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# IHARA'S LEMMA AND LEVEL RISING IN HIGHER DIMENSION

*by*

Boyer Pascal

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**Abstract.** — A key ingredient in the Taylor-Wiles proof of Fermat last theorem is the classical Ihara's lemma which is used to rise the modularity property between some congruent galoisian representations. In their work on Sato-Tate, Clozel-Harris-Taylor proposed a generalization of the Ihara's lemma in higher dimension for some similitude groups. The main aim of this paper is then to prove some new instances of this generalized Ihara's lemma by considering some particular non pseudo Eisenstein maximal ideals of unramified Hecke algebras. As a consequence, we prove a level rising statement.

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### Introduction

Let  $F = F^+E$  be a CM field with  $E/\mathbb{Q}$  quadratic imaginary. For  $\overline{B}/F$  a central division algebra with dimension  $d^2$  equipped with a involution of second specie  $*$  and  $\beta \in \overline{B}^{*-1}$ , consider the similitude group  $\overline{G}/\mathbb{Q}$  defined for any  $\mathbb{Q}$ -algebra  $R$  by

$$\overline{G}(R) := \{(\lambda, g) \in R^\times \times (\overline{B}^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp\beta} = \lambda\}$$

with  $\overline{B}^{op} = \overline{B} \otimes_{F,c} F$  where  $c = *_|_F$  is the complex conjugation and  $\sharp_\beta$  the involution  $x \mapsto x^{\sharp\beta} = \beta x^* \beta^{-1}$ . For  $p = uu^c$  decomposed in  $E$ , we have

$$\overline{G}(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (\overline{B}_w^{op})^\times$$

where  $w$  describes the places of  $F$  above  $u$ . We suppose:

- the associated unitary group  $\overline{G}_0(\mathbb{R})$  being compact,
- for any place  $x$  of  $\mathbb{Q}$  inert or ramified in  $E$ , then  $G(\mathbb{Q}_x)$  is quasi-split.
- There exists a place  $v_0$  of  $F$  above  $u$  such that  $\overline{B}_{v_0} \simeq D_{v_0,d}$  is the central division algebra over the completion  $F_{v_0}$  of  $F$  at  $v_0$ , with invariant  $\frac{1}{d}$ .

Fix a prime number  $l \neq p$  and consider a finite set  $S$  of places of  $F$  containing the ramification places  $\text{Bad}$  of  $\overline{B}$ . Let denote  $\mathbb{T}_S/\overline{\mathbb{Z}}_l$  the unramified Hecke algebra of  $G$  outside  $S$ . For a cohomological minimal prime ideal  $\tilde{\mathfrak{m}}$  of  $\mathbb{T}_S$ , we can associate both a near equivalence class of  $\overline{\mathbb{Q}}_l$ -automorphic representation  $\Pi_{\tilde{\mathfrak{m}}}$  and a Galois representation

$$\rho_{\tilde{\mathfrak{m}}} : G_F := \text{Gal}(\overline{F}/F) \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

such that the eigenvalues of the Frobenius morphism at an unramified place  $w$  are given by the Satake's parameter of the local component  $\Pi_{\tilde{\mathfrak{m}},w}$  of  $\Pi_{\tilde{\mathfrak{m}}}$ . The semi-simple class of the reduction modulo  $l$  of  $\rho_{\tilde{\mathfrak{m}}}$  depends only of the maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}$  containing  $\tilde{\mathfrak{m}}$ .

**Conjecture.** — (*Generalized Ihara's lemma by Clozel-Harris-Taylor*) Consider

- an open compact subgroup  $\overline{U}$  of  $\overline{G}(\mathbb{A})$  such that outside  $S$ , its local component is the maximal compact subgroup;
- a place  $w_0 \notin S$  decomposed in  $E$ ;
- a maximal  $\mathfrak{m}$  of  $\mathbb{T}_S$  such that  $\overline{\rho}_{\mathfrak{m}}$  is absolutely irreducible.

Let  $\overline{\pi}$  be an irreducible sub-representation of  $\mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}) / \overline{U}^{w_0}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ , where  $\overline{U} = \overline{U}_{w_0} \overline{U}^{w_0}$ , then its local component  $\overline{\pi}_{w_0}$  at  $w_0$  is generic.

*Remark:* In its classical version for  $GL_2$ , Ihara's lemma is used to rise the modularity property between some congruent galoisian representations and so was the role of this higher dimensional version in Clozel-Harris-Taylor paper on the Sato-Tate conjecture. Shortly after Taylor found an argument to avoid Ihara's lemma. However this conjecture remains highly interesting, see for example works of Clozel-Thorne [9], or Emerton-Helm [12].

The main result of this paper is the following instances of the previous conjecture.

**Theorem.** — *The previous generalized Ihara's conjecture is true if the maximal ideal  $\mathfrak{m}$  verifies the following extra properties.*

- (H1) *There exists a place  $v_1 \in \text{Spl}$  such that the restriction  $\overline{\rho}_{\mathfrak{m}, v_1}$  of  $\overline{\rho}_{\mathfrak{m}}$  to the decomposition group at  $v_1$  is absolutely irreducible.*
- (H2) *There exists a place  $w_1$  of  $F$  outside  $S$  such that  $w_1 \mid_{\mathbb{Q}}$  splits in  $E$  and such that the set of eigenvalues of  $\overline{\rho}_{\mathfrak{m}}(\text{Frob}_{w_1})$  does not contain a subset of the form  $\{\alpha, q_{w_1} \alpha\}$  where  $q_{w_1}$  is the cardinal of the residual field of  $F_{w_1}$ .<sup>(1)</sup>*
- (H3) *The image of  $\overline{\rho}_{\mathfrak{m}, w_0}$  in its Grothendieck group is multiplicity free and<sup>(2)</sup> does not contain any full Zelevinsky line.*

*Remark:* concerning (H2), you can replace it by any hypothesis which insures the freeness of the cohomology of the Kottwitz-Harris-Taylor Shimura variety of §1.1, cf. for example point (2) before theorem 2.2.1.

To prove this result, we first translate such a property to the cohomology group of middle degree of the Kottwitz-Harris-Taylor Shimura variety  $X_{\mathcal{I}}$  associated to the similitude group  $G/\mathbb{Q}$  such that

<sup>(1)</sup>In particular  $q_{w_1}$  can't be congruent to 1 modulo  $l$ .

<sup>(2)</sup>Using the main result of [8] we could take off the condition about not containing a full Zelevinsky line, cf. the last remark of §3.2.

