_2}{\partial x_t} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z}\\
&(\eta - \eta_t)\psi_3 - \eta_x\psi_0 - j\eta_y\psi_0 + \eta_z\psi_1 = -j\frac{\partial\psi_3}{\partial x_t} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z}
\end{aligned} \qquad \text{(XII-7)}$$

It appears as a very complicated differential system.

One may attempt to find the wave functions ψ_i (i = 0, 1, 2, 3) solutions of this system as it was made in chapter VI, in the form:

$$\psi_i = \{a_i S_x S_y S_z + b_i S_x C_y S_z + c_i C_x S_y S_z + d_i C_x C_y S_z + e_i S_x S_y C_z + f_i S_x C_y C_z + g_i C_x S_y C_z + h_i C_x C_y C_z\} C_t$$
$$+ \{i_i S_x S_y S_z + j_i S_x C_y S_z + k_i C_x S_y S_z + l_i C_x C_y S_z + m_i S_x S_y C_z + n_i S_x C_y C_z + o_i C_x S_y C_z + p_i C_x C_y C_z\} S_t \quad \text{(XII-8)}$$

with the usual notation:

$$S_x = \sin(k_x x)\ S_y = \sin(k_y y)\ S_z = \sin(k_z z)\ S_t = \sin(k_t ct) \quad \text{(XII-9)}$$

$$C_x = \cos(k_x x)\ C_y = \cos(k_y y)\ C_z = \cos(k_z z)\ C_t = \cos(k_t ct) \quad \text{(XII-10)}$$

Using the methodology set out in chapter VI, we are led to build a homogeneous system of 64 equations with 64 unknowns which is described below:

$$\begin{aligned}
&-jka_0 - jk_x k_3 - k_y j_3 - jk_z m_2 - (\eta + \eta_t) i_0 - (\eta_x - j\eta_y) i_3 - \eta_z i_2 = 0\\
&-jkb_0 - jk_x l_3 + k_y i_3 - jk_z n_2 - (\eta + \eta_t) j_0 - (\eta_x - j\eta_y) j_3 - \eta_z j_2 = 0\\
&-jkc_0 + jk_x i_3 - k_y l_3 - jk_z o_2 - (\eta + \eta_t) k_0 - (\eta_x - j\eta_y) k_3 - \eta_z k_2 = 0\\
&-jkd_0 + jk_x j_3 + k_y k_3 - jk_z p_2 - (\eta + \eta_t) l_0 - (\eta_x - j\eta_y) l_3 - \eta_z l_2 = 0\\
&-jke_0 - jk_x o_3 - k_y n_3 + jk_z i_2 - (\eta + \eta_t) m_0 - (\eta_x - j\eta_y) m_3 - \eta_z m_2 = 0\\
&-jkf_0 - jk_x p_3 + k_y m_3 + jk_z j_2 - (\eta + \eta_t) n_0 - (\eta_x - j\eta_y) n_3 - \eta_z n_2 = 0\\
&-jkg_0 + jk_x m_3 - k_y p_3 + jk_z k_2 - (\eta + \eta_t) o_0 - (\eta_x - j\eta_y) o_3 - \eta_z o_2 = 0\\
&-jkh_0 + jk_x n_3 + k_y o_3 + jk_z l_2 - (\eta + \eta_t) p_0 - (\eta_x - j\eta_y) p_3 - \eta_z p_2 = 0\\
&jki_0 - jk_x c_3 - k_y b_3 - jk_z e_2 - (\eta + \eta_t) a_0 - (\eta_x - j\eta_y) a_3 - \eta_z a_2 = 0\\
&jkj_0 - jk_x d_3 + k_y a_3 - jk_z f_2 - (\eta + \eta_t) b_0 - (\eta_x - j\eta_y) b_3 - \eta_z b_2 = 0\\
&jkk_0 + jk_x a_3 - k_y d_3 - jk_z g_2 - (\eta + \eta_t) c_0 - (\eta_x - j\eta_y) c_3 - \eta_z c_2 = 0\\
&jkl_0 + jk_x b_3 + k_y c_3 - jk_z h_2 - (\eta + \eta_t) d_0 - (\eta_x - j\eta_y) d_3 - \eta_z d_2 = 0\\
&jkm_0 - jk_x g_3 - k_y f_3 + jk_z a_2 - (\eta + \eta_t) e_0 - (\eta_x - j\eta_y) e_3 - \eta_z e_2 = 0\\
&jkn_0 - jk_x h_3 + k_y e_3 + jk_z b_2 - (\eta + \eta_t) f_0 - (\eta_x - j\eta_y) f_3 - \eta_z f_2 = 0\\
&jko_0 + jk_x e_3 - k_y h_3 + jk_z c_2 - (\eta + \eta_t) g_0 - (\eta_x - j\eta_y) g_3 - \eta_z g_2 = 0\\
&jkp_0 + jk_x f_3 + k_y g_3 + jk_z d_2 - (\eta + \eta_t) h_0 - (\eta_x - j\eta_y) h_3 - \eta_z h_2 = 0
\end{aligned} \quad \text{(XII-11)}$$

$$
\begin{aligned}
&-jka_1 - jk_x k_2 + k_y j_2 + jk_z m_3 - (\eta + \eta_t) i_1 - (\eta_x + j\eta_y) i_2 + \eta_z i_3 = 0\\
&-jkb_1 - jk_x l_2 - k_y i_2 + jk_z n_3 - (\eta + \eta_t) j_1 - (\eta_x + j\eta_y) j_2 + \eta_z j_3 = 0\\
&-jkc_1 + jk_x i_2 + k_y l_2 + jk_z o_3 - (\eta + \eta_t) k_1 - (\eta_x + j\eta_y) k_2 + \eta_z k_3 = 0\\
&-jkd_1 + jk_x j_2 - k_y k_2 + jk_z p_3 - (\eta + \eta_t) l_1 - (\eta_x + j\eta_y) l_2 + \eta_z l_3 = 0\\
&-jke_1 - jk_x o_2 + k_y n_2 - jk_z i_3 - (\eta + \eta_t) m_1 - (\eta_x + j\eta_y) m_2 + \eta_z m_3 = 0\\
&-jkf_1 - jk_x p_2 - k_y m_2 - jk_z j_3 - (\eta + \eta_t) n_1 - (\eta_x + j\eta_y) n_2 + \eta_z n_3 = 0\\
&-jkg_1 + jk_x m_2 + k_y p_2 - jk_z k_3 - (\eta + \eta_t) o_1 - (\eta_x + j\eta_y) o_2 + \eta_z o_3 = 0\\
&-jkh_1 + jk_x n_2 - k_y o_2 - jk_z l_3 - (\eta + \eta_t) p_1 - (\eta_x + j\eta_y) p_2 + \eta_z p_3 = 0\\
&jki_1 - jk_x c_2 + k_y b_2 + jk_z e_3 - (\eta + \eta_t) a_1 - (\eta_x + j\eta_y) a_2 + \eta_z a_3 = 0\\
&jkj_1 - jk_x d_2 - k_y a_2 + jk_z f_3 - (\eta + \eta_t) b_1 - (\eta_x + j\eta_y) b_2 + \eta_z b_3 = 0\\
&jkk_1 + jk_x a_2 + k_y d_2 + jk_z g_3 - (\eta + \eta_t) c_1 - (\eta_x + j\eta_y) c_2 + \eta_z c_3 = 0\\
&jkl_1 + jk_x b_2 - k_y c_2 + jk_z h_3 - (\eta + \eta_t) d_1 - (\eta_x + j\eta_y) d_2 + \eta_z d_3 = 0\\
&jkm_1 - jk_x g_2 + k_y f_2 - jk_z a_3 - (\eta + \eta_t) e_1 - (\eta_x + j\eta_y) e_2 + \eta_z e_3 = 0\\
&jkn_1 - jk_x h_2 - k_y e_2 - jk_z b_3 - (\eta + \eta_t) f_1 - (\eta_x + j\eta_y) f_2 + \eta_z f_3 = 0\\
&jko_1 + jk_x e_2 + k_y h_2 - jk_z c_3 - (\eta + \eta_t) g_1 - (\eta_x + j\eta_y) g_2 + \eta_z g_3 = 0\\
&jkp_1 + jk_x f_2 - k_y g_2 - jk_z d_3 - (\eta + \eta_t) h_1 - (\eta_x + j\eta_y) h_2 + \eta_z h_3 = 0
\end{aligned}
\qquad \text{(XII-12)}
$$

$$
\begin{aligned}
&jka_2 + jk_x k_1 + k_y j_1 + jk_z m_0 - (\eta - \eta_t) i_2 + (\eta_x - j\eta_y) i_1 + \eta_z i_0 = 0\\
&jkb_2 + jk_x l_1 - k_y i_1 + jk_z n_0 - (\eta - \eta_t) j_2 + (\eta_x - j\eta_y) j_1 + \eta_z j_0 = 0\\
&jkc_2 - jk_x i_1 + k_y l_1 + jk_z o_0 - (\eta - \eta_t) k_2 + (\eta_x - j\eta_y) k_1 + \eta_z k_0 = 0\\
&jkd_2 - jk_x j_1 - k_y k_1 + jk_z p_0 - (\eta - \eta_t) l_2 + (\eta_x - j\eta_y) l_1 + \eta_z l_0 = 0\\
&jke_2 + jk_x o_1 + k_y n_1 - jk_z i_0 - (\eta - \eta_t) m_2 + (\eta_x - j\eta_y) m_1 + \eta_z m_0 = 0\\
&jkf_2 + jk_x p_1 - k_y m_1 - jk_z j_0 - (\eta - \eta_t) n_2 + (\eta_x - j\eta_y) n_1 + \eta_z n_0 = 0\\
&jkg_2 - jk_x m_1 + k_y p_1 - jk_z k_0 - (\eta - \eta_t) o_2 + (\eta_x - j\eta_y) o_1 + \eta_z o_0 = 0\\
&jkh_2 - jk_x n_1 - k_y o_1 - jk_z l_0 - (\eta - \eta_t) p_2 + (\eta_x - j\eta_y) p_1 + \eta_z p_0 = 0\\
&-jki_2 + jk_x c_1 + k_y b_1 + jk_z e_0 - (\eta - \eta_t) a_2 + (\eta_x - j\eta_y) a_1 + \eta_z a_0 = 0\\
&-jkj_2 + jk_x d_1 - k_y a_1 + jk_z f_0 - (\eta - \eta_t) b_2 + (\eta_x - j\eta_y) b_1 + \eta_z b_0 = 0\\
&-jkk_2 - jk_x a_1 + k_y d_1 + jk_z g_0 - (\eta - \eta_t) c_2 + (\eta_x - j\eta_y) c_1 + \eta_z c_0 = 0\\
&-jkl_2 - jk_x b_1 - k_y c_1 + jk_z h_0 - (\eta - \eta_t) d_2 + (\eta_x - j\eta_y) d_1 + \eta_z d_0 = 0\\
&-jkm_2 + jk_x g_1 + k_y f_1 - jk_z a_0 - (\eta - \eta_t) e_2 + (\eta_x - j\eta_y) e_1 + \eta_z e_0 = 0\\
&-jkn_2 + jk_x h_1 - k_y e_1 - jk_z b_0 - (\eta - \eta_t) f_2 + (\eta_x - j\eta_y) f_1 + \eta_z f_0 = 0\\
&-jko_2 - jk_x e_1 + k_y h_1 - jk_z c_0 - (\eta - \eta_t) g_2 + (\eta_x - j\eta_y) g_1 + \eta_z g_0 = 0\\
&-jkp_2 - jk_x f_1 - k_y g_1 - jk_z d_0 - (\eta - \eta_t) h_2 + (\eta_x - j\eta_y) h_1 + \eta_z h_0 = 0
\end{aligned}
\qquad \text{(XII-13)}
$$

$$
\begin{aligned}
&jka_3 + jk_x k_0 - k_y j_0 - jk_z m_1 - (\eta - \eta_t) i_3 + (\eta_x + j\eta_y) i_0 - \eta_z i_1 = 0\\
&jkb_3 + jk_x l_0 + k_y i_0 - jk_z n_1 - (\eta - \eta_t) j_3 + (\eta_x + j\eta_y) j_0 - \eta_z j_1 = 0\\
&jkc_3 - jk_x i_0 - k_y l_0 - jk_z o_1 - (\eta - \eta_t) k_3 + (\eta_x + j\eta_y) k_0 - \eta_z k_1 = 0\\
&jkd_3 - jk_x j_0 + k_y k_0 - jk_z p_1 - (\eta - \eta_t) l_3 + (\eta_x + j\eta_y) l_0 - \eta_z l_1 = 0\\
&jke_3 + jk_x o_0 - k_y n_0 + jk_z i_1 - (\eta - \eta_t) m_3 + (\eta_x + j\eta_y) m_0 - \eta_z m_1 = 0\\
&jkf_3 + jk_x p_0 + k_y m_0 + jk_z j_1 - (\eta - \eta_t) n_3 + (\eta_x + j\eta_y) n_0 - \eta_z n_1 = 0\\
&jkg_3 - jk_x m_0 - k_y p_0 + jk_z k_1 - (\eta - \eta_t) o_3 + (\eta_x + j\eta_y) o_0 - \eta_z o_1 = 0\\
&jkh_3 - jk_x n_0 + k_y o_0 + jk_z l_1 - (\eta - \eta_t) p_3 + (\eta_x + j\eta_y) p_0 - \eta_z p_1 = 0\\
&-jki_3 + jk_x c_0 - k_y b_0 - jk_z e_1 - (\eta - \eta_t) a_3 + (\eta_x + j\eta_y) a_0 - \eta_z a_1 = 0\\
&-jkj_3 + jk_x d_0 + k_y a_0 - jk_z f_1 - (\eta - \eta_t) b_3 + (\eta_x + j\eta_y) b_0 - \eta_z b_1 = 0\\
&-jkk_3 - jk_x a_0 - k_y d_0 - jk_z g_1 - (\eta - \eta_t) c_3 + (\eta_x + j\eta_y) c_0 - \eta_z c_1 = 0\\
&-jkl_3 - jk_x b_0 + k_y c_0 - jk_z h_1 - (\eta - \eta_t) d_3 + (\eta_x + j\eta_y) d_0 - \eta_z d_1 = 0\\
&-jkm_3 + jk_x g_0 - k_y f_0 + jk_z a_1 - (\eta - \eta_t) e_3 + (\eta_x + j\eta_y) e_0 - \eta_z e_1 = 0\\
&-jkn_3 + jk_x h_0 + k_y e_0 + jk_z b_1 - (\eta - \eta_t) f_3 + (\eta_x + j\eta_y) f_0 - \eta_z f_1 = 0\\
&-jko_3 - jk_x e_0 - k_y h_0 + jk_z c_1 - (\eta - \eta_t) g_3 + (\eta_x + j\eta_y) g_0 - \eta_z g_1 = 0\\
&-jkp_3 - jk_x f_0 + k_y g_0 + jk_z d_1 - (\eta - \eta_t) h_3 + (\eta_x + j\eta_y) h_0 - \eta_z h_1 = 0
\end{aligned}
\qquad \text{(XII-14)}
$$

The first idea is to build on the results of Chapter VI, it means to assign a mode to ψ_0, and search patterns that are solutions of the system for wave functions ψ_1, ψ_2, ψ_3. A long and tedious mathematical work has not allowed to express an exact solution for ψ_1, ψ_2, ψ_3. This work has not led to show that such a solution does not exist, and so the problem remains open. Complementary indications will be given in chapter XXII.

Progress towards a possible solution will come from a physical analysis of this system.

In first place, the determinant of this system should be null. We hypothesize that the condition of nullity is provided by the equation of energy conservation.

We must therefore establish this new equation of conservation on the basis of the energy provided to the charged particle by the presence of the electromagnetic potential. This energy is of two kinds:

- the energy provided by the scalar potential ϕ allows to increase the kinetic energy of the charged particle. Relativity suggests that this increase in energy is transformed in mass energy.
- the energy provided by the vector potential (A_x, A_y, A_z) allows only to change the direction of the trajectory of the particle, without kinetic energy supply: this is therefore a purely impulse energy.

Outside the presence of the electromagnetic field, the equation of conservation of energy is written:

$$-k_t^2 + k_x^2 + k_y^2 + k_z^2 + \eta^2 = 0 \qquad \text{(XII-15)}$$

Inputs of energy of the electromagnetic field in this relationship can be introduced in many ways without being trivial to make a priori choice among all possible formulations.

Reflection led to conclude that the correct form is as follows:

$$-(k_t+\eta_t)^2+(k_x+\eta_x)^2+(k_y+\eta_y)^2+(k_z+\eta_z)^2+\eta^2=0 \qquad \text{(XII-16)}$$

It should be noted in particular that the kinetic energy η_t provided by the scalar potential ϕ is not associated with the mass energy η as it could think intuitively.

The rigorous justification is that the sum of the wave four-vector and the electromagnetic potential four-vector gives a four-vector too:

$$\begin{pmatrix} k_x \\ k_y \\ k_z \\ k_t \end{pmatrix}+\begin{pmatrix} \eta_x \\ \eta_y \\ \eta_z \\ \eta_t \end{pmatrix}=\begin{pmatrix} k_x+\eta_x \\ k_y+\eta_y \\ k_z+\eta_z \\ k_t+\eta_t \end{pmatrix} \qquad \text{(XII-17)}$$

And since the pseudo-norm of the four-vector sum is constant, this leads directly to the equation of conservation of energy (XII-16).

Therefore, we hypothesize that the determinant of the overall system of 64 equations with 64 unknowns is zero when the equation of conservation of energy (XII-16) is checked.

But, it does not provide track to move towards a solution of the generalized electromagnetic interacting DIRAC equation system.

A detailed analysis of the system of DIRAC allows however to see if one excite a wave function with a mode in time quadrature for the scalar potential and and in space quadrature for the vector potential, it manages to get some solutions relative to the stationary modes (XII-11,12,13,14). Quadrature modes must also express exchanges of energy between positive and negative energy which introduced so the quantity j = sqrt(-1) between expressions of these modes.

In summary, the presence of the electromagnetic field don't excite wave functions in the form of independent stationary modes, but in the form of combinations of modes related in time and space quadrature, and they reflect exchanges of energy between positive energy and negative energy.

These modes will therefore present themselves in the form of combinations of functions of x, y, z, t with amplitude A, which have the following expression:

$$A\exp(\pm jk_x x)\exp(\pm jk_y y)\exp(\pm k_z z)\exp(\pm jk_t x_t)=A\exp\left\{j\left(\pm k_t x_t\pm k_x x\pm k_y y\pm k_z z\right)\right\} \qquad \text{(XII-18)}$$

The signs + and - present in this expression will affect the relationship of conservation of energy which should take them into account in the form:

$$-(\eta_t \pm k_t)^2 + (\eta_x \pm k_x)^2 + (\eta_y \pm k_y)^2 + (\eta_z \pm k_z)^2 + \eta^2 = 0 \qquad \text{(XII-19)}$$

These considerations are illustrated on this particular example, which corresponds to an exact solution of the DIRAC equation generalized to electromagnetic interacting.

$$\begin{aligned}
\psi_0 &= -(k_z + \eta_z)\exp\{j(-k_t x_t + k_x x + k_y y + k_z z)\} \\
\psi_1 &= -(k_x + \eta_x)\exp\{j(-k_t x_t + k_x x + k_y y + k_z z)\} - j(k_y + \eta_y)\exp\{j(-k_t x_t + k_x x + k_y y + k_z z)\} \\
\psi_2 &= (\eta - k_t + \eta_t)\exp\{j(-k_t x_t + k_x x + k_y y + k_z z)\} \\
\psi_3 &= 0
\end{aligned} \qquad \text{(XII-20)}$$

This solution is associated with the conservation of energy equation:

$$-(\eta_t - k_t)^2 + (\eta_x + k_x)^2 + (\eta_y + k_y)^2 + (\eta_z + k_z)^2 + \eta^2 = 0 \qquad \text{(XII-21)}$$

It is suitable to be convincing, to detail checks of these properties. We skip in the calculations below exponential coming in factor with all terms.

<u>First generalized DIRAC equation:</u>

$$(\eta + \eta_t)\psi_0 + \eta_x\psi_3 - j\eta_y\psi_3 + \eta_z\psi_2 = j\frac{\partial\psi_0}{\partial x_t} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z} \qquad \text{(XII-22)}$$

$$-(\eta + \eta_t)(k_z + \eta_z) + 0 + 0 + \eta_z(\eta - k_t + \eta_t) = -(k_z + \eta_z)k_t + 0 + 0 - (\eta - k_t + \eta_t)k_z \qquad \text{(XII-23)}$$

<u>Second generalized DIRAC equation:</u>

$$(\eta + \eta_t)\psi_1 + \eta_x\psi_2 + j\eta_y\psi_2 - \eta_z\psi_3 = j\frac{\partial\psi_1}{\partial x_t} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z} \qquad \text{(XII-24)}$$

$$\begin{aligned}
&(\eta + \eta_t)\{-(k_x + \eta_x) - j(k_y + \eta_y)\} + \eta_x(\eta - k_t + \eta_t) + j\eta_y(\eta - k_t + \eta_t) - 0 = \\
&k_t\{-(k_x + \eta_x) - j(k_y + \eta_y)\} - k_x(\eta - k_t + \eta_t) - jk_y(\eta - k_t + \eta_t) - 0
\end{aligned} \qquad \text{(XII-25)}$$

<u>Third generalized DIRAC equation:</u>

$$(\eta - \eta_t)\psi_2 - \eta_x\psi_1 + j\eta_y\psi_1 - \eta_z\psi_0 = -j\frac{\partial\psi_2}{\partial x_t} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z} \qquad \text{(XII-26)}$$

$$\begin{aligned}
&(\eta - \eta_t)(\eta - k_t + \eta_t) + \eta_x\{(k_x + \eta_x) + j(k_y + \eta_y)\} - j\eta_y\{(k_x + \eta_x) + j(k_y + \eta_y)\} + \eta_z(k_z + \eta_z) = \\
&-(\eta - k_t + \eta_t)k_t - k_x\{(k_x + \eta_x) + j(k_y + \eta_y)\} + jk_y\{(k_x + \eta_x) + j(k_y + \eta_y)\} - k_z(k_z + \eta_z)
\end{aligned} \qquad \text{(XII-27)}$$

The imaginary terms vanish, and it remains:

$$-(\eta_t - k_t)^2 + (\eta_x + k_x)^2 + (\eta_y + k_y)^2 + (\eta_z + k_z)^2 + \eta^2 = 0 \qquad \text{(XII-28)}$$

or so the equation of conservation of energy.

Fourth generalized DIRAC equation:

$$(\eta - \eta_t)\psi_3 - \eta_x\psi_0 - j\eta_y\psi_0 + \eta_z\psi_1 = -j\frac{\partial\psi_3}{\partial x_t} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z} \qquad \text{(XII-29)}$$

$$\begin{aligned} &0 + \eta_x(k_z + \eta_z) + j\eta_y(k_z + \eta_z) - \eta_z\{(k_x + \eta_x) + j(k_y + \eta_y)\} = \\ &0 - k_x(k_z + \eta_z) - jk_y(k_z + \eta_z) + k_z\{(k_x + \eta_x) + j(k_y + \eta_y)\} \end{aligned} \qquad \text{(XII-30)}$$

Verification that the (XII-20) solution is an exact solution to the generalized DIRAC system is completed.

We would think that on the basis of the exact solution (XII-20), it is possible to construct purely real stationary solutions for example by summing two solutions in exp (jωt) and exp (-jωt), which would be in contradiction with the previous statement on the impossibility to obtain such solutions in the generalized DIRAC system.

In fact, it is impossible to sum these solutions, because they are relative each to a different energy conservation equation, and thus to a different condition of nullity of the determinant of the system.

One can however mix real stationary modes and complex stationary modes, as shown in the exact solution below, in which the presence of the electromagnetic potential is reduced to the scalar potential:

$$\begin{aligned} \psi_0 &= 0 \\ \psi_1 &= -(\eta - k_t - \eta_t)\sin(k_x x)\cos(k_y y)\cos(k_z z)\exp(jk_t x_t) \\ \psi_2 &= -k_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\exp(jk_t x_t) + jk_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\exp(jk_t x_t) \\ \psi_3 &= jk_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\exp(jk_t x_t) \end{aligned} \qquad \text{(XII-31)}$$

It is associated with the conservation of energy equation:

$$(k_t + \eta_t)^2 - k_x^2 - k_y^2 - k_z^2 - \eta^2 = 0 \qquad \text{(XII-32)}$$

XIII

The currents of DIRAC (2)

DIRAC currents related to the solutions of the generalized equation to the presence of an electromagnetic field have a remarkable property which will be illustrated on the example of solution proposed earlier and recalled to memory:

$$\begin{aligned}
\psi_0 &= -(k_z+\eta_z)\exp\{j(-k_t x_t + k_x x + k_y y + k_z z)\}\\
\psi_1 &= -(k_x+\eta_x)\exp\{j(-k_t x_t + k_x x + k_y y + k_z z)\} - j(k_y+\eta_y)\exp\{j(-k_t x_t + k_x x + k_y y + k_z z)\}\\
\psi_2 &= (\eta - k_t + \eta_t)\exp\{j(-k_t x_t + k_x x + k_y y + k_z z)\}\\
\psi_3 &= 0
\end{aligned} \qquad \text{(XIII-1)}$$

The expression of these currents is recalled below. Multiplied by a constant adequate, they become homogeneous to a volumetric energy density, and the term J^0 represents the total energy volume density.

$$\begin{aligned}
J^0 &= \psi_0^*\psi_0 + \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3\\
J^1 &= \psi_3^*\psi_0 + \psi_2^*\psi_1 + \psi_1^*\psi_2 + \psi_0^*\psi_3\\
J^2 &= j\psi_3^*\psi_0 - j\psi_2^*\psi_1 + j\psi_1^*\psi_2 - j\psi_0^*\psi_3\\
J^3 &= \psi_2^*\psi_0 - \psi_3^*\psi_1 + \psi_0^*\psi_2 - \psi_1^*\psi_3
\end{aligned} \qquad \text{(XIII-2)}$$

In addition, these currents must check the local conservation of energy equation:

$$\frac{\partial J^0}{\partial x_t} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z} = 0 \qquad \text{(XIII-3)}$$

One obtains, in a straightforward way, from wave functions (XIII-1) and currents (XIII-2), the following expressions:

$$\begin{aligned}
J^0 &= (k_x+\eta_x)^2 + (k_z+\eta_z)^2 + (k_z+\eta_z)^2 + (\eta - k_t + \eta_t)^2\\
J^1 &= -2(\eta - k_t + \eta_t)(k_x+\eta_x)\\
J^2 &= -2(\eta - k_t + \eta_t)(k_y+\eta_y)\\
J^3 &= -2(\eta - k_t + \eta_t)(k_z+\eta_z)
\end{aligned} \qquad \text{(XIII-4)}$$

The surprising result that appears in the expression of these currents is that they depend on neither time nor space. In other words, the volume density of total energy represented by J^0 is uniform inside the block that contains the energy of the particle.

This result presents a greater analogy to the example of a particle which has twice its mass energy which is presented in chapter VI, with however a difference subject to the assumptions made in the preparation of the solutions in the presence of electromagnetic field.

If we accept that there is no purely real solutions for the wave functions obtained in the presence of an electromagnetic field, this particular scheme of uniform energy within the particle density is imposed by the presence of the electromagnetic field, while nothing requires it for the free particle.

The normalization constant C is obtained by writing that J^0 represents the volume density of total energy of the particle placed in an electromagnetic field, and then the total energy contained in volume V that delimits the particle:

$$E = \left(\hbar^2 c^2\right)\left(k_t + \eta_t\right)^2 V \qquad \text{(XIII-5)}$$

In equating this relationship with (XIII-4) multiplied by the normalizing constant C^2, one obtains:

$$E = C^2\left\{\left(k_x + \eta_x\right)^2 + \left(k_z + \eta_z\right)^2 + \left(k_z + \eta_z\right)^2 + \left(\eta - k_t + \eta_t\right)^2\right\} = \left(\hbar^2 c^2\right)\left(k_t + \eta_t\right)^2 V \qquad \text{(XIII-6)}$$

or still, by introducing the equation of conservation of energy called for memory:

$$\left(k_t + \eta_t\right)^2 = \left(k_x + \eta_x\right)^2 + \left(k_y + \eta_y\right)^2 + \left(k_z + \eta_z\right)^2 + \eta^2 \qquad \text{(XIII-7)}$$

$$C^2 = \frac{\left(\hbar^2 c^2\right)\left(k_t + \eta_t\right)^2 V}{\left(k_t + \eta_t\right)^2 - \eta^2 + \left(\eta - k_t + \eta_t\right)^2} \qquad \text{(XIII-8)}$$

XIV

Conclusion of the second part

We know that solutions to the DIRAC equation which correspond to the reality of the observations are solutions developed using spherical modes and in this context, one may wonder what is the interest to work on solutions obtained in Cartesian coordinates.

It appears that Cartesian solutions are more quickly attainable, and that they can learn to us valuable information about the behavior of energy which the particle is constituted.

On the basis of an equation of conservation of energy which is a fundamental physical reasoning and on the assumption of an internal evolution of energy based on exchanges between stationary modes, the exact solutions to the DIRAC equation deliver new elements likely to describe the physics of the infinitely small.

There is no assumption on the spatial extent of the modes that are supposed to describe the behavior of the particle, but it is legitimate to think that this scope exceeds the size given in classical physics if you want for example to be able to explain the phenomena of interference. This interpretation was already present in the thought of Louis DE BROGLIE during his thesis: ”Do we assume the localized periodic phenomenon *inside* the piece of energy? This is not necessary and will result in paragraph (III) it is probably spread in a large region of space.”

In opposition to the Copenhagen school, the energy interpretation of stationary solutions is perfectly deterministic, but it does not contradict the experiences of the probabilistic vision of quantum mechanics.

It justifies the wave particle duality in indicating in what manner the internal energy to the particle alternately goes in the form of mass energy and wave energy.

It shows how the HEISENBERG uncertainty principle is interpreted by indicating how the mass energy and impulse energy are not simultaneously present in the same place.

Finally, it is fully compatible with the interpretation of a wave phenomenon associated with the particle following the DE BROGLIE wavelength.

Third part

Energy approach of Dirac equation and its exact solutions in spherical coordinates

XV

DIRAC equation in spherical coordinates

The analysis of stationary solutions of the DIRAC equation in Cartesian coordinates allowed to highlight the properties which, while being backed by a perfectly deterministic theory, are in agreement with all the results obtained in the statistical interpretation of the Copenhagen school.

If stationary modes are able to represent the exchange of energy within particles, there are little chance that it is in the shape of a parallelepiped. Everything indicates, in particular solutions of the SCHRÖDINGER equation, that the coordinate system the most suitable, one that provides solutions in agreement with experimental observations, is the system of spherical coordinates (Figure XV-1)

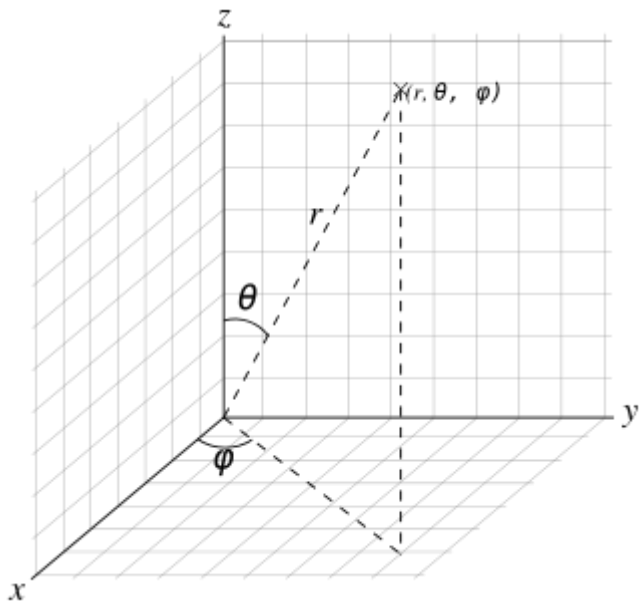


Figure (XV-1): *representation of the spherical coordinate system*

It is expected to appear in exact solutions to the DIRAC equation in spherical coordinates, informations that allow to better understand how the spin of the electron is related to internal rotation of energy.

Even before discussing the search for solutions, we must transform the DIRAC equation in spherical coordinates.

The starting point is given by the link relations between the cartesian and spherical coordinates:

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \qquad \text{(XV-1)}$$

We deduce the differential relations:

$$\begin{aligned} dx &= dr\sin\theta\cos\varphi + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi \\ dy &= dr\sin\theta\sin\varphi + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi \\ dz &= dr\cos\theta - r\sin\theta d\theta \end{aligned} \qquad \text{(XV-2)}$$

Or again using matrix writing:

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix} \qquad \text{(XV-3)}$$

And then, by matrix inversion:

$$\begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \dfrac{\cos\theta\cos\varphi}{r} & \dfrac{\cos\theta\sin\varphi}{r} & \dfrac{-\sin\theta}{r} \\ \dfrac{-\sin\varphi}{r\sin\theta} & \dfrac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \qquad \text{(XV-4)}$$

The DIRAC equation in Cartesian coordinates is recalled below:

$$j\left[\gamma_0\left(\frac{\partial}{\partial x_t}\right) + \gamma_1\left(\frac{\partial}{\partial x}\right) + \gamma_2\left(\frac{\partial}{\partial y}\right) + \gamma_3\left(\frac{\partial}{\partial z}\right)\right](\psi) = \eta(\psi) \qquad \text{(XV-5)}$$

where matrices γ_i are of the form:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -j \\ 0 & 0 & j & 0 \\ 0 & j & 0 & 0 \\ -j & 0 & 0 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \text{(XV-6)}$$

The objective is to transform this equation to write it with the partial derivatives with respect to spherical variables r, θ, φ:

$$j\left[\gamma_0\left(\frac{\partial}{\partial x_t}\right)+\gamma_r\left(\frac{\partial}{\partial r}\right)+\gamma_\theta\frac{1}{r}\left(\frac{\partial}{\partial\theta}\right)+\gamma_\varphi\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\varphi}\right)\right](\psi)=\eta(\psi) \qquad \text{(XV-7)}$$

The issue is the determination of the new matrices γ_r, γ_θ, γ_φ. This requires, as a first step, to formalize the passage of the partial derivatives with respect to x, y, z into partial derivatives with respect to r, θ, φ.

To establish these relationships, we can use the total differential, which is a constant of the transformation:

$$d\psi=\frac{\partial\psi}{\partial r}dr+\frac{\partial\psi}{\partial\theta}d\theta+\frac{\partial\psi}{\partial\varphi}d\varphi \qquad \text{(XV-8)}$$

By introducing in this relation the differentials dr, dθ, dφ given in (XV-4), we obtain:

$$\begin{aligned}d\psi=&\frac{\partial\psi}{\partial r}(\sin\theta\cos\varphi dx+\sin\theta\sin\varphi dy+\cos\theta dz)+\\&\frac{\partial\psi}{\partial\theta}\frac{1}{r}(\cos\theta\cos\varphi dx+\cos\theta\sin\varphi dy-\sin\theta dz)\\&+\frac{\partial\psi}{\partial\varphi}\frac{1}{r\sin\theta}(-\sin\varphi dx+\cos\varphi dy)\end{aligned} \qquad \text{(XV-9)}$$

We gather terms which are linked to Cartesian differentials dx, dy, dz:

$$\begin{aligned}d\psi=&dx\left\{\frac{\partial\psi}{\partial r}\sin\theta\cos\varphi+\frac{\partial\psi}{\partial\theta}\frac{1}{r}\cos\theta\cos\varphi-\frac{\partial\psi}{\partial\varphi}\frac{1}{r\sin\theta}\sin\varphi\right\}\\&+dy\left\{\frac{\partial\psi}{\partial r}\sin\theta\sin\varphi+\frac{\partial\psi}{\partial\theta}\frac{1}{r}\cos\theta\sin\varphi+\frac{\partial\psi}{\partial\varphi}\frac{1}{r\sin\theta}\cos\varphi\right\}\\&+dz\left\{\frac{\partial\psi}{\partial r}\cos\theta-\frac{\partial\psi}{\partial\theta}\frac{1}{r}\sin\theta\right\}\end{aligned} \qquad \text{(XV-10)}$$

We then identify this expression with the total differential in Cartesian coordinates, which is a constant of the transformation:

$$d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy+\frac{\partial\psi}{\partial z}dz \qquad \text{(XV-11)}$$

This identification provides the searched transformation between the partial derivatives in spherical coordinates and the partial derivatives in Cartesian coordinates:

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial r}\sin\theta\cos\varphi+\frac{\partial\psi}{\partial\theta}\frac{1}{r}\cos\theta\cos\varphi-\frac{\partial\psi}{\partial\varphi}\frac{1}{r\sin\theta}\sin\varphi$$
$$\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial r}\sin\theta\sin\varphi+\frac{\partial\psi}{\partial\theta}\frac{1}{r}\cos\theta\sin\varphi+\frac{\partial\psi}{\partial\varphi}\frac{1}{r\sin\theta}\cos\varphi \qquad \text{(XV-12)}$$
$$\frac{\partial\psi}{\partial z}=\frac{\partial\psi}{\partial r}\cos\theta-\frac{\partial\psi}{\partial\theta}\frac{1}{r}\sin\theta$$

It is then possible to change the coordinate system by reporting these equalities in DIRAC equation in Cartesian coordinates:

$$j\left[\gamma_0\left(\frac{\partial}{\partial x_t}\right)+\gamma_1\left(\frac{\partial}{\partial x}\right)+\gamma_2\left(\frac{\partial}{\partial y}\right)+\gamma_3\left(\frac{\partial}{\partial z}\right)\right](\psi)=\frac{m_0 c}{\hbar}(\psi) \qquad \text{(XV-13)}$$

The explicit formulation is fully developed below:

$$\eta\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=j\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial x_t}\\ \frac{\partial\psi_1}{\partial x_t}\\ \frac{\partial\psi_2}{\partial x_t}\\ \frac{\partial\psi_3}{\partial x_t}\end{pmatrix}$$

$$+j\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial r}\sin\theta\cos\varphi+\frac{\partial\psi_0}{\partial\theta}\frac{1}{r}\cos\theta\cos\varphi-\frac{\partial\psi_0}{\partial\varphi}\frac{1}{r\sin\theta}\sin\varphi\\ \frac{\partial\psi_1}{\partial r}\sin\theta\cos\varphi+\frac{\partial\psi_1}{\partial\theta}\frac{1}{r}\cos\theta\cos\varphi-\frac{\partial\psi_1}{\partial\varphi}\frac{1}{r\sin\theta}\sin\varphi\\ \frac{\partial\psi_2}{\partial r}\sin\theta\cos\varphi+\frac{\partial\psi_2}{\partial\theta}\frac{1}{r}\cos\theta\cos\varphi-\frac{\partial\psi_2}{\partial\varphi}\frac{1}{r\sin\theta}\sin\varphi\\ \frac{\partial\psi_3}{\partial r}\sin\theta\cos\varphi+\frac{\partial\psi_3}{\partial\theta}\frac{1}{r}\cos\theta\cos\varphi-\frac{\partial\psi_3}{\partial\varphi}\frac{1}{r\sin\theta}\sin\varphi\end{pmatrix}$$

$$+j\begin{pmatrix}0&0&0&-j\\0&0&j&0\\0&j&0&0\\-j&0&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial r}\sin\theta\sin\varphi+\frac{\partial\psi_0}{\partial\theta}\frac{1}{r}\cos\theta\sin\varphi+\frac{\partial\psi_0}{\partial\varphi}\frac{1}{r\sin\theta}\cos\varphi\\ \frac{\partial\psi_1}{\partial r}\sin\theta\sin\varphi+\frac{\partial\psi_1}{\partial\theta}\frac{1}{r}\cos\theta\sin\varphi+\frac{\partial\psi_1}{\partial\varphi}\frac{1}{r\sin\theta}\cos\varphi\\ \frac{\partial\psi_2}{\partial r}\sin\theta\sin\varphi+\frac{\partial\psi_2}{\partial\theta}\frac{1}{r}\cos\theta\sin\varphi+\frac{\partial\psi_2}{\partial\varphi}\frac{1}{r\sin\theta}\cos\varphi\\ \frac{\partial\psi_3}{\partial r}\sin\theta\sin\varphi+\frac{\partial\psi_3}{\partial\theta}\frac{1}{r}\cos\theta\sin\varphi+\frac{\partial\psi_3}{\partial\varphi}\frac{1}{r\sin\theta}\cos\varphi\end{pmatrix}$$

$$+j\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial r}\cos\theta-\frac{\partial\psi_0}{\partial\theta}\frac{1}{r}\sin\theta\\ \frac{\partial\psi_1}{\partial r}\cos\theta-\frac{\partial\psi_1}{\partial\theta}\frac{1}{r}\sin\theta\\ \frac{\partial\psi_2}{\partial r}\cos\theta-\frac{\partial\psi_2}{\partial\theta}\frac{1}{r}\sin\theta\\ \frac{\partial\psi_3}{\partial r}\cos\theta-\frac{\partial\psi_3}{\partial\theta}\frac{1}{r}\sin\theta\end{pmatrix} \qquad \text{(XV-14)}$$

We organize terms around the partial derivative in r, θ, φ, in order to obtain a matrix relation:

$$j\left[\gamma_0\left(\frac{\partial}{\partial x_t}\right)+\gamma_r\left(\frac{\partial}{\partial r}\right)+\gamma_\theta\frac{1}{r}\left(\frac{\partial}{\partial\theta}\right)+\gamma_\varphi\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\varphi}\right)\right](\psi)=\eta(\psi) \qquad \text{(XV-15)}$$

This grouping led by identification to the searched matrices γ_r, γ_θ, γ_φ:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma_r = \begin{pmatrix} 0 & 0 & \cos\theta & \sin\theta e^{-j\varphi} \\ 0 & 0 & \sin\theta e^{j\varphi} & -\cos\theta \\ -\cos\theta & -\sin\theta e^{-j\varphi} & 0 & 0 \\ -\sin\theta e^{j\varphi} & \cos\theta & 0 & 0 \end{pmatrix}$$

$$\gamma_\theta = \begin{pmatrix} 0 & 0 & -\sin\theta & \cos\theta e^{-j\varphi} \\ 0 & 0 & \cos\theta e^{j\varphi} & \sin\theta \\ \sin\theta & -\cos\theta e^{-j\varphi} & 0 & 0 \\ -\cos\theta e^{j\varphi} & -\sin\theta & 0 & 0 \end{pmatrix} \qquad \gamma_\varphi = \begin{pmatrix} 0 & 0 & 0 & -je^{-j\varphi} \\ 0 & 0 & je^{j\varphi} & 0 \\ 0 & je^{-j\varphi} & 0 & 0 \\ -je^{j\varphi} & 0 & 0 & 0 \end{pmatrix}$$

(XV-16)

It should be noted in particular that the obtained matrices check the general properties of DIRAC matrices:

$$(\gamma_0)^2 = 1, \quad (\gamma_r)^2 = (\gamma_\theta)^2 = (\gamma_\varphi)^2 = -1 \qquad \text{(XV-17)}$$

Developed system of DIRAC matrix writing takes the form:

$$\eta\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=j\begin{pmatrix}1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial x_t}\\ \frac{\partial\psi_1}{\partial x_t}\\ \frac{\partial\psi_2}{\partial x_t}\\ \frac{\partial\psi_3}{\partial x_t}\end{pmatrix}$$

$$+j\begin{pmatrix}0&0&\cos\theta&\sin\theta e^{-j\varphi}\\ 0&0&\sin\theta e^{j\varphi}&-\cos\theta\\ -\cos\theta&-\sin\theta e^{-j\varphi}&0&0\\ -\sin\theta e^{j\varphi}&\cos\theta&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial r}\\ \frac{\partial\psi_1}{\partial r}\\ \frac{\partial\psi_2}{\partial r}\\ \frac{\partial\psi_3}{\partial r}\end{pmatrix}$$

$$+j\frac{1}{r}\begin{pmatrix}0&0&-\sin\theta&\cos\theta e^{-j\varphi}\\ 0&0&\cos\theta e^{j\varphi}&\sin\theta\\ \sin\theta&-\cos\theta e^{-j\varphi}&0&0\\ -\cos\theta e^{j\varphi}&-\sin\theta&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial\theta}\\ \frac{\partial\psi_1}{\partial\theta}\\ \frac{\partial\psi_2}{\partial\theta}\\ \frac{\partial\psi_3}{\partial\theta}\end{pmatrix}$$

$$+j\frac{1}{r\sin\theta}\begin{pmatrix}0&0&0&-je^{-j\varphi}\\ 0&0&je^{j\varphi}&0\\ 0&je^{-j\varphi}&0&0\\ -je^{j\varphi}&0&0&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_0}{\partial\varphi}\\ \frac{\partial\psi_1}{\partial\varphi}\\ \frac{\partial\psi_2}{\partial\varphi}\\ \frac{\partial\psi_3}{\partial\varphi}\end{pmatrix}+ \qquad \text{(XV-18)}$$

We can deduce the 4 equations with partial derivatives expressing the DIRAC system in spherical coordinates:

$$\eta\psi_0 = j\left\{\frac{\partial\psi_0}{\partial x_t} + \cos\theta\frac{\partial\psi_2}{\partial r} + \sin\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(-\sin\theta\frac{\partial\psi_2}{\partial\theta} + \cos\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial\theta}\right) - \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_3}{\partial\varphi}\right\}$$

$$\eta\psi_1 = j\left\{\frac{\partial\psi_1}{\partial x_t} + \sin\theta e^{j\varphi}\frac{\partial\psi_2}{\partial r} - \cos\theta\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(\cos\theta e^{j\varphi}\frac{\partial\psi_2}{\partial\theta} + \sin\theta\frac{\partial\psi_3}{\partial\theta}\right) + \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_2}{\partial\varphi}\right\}$$

$$\eta\psi_2 = j\left\{-\frac{\partial\psi_2}{\partial x_t} - \cos\theta\frac{\partial\psi_0}{\partial r} - \sin\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(\sin\theta\frac{\partial\psi_0}{\partial\theta} - \cos\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial\theta}\right) + \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_1}{\partial\varphi}\right\}$$

$$\eta\psi_3 = j\left\{-\frac{\partial\psi_3}{\partial x_t} - \sin\theta e^{j\varphi}\frac{\partial\psi_0}{\partial r} + \cos\theta\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(-\cos\theta e^{j\varphi}\frac{\partial\psi_0}{\partial\theta} - \sin\theta\frac{\partial\psi_1}{\partial\theta}\right) - \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_0}{\partial\varphi}\right\}$$

(XV-19)

It is a non-linear system, and the method used with the Cartesian coordinates is no longer applicable.

The complexity of this system is such that it is difficult (impossible?) to consider a purely mathematical method allowing to lead to an exact solution. We will show in the following chapters that a physical approach based on exchanges of energy between spherical modes allows to progress towards such solutions.

XVI

KLEIN-GORDON equation in spherical coordinates

In the process of finding exact solutions to the DIRAC equation in spherical coordinates, we hypothesize that the conservation equation of energy between the mass energy, wave energy, and impulse energy will be given to us by the KLEIN-GORDON equation in spherical coordinates:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-\frac{\partial^2}{\partial x_t^2}\right](\psi)=\eta^2(\psi) \qquad \text{(XVI-1)}$$

The modal solutions of this equation are presented in the form of a product of three separable functions in (r), (x_t), and (θ, φ).

I - Separable solution (θ, φ)

This solution is based on spherical harmonics $Y_{\ell m}$ that are functions of two parameters:

- ℓ is called the harmonic degree of $Y_{\ell m}$, and it is a natural number.
- m is known as the spherical harmonic order: it is an integer such that $|m| <= \ell$ and therefore: $m = -\ell, -\ell+1, \ldots, \ell$.

$$Y_{\ell m}(\theta,\varphi)=(-1)^{\frac{m+|m|}{2}}\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_\ell^{|m|}(\cos\theta)\exp(j m\varphi) \qquad \text{(XVI-2)}$$

$0 \le \theta \le \pi$ et $0 \le \varphi \le 2\pi$

P_ℓ^m is a LEGENDRE polynomial raised to power m.

Spherical harmonics are solutions of the eigenvalue equation:

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{\ell m}(\theta,\varphi)=-\ell(\ell+1)Y_{\ell m}(\theta,\varphi) \qquad \text{(XVI-3)}$$

The first standard spherical harmonics are the following:

$$
\begin{aligned}
&Y_{00} = \frac{1}{\sqrt{4\pi}} \\
&Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \quad Y_{1-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-j\varphi} \quad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{j\varphi} \\
&Y_{20} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \quad Y_{2-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-j\varphi} \quad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{j\varphi} \\
&\qquad\qquad Y_{2-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-j2\varphi} \quad Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{j2\varphi} \\
&Y_{30} = \sqrt{\frac{7}{16\pi}}\left(5\cos^2\theta - 1\right)\cos\theta \qquad Y_{3\mp1} = \pm\sqrt{\frac{35}{64\pi}}\left(5\cos^2\theta - 1\right)\sin\theta e^{\mp j\varphi} \\
&Y_{3\mp2} = \sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta e^{\mp j2\varphi} \qquad Y_{3\mp3} = \pm\sqrt{\frac{35}{64\pi}}\sin^3\theta e^{\mp j3\varphi}
\end{aligned}
\tag{XVI-4}
$$

II – The separable solution in r

This solution is given by any linear combination of spherical BESSEL functions $j_n(r)$ and $y_n(r)$ where the index n is a natural number. These functions can be defined from the BESSEL functions:

$$
\begin{aligned}
&j_n(r) = \sqrt{\frac{\pi}{2r}}J_{n+\frac{1}{2}}(r) \\
&y_n(r) = \sqrt{\frac{\pi}{2r}}N_{n+\frac{1}{2}}(r)
\end{aligned}
\tag{XVI-5}
$$

or from generators, more convenient to determine the explicit formulations:

$$
\begin{aligned}
&j_n(r) = +(-1)^n r^n\left(\frac{1}{r}\frac{d}{dr}\right)^n\left(\frac{\sin r}{r}\right) \\
&y_n(r) = -(-1)^n r^n\left(\frac{1}{r}\frac{d}{dr}\right)^n\left(\frac{\cos r}{r}\right)
\end{aligned}
\tag{XVI-6}
$$

The first spherical BESSEL functions are as follows:

$$
\begin{aligned}
&j_0(r) = \frac{\sin r}{r} && y_0(r) = -\frac{\cos r}{r} \\
&j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} && y_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \\
&j_2(r) = \left(\frac{3}{r^3} - \frac{1}{r}\right)\sin r - 3\frac{\cos r}{r^2} && y_2(r) = -\left(\frac{3}{r^3} - \frac{1}{r}\right)\cos r - 3\frac{\sin r}{r^2} \\
&j_3(r) = \left(\frac{15}{r^4} - \frac{6}{r^2}\right)\sin r - \left(\frac{15}{r^3} - \frac{1}{r}\right)\cos r && y_3(r) = -\left(\frac{15}{r^4} - \frac{6}{r^2}\right)\cos r - \left(\frac{15}{r^3} - \frac{1}{r}\right)\sin r
\end{aligned}
\tag{XVI-7}
$$

If we refer to $f_n(r)$ the general function representative of $j_n(r)$ or $y_n(r)$, $f_n(r)$ is solution of the spherical BESSEL differential equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)f_n(r)+\left[1-\frac{n(n+1)}{r^2}\right]f_n(r)=0 \qquad \text{(XVI-8)}$$

Or in an equivalent way:

$$\frac{\partial^2 f_n(r)}{\partial r^2}+\frac{2}{r}\frac{\partial f_n(r)}{\partial r}+\left[1-\frac{n(n+1)}{r^2}\right]f_n(r)=0 \qquad \text{(XVI-9)}$$

Spherical cavity modes dependent only from r are called the pulsed modes. They are functions of a radial propagation constant or radial space pulsation that we refer to by k_r and who plays a role analogous to propagation constants k_x, k_y, or k_z for rectangular cavities.

By introducing the variable $R = k_r r$ in spherical BESSEL equation (XVI-19), we get successively:

$$\begin{aligned}
&\frac{\partial^2 f_n(R)}{\partial R^2}+\frac{2}{R}\frac{\partial f_n(R)}{\partial R}+\left[1-\frac{n(n+1)}{R^2}\right]f_n(R)=0\\
&\frac{1}{k_r^2}\frac{\partial^2 f_n(k_r r)}{\partial r^2}+\frac{1}{k_r^2}\frac{2}{r}\frac{\partial f_n(k_r r)}{\partial r}+\left[1-\frac{n(n+1)}{(k_r r)^2}\right]f_n(k_r r)=0\\
&\frac{\partial^2 f_n(k_r r)}{\partial r^2}+\frac{2}{r}\frac{\partial f_n(k_r r)}{\partial r}+\left[k_r^2-\frac{n(n+1)}{r^2}\right]f_n(k_r r)=0\\
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)f_n(k_r r)+\left[k_r^2-\frac{n(n+1)}{r^2}\right]f_n(k_r r)=0
\end{aligned} \qquad \text{(XVI-10)}$$

III – The separable solution in x_t

If we refer to this solution by u, it is solution of the eigenvalue equation:

$$\frac{\partial^2}{\partial x_t^2}u(k_t x_t)+k_t^2 u(k_t x_t)=0 \qquad \text{(XVI-11)}$$

It is thus constituted by any linear combination of the trigonometric functions $\cos(k_t x_t)$ and $\sin(k_t x_t)$.

IV - The complete solution in stationary modes separated in (r, θ, φ and x_t)

The KLEIN-GORDON equation in spherical coordinates is recalled for memory:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-\frac{\partial^2}{\partial x_t^2}\right](\psi)=\eta^2(\psi) \qquad \text{(XVI-12)}$$

Solutions in stationary modes appear under the form of a product of independent functions in r, (θ, φ), and x_t:

$$\psi(r,\theta,\varphi,t) = f_\ell(k_r r)Y\ell m(\theta,\varphi)u(k_t x_t) \qquad \text{(XVI-13)}$$

The introduction of this form of solution in the KLEIN-GORDON equation in spherical coordinates leads to the following substitutions:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi(r,\theta,\varphi,t) = -\left[k_r^2 - \frac{\ell(\ell+1)}{r^2}\right]f_\ell(k_r r)Y\ell m(\theta,\varphi)u(k_t x_t)$$

$$\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi(r,\theta,\varphi,t) = -\frac{\ell(\ell+1)}{r^2}f_\ell(k_r r)Y\ell m(\theta,\varphi)u(k_t x_t)$$

$$-\frac{\partial^2}{\partial x_t^2}\psi(r,\theta,\varphi,t) = k_t^2 f_\ell(k_r r)Y\ell m(\theta,\varphi)u(k_t x_t) \qquad \text{(XVI-14)}$$

By substitution of these 3 relationships in (XVI-12), yields the relationship of energy:

$$k_t^2 = \eta^2 + k_r^2 \qquad \text{(XVI-15)}$$

After multiplication by the constant $(\hbar c)^2$, this relation becomes:

$$(\hbar c k_t)^2 = (\hbar c\eta)^2 + (\hbar c k_r)^2 \qquad \text{(XVI-16)}$$

or again :

$$(\hbar\omega)^2 = (m_0 c^2)^2 + (\hbar c k_r)^2 \qquad \text{(XVI-17)}$$

As in Cartesian coordinates, it identifies in this relationship three kinds of energy: wave energy $(\hbar\omega)$, masse energy (m_0c^2) and impulse energy $(\hbar c k_r)$.

It should be noted that this relationship of conservation is independent of the excited modes depending on θ and φ. This means that the distribution between mass energy, wave energy, and impulse energy is not depending on modes excited in rotation following θ and φ.

As in the case of Cartesian coordinates, this relationship of energy conservation will play a fundamental role in obtaining stationary solutions to the DIRAC equation in spherical coordinates.

XVII

Exact solutions of the DIRAC equation in spherical coordinates

In previous chapters, we have shown that the exact solutions of the Dirac equation in spherical coordinates must check the system:

$$\eta\psi_0 = j\left\{\frac{\partial\psi_0}{\partial x_t} + \cos\theta\frac{\partial\psi_2}{\partial r} + \sin\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(-\sin\theta\frac{\partial\psi_2}{\partial\theta} + \cos\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial\theta}\right) - \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_3}{\partial\varphi}\right\}$$

$$\eta\psi_1 = j\left\{\frac{\partial\psi_1}{\partial x_t} + \sin\theta e^{j\varphi}\frac{\partial\psi_2}{\partial r} - \cos\theta\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(\cos\theta e^{j\varphi}\frac{\partial\psi_2}{\partial\theta} + \sin\theta\frac{\partial\psi_3}{\partial\theta}\right) + \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_2}{\partial\varphi}\right\}$$

$$\eta\psi_2 = j\left\{-\frac{\partial\psi_2}{\partial x_t} - \cos\theta\frac{\partial\psi_0}{\partial r} - \sin\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(\sin\theta\frac{\partial\psi_0}{\partial\theta} - \cos\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial\theta}\right) + \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_1}{\partial\varphi}\right\}$$

$$\eta\psi_3 = j\left\{-\frac{\partial\psi_3}{\partial x_t} - \sin\theta e^{j\varphi}\frac{\partial\psi_0}{\partial r} + \cos\theta\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(-\cos\theta e^{j\varphi}\frac{\partial\psi_0}{\partial\theta} - \sin\theta\frac{\partial\psi_1}{\partial\theta}\right) - \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_0}{\partial\varphi}\right\}$$

(XVII-1)

We have also formulated the hypothesis that if exact solutions exist in the form of stationary modes describing energy exchange within the particle, these solutions must be compatible with the equation of conservation of energy established using the KLEIN-GORDON equation:

$$k_t^2 = \eta^2 + k_r^2 \qquad \text{(XVII-2)}$$

However, these elements are insufficient to advance in the search for solutions to the above system (XVII-1).

We must therefore find new features able, in a heuristic approach, to restrict the field of possible solutions.

We are going to do this using two observations of the exact solutions obtained in Cartesian coordinates. Consider for example the following solution:

$$\psi_0 = 0$$
$$\psi_1 = -(\hbar\omega)\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j(m_0 c^2)\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_2 = j\hbar c k_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \hbar c k_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$
$$\psi_3 = \hbar c k_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

(XVII-3)

The first observation to be noted is that the bi-spinor representing some of these solutions can be broken down as follows:

- the first spinor expresses on one of its components exchange between mass energy and wave energy.

- the second spinor expresses on its two components impulse energy exchange.

The second observation is linked to the wave-particle duality. It appears in exact solutions in the form of a same spatial mode, one with energy-positive, the other with negative energy (because the presence of $j = \sqrt{-1}$), excited in time quadrature, which indicates that when the mass energy is maximum, the wave energy is zero, and vice versa. Modes that are present in this exchange of energy must be solution of the KLEIN-GORDON equation that appears systematically in a system of DIRAC equations.

Based on these two observations, empirical tests have shown that there is indeed exact spherical solutions to DIRAC system (XVII-1) in the form of stationary modes, with respect to the equation of conservation of energy (XVII-2)

Some spherical modes do not allow to obtain exact solutions to the DIRAC equation. The empirical methodology proposed here does not say whether there is a mathematical absence of these solutions, or if these modes-related solutions are achievable by other methods

The first spherical mode $Y_{00}(\theta, \varphi)$ solution of the KLEIN-GORDON equation is equal to a constant, so it is independent of θ, and φ. It doesn't lead to an exact solution of the Dirac equation.

We will develop in detail in the following chapters, exact solutions for some modes of the spherical solutions of the KLEIN-GORDON equation.

XVIII

Exact solutions of DIRAC equation on modes Y_{n-n} and Y_{nn}

We are interested in this chapter to modes solutions of the KLEIN-GORDON equation for which the parameters ℓ and |m| are equal, and we shall put $\ell = |m| = n \neq 0$.

The angular description of these modes is given by the following spherical harmonics, in which the normalization constant, which plays no role to establish the validity of the solutions has been omitted:

$$Y_{n-n} = \sin^n \theta e^{-jn\varphi} \quad Y_{nn} = \sin^n \theta e^{jn\varphi} \tag{XVIII-1}$$

We will work on the mode Y_{n-n}, before deriving, by simple considerations, the expression of the Y_{nn} mode solutions.

The spherical harmonic Y_{n-n} is associated with the radial function given by two spherical BESSEL functions of order n:

$$f_n(k_r r) = j_n(k_r r) = \sqrt{\frac{\pi}{2r}} J_{n+\frac{1}{2}}(k_r r) \tag{XVIII-2}$$

or

$$f_n(k_r r) = y_n(k_r r) = \sqrt{\frac{\pi}{2r}} N_{n+\frac{1}{2}}(k_r r) \tag{XVIII-3}$$

Modal solutions of the KLEIN-GORDON equation expressed under the form:

$$\psi(r,\theta,\varphi,t) = f_n(k_r r) Y_{n-n}(\theta,\varphi) u(k_t x_t) \tag{XVIII-4}$$

where the time dependence is given by any linear combination of functions: $u(k_t x_t) = \cos(k_t x_t)$ ou $u(k_t x_t) = \sin(k_t x_t)$

The approach discussed in the previous chapter incentive to propose the following solution which will prove to be be an exact solution:

$$\psi_0 = \eta f_n(k_r r)\sin^n\theta e^{-nj\varphi}\sin(k_t x_t) + jk_t f_n(k_r r)\sin^n\theta e^{-nj\varphi}\cos(k_t x_t)$$
$$\psi_1 = 0$$
$$\psi_2 = j\cos\theta\sin^n\theta e^{-nj\varphi}\sin(k_t x_t)\left\{-k_r f_n'(k_r r) + n\frac{f_n(k_r r)}{r}\right\} \qquad \text{(XVIII-5)}$$
$$\psi_3 = j\sin(k_t x_t)\sin^{n-1}\theta e^{-(n-1)j\varphi}\left\{-\sin^2\theta k_r f_n'(k_r r) - n\left(\cos^2\theta + 1\right)\frac{f_n(k_r r)}{r}\right\}$$

in which we have adopted the simplified notation: $f_n'(k_r r) = df_n(k_r r)/d(k_r r)$.

The first spinor describes the duality wave-corpuscle through the wave function ψ_0. Mass energy is exchanged with the wave energy on the spatial mode defined by the spherical harmonic $Y_{n\text{-}n}$. It responds to the approach discussed in the previous chapter.

The second spinor exchanges impulse energy to which we will return later. It is inferred from the spherical DIRAC system after substitution of the first spinor.

The equation of conservation of energy associated with this solution is recalled for memory:

$$k_t^2 = \eta^2 + k_r^2 \qquad \text{(XVIII-6)}$$

This relationship is independent of the nature of the excited spherical modes.

In previous chapters, we have shown that the exact solutions of the DIRAC equation in spherical coordinates must check the system:

$$\eta\psi_0 = j\left\{\frac{\partial\psi_0}{\partial x_t} + \cos\theta\frac{\partial\psi_2}{\partial r} + \sin\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(-\sin\theta\frac{\partial\psi_2}{\partial\theta} + \cos\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial\theta}\right) - \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_3}{\partial\varphi}\right\}$$
$$\eta\psi_1 = j\left\{\frac{\partial\psi_1}{\partial x_t} + \sin\theta e^{j\varphi}\frac{\partial\psi_2}{\partial r} - \cos\theta\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(\cos\theta e^{j\varphi}\frac{\partial\psi_2}{\partial\theta} + \sin\theta\frac{\partial\psi_3}{\partial\theta}\right) + \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_2}{\partial\varphi}\right\}$$
$$\eta\psi_2 = j\left\{-\frac{\partial\psi_2}{\partial x_t} - \cos\theta\frac{\partial\psi_0}{\partial r} - \sin\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(\sin\theta\frac{\partial\psi_0}{\partial\theta} - \cos\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial\theta}\right) + \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_1}{\partial\varphi}\right\}$$
$$\eta\psi_3 = j\left\{-\frac{\partial\psi_3}{\partial x_t} - \sin\theta e^{j\varphi}\frac{\partial\psi_0}{\partial r} + \cos\theta\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(-\cos\theta e^{j\varphi}\frac{\partial\psi_0}{\partial\theta} - \sin\theta\frac{\partial\psi_1}{\partial\theta}\right) - \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_0}{\partial\varphi}\right\}$$
(XVIII-7)

This system verification requires only basic calculations, but it is laborious. It is however a decisive argument to convince of the validity of the proposed solution, and aid is proposed below giving the explicit formulation of each of the terms of these equations.

The first equation, more complex because it contains the KLEIN-GORDON equation, will be treated as a last, and we propose to give first the terms relating to equations 2, 3, and 4. Verification of the sums is left to the reader care.

Verification elements of equation 2 of the spherical DIRAC system:

$$\eta\psi_1 = j\left\{\frac{\partial\psi_1}{\partial x_t} + \sin\theta e^{j\varphi}\frac{\partial\psi_2}{\partial r} - \cos\theta\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(\cos\theta e^{j\varphi}\frac{\partial\psi_2}{\partial\theta} + \sin\theta\frac{\partial\psi_3}{\partial\theta}\right) + \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_2}{\partial\varphi}\right\}$$

(XVIII-8)

$$\eta\psi_1 = 0$$

$$j\frac{\partial\psi_1}{\partial x_t} = 0$$

$$j\sin\theta e^{j\varphi}\frac{\partial\psi_2}{\partial r} = -\cos\theta\sin^{n+1}\theta\sin(k_t x_t)e^{-(n-1)j\varphi}\left\{-k_r^2 f_n''(k_r r) + n\left(\frac{k_r f_n'(k_r r)}{r} - \frac{f_n(k_r r)}{r^2}\right)\right\}$$

$$-j\cos\theta\frac{\partial\psi_3}{\partial r} = \cos\theta\sin^{n-1}\theta\sin(k_t x_t)e^{-(n-1)j\varphi}\left\{-\sin^2\theta k_r^2 f_n''(k_r r) - n\left(\cos^2\theta + 1\right)\left(\frac{k_r f_n'(k_r r)}{r} - \frac{f_n(k_r r)}{r^2}\right)\right\}$$

$$j\frac{\cos\theta e^{j\varphi}}{r}\frac{\partial\psi_2}{\partial\theta} = -\cos\theta\sin^{n-1}\theta\left(n\cos^2\theta - \sin^2\theta\right)\sin(k_t x_t)e^{-(n-1)j\varphi}\left\{-\frac{k_r f_n'(k_r r)}{r} + n\frac{f_n(k_r r)}{r^2}\right\}$$

$$j\frac{\sin\theta}{r}\frac{\partial\psi_3}{\partial\theta} = \sin(k_t x_t)\sin^{n-1}\theta e^{-(n-1)j\varphi}\left\{\begin{array}{l}(n+1)\sin^2\theta\cos\theta\dfrac{k_r f_n'(k_r r)}{r} \\ + n\left((n-1)\cos^3\theta - 2\sin^2\theta\cos\theta + (n-1)\cos\theta\right)\dfrac{f_n(k_r r)}{r^2}\end{array}\right\}$$

$$-\frac{e^{j\varphi}}{r\sin\theta}\frac{\partial\psi_2}{\partial\varphi} = -n\cos\theta\sin^{n-1}\theta\sin(k_t x_t)e^{-(n-1)j\varphi}\left\{-\frac{k_r f_n'(k_r r)}{r} + n\frac{f_n(k_r r)}{r^2}\right\}$$

(XVIII-9)

Verification elements of equation 3 of the spherical DIRAC system:

$$\eta\psi_2 = j\left\{-\frac{\partial\psi_2}{\partial x_t} - \cos\theta\frac{\partial\psi_0}{\partial r} - \sin\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(\sin\theta\frac{\partial\psi_0}{\partial\theta} - \cos\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial\theta}\right) + \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_1}{\partial\varphi}\right\}$$

(XVIII-10)

$$\eta\psi_2 = j\eta\cos\theta\sin^n\theta e^{-nj\varphi}\sin(k_t x_t)\left\{-k_r f_n'(k_r r) + n\frac{f_n(k_r r)}{r}\right\}$$

$$-j\frac{\partial\psi_2}{\partial x_t} = k_t\cos\theta\sin^n\theta e^{-nj\varphi}\cos(k_t x_t)\left\{-k_r f_n'(k_r r) + n\frac{f_n(k_r r)}{r}\right\}$$

$$-j\cos\theta\frac{\partial\psi_0}{\partial r} = -j\eta k_r f_n'(k_r r)\cos\theta\sin^n\theta e^{-nj\varphi}\sin(k_t x_t) + k_t k_r f_n'(k_r r)\cos\theta\sin^n\theta e^{-nj\varphi}\cos(k_t x_t)$$

$$-j\sin\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial r} = 0$$

$$j\frac{\sin\theta}{r}\frac{\partial\psi_0}{\partial\theta} = j\eta\frac{f_n(k_r r)}{r}n\sin^n\theta\cos\theta e^{-nj\varphi}\sin(k_t x_t) - k_t\frac{f_n(k_r r)}{r}n\sin^n\theta\cos\theta e^{-nj\varphi}\cos(k_t x_t)$$

$$-j\frac{\cos\theta}{r}e^{-j\varphi}\frac{\partial\psi_1}{\partial\theta} = 0$$

$$-\frac{e^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_1}{\partial\varphi} = 0$$

(XVIII-11)

Verification elements of equation 4 of the spherical DIRAC system:

$$\eta\psi_3 = j\left\{-\frac{\partial\psi_3}{\partial x_t} - \sin\theta e^{j\varphi}\frac{\partial\psi_0}{\partial r} + \cos\theta\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(-\cos\theta e^{j\varphi}\frac{\partial\psi_0}{\partial\theta} - \sin\theta\frac{\partial\psi_1}{\partial\theta}\right) - \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_0}{\partial\varphi}\right\}$$

(XVIII-12)

$$\eta\psi_3 = j\eta\sin(k_t x_t)\sin^{n-1}\theta e^{-(n-1)j\varphi}\left\{-\sin^2\theta k_r f_n'(k_r r) - n\left(\cos^2\theta + 1\right)\frac{f_n(k_r r)}{r}\right\}$$

$$-j\frac{\partial\psi_3}{\partial x_t} = k_t\cos(k_t x_t)\sin^{n-1}\theta e^{-(n-1)j\varphi}\left\{-\sin^2\theta k_r f_n'(k_r r) - n\left(\cos^2\theta + 1\right)\frac{f_n(k_r r)}{r}\right\}$$

$$-j\sin\theta e^{j\varphi}\frac{\partial\psi_0}{\partial r} = -j\eta k_r f_n'(k_r r)\sin^{n+1}\theta e^{-(n-1)j\varphi}\sin(k_t x_t) + k_r k_t f_n'(k_r r)\sin^{n+1}\theta e^{-(n-1)j\varphi}\cos(k_t x_t)$$

$$j\cos\theta\frac{\partial\psi_1}{\partial r} = 0$$

$$-j\frac{\cos\theta e^{j\varphi}}{r}\frac{\partial\psi_0}{\partial\theta} = \left\{\begin{array}{l} -j\eta\dfrac{f_n(k_r r)}{r}n\sin^{n-1}\theta\cos^2\theta e^{-(n-1)j\varphi}\sin(k_t x_t) \\ +k_t\dfrac{f_n(k_r r)}{r}n\sin^{n-1}\theta\cos^2\theta e^{-(n-1)j\varphi}\cos(k_t x_t)\end{array}\right\}$$

$$-j\frac{\sin\theta}{r}\frac{\partial\psi_1}{\partial\theta} = 0$$

$$\frac{e^{j\varphi}}{r\sin\theta}\frac{\partial\psi_0}{\partial\varphi} = -jn\eta\frac{f_n(k_r r)}{r}\sin^{n-1}\theta e^{-(n-1)j\varphi}\sin(k_t x_t) + nk_t\frac{f_n(k_r r)}{r}\sin^{n-1}\theta e^{-(n-1)j\varphi}\cos(k_t x_t)$$

(XVIII-13)

Verification elements of equation 1 of the spherical DIRAC system:

As stated previously, this verification justifies special attention because it contains the equation of conservation of energy derived from the KLEIN-GORDON equation.

$$\eta\psi_0 = j\left\{\frac{\partial\psi_0}{\partial x_t} + \cos\theta\frac{\partial\psi_2}{\partial r} + \sin\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(-\sin\theta\frac{\partial\psi_2}{\partial\theta} + \cos\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial\theta}\right) - \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_3}{\partial\varphi}\right\}$$

(XVIII-14)

The explicit calculation of each of the terms is given below:

$$\eta\psi_0 = \eta^2 f_n(k_r r)\sin^n\theta e^{-nj\varphi}\sin(k_t x_t) + j\eta k_t f_n(k_r r)\sin^n\theta e^{-nj\varphi}\cos(k_t x_t)$$

$$j\frac{\partial\psi_0}{\partial x_t} = j\eta k_t f_n(k_r r)\sin^n\theta e^{-nj\varphi}\cos(k_t x_t) + k_t^2 f_n(k_r r)\sin^n\theta e^{-nj\varphi}\sin(k_t x_t)$$

$$j\cos\theta\frac{\partial\psi_2}{\partial r} = -\cos^2\theta\sin^n\theta\sin(k_t x_t)e^{-nj\varphi}\left\{-k_r^2 f_n''(k_r r) + n\left(\frac{k_r f_n'(k_r r)}{r} - \frac{f_n(k_r r)}{r^2}\right)\right\}$$

$$j\sin\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial r} = -\sin^n\theta\sin(k_t x_t)e^{-nj\varphi}\left\{-\sin^2\theta k_r^2 f_n''(k_r r) - n\left(\cos^2\theta+1\right)\left(\frac{k_r f_n'(k_r r)}{r} - \frac{f_n(k_r r)}{r^2}\right)\right\}$$

$$-j\frac{\sin\theta}{r}\frac{\partial\psi_2}{\partial\theta} = \sin^n\left(n\cos^2\theta - \sin^2\theta\right)\sin(k_t x_t)e^{-nj\varphi}\left\{-\frac{k_r f_n'(k_r r)}{r} + n\frac{f_n(k_r r)}{r^2}\right\}$$

$$j\frac{\cos\theta}{r}e^{-j\varphi}\frac{\partial\psi_3}{\partial\theta} = \sin(k_t x_t)\cos\theta e^{-nj\varphi}\left\{\begin{array}{l}(n+1)\sin^n\theta\cos\theta\dfrac{k_r f_n'(k_r r)}{r} \\ +n\left((n-1)\sin^{n-2}\theta\cos^3\theta - 2\sin^n\theta\cos\theta + (n-1)\sin^{n-2}\theta\cos\theta\right)\dfrac{f_n(k_r r)}{r^2}\end{array}\right\}$$

$$\frac{e^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_3}{\partial\varphi} = (n-1)\sin(k_t x_t)\sin^{n-2}\theta e^{-nj\varphi}\left\{-\sin^2\theta k_r\frac{f_n'(k_r r)}{r} - n\frac{\cos^2\theta+1}{r^2}f_n(k_r r)\right\}$$

(XVIII-15)

After simplification by $\sin(k_t x_t).e^{-j\varphi}$, the first equation of the DIRAC system is written:

$$\begin{array}{l}\eta^2 f_n(k_r r)\sin^n\theta = \sin^n\theta k_t^2 f_n(k_r r) + \sin^n\theta k_r^2 f_n''(k_r r) \\ +\dfrac{k_r f_n'(k_r r)}{r}\left\{\begin{array}{l}-n\cos^2\theta\sin^n\theta + n\sin^n\theta\left(\cos^2\theta+1\right) - \sin^n\left(n\cos^2\theta - \sin^2\theta\right) \\ +(n+1)\sin^n\theta\cos^2\theta - (n-1)\sin^n\theta\end{array}\right\} \\ +\dfrac{f_n(k_r r)}{r^2}\left\{\begin{array}{l}n\cos^2\theta\sin^n\theta - n\sin^n\theta\left(\cos^2\theta+1\right) + n\sin^n\left(n\cos^2\theta - \sin^2\theta\right) \\ +n\left((n-1)\sin^{n-2}\theta\cos^4\theta - 2\sin^n\theta\cos^2\theta + (n-1)\sin^{n-2}\theta\cos^2\theta\right) \\ -n(n-1)\sin^{n-2}\theta\left(\cos^2\theta+1\right)\end{array}\right\}\end{array}$$

(XVIII-16)

Terms within brackets reduce in a remarkable manner to obtain:

$$\eta^2 f_n(k_r r) = k_t^2 f_n(k_r r) + k_r^2 f_n''(k_r r) + 2\frac{k_r f_n'(k_r r)}{r} - n(n+1)\frac{f_n(k_r r)}{r^2}$$

(XVIII-17)

We can rewrite this relationship by inserting the partial derivatives with respect to r:

$$\eta^2 f_n(k_r r) = k_t^2 f_n(k_r r) + \frac{\partial^2}{\partial r^2}f_n(k_r r) + \frac{2}{r}\frac{\partial}{\partial r}f_n(k_r r) - n(n+1)\frac{f_n(k_r r)}{r^2}$$

(XVIII-18)

Or again:

$$\eta^2 f_n(k_r r) = k_t^2 f_n(k_r r) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)f_n(k_r r) - n(n+1)\frac{f_n(k_r r)}{r^2}$$

(XVIII-19)

Using the property of the spherical BESSEL function established in (XVI-10) and recalled for memory:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)f_\ell(k_r r)+\left[k_r^2-\frac{\ell(\ell+1)}{r^2}\right]f_\ell(k_r r)=0 \qquad \text{(XVIII-20)}$$

we obtain for $\ell = n$:

$$\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)f_n(k_r r)+\left[k_r^2-\frac{n(n+1)}{r^2}\right]f_n(k_r r)=0\\&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)f_n(k_r r)-\frac{n(n+1)}{r^2}f_n(k_r r)=-k_r^2 f_n(k_r r)\end{aligned} \qquad \text{(XVIII-21)}$$

By postponing this last result in the below mentioned (XVIII-19) relationship:

$$\eta^2 f_n(k_r r)=k_t^2 f_n(k_r r)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)f_n(k_r r)-n(n+1)\frac{f_n(k_r r)}{r^2} \qquad \text{(XVIII-22)}$$

We obtain the relationship of conservation of energy:

$$\begin{aligned}&\eta^2 f_n(k_r r)=k_t^2 f_n(k_r r)-k_r^2 f_n(k_r r)\\&k_t^2=\eta^2+k_r^2\end{aligned} \qquad \text{(XVIII-23)}$$

Verification of the exact solution to the DIRAC equation in spherical coordinates is completed. As in Cartesian coordinates, to be valid, this solution must be associated with the equation of conservation of energy in spherical coordinates.

At the conclusion of this chapter, we look at the similar mode in Y_{nn}:

$$Y_{nn}=\sin^n\theta e^{jn\varphi} \qquad \text{(XVIII-24)}$$

If it ignores the normalization constant that is not involved in the calculations, the only difference from the spherical harmonic Y_{n-n} of the preceding paragraph is in the sign of φ.