- $G(\mathbb{A}^\infty) = \overline{G}(\mathbb{A}^{\infty,p}) \times GL_d(F_{v_0}) \times \prod_{w|u, w \neq v_0} (\overline{B}_w^{op})^\times$ ,
- the signatures of  $G(\mathbb{R})$  are  $(1, d-1) \times (0, d) \times \cdots \times (0, d)$ .

In particular to each prime ideal  $\tilde{\mathfrak{m}}$  of  $\mathbb{T}_S$ , is associated a  $\overline{\mathbb{Q}}_l$ -irreducible automorphic representation  $\Pi_{\tilde{\mathfrak{m}}}$  of  $G(\mathbb{A}_{\mathbb{Q}})$  whose Satake parameters at finite places outside  $S$  are prescribed by  $\tilde{\mathfrak{m}}$ . We then compute the cohomology groups of the geometric generic fiber of  $X_{\mathcal{I}}$  through the spectral sequence of vanishing cycles at the place  $v_0$ . Thanks to (H2), we know, cf. [7], the  $H^i(X_U, \overline{\mathbb{Z}}_l)_m$  to be free and so,  $H^i(X_U, \overline{\mathbb{F}}_l)_m = (0)$  for  $i \neq d-1$ . The contribution of the supersingular points of the special fiber at  $v_0$ , allows us to associate to an irreducible sub-representation  $\bar{\pi}$  of  $\mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}) / \overline{U}^{w_0}, \overline{\mathbb{F}}_l)_m$ , an irreducible sub-representation  $\pi$  of  $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_m$ , such that  $\pi^{\infty, v_0} \simeq \bar{\pi}^{\infty, v_0}$ . We then try to prove the genericness of  $\pi_{w_0}$  by proving, using (H1) and (H3), such genericness property of irreducible submodules of  $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_m$ . More precisely, (H1) allows us to show, cf. §3.1, that the lattices of isotypic components of  $H^{d-1}(X_U, \overline{\mathbb{Q}}_l)_m$  given by the  $\overline{\mathbb{Z}}_l$ -cohomology, can be written as a tensorial product of stable lattices for the  $G(\mathbb{A}^\infty)$  and the Galois actions. Finally (H3) is needed to control the combinatorics and to avoid the issue between cuspidal representations which are not supercuspidal.

*Remark:* we can understand the need of (H1), (H2) and (H3) by the fact we prove the genericness property of irreducible submodule of the whole of  $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_m$  instead of just the part given by the supersingular stratum to this cohomology group.

To state our application to level rising, denote  $\mathcal{S}_{w_0}(\mathfrak{m})$  the supercuspidal support of the modulo  $l$  reduction of  $\Pi_{\tilde{\mathfrak{m}}, w_0}$  for any prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ : it depends only on  $\mathfrak{m}$ . By (H3) this support is multiplicity free and we first break it  $\mathcal{S}_{w_0}(\mathfrak{m}) = \coprod_{\varrho} \mathcal{S}_{\varrho}(\mathfrak{m})$  according to the inertial equivalence classes  $\varrho$  of irreducible  $\overline{\mathbb{F}}_l$ -representations of some  $GL_{g(\varrho)}(F_{w_0})$  with  $1 \leq g(\varrho) \leq d$ . For any such  $\varrho$  we then denote  $l_1(\varrho) \geq \cdots \geq l_{r(\varrho)}(\varrho) \geq 1$ , such that  $\mathcal{S}_{\varrho}(\mathfrak{m})$  can be written as the union of  $r(\varrho)$  Zelevinsky unlinked segments of length  $l_i(\varrho)$

$$[\varrho\nu_i^k, \bar{\rho}\nu^{k+l_i(\varrho)}] = \{\varrho\nu^k, \varrho\nu^{k+1}, \dots, \varrho\nu^{k+l_i(\varrho)}\}.$$

Then for any minimal prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ , and  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$ , we write its local component  $\Pi_{w_0} \simeq \times_{\varrho} \Pi_{w_0}(\varrho)$  and  $\Pi_{w_0}(\varrho) \simeq \times_{i=1}^{r(\varrho)} \Pi_{w_0}(\varrho, i)$  where for each  $1 \leq i \leq r(\varrho)$ , the modulo  $l$  reduction of the supercuspidal support

of  $\Pi_{w_0}(\varrho, i)$  is, with the notations of the previous section, those of the Zelevinsky segment  $[\varrho\nu^{\delta_i}, \varrho\nu^{\delta_i+l_i(\varrho)}]$ .

**Proposition.** — *Take a maximal ideal  $\mathfrak{m}$  verifying the hypothesis (H1), (H2) and (H3). Let  $\varrho_0$  such that  $\mathcal{S}_{\varrho_0}(\mathfrak{m})$  is non empty and consider  $1 \leq i \leq r(\varrho_0)$ . Then there exists a minimal prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  and an automorphic representation  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$  such that with the previous notation  $\Pi_{w_0}(\varrho_0, i)$  is non degenerate, i.e. isomorphic to  $\text{St}_{l_i(\varrho)}(\pi_{w_0})$  for some irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representation  $\pi_{w_0}$ .*

In particular when there is only one segment, which is always the case for  $GL_2$ , then the result is optimal.

## 1. Shimura variety of Kottwitz-Harris-Taylor type

**1.1. Geometry.** — Recall from the introduction that a prime number  $l$  is fixed, distinct from all others prime numbers which will be considered in the following. Let  $F = F^+E$  be a CM field with  $E/\mathbb{Q}$  imaginary quadratic such that  $l$  is unramified, and  $F^+$  totally real with a fixed embedding  $\tau : F^+ \hookrightarrow \mathbb{R}$ . For a place  $v$  of  $F$ , we denote  $F_v$  the completion of  $F$  at  $v$ , with ring of integers  $\mathcal{O}_v$ , uniformizer  $\varpi_v$  and residual field  $\kappa(v)$  with cardinal  $q_v$ .