This induces an immediate consequence: we can no longer find exact solution by shifting the duality wave-corpuscle by the wave function ψ_0. The exact solution to the spherical DIRAC system can only be constructed by carriing this duality by the wave function ψ_1. We then get a similar solution to the previous one, in which the wave functions of the spinor that carries the pulse energy is exchanged:

$$\begin{aligned}&\psi_0=0\\&\psi_1=\eta f_n(k_r r)\sin^n\theta e^{njφ}\sin(k_t x_t)+jk_t f_n(k_r r)\sin^n\theta e^{njφ}\cos(k_t x_t)\\&\psi_2=j\sin(k_t x_t)\sin^{n-1}\theta e^{(n-1)jφ}\left\{-\sin^2\theta k_r f_n'(k_r r)-n(\cos^2\theta+1)\frac{f_n(k_r r)}{r}\right\}\\&\psi_3=j\cos\theta\sin^n\theta e^{njφ}\sin(k_t x_t)\left\{k_r f_n'(k_r r)-n\frac{f_n(k_r r)}{r}\right\}\end{aligned} \qquad \text{(XVIII-25)}$$

We will see in a next chapter that the sign of φ determines the direction of rotation of the energy. Using the conventional rules of orientation of the spin (rule of the corkscrew for example), we can associate a spin direction to each of the proposed solutions.

The spinor:

$$\begin{aligned}\psi_0 &= \eta f_n(k_r r)\sin^n\theta e^{-njφ}\sin(k_t x_t) + jk_t f_n(k_r r)\sin^n\theta e^{-njφ}\cos(k_t x_t)\\ \psi_1 &= 0\end{aligned} \qquad \text{(XVIII-26)}$$

can be associated with the positive z-oriented spin and it is usually called spin "up".

The spinor:

$$\begin{aligned}\psi_0 &= 0\\ \psi_1 &= \eta f_n(k_r r)\sin^n\theta e^{njφ}\sin(k_t x_t) + jk_t f_n(k_r r)\sin^n\theta e^{njφ}\cos(k_t x_t)\end{aligned} \qquad \text{(XVIII-27)}$$

can be associated with the negative z-oriented spin and it is usually called spin "down".

XIX

Other exact solutions

I - The rotation modes

Basic exact solutions presented in the previous chapter do not highlight the rotation of power.

In doing similarly looking for propagative solution during the study in Cartesian coordinates, it is possible to find exact solutions that express this rotation.

Spherical DIRAC system is linear with respect to the sine and cosine functions of x_t, then we can build new exact solutions by summing solutions whose modes variations cover x_t.

We choose to work in a way representing a 'down' spin and we call (solution 1) the solution obtained in (XVIII-25):

$$\begin{aligned}
&\psi_0 = 0\\
&\psi_1 = \eta f_n(k_r r)\sin^n\theta e^{njq}\sin(k_t x_t) + jk_t f_n(k_r r)\sin^n\theta e^{njq}\cos(k_t x_t)\\
&\psi_2 = j\sin(k_t x_t)\sin^{n-1}\theta e^{(n-1)jq}\left\{-\sin^2\theta k_r f_n'(k_r r) - n\left(\cos^2\theta + 1\right)\frac{f_n(k_r r)}{r}\right\}\\
&\psi_3 = j\cos\theta\sin^n\theta e^{njq}\sin(k_t x_t)\left\{k_r f_n'(k_r r) - n\frac{f_n(k_r r)}{r}\right\}
\end{aligned} \qquad \text{(XIX-1)}$$

We call (solution 2) the exact solution obtained by exchanging the sine and cosine in the wave function ψ_1 :

$$\begin{aligned}
&\psi_0 = 0\\
&\psi_1 = \eta f_n(k_r r)\sin^n\theta e^{njq}\cos(k_t x_t) - jk_t f_n(k_r r)\sin^n\theta e^{njq}\sin(k_t x_t)\\
&\psi_2 = j\cos(k_t x_t)\sin^{n-1}\theta e^{(n-1)jq}\left\{-\sin^2\theta k_r f_n'(k_r r) - n\left(\cos^2\theta + 1\right)\frac{f_n(k_r r)}{r}\right\}\\
&\psi_3 = j\cos\theta\sin^n\theta e^{njq}\cos(k_t x_t)\left\{k_r f_n'(k_r r) - n\frac{f_n(k_r r)}{r}\right\}
\end{aligned} \qquad \text{(XIX-2)}$$

We build the exact solution obtained by the linear combination: (solution 2) + j (solution 1):

$$\psi_0 = 0$$
$$\psi_1 = \eta f_n(k_r r)\sin^n\theta \exp\, j(k_t x_t + n\varphi) - k_t f_n(k_r r)\sin^n\theta \exp\, j(k_t x_t + n\varphi)$$
$$\psi_2 = j\exp\, j(k_t x_t + (n-1)\varphi)\sin^{n-1}\theta\left\{-\sin^2\theta k_r f_n'(k_r r) - n(\cos^2\theta + 1)\frac{f_n(k_r r)}{r}\right\} \qquad \text{(XIX-3)}$$
$$\psi_3 = j\cos\theta\sin^n\theta \exp\, j(k_t x_t + n\varphi)\left\{k_r f_n'(k_r r) - n\frac{f_n(k_r r)}{r}\right\}$$

Terms with exp j($k_t x_t$ + nφ) and exp j($k_t x_t$ + (n-1)φ) represent both temporal rotation energy depending on the angle φ and exchanges between positive and negative energy shown by the presence of the imaginary term j.

The direction of rotation can be evaluated in the same way that the meaning of a progressive wave propagation.

A wave as (ωt – kx) is moving towards the positive x: a wave as ($k_t x_t$ - nφ) is moving towards positive φ.
A wave as (ωt + kx) is progressing towards the negative x: a wave as ($k_t x_t$ + nφ) moves towards the negative φ, which justifies the name of spin 'down' for the solution (XIX-3).

It appears impossible to construct exact solutions in rotation as ($k_t x_t$ + θ) because there are no similar solutions in sinθ and cosθ which are necessary to obtain by combination with temporal sin ($k_t x_t$) and cos ($k_t x_t$) functions of rotation by θ.

II – Other solutions

Among the modal solutions of the KLEIN-GORDON equation, for an order ℓ given, |m| can take all values between 0 and ℓ. The previous chapters study show that the modes such as ℓ = |m| = n ≠ 0 lead to exact solutions of the DIRAC equation.

It turns out that all modes do not result in exact solutions.

A specific test on the Y_{2-1} and Y_{21} modes shows that these modes allow exact solutions which are reported below.

II-1 - mode Y_{2-1}

It is associated with the spherical harmonic Y_{2-1}:

$$Y_{2-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-j\varphi} \qquad \text{(XIX-4)}$$

The radial function is given by two spherical BESSEL functions of order 2:

$$f_2(k_r r) = j_2(k_r r) = \left(\frac{3}{(k_r r)^3} - \frac{1}{(k_r r)}\right)\sin(k_r r) - 3\frac{\cos(k_r r)}{(k_r r)^2} \qquad \text{(XIX-5)}$$

or

$$f_2(k_r r) = y_2(k_r r) = -\left(\frac{3}{(k_r r)^3} - \frac{1}{(k_r r)}\right)\cos(k_r r) - 3\frac{\sin(k_r r)}{(k_r r)^2} \qquad \text{(XIX-6)}$$

Modal solutions of the KLEIN-GORDON equation expressed in the form:

$$\psi(r,\theta,\varphi,t) = f_2(k_r r)Y_{2-1}(\theta,\varphi)u(k_t x_t) \qquad \text{(XIX-7)}$$

where the time dependence is given by any linear combination of functions: $u(k_t x_t) = \cos(k_t x_t)$ or $u(k_t x_t) = \sin(k_t x_t)$

The way proposed in the previous chapters allows to get the following exact solution:

$$\begin{aligned}
\psi_0 &= \eta f_2(k_r r)\sin\theta\cos\theta e^{-j\varphi}\sin(k_t x_t) + jk_t f_2(k_r r)\sin\theta\cos\theta e^{-j\varphi}\cos(k_t x_t)\\
\psi_1 &= 0\\
\psi_2 &= j\sin\theta e^{-j\varphi}\sin(k_t x_t)\left\{-\cos^2\theta k_r f_2'(k_r r) + (1-2\sin^2\theta)\frac{f_2(k_r r)}{r}\right\}\\
\psi_3 &= j\sin(k_t x_t)\cos\theta\left\{-\sin^2\theta k_r f_2'(k_r r) - 2\cos^2\theta\frac{f_2(k_r r)}{r}\right\}
\end{aligned} \qquad \text{(XIX-8)}$$

in which we adopted the notation: $f_2'(k_r r) = df_2(k_r r)/d(k_r r)$.

II-2 - mode Y_{21}

It is associated with the spherical harmonic Y_{21} recalled below:

$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{j\varphi} \qquad \text{(XIX-9)}$$

The normalizing constant being not involved in calculations, the only difference from the spherical harmonic $Y_{2\text{-}1}$ of the preceding paragraph is in the sign of φ. This indicates that the direction of rotation of the energy is reversed from the $Y_{2\text{-}1}$ mode.

The following exact solution is obtained:

$$\begin{aligned}
\psi_0 &= 0\\
\psi_1 &= \eta f_2(k_r r)\sin\theta\cos\theta e^{j\varphi}\sin(k_t x_t) + jk_t f_2(k_r r)\sin\theta\cos\theta e^{j\varphi}\cos(k_t x_t)\\
\psi_2 &= -j\sin(k_t x_t)\cos\theta\left\{\sin^2\theta k_r f_2'(k_r r) + 2\cos^2\theta\frac{f_2(k_r r)}{r}\right\}\\
\psi_3 &= j\sin(k_t x_t)\sin\theta e^{j\varphi}\left\{\cos^2\theta k_r f_2'(k_r r) - (1-2\sin^2\theta)\frac{f_2(k_r r)}{r}\right\}
\end{aligned} \qquad \text{(XIX-10)}$$

II-3 – Other modes

It has not been discoved simple rule for determining if a mode is an exact solution or not when $\ell > |m|$, forcing a search to specific checks which becomes particularly laborious when

the order of the modes increases. It is an open problem: If some modes do not lead to exact solutions of the DIRAC equation, there is necessarily a physical reason that prevents the installation of these modes and which should be elucidated.

XX

Some considerations on exact solutions in spherical coordinates

The passage of the Cartesian coordinates to spherical coordinates allows to retrieve properties that have been highlighted in the analysis of exact solutions in Cartesian coordinates, in a completely deterministic approach.

The major advantage of a formulation of exact solutions to the DIRAC equation in spherical coordinates lies in the fact that we'll be able express analytically the rotation of power.

Some first elements of reflection are proposed below. These elements seem to confirm that the developed solutions are in agreement with the main observed physical phenomena. However we will have to move further in this reflection before concluding or not to the final validity of the proposed solutions.

I – Particle of spin 1

Let us consider an exact solution obtained on mode Y_{1-1}:

$$\psi_0 = \eta f_1(k_r r)\sin\theta e^{-j\varphi}\sin(k_t x_t) + jk_t f_1(k_r r)\sin\theta e^{-j\varphi}\cos(k_t x_t)$$
$$\psi_1 = 0$$
$$\psi_2 = j\cos\theta\sin\theta e^{-j\varphi}\sin(k_t x_t)\left\{-k_r f_1'(k_r r) + \frac{f_1(k_r r)}{r}\right\} \qquad \text{(XX-1)}$$
$$\psi_3 = j\sin(k_t x_t)\left\{-\sin^2\theta k_r f_1'(k_r r) - (\cos^2\theta + 1)\frac{f_1(k_r r)}{r}\right\}$$

where f_1 is a spherical BESSEL function of order 1:

$$j_1(k_r r) = \frac{\sin(k_r r)}{(k_r r)^2} - \frac{\cos(k_r r)}{(k_r r)} \qquad y_1(k_r r) = -\frac{\cos(k_r r)}{(k_r r)^2} - \frac{\sin(k_r r)}{(k_r r)} \qquad \text{(XX-2)}$$

and where we adopted the notation $f_1'(k_r r) = df_1(k_r r)/d(k_r r)$.

After multiplication by the constant $\hbar c$, each of the wave functions has the dimension of energy:

$$\psi_0 = (m_0 c^2) f_1(k_r r)\sin\theta e^{-j\varphi}\sin(k_t x_t) + j(\hbar\omega) f_1(k_r r)\sin\theta e^{-j\varphi}\cos(k_t x_t)$$
$$\psi_1 = 0$$
$$\psi_2 = j(\hbar c k_r)\cos\theta\sin\theta e^{-j\varphi}\sin(k_t x_t)\left\{-f_1'(k_r r) + \frac{f_1(k_r r)}{k_r r}\right\} \qquad \text{(XX-3)}$$
$$\psi_3 = j(\hbar c k_r)\sin(k_t x_t)\left\{-\sin^2\theta f_1'(k_r r) - (\cos^2\theta + 1)\frac{f_1(k_r r)}{k_r r}\right\}$$

The functions above have a finite value in the vicinity of r = 0 only when f_1 is a spherical Bessel function of the first kind J_1. We will maintain despite all subsequently the name f_1 in order to discuss solutions in their greater generality.

We ignore in the discussion that follows the multiplicative constant for standardization. It is recalled that this constant has a dual role: make so that each wave function has the dimension of a square root of volume energy density, and that the integral of the density on the volume containing the energy gives the total energy contained in the particle.

The first spinor formed by ψ_0 and ψ_1 carries the exchange of energy between mass energy and wave energy.

The second spinor formed by ψ_2 and ψ_3 carries spatial and temporal pulse energy exchanges, which is confirmed by the presence of the expression $(\hbar c k_r)$. This impulse energy can be firstly assigned to each of the variables of space r, θ, φ.

The movement of rotation which is highlighted in exact solutions takes place around the Oz axis: it induces an impulse energy along φ.

The variable space r plays a role analogous to variables space x, y and z in Cartesian coordinates: it induces an impulse energy following r.

The θ variable appears to play no role in impulse terms, and we will make the assumption that the impulse energy associated with this variable is zero.

The wave function ψ_3 cannot be associated with a rotating motion along φ: let us assume that it is relative to r impulse energy. We will therefore associate the ψ_2 wave function to impulse energy of rotation. This hypothesis is supported by the fact that the ψ_2 wave function changes sign when passing from a spin "up" for a spin "down". It is also confirmed by the spatial distribution of energy depending on θ, as we shall see later.

When performing a rotation by an angle $\varphi = 2\pi$, the bispinor (XX-1) found its initial position, which suggests that we can associate this mode with a particle of spin 1. We know that in this case the kinetic angular momentum has the maximum chance of making an angle $\theta = 45°$ and $\theta = 135°$ with the Oz axis.

We can link this property to the pulse rotational energy carried by the ψ_2 wave function:

$$\psi_2 = j(\hbar c k_r)\cos\theta\sin\theta e^{-j\varphi}\sin(k_t x_t)\left\{-f_1'(k_r r)+\frac{f_1(k_r r)}{k_r r}\right\} \qquad \text{(XX-4)}$$

From the point of view of the volumetric energy density, the θ dependence is given by the function:

$$A(\theta) = (\cos\theta\sin\theta)^2 \qquad \text{(XX-5)}$$

This function is shown in figure (XX-1) below:

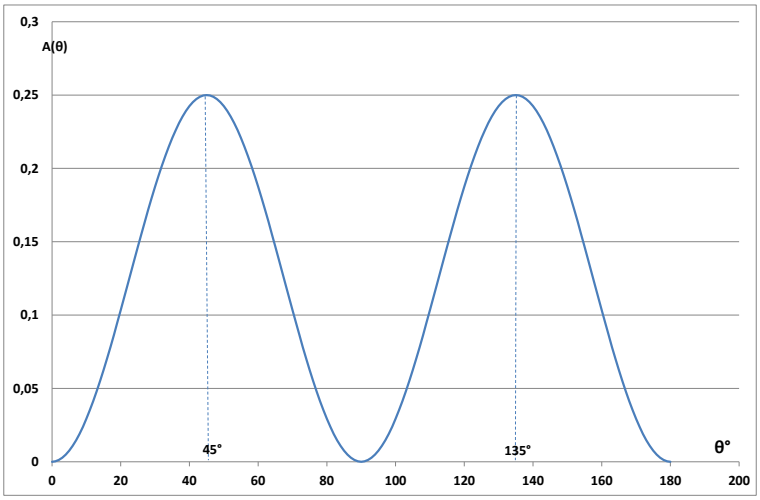


Figure (XX-1): *Representation of the distribution range of volumetric density of impulse energy versus θ for the Y_{11} or Y_{1-1} mode likely to represent a particle of spin 1*

The rotational impulse energy density shows a maximum in the θ = 45 ° and θ = 135 ° directions. It is recalled that in an energy and deterministic approach, distribution around these values no longer represents a probability density, but an energy volume density.

Impulse energy along r is carried by the wave ψ_3 function:

$$\psi_3 = j(\hbar c k_r)\sin(k_t x_t)\left\{-\sin^2\theta f_1'(k_r r)-(\cos^2\theta+1)\frac{f_1(k_r r)}{k_r r}\right\} \qquad \text{(XX-6)}$$

None of the terms separately seems to be subject to a simple physical interpretation.

The photon is an important special case of particle of spin 1. It's a particle whose rest mass is null and in this energy approach, wave functions which describe this particle obey the following relationships to a multiplicative constant close:

$$\psi_0 = (\hbar\omega) f_1(k_r r)\sin\theta e^{-j\varphi}\cos(k_t x_t)$$
$$\psi_1 = 0$$
$$\psi_2 = (\hbar c k_r)\cos\theta\sin\theta e^{-j\varphi}\sin(k_t x_t)\left\{-f_1'(k_r r)+\frac{f_1(k_r r)}{k_r r}\right\} \qquad \text{(XX-7)}$$
$$\psi_3 = (\hbar c k_r)\sin(k_t x_t)\left\{-\sin^2\theta f_1'(k_r r)-(\cos^2\theta+1)\frac{f_1(k_r r)}{k_r r}\right\}$$

The energy exchanges occur between the wave energy brought by the first spinor, and impulse energy carried by the second spinor: when the first is maximum ($|\cos(k_t x_t)| = 1$), the second is null ($\sin(k_t x_t) = 0$) and vice versa.

In the energy approach, the mass energy of the photon being zero, it is impossible to know its position. During its meeting with other particles, it can only share pulse energy.

II – Particle of spin 1/2

We know that the DIRAC equation is the equation whose solutions are the closest to the physical behavior of the electron. We must therefore find in exact solutions to this equation in spherical coordinates, some key properties highlighting both theoretical and experimental behavior of this particle.

The solution which appears to have the greatest analogy is one based on mode Y_{22} or $Y_{2\text{-}2}$:

$$\psi_0 = \eta f_2(k_r r)\sin^2\theta e^{-2j\varphi}\sin(k_t x_t) + jk_t f_2(k_r r)\sin^2\theta e^{-2j\varphi}\cos(k_t x_t)$$
$$\psi_1 = 0$$
$$\psi_2 = jk_r\cos\theta\sin^2\theta e^{-2j\varphi}\sin(k_t x_t)\left\{-f_2'(k_r r)+2\frac{f_2(k_r r)}{k_r r}\right\} \qquad \text{(XX-8)}$$
$$\psi_3 = jk_r\sin(k_t x_t)\sin\theta e^{-j\varphi}\left\{-\sin^2\theta f_2'(k_r r)-2(\cos^2\theta+1)\frac{f_2(k_r r)}{k_r r}\right\}$$

where f_2 is a spherical BESSEL function of order 2:

$$j_2(k_r r) = \left(\frac{3}{(k_r r)^3}-\frac{1}{(k_r r)}\right)\sin(k_r r)-3\frac{\cos(k_r r)}{(k_r r)^2} \qquad \text{(XX-9)}$$
$$y_2(k_r r) = -\left(\frac{3}{(k_r r)^3}-\frac{1}{(k_r r)}\right)\cos(k_r r)-3\frac{\sin(k_r r)}{(k_r r)^2} \qquad \text{(XX-10)}$$

and where we adopted the notation $f_2'(k_r r) = df_2(k_r r)/d(k_r r)$.

After multiplication by the constant $\hbar c$, each of the wave functions has the dimension of energy, and we will do as previously, abstraction of the constant of standardization in the analysis of this solution:

$$\psi_0 = \left(m_0 c^2\right) f_2(k_r r)\sin^2\theta e^{-2j\varphi}\sin(k_t x_t) + j(\hbar\omega) f_2(k_r r)\sin^2\theta e^{-2j\varphi}\cos(k_t x_t)$$
$$\psi_1 = 0$$
$$\psi_2 = j(\hbar c k_r)\cos\theta\sin^2\theta e^{-2j\varphi}\sin(k_t x_t)\left\{-f_2'(k_r r) + 2\frac{f_2(k_r r)}{k_r r}\right\} \qquad \text{(XX-11)}$$
$$\psi_3 = j(\hbar c k_r)\sin(k_t x_t)\sin\theta e^{-j\varphi}\left\{-\sin^2\theta f_2'(k_r r) - 2\left(\cos^2\theta + 1\right)\frac{f_2(k_r r)}{k_r r}\right\}$$

The first important element concerns the rotation angle φ: if the wave function ψ_3 rotates of 2π, then ψ_0 and ψ_2 wave functions are rotated 4π.

The spin ½ of the electron is often presented by explaining that the bi-spinor must rotate 4π before returning to its original position. It appears in the solution (XX-8) new elements which indicate that all components of a spinor rotation do not necessarily vary with the same angular range.

Seen under this aspect, it is necessary that ψ_0 and ψ_2 components turn of 4π so that the component ψ_3 returned to its initial state.

The second element which suggests that the (XX-8) solution may characterize the inner workings of the electron is contained within the ψ_2 wave function:

$$\psi_2 = j(\hbar c k_r)\cos\theta\sin^2\theta e^{-2j\varphi}\sin(k_t x_t)\left\{-f_2'(k_r r) + 2\frac{f_2(k_r r)}{k_r r}\right\} \qquad \text{(XX-12)}$$

This wave function carries impulse energy along φ. Since the θ direction is separable, we can find out in which direction this energy is maximum along φ, hence we have to solve:

$$\frac{d}{d\theta}\left(\cos\theta\sin^2\theta\right) = -\sin^3\theta + 2\sin\theta\cos^2\theta = 0 \qquad \text{(XX-13)}$$

what gives an angle θ_{max} such as:

$$tg(\theta_{max}) = \sqrt{2} \qquad \text{(XX-14)}$$

And then:

$$\theta_{max} = 54{,}73° \qquad \text{(XX-15)}$$

From the point of view of the volumetric energy density, the θ dependence is given by the function:

$$A(\theta) = \left(\cos\theta\sin^2\theta\right)^2 \qquad \text{(XX-16)}$$

This function is shown in figure (XX-2) below:

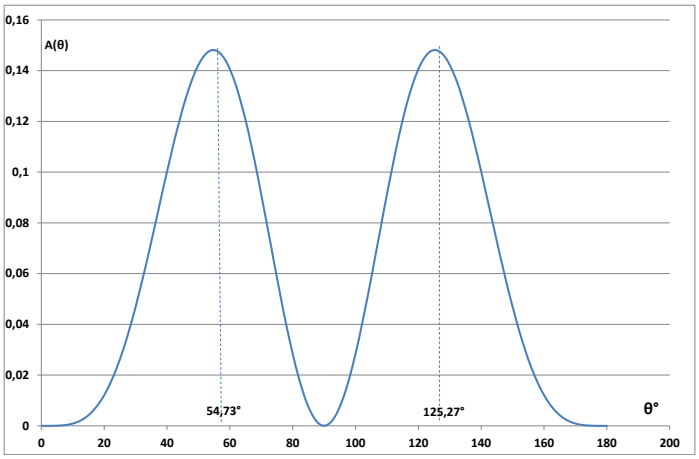


Figure (XX-2): *Representation of the distribution of volumetric density of impulse energy versus θ for Y_{22} or Y_{2-2} modes likely to represent a particle of spin ½*

It appears that these directions correspond to the directions of the angular momentum of the electron.

III – Spin particle « 1/n »

One can search in the exact solutions to the DIRAC equation on modes Y_{nn} or Y_{n-n}, the general expression of the direction in which the angular momentum of spin is maximum.

Impulse energy of spin is carried by the wave function ψ_2 who is remembered for memory:

$$\psi_2 = j\cos\theta\sin^n\theta e^{-nj\varphi}\sin(k_t x_t)\left\{-k_r f_n'(k_r r)+n\frac{f_n(k_r r)}{r}\right\} \qquad \text{(XX-17)}$$

Since this expression is separable in θ, one can determine the θ_{max} value for which this impulse energy is maximum. This is equivalent to solving the equation:

$$\frac{d}{d\theta}\left(\cos\theta\sin^n\theta\right)=-\sin^{n+1}\theta+n\sin^{n-1}\theta\cos^2\theta=0 \qquad \text{(XX-18)}$$

What gives an angle θ_{max} such:

$$tg^2\left(\theta_{max}\right)=n \qquad \text{(XX-19)}$$

It turns out that this direction is identical to that provided by the quantization of angular momentum of spin in classical mechanics for a spin in “1/n”. The quotation marks mean that we enter here in a field for which there is, to the knowledge of the author, no particle known to date with this property for n different from 1 and 2.

To show it, we adopt a quantization of angular momentum of spin along the Oz axis in the form:

$$S_z = \frac{1}{n}\hbar \qquad \text{(XX-20)}$$

Standard angular momentum S in quantum mechanics is written:

$$S = \sqrt{\frac{1}{n}\left(\frac{1}{n}+1\right)}\hbar \qquad \text{(XX-21)}$$

And the angle θ_{max} is given by the relationship:

$$\cos(\theta_{max}) = \frac{S_z}{S} = \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}\left(\frac{1}{n}+1\right)}} = \frac{1}{\sqrt{n+1}} \qquad \text{(XX-22)}$$

Equality with the angle (XX-19) provided by the energy approach and the angle (XX-22) provided by classical quantization of the spin angular momentum is given by the trigonometric relationship:

$$\frac{1}{\cos^2\theta} = tg^2\theta + 1 \qquad \text{(XX-23)}$$

XXI

Spherical DIRAC currents

Exact solutions to the DIRAC equation can only represent physical solutions if they comply with the local conservation of energy, which the mathematical translation is carried out by expressing that the four-divergence of volume energy density must be null.

$$\frac{\partial}{\partial x^{\mu}} J^{\mu} = \partial_{\mu} J^{\mu} = 0 \qquad \text{(XXI-1)}$$

We have seen in Cartesian coordinates that one could extract from DIRAC equations the following relationship:

$$\frac{\partial}{\partial x^{\mu}} \left(\overline{\psi} \gamma^{\mu} \psi \right) = 0 \qquad \text{(XXI-2))}$$

In which:

$$\overline{\psi} = \left(\psi^{*} \right)^{T} \gamma^{0} = \left(\psi_0^{*}, \psi_1^{*}, \psi_2^{*}, \psi_3^{*} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \left(\psi_0^{*}, \psi_1^{*}, -\psi_2^{*}, -\psi_3^{*} \right) \qquad \text{(XXI-3)}$$

By identifying (XXI-2) with (XXI-1), we infer the four-vector current of DIRAC:

$$J^{\mu} = \overline{\psi} \gamma^{\mu} \psi \qquad \text{(XXI-4)}$$

where γ^{μ} are the DIRAC matrices.

To establish expressions of these currents in spherical coordinates, we must use the DIRAC matrices obtained in this coordinate system. These matrices are index by the letters t, r, θ, φ which represent differential to which they apply.

$$\gamma_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{XXI-5}$$

$$\gamma_r = \begin{pmatrix} 0 & 0 & \cos\theta & \sin\theta e^{-j\varphi} \\ 0 & 0 & \sin\theta e^{j\varphi} & -\cos\theta \\ -\cos\theta & -\sin\theta e^{-j\varphi} & 0 & 0 \\ -\sin\theta e^{j\varphi} & \cos\theta & 0 & 0 \end{pmatrix} \tag{XXI-6}$$

$$\gamma_\theta = \begin{pmatrix} 0 & 0 & -\sin\theta & \cos\theta e^{-j\varphi} \\ 0 & 0 & \cos\theta e^{j\varphi} & \sin\theta \\ \sin\theta & -\cos\theta e^{-j\varphi} & 0 & 0 \\ -\cos\theta e^{j\varphi} & -\sin\theta & 0 & 0 \end{pmatrix} \tag{XXI-7}$$

$$\gamma_\varphi = \begin{pmatrix} 0 & 0 & 0 & -je^{-j\varphi} \\ 0 & 0 & je^{j\varphi} & 0 \\ 0 & je^{-j\varphi} & 0 & 0 \\ -je^{j\varphi} & 0 & 0 & 0 \end{pmatrix} \tag{XXI-8}$$

Details of the analytical expressions of these currents are shown below:

index component t:

$$J^t = \overline{\psi}\gamma_0\psi = \left(\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \left(\psi_0^*, \psi_1^*, \psi_2^*, \psi_3^*\right) \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$J^t = \psi_0^*\psi_0 + \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 \tag{XXI-9}$$

index component r:

$$J^r = \overline{\psi}\gamma_r\psi = \left(\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*\right)\begin{pmatrix} 0 & 0 & \cos\theta & \sin\theta e^{-j\varphi} \\ 0 & 0 & \sin\theta e^{j\varphi} & -\cos\theta \\ -\cos\theta & -\sin\theta e^{-j\varphi} & 0 & 0 \\ -\sin\theta e^{j\varphi} & \cos\theta & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$= \left(\psi_2^*\cos\theta + \psi_3^*\sin\theta e^{j\varphi}, \psi_2^*\sin\theta e^{-j\varphi} - \psi_3^*\cos\theta, \psi_0^*\cos\theta + \psi_1^*\sin\theta e^{j\varphi}, \psi_0^*\sin\theta e^{-j\varphi} - \psi_1^*\cos\theta\right)\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$J^r = \psi_0\left(\psi_2^*\cos\theta + \psi_3^*\sin\theta e^{j\varphi}\right) + \psi_1\left(\psi_2^*\sin\theta e^{-j\varphi} - \psi_3^*\cos\theta\right)$$
$$+ \psi_2\left(\psi_0^*\cos\theta + \psi_1^*\sin\theta e^{j\varphi}\right) + \psi_3\left(\psi_0^*\sin\theta e^{-j\varphi} - \psi_1^*\cos\theta\right)$$

(XXI-10)

index component θ:

$$J^\theta = \overline{\psi}\frac{\gamma_\theta}{r}\psi = \left(\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*\right)\frac{1}{r}\begin{pmatrix} 0 & 0 & -\sin\theta & \cos\theta e^{-j\varphi} \\ 0 & 0 & \cos\theta e^{j\varphi} & \sin\theta \\ \sin\theta & -\cos\theta e^{-j\varphi} & 0 & 0 \\ -\cos\theta e^{j\varphi} & -\sin\theta & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$= \frac{1}{r}\left(-\psi_2^*\sin\theta + \psi_3^*\cos\theta e^{j\varphi}, \psi_2^*\cos\theta e^{-j\varphi} + \psi_3^*\sin\theta, -\psi_0^*\sin\theta + \psi_1^*\cos\theta e^{j\varphi}, \psi_0^*\cos\theta e^{-j\varphi} + \psi_1^*\sin\theta\right)\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$rJ^\theta = \psi_0\left(-\psi_2^*\sin\theta + \psi_3^*\cos\theta e^{j\varphi}\right) + \psi_1\left(\psi_2^*\cos\theta e^{-j\varphi} + \psi_3^*\sin\theta\right)$$
$$+ \psi_2\left(-\psi_0^*\sin\theta + \psi_1^*\cos\theta e^{j\varphi}\right) + \psi_3\left(\psi_0^*\cos\theta e^{-j\varphi} + \psi_1^*\sin\theta\right)$$

(XXI-11)

index component φ:

$$J^\varphi = \overline{\psi}\frac{\gamma_\varphi}{r\sin\theta}\psi = \left(\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*\right)\frac{1}{r\sin\theta}\begin{pmatrix} 0 & 0 & 0 & -je^{-j\varphi} \\ 0 & 0 & je^{j\varphi} & 0 \\ 0 & je^{-j\varphi} & 0 & 0 \\ -je^{j\varphi} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$J^\varphi = \frac{j}{r\sin\theta}\left(\psi_3^*e^{j\varphi}, -\psi_2^*e^{-j\varphi}, \psi_1^*e^{j\varphi}, -\psi_0^*e^{-j\varphi}\right)\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

(XXI-12)

$$r\sin\theta J^\varphi = j\left(\psi_0\psi_3^*e^{j\varphi} - \psi_1\psi_2^*e^{-j\varphi} + \psi_2\psi_1^*e^{j\varphi} - \psi_3\psi_0^*e^{-j\varphi}\right)$$

These analytical expressions allow to check how exact solutions obtained comply with the local conservation of energy. The calculations may be particularly laborious. Some elements of calculations are presented below, on the example of the rotating solution obtained in a previous chapter:

$$\begin{aligned}
&\psi_0 = 0\\
&\psi_1 = (\eta - k_t) f_n(k_r r)\sin^n\theta \exp\, j(k_t x_t + n\varphi)\\
&\psi_2 = j\exp\, j(k_t x_t + (n-1)\varphi)\sin^{n-1}\theta\left\{-\sin^2\theta k_r f_n'(k_r r) - n(\cos^2\theta + 1)\frac{f_n(k_r r)}{r}\right\}\\
&\psi_3 = j\cos\theta\sin^n\theta\exp\, j(k_t x_t + n\varphi)\left\{k_r f_n'(k_r r) - n\frac{f_n(k_r r)}{r}\right\}
\end{aligned} \qquad \text{(XXI-13)}$$

Calculation of J^t:

$$\begin{aligned}
&J^t = \psi_0^*\psi_0 + \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi\\
&J^t = (\eta - k_t)^2\left\{f_n(k_r r)\sin^n\theta\right\}^2 + k_r^2\left\{-\sin^{n+1}\theta f_n'(k_r r) - n\sin^{n-1}\theta(\cos^2\theta + 1)\frac{f_n(k_r r)}{k_r r}\right\}^2\\
&+ (\cos\theta\sin^n\theta)^2 k_r^2\left\{f_n'(k_r r) - n\frac{f_n(k_r r)}{k_r r}\right\}^2
\end{aligned} \qquad \text{(XXI-14)}$$

On this particular mode, the volume density of total energy depends neither time nor of the space variable φ

Calculation of J^r:

$$\begin{aligned}
J^r = {}& \psi_0(\psi_2^*\cos\theta + \psi_3^*\sin\theta e^{j\varphi}) + \psi_1(\psi_2^*\sin\theta e^{-j\varphi} - \psi_3^*\cos\theta)\\
&+ \psi_2(\psi_0^*\cos\theta + \psi_1^*\sin\theta e^{j\varphi}) + \psi_3(\psi_0^*\sin\theta e^{-j\varphi} - \psi_1^*\cos\theta)
\end{aligned} \qquad \text{(XXI-15)}$$

For the reporting solution, we have $\psi_0 = 0$:

$$J^r = \psi_1(\psi_2^*\sin\theta e^{-j\varphi} - \psi_3^*\cos\theta) + \psi_2(\psi_1^*\sin\theta e^{j\varphi}) + \psi_3(-\psi_1^*\cos\theta) \qquad \text{(XXI-16)}$$

The exponential terms vanish by conjugation in the products of the wave functions. After this cancellation, it remains in ψ_2 and ψ_3 purely imaginary terms. We can deduce:

$$J^r = 0 \qquad \text{(XXI-17)}$$

Calculation of J^θ:

$$\begin{aligned}
rJ^\theta = {}& \psi_0(-\psi_2^*\sin\theta + \psi_3^*\cos\theta e^{j\varphi}) + \psi_1(\psi_2^*\cos\theta e^{-j\varphi} + \psi_3^*\sin\theta)\\
&+ \psi_2(-\psi_0^*\sin\theta + \psi_1^*\cos\theta e^{j\varphi}) + \psi_3(\psi_0^*\cos\theta e^{-j\varphi} + \psi_1^*\sin\theta)
\end{aligned} \qquad \text{(XXI-18)}$$

For the reporting solution, we have $\psi_0 = 0$:

$$rJ^{\theta} = \psi_1\left(\psi_2^* \cos\theta e^{-j\varphi} + \psi_3^* \sin\theta\right) + \psi_2\left(\psi_1^* \cos\theta e^{j\varphi}\right) + \psi_3\left(\psi_1^* \sin\theta\right) \qquad \text{(XXI-19)}$$

The exponential terms vanish by conjugation in the products of the wave functions. After this cancellation, it remains in ψ_2 and ψ_3 purely imaginary terms. We can deduce:

$$J^{\theta} = 0 \qquad \text{(XXI-20)}$$

Calculation of J^{φ}:

$$r\sin\theta J^{\varphi} = j\left(\psi_0\psi_3^* e^{j\varphi} - \psi_1\psi_2^* e^{-j\varphi} + \psi_2\psi_1^* e^{j\varphi} - \psi_3\psi_0^* e^{-j\varphi}\right) \qquad \text{(XXI-21)}$$

For the reporting solution, we have $\psi_0 = 0$:

$$r\sin\theta J^{\varphi} = j\left(-\psi_1\psi_2^* e^{-j\varphi} + \psi_2\psi_1^* e^{j\varphi}\right) \qquad \text{(XXI-22)}$$

We obtain:

$$J^{\varphi} = 2\left(k_t - \eta\right)\frac{f_n(k_r r)}{r}\sin^{2n-2}\theta\left\{\sin^2\theta k_r f_n'(k_r r) + n\left(\cos^2\theta + 1\right)\frac{f_n(k_r r)}{r}\right\} \qquad \text{(XXI-23)}$$

Since we are in possession of the currents of DIRAC, we can now check if these currents that can be associated to the evolution of the density of energy in time and space, well check the relationship of local conservation of energy.

This relationship is written in spherical coordinates:

$$\frac{\partial J^t}{\partial x_t} + \frac{1}{r^2}\frac{\partial\left(r^2 J^r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta J^{\theta}\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial J^{\varphi}}{\partial\varphi} = 0 \qquad \text{(XXI-24)}$$

It is verified, because each of the terms of this divergence, taken separately, is zero. A somewhat more detailed analysis shows that about these modes in rotation:

- Total volume energy density J^t depends neither on time, nor on variable spatial φ. On the other hand, it varies following r and θ, and is depending on the modes that are excited.
- Currents of DIRAC following r and θ being zero, there is no flow of energy following these directions.
- The Dirac current along φ is non-zero, reflecting the existence of a flow of energy linked with the angular rotation of energy in that angular direction. This flow of energy is uniform because it does not depend on φ. It can be attached to the rotation, with a constant speed, of energy following this direction.

XXII

Generalized DIRAC equation in spherical coordinates

Obtaining solutions to the Dirac equation in the form of spherical modes leads naturally to wonder about the existence of such solutions when the particle is immersed in a constant and uniform four-potential.

This problem has already been processed in Cartesian coordinates in chapter XII. We propose to resume it in spherical coordinates in a somewhat different approach.

We must first determine the equation of conservation of energy between two frames linked by relativity.

We adopt as a starting point a Cartesian coordinate system and the form of the KLEIN-GORDON equation invariant under the LORENTZ transformation:

$$\left\{\left(j\frac{\partial}{\partial x^{\mu}}-\frac{q}{\hbar}A_{\mu}\right)\left(j\frac{\partial}{\partial x^{\mu}}-\frac{q}{\hbar}A_{\mu}\right)\right\}(\psi)=\left(\frac{m_0 c}{\hbar}\right)^2(\psi) \qquad \text{(XXII-1)}$$

In this relationship, the A_{μ} represent the components of the four-potential (ϕ/c A_x, A_y, A_z). In order to lighten writing, we put, as in chapter XII:

$$\eta_t=\frac{q\phi}{\hbar c} \qquad \eta_x=\frac{qA_x}{\hbar} \qquad \eta_y=\frac{qA_y}{\hbar} \qquad \eta_z=\frac{qA_z}{\hbar} \qquad \eta=\frac{m_0 c}{\hbar} \qquad \text{(XXII-2)}$$

The development of the equation (XXII-1) in a metric (+,-,-, -) gives us:

$$\left\{\begin{array}{l}\left(-\frac{\partial^2}{\partial x_t^2}-2j\eta_t\frac{\partial}{\partial x_t}+\eta_t^2\right)-\left(-\frac{\partial^2}{\partial x^2}-2j\eta_x\frac{\partial}{\partial x}+\eta_x^2\right)\\ -\left(-\frac{\partial^2}{\partial y^2}-2j\eta_y\frac{\partial}{\partial y}+\eta_y^2\right)-\left(-\frac{\partial^2}{\partial z^2}-2j\eta_z\frac{\partial}{\partial z}+\eta_z^2\right)\end{array}\right\}(\psi)=\eta^2(\psi) \qquad \text{(XXII-3)}$$

Or still :

$$\left(-\frac{\partial^2}{\partial x_t^2}-2j\eta_t\frac{\partial}{\partial x_t}+\eta_t^2\right)(\psi)=\left\{\begin{array}{l}\left(-\frac{\partial^2}{\partial x^2}-2j\eta_x\frac{\partial}{\partial x}+\eta_x^2\right)+\\ \left(-\frac{\partial^2}{\partial y^2}-2j\eta_y\frac{\partial}{\partial y}+\eta_y^2\right)+\\ \left(-\frac{\partial^2}{\partial z^2}-2j\eta_z\frac{\partial}{\partial z}+\eta_z^2\right)\end{array}\right\}(\psi)+\eta^2(\psi) \qquad \text{(XXII-4)}$$

The scalar potential affects the term on the left of the equal sign, whereas the vector potential affects the right term. We will consider these two cases separately.

I – The scalar potential

We are interested in this part to the effect of the scalar potential, and we are working with a null vector potential, which leads to put $\eta_x = \eta_y = \eta_z = 0$ in the equation (XXII-4):

$$\left(-\frac{\partial^2}{\partial x_t^2}-2j\eta_t\frac{\partial}{\partial x_t}+\eta_t^2\right)(\psi)=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\psi)+\eta^2(\psi) \qquad \text{(XXII-5)}$$

The introduction of spherical coordinates provides the Laplacian in the form:

$$\left(-\frac{\partial^2}{\partial x_t^2}-2j\eta_t\frac{\partial}{\partial x_t}+\eta_t^2\right)(\psi)=-\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}(\psi)+\eta^2(\psi) \qquad \text{(XXII-6)}$$

It appears from the analysis of spherical modes of Chapter XVI that stationary modes should check the relationship:

$$\left(-\frac{\partial^2}{\partial x_t^2}-2j\eta_t\frac{\partial}{\partial x_t}+\eta_t^2\right)(\psi)=\left(k_r^2+\eta^2\right)(\psi) \qquad \text{(XXII-7)}$$

It appears a note already formulated in another way in chapter XII on the basis of observations relating to the linear system XII-11, 12, 13, 14: it is impossible to find a stationary solution in sin ($k_t x_t$) or cos($k_t x_t$) that allows to obtain an equation of conservation of energy derived from (XXII-7).

Possible solutions appear in the form:

$$\cos(k_t x_t)\pm j\sin(k_t x_t)=\exp(\pm jk_t x_t) \qquad \text{(XXII-8)}$$

They reflect the excitement of modes in quadrature which exchange positive and negative energy.

By postponing this relationship (XXII-8) in (XXII-7), the relationship of conservation of energy which must be verified by the system of Dirac in spherical coordinates in a scalar potential is obtained:

$$\left(k_t^2 \pm 2k_t\eta_t + \eta_t^2\right) = \left(k_r^2 + \eta^2\right) \qquad \text{(XXII-9)}$$

Or still:

$$\left(k_t \pm \eta_t\right)^2 = \left(k_r^2 + \eta^2\right) \qquad \text{(XXII-10)}$$

It's the relation obtained in chapter XII, transposed in spherical coordinates.

The system of Dirac generalized to a scalar potential in spherical coordinates is written:

$$\begin{aligned}
(\eta+\eta_t)\psi_0 &= j\left\{\frac{\partial\psi_0}{\partial x_t} + \cos\theta\frac{\partial\psi_2}{\partial r} + \sin\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(-\sin\theta\frac{\partial\psi_2}{\partial\theta} + \cos\theta e^{-j\varphi}\frac{\partial\psi_3}{\partial\theta}\right) - \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_3}{\partial\varphi}\right\}\\
(\eta+\eta_t)\psi_1 &= j\left\{\frac{\partial\psi_1}{\partial x_t} + \sin\theta e^{j\varphi}\frac{\partial\psi_2}{\partial r} - \cos\theta\frac{\partial\psi_3}{\partial r} + \frac{1}{r}\left(\cos\theta e^{j\varphi}\frac{\partial\psi_2}{\partial\theta} + \sin\theta\frac{\partial\psi_3}{\partial\theta}\right) + \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_2}{\partial\varphi}\right\}\\
(\eta-\eta_t)\psi_2 &= j\left\{-\frac{\partial\psi_2}{\partial x_t} - \cos\theta\frac{\partial\psi_0}{\partial r} - \sin\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(\sin\theta\frac{\partial\psi_0}{\partial\theta} - \cos\theta e^{-j\varphi}\frac{\partial\psi_1}{\partial\theta}\right) + \frac{je^{-j\varphi}}{r\sin\theta}\frac{\partial\psi_1}{\partial\varphi}\right\}\\
(\eta-\eta_t)\psi_3 &= j\left\{-\frac{\partial\psi_3}{\partial x_t} - \sin\theta e^{j\varphi}\frac{\partial\psi_0}{\partial r} + \cos\theta\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\left(-\cos\theta e^{j\varphi}\frac{\partial\psi_0}{\partial\theta} - \sin\theta\frac{\partial\psi_1}{\partial\theta}\right) - \frac{je^{j\varphi}}{r\sin\theta}\frac{\partial\psi_0}{\partial\varphi}\right\}
\end{aligned} \qquad \text{(XXII-11)}$$

Using the exact solution (XXI-13) for example, the exact solution in a scalar potential is obtained in the form

$$\begin{aligned}
\psi_0 &= 0\\
\psi_1 &= (\eta - k_t - \eta_t) f_n(k_r r)\sin^n\theta\exp j(k_t x_t + n\varphi)\\
\psi_2 &= -j\exp j(k_t x_t + (n-1)\varphi)\sin^{n-1}\theta\left\{\sin^2\theta k_r f_n'(k_r r) + n\left(\cos^2\theta + 1\right)\frac{f_n(k_r r)}{r}\right\}\\
\psi_3 &= j\cos\theta\sin^n\theta\exp j(k_t x_t + n\varphi)\left\{k_r f_n'(k_r r) - n\frac{f_n(k_r r)}{r}\right\}
\end{aligned} \qquad \text{(XXII-12)}$$

It is associated with the relation of energy conservation:

$$\left(k_t + \eta_t\right)^2 = \left(k_r^2 + \eta^2\right) \qquad \text{(XXII-13)}$$

II – The vector potential

From (XXII-4), we deduct the KLEIN-GORDON equation written in Cartesian coordinates for a scalar potential equal to zero and a constant vector potential:

$$-\frac{\partial^2}{\partial x_t^2}(\psi)=\left\{\left(-\frac{\partial^2}{\partial x^2}-2j\eta_x\frac{\partial}{\partial x}+\eta_x^2\right)+\left(-\frac{\partial^2}{\partial y^2}-2j\eta_y\frac{\partial}{\partial y}+\eta_y^2\right)+\left(-\frac{\partial^2}{\partial z^2}-2j\eta_z\frac{\partial}{\partial z}+\eta_z^2\right)+\eta^2\right\}(\psi)$$

(XXII-14)

Or still, by rearranging the terms:

$$-\frac{\partial^2}{\partial x_t^2}(\psi)=\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}-2j\frac{\partial}{\partial x}\eta_x-2j\frac{\partial}{\partial y}\eta_y-2j\frac{\partial}{\partial z}\eta_z+\eta_x^2+\eta_y^2+\eta_z^2+\eta^2\right)(\psi)$$

(XXII-15)

The passage in spherical coordinates is made by substituting the Laplacian and the partial derivatives already encountered in Chapter XV

$$\begin{aligned}
\frac{\partial\psi}{\partial x}&=\frac{\partial\psi}{\partial r}\sin\theta\cos\varphi+\frac{\partial\psi}{\partial\theta}\frac{1}{r}\cos\theta\cos\varphi-\frac{\partial\psi}{\partial\varphi}\frac{1}{r\sin\theta}\sin\varphi\\
\frac{\partial\psi}{\partial y}&=\frac{\partial\psi}{\partial r}\sin\theta\sin\varphi+\frac{\partial\psi}{\partial\theta}\frac{1}{r}\cos\theta\sin\varphi+\frac{\partial\psi}{\partial\varphi}\frac{1}{r\sin\theta}\cos\varphi\\
\frac{\partial\psi}{\partial z}&=\frac{\partial\psi}{\partial r}\cos\theta-\frac{\partial\psi}{\partial\theta}\frac{1}{r}\sin\theta
\end{aligned}$$

(XXII-16)

The vector potential components are transformed following relations:

$$\begin{aligned}
\eta_x&=\frac{qA_x}{\hbar}=\frac{qA_r}{\hbar}\sin\theta\cos\varphi=\eta_r\sin\theta\cos\varphi\\
\eta_y&=\frac{qA_y}{\hbar}=\frac{qA_r}{\hbar}\sin\theta\sin\varphi=\eta_r\sin\theta\cos\varphi\\
\eta_z&=\frac{qA_z}{\hbar}=\frac{qA_r}{\hbar}\cos\theta=\eta_r\cos\theta
\end{aligned}$$

(XXII-17)

We can deduce the KLEIN-GORDON equation in spherical coordinates in a vector potential

$$-\frac{\partial^2}{\partial x_t^2}(\psi)=\left(\begin{aligned}
&-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\\
&-2j\left(\frac{\partial}{\partial r}\sin\theta\cos\varphi+\frac{\partial}{\partial\theta}\frac{1}{r}\cos\theta\cos\varphi-\frac{\partial}{\partial\varphi}\frac{1}{r\sin\theta}\sin\varphi\right)\eta_r\sin\theta\cos\varphi\\
&-2j\left(\frac{\partial}{\partial r}\sin\theta\sin\varphi+\frac{\partial}{\partial\theta}\frac{1}{r}\cos\theta\sin\varphi+\frac{\partial}{\partial\varphi}\frac{1}{r\sin\theta}\cos\varphi\right)\eta_r\sin\theta\sin\varphi\\
&-2j\left(\frac{\partial}{\partial r}\cos\theta-\frac{\partial}{\partial\theta}\frac{1}{r}\sin\theta\right)\eta_r\cos\theta\\
&+(\eta_r\sin\theta\cos\varphi)^2+(\eta_r\sin\theta\sin\varphi)^2+(\eta_r\cos\theta)^2+\eta^2
\end{aligned}\right)(\psi)$$

(XXII-18)

After reduction of the terms which vanish, we get:

$$-\frac{\partial^2}{\partial x_t^2}(\psi)=\left(\begin{array}{l}-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\\-2j\frac{\partial}{\partial r}\eta_r+\eta_r^2+\eta^2\end{array}\right)(\psi) \qquad \text{(XXII-19)}$$

If we require that the wave function ψ comes in the form of spherical stationary modes, this leads to the following expression of conservation of energy:

$$k_t^2(\psi)=\left(k_r^2-2j\frac{\partial}{\partial r}\eta_r+\eta_r^2+\eta^2\right)(\psi) \qquad \text{(XXII-20)}$$

This relationship is completely analogous to the relationship (XXII-7) obtained for the scalar potential.

It becomes a problem that could not be overcome by the author: dependence in r of the spherical BESSEL functions in the stationary solution r does not make the relationship of conservation of energy (XXII-20) independent of r. It follows that no formulation of an exact solution to the DIRAC equation in spherical coordinates and under a constant vector potential could be formulated.

XXIII

Conclusion of the third part

I – On the conservation of energy

The importance of this conservation was repeatedly recalled in this manuscript. It is of two kinds: one is conservation of energy during one change of inertial frame, the other on local conservation of energy. Both contribute to the coherence of the presented solutions.

I-1 Conservation of energy for change of frame

It is linked to the fact that the pseudo-norm of the pulse energy four vector:

$$\tilde{P} = \begin{pmatrix} m\frac{dx}{dt} \\ m\frac{dy}{dt} \\ m\frac{dz}{dt} \\ \frac{E}{c} \end{pmatrix} = \begin{pmatrix} mv_x \\ mv_y \\ mv_z \\ \frac{E}{c} \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ \frac{E}{c} \end{pmatrix} \qquad \text{(XXIII-1)}$$

expressed explicitly under the form :

$$E^2 - p^2c^2 = cte \qquad \text{(XXIII-2)}$$

is a constant that does not depend on the inertial frame in which it is assessed. This constant is equal to the energy of the rest mass squared, it is inferred that in any inertial frame in which the particle is likely to evolve, the following relationship holds:

$$E^2 = (pc)^2 + (m_0c^2)^2 \qquad \text{(XXIII-3)}$$

In the frame in which it is at rest, its impulse energy is zero, and its total energy is equal to its mass energy. In a frame where it is no more at rest, its impulse energy takes a finite value that increases its total energy in the report given by the respect for the relationship (XXIII-2) or energy (XXIII-3) above.

When we transpose these concepts to quantum mechanics, one obtains the relationship that translates the same phenomena of energy conservation by changing frame:

$$\hbar^2 c^2 k_t^2 = \hbar^2 c^2 \left(k_x^2 + k_y^2 + k_z^2\right) + \left(m_0 c^2\right)^2 \qquad \text{(XXIII-4)}$$

or still after division of the two members of equality by $\hbar^2 c^2$;

$$k_t^2 = \left(k_x^2 + k_y^2 + k_z^2\right) + \eta^2 \qquad \text{(XXIII-5)}$$

When switching from one inertial frame to another Galilean frame, this relationship expresses the fact that during changes in 3 kinds of different energies, the mass energy remains constant, while the impulse energy and wave energy vary in the proportions given by the relationship (XXIII-4)

Because the relativistic DIRAC equation:

$$\left(j\gamma^\mu \frac{\partial}{\partial x^\mu} - \frac{m_0 c}{\hbar}\right)\psi\left(x^\mu\right) = 0 \qquad \mu = 0, 1, 2, 3 \qquad \text{(XXIII-6)}$$

is invariant under change of inertial frame, this imposes that solutions that are expressed on the basis of k_t, k_x, k_y, k_z, m_0, check the equation of conservation of energy (XXIII-4).

II – On the imaginary form of the DIRAC equation solutions

The solutions of the DIRAC equation are represented by a bispinor. It is recognized that each of these spinors represents for one, a particle, for the other, its antiparticle.

Once the particle is moving, it appears that the two spinors are no more independent, but closely correlated in a relationship whose physical meaning escapes to classical quantum mechanics. In particular, the presence of imaginary terms found no satisfactory explanation, if it is accepted that the squared wave functions represent probability densities.

In an energy approach, the squared wave functions are homogeneous to a volumetric energy density. All physics is built around a signed energy representation, which can be positive or negative. For example, the total energy of a satellite in orbit around the Earth is negative, and the negative sign is justified there because he cannot escape the gravity. The analysis presented in the first part also shows that two spinors which are solutions of the DIRAC equation have energy of mass opposite when the particle is at rest. Therefore, if we consider a quantity that represents the square root of this energy, or more exactly the square root of the energy volume density, there is no problem to see the complex number j = sqrt(-1), which simply indicates that it is a quantity linked with a negative energy. Thus, in the energy exchanges that reflect the solutions in the form of stationary modes, the presence of the complex number j = sqrt(-1) simply refers to an energy that must be counted negatively.

In an energy approach, DIRAC currents take a clear physical meaning: they are homogeneous to a volumetric energy density, and thus naturally check the local conservation of energy equation.

III – On the notion of negative energy

If we can be satisfied by the previous paragraph that it is possible to use a representation of signed energy, it appears that the concept of negative energy that is used in this energy approach must be explained more because its physical meaning is not clear.

In this document, we found it for the first time in the classic formalism with relations (III-12) and (III-13) that connect two spinors when the particle is at rest. These two relationships are obtained in the case where the time dependence of spinors is in exp(-jωt). They are recalled below:

$$E\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=\begin{pmatrix}mc^2 & 0\\ 0 & -mc^2\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \qquad \text{(XXIII-7)}$$

or still :

$$\begin{aligned}E(\varphi)&=mc^2(\varphi)\\ E(\chi)&=-mc^2(\chi)\end{aligned} \qquad \text{(XXIII-8)}$$

The minus sign which appears with the energy associated with the second spinor leads to think that this spinor describes the behavior of the antiparticle. However, the experimental behavior of the antiparticle of the electron shows that its mass energy is identical to that of the electron: its load alone has changed sign.

Then, the sign - which is present before the mass energy seems not to be able to be associated with the description of the mass energy of a particle of antimatter.

Consequently, arises clearly, the physical meaning to be given to negative energy in this energy approach.

We propose to develop an interpretation based on a simple, concrete example of classical physics. This interpretation should be considered with caution as long as it has not received additional evidence in his favour on the part of the scientific community.

The example that will be developed concerns electrical energy.

Let us consider a classical RLC circuit. If we load the capacity before the closure of this circuit, the energy stored by the capacity performs round-trips between capacity and self. At each round trip, a fraction of the energy is dissipated in the resistance R until the total disappearance of energy and thus oscillations.

One ways for the continuation of the oscillations is to include in this circuit a negative resistance (-R). This resistance can be constructed using an operational amplifier that delivers a voltage proportional to the intensity in a fraction (-R): it is therefore a very practical device.

The power P_R dissipated in the resistance R at each time t is simply written:

$$P_R(t) = R\, I^2(t) \qquad \text{(XXIII-9)}$$

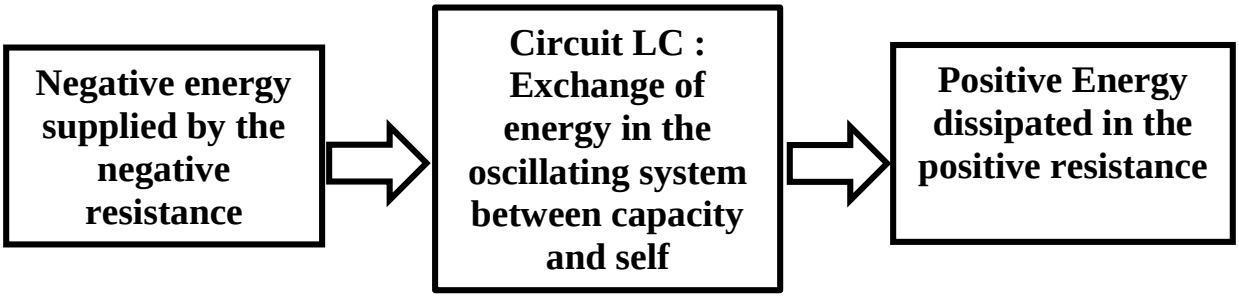
and it is apparent that this power is counted positively.

Power $P_{(-R)}$ provided by the negative resistance at each time t is written thus:

$$P_{(-R)}(t) = -R\, I^2(t) \qquad \text{(XXIII-10)}$$

and it is apparent that this power is counted negatively.

Such a device can be summarised as follows:



The physical interpretation that can be made is as follows: a positive power expresses a power that disappears from the system that exchanges of energy, while a negative power is a power that is fed into this system. In this simple case, the conservation of energy introduced in the system at any moment implies:

$$P_R(t) + P_{(-R)}(t) = R\, I^2(t) - R\, I^2(t) = 0 \qquad \text{(XXIII-11)}$$

If now, by analogy with the wave functions, we are interested in quantities ψ_+ and ψ_- that represent the square root of power, these quantities are defined by the relationship:

$$\begin{aligned} \psi_+ &= \sqrt{P_R(t)} = \sqrt{RI^2(t)} \\ \psi_- &= \sqrt{P_{(-R)}(t)} = \sqrt{-RI^2(t)} = j\sqrt{RI^2(t)} \end{aligned} \qquad \text{(XXIII-12)}$$

One can no longer simply express the relationship of conservation of energy above using ψ_+ and ψ_- quantities. If one wants to nevertheless express the sum or the difference between these quantities that are exchanged in the system, one is led to consider expressions of the type:

$$\begin{aligned} \psi &= \psi_+ + \psi_- = \sqrt{P_R(t)} + \sqrt{P_{(-R)}(t)} = \sqrt{RI^2(t)} + j\sqrt{RI^2(t)} \\ \psi^* &= \psi_+ - \psi_- = \sqrt{P_R(t)} - \sqrt{P_{(-R)}(t)} = \sqrt{RI^2(t)} - j\sqrt{RI^2(t)} \end{aligned} \qquad \text{(XXIII-13)}$$

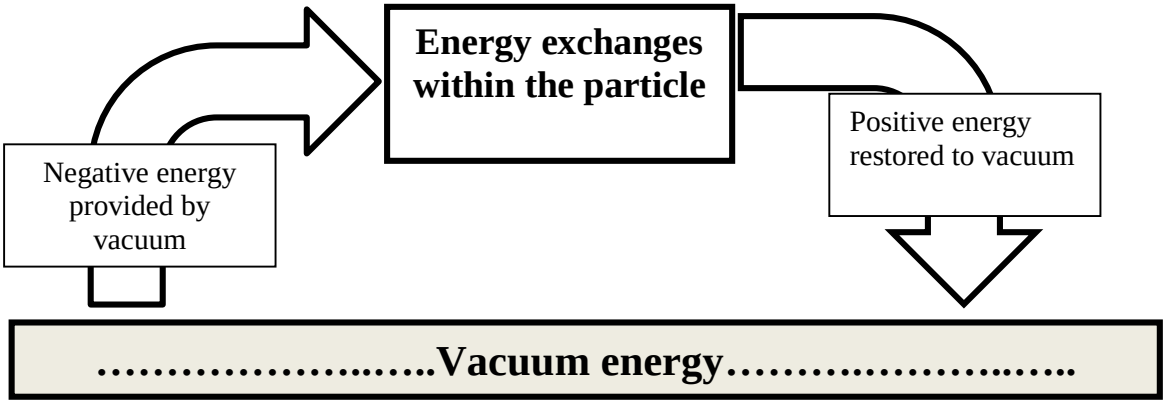
The exchanged power P(t) may be expressed by multiplying the previous imaginary quantity by its conjugate:

$$P(t) = \frac{1}{2}\psi\psi^* = RI^2(t) \qquad \text{(XXIII-14)}$$

By looking at this expression, we observe that the calculation of this exchanged power is always positive: we get a quantity whose behavior is analogous to Dirac currents.

When one attempt to transpose the energy interpretation above towards quantum energy exchanges, it immediately comes an inevitable question: where is the origin of the energy introduced in the particle (negative energy), and where is going the energy that escapes (positive energy)?

It seems that the only possible response is an exchanged with the energy of the vacuum, whose existence seems to be confirmed. Under this hypothesis, the exchange of energy could be represented schematically as follows:



A scheme of this nature seems consistent with the current knowledge of the quantum fluctuations of the vacuum, and gives a physical meaning clear and unambiguous to the notion of positive energy and negative energy. It expresses the fact that in the quantum world, particles exist and propagate by means of permanent exchange of energy with the energy of the vacuum. We could see in these exchanges support spread allowing photons to propagate in vacuum over distances of several light-years.

IV – general conclusion

Quantum mechanics cannot be satisfied a very long time yet the impasse in which it is maintained by the ignorance of the underlying physical phenomena to the wave-particle duality.

Because the DIRAC equation is the equation that describes the best to date, the behavior of the particles which compose the infinitely small world, we can think learn significant informations from it, if one is able to extract exact solutions.

The general treatment of this equation as it is adopted today, and as it is developed in the first part, does not give all usable informations from this equation.

Some solutions are achievable only at the cost of additional conditions relating to the conservation of energy. Combining these conditions to the hypothesis of solutions in the form of stationary modes, it is possible to show that one can construct a deterministic physics and energy vision of the infinitely small physics.

The analysis of the solutions in Cartesian coordinates has allowed to check the complete consistency of these solutions with the vision of the Copenhagen school.

Solutions in spherical coordinates confirms this consistency with various theoretical and experimental elements.

The wave-particle duality is confirmed as an exchange of energy between mass energy and wave energy. The presence of imaginary terms in these exchanges shows that there is continuously transfer of energy between positive energy and negative energy. It is this exchange, taken as a plausible hypothesis, which allowed to access by a heuristic reasoning to exact solutions in spherical coordinates.

The concept of spin ½, and its strange rotation of 4π to recover the initial state has an extremely simple physical explanation in one of exact solutions: when the component that carries the impulse energy following r rotates 2π to return to its starting point, the component that carries the impulse energy depending on φ and the component carrying the wave-particle duality turn 4π.

The uncertainty principle is not as exclusive as in Cartesian coordinates, but it may be noted that at any given time, mass energy and impulse energy are never maximum at the same place, which is sufficient to show that in a point in space we cannot have all of the information concerning these two kinds of energy. Since we can only have a measure of the momentum or position by an exchange of energy with these two quantities, these measures are necessarily tainted uncertainty when they occur in the same point of space.

Fourth part

Complements

XXIV

The DIRAC equation in a variable potential: classical approach

Modal solutions to the generalized DIRAC equation proposed in previous chapters have been developed in a very simplistic framework that is a uniform and constant potential, i.e. independent of space and time. We know that in these circumstances the associated electromagnetic field is null. So far, the presence of the potential causes changes in the particle energy that are highlighted in the exact solutions that have been developed.

Now, we want to move towards exact solutions when the particle is immersed in a variable potential in space and in time. Such solutions should enable us to apprehend the modal changes that occur in the particle in the presence of an electric and magnetic field, uniform or variable in space and time.

To move towards such solutions, it is necessary to lean on the existing classical formalism, in which the constraints imposed by the modal solutions will be introduced. This formalism is most often presented in a very condensed form which enables an overall vision of the phenomena, but often mask elements of great complexity. These elements are listed in detail below, in order to introduce changes induced by the stationary solutions in the following chapters.

We take as starting point the system of DIRAC for a particle in a variable potential, which means that each component of the potential may depend on the variables of space and time:

$$\eta\psi_0 = j\left(\frac{\partial\psi_0}{\partial x_t} + j\frac{q\phi}{\hbar}\psi_0\right) + j\left(\frac{\partial\psi_3}{\partial x} + j\frac{qA_x}{\hbar}\psi_3\right) + \left(\frac{\partial\psi_3}{\partial y} + j\frac{qA_y}{\hbar}\psi_3\right) + j\left(\frac{\partial\psi_2}{\partial z} + j\frac{qA_z}{\hbar}\psi_2\right)$$

$$\eta\psi_1 = j\left(\frac{\partial\psi_1}{\partial x_t} + j\frac{q\phi}{\hbar}\psi_1\right) + j\left(\frac{\partial\psi_2}{\partial x} + j\frac{qA_x}{\hbar}\psi_2\right) - \left(\frac{\partial\psi_2}{\partial y} + j\frac{qA_y}{\hbar}\psi_2\right) - j\left(\frac{\partial\psi_3}{\partial z} + j\frac{qA_z}{\hbar}\psi_3\right)$$

$$\eta\psi_2 = -j\left(\frac{\partial\psi_2}{\partial x_t} + j\frac{q\phi}{\hbar}\psi_2\right) - j\left(\frac{\partial\psi_1}{\partial x} + j\frac{qA_x}{\hbar}\psi_1\right) - \left(\frac{\partial\psi_1}{\partial y} + j\frac{qA_y}{\hbar}\psi_1\right) - j\left(\frac{\partial\psi_0}{\partial z} + j\frac{qA_z}{\hbar}\psi_0\right)$$

$$\eta\psi_3 = -j\left(\frac{\partial\psi_3}{\partial x_t} + j\frac{q\phi}{\hbar}\psi_3\right) - j\left(\frac{\partial\psi_0}{\partial x} + j\frac{qA_x}{\hbar}\psi_0\right) + \left(\frac{\partial\psi_0}{\partial y} + j\frac{qA_y}{\hbar}\psi_0\right) + j\left(\frac{\partial\psi_1}{\partial z} + j\frac{qA_z}{\hbar}\psi_1\right)$$

(XXIV-1)

As in the rest of the document, it retains the abridged notation:

$$\eta_t = \frac{q\phi}{\hbar c} \qquad \eta_x = \frac{qA_x}{\hbar} \qquad \eta_y = \frac{qA_y}{\hbar} \qquad \eta_z = \frac{qA_z}{\hbar} \tag{XXIV-2}$$

In this notation, the components of the potential are related to the electromagnetic field by the following relationships:

- For the magnetic field:

$$\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \overrightarrow{Rot}\left(\vec{A}\right) = \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} \tag{XXIV-3}$$

For the electric field:

$$\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -\overrightarrow{Grad}(\phi) - \frac{\partial \vec{A}}{\partial t} = \begin{pmatrix} -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} \\ -\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t} \\ -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} \end{pmatrix} \tag{XXIV-4}$$

In order to highlight the behaviour of the bi-spinor, we rearrange the system (XXIV-1) above in the form:

$$\begin{aligned}
&j\frac{\partial \psi_0}{\partial x_t} - \eta_t\psi_0 = \eta\psi_0 - j\frac{\partial \psi_3}{\partial x} + \eta_x\psi_3 - \frac{\partial \psi_3}{\partial y} - j\eta_y\psi_3 - j\frac{\partial \psi_2}{\partial z} + \eta_z\psi_2 \\
&j\frac{\partial \psi_1}{\partial x_t} - \eta_t\psi_1 = \eta\psi_1 - j\frac{\partial \psi_2}{\partial x} + \eta_x\psi_2 + \frac{\partial \psi_2}{\partial y} + j\eta_y\psi_2 + j\frac{\partial \psi_3}{\partial z} - \eta_z\psi_3 \\
&j\frac{\partial \psi_2}{\partial x_t} - \eta_t\psi_2 = -\eta\psi_2 - j\frac{\partial \psi_1}{\partial x} + \eta_x\psi_1 - \frac{\partial \psi_1}{\partial y} - j\eta_y\psi_1 - j\frac{\partial \psi_0}{\partial z} + \eta_z\psi_0 \\
&j\frac{\partial \psi_3}{\partial x_t} - \eta_t\psi_3 = -\eta\psi_3 - j\frac{\partial \psi_0}{\partial x} + \eta_x\psi_0 + \frac{\partial \psi_0}{\partial y} + j\eta_y\psi_0 + j\frac{\partial \psi_1}{\partial z} - \eta_z\psi_1
\end{aligned} \tag{XXIV-5}$$

Therefore, if we put:

$$\psi=\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\qquad\text{with }\varphi=\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}\qquad\text{et}\qquad\chi=\begin{pmatrix}\psi_2\\ \psi_3\end{pmatrix}\tag{XXIV-6}$$

Dirac system breaks down into two coupled systems:

$$\begin{aligned}
&j\left(\frac{\partial}{\partial x_t}+j\eta_t\right)\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}=\eta\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}-j\begin{pmatrix}\left(\frac{\partial}{\partial z}+j\eta_z\right) & \left(\frac{\partial}{\partial x}+j\eta_x\right)-j\left(\frac{\partial}{\partial y}+j\eta_y\right)\\ \left(\frac{\partial}{\partial x}+j\eta_x\right)+j\left(\frac{\partial}{\partial y}+j\eta_y\right) & -\left(\frac{\partial}{\partial z}+j\eta_z\right)\end{pmatrix}\begin{pmatrix}\psi_2\\ \psi_3\end{pmatrix}\\
&j\left(\frac{\partial}{\partial x_t}+j\eta_t\right)\begin{pmatrix}\psi_2\\ \psi_3\end{pmatrix}=-\eta\begin{pmatrix}\psi_2\\ \psi_3\end{pmatrix}-j\begin{pmatrix}\left(\frac{\partial}{\partial z}+j\eta_z\right) & \left(\frac{\partial}{\partial x}+j\eta_x\right)-j\left(\frac{\partial}{\partial y}+j\eta_y\right)\\ \left(\frac{\partial}{\partial x}+j\eta_x\right)+j\left(\frac{\partial}{\partial y}+j\eta_y\right) & -\left(\frac{\partial}{\partial z}+j\eta_z\right)\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}
\end{aligned}\tag{XXIV-7}$$

We put for easy writing:

$$M=-j\begin{pmatrix}\left(\frac{\partial}{\partial z}+j\eta_z\right) & \left(\frac{\partial}{\partial x}+j\eta_x\right)-j\left(\frac{\partial}{\partial y}+j\eta_y\right)\\ \left(\frac{\partial}{\partial x}+j\eta_x\right)+j\left(\frac{\partial}{\partial y}+j\eta_y\right) & -\left(\frac{\partial}{\partial z}+j\eta_z\right)\end{pmatrix}\tag{XXIV-8}$$

The coupled system takes a simplified form:

$$\begin{aligned}
&j\left(\frac{\partial}{\partial x_t}+j\eta_t\right)(\varphi)=\eta(\varphi)+M(\chi)\\
&j\left(\frac{\partial}{\partial x_t}+j\eta_t\right)(\chi)=-\eta(\varphi)+M(\varphi)
\end{aligned}\tag{XXIV-9}$$

Or still:

$$\begin{aligned}
&\left(j\frac{\partial}{\partial x_t}-(\eta_t+\eta)\right)(\varphi)=M(\chi)\\
&\left(j\frac{\partial}{\partial x_t}-(\eta_t-\eta)\right)(\chi)=M(\varphi)
\end{aligned}\tag{XXIV-10}$$

From which we deduce formally:

$$\begin{aligned}\left(j\frac{\partial}{\partial x_t}-(\eta_t+\eta)\right)\left(j\frac{\partial\varphi}{\partial x_t}-(\eta_t-\eta)\varphi\right)&=M^2\varphi\\ \left(j\frac{\partial}{\partial x_t}-(\eta_t-\eta)\right)\left(j\frac{\partial\chi}{\partial x_t}-(\eta_t+\eta)\chi\right)&=M^2\chi\end{aligned}\qquad\text{(XXIV-11)}$$

Yields by developing the left member of equality:

$$\begin{aligned}-\frac{\partial^2\varphi}{\partial x_t^2}-2j\frac{\partial\varphi}{\partial x_t}\eta_t+\eta_t^2\varphi-j\frac{\partial\eta_t}{\partial x_t}\varphi-\eta^2\varphi&=M^2\varphi\\ -\frac{\partial^2\chi}{\partial x_t^2}-2j\frac{\partial\chi}{\partial x_t}\eta_t+\eta_t^2\chi-j\frac{\partial\eta_t}{\partial x_t}\chi-\eta^2\chi&=M^2\chi\end{aligned}\qquad\text{(XXIV-12)}$$

To make progress towards a possible solution, we must now establish the expression of matrix M².

If we adopt the following notation for PAULI matrixes and the operators involved in matrix computations:

$$\vec{\sigma}=\begin{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ \begin{pmatrix}0&-j\\j&0\end{pmatrix}\\ \begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{pmatrix}\qquad \vec{p}=\begin{pmatrix}-j\hbar\frac{\partial}{\partial x}\\ -j\hbar\frac{\partial}{\partial y}\\ -j\hbar\frac{\partial}{\partial z}\end{pmatrix}\qquad \vec{\pi}=\left(\vec{p}+q\vec{A}\right)=\begin{pmatrix}-j\hbar\frac{\partial}{\partial x}+qA_x\\ -j\hbar\frac{\partial}{\partial y}+qA_y\\ -j\hbar\frac{\partial}{\partial z}+qA_z\end{pmatrix}=\begin{pmatrix}-j\hbar\left(\frac{\partial}{\partial x}+j\eta_x\right)\\ -j\hbar\left(\frac{\partial}{\partial y}+j\eta_y\right)\\ -j\hbar\left(\frac{\partial}{\partial z}+j\eta_z\right)\end{pmatrix}\qquad\text{(XXIV-13)}$$

the matrix M established in (XXIV-8) is expressed as follows:

$$\hbar M(\varphi)=\left[\vec{\sigma}.\left(\vec{p}+q\vec{A}\right)\right](\varphi)=-j\hbar\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\left(\frac{\partial}{\partial x}+j\eta_x\right)+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\left(\frac{\partial}{\partial y}+j\eta_y\right)+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\left(\frac{\partial}{\partial z}+j\eta_z\right)\right](\varphi)$$

$$M=-j\begin{pmatrix}\left(\frac{\partial}{\partial z}+j\eta_z\right) & \left(\frac{\partial}{\partial x}+j\eta_x\right)-j\left(\frac{\partial}{\partial y}+j\eta_y\right)\\ \left(\frac{\partial}{\partial x}+j\eta_x\right)+j\left(\frac{\partial}{\partial y}+j\eta_y\right) & -\left(\frac{\partial}{\partial z}+j\eta_z\right)\end{pmatrix}\qquad\text{(XXIV-14)}$$

The development of the matrix M² is a bit laborious. The reader will take care that each of the terms of the matrix represents an operator and must therefore be treated as such in the operations of derivations.