Let  $B$  be a central division algebra over  $F$  of dimension  $d^2$  such that at any place  $x$  of  $F$ , either  $B_x$  is split either it's a division algebra. We moreover suppose the existence of an involution of second specie  $*$  on  $B$  such that  $*|_F$  is the complex conjugation  $c$ . For  $\beta \in B^{*-1}$ , we denote  $\sharp_\beta : x \mapsto \beta x^* \beta^{-1}$  and let  $G/\mathbb{Q}$  such that for any  $\mathbb{Q}$ -algebra  $R$ ,

$$G(R) = \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\},$$

with  $B^{op} = B \otimes_c F$ . If  $x = yy^c$  is split in  $E$  then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times$$

where, identifying the places of  $F^+$  above  $x$  with those of  $F$  above  $y$ , we write  $x = \prod_i z_i$ . Moreover we can impose that

- if  $x$  is inert in  $E$  then  $G(\mathbb{Q}_x)$  is quasi-split,
- the signature of  $G(\mathbb{R})$  are  $(1, d-1) \times (0, d) \times \cdots \times (0, d)$ .

With the notations of the introduction we have

$$G(\mathbb{A}^\infty) = \overline{G}(\mathbb{A}^{\infty,p}) \times \left( \mathbb{Q}_{p_{v_0}}^\times GL_d(F_{v_0}) \times \prod_{\substack{w|u \\ w \neq v_0}} (\overline{B}_w^{op})^\times \right).$$

**1.1.1. Definition.** — We denote  $\text{Bad}$  the set of places  $w$  of  $F$  such that  $B_w$  is non split. Let  $\text{Spl}$  the set  $w$  of finite places of  $F$  not in  $\text{Bad}$  such that  $w|_{\mathbb{Q}}$  is split in  $E$ . For such a place  $w$  with  $p = w|_{\mathbb{Q}}$ , we write abusively

$$G(\mathbb{A}^w) = G(\mathbb{A}^p) \times \mathbb{Q}_p^\times \times \prod_{\substack{u|p \\ u \neq w}} (B_u^{op})^\times,$$

and  $G(F_w) = GL_d(F_w)$ .

*Remark:* with the notations of the introduction, the role of  $w$  in the previous definition will be taken by  $v_0$ ,  $v_1$  or  $w_0$ .

**1.1.1. Notation.** — For all open compact subgroup  $U^p$  of  $G(\mathbb{A}^{\infty,p})$  and  $m = (m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r$ , we consider

$$U^p(m) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^r \text{Ker}(\mathcal{O}_{B_{v_i}}^\times \longrightarrow (\mathcal{O}_{B_{v_i}}/\mathcal{P}_{v_i}^{m_i})^\times).$$

For  $w_0$  one of the  $v_i$  and  $n \in \mathbb{N}$ , we also introduce  $U^{w_0}(n) := U^p(0, \dots, 0, n, 0, \dots, 0)$ .

We then denote  $\mathcal{I}$  for the set of these  $U^p(m)$  such that there exists a place  $x$  for which the projection from  $U^p$  to  $G(\mathbb{Q}_x)$  doesn't contain any element with finite order except the identity, cf. [13] below of page 90.

Attached to each  $I \in \mathcal{I}$  is a Shimura variety  $X_I \rightarrow \text{Spec } \mathcal{O}_v$  of type Kottwitz-Harris-Taylor and we denote  $X_{\mathcal{I}} = (X_I)_{I \in \mathcal{I}}$  the projective system: recall that the transition morphisms  $r_{J,I} : X_J \rightarrow X_I$  are finite flat and even etale when  $m_1(J) = m_1(I)$ . This projective system is then equipped with a Hecke action of  $G(\mathbb{A}^\infty) \times \mathbb{Z}$ , where the action of  $z$  in the Weil group  $W_v$  of  $F_v$  is given par  $-\text{deg}(z) \in \mathbb{Z}$  where  $\text{deg} = \text{val} \circ \text{Art}_v^{-1}$ , where  $\text{Art}_v^{-1} : W_v^{ab} \simeq F_v^\times$  is the Artin isomorphism which sends geometric Frobenius to uniformizers.

**1.1.2. Notations.** — (cf. [2] §1.3) Let  $I \in \mathcal{I}$ ,

- the special fiber of  $X_I$  will be denoted  $X_{I,s}$  and its geometric special fiber  $X_{I,\bar{s}} := X_{I,s} \times \text{Spec } \bar{\mathbb{F}}_p$ .
- For  $1 \leq h \leq d$ , let  $X_{I,\bar{s}}^{\geq h}$  (resp.  $X_{I,\bar{s}}^{=h}$ ) be the closed (resp. open) Newton stratum of height  $h$ , defined as the subscheme where the connected component of the universal Barsotti-Tate group is of rank greater or equal to  $h$  (resp. equal to  $h$ ).

*Remark:*  $X_{I,\bar{s}}^{\geq h}$  is of pure dimension  $d - h$ . For  $1 \leq h < d$ , the Newton stratum  $X_{I,\bar{s}}^{=h}$  is geometrically induced under the action of the parabolic subgroup  $P_{h,d-h}(F_v)$ , defined as the stabilizer of the first  $h$  vectors of the canonical basis of  $F_v^d$ . Concretely this means there exists a closed subscheme  $X_{I,\bar{s},1_h}^{=h}$  stabilized by the Hecke action of  $P_{h,d-h}(F_v)$  and such that

$$X_{I,\bar{s}}^{=h} \simeq X_{I,\bar{s},1_h}^{=h} \times_{P_{h,d-h}(F_v)} GL_d(F_v).$$

**1.1.3. Notations.** — Let denote

$$i^h : X_{I,\bar{s}}^{\geq h} \hookrightarrow X_{I,\bar{s}}^{\geq 1}, \quad j^{\geq h} : X_{I,\bar{s}}^{=h} \hookrightarrow X_{I,\bar{s}}^{\geq h}$$

and  $j^{=h} = i^h j^{\geq h}$ .

**1.2. Jacquet-Langlands correspondence and  $\varrho$ -type.** — For a representation  $\pi_v$  of  $GL_d(F_v)$  and  $n \in \frac{1}{2}\mathbb{Z}$ , set  $\pi_v\{n\} := \pi_v \otimes q_v^{-n \text{ val} \circ \det}$ . Recall that the normalized induction of two representations  $\pi_{v,1}$  and  $\pi_{v,2}$  of respectively  $GL_{n_1}(F_v)$  and  $GL_{n_2}(F_v)$  is

$$\pi_1 \times \pi_2 := \text{ind}_{P_{n_1, n_1+n_2}(F_v)}^{GL_{n_1+n_2}(F_v)} \pi_{v,1}\left\{\frac{n_2}{2}\right\} \otimes \pi_{v,2}\left\{-\frac{n_1}{2}\right\}.$$

A representation  $\pi_v$  of  $GL_d(F_v)$  is called *cuspidal* (resp. *supercuspidal*) if it's not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true for  $\bar{\mathbb{F}}_l$ .

*Remark:* the modulo  $l$  reduction of an irreducible  $\bar{\mathbb{Q}}_l$ -representation is still irreducible and cuspidal but not necessary supercuspidal. In this last case, its supercuspidal support is a Zelevinsky segment associated to some unique inertial equivalent classe  $\varrho$ , where  $\varrho$  is an irreducible supercuspidal  $\bar{\mathbb{F}}_l$ -representation. Thanks to (H3) we will not be concerned by this subtlety.



**1.2.1. Definition.** — We say that  $\pi_v$  is of type  $\varrho$  when the supercuspidal support of its modulo  $l$  reduction is contained in the Zelevinsky line of  $\varrho$ .

**1.2.1. Definition.** — (see [17] §9 and [3] §1.4) Let  $g$  be a divisor of  $d = sg$  and  $\pi_v$  an irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representation of  $GL_g(F_v)$ . Then the normalized induced representation

$$\pi_v\left\{\frac{1-s}{2}\right\} \times \pi_v\left\{\frac{3-s}{2}\right\} \times \cdots \times \pi_v\left\{\frac{s-1}{2}\right\}$$

holds a unique irreducible quotient (resp. subspace) denoted  $\text{St}_s(\pi_v)$  (resp.  $\text{Speh}_s(\pi_v)$ ); it's a generalized Steinberg (resp. Speh) representation.

*Remark:* if  $\chi_v$  is a character of  $F_v^\times$  then  $\text{Speh}_s(\chi_v) = \chi_v \circ \det$ .

The local Jacquet-Langlands correspondance is a bijection between irreducible essentially square integrable representations of  $GL_d(F_v)$ , i.e. representations of the type  $\text{St}_s(\pi_v)$  for  $\pi_v$  cuspidal, and irreducible representations of  $D_{v,d}^\times$  where  $D_{v,d}$  is the central division algebra over  $F_v$  with invariant  $\frac{1}{d}$ .

**1.2.2. Notation.** — We will denote  $\pi_v[s]_D$  the irreducible representation of  $D_{v,d}^\times$  associated to  $\text{St}_s(\pi_v)$  by the local Jacquet-Langlands correspondance.

We denote  $\mathcal{R}_{\overline{\mathbb{F}}_l}(d)$  the set of equivalence classes of irreducible  $\overline{\mathbb{F}}_l$ -representations of  $D_{v,d}^\times$ . For  $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(d)$ , let  $\mathcal{C}_{\bar{\tau}}$  be the sub-category of smooth  $\mathbb{Z}_l^{nr}$ -representations of  $D_{v,d}^\times$  with objects whose irreducible sub-quotients are isomorphic to a sub-quotient of  $\bar{\tau}|_{D_{v,d}^\times}$ . Note that  $\mathcal{C}_{\bar{\tau}}$  is a direct factor inside  $\text{Rep}_{\mathbb{Z}_l^{nr}}^\infty(D_{v,d}^\times)$  so that every  $\mathbb{Z}_l^{nr}$ -representation  $V_{\mathbb{Z}_l^{nr}}$  of  $D_{v,d}^\times$  can be decomposed as a direct sum

$$V_{\mathbb{Z}_l^{nr}} \simeq \bigoplus_{\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(d)} V_{\mathbb{Z}_l^{nr}, \bar{\tau}}$$

where  $V_{\mathbb{Z}_l^{nr}, \bar{\tau}}$  is an object of  $\mathcal{C}_{\bar{\tau}}$ .

Let  $\pi_v$  be an irreducible cuspidal representation of  $GL_g(F_v)$  and fix an integer  $s \geq 1$ . Then the modulo  $l$  reduction of  $\text{Speh}_s(\pi)$  is irreducible, cf. [11] §2.2.3.

**1.2.3. Notation.** — When the modulo  $l$  reduction of  $\pi$ , denoted  $\varrho$ , is supercuspidal, then we will denote  $\text{Speh}_s(\varrho)$  the modulo  $l$  reduction of  $\text{Speh}_s(\pi)$ : we call it a  $\overline{\mathbb{F}}_l$ -superSpeh representation.

By [11] 3.1.4, we have a bijection

$$\begin{aligned} & \left\{ \overline{\mathbb{F}}_l\text{-superSpeh irreducible representations of } GL_d(F_v) \right\} \\ & \simeq \\ & \left\{ \overline{\mathbb{F}}_l\text{-representations irreducible of } D_{v,d}^\times \right\} \quad (1.2.1) \end{aligned}$$

compatible with the modulo  $l$  reduction in the sense that if  $\pi_v$  is a lifting of  $\varrho$ , then the modulo  $l$  reduction of  $\pi^\vee[s]_D$  matches through the previous bijection, with the superSpeh  $\text{Speh}_s(\varrho)$ .

**1.2.2. Definition.** — A  $\overline{\mathbb{F}}_l$ -representation of  $D_{v,d}^\times$  (resp. an irreducible cuspidal representation of  $GL_d(F_v)$ ) is said of type  $\varrho$  if all its irreducible sub-quotient are, through the previous bijection, associated to some superSpeh  $\text{Speh}_s(\varrho)$  (resp. its supercuspidal support belongs to the Zelevinsky line of  $\varrho$ ).

Recall that if  $\epsilon(\varrho)$  is the cardinal of the Zelevinsky line associated to  $\varrho$  and, cf. [16] p.51, then let

$$m(\varrho) = \begin{cases} \epsilon(\varrho), & \text{si } \epsilon(\varrho) > 1; \\ l, & \text{sinon.} \end{cases}$$

**1.2.4. Notation.** — Let  $r(\varrho)$  the biggest integer  $i$  such that  $l^i$  divides  $\frac{d}{m(\varrho)g}$  and define

$$g_{-1}(\varrho) = g \quad \text{and} \quad \forall 0 \leq i \leq r(\varrho), \quad g_i(\varrho) = m(\varrho)l^i g.$$

We also denote  $s_i(\varrho) := \lfloor \frac{d}{g_i(\varrho)} \rfloor$ .

Then if  $\pi_v$  is an irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representation of  $GL_k(F_v)$  with type  $\varrho$ , then there exists  $i$  such that  $k = g_i$ . We say that  $\pi_v$  is of  $\varrho$ -type  $i$  and we denote  $\text{Scusp}_i(\varrho)$  the set of inertial equivalence classes of irreducible cuspidal representations of  $\varrho$ -type  $i$ .