If we point out each of the terms of this matrix by:

$$M^2=\begin{pmatrix}M_{11}&M_{12}\\M_{21}&M_{22}\end{pmatrix}\qquad\text{(XXIV-15)}$$

Yields for example for M_{11} and M_{12}:

$$M_{11}=-\left\{\begin{array}{l}\left(\frac{\partial^2\varphi}{\partial x^2}+2j\eta_x\frac{\partial\varphi}{\partial x}-\eta_x^2\varphi+j\varphi\frac{\partial\eta_x}{\partial x}\right)+\left(\frac{\partial^2\varphi}{\partial y^2}+2j\eta_y\frac{\partial\varphi}{\partial y}-\eta_y^2\varphi+j\varphi\frac{\partial\eta_y}{\partial y}\right)\\+\left(\frac{\partial^2\varphi}{\partial z^2}+2j\eta_z\frac{\partial\varphi}{\partial z}-\eta_z^2\varphi+j\varphi\frac{\partial\eta_z}{\partial z}\right)+j\left\{\left(\frac{\partial}{\partial x}+j\eta_x\right)\left(\frac{\partial\varphi}{\partial y}+j\eta_y\varphi\right)-\left(\frac{\partial}{\partial y}+j\eta_y\right)\left(\frac{\partial\varphi}{\partial x}+j\eta_x\varphi\right)\right\}\end{array}\right\}$$

$$M_{11}=-\left\{\begin{array}{l}\left(\frac{\partial^2\varphi}{\partial x^2}+2j\eta_x\frac{\partial\varphi}{\partial x}-\eta_x^2\varphi+j\varphi\frac{\partial\eta_x}{\partial x}\right)+\left(\frac{\partial^2\varphi}{\partial y^2}+2j\eta_y\frac{\partial\varphi}{\partial y}-\eta_y^2\varphi+j\varphi\frac{\partial\eta_y}{\partial y}\right)\\+\left(\frac{\partial^2\varphi}{\partial z^2}+2j\eta_z\frac{\partial\varphi}{\partial z}-\eta_z^2\varphi+j\varphi\frac{\partial\eta_z}{\partial z}\right)-\left\{\frac{\partial\eta_y}{\partial x}\varphi-\frac{\partial\eta_x}{\partial y}\varphi\right\}\end{array}\right\}$$

(XXIV-16)

$$M_{12}=-\left\{\left(\frac{\partial}{\partial z}+j\eta_z\right)\left\{\left(\frac{\partial\varphi}{\partial x}+j\eta_x\varphi\right)-j\left(\frac{\partial\varphi}{\partial y}+j\eta_y\varphi\right)\right\}-\left\{\left(\frac{\partial}{\partial x}+j\eta_x\right)-j\left(\frac{\partial}{\partial y}+j\eta_y\right)\right\}\left(\frac{\partial\varphi}{\partial z}+j\eta_z\varphi\right)\right\}$$

$$M_{12}=-\left\{j\frac{\partial\eta_x}{\partial z}\varphi+\frac{\partial\eta_y}{\partial z}\varphi-\left(j\frac{\partial\eta_z}{\partial x}\varphi+\frac{\partial\eta_z}{\partial y}\varphi\right)\right\}=\left\{\left(\frac{\partial\eta_z}{\partial y}-\frac{\partial\eta_y}{\partial z}\right)\varphi-j\left(\frac{\partial\eta_x}{\partial z}-\frac{\partial\eta_z}{\partial x}\right)\varphi\right\}$$

(XXIV-17)

The complete matrix is explicitly detailed below:

$$M^2=\begin{pmatrix}-\left\{\begin{array}{l}\left(\frac{\partial^2\varphi}{\partial x^2}+2j\eta_x\frac{\partial\varphi}{\partial x}-\eta_x^2\varphi+j\varphi\frac{\partial\eta_x}{\partial x}\right)\\+\left(\frac{\partial^2\varphi}{\partial y^2}+2j\eta_y\frac{\partial\varphi}{\partial y}-\eta_y^2\varphi+j\varphi\frac{\partial\eta_y}{\partial y}\right)\\+\left(\frac{\partial^2\varphi}{\partial z^2}+2j\eta_z\frac{\partial\varphi}{\partial z}-\eta_z^2\varphi+j\varphi\frac{\partial\eta_z}{\partial z}\right)\\-\left\{\frac{\partial\eta_y}{\partial x}\varphi-\frac{\partial\eta_x}{\partial y}\varphi\right\}\end{array}\right\} & \left(\frac{\partial\eta_z}{\partial y}-\frac{\partial\eta_y}{\partial z}\right)\varphi-j\left(\frac{\partial\eta_x}{\partial z}-\frac{\partial\eta_z}{\partial x}\right)\varphi\\ \left(\frac{\partial\eta_z}{\partial y}-\frac{\partial\eta_y}{\partial z}\right)\varphi+j\left(\frac{\partial\eta_x}{\partial z}-\frac{\partial\eta_z}{\partial x}\right)\varphi & -\left\{\begin{array}{l}\left(\frac{\partial^2\varphi}{\partial x^2}+2j\eta_x\frac{\partial\varphi}{\partial x}-\eta_x^2\varphi+j\varphi\frac{\partial\eta_x}{\partial x}\right)\\+\left(\frac{\partial^2\varphi}{\partial y^2}+2j\eta_y\frac{\partial\varphi}{\partial y}-\eta_y^2\varphi+j\varphi\frac{\partial\eta_y}{\partial y}\right)\\+\left(\frac{\partial^2\varphi}{\partial z^2}+2j\eta_z\frac{\partial\varphi}{\partial z}-\eta_z^2\varphi+j\varphi\frac{\partial\eta_z}{\partial z}\right)\\+\left\{\frac{\partial\eta_y}{\partial x}\varphi-\frac{\partial\eta_x}{\partial y}\varphi\right\}\end{array}\right\}\end{pmatrix}$$

(XXIV-18)

If we now take the relationship (XXIV-12) in matrix form, we obtain the following equality which should be checked for any spinor:

$$M^2 = \begin{pmatrix} -\frac{\partial^2\varphi}{\partial x_t^2} - 2j\frac{\partial\varphi}{\partial x_t}\eta_t + \eta_t^2\varphi - j\frac{\partial\eta_t}{\partial x_t}\varphi - \eta^2\varphi & 0 \\ 0 & -\frac{\partial^2\varphi}{\partial x_t^2} - 2j\frac{\partial\varphi}{\partial x_t}\eta_t + \eta_t^2\varphi - j\frac{\partial\eta_t}{\partial x_t}\varphi - \eta^2\varphi \end{pmatrix} \quad \text{(XXIV-19)}$$

It appears that equality above can be checked only if:

$$\left(\frac{\partial\eta_z}{\partial y} - \frac{\partial\eta_y}{\partial z}\right)\varphi + j\left(\frac{\partial\eta_x}{\partial z} - \frac{\partial\eta_z}{\partial x}\right)\varphi = 0 \quad \text{(XXIV-20)}$$

For this relationship to be true regardless of any spinor φ, this requires:

$$\begin{aligned} \left(\frac{\partial\eta_z}{\partial y} - \frac{\partial\eta_y}{\partial z}\right) &= 0 \\ \left(\frac{\partial\eta_x}{\partial z} - \frac{\partial\eta_z}{\partial x}\right) &= 0 \end{aligned} \quad \text{(XXIV-21)}$$

When the particle is immersed in a magnetic field, it is deducted from (XXIV-21) and (XXIV-3) that the system of DIRAC have only solutions when the magnetic field is directed along the Oz axis.

Since angular momentum is also oriented along the axis Oz, this leads to a quantification of the spin when the particle is immersed in a magnetic field, with only two possible States.

Specifically, when a free particle is introduced in a STERN and GERLACH device for example, this necessarily induced a spatial evolution of the internal modes to the particle. This spatial evolution will put in alignment the magnetic moment of the electron with the magnetic field generated by the device.

It can be assumed that this alignment will be done on an extremely low duration. If it is estimated empirically on a time τ equal to a hundred period, it gets an order of magnitude of time τ as followed:

$$\tau = 100.\frac{2\pi}{\omega_0} = 200\pi\frac{\hbar}{m_0c^2} = 200\pi\frac{1{,}05.10^{-34}}{\left(9{,}11.10^{-31}\right)\left(3.10^8\right)^2} \approx 8{,}09.10^{-19}\text{s} \quad \text{(XXIV-22)}$$

By comparing this value to the duration t of the path of the electron on a distance of 1 meter, with the limit speed of light:

$$t = \frac{1}{3.10^8} = 0{,}33.10^{-8}\text{s} \quad \text{(XXIV-23)}$$

It can be concluded that the alignment of the magnetic moment of the electron with the magnetic field generated by the device of STERN and GERLACH is almost instantly to its

entry into the device. It thus gets a description of the concept of spatial quantization, a concept which is extremely difficult to have a physical representation when addressing quantum mechanics.

We propose to finish this chapter to remember the relationship established by DIRAC, which highlights the role of the magnetic field on spinors:

$$\hbar^2 M^2(\varphi) = \left[\vec{\sigma}.\left(\vec{p}+q\vec{A}\right)\right]^2(\varphi) = \left(\vec{p}+q\vec{A}\right)^2(\varphi) + \vec{\sigma}.\overrightarrow{Rot}\left(q\vec{A}\right)(\varphi) = \left(\vec{p}+q\vec{A}\right)^2(\varphi) + \vec{\sigma}.q\vec{B}(\varphi) \qquad \text{(XXIV-24)}$$

Matrix M^2 left of equality has already been expressed in (XXIV-18). The calculation of the terms located to the right of equality is detailed below:

$$\left(\vec{p}+q\vec{A}\right)^2(\varphi) = \begin{pmatrix} -j\hbar\dfrac{\partial}{\partial x}+qA_x \\ -j\hbar\dfrac{\partial}{\partial y}+qA_y \\ -j\hbar\dfrac{\partial}{\partial z}+qA_z \end{pmatrix}\begin{pmatrix} -j\hbar\dfrac{\partial\phi}{\partial x}+qA_x\varphi \\ -j\hbar\dfrac{\partial\phi}{\partial y}+qA_y\varphi \\ -j\hbar\dfrac{\partial\phi}{\partial z}+qA_z\varphi \end{pmatrix} = -\hbar^2\begin{pmatrix} \dfrac{\partial}{\partial x}+j\eta_x \\ \dfrac{\partial}{\partial y}+j\eta_y \\ \dfrac{\partial}{\partial z}+j\eta_z \end{pmatrix}\begin{pmatrix} \dfrac{\partial\phi}{\partial x}+j\eta_x\varphi \\ \dfrac{\partial\phi}{\partial y}+j\eta_y\varphi \\ \dfrac{\partial\phi}{\partial z}+j\eta_z\varphi \end{pmatrix}$$

$$\left(\vec{p}+q\vec{A}\right)^2(\varphi) = -\hbar^2\left\{\begin{array}{l} \left(\dfrac{\partial^2\varphi}{\partial x^2}+2j\eta_x\dfrac{\partial\varphi}{\partial x}-\eta_x^2\varphi+j\varphi\dfrac{\partial\eta_x}{\partial x}\right)+\left(\dfrac{\partial^2\varphi}{\partial y^2}+2j\eta_y\dfrac{\partial\varphi}{\partial y}-\eta_y^2\varphi+j\varphi\dfrac{\partial\eta_y}{\partial y}\right) \\ +\left(\dfrac{\partial^2\varphi}{\partial z^2}+2j\eta_z\dfrac{\partial\varphi}{\partial z}-\eta_z^2\varphi+j\varphi\dfrac{\partial\eta_z}{\partial z}\right) \end{array}\right\} \qquad \text{(XXIV-25)}$$

$$\vec{\sigma}.\overrightarrow{Rot}\left(q\vec{A}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(\frac{\partial qA_z}{\partial y}-\frac{\partial qA_y}{\partial z}\right)+\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}\left(\frac{\partial qA_x}{\partial z}-\frac{\partial qA_z}{\partial x}\right)+\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{\partial qA_y}{\partial x}-\frac{\partial qA_x}{\partial y}\right)$$

$$\vec{\sigma}.\overrightarrow{Rot}\left(q\vec{A}\right) = \begin{pmatrix} \dfrac{\partial qA_y}{\partial x}-\dfrac{\partial qA_x}{\partial y} & \left(\dfrac{\partial qA_z}{\partial y}-\dfrac{\partial qA_y}{\partial z}\right)-j\left(\dfrac{\partial qA_x}{\partial z}-\dfrac{\partial qA_z}{\partial x}\right) \\ \left(\dfrac{\partial qA_z}{\partial y}-\dfrac{\partial qA_y}{\partial z}\right)+j\left(\dfrac{\partial qA_x}{\partial z}-\dfrac{\partial qA_z}{\partial x}\right) & -\left(\dfrac{\partial qA_y}{\partial x}-\dfrac{\partial qA_x}{\partial y}\right) \end{pmatrix}$$

$$\vec{\sigma}.\overrightarrow{Rot}\left(q\vec{A}\right) = \hbar^2\begin{pmatrix} \dfrac{\partial\eta_y}{\partial x}-\dfrac{\partial\eta_x}{\partial y} & \left(\dfrac{\partial\eta_z}{\partial y}-\dfrac{\partial\eta_y}{\partial z}\right)-j\left(\dfrac{\partial\eta_x}{\partial z}-\dfrac{\partial\eta_z}{\partial x}\right) \\ \left(\dfrac{\partial\eta_z}{\partial y}-\dfrac{\partial\eta_y}{\partial z}\right)+j\left(\dfrac{\partial\eta_x}{\partial z}-\dfrac{\partial\eta_z}{\partial x}\right) & -\left(\dfrac{\partial\eta_y}{\partial x}-\dfrac{\partial\eta_x}{\partial y}\right) \end{pmatrix} \qquad \text{(XXIV-26)}$$

The relationship:

$$\hbar^2 M^2(\varphi) = \left[\vec{\sigma}.\left(\vec{p}+q\vec{A}\right)\right]^2(\varphi) = \left(\vec{p}+q\vec{A}\right)^2(\varphi) + \vec{\sigma}.\overrightarrow{Rot}\left(q\vec{A}\right)(\varphi) \qquad \text{(XXIV-27)}$$

is checked by introducing the identity matrix in factor with the term $\left(\vec{p}+q\vec{A}\right)^2$.

XXV

Exact solutions of the Dirac equation in a variable potential in Cartesian coordinates

We discuss in this chapter looking for exact solutions to the DIRAC equation in a variable potential. We know from the previous chapter that this potential can have a magnetic field, but only directed along the Oz axis.

The presence of the electromagnetic field will induce changes on stationary modes, and so on the components of the wave vector k_t, k_x, k_y, k_z, which will depend on the components of the potential, and there are several ways to take into account these changes.

We adopt the following search strategy: on a point of space and time, the potential will depend on the spatial and temporal variables x, y, z, t. We seek modal solutions in the form of imaginary exponentials $\exp[j(k_t x_t + k_x x + k_y y + k_z z)]$.

The changes with respect to the modes of free space (without potential) will carry on modifications on the amplitude of each wave functions, and on the components of the wave vector k_t, k_x, k_y, k_z, through the conservation of energy equation.

In the previous chapter, we have established the following equality which must be valid for any spinor:

$$M^2 = \begin{pmatrix} -\frac{\partial^2\varphi}{\partial x_t^2} - 2j\frac{\partial\varphi}{\partial x_t}\eta_t + \eta_t^2\varphi - j\frac{\partial\eta_t}{\partial x_t}\varphi - \eta^2\varphi & 0 \\ 0 & -\frac{\partial^2\varphi}{\partial x_t^2} - 2j\frac{\partial\varphi}{\partial x_t}\eta_t + \eta_t^2\varphi - j\frac{\partial\eta_t}{\partial x_t}\varphi - \eta^2\varphi \end{pmatrix} \quad \text{(XXV-1)}$$

The particular shape of relation (XXV-1) shows that the two spinors are decoupled. Developing writing, we obtain the equation of conservation of energy in its most general form by gathering the influence of the vector potential under the two sign + and -:

$$\frac{\partial^2\varphi}{\partial x_t^2}+2j\frac{\partial\varphi}{\partial x_t}\eta_t-\eta_t^2\varphi+j\frac{\partial\eta_t}{\partial x_t}\varphi+\eta^2\varphi=$$
$$\left\{\left(\frac{\partial^2\varphi}{\partial x^2}+2j\eta_x\frac{\partial\varphi}{\partial x}-\eta_x^2\varphi+j\varphi\frac{\partial\eta_x}{\partial x}\right)+\left(\frac{\partial^2\varphi}{\partial y^2}+2j\eta_y\frac{\partial\varphi}{\partial y}-\eta_y^2\varphi+j\varphi\frac{\partial\eta_y}{\partial y}\right)\right.$$
$$\left.+\left(\frac{\partial^2\varphi}{\partial z^2}+2j\eta_z\frac{\partial\varphi}{\partial z}-\eta_z^2\varphi+j\varphi\frac{\partial\eta_z}{\partial z}\right)\pm\left\{\frac{\partial\eta_y}{\partial x}\varphi-\frac{\partial\eta_x}{\partial y}\varphi\right\}\right\} \quad \text{(XXV-2)}$$

There are solutions only if we are in a magnetic field oriented along Oz, which imposes:

$$\left(\frac{\partial\eta_z}{\partial y}-\frac{\partial\eta_y}{\partial z}\right)=0$$
$$\left(\frac{\partial\eta_x}{\partial z}-\frac{\partial\eta_z}{\partial x}\right)=0 \quad \text{(XXV-3)}$$

We will restrict the search of solutions field placing us in the following simplifying assumptions:

$$\frac{\partial\eta_t}{\partial x_t}=\frac{\partial\eta_x}{\partial x}=\frac{\partial\eta_y}{\partial y}=\eta_z=0 \quad \text{(XXV-4)}$$

We obtain after simplification:

$$\frac{\partial^2\varphi}{\partial x_t^2}+2j\frac{\partial\varphi}{\partial x_t}\eta_t-\eta_t^2\varphi+\eta^2\varphi=\left\{\begin{array}{l}\left(\frac{\partial^2\varphi}{\partial x^2}+2j\eta_x\frac{\partial\varphi}{\partial x}-\eta_x^2\varphi\right)+\left(\frac{\partial^2\varphi}{\partial y^2}+2j\eta_y\frac{\partial\varphi}{\partial y}-\eta_y^2\varphi\right)\\+\frac{\partial^2\varphi}{\partial z^2}\pm\left\{\frac{\partial\eta_y}{\partial x}\varphi-\frac{\partial\eta_x}{\partial y}\varphi\right\}\end{array}\right\} \quad \text{(XXV-5)}$$

We introduce the particular form of the solution:

$$\varphi=\exp\, j\left(k_t x_t+k_x x+k_y y+k_z z\right) \quad \text{(XXV-6)}$$

That gives the energy conservation equation relative to (XXV-6) in a variable potential:

$$\left(k_t+\eta_t\right)^2=\left(k_x+\eta_x\right)^2+\left(k_y+\eta_y\right)^2+k_z^2+\eta^2\pm\left(\frac{\partial\eta_y}{\partial x}-\frac{\partial\eta_x}{\partial y}\right) \quad \text{(XXV-7)}$$

It is now possible to begin the search for solutions itself. For a null potential, DIRAC system is written:

$$\begin{aligned}
\eta\psi_0 &= j\frac{\partial\psi_0}{\partial x_t} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z}\\
\eta\psi_1 &= j\frac{\partial\psi_1}{\partial x_t} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z}\\
\eta\psi_2 &= -j\frac{\partial\psi_2}{\partial x_t} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z}\\
\eta\psi_3 &= -j\frac{\partial\psi_3}{\partial x_t} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z}
\end{aligned} \qquad \text{(XXV-8)}$$

A possible solution to this system in an exponential form has already been developed in the previous chapters:

$$\begin{aligned}
\psi_0 &= (\eta - k_t)\exp\, j(k_t x_t + k_x x + k_y y + k_z z)\\
\psi_1 &= 0\\
\psi_2 &= k_z \exp\, j(k_t x_t + k_x x + k_y y + k_z z)\\
\psi_3 &= (k_x + jk_y)\exp\, j(k_t x_t + k_x x + k_y y + k_z z)
\end{aligned} \qquad \text{(XXV-9)}$$

In a variable potential, DIRAC system is amended as follows:

$$\begin{aligned}
(\eta + \eta_t)\psi_0 + \eta_x\psi_3 - j\eta_y\psi_3 + \eta_z\psi_2 &= j\frac{\partial\psi_0}{\partial x_t} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z}\\
(\eta + \eta_t)\psi_1 + \eta_x\psi_2 + j\eta_y\psi_2 - \eta_z\psi_3 &= j\frac{\partial\psi_1}{\partial x_t} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z}\\
(\eta - \eta_t)\psi_2 - \eta_x\psi_1 + j\eta_y\psi_1 - \eta_z\psi_0 &= -j\frac{\partial\psi_2}{\partial x_t} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z}\\
(\eta - \eta_t)\psi_3 - \eta_x\psi_0 - j\eta_y\psi_0 + \eta_z\psi_1 &= -j\frac{\partial\psi_3}{\partial x_t} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z}
\end{aligned} \qquad \text{(XXV-10)}$$

A possible solution to the system of DIRAC in a variable potential appears in the following form:

$$\begin{aligned}
\psi_0 &= (\eta - k_t - \eta_t)\exp\, j(k_t x_t + k_x x + k_y y + k_z z)\\
\psi_1 &= 0\\
\psi_2 &= k_z \exp\, j(k_t x_t + k_x x + k_y y + k_z z)\\
\psi_3 &= \{k_x + \eta_x + j(k_y + \eta_y)\}\exp\, j(k_t x_t + k_x x + k_y y + k_z z)
\end{aligned} \qquad \text{(XXV-11)}$$

It made the observation that the form of the solution remains identical to that which was obtained for a uniform potential. The introduction of a variable potential comes only to change the equation of energy conservation.

We give below a few elements of check. For this solution, the first equation of DIRAC system provides energy conservation equation.

First equation of the DIRAC system:

$$(\eta+\eta_t)\psi_0+\eta_x\psi_3-j\eta_y\psi_3+\eta_z\psi_2=j\frac{\partial\psi_0}{\partial x_t}+j\frac{\partial\psi_3}{\partial x}+\frac{\partial\psi_3}{\partial y}+j\frac{\partial\psi_2}{\partial z}$$

$$(\eta+\eta_t)\psi_0+\eta_x\psi_3-j\eta_y\psi_3=(\eta+\eta_t)(\eta-k_t-\eta_t)+\eta_x\{k_x+\eta_x+j(k_y+\eta_y)\}-j\eta_y\{k_x+\eta_x+j(k_y+\eta_y)\}$$

$$j\frac{\partial\psi_0}{\partial x_t}=-k_t(\eta-k_t-\eta_t)$$

$$j\frac{\partial\psi_3}{\partial x}=-k_x\{k_x+\eta_x+j(k_y+\eta_y)\}-\frac{\partial\eta_y}{\partial x}$$

$$\frac{\partial\psi_3}{\partial y}=jk_y\{k_x+\eta_x+j(k_y+\eta_y)\}+\frac{\partial\eta_x}{\partial y}$$

$$j\frac{\partial\psi_2}{\partial z}=-k_z^2$$

(XXV-12)

The sums are left to the care of the reader. We get to the first equation:

$$(k_t+\eta_t)^2=(k_x+\eta_x)^2+(k_y+\eta_y)^2+k_z^2+\eta^2+\left(\frac{\partial\eta_y}{\partial x}-\frac{\partial\eta_x}{\partial y}\right)\qquad\text{(XXV-13)}$$

Second equation of the DIRAC system:

$$(\eta+\eta_t)\psi_1+\eta_x\psi_2+j\eta_y\psi_2-\eta_z\psi_3=j\frac{\partial\psi_1}{\partial x_t}+j\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_2}{\partial y}-j\frac{\partial\psi_3}{\partial z}$$

$$\eta_x\psi_2+j\eta_y\psi_2=\eta_xk_z+j\eta_yk_z$$

$$j\frac{\partial\psi_2}{\partial x}=-k_xk_z\qquad\text{(XXV-14)}$$

$$-\frac{\partial\psi_2}{\partial y}=-jk_yk_z$$

$$-j\frac{\partial\psi_3}{\partial z}=k_z\{k_x+\eta_x+j(k_y+\eta_y)\}$$

Third equation of the DIRAC system:

$$(\eta-\eta_t)\psi_2-\eta_x\psi_1+j\eta_y\psi_1-\eta_z\psi_0=-j\frac{\partial\psi_2}{\partial x_t}-j\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_1}{\partial y}-j\frac{\partial\psi_0}{\partial z}$$

$$(\eta-\eta_t)\psi_2=(\eta-\eta_t)k_z$$

$$-j\frac{\partial\psi_2}{\partial x_t}=k_tk_z\qquad\text{(XXV-15)}$$

$$-j\frac{\partial\psi_0}{\partial z}=k_z(\eta-k_t-\eta_t)$$

Fourth equation of the DIRAC system:

$$(\eta-\eta_t)\psi_3-\eta_x\psi_0-j\eta_y\psi_0+\eta_z\psi_1=-j\frac{\partial\psi_3}{\partial x_t}-j\frac{\partial\psi_0}{\partial x}+\frac{\partial\psi_0}{\partial y}+j\frac{\partial\psi_1}{\partial z}$$

$$(\eta-\eta_t)\psi_3-\eta_x\psi_0-j\eta_y\psi_0=(\eta-\eta_t)\{k_x+\eta_x+j(k_y+\eta_y)\}-\eta_x(\eta-k_t-\eta_t)-j\eta_y(\eta-k_t-\eta_t)$$

$$-j\frac{\partial\psi_3}{\partial x_t}=k_t\{k_x+\eta_x+j(k_y+\eta_y)\}$$

$$-j\frac{\partial\psi_0}{\partial x}=k_x(\eta-k_t-\eta_t)$$

$$\frac{\partial\psi_0}{\partial y}=jk_y(\eta-k_t-\eta_t)$$

(XXV-16)

XXVI

Elements of reflections on entanglement

Quantum entanglement is a lot of interest, both on the theoretical level, and by developing applications that seem promising.

In a general way, we can sketch the following definition: two particles which, at some point, have dependent physical properties from another in terms of energy, momentum, or angular momentum become entangled. This entanglement can take place during the creation of these particles and concrete examples are commonly implemented with pairs of photons.

Research team of Alain ASPECT has been interested very early with experimental properties of the intricate particles. The experiences implemented, which one can only underline the difficulty, the rigor and precision, helped provide experimental results of high reliability, on which we can rely in a very safe way to conduct a reflection on the robustness of any quantum theory.

The interpretation that prevails is consistent with the vision of the Copenhagen school. Application of BELL's theorem to experimental results allows to conclude with certainty that there is no hidden variables that may supplement the quantum behavior of the particles.

On the other hand, the probabilistic quantum theory provides a certain correlation between the properties of two intricate particles, correlation which is confirmed experimentally.

The uncertainty principle, in its interpretation of Copenhagen school, states that before the measurement, the state of every particle is indeterminate.

It appears only one possible conclusion: the measurement of a particle induces instantly and remote, the total correlation of the second particle. One is thus led to conclude to the no locality of quantum physics.

It is clear that such a conclusion calls into question the notion of time and space that founded the classical physics. No more phenomenon can be described with a differential equation, which represents by definition local changes in the physical properties.

One is then led to the following question: How can a theory that contains a differential equation that governs the evolution of energy in its fundamental assumptions can have as conclusion a non-local quantum physics? This is not something trivial, but rather a fundamental contradiction between a hypothesis and the conclusion of a reasoning including this hypothesis. As long as this contradiction has not been clarified by convincing explanations, there will a doubt on the way that leads from hypothesis to conclusion. We will try to provide some additional explanations on this issue later in this chapter.

I – Energy and deterministic approach

We propose to show in this paragraph that an energy and deterministic approach provides predictions consistent with the measured properties of the intricate particles.

It should be at first to try to define the properties that should allow to characterize the behavior of the intricate particles. It seems that they are essentially two:

Property 1: Experimentation shows, thanks to BELL's theorem in the Copenhagen interpretation , that there is no hidden variables that may complete the behavior of the particle.

Property 2: Experimentation shows that there is a certain correlation between the two particles that spread.

Regarding property 1, the energy approach is based on exact solutions to the Dirac equation. The wavefunctions implemented are different from those of the Copenhagen school only by a constant multiplicative factor, and there is, therefore, no hidden variable in the description of the functioning of the particle.

Regarding property 2, the modes installed during the creation of the particles spread theoretically without distortion, implying that if these particles have dependent modal properties, they will retain these properties during their displacement or spread, inducing a certain correlation between these particles.

So, with no additional postulate to classical physics, energy and deterministic approach shows in agreement with the measured results of the intricate particles.

If it turns out that other properties are necessary for the characterization of the intricate particles, then it should be ensured that these properties are well compatible with energy and deterministic approach.

II – The Copenhagen school interpretation

In view of the above, one wonders where in the reasoning, the Copenhagen interpretation imposes no locality of quantum physics.

It is acknowledged that the two particles respect the laws of classical physics during their creation: conservation of energy, momentum, and angular momentum. We must therefore admit that when they are creating, the two particles are in a certain correlation.

After propagation, the two particles are also detected in a certain correlation.

It is so by imposing an indeterminate state of the particle between its creation and its detection that the Copenhagen school induces the non-locality of quantum physics.

This undetermined state is justified by the principle of indeterminacy of HEISENBERG. Impossible to know simultaneously some properties of the particles is interpreted as an intrinsic quantum indeterminacy, which leads to assert that, during its spread, the particle is in an inherently indeterminate state.

It is on this last point that the energy and deterministic approach is fundamentally different from the Copenhagen school. A more detailed analysis requires making a return on the principle of indeterminacy.

III – Back on the principle of indeterminacy

The principle of indeterminacy is obtained by equating the particle at a point in space, the point being defined in the mathematical sense of the term, i.e. with no spatial extent.

The interpretation of the principle of indeterminacy in Copenhagen school indicates that, for this mathematical point assigned to a mass m, one can define both a position x and a momentum p, but that these two quantities are obeying the formulation of HEISENBERG uncertainty:

$$\Delta x.\Delta p_x > \frac{\hbar}{2} \qquad \text{(XXVI-1)}$$

From this relationship of uncertainty with respect to a point without spatial extension, it is deducted a general and intrinsic property of indeterminacy of quantum world.

Energy and deterministic approach proposes instead to give the particle a certain spatial extension. This vision seems more in keeping with the experimental observations. If for example, in an experiment of diffraction, the photon takes a random direction, but who gradually rebuilt the figure of diffraction, presumably wavelike energy is sensitive to the presence of the opening, which implies a certain spatial extension of the particle.

If we place at a mathematical point (without spatial extension) inside the particle, the exact solutions to the DIRAC equation shows that it is impossible to have at this point of the entire energy of mass relative to the operator position, and impulse energy. We are in agreement with the principle of HEISENBERG's indeterminacy (XXVI-1).

It doesn't lead to uncertainty of the particle position, which can be assumed to be localized in a parallelepiped rectangle (in Cartesian coordinates), with dimension, position, and speed perfectly defined. It is in this sense that the energy and deterministic approach is fundamentally different from the Copenhagen school.

We can try to illustrate this difference on a concrete example. We choose for the sake of clarity and simplification, an exact solution to one dimension (according to the z axis) of the DIRAC equation.

$$\begin{aligned}
\psi_0 &= C(\hbar\omega)\cos(k_z z)\sin(k_t x_t) + jC(m_0 c^2)\cos(k_z z)\cos(k_t x_t) \\
\psi_1 &= 0 \\
\psi_2 &= -C(\hbar c k_z)\sin(k_z z)\cos(k_t x_t) \\
\psi_3 &= 0
\end{aligned} \qquad \text{(XXVI-2)}$$

In this solution, the normalization constant C was introduced to remind that the wave function has the dimension of the square root of a linear density of energy.

If we place at a moment where the linear densities of mass energy and impulse are maximum, this solution becomes:

$$\begin{aligned}\psi_0 &= jC\left(m_0c^2\right)\cos(k_z z)\\ \psi_1 &= 0\\ \psi_2 &= -C\left(\hbar c k_z\right)\sin(k_z z)\\ \psi_3 &= 0\end{aligned} \qquad \text{(XXVI-3)}$$

The linear densities of mass energy ΔE_{masse} and impulse energy $\Delta E_{impulsionnelle}$ are written respectively

$$\begin{aligned}\Delta E_{masse} &= C^2\left(m_0c^2\right)^2\cos^2(k_z z)\\ \Delta E_{impulsionnelle} &= C^2\left(\hbar c k_z\right)^2\sin^2(k_z z)\end{aligned} \qquad \text{(XXVI-4)}$$

If it gives the particle a $2L_z$ dimension, and a spatial location between - L_z and + L_z, and if it is assumed it is excited on a fundamental mode, then the energy densities become:

$$\begin{aligned}\Delta E_{masse} &= C^2\left(m_0c^2\right)^2\cos^2\left(\frac{\pi}{2L_z}z\right)\\ \Delta E_{impulsionnelle} &= C^2\left(\hbar c k_z\right)^2\sin^2\left(\frac{\pi}{2L_z}z\right)\end{aligned} \qquad \text{(XXVI-5)}$$

We can give the following graphical representation:

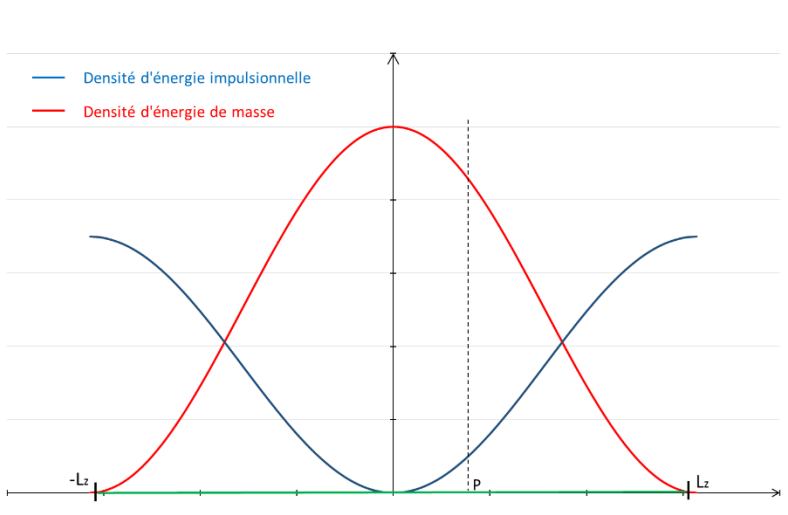


Figure (XXVI-1) : *Representation of the spatial extension of a particle (in green), with its power energy density of impulse and mass.*

The spatial extension of the particle is represented in green. When the particle is moving, the classical physics admits that it is generally all the spatial extension which moves.

While the position and speed of this particle obey classical physics, if we place ourselves in an internal observation point P to the particle, it is impossible to get the full information to the mass energy and impulse energy simultaneously.

If we try to formalize the same representation in Copenhagen school, we faced a difficulty. In this formalism, the equation of evolution of the wave function is given by the SCHRÖDINGER equation. But this equation does not distinguish the share of mass energy and

the share of impulse energy inside the particle. It provides a global representation which is condensed in a point of the space where the particle without spatial extension is supposed to be localized.

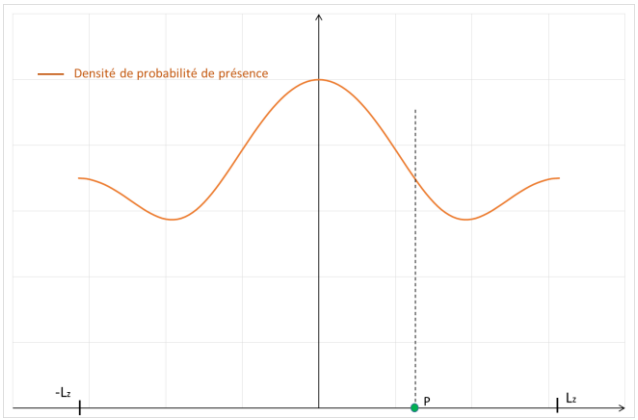


Figure (XXVI-2): *Representation of a point wise particle (in green), with its density of probability of presence which corresponds to the whole energy density in the energy approach.*

Since we can't determine the share of energy of mass and energy impulse at this point, this leads to assign to this point wise particle and without spatial extension, uncertainty about its position and its pulse.

In this comparison, the representation of the Copenhagen school, which admits the SCHRÖDINGER equation as one of its founding assumptions, appears in trouble.

From a physical point of view, it seems unrealistic to represent a particle by a point without spatial dimension. The classical mechanics allows this assimilation only in the case of a spherical symmetry, which is not the case of quantum particles. It is also very difficult to define a kinetic moment for a strictly point particle.

Finally, a point wise representation forbids any investigative approach to seek to understand the inner workings of a quantum particle.

DIRAC equation and its stationary solutions do not suffer from these limitations, because they implicitly assume that the particle has a certain spatial dimension. It also checks a relativistic invariance that lacks SCHRÖDINGER equation. DIRAC equation therefore has advantages that predispose it much more than SCHRÖDINGER equation to establish a theoretical approach to quantum mechanics.

XXVII

Deterministic approach of the diffraction and interference of particles

When, in a physics experiment appear characteristic observations of a diffraction or interference, you can consider with certainty that there is a wave phenomenon associated with this experience. In this way, DAVISSON and GERMER were able to confirm Louis DE BROGLIE hypothesis which associates to each particle of mass m and speed v, a matter wave of wavelength $\lambda = h/mv$.

The electron and photon that can be considered as a particle, it is possible to send these particles on a slot of width Δx one by one. We can see that the trajectory of the particle is unpredictable at the exit of the slot: the particle can go in all directions after the crossing of the opening.

For an important dimension of slot, the particles pass through with very little change of direction. When the dimension of the slit is reduced, the lobe of diffraction increases, and particles can go in very different directions.

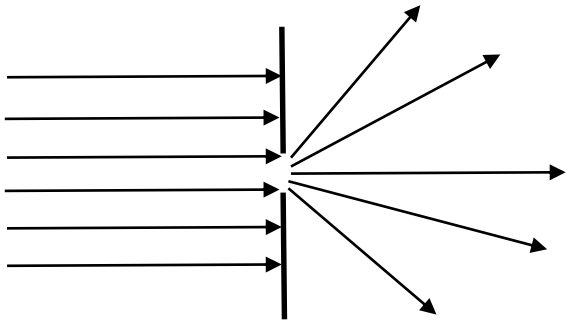


Figure XXVII-1 : *Diffraction of a beam of particles through a slit of width Δx.*

The Copenhagen school associate a confirmation of the principle of indeterminacy of HEISENBERG in the following way:

- More we try to identify the position of the particle by reducing the dimension of the slot Δx, more we impose uncertainty on the component Δv_x of speed depending on x.
- More we have uncertainty about the position of the particle Δx by increasing the size of the slot, and more uncertainty on the component of Δv_x speed is low: the path of the particles can be only weakly diverging from the original path.

It is a convincing argument that accredits strongly the idea of intrinsic uncertainty of the quantum method, and which is commonly used to illustrate the principle of indeterminacy. We can however note that the probabilistic explanation described above is valid only for the main

lobe of diffraction. It does not explain why zeros and lateral side lobes appear in the figure of complete diffraction.

When there are questions about the possibility of a deterministic quantum physics, it is necessary to think about other possible interpretations of these phenomena, and to search for deterministic explanations that we would at least as compelling as those of the Copenhagen school.

Until recently (2005), no known deterministic phenomenon was comparable to that which is observed during the diffraction or interference of individual quantum particles. It will be the work of the team of Yves COUDER on walkers droplets to prove that a particle in symbiosis with a wave can have behavior that presents major analogies with the diffraction and interference of quantum particles. The experiments on these macroscopic nature droplets provide the certainty that it is not necessary to introduce a principle of indeterminacy to provide an explanation for the phenomena of interference and diffractions of quantum particles.

Let us remember in a few words how these walkers droplets are obtained. A container of silicone oil is set in vertical vibration at a frequency f by a suitable device. When the right conditions of frequency and amplitude of vibration of the device are established, it can be created on the surface of the liquid droplets bouncing indefinitely, and that can get running spontaneously in moving on the surface of the liquid. More details can be found in the references to this chapter at the end of this document.

The resulting droplets exist symbiotically with the wave they generate: the disappearance of one results in the loss of the other. From a physical point of view, we can describe this symbiosis by a permanent exchange between the mechanical energy in the droplet and wave energy which is visible on the surface of the oil bath. The energy required for these exchanges is provided by the vibrating device.

The description of the operation of this device suggests already some analogies with the energy approach of this document. But these analogies can be pushed much more, particularly in the area of diffraction and interference that interests us in this chapter. The article by Y. COUDER and E. FORT 'Single-Particle Diffraction and Interference at a search Scale' is devoted entirely to the study of these phenomena associated with the walkers droplets. All of the results discussed later in this chapter are taken from this publication.

The diffraction results are as follows: Walker droplet is created in normal impact compared to a slot towards it travels (Figure 1A). This droplet through the slot, but there is a remarkable phenomenon: after the crossing of the slot, the droplet takes a direction that seems random (figure 2A and 2B of the publication reproduced below).