**1.2.5. Notation.** — For  $\varrho$  an irreducible supercuspidal  $\overline{\mathbb{F}}_l$ -representation of  $GL_g(F_v)$ , we denote  $\mathcal{R}_\varrho = \coprod_{s \geq 1} \mathcal{R}_\varrho(sg)$  where  $\mathcal{R}_\varrho(sg)$  is the set of equivalence classes of irreducible  $\overline{\mathbb{F}}_l$ -representations of  $D_{v,sg}^\times$  of type  $\varrho$ .

**1.3. Harris-Taylor local systems.** — Let  $\pi_v$  be an irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representation of  $GL_g(F_v)$  and fix  $t \geq 1$  such that  $tg \leq d$ . Thanks to Igusa varieties, Harris and Taylor constructed a local system on  $X_{\mathcal{I}, \bar{s}, \overline{1}_h}^{=tg}$

$$\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1}_h} = \bigoplus_{i=1}^{e_{\pi_v}} \mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_{v,i})_{\overline{1}_h}$$

where  $(\pi_v[t]_D)_{|D_{v,h}^\times} = \bigoplus_{i=1}^{e_{\pi_v}} \rho_{v,i}$  with  $\rho_{v,i}$  irreducible. The Hecke action of  $P_{tg, d-tg}(F_v)$  is then given through its quotient  $GL_{d-tg} \times \mathbb{Z}$ . These local systems have stable  $\overline{\mathbb{Z}}_l$ -lattices and we will write simply  $\mathcal{L}(\pi_v[t]_D)_{\overline{1}_h}$  for any  $\overline{\mathbb{Z}}_l$ -stable lattice that we don't want to specify.

**1.3.1. Notations.** — For  $\Pi_t$  any representation of  $GL_{tg}$  and  $\Xi : \frac{1}{2}\mathbb{Z} \rightarrow \overline{\mathbb{Z}}_l^\times$  defined by  $\Xi(\frac{1}{2}) = q^{1/2}$ , we introduce

$$\widetilde{HT}_1(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]_D)_{\overline{1}_h} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

and its induced version

$$\widetilde{HT}(\pi_v, \Pi_t) := \left( \mathcal{L}(\pi_v[t]_D)_{\overline{1}_h} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}} \right) \times_{P_{tg, d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of  $P_{tg, d-tg}(F_v)$  acts trivially and the action of

$$(g^{\infty, v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty, v}) \times P_{tg, d-tg}(F_v) \times W_v$$

is given

- by the action of  $g_v^c$  on  $\Pi_t$  and  $\deg(\sigma_v) \in \mathbb{Z}$  on  $\Xi^{\frac{tg-d}{2}}$ , and
- the action of  $(g^{\infty, v}, g_v^{et}, \text{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty, v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$  on  $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1}_h} \otimes \Xi^{\frac{tg-d}{2}}$ .

We also introduce

$$HT(\pi_v, \Pi_t)_{\overline{1}_h} := \widetilde{HT}(\pi_v, \Pi_t)_{\overline{1}_h}[d - tg],$$

and the perverse sheaf

$$P(t, \pi_v)_{\overline{1}_h} := j_{\overline{1}_h, !}^{=tg} HT(\pi_v, \text{St}_t(\pi_v))_{\overline{1}_h} \otimes \mathbb{L}(\pi_v),$$

and their induced version,  $HT(\pi_v, \Pi_t)$  and  $P(t, \pi_v)$ , where

$$j = h = i^h \circ j^{\geq h} : X_{\mathcal{I}, \bar{s}}^{\leq h} \hookrightarrow X_{\mathcal{I}, \bar{s}}^{\geq h} \hookrightarrow X_{\mathcal{I}, \bar{s}}$$

and  $\mathbb{L}^\vee$  is the local Langlands correspondence.

*Remark:* recall that  $\pi'_v$  is said inertially equivalent to  $\pi_v$  if there exists a character  $\zeta : \mathbb{Z} \rightarrow \overline{\mathbb{Q}}_l^\times$  such that  $\pi'_v \simeq \pi_v \otimes (\zeta \circ \text{val} \circ \det)$ . Note, cf. [2] 2.1.4, that  $P(t, \pi_v)$  depends only on the inertial class of  $\pi_v$  and

$$P(t, \pi_v) = e_{\pi_v} \mathcal{P}(t, \pi_v)$$

where  $\mathcal{P}(t, \pi_v)$  is an irreducible perverse sheaf. When we want to speak of the  $\overline{\mathbb{Q}}_l$ -versions we will add it on the notations.

**1.3.1. Definition.** — We will say that  $HT(\pi_v, \Pi_t)$  or  $\mathcal{P}(t, \pi_v)$  is of type  $\rho$  if  $\pi_v$  is.

**1.3.2. Lemma.** — If  $\rho \otimes \sigma$  is a  $GL_d(F_v) \times W_v$  equivariant irreducible sub-quotient of  $H^i(X_{\mathcal{I}, \bar{s}_v}, \mathcal{P}(\pi_v, t) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$  then

- $\sigma$  is an irreducible sub-quotient of the modulo  $l$  reduction of  $\mathbb{L}(\pi_v \otimes \chi_v)$ , where  $\chi_v$  is an unramified character of  $F_v$ , and
- $\rho$  is an irreducible sub-quotient of the modulo  $l$  reduction of a induced representation of the following shape  $\text{St}_t(\pi_v \otimes \chi_v) \times \psi_v$  where  $\psi_v$  is an entire irreducible representation of  $GL_{d-tg}(F_v)$ .

*Proof.* — The result follows directly from the description of the actions given previously.  $\square$

As usually for  $\sigma$  a representation of  $W_v$  and  $n \in \frac{1}{2}\mathbb{Z}$ , we will denote  $\sigma(n)$  the twisted representation  $g \mapsto \sigma(g) | \text{Art}_v^{-1}(g) |^n$  where  $| - |$  is the absolute value of  $F_v$ .

**1.4. Free perverse sheaf.** — Let  $S = \text{Spec } \mathbb{F}_q$  and  $X/S$  of finite type, then the usual  $t$ -structure on  $\mathcal{D}(X, \overline{\mathbb{Z}}_l) := D_c^b(X, \overline{\mathbb{Z}}_l)$  is

$$\begin{aligned} A \in {}^p\mathcal{D}^{\leq 0}(X, \overline{\mathbb{Z}}_l) &\Leftrightarrow \forall x \in X, \mathcal{H}^k i_x^* A = 0, \forall k > -\dim \overline{\{x\}} \\ A \in {}^p\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l) &\Leftrightarrow \forall x \in X, \mathcal{H}^k i_x^! A = 0, \forall k < -\dim \overline{\{x\}} \end{aligned}$$

where  $i_x : \text{Spec } \kappa(x) \hookrightarrow X$  and  $\mathcal{H}^k(K)$  is the  $k$ -th sheaf of cohomology of  $K$ .