This random direction cannot be attributed to an inherent indeterminacy of the phenomena studied without denying completely the classical physics. So it has a deterministic cause we will try to identify further later in this chapter.

The idea of the authors of the publication is to perform a statistical count N(α) of particles that go in each direction α of space after going through the slot, to the manner in which we can perform a count photons in a phenomenon of diffraction.

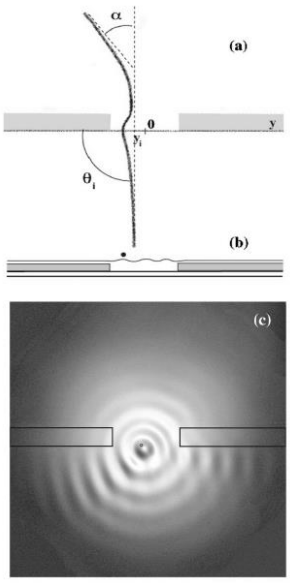


FIG 1

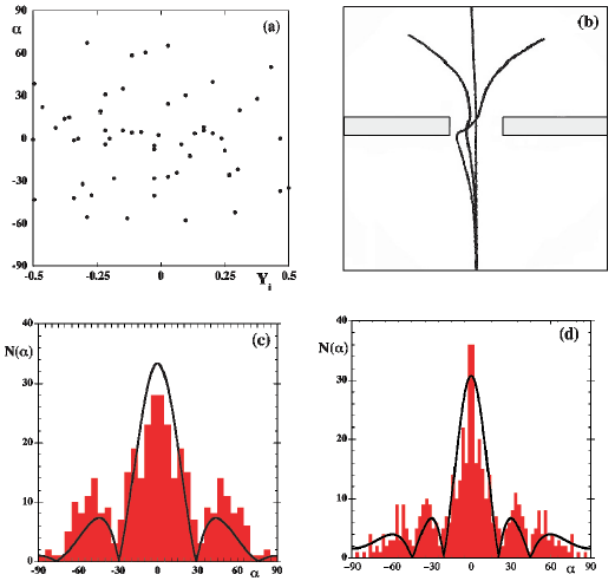


FIG 2

Source : PRL **97,** 154101-2 (13 OCTOBER 2006)

The result reported in figures 2c and 2d lead convincingly towards interpretation associated with a diffraction of an undulatory phenomenon.

If there was a doubt about the fact that these figures are obtained without any intrinsic uncertainty to the physics of the phenomenon, the authors eliminate dfinitively this possibility by a simplified numerical simulation of the phenomenon which confirms that the droplets build one by one the figure of diffraction of an undulatory phenomenon.

But then, how can we explain that the droplets are 'diffracted' one by one, in any direction, but with a certain probability? and if this probability is not linked to an intrinsic uncertainty of the phenomenon, where is the origin?

The movement of the droplet is influenced by any change in the wave associated with it. In particular, during the approach of the slot, there is a diffracted wave coming to change the incident wave and therefore modify the path. But this change of trajectory induced in turn, when the drop falls, a modification of the generated wave, and therefore of the new wave diffracted. These phenomena are cumulative on several jumps of the droplet. This relationship between the droplet and its associated wave, both recursive and cumulative, shows that we have all the ingredients leading to a chaotic phenomenon. The reflections of this paragraph are not evidence of this evolution toward chaos, but a track that seems consistent with what is observed around the diffraction and interference of the walkers droplets.

Especially, it may be noted in figure 2A, at the experimental uncertainty, that droplets coming into the slot with an identical offset y_i are likely to cross the slot with a totally different exit angle. This phenomenon can be interpreted as an infinite sensitivity to initial conditions, which is known as typical of chaotic phenomena.

One of the indisputable contributions of Y. COUDER team studies lies in the fact that we can observe and record these chaotic trajectories. Several examples of droplets who "seek" their way during a diffraction by two slots are given in a video given in reference. It is remarkable how they choose one of the two openings, guided by the associated wave, which it, interfered through two openings at once.

The output of these chaotic trajectories is done randomly, but with a different probability in every direction of space. The essential fact is that this probability is no more related to an intrinsic uncertainty of phenomena which one can provide no explanation. It can be interpreted as the output of a chaotic phenomenon which is infinitely sensitive to initial conditions, and is generally of complexity such as attractors only will make sense to the output path.

Energy and deterministic approach proposed in this document present two important analogies with the experiments on the walkers droplets.

It requires energy support that is in case the energy of the vacuum, and in the other the vibrant energy of the tray which supports the container filled with oil.

It is based on exchanges of energy between mass energy, impulse, and wave energy. Walkers droplets exchange of gravitational potential energy, kinetic energy and wave energy.

It has also some differences: particularly in a case we got a stationary wave, while the other considered essentially progressive waves.

However, deterministic explanations that have been proposed to explain the diffraction of the walkers droplets, can be reproduced identically to justify the diffraction of the quantum particle.

Robert BRADY and Ross ANDERSON article given as reference goes even further in this direction, since it suggests in a simplified energy approach, that the phenomenon of the walkers droplets can be described by a SCHRÖDINGER or KLEIN-GORDON equation analogous to quantum mechanics.

An analysis in this way opens the door on a possible deterministic explanation of diffraction and interference of the quantum particles, door that was previously locked, double-locked by the Copenhagen school in a probabilistic approach.

XXVIII

Can the DIRAC equation be admitted as a founder equation of quantum mechanics?

Quantum mechanics which is built for almost a century differs significantly from classical physics. Under the leadership of Max BORN, the description of quantum phenomena took an essentially probabilistic way that makes it incompatible with the deterministic vision of classical physics.

This new vision has led to a deeper reflection on the way in which we could give it a consistent base. This consistency is needed from an internal point of view in the quantum world, and on the other hand in the transition to the macroscopic world, since in this approach, the two worlds are governed by different laws.

This base of coherence has been defined using six postulates that are recalled, in a very general manner and without going into the details, in the following lines:

Postulate 1 : Definition of the quantum state

Knowledge of the State of a quantum system is completely contained, at time t, in a normalisable vector of the States space H. It is usually noted in the form of a ket $|\psi(t)>$

Postulate 2 : Principle of correspondence

Any observable property, for example the position, or energy, or the spin corresponds to a linear hermitian operator acting on the Hilbert space vectors H. This operator is named observable.

Postulate 3 : Measure: possible values of an observable

The measurement of a physical quantity represented by the observable can only provide one of the eigenvalues of A.

Postulate 4 : BORN postulate: probabilistic interpretation of the wave function

The measurement of a physical quantity represented by the observable, made on the standard quantum state $|\psi(t)>$ *, gives the result* a_n*, with the probability* P_n *equal to* $|c^n|^2$.

Postulate 5 : Measure: reduction of the wave packet; getting a unique value; projection of the quantum state

If the measure of the physical quantity A, at time t, on a system that is represented by the vector $|\psi(t)>$ *gives as result the eigenvalue* a_n*, then the State of the system immediately after the measurement is projected onto own subspace associated with* a_n.

Postulate 6 : Time quantum state evolution

State |ψ(t)> of any no-relativist quantum System is a solution of time dependent Schrödinger equation.

The data of these six postulates helped provide a framework in which quantum physics was able to develop and explain most of the phenomena observed and measured. The success achieved by the predictions of this theory led physicists to adopt it as the best representation of the quantum world, without however lead to a full and complete satisfaction.

For what reasons?

Essentially because the representation of Copenhagen leaves in the shadow part of the physical phenomena of the quantum world where it does not provide a satisfactory explanation. This situation puts the physicists in an uncomfortable position, which is to say that there is a theory can have a very high precision in some predictions, but unable to provide a coherent explanation for phenomena as basic as wave-corpuscle duality.

This discomfort is palpable in some situations, which for example, dealing with the intricate particles.

We have seen that when we adopt the vision of Copenhagen school unconditionally and without a doubt on its universal scope, we are led to interpret the phenomenon of entanglement as a non-local phenomenon, and so is another physics than that based on our current knowledge. A more rigorous reasoning would pose the problem in these terms:

- Either we admit the universal scope of the postulates of the Copenhagen school, and so entanglement leads to physics that have nothing to do with the classical physics.
- Either we admit that the phenomenon of entanglement is part of classical physics, and we should wonder about the scope and limits of the postulates of the Copenhagen school.

The second hypothesis is never discussed. The postulate n°6 of the Copenhagen school yet has weaknesses that questions us because this postulate governs the energy evolution of the quantum world. We recall that the disability of one of the postulates leads at least to a partial invalidity of the theory.

I - The assets of the assumption n° 6

If SCHRÖDINGER equation was able to acquire the status of a postulate, it's because it has assets that have inspired confidence to physicists.

In the first place, in an electrostatic potential in spherical symmetry, it allows to find with great precision the various series (BALMER, LYMAN, PASCHEN) of the emission lines of the hydrogen atom, and the RYDBERG constant

Time dependent SCHRÖDINGER equation takes the form:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi+E_p\psi=\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2+E_p\right)\psi=\hat{H}\psi$$

where E_p means the potential energy and $\hat{H}$ the Hamiltonian operator, which is associated with the total energy of the particle or system.

In this representation, the left term can be associated with the variation of the wave function $d\psi$ during an elementary period dt.

The term on the right indicates that this evolution is also provided by a Hamiltonian operator applied to the wave function.

It is concluded that the temporal variation of the wave function is described only based on the total energy of the particle or system.

From the point of view of physics, this description is fully satisfactory, and can lead to admit the SCHRÖDINGER equation as the equation that represents the energy evolution of quantum systems.

II - The weaknesses of postulate N° 6

A postulate is admitted and no demonstrated ownership by definition. In this, a postulate is always a weak link in a theory, because it is always likely to be questioned.

For this property to be admitted as founding, it must have obvious physical nature that makes no doubt in any of the field on which it extends.

The postulate N° 6 based interpretation of the Copenhagen school on the SCHRÖDINGER equation. However, it is known that this equation is in trouble on many elements in the description of the quantum world:

- This equation does not, in a situation that is extremely simple, give the good emission lines of the fine structure of the hydrogen atom.

- It is admitted that a necessary condition for the validity of a physical law is its invariance by changing frame. The SCHRÖDINGER equation does not have this property

- The spin cannot be properly described by the postulate N° 6. Yet, it is a fundamental property of quantum particles.

On the simple basis of the three previous observations, one wonders how this postulate can claim universal validity in the quantum world, and how we can use unquestionably SCHRÖDINGER equation to predict all of the properties of quantum particles. There is, at the very least, some properties that it is unable to predict.

III – DIRAC equation

In view of the weaknesses of the postulate N° 6 which is based on the SCHRÖDINGER equation, one would be tempted to substitute the DIRAC equation. Indeed, the latter provides the correct solutions to the fine structure of the hydrogen atom, it is invariant under the LORENTZ transformation and it correctly describes the spin of the electron.

The Copenhagen school cannot implement this substitution because it gives to the wave-squared function the significance of volume density of probability of presence. Since the solutions of the DIRAC equation have 4 terms grouped in the form of 2-spinors, it becomes impossible to associate a physical meaning of probability of presence to each of the terms.

The Copenhagen school is thus installed in this curious behavior: it accepts and uses the DIRAC equation, which has a general level of description of the quantum world superior to the SCHRÖDINGER equation, but without promoting it to the rank of founding equation of quantum mechanics because it can't give probabilistic physical meaning to each of its terms.

When we adopt an energy point of view, we better identifies the respective contributions of these two equations.

The SCHRÖDINGER equation is correct to describe all the phenomena that are the result of the only implementation of the total energy of the particle. This total energy is directly linked to the probability of presence of the particle by the Copenhagen school.

The DIRAC equation is more subtle, as it distinguished the share of mass energy and the share of impulse energy in the total energy. It allows a finer analysis of the phenomena, but makes it impossible for the assimilation of the squared wave function to a volume density of probability of presence of the particle.

We can summarize the situation in the following way: the DIRAC equation cannot be admitted by the Copenhagen school as a fundamental equation of quantum physics, because compliance of postulates 1 to 5 with this equation poses insurmountable problems in a probabilistic approach.

There are then two possible attitudes:

Either we keep postulates 1 to 5 and one rejects the DIRAC equation of the founding assumptions without justification on its rejection: it is the choice of the school of Copenhagen,

Either we admit that an equation invariant under the Lorentz transformation presents physical safeguards over and above an equation that does not have this invariance (SCHRÖDINGER equation), and we look at the consequences of this choice on the postulates 1 to 5.

This is the second path we choose to explore in the next paragraph.

IV - Can we define a founding base of quantum mechanics on the basis of the DIRAC equation?

Is it possible to promote the DIRAC equation to the rank of fundamental equation of quantum mechanics, in replacement of the SCHRÖDINGER equation and under what conditions?

On the basis of exact solutions to the DIRAC equation in the form of standing waves, the quantum state of a particle can be defined by the knowledge of its energy state at each moment.

The concept of observable property, such as for example the position or momentum, requires no special precautions, since these properties are present within the particle, but in different places. The HEISENBERG uncertainty principle is not questioned, but it must have an energy interpretation which is not probabilistic.

The concept of measure should be redefined in terms of energy exchanges. A measurement on a quantum object N° 1, is to interact this quantum object N° 1 with another quantum object N° 2 which has known properties, and deduct from measured results of the interaction, information related to the quantum object N° 1 at the time of the interaction.

A concrete example is suggested by the COMPTON effect.

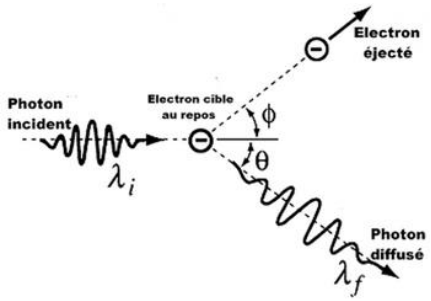


FIG XXVIII-1 : *representation of the COMPTON effect*

If we suppose known the direction of the incident photon and its wavelength λ_i, the direction of the diffuse photon and its wavelength λ_f, then conservation of energy of the classical mechanics rules allows to know the position of the electron during the interaction, as well as the speed and direction in which it is ejected. This is an ordinary way in classical physic.

It should also be noted the delicate concept of reduction of the wave packet and decoherence.

For the Copenhagen school, the reduction of the wave packet stipulates that after a measure, a physical system sees its condition entirely reduced to that which has been measured. This feature is made necessary because of the indeterminacy of the quantum world which assumes the superposition of an infinite number of quantum state before the act of measurement. This notion is not useful in an energy and deterministic approach where the quantum state is supposed to be perfectly known at every time.

Quantum decoherence is a theory to explain the transition between the physical quantum and classical physical rules as we know them, at a macroscopic level. Since energy and deterministic approach relies only on the postulates of the classical physics, the problem of decoherence is not useful in this approach.

It appears that no additional assumption is needed to move from classical physics to the deterministic quantum physics deducted from the DIRAC equation: the transition is carried out using the single postulate of conservation of energy.

XXIX

On the invariance of the laws of physics by change of frame

Among the arguments that argue in favour of the introduction of the DIRAC equation as a fundamental equation of quantum physics, there is that of its invariance by change of frame.

It is an extremely strong argument that express the absolute consistency between what is known about relativity, and all that can be inferred from the DIRAC equation.

Let us consider two frames we shall refer to by (R) and (R'). The frame (R') is presumed to be in translational movement at constant speed compared to the frame (R).

The coordinates of time and space of the frame (R') are connected at the frame (R) space-time coordinates by a LORENTZ transformation.

The bi-spinor ψ that represents the state of the particle in the frame (R) is represented in the frame (R') by another bi-spinor ψ'.

The invariance of the DIRAC equation by frame change express the fact that there is the same relationship between the components of the bi-spinor ψ in the frame (R), as between the components of the bi-spinor ψ' in the frame (R'). These two relationships are provided by the DIRAC equation expressed in the frame (R), and the DIRAC equation expressed in the frame (R'):

$$\begin{aligned}&\left(j\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}-\frac{m_0 c}{\hbar}\right)\psi\left(x^{\mu}\right)=0\\&\left(j\gamma^{\mu}\frac{\partial}{\partial x'^{\mu}}-\frac{m_0 c}{\hbar}\right)\psi'\left(x'^{\mu}\right)=0\end{aligned}\qquad\text{(XXIX-1)}$$

We propose to show how this invariance is verified, in detailing the heart of the DIRAC system. It is essentially an educational exercise, because a purely mathematical demonstration may be obtained more quickly by tensorial analysis. Its justification lies in the fact that extremely concise writing of tensorial analysis masks elements the physicist needs to his detailed understanding of phenomena.

I –LORENTZ transformation

We will work with the classic transformation where two frames (R) and (R') are in translation at constant speed along the x axis. In these conditions, the LORENTZ transformation is written:

$$dx_t' = \frac{dx_t - \frac{v}{c}dx}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\left(dx_t - \beta dx\right)$$

$$dx' = \frac{dx - \frac{v}{c}dx_t}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\left(dx - \beta dx_t\right) \qquad \text{(XXIX-2)}$$

$$dy' = dy$$

$$dz' = dz$$

We put, in usual notation:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{et} \quad \beta = \frac{v}{c} \qquad \text{(XXIX-3)}$$

Which leads to a more condensed matrix writing:

$$\begin{pmatrix} dx_t' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx_t \\ dx \\ dy \\ dz \end{pmatrix} = (\Lambda) \begin{pmatrix} dx_t \\ dx \\ dy \\ dz \end{pmatrix} \qquad \text{(XXIX-4)}$$

We deduce relations between the partial derivatives:

$$\frac{\partial}{\partial x_t} = \gamma \frac{\partial}{\partial x_t'} - \beta\gamma \frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial x} = -\beta\gamma \frac{\partial}{\partial x_t'} + \gamma \frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

or still:

$$\begin{pmatrix} \frac{\partial}{\partial x_t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = (\Lambda) \begin{pmatrix} \frac{\partial}{\partial x_t'} \\ \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix} \qquad \text{(XXIX-5)}$$

As well as the reciprocal relations:

$$\frac{\partial}{\partial x_t'} = \gamma\frac{\partial}{\partial x_t} + \beta\gamma\frac{\partial}{\partial x}$$
$$\frac{\partial}{\partial x'} = \beta\gamma\frac{\partial}{\partial x_t} + \gamma\frac{\partial}{\partial x}$$
$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

or still:
$$\begin{pmatrix} \frac{\partial}{\partial x_t'} \\ \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix} = \left(\Lambda^{-1}\right)\begin{pmatrix} \frac{\partial}{\partial x_t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \qquad \text{(XXIX-6)}$$

We recall that the determinant of the matrix Λ is equal to 1:

$$\mathrm{Det}(\Lambda) = \gamma^2 - \beta^2\gamma^2 = \gamma^2\left(1-\beta^2\right) = 1 \qquad \text{(XXIX-7)}$$

II – The transformation of the bi-spinor

We can consider that it is the main difficulty in the search for an invariant formulation of the DIRAC equation.

To give a meaning to this research, we admit that the components of the transformed bi-spinor ψ' of (R') are necessarily written as a linear combination of the components of the bi-spinor ψ of (R), which leads to the determination of a matrix S with constant coefficients:

$$\left(\psi'\right) = \begin{pmatrix} \psi_0' \\ \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} S_{00} & S_{01} & S_{02} & S_{03} \\ S_{10} & S_{11} & S_{12} & S_{13} \\ S_{20} & S_{21} & S_{22} & S_{23} \\ S_{30} & S_{31} & S_{32} & S_{33} \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} S_{00}\psi_0 + S_{01}\psi_1 + S_{02}\psi_2 + S_{03}\psi_3 \\ S_{10}\psi_0 + S_{11}\psi_1 + S_{12}\psi_2 + S_{13}\psi_3 \\ S_{20}\psi_0 + S_{21}\psi_1 + S_{22}\psi_2 + S_{23}\psi_3 \\ S_{30}\psi_0 + S_{31}\psi_1 + S_{32}\psi_2 + S_{33}\psi_3 \end{pmatrix} \qquad \text{(XXIX-8)}$$

We will temporarily admit (the demonstration is given later in this chapter), that for the LORENTZ transformation (XXIX-2, 3, 4), the shape of the S matrix is as follows:

$$S = \begin{pmatrix} S_{00} & S_{01} & S_{02} & S_{03} \\ S_{10} & S_{11} & S_{12} & S_{13} \\ S_{20} & S_{21} & S_{22} & S_{23} \\ S_{30} & S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} \gamma+1 & 0 & 0 & -\beta\gamma \\ 0 & \gamma+1 & -\beta\gamma & 0 \\ 0 & -\beta\gamma & \gamma+1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma+1 \end{pmatrix} \qquad \text{(XXIX-9)}$$

We can deduce relationships between the components of the bi-spinor ψ and the components of the bi-spinor ψ':

$$\left(\psi'\right) = \begin{pmatrix} \psi_0' \\ \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} \gamma+1 & 0 & 0 & -\beta\gamma \\ 0 & \gamma+1 & -\beta\gamma & 0 \\ 0 & -\beta\gamma & \gamma+1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma+1 \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} (\gamma+1)\psi_0 - \beta\gamma\psi_3 \\ (\gamma+1)\psi_1 - \beta\gamma\psi_2 \\ -\beta\gamma\psi_1 + (\gamma+1)\psi_2 \\ -\beta\gamma\psi_0 + (\gamma+1)\psi_3 \end{pmatrix} \qquad \text{(XXIX-10)}$$

One can check that in this formulation, the two spinor ψ' appear as a linear combination of the two spinor ψ.

$$\begin{pmatrix}\psi_0'\\ \psi_1'\end{pmatrix}=(\gamma+1)\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}-\beta\gamma\begin{pmatrix}\psi_3\\ \psi_2\end{pmatrix}$$
$$\begin{pmatrix}\psi_2'\\ \psi_3'\end{pmatrix}=-\beta\gamma\begin{pmatrix}\psi_1\\ \psi_0\end{pmatrix}+(\gamma+1)\begin{pmatrix}\psi_2\\ \psi_3\end{pmatrix} \qquad \text{(XXIX-11)}$$

There is an inversion of the order of the components of certain spinors, without consequence on the fact that it is an internal transformation to the spinor space.

With this transformation, and the transformation of partial differential related to the LORENTZ transformation, we have all the elements to verify the invariance of the DIRAC system by changing frame.

III – Invariance of the DIRAC system by changing frame

We adopt as a starting point, the DIRAC equation developed in the frame (R'):

$$\begin{aligned}
\eta\psi_0' &= j\frac{\partial\psi_0'}{\partial x_t'}+j\frac{\partial\psi_3'}{\partial x'}+\frac{\partial\psi_3'}{\partial y'}+j\frac{\partial\psi_2'}{\partial z'}\\
\eta\psi_1' &= j\frac{\partial\psi_1'}{\partial x_t'}+j\frac{\partial\psi_2'}{\partial x'}-\frac{\partial\psi_2'}{\partial y'}-j\frac{\partial\psi_3'}{\partial z'}\\
\eta\psi_2' &= -j\frac{\partial\psi_2'}{\partial x_t'}-j\frac{\partial\psi_1'}{\partial x'}-\frac{\partial\psi_1'}{\partial y'}-j\frac{\partial\psi_0'}{\partial z'}\\
\eta\psi_3' &= -j\frac{\partial\psi_3'}{\partial x_t'}-j\frac{\partial\psi_0'}{\partial x'}+\frac{\partial\psi_0'}{\partial y'}+j\frac{\partial\psi_1'}{\partial z'}
\end{aligned} \qquad \text{(XXIX-12)}$$

In a first step, we perform the transformation of the wavefunctions of the frame (R') based on the wavefunctions of the frame (R), using the relationship (XXIX-10):

$$\eta\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}=\left\{\begin{array}{l} j\dfrac{\partial\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}}{\partial x_t'}+j\dfrac{\partial\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}}{\partial x'} \\ +\dfrac{\partial\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}}{\partial y'}+j\dfrac{\partial\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}}{\partial z'} \end{array}\right\}$$

$$\eta\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}=\left\{\begin{array}{l} j\dfrac{\partial\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}}{\partial x_t'}+j\dfrac{\partial\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}}{\partial x'} \\ -\dfrac{\partial\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}}{\partial y'}-j\dfrac{\partial\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}}{\partial z'} \end{array}\right\}$$

$$\eta\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}=\left\{\begin{array}{l} -j\dfrac{\partial\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}}{\partial x_t'}-j\dfrac{\partial\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}}{\partial x'} \\ -\dfrac{\partial\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}}{\partial y'}-j\dfrac{\partial\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}}{\partial z'} \end{array}\right\}$$

$$\eta\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}=\left\{\begin{array}{l} -j\dfrac{\partial\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}}{\partial x_t'}-j\dfrac{\partial\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}}{\partial x'} \\ +\dfrac{\partial\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}}{\partial y'}+j\dfrac{\partial\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}}{\partial z'} \end{array}\right\} \quad \text{(XXIX-13)}$$

In a second step, we replace the partial derivatives of the frame (R') by their expression based on the partial derivatives of the frame (R) using the relationship (XXIX-6):

$$\eta\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}=\left\{\begin{array}{l}j\left(\gamma\dfrac{\partial}{\partial x_t}+\beta\gamma\dfrac{\partial}{\partial x}\right)\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}\\+j\left(\beta\gamma\dfrac{\partial}{\partial x_t}+\gamma\dfrac{\partial}{\partial x}\right)\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}\\+\dfrac{\partial\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}}{\partial y}+j\dfrac{\partial\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}}{\partial z}\end{array}\right\}$$

$$\eta\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}=\left\{\begin{array}{l}j\left(\gamma\dfrac{\partial}{\partial x_t}+\beta\gamma\dfrac{\partial}{\partial x}\right)\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}\\+j\left(\beta\gamma\dfrac{\partial}{\partial x_t}+\gamma\dfrac{\partial}{\partial x}\right)\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}\\-\dfrac{\partial\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}}{\partial y}-j\dfrac{\partial\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}}{\partial z}\end{array}\right\}$$

$$\eta\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}=\left\{\begin{array}{l}-j\left(\gamma\dfrac{\partial}{\partial x_t}+\beta\gamma\dfrac{\partial}{\partial x}\right)\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}\\-j\left(\beta\gamma\dfrac{\partial}{\partial x_t}+\gamma\dfrac{\partial}{\partial x}\right)\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}\\-\dfrac{\partial\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}}{\partial y}-j\dfrac{\partial\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}}{\partial z}\end{array}\right\}$$

$$\eta\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}=\left\{\begin{array}{l}-j\left(\gamma\dfrac{\partial}{\partial x_t}+\beta\gamma\dfrac{\partial}{\partial x}\right)\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}\\-j\left(\beta\gamma\dfrac{\partial}{\partial x_t}+\gamma\dfrac{\partial}{\partial x}\right)\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}\\+\dfrac{\partial\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}}{\partial y}+j\dfrac{\partial\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}}{\partial z}\end{array}\right\}$$

(XXIX-14)

In the third step, we develop the differential expressions:

$$\eta\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}=\left\{\begin{array}{l}\mathrm{j}\left(\gamma\dfrac{\partial(\gamma+1)\psi_0}{\partial x_t}+\beta\gamma\dfrac{\partial(\gamma+1)\psi_0}{\partial x}\right)+\mathrm{j}\left(-\gamma\dfrac{\partial\beta\gamma\psi_3}{\partial x_t}-\beta\gamma\dfrac{\partial\beta\gamma\psi_3}{\partial x}\right)\\+\mathrm{j}\left(-\beta\gamma\dfrac{\partial\beta\gamma\psi_0}{\partial x_t}-\gamma\dfrac{\partial\beta\gamma\psi_0}{\partial x}\right)+\mathrm{j}\left(\beta\gamma\dfrac{\partial(\gamma+1)\psi_3}{\partial x_t}+\gamma\dfrac{\partial(\gamma+1)\psi_3}{\partial x}\right)\\+\dfrac{\partial\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}}{\partial y}+\mathrm{j}\dfrac{\partial\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}}{\partial z}\end{array}\right\}$$

$$\eta\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}=\left\{\begin{array}{l}\mathrm{j}\left(\gamma\dfrac{\partial(\gamma+1)\psi_1}{\partial x_t}+\beta\gamma\dfrac{\partial(\gamma+1)\psi_1}{\partial x}\right)+\mathrm{j}\left(-\gamma\dfrac{\partial\beta\gamma\psi_2}{\partial x_t}-\beta\gamma\dfrac{\partial\beta\gamma\psi_2}{\partial x}\right)\\+\mathrm{j}\left(-\beta\gamma\dfrac{\partial\beta\gamma\psi_1}{\partial x_t}-\gamma\dfrac{\partial\beta\gamma\psi_1}{\partial x}\right)+\mathrm{j}\left(\beta\gamma\dfrac{\partial(\gamma+1)\psi_2}{\partial x_t}+\gamma\dfrac{\partial(\gamma+1)\psi_2}{\partial x}\right)\\-\dfrac{\partial\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}}{\partial y}-\mathrm{j}\dfrac{\partial\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}}{\partial z}\end{array}\right\}$$

$$\eta\{-\beta\gamma\psi_1+(\gamma+1)\psi_2\}=\left\{\begin{array}{l}-\mathrm{j}\left(-\gamma\dfrac{\partial\beta\gamma\psi_1}{\partial x_t}-\beta\gamma\dfrac{\partial\beta\gamma\psi_1}{\partial x}\right)-\mathrm{j}\left(\gamma\dfrac{\partial(\gamma+1)\psi_2}{\partial x_t}+\beta\gamma\dfrac{\partial(\gamma+1)\psi_2}{\partial x}\right)\\-\mathrm{j}\left(\beta\gamma\dfrac{\partial(\gamma+1)\psi_1}{\partial x_t}+\gamma\dfrac{\partial(\gamma+1)\psi_1}{\partial x}\right)-\mathrm{j}\left(-\beta\gamma\dfrac{\partial\beta\gamma\psi_2}{\partial x_t}-\gamma\dfrac{\partial\beta\gamma\psi_2}{\partial x}\right)\\-\dfrac{\partial\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}}{\partial y}-\mathrm{j}\dfrac{\partial\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}}{\partial z}\end{array}\right\}$$

$$\eta\{-\beta\gamma\psi_0+(\gamma+1)\psi_3\}=\left\{\begin{array}{l}-\mathrm{j}\left(-\gamma\dfrac{\partial\beta\gamma\psi_0}{\partial x_t}-\beta\gamma\dfrac{\partial\beta\gamma\psi_0}{\partial x}\right)-\mathrm{j}\left(\gamma\dfrac{\partial(\gamma+1)\psi_3}{\partial x_t}+\beta\gamma\dfrac{\partial(\gamma+1)\psi_3}{\partial x}\right)\\-\mathrm{j}\left(\beta\gamma\dfrac{\partial(\gamma+1)\psi_0}{\partial x_t}+\gamma\dfrac{\partial(\gamma+1)\psi_0}{\partial x}\right)-\mathrm{j}\left(-\beta\gamma\dfrac{\partial\beta\gamma\psi_3}{\partial x_t}-\gamma\dfrac{\partial\beta\gamma\psi_3}{\partial x}\right)\\+\dfrac{\partial\{(\gamma+1)\psi_0-\beta\gamma\psi_3\}}{\partial y}+\mathrm{j}\dfrac{\partial\{(\gamma+1)\psi_1-\beta\gamma\psi_2\}}{\partial z}\end{array}\right\}$$

(XXIX-15)

Making use of the relationship given by the determinant of the passage matrix:

$$\gamma^2\left(1-\beta^2\right)=1 \qquad \text{(XXIX-16)}$$

We get by combining like terms:

$$
\eta\begin{Bmatrix}(\gamma+1)\psi_0\\ -\beta\gamma\psi_3\end{Bmatrix}=\begin{Bmatrix}j\frac{\partial(\gamma+1)\psi_0}{\partial x_t}+j\frac{\partial(\gamma+1)\psi_3}{\partial x}+\frac{\partial(\gamma+1)\psi_3}{\partial y}+j\frac{\partial(\gamma+1)\psi_2}{\partial z}\\ j\frac{\partial\beta\gamma\psi_3}{\partial x_t}+j\frac{\partial\beta\gamma\psi_0}{\partial x}-\frac{\partial\beta\gamma\psi_0}{\partial y}-j\frac{\partial\beta\gamma\psi_1}{\partial z}\end{Bmatrix}
$$

$$
\eta\begin{Bmatrix}(\gamma+1)\psi_1\\ -\beta\gamma\psi_2\end{Bmatrix}=\begin{Bmatrix}j\frac{\partial(\gamma+1)\psi_1}{\partial x_t}+j\frac{\partial(\gamma+1)\psi_2}{\partial x}-\frac{\partial(\gamma+1)\psi_2}{\partial y}-j\frac{\partial(\gamma+1)\psi_3}{\partial z}\\ j\frac{\partial\beta\gamma\psi_2}{\partial x_t}+j\frac{\partial\beta\gamma\psi_1}{\partial x}+\frac{\partial\beta\gamma\psi_1}{\partial y}+j\frac{\partial\beta\gamma\psi_0}{\partial z}\end{Bmatrix}
$$

$$
\eta\begin{Bmatrix}(\gamma+1)\psi_2\\ -\beta\gamma\psi_1\end{Bmatrix}=\begin{Bmatrix}-j\frac{\partial(\gamma+1)\psi_2}{\partial x_t}-j\frac{\partial(\gamma+1)\psi_1}{\partial x}-\frac{\partial(\gamma+1)\psi_1}{\partial y}-j\frac{\partial(\gamma+1)\psi_0}{\partial z}\\ -j\frac{\partial\beta\gamma\psi_1}{\partial x_t}-j\frac{\partial\beta\gamma\psi_2}{\partial x}+\frac{\partial\beta\gamma\psi_2}{\partial y}+j\frac{\partial\beta\gamma\psi_3}{\partial z}\end{Bmatrix}
$$

$$
\eta\begin{Bmatrix}(\gamma+1)\psi_3\\ -\beta\gamma\psi_0\end{Bmatrix}=\begin{Bmatrix}-j\frac{\partial(\gamma+1)\psi_3}{\partial x_t}-j\frac{\partial(\gamma+1)\psi_0}{\partial x}+\frac{\partial(\gamma+1)\psi_0}{\partial y}+j\frac{\partial(\gamma+1)\psi_1}{\partial z}\\ -j\frac{\partial\beta\gamma\psi_0}{\partial x_t}-j\frac{\partial\beta\gamma\psi_3}{\partial x}-\frac{\partial\beta\gamma\psi_3}{\partial y}-j\frac{\partial\beta\gamma\psi_2}{\partial z}\end{Bmatrix}
\qquad \text{(XXIX-17)}
$$

To be valid for any γ and β, this last relationship requires:

$$
\begin{aligned}
\eta\psi_0 &= j\frac{\partial\psi_0}{\partial x_t}+j\frac{\partial\psi_3}{\partial x}+\frac{\partial\psi_3}{\partial y}+j\frac{\partial\psi_2}{\partial z}\\
\eta\psi_1 &= j\frac{\partial\psi_1}{\partial x_t}+j\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_2}{\partial y}-j\frac{\partial\psi_3}{\partial z}\\
\eta\psi_2 &= -j\frac{\partial\psi_2}{\partial x_t}-j\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_1}{\partial y}-j\frac{\partial\psi_0}{\partial z}\\
\eta\psi_3 &= -j\frac{\partial\psi_3}{\partial x_t}-j\frac{\partial\psi_0}{\partial x}+\frac{\partial\psi_0}{\partial y}+j\frac{\partial\psi_1}{\partial z}
\end{aligned}
\qquad \text{(XXIX-18)}
$$

That concludes the demonstration of the invariance of the DIRAC system under LORENTZ transformation.

One might think that it is a trivial property, which could be reproduced for any transformation of the bi-spinor of DIRAC. Indeed, it isn't. The research of a matrix S defined by (XXIX-8) such as:

$$(\psi') = S\,(\psi) \qquad \text{(XXIX-19)}$$

leads to the resolution of a system of 64 equations and 16 unknowns, and which therefore puts on this matrix very strong compatibility constraints between the transformation of coordinates and the transformation of the bi-spinor.

IV – Research of the transformation of the bi-spinor S matrix

The transformation matrix S of the bi-spinor (XXIX-9) depends on the transformation of coordinates (XXIX-4) between the two frames (R) and (R').

We indicate in the following lines, how the S matrix is obtained in the case of a LORENTZ transformation between two frames in translation at constant speed along the Ox axis. The method is directly applicable to other transformations, provided that these transformations allow the invariance of the DIRAC system.

In the frame (R'), the DIRAC equation is written:

$$\left(j\gamma^{\mu} \frac{\partial}{\partial x'^{\mu}} - \frac{m_0 c}{\hbar} \right) \psi'\left(x'^{\mu}\right) = 0 \qquad \text{(XXIX-20)}$$

That is, in developed writing, where the DIRAC matrices are represented by γ^0, γ^1, γ^2, γ^3 :

$$j\gamma^0 \begin{pmatrix} \frac{\partial \psi_0'}{\partial x_t'} \\ \frac{\partial \psi_1'}{\partial x_t'} \\ \frac{\partial \psi_2'}{\partial x_t'} \\ \frac{\partial \psi_3'}{\partial x_t'} \end{pmatrix} + j\gamma^1 \begin{pmatrix} \frac{\partial \psi_0'}{\partial x'} \\ \frac{\partial \psi_1'}{\partial x'} \\ \frac{\partial \psi_2'}{\partial x'} \\ \frac{\partial \psi_3'}{\partial x'} \end{pmatrix} + j\gamma^2 \begin{pmatrix} \frac{\partial \psi_0'}{\partial y'} \\ \frac{\partial \psi_1'}{\partial y'} \\ \frac{\partial \psi_2'}{\partial y'} \\ \frac{\partial \psi_3'}{\partial y'} \end{pmatrix} + j\gamma^3 \begin{pmatrix} \frac{\partial \psi_0'}{\partial z'} \\ \frac{\partial \psi_1'}{\partial z'} \\ \frac{\partial \psi_2'}{\partial z'} \\ \frac{\partial \psi_3'}{\partial z'} \end{pmatrix} - \frac{m_0 c}{\hbar} \begin{pmatrix} \psi_0' \\ \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = 0 \qquad \text{(XXIX-21)}$$

We make the transformation of the bi-spinors between frame (R) and frame (R') by using the S matrix, and then by using the relationship:

$$(\psi') = S\,(\psi) \qquad \text{(XXIX-22)}$$

Partial derivatives through the matrix to apply to the components of the bi-spinor expressed in the frame (R):

$$j\gamma^0 S \begin{pmatrix} \frac{\partial \psi_0}{\partial x_t'} \\ \frac{\partial \psi_1}{\partial x_t'} \\ \frac{\partial \psi_2}{\partial x_t'} \\ \frac{\partial \psi_3}{\partial x_t'} \end{pmatrix} + j\gamma^1 S \begin{pmatrix} \frac{\partial \psi_0}{\partial x'} \\ \frac{\partial \psi_1}{\partial x'} \\ \frac{\partial \psi_2}{\partial x'} \\ \frac{\partial \psi_3}{\partial x'} \end{pmatrix} + j\gamma^2 S \begin{pmatrix} \frac{\partial \psi_0}{\partial y'} \\ \frac{\partial \psi_1}{\partial y'} \\ \frac{\partial \psi_2}{\partial y'} \\ \frac{\partial \psi_3}{\partial y'} \end{pmatrix} + j\gamma^3 S \begin{pmatrix} \frac{\partial \psi_0}{\partial z'} \\ \frac{\partial \psi_1}{\partial z'} \\ \frac{\partial \psi_2}{\partial z'} \\ \frac{\partial \psi_3}{\partial z'} \end{pmatrix} - \frac{m_0 c}{\hbar} S \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0 \qquad \text{(XXIX-23)}$$

The change of differential variable is made. In the particular case of the coordinate's transformation relative to a LORENTZ transformation along the Ox axis (XXIX-6), one obtains:

$$j\gamma^0 S\begin{pmatrix}\gamma\frac{\partial\psi_0}{\partial x_t}+\beta\gamma\frac{\partial\psi_0}{\partial x}\\ \gamma\frac{\partial\psi_1}{\partial x_t}+\beta\gamma\frac{\partial\psi_1}{\partial x}\\ \gamma\frac{\partial\psi_2}{\partial x_t}+\beta\gamma\frac{\partial\psi_2}{\partial x}\\ \gamma\frac{\partial\psi_3}{\partial x_t}+\beta\gamma\frac{\partial\psi_3}{\partial x}\end{pmatrix}+j\gamma^1 S\begin{pmatrix}\beta\gamma\frac{\partial\psi_0}{\partial x_t}+\gamma\frac{\partial\psi_0}{\partial x}\\ \beta\gamma\frac{\partial\psi_1}{\partial x_t}+\gamma\frac{\partial\psi_1}{\partial x}\\ \beta\gamma\frac{\partial\psi_2}{\partial x_t}+\gamma\frac{\partial\psi_2}{\partial x}\\ \beta\gamma\frac{\partial\psi_3}{\partial x_t}+\gamma\frac{\partial\psi_3}{\partial x}\end{pmatrix}+j\gamma^2 S\begin{pmatrix}\frac{\partial\psi_0}{\partial y}\\ \frac{\partial\psi_1}{\partial y}\\ \frac{\partial\psi_2}{\partial y}\\ \frac{\partial\psi_3}{\partial y}\end{pmatrix}+j\gamma^3 S\begin{pmatrix}\frac{\partial\psi_0}{\partial z}\\ \frac{\partial\psi_1}{\partial z}\\ \frac{\partial\psi_2}{\partial z}\\ \frac{\partial\psi_3}{\partial z}\end{pmatrix}-\frac{mc}{\hbar}S\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=0$$

(XXIX-24)

It then appears wise to left multiply all of the DIRAC equation by the S^{-1} matrix, inverse of S matrix:

$$jS^{-1}\gamma^0 S\begin{pmatrix}\gamma\frac{\partial\psi_0}{\partial x_t}+\beta\gamma\frac{\partial\psi_0}{\partial x}\\ \gamma\frac{\partial\psi_1}{\partial x_t}+\beta\gamma\frac{\partial\psi_1}{\partial x}\\ \gamma\frac{\partial\psi_2}{\partial x_t}+\beta\gamma\frac{\partial\psi_2}{\partial x}\\ \gamma\frac{\partial\psi_3}{\partial x_t}+\beta\gamma\frac{\partial\psi_3}{\partial x}\end{pmatrix}+jS^{-1}\gamma^1 S\begin{pmatrix}\beta\gamma\frac{\partial\psi_0}{\partial x_t}+\gamma\frac{\partial\psi_0}{\partial x}\\ \beta\gamma\frac{\partial\psi_1}{\partial x_t}+\gamma\frac{\partial\psi_1}{\partial x}\\ \beta\gamma\frac{\partial\psi_2}{\partial x_t}+\gamma\frac{\partial\psi_2}{\partial x}\\ \beta\gamma\frac{\partial\psi_3}{\partial x_t}+\gamma\frac{\partial\psi_3}{\partial x}\end{pmatrix}+jS^{-1}\gamma^2 S\begin{pmatrix}\frac{\partial\psi_0}{\partial y}\\ \frac{\partial\psi_1}{\partial y}\\ \frac{\partial\psi_2}{\partial y}\\ \frac{\partial\psi_3}{\partial y}\end{pmatrix}+jS^{-1}\gamma^3 S\begin{pmatrix}\frac{\partial\psi_0}{\partial z}\\ \frac{\partial\psi_1}{\partial z}\\ \frac{\partial\psi_2}{\partial z}\\ \frac{\partial\psi_3}{\partial z}\end{pmatrix}-\frac{mc}{\hbar}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=0$$

(XXIX-25)

We gather then each differential columns:

$$j\gamma\left\{S^{-1}\gamma^0 S+S^{-1}\gamma^1 S\beta\right\}\begin{pmatrix}\frac{\partial\psi_0}{\partial x_t}\\ \frac{\partial\psi_1}{\partial x_t}\\ \frac{\partial\psi_2}{\partial x_t}\\ \frac{\partial\psi_3}{\partial x_t}\end{pmatrix}+j\gamma\left\{S^{-1}\gamma^0 S\beta+S^{-1}\gamma^1 S\right\}\begin{pmatrix}\frac{\partial\psi_0}{\partial x}\\ \frac{\partial\psi_1}{\partial x}\\ \frac{\partial\psi_2}{\partial x}\\ \frac{\partial\psi_3}{\partial x}\end{pmatrix}$$

(XXIX-26)

$$+jS^{-1}\gamma^2 S\begin{pmatrix}\frac{\partial\psi_0}{\partial y}\\ \frac{\partial\psi_1}{\partial y}\\ \frac{\partial\psi_2}{\partial y}\\ \frac{\partial\psi_3}{\partial y}\end{pmatrix}+jS^{-1}\gamma^3 S\begin{pmatrix}\frac{\partial\psi_0}{\partial z}\\ \frac{\partial\psi_1}{\partial z}\\ \frac{\partial\psi_2}{\partial z}\\ \frac{\partial\psi_3}{\partial z}\end{pmatrix}=\frac{mc}{\hbar}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$$

We'll take care of make the difference in these expressions, between γ which designates a coefficient of the transformation of LORENTZ set at (XXIX-3), and γ^0, γ^1, γ^2, γ^3, which designate matrices of DIRAC.

The invariance of the DIRAC system will be achieved when:

$$\begin{aligned}
&\gamma\left\{S^{-1}\gamma^0 S + S^{-1}\gamma^1 S\beta\right\} = \gamma^0\\
&\gamma\left\{S^{-1}\gamma^0 S\beta + S^{-1}\gamma^1 S\right\} = \gamma^1\\
&S^{-1}\gamma^2 S = \gamma^2\\
&S^{-1}\gamma^3 S = \gamma^3
\end{aligned} \qquad \text{(XXIX-27)}$$

It appears again wise to multiply left by the transformation of the bi-spinors S matrix, to get a system of equations that contains only the terms of this matrix

$$\begin{aligned}
&\gamma\left\{\gamma^0 S + \gamma^1 S\beta\right\} = S\gamma^0\\
&\gamma\left\{\gamma^0 S\beta + \gamma^1 S\right\} = S\gamma^1\\
&\gamma^2 S = S\gamma^2\\
&\gamma^3 S = S\gamma^3
\end{aligned} \qquad \text{(XXIX-28)}$$

It is a system of 64 equations, with 16 unknowns which are the terms of the S matrix. Its writing, with the developed DIRAC matrices, is given below:

$$\begin{aligned}
&\gamma\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}S + \beta\gamma\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}S = S\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\\
&\beta\gamma\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}S + \gamma\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}S = S\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\\
&\begin{pmatrix}0&0&0&-j\\0&0&j&0\\0&j&0&0\\-j&0&0&0\end{pmatrix}S = S\begin{pmatrix}0&0&0&-j\\0&0&j&0\\0&j&0&0\\-j&0&0&0\end{pmatrix}\\
&\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}S = S\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}
\end{aligned} \qquad \text{(XXIX-29)}$$

This system, oversized compared to the number of unknown, shows that all the transformations of coordinates are not eligible so that there are solutions. The same remark applies to the transformations of the bi-spinors.

The last two relationships of the system (XXIX-28) or (XXIX-29) express, in the particular case that is processed, the matrices of DIRAC γ^2 and γ^3 must switch with the S matrix. We deduce from these 32 equations that the S matrix must be of the form:

$$S=\begin{pmatrix} S_{00} & S_{01} & S_{02} & S_{03} \\ S_{10} & S_{11} & S_{12} & S_{13} \\ S_{20} & S_{21} & S_{22} & S_{23} \\ S_{30} & S_{31} & S_{32} & S_{33} \end{pmatrix}=\begin{pmatrix} d & a & b & c \\ a & d & c & b \\ -b & c & d & -a \\ c & -b & -a & d \end{pmatrix} \qquad \text{(XXIX-30)}$$

By injecting this form of matrix in the 32 equations remaining system (XXIX-28) or (XXIX-29), we are led to see the nullity of the coefficients a and b, and we can deduce the desired matrix:

$$S=\begin{pmatrix} S_{00} & S_{01} & S_{02} & S_{03} \\ S_{10} & S_{11} & S_{12} & S_{13} \\ S_{20} & S_{21} & S_{22} & S_{23} \\ S_{30} & S_{31} & S_{32} & S_{33} \end{pmatrix}=\begin{pmatrix} \gamma+1 & 0 & 0 & -\beta\gamma \\ 0 & \gamma+1 & -\beta\gamma & 0 \\ 0 & -\beta\gamma & \gamma+1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma+1 \end{pmatrix} \qquad \text{(XXIX-31)}$$

This matrix is defined to a multiplicative constant close that we shall refer to by K (β, γ). This constant, which plays no role to establish the invariance of the DIRAC system, has been omitted so far. More generally, we have to put:

$$S(\beta,\gamma)=K(\beta,\gamma)\begin{pmatrix} \gamma+1 & 0 & 0 & -\beta\gamma \\ 0 & \gamma+1 & -\beta\gamma & 0 \\ 0 & -\beta\gamma & \gamma+1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma+1 \end{pmatrix} \qquad \text{(XXIX-32)}$$

When we switch the role of frames (R) and (R'), this is equivalent to exchange the sign of the relative speed, and so β.

The composition of two translations of the frames with opposite speeds must allow to find identical spinors, and we can then write:

$$K(\beta,\gamma)\begin{pmatrix} \gamma+1 & 0 & 0 & -\beta\gamma \\ 0 & \gamma+1 & -\beta\gamma & 0 \\ 0 & -\beta\gamma & \gamma+1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma+1 \end{pmatrix}K(-\beta,\gamma)\begin{pmatrix} \gamma+1 & 0 & 0 & \beta\gamma \\ 0 & \gamma+1 & \beta\gamma & 0 \\ 0 & \beta\gamma & \gamma+1 & 0 \\ \beta\gamma & 0 & 0 & \gamma+1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(XXIX-33)}$$

Or still:

$$K(\beta,\gamma)K(-\beta,\gamma)\begin{pmatrix} 2\gamma+2 & 0 & 0 & 0 \\ 0 & 2\gamma+2 & 0 & 0 \\ 0 & 0 & 2\gamma+2 & 0 \\ 0 & 0 & 0 & 2\gamma+2 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(XXIX-34)}$$

We deduce the K constant:

$$K(\beta,\gamma)=K(-\beta,\gamma)=\pm\frac{1}{\sqrt{2\gamma+2}} \qquad \text{(XXIX-35)}$$

And the final shape of the S matrix:

$$S(\beta,\gamma)=\pm\frac{1}{\sqrt{2\gamma+2}}\begin{pmatrix}\gamma+1 & 0 & 0 & -\beta\gamma\\ 0 & \gamma+1 & -\beta\gamma & 0\\ 0 & -\beta\gamma & \gamma+1 & 0\\ -\beta\gamma & 0 & 0 & \gamma+1\end{pmatrix} \qquad \text{(XXIX-36)}$$

We infer the inverse matrix S^{-1}:

$$S^{-1}(\beta,\gamma)=\pm\frac{1}{\sqrt{2\gamma+2}}\begin{pmatrix}\gamma+1 & 0 & 0 & \beta\gamma\\ 0 & \gamma+1 & \beta\gamma & 0\\ 0 & \beta\gamma & \gamma+1 & 0\\ \beta\gamma & 0 & 0 & \gamma+1\end{pmatrix} \qquad \text{(XXIX-37)}$$

V - Non invariance of the SCHRÖDINGER equation

In a null potential, the SCHRÖDINGER equation for a free particle is written in the frame (R')

$$j\hbar\frac{\partial\psi'}{\partial x_t'}=-\frac{\hbar^2}{2mc}\left(\frac{\partial^2\psi'}{\partial x'^2}+\frac{\partial^2\psi'}{\partial y'^2}+\frac{\partial^2\psi'}{\partial z'^2}\right) \qquad \text{(XXIX-38)}$$

By substituting the partial derivatives, deducted from the LORENTZ transformation (XXIX-6), one obtains successively

$$\begin{aligned} j\hbar\left(\gamma\frac{\partial\psi'}{\partial x_t}+\beta\gamma\frac{\partial\psi'}{\partial x}\right)&=-\frac{\hbar^2}{2mc}\left(\left(\beta\gamma\frac{\partial}{\partial x_t}+\gamma\frac{\partial}{\partial x}\right)\left(\beta\gamma\frac{\partial\psi'}{\partial x_t}+\gamma\frac{\partial\psi'}{\partial x}\right)+\frac{\partial^2\psi'}{\partial y^2}+\frac{\partial^2\psi'}{\partial z^2}\right)\\ j\hbar\left(\gamma\frac{\partial\psi'}{\partial x_t}+\beta\gamma\frac{\partial\psi'}{\partial x}\right)&=-\frac{\hbar^2}{2mc}\left(\left(\beta^2\gamma^2\frac{\partial^2\psi'}{\partial x_t^2}+2\beta\gamma^2\frac{\partial^2\psi'}{\partial x_t\partial x}+\gamma^2\frac{\partial^2\psi'}{\partial x^2}\right)+\frac{\partial^2\psi'}{\partial y^2}+\frac{\partial^2\psi'}{\partial z^2}\right)\end{aligned} \qquad \text{(XXIX-39)}$$

Invariance of this equation will be checked if it is possible to find a linear relationship between ψ and ψ', such that the equation above can be put in the form:

$$j\hbar\frac{\partial\psi}{\partial x_t}=-\frac{\hbar^2}{2mc}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right) \qquad \text{(XXIX-40)}$$

Such equality is possible only in imposing $\beta = 0$ and $\gamma = 1$, which indicates that the two frames (R) and (R') are identical.

It is not surprising that the SCHRÖDINGER equation is not invariant under the LORENTZ transformation, because its total energy is set from the kinetic and potential energy issued from non-relativistic mechanics.

VI – Conclusion

The invariance of physical laws by changing frame is based on the following reasoning: the frame which is chosen to describe a physics experiment cannot have any influence on its conduct.

If one observes the fall of a rock from a balcony or from a passing train, the frame of observation should not have any influence on the phenomenon which is happening.

Physical quantities will be "dressed" differently depending on the frame, but the physical reality that will be behind this “covering” will be the same.

If this property is not checked, it is that there is a problem either in the "covering" of the physical quantities viewed in each frame, either in the physical law itself.

This view is completely analogous to that which is associated with the vector representation. A vector represents a unique quantity, but the "dressing" depends on the coordinate system in which it is described: Cartesian, cylindrical, or spherical or other.

However, if we make the sum of two vectors written in Cartesian coordinates, the result will be a vector identical to that obtained as the sum of these two vectors written in cylindrical coordinates. The law of addition written in a frame is identical in the other frame, and it is therefore no longer needed to specify the coordinate system in which it is written.

When an equation or a relationship between physical quantities is invariant by change of frame, this ensures that the observed phenomenon is seen in the same way in each of the frames.

This extremely powerful concept can be put forward to give a preference to the DIRAC equation rather than the SCHRÖDINGER equation in the description of the quantum world.

XXX

Reflections on the concept of indeterminate states

When trying to understand how quantum physics, and especially all of the assumptions underpinning the Copenhagen school are built, we can suppose that the thought process has been pretty close the following:

Any measure that tries to identify the quantum state of a particle gives a result including a share of random. Without more information about phenomena which take place, a theoretical modeling can only be implemented in the form of a probabilistic approach.

This probabilistic approach must make to coexist two essential elements: the HEISENBERG uncertainty principle, and certain quantum state which is detected during a measurement. The first element is the cornerstone that supports all theoretical reflections on quantum mechanics, the second is a fact of experience that shows that a single quantum measure gives a single observation.

The association of these two elements is a problem.

If, after the measurement, the particle is in a certain state, the postulate N° 6 governing the evolution of the state of the particle must allow to go back in time, and so to find the state of the particle before measurement. However the uncertainty principle states that the state can not be known with certainty.

If we want to preserve despite all, certain quantum state after the measurement, and the indeterminate quantum state before the measurement, it is necessary to insert between a phenomenon peculiar to the quantum world which is called phenomenon of decoherence. This phenomenon of decoherence introduced a border of unknown nature between the functioning of the quantum world and the functioning of the deterministic world.

Once this border has been put, the quantum theory of the Copenhagen school is interpreted fairly simple. Before the measurement, the quantum particle is in an indeterminate state, which can, from a purely mathematical point of view, be seen as an infinite number of superimposed states. At the time of the measurement, a phenomenon specific to the quantum world, called the wave packet reduction, chooses randomly from these undetermined states what will appear on the measuring apparatus.

If this contrintuitive vision of physics is imposed after going through a century of turmoil and criticism, it is because it provides observations and measures of quantum properties

consistent with the theoretical model developed by the Copenhagen school: this state of fact makes it no questionable.

To be convincing, a deterministic approach must be able to provide additional explanations to this phenomenon which has emerged as a strong and inexplicable constraint to the founders of quantum mechanics. The essential question raised by this interpretation can be summarized in the following way: why a particle is in a quantum state which cannot be known before the act of measurement?

On this point, we are going to show that an energy and deterministic approach is compatible with the point of view of the Copenhagen school.

In a simplified approach, we will only consider the indeterminacy between wave and corpuscular nature of quantum particles.

For the Copenhagen school, a particle in a corpuscular state is represented by a mathematical point which is attributed a position and momentum by the uncertainty principle. A particle in wave state is represented by a wave that surrounds this mathematical point. As long as it is not materialized, this particle is in an undefined state between wave and particle, or a superposition of these two states.

In an energy and deterministic approach, the state of the particle is given to any time by exact stationary solutions of the DIRAC equation. In Cartesian coordinates, for a particle at rest, the spinor that carries the pulse energy is zero, and a possible solution comes in the form:

$$\begin{aligned}\psi_0 &= \left(m_0c^2\right)\cos(\omega t) - j\left(\hbar\omega\right)\sin(\omega t)\\ \psi_1 &= 0\\ \psi_2 &= 0\\ \psi_3 &= 0\end{aligned} \qquad \text{(XXX-1)}$$

In this particular case, the conservation of energy equation indicates that the mass energy is equal to the wave energy.

There where the Copenhagen school says that the particle is in an indeterminate state between wave and matter, what says a deterministic approach?

That this state is actually indefinite in the meaning where it alternates between mass energy and wave energy to the ω pulse. For an electron at rest, the order of magnitude of this pulse is given by the relationship of conservation of energy:

$$\hbar\omega = m_0c^2 \qquad \text{(XXX-2)}$$

The periodic phenomenon at play in these exchanges has a period T:

$$T = \frac{2\pi}{\omega} = \frac{h}{m_0c^2} \approx \frac{6{,}62.10^{-34}}{\left(9{,}11.10^{-31}\right)\left(3.10^{8}\right)^2} \approx 8{,}07.10^{-21}s \qquad \text{(XXX-3)}$$

There is so far no experimental way to identify exactly the state of the particle at time t.

It is concluded that the founders of quantum mechanics have defined the best representation of the quantum world that it was possible to formulate, on the basis of what is actually observed and consistent with the HEISENBERG uncertainty principle.

The essential difference between the Copenhagen school and the deterministic approach appears in the reduction of the wave packet.

The Copenhagen school is trapped in the principle of indeterminacy that requires it to introduce a border between the indeterminacy that exists prior to the reduction of the wave packet and determinism that exists after the reduction of the wave packet.

Efforts to a better understanding of these phenomena are considerable, notably through the works of the Nobel Prize Serge HAROCHE and David J. WINELAND and their teams. But these works also show that this border seems extremely difficult (impossible?) to clearly highlight, both on the experimental plan and on a theoretical level. This border crossing is done by a draw that quantum mechanics cannot explain.

In an energy and deterministic approach, there is no border of this nature. There are extremely fast energy exchanges that allow an electron to absorb or emit a photon, or materialize in another way by transferring its energy. The random phenomena that are observed during these exchanges are not due, as for the Copenhagen school, to an intrinsic indeterminacy of quantum world. They are due to the fact that the energy state of the particle varies with speed such that it is impossible to make a material other than statistical observation.

Even in a deterministic approach, at the current state of the science, it seems impossible to know the exact quantum state of a particle (which would require a temporal precision $< 10^{-21}$s), before this particle has transferred its energy to a system with sufficient stability in time to allow an observation.

The energy and deterministic approach does not contradict the Copenhagen school in its statistical vision of quantum phenomena. It brings supplements that may justify a deterministic approach to the places where the Copenhagen school proved powerless to progress in a more detailed knowledge of the quantum phenomena.

Fifth part

Elements of coherence

In this final part, we try to convince of the interest of the approach presented by highlighting some elements of consistency with previous works or elements of coherence with other areas of Physics: electromagnetism, quantum fields, mechanical theory...

XXXI

Comparison with another exact solution of the DIRAC equation

It appeared over the chapters developed in this document, that stationary solutions could be exact solutions of the DIRAC equation. Very quickly, a special form of these solutions has been privileged, without no argument to justify this preference. A non-normalized example of these solutions is recalled below:

$$\begin{aligned}
\psi_0 &= \cos(k_x x)\cos(k_y y)\cos(k_z z)\left\{j(\hbar\omega)\cos(k_t x_t) + (m_0 c^2)\sin(k_t x_t)\right\} \\
\psi_1 &= 0 \\
\psi_2 &= j(\hbar c k_z)\cos(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t) \\
\psi_3 &= j(\hbar c k_x)\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) + (\hbar c k_y)\cos(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t)
\end{aligned} \qquad \text{(XXXI-1)}$$

The solution consists of two spinors, the first is associated with exchanges between wave energy and the energy of mass, while the second is associated with impulse energy exchanges.

No mathematical constraints does focus on one solution rather than another of the many possibilities presented in Chapter VII. If the particular form given by (XXXI-1) above has been favored, it is because it allowed a simple physical interpretation of the energy exchanges that were conjectured within the particle.

The conviction that it was an interesting formulation was strengthened by obtaining new solutions with the same structure. That was the case, at first, with similar solutions in the presence of an electromagnetic potential, then in a second time, with similar solutions in spherical coordinates. But this is not a serious scientific justification of the preference that was given to this type of solution.

We show that, in very different ways, Walter GREINER had already established that the first spinor express exchanges between wave energy and mass energy, while the second spinor express impulse energy exchanges. These exchanges were not highlighted in the proposed formulation, and therefore we'll modify (very slightly) this formulation to make them appear.

The rest of this discussion relates to the passage from the book of Walter GREINER: RELATIVISTIC QUANTUM MECHANICS, Wave equations, third edition, Springer, 2000, Chapter 8: Wave Packets of Plane Waves Dirac, equation (14) page 195.

Exercise 8.5 which begins on page 191, examines the temporal evolution of a plane wave Paquet with a Gaussian amplitude distribution. In the notation of the author, this wave packet is represented by the relationship

$$\psi'(x,0,s) := \frac{1}{\left(\pi d^2\right)^{3/4}} e^{-|x|^2/2d^2} \omega^1(0) \qquad \text{(XXXI-2)}$$

With :

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{(XXXI-3)}$$

Physically, this indicates that the wave packet (XXXI-2) is centered on the origin at time t = 0, and it is carried by the first component of the DIRAC bispinor at time t = 0

Following a difficult demonstration, W. GREINER shows that a rigorous resolution of the system of DIRAC, leads to a temporal evolution of the bispinor, rewritten verbatim below in the scoring of the author, after correction of some minor typographical errors $\left(h \to \hbar \;\; , \; p_y - ip_x \to \left(p_y c - ip_x c\right)\right)$:

$$c_1(\vec{x},t) := \int \frac{d^3p}{(2\pi\hbar)^{3/2}} e^{i\vec{p}.\vec{x}/\hbar} \frac{1}{\pi^{3/4}} \left(\frac{d}{\hbar}\right)^{3/2} e^{-|\vec{p}|^2 d^2/2\hbar^2} \left(\cos \frac{\sqrt{m_0^2c^4 + \vec{p}^2c^2}}{\hbar} t - i \frac{m_0c^2}{\sqrt{m_0^2c^4 + \vec{p}^2c^2}} \sin \frac{\sqrt{m_0^2c^4 + \vec{p}^2c^2}}{\hbar} t \right)$$

$$c_2(\vec{x},t) := 0$$

$$c_3(\vec{x},t) := -i \int \frac{d^3p}{(2\pi\hbar)^{3/2}} e^{i\vec{p}.\vec{x}/\hbar} \frac{1}{\pi^{3/4}} \left(\frac{d}{\hbar}\right)^{3/2} e^{-|\vec{p}|^2 d^2/2\hbar^2} \left(\frac{p_z c}{\sqrt{m_0^2c^4 + \vec{p}^2c^2}} \sin \frac{\sqrt{m_0^2c^4 + \vec{p}^2c^2}}{\hbar} t \right)$$

$$c_4(\vec{x},t) := \int \frac{d^3p}{(2\pi\hbar)^{3/2}} e^{i\vec{p}.\vec{x}/\hbar} \frac{1}{\pi^{3/4}} \left(\frac{d}{\hbar}\right)^{3/2} e^{-|\vec{p}|^2 d^2/2\hbar^2} \left(\frac{(p_y - ip_x)c}{\sqrt{m_0^2c^4 + \vec{p}^2c^2}} \sin \frac{\sqrt{m_0^2c^4 + \vec{p}^2c^2}}{\hbar} t \right)$$

(XXXI-4)

In order to compare this result with exact solutions obtained in this document, we carry out substitutions:

$$\frac{\sqrt{m_0^2c^4+\vec{p}^2c^2}}{\hbar}t=\frac{\hbar\omega}{\hbar}t=\omega t=k_t x_t$$
$$p_x=\hbar k_x \qquad \text{(XXXI-5)}$$
$$p_y=\hbar k_y$$
$$p_z=\hbar k_z$$

After multiplying each term by $\hbar\omega=\sqrt{m_0^2c^4+\vec{p}^2c^2}$, we get:

$$(\hbar\omega)c_1(\vec{x},t):=\int\frac{d^3p}{(2\pi\hbar)^{3/2}}e^{i\vec{p}.\vec{x}/\hbar}\frac{1}{\pi^{3/4}}\left(\frac{d}{\hbar}\right)^{3/2}e^{-|\vec{p}|^2d^2/2\hbar^2}\left((\hbar\omega)\cos(k_t x_t)-i(m_0c^2)\sin(k_t x_t)\right)$$
$$(\hbar\omega)c_2(\vec{x},t):=0$$
$$(\hbar\omega)c_3(\vec{x},t):=-i\int\frac{d^3p}{(2\pi\hbar)^{3/2}}e^{i\vec{p}.\vec{x}/\hbar}\frac{1}{\pi^{3/4}}\left(\frac{d}{\hbar}\right)^{3/2}e^{-|\vec{p}|^2d^2/2\hbar^2}(\hbar ck_z)\sin(k_t x_t)$$
$$(\hbar\omega)c_4(\vec{x},t):=\int\frac{d^3p}{(2\pi\hbar)^{3/2}}e^{i\vec{p}.\vec{x}/\hbar}\frac{1}{\pi^{3/4}}\left(\frac{d}{\hbar}\right)^{3/2}e^{-|\vec{p}|^2d^2/2\hbar^2}(\hbar ck_y-i\hbar ck_x)\sin(k_t x_t)$$

(XXXI-6)

From the energy point of view, the result of W. GREINER revealed two remarkable properties.

It is confirmed that the first spinor carry exchanges between mass energy and wave energy, and that these exchanges have place with a temporal quadrature.

It is confirmed that the second spinor carry the impulse energy exchanges. These exchanges are in phase with the evolution of the mass energy, and in temporal quadrature with the evolution of wave energy

The solution established by W. GREINER was obtained through channels that seem completely disjoint from those that led to the stationary solutions. We may see a confirmation of the formalism which has been developed throughout this document.

XXXII

Photon of DIRAC and MAXWELL's equations

The DIRAC equation is often presented as an equation that only allows to determine the characteristics of the particles of spin ½. We conjectured to chapter XX, that some stationary solutions seemed likely to describe the behavior of particles of spin 1, and so among them, the photon.

In this chapter, we seek to support this hypothesis by examining whether there is a link between the structure of standing waves that build the photon, and the very near field as it can be calculated exactly using Maxwell's equations. These solutions are few, and we will use the relationships established by KOTTLER.

We show that one of these solutions brings out some links between the exact stationary solutions to the DIRAC equation and exact solutions to MAXWELL's equations. Specifically, we show that near-field structure deduced from MAXWELL's equations presents great analogies with structure from standing waves of solutions to the DIRAC equation.

I – KOTTLER formulas

The electromagnetic field radiated by a current density distribution can be obtained accurately at any distance of sources, by using one of the variants of KOTTLER formulas:

$$\vec{E}(P)=\frac{k^2}{4\pi j\omega\varepsilon_0}\iiint_V\left\{\left(1+\frac{1}{jkr}-\frac{1}{k^2r^2}\right)\vec{J}(M_0)-\left(1+\frac{3}{jkr}-\frac{3}{k^2r^2}\right)\left(\vec{J}(M_0).\vec{U}\right)\vec{U}\right\}\frac{e^{-jkr}}{r}dv$$
$$\vec{B}(P)=\frac{jk\mu_0}{4\pi}\iiint_V\left(1+\frac{1}{jkr}\right)\left(\vec{J}(M_0)\Lambda\vec{U}\right)\frac{e^{-jkr}}{r}dv \qquad \text{(XXXII-1)}$$

These relationships are valid in harmonic evolution, to the ω pulse, with a time dependence in exp(jωt), and with a propagation constant k = ω/c. We will not work in this chapter on the temporal evolution of the fields, and it will be omitted later in the expression of electromagnetic fields.

The various parameters involved in these relationships are shown in figure 1 below.

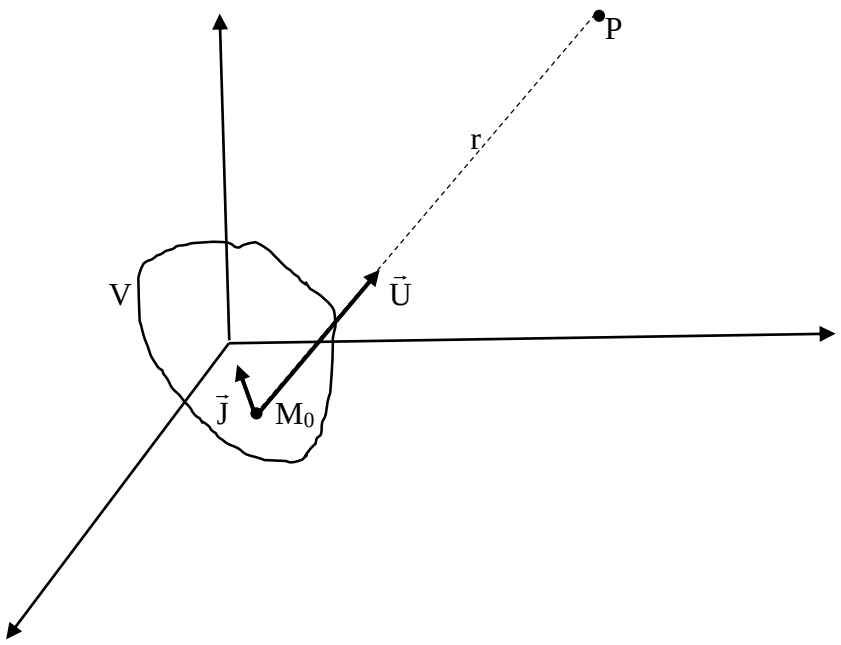


Figure XXXII-1: *Representation of the parameters used in the formulation of KOTTLER.*

P is the point of observation and calculation of the field.

M_0 refers to the point where we found the current element $\vec{J}(M_0)$.

V represents the volume containing all the sources of currents that contribute to the electromagnetic field calculated at observation point P.

$\vec{U}$ is a unit vector in the direction $\overrightarrow{M_0P}$

r represents the distance M_0P.

We will consider a single element of current $\vec{J}$, centered on the origin, and we will look at the structure of the electromagnetic field around this infinitely small element.

This current element will have a fixed direction without dependence of time. From an electromagnetic point of view, this implies that the field radiated at a great distance will have a linear polarization.

We do not need to integrate on volume V, and we get the complex amplitude of the near field in the simplified form:

$$\vec{E}(P) = \frac{k^2}{4\pi j\omega\varepsilon_0}\left\{\left(1+\frac{1}{jkr}-\frac{1}{k^2r^2}\right)\vec{J}-\left(1+\frac{3}{jkr}-\frac{3}{k^2r^2}\right)(\vec{J}.\vec{U})\vec{U}\right\}\frac{e^{-jkr}}{r}$$
$$\vec{B}(P) = \frac{jk\mu_0}{4\pi}\left(1+\frac{1}{jkr}\right)(\vec{J}\Lambda\vec{U})\frac{e^{-jkr}}{r} \qquad \text{(XXXII-2)}$$

Or still after development:

$$\vec{E}(P) = -j\frac{k^2}{4\pi\omega\varepsilon_0}\left\{\left(\frac{e^{-jkr}}{r} - j\frac{e^{-jkr}}{kr^2} - \frac{e^{-jkr}}{k^2r^3}\right)\vec{J} - \left(\frac{e^{-jkr}}{r} - 3j\frac{e^{-jkr}}{kr^2} - 3\frac{e^{-jkr}}{k^2r^3}\right)(\vec{J}.\vec{U})\vec{U}\right\}$$
$$\vec{B}(P) = \frac{jk\mu_0}{4\pi}\left(\frac{e^{-jkr}}{r} - j\frac{e^{-jkr}}{kr^2}\right)(\vec{J}\Lambda\vec{U}) \qquad \text{(XXXII-3)}$$

For a homogeneous notation in power of (kr), we factorize k:

$$\vec{E}(P) = \frac{k^3}{4\pi\omega\varepsilon_0}\left\{\left(-j\frac{e^{-jkr}}{kr} - \frac{e^{-jkr}}{k^2r^2} + j\frac{e^{-jkr}}{k^3r^3}\right)\vec{J} - \left(-j\frac{e^{-jkr}}{kr} - 3\frac{e^{-jkr}}{k^2r^2} + 3j\frac{e^{-jkr}}{k^3r^3}\right)(\vec{J}.\vec{U})\vec{U}\right\}$$
$$\vec{B}(P) = \frac{k^2\mu_0}{4\pi}\left(j\frac{e^{-jkr}}{kr} + \frac{e^{-jkr}}{k^2r^2}\right)(\vec{J}\Lambda\vec{U}) \qquad \text{(XXXII-4)}$$

Without changing the notation for electromagnetic fields, we go from complex representation to physical representation by taking the real part of the expression above:

$$\vec{E}(P) = \frac{k^3}{4\pi\omega\varepsilon_0}\left\{\left(-\frac{\sin(kr)}{kr} - \frac{\cos(kr)}{k^2r^2} + \frac{\sin(kr)}{k^3r^3}\right)\vec{J} + \left(\frac{\sin(kr)}{kr} + 3\frac{\cos(kr)}{k^2r^2} - 3\frac{\sin(kr)}{k^3r^3}\right)(\vec{J}.\vec{U})\vec{U}\right\}$$
$$\vec{B}(P) = \frac{k^2\mu_0}{4\pi}\left(\frac{\sin(kr)}{kr} + \frac{\cos(kr)}{k^2r^2}\right)(\vec{J}\Lambda\vec{U}) \qquad \text{(XXXII-5)}$$

We will show in the next paragraphs that (kr) dependence of the field in the vicinity of origin has significant similarities to (kr) dependence of the wavefunctions issued from exact solutions to the DIRAC equation.

II – The photon of DIRAC

We have speculated in chapter XX, that the modes associated with the description of the photon might be represented by the following solution:

$$\psi_0 = (\hbar\omega)f_1(k_r r)\sin\theta e^{-j\varphi}\cos(k_t x_t)$$
$$\psi_1 = 0$$
$$\psi_2 = (\hbar c k_r)\cos\theta\sin\theta e^{-j\varphi}\sin(k_t x_t)\left\{-f_1'(k_r r) + \frac{f_1(k_r r)}{k_r r}\right\} \qquad \text{(XXXII-6)}$$
$$\psi_3 = (\hbar c k_r)\sin(k_t x_t)\left\{-\sin^2\theta f_1'(k_r r) - (\cos^2\theta + 1)\frac{f_1(k_r r)}{k_r r}\right\}$$

Where f_1 is a spherical BESSEL function of order 1:

$$j_1(k_r r) = \frac{\sin(k_r r)}{(k_r r)^2} - \frac{\cos(k_r r)}{(k_r r)} \qquad y_1(k_r r) = -\frac{\cos(k_r r)}{(k_r r)^2} - \frac{\sin(k_r r)}{(k_r r)} \qquad \text{(XXXII-7)}$$

and where we adopted the notation $f_1'(k_r r) = df_1(k_r r)/d(k_r r)$.

There was, in truth, few arguments to support this hypothesis. We will reconsider it, seen under the angle of MAXWELL's equations.

This solution is associated with the conservation of energy equation:

$$k_t^2 = \eta^2 + k_r^2 \qquad \text{(XXXII-8)}$$

And since the photon has no mass energy, this imposes $\eta = 0$, and we will put in the following:

$$k_t = k_r = k \qquad \text{(XXXII-9)}$$

The exact solution to the DIRAC equation is written with this notation:

$$\begin{aligned}
\psi_0 &= (\hbar\omega) f_1(kr) \sin\theta e^{-j\varphi} \cos(\omega t) \\
\psi_1 &= 0 \\
\psi_2 &= (\hbar\omega) \cos\theta \sin\theta e^{-j\varphi} \sin(\omega t) \left\{ -f_1'(kr) + \frac{f_1(kr)}{kr} \right\} \\
\psi_3 &= (\hbar\omega) \sin(\omega t) \left\{ -\sin^2\theta f_1'(kr) - (\cos^2\theta + 1) \frac{f_1(kr)}{kr} \right\}
\end{aligned} \qquad \text{(XXXII-10)}$$

We can build two sets of solutions, one such as $f_1 = \alpha\, j_1$, the other such as $f_1 = \alpha\, y_1$, where α can be a real or complex constant.