**1.4.1. Notation.** — Let denote  ${}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l)$  the heart of this  $t$ -structure with associated cohomology functors  ${}^p\mathcal{H}^i$ . For a functor  $T$  we denote  ${}^pT := {}^p\mathcal{H}^0 \circ T$ .

The category  ${}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l)$  is abelian equipped with a torsion theory  $(\mathcal{T}, \mathcal{F})$  where  $\mathcal{T}$  (resp.  $\mathcal{F}$ ) is the full subcategory of objects  $T$  (resp.  $F$ ) such that  $l^N 1_T$  is trivial for some large enough  $N$  (resp.  $l.1_F$  is a monomorphism). Applying Grothendieck-Verdier duality, we obtain

$$\begin{aligned} {}^{p+}\mathcal{D}^{\leq 0}(X, \overline{\mathbb{Z}}_l) &:= \{A \in {}^p\mathcal{D}^{\leq 1}(X, \overline{\mathbb{Z}}_l) : {}^p\mathcal{H}^1(A) \in \mathcal{T}\} \\ {}^{p+}\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l) &:= \{A \in {}^p\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l) : {}^p\mathcal{H}^0(A) \in \mathcal{F}\} \end{aligned}$$

with heart  ${}^{p+}\mathcal{C}(X, \overline{\mathbb{Z}}_l)$  equipped with its torsion theory  $(\mathcal{F}, \mathcal{T}[-1])$ .

**1.4.2. Definition.** — (cf. [5] §1.3) Let

$$\mathcal{F}(X, \overline{\mathbb{Z}}_l) := {}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{C}(X, \overline{\mathbb{Z}}_l) = {}^p\mathcal{D}^{\leq 0}(X, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l)$$

the quasi-abelian category of free perverse sheaves over  $X$ .

*Remark:* for an object  $L$  of  $\mathcal{F}(X, \overline{\mathbb{Z}}_l)$ , we will consider filtrations

$$L_1 \subset L_2 \subset \dots \subset L_e = L$$

such that for every  $1 \leq i \leq e-1$ ,  $L_i \hookrightarrow L_{i+1}$  is a strict monomorphism, i.e.  $L_{i+1}/L_i$  is an object of  $\mathcal{F}(X, \overline{\mathbb{Z}}_l)$ .

For a free  $L \in \mathcal{F}(X, \Lambda)$ , we consider the following diagram

$$\begin{array}{ccccc} & & L & & \\ & \nearrow \text{can}_{!,L} & & \searrow \text{can}_{*,L} & \\ {}^{p+}j_!j^*L & \twoheadrightarrow & {}^p j_{!*}j^*L & \xrightarrow{+} & {}^{p+}j_{!*}j^*L & \twoheadrightarrow & {}^p j_*j^*L \end{array}$$

where below is, cf. the remark following 1.3.12 of [5], the canonical factorisation of  ${}^{p+}j_!j^*L \longrightarrow {}^p j_*j^*L$  and where the maps  $\text{can}_{!,L}$  and  $\text{can}_{*,L}$  are given by the adjunction property. Consider now  $X$  equipped with a stratification

$$X = X^{\geq 1} \supset X^{\geq 2} \supset \dots \supset X^{\geq d},$$

and let  $L \in \mathcal{F}(X, \overline{\mathbb{Z}}_l)$ . For  $1 \leq h < d$ , let denote  $X^{1 \leq h} := X^{\geq 1} - X^{\geq h+1}$  and  $j^{1 \leq h} : X^{1 \leq h} \hookrightarrow X^{\geq 1}$ . We then define

$$\text{Fil}_!^r(L) := \text{Im}_{\mathcal{F}} \left( {}^{p+}j_!^{1 \leq r} j^{1 \leq r,*} L \longrightarrow L \right),$$

which gives a filtration

$$0 = \mathrm{Fil}_!^0(L) \subset \mathrm{Fil}_!^1(L) \subset \mathrm{Fil}_!^1(L) \cdots \subset \mathrm{Fil}_!^{d-1}(L) \subset \mathrm{Fil}_!^d(L) = L.$$

Dually  $\mathrm{CoFil}_{*,r}(L) = \mathrm{Coim}_{\mathcal{F}}\left(L \rightarrow {}^p j_*^{1 \leq r} j^{1 \leq r,*} L\right)$ , define a cofiltration

$$\begin{aligned} L = \mathrm{CoFil}_{\mathfrak{S},*,d}(L) &\rightarrow \mathrm{CoFil}_{\mathfrak{S},*,d-1}(L) \rightarrow \cdots \\ &\cdots \rightarrow \mathrm{CoFil}_{\mathfrak{S},*,1}(L) \rightarrow \mathrm{CoFil}_{\mathfrak{S},*,0}(L) = 0, \end{aligned}$$

and a filtration

$$0 = \mathrm{Fil}_*^{-d}(L) \subset \mathrm{Fil}_*^{1-d}(L) \subset \cdots \subset \mathrm{Fil}_*^0(L) = L$$

where  $\mathrm{Fil}_*^{-r}(L) := \mathrm{Ker}_{\mathcal{F}}(L \rightarrow \mathrm{CoFil}_{*,r}(L))$ .

*Remark:* these two constructions are exchanged by Grothendieck-Verdier duality,  $D\left(\mathrm{CoFil}_{\mathfrak{S},!, -r}(L)\right) \simeq \mathrm{Fil}_{\mathfrak{S},*}^{-r}(D(L))$  and  $D\left(\mathrm{CoFil}_{\mathfrak{S},*,r}(L)\right) \simeq \mathrm{Fil}_{\mathfrak{S},!}^r(D(L))$ .

We can also refine the previous filtrations, cf. [5] proposition 2.3.3, to obtain exhaustive filtrations

$$\begin{aligned} 0 = \mathrm{Fill}_!^{-2^{d-1}}(L) &\subset \mathrm{Fill}_!^{-2^{d-1}+1}(L) \subset \cdots \\ &\cdots \subset \mathrm{Fill}_!^0(L) \subset \cdots \subset \mathrm{Fill}_!^{2^{d-1}-1}(L) = L, \quad (1.4.0) \end{aligned}$$

such that the graduate  $\mathrm{gr}^k(L)$  are simple over  $\overline{\mathbb{Q}}_l$ , i.e. verify  ${}^p j_*^{j=h,*} \mathrm{gr}^k(L) \hookrightarrow_+ \mathrm{gr}^k(L)$  for some  $h$ . Dually we can construct a cofiltration

$$L = \mathrm{CoFill}_{*,2^{d-1}}(L) \rightarrow \mathrm{CoFill}_{*,2^{d-1}-1}(L) \rightarrow \cdots \rightarrow \mathrm{CoFill}_{*,0}(L) = 0$$

and a filtration  $\mathrm{Fill}_*^{-r}(L) := \mathrm{Ker}_{\mathcal{F}}(L \rightarrow \mathrm{CoFill}_{*,r}(L))$ .

## 1.5. Vanishing cycles perverse sheaf. —

**1.5.1. Notation.** — For  $I \in \mathcal{I}$ , let

$$\Psi_{I,\Lambda} := R\Psi_{\eta_v,I}(\Lambda[d-1])\left(\frac{d-1}{2}\right)$$

be the vanishing cycle autodual perverse sheaf on  $X_{I,\bar{s}_v}$ . When  $\Lambda = \overline{\mathbb{Z}}_l$ , we will simply write  $\Psi_{\mathcal{I}}$ .

Recall the following result of [13] relating  $\Psi_{\mathcal{I}}$  with Harris-Taylor local systems.

**1.5.2. Proposition.** — (cf. [13] proposition IV.2.2 and §2.4 of [2])  
 There is an isomorphism  $G(\mathbb{A}^{\infty,v}) \times P_{h,d-h}(F_v) \times W_v$ -equivariant

$$\mathrm{ind}_{(D_{v,h}^\times)^0 \varpi_v^{\mathbb{Z}}}^{D_{v,h}^\times} (\mathcal{H}^{h-d-i} \Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l})|_{X_{\mathcal{I}, \overline{s}, \overline{1}_h} = h} \simeq \bigoplus_{\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)} \mathcal{L}_{\overline{\mathbb{Z}}_l, \overline{1}_h}(\mathcal{U}_{\bar{\tau}, \mathbb{N}}^{h-1-i}),$$

where

- $\mathcal{R}_{\overline{\mathbb{F}}_l}(h)$  is the set of equivalence classes of irreducible  $\overline{\mathbb{F}}_l$ -representations of  $D_{v,h}^\times$ ;
- for  $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)$  and  $V$  a  $\overline{\mathbb{Z}}_l$ -representation of  $D_{v,h}^\times$ , then  $V_{\bar{\tau}}$  denotes, cf. [10] §B.2, the direct factor of  $V$  whose irreducible subquotients are isomorphic to a subquotient of  $\bar{\tau}|_{\mathcal{D}_{v,h}^\times}$  where  $\mathcal{D}_{v,h}$  is the maximal order of  $D_{v,h}$ .
- With the previous notation,  $\mathcal{U}_{\bar{\tau}, \mathbb{N}}^i := (\mathcal{U}_{F_v, \overline{\mathbb{Z}}_l, d}^i)_{\bar{\tau}}$ .
- The matching between the system indexed by  $\mathcal{I}$  and those by  $\mathbb{N}$  is given by the application  $m_1 : \mathcal{I} \rightarrow \mathbb{N}$ .

*Remark:* for  $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)$ , and a lifting  $\tau$  which by Jacquet-Langlands correspondence can be written  $\tau \simeq \pi[t]_D$  for  $\pi$  irreducible cuspidal, let  $\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$  be in the supercuspidal support. Then the inertial class of  $\varrho$  depends only on  $\bar{\tau}$  and we will use the following notation.

**1.5.3. Notation.** — With the previous notation, we denote  $V_\varrho$  for  $V_{\bar{\tau}}$ .

*Remark:*  $\Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l}$  is an object of  $\mathcal{F}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l)$ . Indeed, by [1] proposition 4.4.2,  $\Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l}$  is an object of  ${}^p\mathcal{D}^{\leq 0}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l)$ . By [14] variant 4.4 of theorem 4.2, we have  $D\Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l} \simeq \Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l}$ , so that

$$\Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l} \in {}^p\mathcal{D}^{\leq 0}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geq 0}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l) = \mathcal{F}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l).$$

**1.5.4. Proposition.** — (cf. [8] §3.2) We have a decomposition

$$\Psi_{\mathcal{I}} \simeq \bigoplus_{g=1}^d \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_\varrho$$

where all the Harris-Taylor perverse sheaves of  $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  are of type  $\varrho$ .

*Remark:* in [2], we decomposed  $\Psi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  as a direct sum  $\bigoplus_{\pi_v} \Psi_{\pi_v}$  where  $\pi_v$  described the set of equivalent inertial classes of irreducible cuspidal representations. The  $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} \Psi_{\pi_v}$  where  $\mathrm{Cusp}(\varrho)$  is the

set of equivalent inertial classes of irreducible cuspidal representation of type  $\varrho$  in the sense of definition 1.2.2.

In [6], we give the precise description of the  $\mathrm{gr}_{\mathfrak{S},!}^r(\Psi_{\mathcal{I},\varrho})$  which is defined over  $\overline{\mathbb{Z}}_l$ . By construction they are supported on  $X_{\mathcal{I},\overline{s}_v}^{\geq r}$  and trivial if  $g$  does not divide  $r$ . Otherwise for  $r = tg$ , we have

$$\mathrm{ind}_{(D_{v,tg}^\times)^0 \varpi_v^{\mathbb{Z}}}^{D_{v,tg}^\times} \left( j^{=tg,*} \mathrm{gr}_{\mathfrak{S},!}^{tg}(\Psi_{\mathcal{I},\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \right) \simeq \bigoplus_{\substack{i=-1 \\ t_i g_i(\varrho)=tg}}^{r(\varrho)} \bigoplus_{\pi_v \in \mathrm{Scusp}_i(\varrho)} HT(\pi_v, \mathrm{St}_{t_i}(\pi_v)) \otimes \mathbb{L}(\pi_v) \left( -\frac{t_i-1}{2} \right).$$