Solution such as $f_1 = -\, j_1$. This solution will be associated with the electric field, and will be indicated by the letter E.

$$\begin{aligned}
\psi_{0E} &= (\hbar\omega) \sin\theta e^{-j\varphi} \cos(\omega t) \left\{ \frac{\cos(kr)}{(kr)} - \frac{\sin(kr)}{(kr)^2} \right\} \\
\psi_{1E} &= 0 \\
\psi_{2E} &= (\hbar\omega) \cos\theta \sin\theta e^{-j\varphi} \sin(\omega t) \left\{ \frac{\sin(kr)}{(kr)} + 3\frac{\cos(kr)}{(kr)^2} - 3\frac{\sin(kr)}{(kr)^3} \right\} \\
\psi_{3E} &= (\hbar\omega) \sin(\omega t) \left\{ \sin^2\theta \frac{\sin(kr)}{(kr)} + \frac{\cos(kr)}{(kr)^2} \left(2\sin^2\theta - \cos^2\theta - 1\right) + \frac{\sin(kr)}{(kr)^3} \left(-2\sin^2\theta + \cos^2\theta + 1\right) \right\}
\end{aligned} \qquad \text{(XXXII-11)}$$

Solution such as $f_1 = -\, y_1$. This solution will be associated with the magnetic field, and will be indicated by the letter B

$$\psi_{0B} = (\hbar\omega)\sin\theta e^{-j\varphi}\cos(\omega t)\left\{\frac{\sin(kr)}{(kr)} + \frac{\cos(kr)}{(kr)^2}\right\}$$

$$\psi_{1B} = 0$$

$$\psi_{2B} = -(\hbar\omega)\cos\theta\sin\theta e^{-j\varphi}\sin(\omega t)\left\{\frac{\cos(kr)}{(kr)} - 3\frac{\sin(kr)}{(kr)^2} - 3\frac{\cos(kr)}{(kr)^3}\right\}$$

$$\psi_{3B} = -(\hbar\omega)\sin(\omega t)\left\{\sin^2\theta\frac{\cos(kr)}{(kr)} + \frac{\sin(kr)}{(kr)^2}\left(-2\sin^2\theta + \cos^2\theta + 1\right) + \frac{\cos(kr)}{(kr)^3}\left(-2\sin^2\theta + \cos^2\theta + 1\right)\right\}$$

(XXXII-12)

III – Comparison between exact solutions to MAXWELL equations and solutions of DIRAC for the photon

We can now examine the exact solutions to MAXWELL's equations, in the vicinity of a current element which emits electromagnetic photons, and compare the structure of the electromagnetic field that is issued to the structure of the photon which is predicted by the stationary solutions to the DIRAC equation

For the electromagnetic field, we have:

$$\vec{E}(P) = \frac{k^3}{4\pi\omega\varepsilon_0}\left\{\left(-\frac{\sin(kr)}{kr} - \frac{\cos(kr)}{k^2r^2} + \frac{\sin(kr)}{k^3r^3}\right)\vec{J} + \left(\frac{\sin(kr)}{kr} + 3\frac{\cos(kr)}{k^2r^2} - 3\frac{\sin(kr)}{k^3r^3}\right)(\vec{J}.\vec{U})\vec{U}\right\}$$

$$\vec{B}(P) = \frac{k^2\mu_0}{4\pi}\left(\frac{\sin(kr)}{kr} + \frac{\cos(kr)}{k^2r^2}\right)(\vec{J}\Lambda\vec{U})$$

(XXXII-13)

For the photon of DIRAC, we have:

$$\psi_{2E} = (\hbar\omega)\cos\theta\sin\theta e^{-j\varphi}\sin(\omega t)\left\{\frac{\sin(kr)}{(kr)} + 3\frac{\cos(kr)}{(kr)^2} - 3\frac{\sin(kr)}{(kr)^3}\right\}$$

$$\psi_{0B} = (\hbar\omega)\sin\theta e^{-j\varphi}\cos(\omega t)\left\{\frac{\sin(kr)}{(kr)} + \frac{\cos(kr)}{(kr)^2}\right\}$$

(XXXII-14)

It appears that for the wave functions ψ_{2E} and ψ_{0B} of bi-spinor of DIRAC that are separable in r and θ, r dependence is identical to terms of the exact solution to MAXWELL's equations.

The variables θ and φ, offer us a prospective analysis that should only be considered as a track of thoughts.

Since the field treated by MAXWELL's equations is linear polarization, we assume that the corresponding solution of the photon is independent of φ, which could be obtained for example assuming that φ = 0 in this situation where there is no rotation of the field.

It remains to analyze the dependence on θ. It is an internal variable to the DIRAC spinor, which does not appear in the solution of MAXWELL's equations. No obvious interpretation

arises spontaneously to understand how this variable is taken into account in MAXWELL's equations. However, we show that we can give it a simple physical sense in two particular situations

1rst case: the current element vector is orthogonal to the direction of propagation vector

For the electromagnetic field, we have:

$$\vec{E}(P) = \frac{k^3}{4\pi\omega\varepsilon_0}\left\{\left(-\frac{\sin(kr)}{kr} - \frac{\cos(kr)}{k^2r^2} + \frac{\sin(kr)}{k^3r^3}\right)\vec{J}\right\}$$
$$\vec{B}(P) = \frac{k^2\mu_0}{4\pi}\left(\frac{\sin(kr)}{kr} + \frac{\cos(kr)}{k^2r^2}\right)(\vec{J}\Lambda\vec{U}) \qquad \text{(XXXII-15)}$$

By putting $\theta = \pi/2$ in solutions of the DIRAC photon associated to the electric field and the magnetic field, we get:

$$\psi_{2E} = 0$$
$$\psi_{3E} = (\hbar\omega)\sin(\omega t)\left\{\frac{\sin(kr)}{(kr)} + \frac{\cos(kr)}{(kr)^2} - \frac{\sin(kr)}{(kr)^3}\right\} \qquad \text{(XXXII-16)}$$
$$\psi_{0B} = (\hbar\omega)e^{-j\varphi}\cos(\omega t)\left\{\frac{\sin(kr)}{(kr)} + \frac{\cos(kr)}{(kr)^2}\right\}$$

We found again in the wave function ψ_{3E}, with a change of sign, the radial dependence of the electric field.

2nd case: the current element vector is parallel to the direction of propagation vector

For the electromagnetic field, we have:

$$\vec{E}(P) = \frac{k^3}{4\pi\omega\varepsilon_0}\left\{\left(-\frac{\sin(kr)}{kr} - \frac{\cos(kr)}{k^2r^2} + \frac{\sin(kr)}{k^3r^3}\right)\vec{J} + \left(\frac{\sin(kr)}{kr} + 3\frac{\cos(kr)}{k^2r^2} - 3\frac{\sin(kr)}{k^3r^3}\right)\vec{J}\right\}$$
$$\vec{B}(P) = 0 \qquad \text{(XXXII-17)}$$

Or still :

$$\vec{E}(P) = \frac{k^3}{4\pi\omega\varepsilon_0}\left(2\frac{\cos(kr)}{k^2r^2} - 2\frac{\sin(kr)}{k^3r^3}\right)\vec{J} \qquad \text{(XXXII-18)}$$
$$\vec{B}(P) = 0$$

By putting $\theta = 0$ in solutions of the DIRAC photon associated with the electric field and the magnetic field, we get

$$\psi_{2E} = 0$$
$$\psi_{3E} = (\hbar\omega)\sin(\omega t)\left\{-2\frac{\cos(kr)}{(kr)^2} + 2\frac{\sin(kr)}{(kr)^3}\right\} \qquad \text{(XXXII-19)}$$
$$\psi_{0B} = 0$$

We found again in the wave function ψ_{3E}, with a change of sign, the radial dependence of the electric field, and the ψ_{0B} wave function associated with a null field magnetic.

IV – Conclusion

When an electron generates a current element by its alternative movement following a constant direction, we know that there is an emission of electromagnetic photons. These photons do not go in a unique direction, as evidenced by the radiation pattern of the dipole, but with a different probability in each direction of space. This observation implies that there is a complex process in the background, that is not described by MAXWELL's equations, or the DIRAC equation.

What seems to show the study developed in this chapter, is that the near-field as it is predicted by MAXWELL's equations presents many analogies with the radial dependence of the stationary modes solution of the DIRAC equation. θ dependence should be linked to the angle between the movement of the dipole and the direction of propagation of the emitted photon, while φ dependence should be linked to the rotation of the polarization.

The physical interpretation of the part of the DIRAC bispineur ψ_3 remains an enigma. We can only notice that the values of θ who cancel the components of near field (components in $1/(kr)^2$ and $1/(kr)^3$) and which are given by the relationship:

$$2\sin^2\theta - \cos^2\theta - 1 = 0$$
$$\cos^2\theta = 1/3 \qquad \text{(XXXII-20)}$$

match the angle that defines the quantization of angular momentum of the electron.

Without being able to draw a final conclusion, this chapter accredits the idea that exchanges of energy in the vicinity of the electron which oscillates in space are directly compatible with exchanges of energy in the photon that is created, such as these exchanges are predicted by the exact stationary solutions to the DIRAC equation.

XXXIII

From planes waves to standing waves

Research of stationary solutions to the DIRAC equation has been developped in this document by the resolution of a system of 64 equations in 64 unknowns. Obtaining solutions is relatively complicated.

Planar waves solutions have long been known, and since a standing wave can always be decomposed as the sum of two plane waves, the objective of this chapter is double:

- give a systematic construction method of stationary solutions from the solutions already established in plane waves
- try to understand why the stationary solutions have not emerged from plane waves solutions that are known and used for nearly a century.

I – Plane waves solutions

We're looking for solutions to the system of DIRAC:

$$\begin{aligned}
\eta\psi_0 &= j\frac{\partial\psi_0}{\partial x_t} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z}\\
\eta\psi_1 &= j\frac{\partial\psi_1}{\partial x_t} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z}\\
\eta\psi_2 &= -j\frac{\partial\psi_2}{\partial x_t} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z}\\
\eta\psi_3 &= -j\frac{\partial\psi_3}{\partial x_t} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z}
\end{aligned} \qquad \text{(XXXIII-1)}$$

Very generally speaking, plane waves seeked are of the form:

$$\begin{aligned}
\psi_0 &= \alpha_0(k_t, k_x, k_y, k_z, \eta)\exp j(\pm k_t x_t \pm k_x x \pm k_y y \pm k_z z)\\
\psi_1 &= \alpha_1(k_t, k_x, k_y, k_z, \eta)\exp j(\pm k_t x_t \pm k_x x \pm k_y y \pm k_z z)\\
\psi_2 &= \alpha_2(k_t, k_x, k_y, k_z, \eta)\exp j(\pm k_t x_t \pm k_x x \pm k_y y \pm k_z z)\\
\psi_3 &= \alpha_3(k_t, k_x, k_y, k_z, \eta)\exp j(\pm k_t x_t \pm k_x x \pm k_y y \pm k_z z)
\end{aligned} \qquad \text{(XXXIII-2)}$$

The signs of the exponential are arbitrary, but they must be the same for the 4 wavefunctions. The coefficients α_0, α_1, α_2, α_3 are constants to be determined, independent of t, x, y, z.

The DIRAC system must ensure the conservation of energy equation:

$$k_t^2 = \left(k_x^2 + k_y^2 + k_z^2\right) + \eta^2 \qquad \text{(XXXIII-3)}$$

To illustrate the search for solutions in plane waves, we arbitrarily choose the signs in the exponential. The deduction of the alternatives is immediate by the change of sign that affects k_t, k_x, k_y, or k_z, and we put:

$$\begin{aligned}
\psi_0 &= \alpha_0 \exp\, j\left(k_t x_t - k_x x - k_y y - k_z z\right) \\
\psi_1 &= \alpha_1 \exp\, j\left(k_t x_t - k_x x - k_y y - k_z z\right) \\
\psi_2 &= \alpha_2 \exp\, j\left(k_t x_t - k_x x - k_y y - k_z z\right) \\
\psi_3 &= \alpha_3 \exp\, j\left(k_t x_t - k_x x - k_y y - k_z z\right)
\end{aligned} \qquad \text{(XXXIII-4)}$$

By injecting this form of solution in the system of DIRAC (XXXIII-1), we get:

$$\begin{aligned}
\eta\alpha_0 &= -\alpha_0 k_t + \alpha_3 k_x - j\alpha_3 k_y + \alpha_2 k_z \\
\eta\alpha_1 &= -\alpha_1 k_t + \alpha_2 k_x + j\alpha_2 k_y - \alpha_3 k_z \\
\eta\alpha_2 &= \alpha_2 k_t - \alpha_1 k_x + j\alpha_1 k_y - \alpha_0 k_z \\
\eta\alpha_3 &= \alpha_3 k_t - \alpha_0 k_x - j\alpha_0 k_y + \alpha_1 k_z
\end{aligned} \qquad \text{(XXXIII-5)}$$

It is a homogeneous system of 4 equations with 4 unknowns:

$$\begin{aligned}
0 &= -\alpha_0\left(k_t + \eta\right) + \alpha_3 k_x - j\alpha_3 k_y + \alpha_2 k_z \\
0 &= -\alpha_1\left(k_t + \eta\right) + \alpha_2 k_x + j\alpha_2 k_y - \alpha_3 k_z \\
0 &= \alpha_2\left(k_t - \eta\right) - \alpha_1 k_x + j\alpha_1 k_y - \alpha_0 k_z \\
0 &= \alpha_3\left(k_t - \eta\right) - \alpha_0 k_x - j\alpha_0 k_y + \alpha_1 k_z
\end{aligned} \qquad \text{(XXXIII-6)}$$

The determinant of this system is equal to:

$$\left(-k_t^2 + k_x^2 + k_y^2 + k_z^2 + \eta^2\right)^2 \qquad \text{(XXXIII-7)}$$

It follows that when the determinant of the system is zero, i.e. when the conservation of energy equation is verified, you can get 4 sets of solutions, by taking successively, as a parametric variable, one of the constants α_0, α_1, α_2, α_3.

$$\begin{array}{llll}
\alpha_0 = \left(k_t - \eta\right) & \alpha_0 = 0 & \alpha_0 = k_z & \alpha_0 = \left(k_x - jk_y\right) \\
\alpha_1 = 0 & \alpha_1 = \left(k_t - \eta\right) & \alpha_1 = \left(k_x + jk_y\right) & \alpha_1 = -k_z \\
\alpha_2 = k_z & \alpha_2 = \left(k_x - jk_y\right) & \alpha_2 = \left(k_t + \eta\right) & \alpha_2 = 0 \\
\alpha_3 = \left(k_x + jk_y\right) & \alpha_3 = -k_z & \alpha_3 = 0 & \alpha_3 = \left(k_t + \eta\right)
\end{array} \qquad \text{(XXXIII-8)}$$

In most of the courses dedicated to the DIRAC equation, these solutions are presented in a slightly different form, so as to standardize the constant which is a function of the mass energy and wave energy in (XXXIII-8):

$$\begin{aligned}
&\alpha_0 = 1 && \alpha_0 = 0 && \alpha_0 = \frac{k_z}{(k_t + \eta)} && \alpha_0 = \frac{(k_x - jk_y)}{(k_t + \eta)} \\
&\alpha_1 = 0 && \alpha_1 = 1 && \alpha_1 = \frac{(k_x + jk_y)}{(k_t + \eta)} && \alpha_1 = \frac{-k_z}{(k_t + \eta)} \\
&\alpha_2 = \frac{k_z}{(k_t - \eta)} && \alpha_2 = \frac{(k_x - jk_y)}{(k_t - \eta)} && \alpha_2 = 1 && \alpha_2 = 0 \\
&\alpha_3 = \frac{(k_x + jk_y)}{(k_t - \eta)} && \alpha_3 = \frac{-k_z}{(k_t - \eta)} && \alpha_3 = 0 && \alpha_3 = 1
\end{aligned} \qquad \text{(XXXIII-9)}$$

These solutions are defined to normalization constant close. This constant which plays no real role in the discussion of this chapter will be temporarily omitted.

It appears already at this stage, that if we embarked on the construction of stationary solutions with this last set of solutions, exchange of energies between mass energy and wave energy will not be out spontaneously. So we'll keep the formulation (XXXIII-8) for the construction of the stationary solutions.

II – From plane waves solutions to standing waves solutions

We propose to show on a particular example, how the combination of several plane waves led to the stationary solutions that have been developed in chapters VI and VII. The method allows to rebuild all of the stationary solutions, but should be adapted in the signs following the stationary solution which is waited.

We aim to build a solution in the form of a product of cosine assigned to wave energy. This will lead us to systematically add solutions in plane waves exp (+ jk…) and exp (- jk …).

We adopt as a starting point the first solution (XXXIII-8) which is in $\exp(jk_t x_t)$. After building the analogous solution in $\exp(-jk_t x_t)$ by changing the sign of k_t, we get successively by summing these two solutions

$$\begin{pmatrix} k_t - \eta \\ 0 \\ k_z \\ k_x + jk_y \end{pmatrix} e^{j(k_t x_t - k_x x - k_y y - k_z z)} + \begin{pmatrix} k_t + \eta \\ 0 \\ k_z \\ k_x + jk_y \end{pmatrix} e^{j(-k_t x_t - k_x x - k_y y - k_z z)}$$

$$\left\{ \begin{pmatrix} k_t - \eta \\ 0 \\ k_z \\ k_x + jk_y \end{pmatrix} e^{j(k_t x_t)} + \begin{pmatrix} k_t + \eta \\ 0 \\ k_z \\ k_x + jk_y \end{pmatrix} e^{j(-k_t x_t)} \right\} e^{j(-k_x x - k_y y - k_z z)} \qquad \text{(XXXIII-10)}$$

$$2 \begin{pmatrix} k_t \cos(k_t x_t) - j\eta \sin(k_t x_t) \\ 0 \\ k_z \cos(k_t x_t) \\ (k_x + jk_y)\cos(k_t x_t) \end{pmatrix} e^{j(-k_x x - k_y y - k_z z)}$$

This operation brings up exchanges between mass energy and wave energy in the first term of the bi-spinor. It remains to repeat this process for each of the spatial components to obtain the complete stationary solution. We routinely skip the 2 coefficient appearing in calculations, and which may be included in a global term of standardization.

Stationary component according to x:

$$\begin{pmatrix} k_t \cos(k_t x_t) - j\eta \sin(k_t x_t) \\ 0 \\ k_z \cos(k_t x_t) \\ (-k_x + jk_y)\cos(k_t x_t) \end{pmatrix} e^{j(k_x x - k_y y - k_z z)} + \begin{pmatrix} k_t \cos(k_t x_t) - j\eta \sin(k_t x_t) \\ 0 \\ k_z \cos(k_t x_t) \\ (k_x + jk_y)\cos(k_t x_t) \end{pmatrix} e^{j(-k_x x - k_y y - k_z z)}$$

$$\left\{ \begin{pmatrix} k_t \cos(k_t x_t) + j\eta \sin(k_t x_t) \\ 0 \\ k_z \cos(k_t x_t) \\ (-k_x + jk_y)\cos(k_t x_t) \end{pmatrix} e^{j(k_x x)} + \begin{pmatrix} k_t \cos(k_t x_t) + j\eta \sin(k_t x_t) \\ 0 \\ k_z \cos(k_t x_t) \\ (k_x + jk_y)\cos(k_t x_t) \end{pmatrix} e^{j(-k_x x)} \right\} e^{j(-k_y y - k_z z)}$$

$$2 \left\{ \begin{pmatrix} k_t \cos(k_t x_t)\cos(k_x x) + j\eta \sin(k_t x_t)\cos(k_x x) \\ 0 \\ k_z \cos(k_t x_t)\cos(k_x x) \\ (-j\sin(k_x x)k_x + jk_y \cos(k_x x))\cos(k_t x_t) \end{pmatrix} \right\} e^{j(-k_y y - k_z z)}$$

(XXXIII-11)

Stationary component according to y:

$$
\begin{Bmatrix}
\begin{pmatrix}
k_t \cos(k_t x_t)\cos(k_x x) - j\eta \sin(k_t x_t)\cos(k_x x) \\
0 \\
k_z \cos(k_t x_t)\cos(k_x x) \\
\left(-j\sin(k_x x)k_x - jk_y \cos(k_x x)\right)\cos(k_t x_t)
\end{pmatrix}
\end{Bmatrix} e^{j(k_y y - k_z z)} +
$$

$$
\begin{Bmatrix}
\begin{pmatrix}
k_t \cos(k_t x_t)\cos(k_x x) - j\eta \sin(k_t x_t)\cos(k_x x) \\
0 \\
k_z \cos(k_t x_t)\cos(k_x x) \\
\left(-j\sin(k_x x)k_x + jk_y \cos(k_x x)\right)\cos(k_t x_t)
\end{pmatrix}
\end{Bmatrix} e^{j(-k_y y - k_z z)}
$$

$$
\begin{Bmatrix}
\begin{pmatrix}
k_t \cos(k_t x_t)\cos(k_x x) - j\eta \sin(k_t x_t)\cos(k_x x) \\
0 \\
k_z \cos(k_t x_t)\cos(k_x x) \\
\left(-j\sin(k_x x)k_x - jk_y \cos(k_x x)\right)\cos(k_t x_t)
\end{pmatrix} e^{j(k_y y)} + \\
\begin{pmatrix}
k_t \cos(k_t x_t)\cos(k_x x) - j\eta \sin(k_t x_t)\cos(k_x x) \\
0 \\
k_z \cos(k_t x_t)\cos(k_x x) \\
\left(-j\sin(k_x x)k_x + jk_y \cos(k_x x)\right)\cos(k_t x_t)
\end{pmatrix} e^{j(-k_y y)}
\end{Bmatrix} e^{j(-k_z z)}
$$

$$
\begin{Bmatrix}
2\begin{pmatrix}
k_t \cos(k_t x_t)\cos(k_x x)\cos(k_y y) - j\eta \sin(k_t x_t)\cos(k_x x)\cos(k_y y) \\
0 \\
k_z \cos(k_t x_t)\cos(k_x x)\cos(k_y y) \\
\left(-j\sin(k_x x)\cos(k_y y)k_x + k_y \cos(k_x x)\sin(k_y y)\right)\cos(k_t x_t)
\end{pmatrix}
\end{Bmatrix} e^{j(-k_z z)}
\qquad \text{(XXXIII-12)}
$$

Stationary component according to z:

$$
\begin{Bmatrix}
\begin{pmatrix}
k_t \cos(k_t x_t)\cos(k_x x)\cos(k_y y) - j\eta \sin(k_t x_t)\cos(k_x x)\cos(k_y y) \\
0 \\
-k_z \cos(k_t x_t)\cos(k_x x)\cos(k_y y) \\
\left(-j\sin(k_x x)\cos(k_y y)k_x + k_y \cos(k_x x)\sin(k_y y)\right)\cos(k_t x_t)
\end{pmatrix}
\end{Bmatrix} e^{j(k_z z)} +
$$

$$
\begin{Bmatrix}
\begin{pmatrix}
k_t \cos(k_t x_t)\cos(k_x x)\cos(k_y y) - j\eta \sin(k_t x_t)\cos(k_x x)\cos(k_y y) \\
0 \\
k_z \cos(k_t x_t)\cos(k_x x)\cos(k_y y) \\
\left(-j\sin(k_x x)\cos(k_y y)k_x + k_y \cos(k_x x)\sin(k_y y)\right)\cos(k_t x_t)
\end{pmatrix}
\end{Bmatrix} e^{j(-k_z z)}
\qquad \text{(XXXIII-13)}
$$

We get the final form of the researched solution:

$$
\begin{pmatrix}
k_t \cos(k_t x_t)\cos(k_x x)\cos(k_y y)\cos(k_z z) - j\eta \sin(k_t x_t)\cos(k_x x)\cos(k_y y)\cos(k_z z) \\
0 \\
-jk_z \cos(k_t x_t)\cos(k_x x)\cos(k_y y)\sin(k_z z) \\
\left\{-jk_x \sin(k_x x)\cos(k_y y) + k_y \cos(k_x x)\sin(k_y y)\right\}\cos(k_z z)\cos(k_t x_t)
\end{pmatrix}
\qquad \text{(XXXIII-14)}
$$

III – Conclusion

Stationary solutions can then be deduced from known exponential solutions for plane waves. It follows that all stationary solutions information is already available in exponential solutions in plane waves. Therefore, we can wonder about the reasons that have placed the stationary solutions in a position of background work on the DIRAC equation.

There probably are several reasons.

From a purely mathematical point of view, since all information is already contained in the plane wave solutions, it appears actually unhelpful to be interested in another representation that will provide no additional information.

Planar waves made from exponential solutions say nothing on the role played by the imaginary term j = racine(-1) and the confusion settles almost systematically with the complex formalism used in other areas of physics.

From a purely mathematical point of view, this confusion remove nothing from the validity of the solutions. For the physicist, this removes understanding and visibility to quantities that he manipulates and this do not encourage him in other approaches to which he attached no physical sense.

Some authors develop a presentation with a positive energy associated with the first spinor, and a negative energy associated with the second spinor. This approach generates another kind of confusion and it becomes very difficult to evolve towards standing waves solutions by using this formalism.

The Copenhagen school assimilates the particle to a point feature. Therefore, there is little interest in moving towards stationary solutions which necessarily assume a certain spatial extension of the particle. There is truly a conceptual difficulty to evolve towards a physical interpretation of the stationary solutions as part of the Copenhagen school. It is in putting deliberately ourselves outside this framework that it becomes possible to show the consistency of the standing waves approach with all of classical physics.

XXXIV

Reflections on the dimension of the wave function

In this chapter, we are looking for elements that could strengthen us in the idea that the wave functions which appear in the formalism of DIRAC have a physical sense that one can assimilate to the root square of a volumetric energy density. We recall that it is a fundamental aspect of this energy and deterministic approach, which proposes to substitute the notion of volumetric density of probability of presence of Copenhagen school, by the notion of volumetric energy density.

We will use MAXWELL's equations for which we know calculate the volumetric energy density attributed to the electromagnetic field. We will do, on a simple example, an analogous treatment in the formalism of DIRAC, specifying the precautions required in the interpretation of solutions, and we will compare this approach to rigorous expressions of MAXWELL's equations.

I – Potential Equations

MAXWELL's equations in the presence of charges ρ and current $\vec{J}$ are often written in function of the electromagnetic fields, and a possible representation of these equations in the time domain is given below:

$$\begin{aligned}
&\vec{\nabla}\Lambda\vec{E} = -\frac{\partial\vec{B}}{\partial t}\\
&\vec{\nabla}\Lambda\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\\
&\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}\\
&\vec{\nabla}.\vec{B} = 0
\end{aligned} \qquad \text{(XXXIV-1)}$$

We can give a representation equivalent to these equations in terms of scalar potential φ and vector potential $\vec{A}$. These potentials are related to electromagnetic fields by the relations:

$$\begin{aligned}
&\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}\\
&\vec{B} = \vec{\nabla}\Lambda\vec{A}
\end{aligned} \qquad \text{(XXXIV-2)}$$

In terms of components, these two relations are written explicitly in cartesian coordinates:

$$\begin{aligned} E_x &= -\left(\frac{\partial \varphi}{\partial x} + \frac{\partial A_x}{\partial t}\right) & B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ E_y &= -\left(\frac{\partial \varphi}{\partial y} + \frac{\partial A_y}{\partial t}\right) & B_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ E_z &= -\left(\frac{\partial \varphi}{\partial z} + \frac{\partial A_z}{\partial t}\right) & B_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{aligned} \qquad \text{(XXXIV-3)}$$

There is a great liberty in the choice of the scalar potential and vector potentials defined in (XXXIV-2). This liberty of choice is called a choice of gauge, and the only rigorously eligible gauge is the LORENZ gauge:

$$\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0 \qquad \text{(XXXIV-4)}$$

Its legitimacy is imposed by the fact that it is the only gauge that leaves invariant equations of potential by changing frame.

The use of this gauge allows to establish potential equations, whose content is strictly equivalent to MAXWELL's equations:

$$\begin{aligned} \vec{\nabla}^2\varphi - \varepsilon_0\mu_0\frac{\partial^2\varphi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \\ \vec{\nabla}^2\vec{A} - \varepsilon_0\mu_0\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0.\vec{J} \end{aligned} \qquad \text{(XXXIV-5)}$$

The properties of the courant four-vector and the perfectly analogous form of these two relationships allow to group them in a four-dimensional formalism: this property will not be used in the following of this chapter.

II – The formalism of DIRAC

This formalism has its origin in the KLEIN-GORDON equation, which is called back to memory:

$$\vec{\nabla}^2(\psi) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\psi) = \frac{m^2c^2}{\hbar^2}(\psi) \qquad \text{(XXXIV-6)}$$

It appears that this formalism cannot apply to the equations of potential (XXXIV-5) since the second member of these equations cannot be considered as an operator who applies to the variable present in the first member.

For a photon of mass zero, the KLEIN-GORDON equation becomes:

$$\vec{\nabla}^2(\psi)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\psi)=0 \qquad \text{(XXXIV-7)}$$

In the absence of charges and currents, which corresponds to the situation of a stationary regime in cavity, the potential equations are written:

$$\begin{aligned}&\vec{\nabla}^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=0\\&\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=0\end{aligned} \qquad \text{(XXXIV-8)}$$

Formally, these wave equations have the same structure as the KLEIN-GORDON equation without second member. We can try to apply the formalism of DIRAC, while keeping in mind to be careful in interpreting the solutions for the above reasons.

III – Example of energy calculation by MAXWELL's equations

We choose a simple stationary example with a null scalar potential, and only one potential component following z.

$$\begin{aligned}&\varphi=0\\&A_x=0\\&A_y=0\\&A_z=\cos(k_t x_t)\cos(k_x x)\cos(k_y y)\end{aligned} \qquad \text{(XXXIV-9)}$$

We omit the constant which would fix the dimension of the vector potential in $V.m^{-1}.s$.
We infer the components of the electromagnetic field:

$$\begin{aligned}&E_x=-\left(\frac{\partial\phi}{\partial x}+\frac{\partial A_x}{\partial t}\right)=0\\&E_y=-\left(\frac{\partial\phi}{\partial y}+\frac{\partial A_y}{\partial t}\right)=0\\&E_z=-\left(\frac{\partial\phi}{\partial z}+\frac{\partial A_z}{\partial t}\right)=k_t\sin(k_t x_t)\cos(k_x x)\cos(k_y y)\\&B_x=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=-k_y\cos(k_t x_t)\cos(k_x x)\sin(k_y y)\\&B_y=\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}=k_x\cos(k_t x_t)\sin(k_x x)\cos(k_y y)\\&B_z=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=0\end{aligned} \qquad \text{(XXXIV-10)}$$

We infer volumetric density of electric energy and magnetic energy:

$$\Delta W_e = \frac{1}{2}\varepsilon_0 E_z^2 = \frac{1}{2}\varepsilon_0\left(\frac{\partial A_z}{\partial t}\right)^2 = \frac{1}{2}\varepsilon_0\omega^2\sin^2(\omega t)\cos^2(k_x x)\cos^2(k_y y)$$

$$\Delta W_e = \frac{1}{2}\frac{1}{\mu_0}\frac{\omega^2}{c^2}\sin^2(\omega t)\cos^2(k_x x)\cos^2(k_y y) = \frac{1}{2}\frac{1}{\mu_0}k_t^2\sin^2(\omega t)\cos^2(k_x x)\cos^2(k_y y)$$

$$\Delta W_m = \frac{1}{2}\frac{1}{\mu_0}\left(B_y^2 + B_x^2\right) = \frac{1}{2}\frac{1}{\mu_0}\left\{\left(\frac{\partial A_z}{\partial x}\right)^2 + \left(\frac{\partial A_z}{\partial y}\right)^2\right\}$$

$$\Delta W_m = \frac{1}{2}\frac{1}{\mu_0}\left\{k_x^2\cos^2(\omega t)\sin^2(k_x x)\cos^2(k_y y) + k_y^2\cos^2(\omega t)\cos^2(k_x x)\sin^2(k_y y)\right\} \qquad \text{(XXXIV-11)}$$

The total volumetric density of energy is expressed by the relation:

$$\Delta W = \Delta W_e + \Delta W_m = \frac{1}{2}\frac{1}{\mu_0}\left\{\begin{array}{l} k_t^2\sin^2(\omega t)\cos^2(k_x x)\cos^2(k_y y) + \\ k_x^2\cos^2(\omega t)\sin^2(k_x x)\cos^2(k_y y) + k_y^2\cos^2(\omega t)\cos^2(k_x x)\sin^2(k_y y)\end{array}\right\} \qquad \text{(XXXIV-12)}$$

It is this expression that we want to compare to the same quantity deducted from the formalism of DIRAC.

IV – Example of energy calculation by using the formalism of DIRAC

DIRAC system without second member takes a simplified form:

$$\begin{aligned} 0 &= j\frac{\partial\psi_0}{\partial x_t} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z} \\ 0 &= j\frac{\partial\psi_1}{\partial x_t} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z} \\ 0 &= -j\frac{\partial\psi_2}{\partial x_t} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z} \\ 0 &= -j\frac{\partial\psi_3}{\partial x_t} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z} \end{aligned} \qquad \text{(XXXIV-13)}$$

We are seeking a z independent solution such as component ψ_0 is equal to the component E_z of the electromagnetic field:

$$\begin{aligned} \psi_0 &= k_t\sin(k_t x_t)\cos(k_x x)\cos(k_y y) \\ \psi_1 &= 0 \\ \psi_2 &= 0 \\ \psi_3 &= -k_x\cos(k_t x_t)\sin(k_x x)\cos(k_y y) - jk_y\cos(k_t x_t)\cos(k_x x)\sin(k_y y) \end{aligned} \qquad \text{(XXXIV-14)}$$

Although the components of the magnetic field appear in ψ_3, it seems that no physical interpretation can be attached to the distribution of these components (the real one and the other complex), for the reasons given above.

The problem of standardization does not seem to play an important role in the example that we develop, and we will not process it. It is sufficient to compare results to a multiplicative constant close.

In energy approach of this document, the quantity that represents the volumetric density of total energy at each point of space is given by the first component of the DIRAC current:

$$\begin{aligned} J^0 &= \psi_0^* \psi_0 + \psi_1^* \psi_1 + \psi_2^* \psi_2 + \psi_3^* \psi_3 \\ J^0 &= \left(k_t \sin(k_t x_t)\cos(k_x x)\cos(k_y y)\right)^2 + \left(k_x \cos(k_t x_t)\sin(k_x x)\cos(k_y y)\right)^2 \\ &+ \left(k_y \cos(k_t x_t)\cos(k_x x)\sin(k_y y)\right)^2 \end{aligned} \qquad \text{(XXXIV-15)}$$

To a multiplicative constant close, it appears that this quantity represents the total volumetric energy density, such that it can be calculated directly from MAXWELL's equations.

V – Conclusion

The equations of quantum mechanics have been built in a heuristic way. One of the surprising consequences of this construction appears in the fact that these equations do not allow to determine the dimension of the wave function. It seems that this is an element that differentiates the quantum equation of all other equations and relationships of classical physics.

In emerging quantum physics, to give a dimension to the wave function, the founders were faced with the problem of its physical interpretation. Max BORN observed the first that, in all experiments where we could materialize the presence of the particles, the probability of presence was directly related to the square of the wave function. This interpretation had a true experimental significance, and had the advantage of allowing the development of a theory coherent with the principle of indeterminacy, without having to specify exactly what was behind the word "particle".

It is impossible to imagine the presence of a particle without affirming that there is a certain energy attached to this particle. Therefore, there is no inconsistency to propose a construction that leads to give to the square of the wave function, the meaning of a volumetric energy density. The objective of this chapter is to support this proposal.

The potential equation is a wave equation which admits stationary modes as solutions. We can deduce the corresponding electromagnetic fields, then the volumetric density of electromagnetic energy presents at every time and at any point of space.

With the reservations that can be made on the method, when dealing with this same wave equation in the formalism of DIRAC, it appears, in the proposed example, that the volumetric density of energy deducted from this treatment is identical (to a multiplicative constant, close), at any time and at any point of space, to which is deduced from MAXWELL's equations.

There is a real difficulty to find a reasoning that allows to decide definitively on the dimension of the wave function, in a rigorous and convincing manner. This example does not contradict the attribution of the dimension of the square root of a volumetric energy density to the wave function.

XXXV

General conclusion

The energy approach to quantum mechanics proposed in this document is in opposition with the probabilistic approach of the Copenhagen school that dominates today, practically in the form of a single thought, the vision of quantum physics.

This probabilistic vision is installed during the discovery of particular physical properties of the infinitely small world. It is imposed under the constraint of the HEISENBERG uncertainty principle, supported by the equivalence of the SCHRÖDINGER equation with this principle. This vision was the only one that allows to account for all experimental results in consistency with quantum uncertainty principle of indeterminacy. It became inescapable as soon as it remained in agreement with experimental results more accurate and refined. Gradually, the physicists accepted it as a safe theory able to predict all of the properties of the quantum world.

When we adopt a critical point of view, the main problem that arises in the evolution of this theory is that it is based on a postulate given by the indeterminacy theorem. Therefore, everything that is developed using this theory is seen through the filter of this indeterminacy and can only be developed in a probabilistic approach. It becomes impossible to get out of this framework to explain specific phenomena of quantum mechanics.

One can only notice the powerlessness of the Copenhagen school to provide an explanation for certain phenomena like the wave-particle duality. After a century of research, and an investment of best physicists in understanding this phenomenon, it can be estimated without too much risk that there is an intrinsic impossibility to explain this duality by using a probabilistic theory.

Other complex phenomena such as entanglement have probabilistic explanation only by using contortions that come out of the classical laws of physics, and so raises some questions.

The theorem of uncertainty itself is accepted as a property of the quantum world, without any attempt of explanation of the physical nature of its interpretation.

Finally, the imaginary nature of the equation of SCHRÖDINGER and its solutions remains Enigma full, and can find no satisfactory explanation in this formalism.

However, to offer an alternative or an evolution of the Copenhagen school is a delicate challenge because any development must remain consistent with all of the achievements of this school, which is an unmovable base of quantum mechanics.

The energy approach presented in this document meets this challenge.

If we consider the DIRAC equation as a refinement of SCHRÖDINGER and KLEIN-GORDON equations, and if we remember the equivalence between the mechanics of the SCHRÖDINGER equation and HEISENBERG matrices, then, exact solutions to the DIRAC equation must be consistent with the overall results built on the principle of HEISENBERG and SCHRÖDINGER equation.

An analysis of exact solutions in the form of standing waves leads to reconsider the physical meaning of the wave function. The Copenhagen school gives their conjugated product the meaning of a volume density of probability of presence of the particle, appellation which hides the ignorance of the underlying physical phenomena to the concept of presence of a particle and which prevents any progress towards a more subtle understanding of the functioning of the particle.

By adopting the meaning of a volumetric energy density, we offer a way in which, without no other hypothesis that energy conservation and the evolution of energy in the form of standing waves, all issues not resolved by the Copenhagen school find a simple explanation.

The internal energy of the particle is broken down into three energies of different nature: energy of mass, impulse, and wave energy. These energies are exchanged between them and in an ultimate way, with the energy of the vacuum. This exchange of energy is signed and takes an opposite sign depending on whether the particle receives or renders the vacuum energy. DIRAC currents gather volume densities of all these energies according to each direction of space, allowing to apply the local conservation of energy principle in calculating their divergence. The imaginary nature of the wave function has no more any problem of interpretation.

There is no longer need to make ad hoc assumptions, or construct convoluted theories to explain the duality wave-matter, since it appears naturally in the stationary solutions as a very fast exchange between mass energy and wave energy.

The mystery of the principle of indeterminacy disappears. The energy exchanges within the particle show that it is impossible to have simultaneously and to the same place, all of the information on the impulse energy, and mass energy that is relative to the position operator.

It remains only a completely deterministic approach in agreement with all achievements of the Copenhagen school.

This research work is proposed to the scientific community interested in this topic to be shared, criticized, amended and/or may be strengthened.

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First part

Quantum mechanics is based on three fundamental equations of SCHRÖDINGER, KLEIN-GORDON and DIRAC and a probabilistic interpretation induced by the HEISENBERG uncertainty principle. The first part of this document focuses on the DIRAC equation and its general treatment as it is presented today by many scientists. The bibliography is abundant at this level. The following internet documents were consulted over the period 2010-2015.

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Second part

The second part deals with searching of exact solutions to the DIRAC equation in the form of stationary modes. It was not found any work having a direct link with the objective of this document. Consulted publications relate to the search for exact solutions to the DIRAC equation associated with diverse potentials or associated with gravitational problems

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