We can then consider the following naive  $\varrho$ -filtration

$$\mathrm{Fil}_{\varrho,r(\varrho),\overline{\mathbb{Q}}_l}^*(\Psi, tg) \subset \cdots \subset \mathrm{Fil}_{\varrho,-1,\overline{\mathbb{Q}}_l}^*(\Psi, tg) = j^{=tg,*} \mathrm{gr}_{\mathfrak{S},!}^{tg}(\Psi_{\mathcal{I},\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$$

where  $\mathrm{ind}_{(D_{v,tg}^\times)^0 \varpi_v^{\mathbb{Z}}}^{D_{v,tg}^\times} \left( \mathrm{Fil}_{\varrho,k,\overline{\mathbb{Q}}_l}^*(\Psi, tg) \right)$  is isomorphic to

$$\bigoplus_{\substack{i=k \\ t_i g_i(\varrho)=tg}}^{r(\varrho)} \bigoplus_{\pi_v \in \mathrm{Scusp}_i(\varrho)} HT(\pi_v, \mathrm{St}_{t_i}(\pi_v)) \otimes \mathbb{L}(\pi_v) \left( -\frac{t_i-1}{2} \right),$$

and the associated entire filtration of  $j^{=tg,*} \mathrm{gr}_{\mathfrak{S},!}^{tg}(\Psi_{\mathcal{I},\varrho})$  defined by pullback

$$\begin{array}{ccc} \mathrm{Fil}_{\varrho,k}^*(\Psi, tg) & \subset \cdots \subset & \mathrm{Fil}_{\varrho,k,\overline{\mathbb{Q}}_l}^*(\Psi, tg) \\ \uparrow & & \downarrow \\ j^{=tg,*} \mathrm{gr}_{\mathfrak{S},!}^{tg}(\Psi_{\mathcal{I},\varrho}) & \hookrightarrow & j^{=tg,*} \mathrm{gr}_{\mathfrak{S},!}^{tg}(\Psi_{\mathcal{I},\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l. \end{array}$$

For  $k = -1, \dots, r(\varrho)$ , the graduates  $\mathrm{gr}_{\varrho,k}(\Psi, tg)$  are then of  $\varrho$ -type  $k$ . We can then refine these filtrations by separating the  $\pi_v \in \mathrm{Scusp}_k(\varrho)$  to obtain

$$(0) = \mathrm{Fil}_{\varrho}^{*,0}(\Psi, tg) \subset \mathrm{Fil}_{\varrho}^{*,1}(\Psi, tg) \subset \cdots \subset \mathrm{Fil}_{\varrho}^{*,r}(\Psi, tg) = j^{=tg,*} \mathrm{gr}_{\mathfrak{S},!}^{tg}(\Psi_{\mathcal{I},\varrho}).$$

By taking the iterated images of  $j_!^{=tg} \mathrm{Fil}_{\varrho}^k(\Psi, tg) \rightarrow \mathrm{gr}_{\mathfrak{S},!}^{tg}(\Psi_{\mathcal{I},\varrho})$ , we then construct a filtration

$$(0) = \mathrm{Fil}_{\varrho}^0(\Psi, tg) \subset \mathrm{Fil}_{\varrho}^1(\Psi, tg) \subset \cdots \subset \mathrm{Fil}_{\varrho}^r(\Psi, tg) = \mathrm{gr}_{\mathfrak{S},!}^{tg}(\Psi_{\mathcal{I},\varrho}).$$

Finally we can filtrate each of these graduate using an exhaustive filtration of stratification to obtain a filtration of  $\Psi_{\mathcal{I},\varrho}$  whose graduates



are the  $\mathfrak{P}(\pi_v, t)(\frac{1-s_i(\varrho)+2k}{2})$  for  $\pi_v \in \text{Scusp}_i(\varrho)$  with  $i \geq -1$ , and  $k = 0, \dots, s_i(\varrho) - 1$ .

## 2. Cohomology of KHT Shimura varieties

### 2.1. Localization at a non pseudo-Eisenstein ideal. —

**2.1.1. Definition.** — Define  $\text{Spl}$  the set of places  $v$  of  $F$  such that  $p_v := v|_{\mathbb{Q}} \neq l$  is split in  $E$  and  $B_v^\times \simeq GL_d(F_v)$ . For each  $I \in \mathcal{I}$ , write  $\text{Spl}(I)$  the subset of  $\text{Spl}$  of places which doesn't divide the level  $I$ .

Let  $\text{Unr}(I)$  be the union of

- places  $q \neq l$  of  $\mathbb{Q}$  inert in  $E$  not below a place of  $\text{Bad}$  and where  $I_q$  is maximal,
- and places  $w \in \text{Spl}(I)$ .

**2.1.2. Notation.** — For  $I \in \mathcal{I}$  a finite level, write

$$\mathbb{T}_I := \prod_{x \in \text{Unr}(I)} \mathbb{T}_x$$

where for  $x$  a place of  $\mathbb{Q}$  (resp.  $x \in \text{Spl}(I)$ ),  $\mathbb{T}_x$  is the unramified Hecke algebra of  $G(\mathbb{Q}_x)$  (resp. of  $GL_d(F_x)$ ) over  $\overline{\mathbb{Z}}_l$ .

*Example:* for  $w \in \text{Spl}(I)$ , we have

$$\mathbb{T}_w = \overline{\mathbb{Z}}_l[T_{w,i} : i = 1, \dots, d],$$

where  $T_{w,i}$  is the characteristic function of

$$GL_d(\mathcal{O}_w) \text{diag}(\overbrace{\varpi_w, \dots, \varpi_w}^i, \overbrace{1, \dots, 1}^{d-i}) GL_d(\mathcal{O}_w) \subset GL_d(F_w).$$

More generally, the Satake isomorphism identifies  $\mathbb{T}_x$  with  $\overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$  where

- $T_x$  is a split torus,
- $W_x$  is the spherical Weyl group
- and  $X^{un}(T_x)$  is the set of  $\overline{\mathbb{Z}}_l$ -unramified characters of  $T_x$ .

Consider a fixed maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}_I$  and for every  $x \in \text{Unr}(I)$  let denote  $S_{\mathfrak{m}}(x)$  be the multi-set<sup>(3)</sup> of modulo  $l$  Satake parameters at  $x$  associated to  $\mathfrak{m}$ .

*Example:* for every  $w \in \text{Spl}(I)$ , the multi-set of Satake parameters at  $w$  corresponds to the roots of the Hecke polynomial

$$P_{\mathfrak{m},w}(X) := \sum_{i=0}^d (-1)^i q_w^{\frac{i(i-1)}{2}} \overline{T_{w,i}} X^{d-i} \in \overline{\mathbb{F}}_l[X]$$

i.e.  $S_{\mathfrak{m}}(w) := \{\lambda \in \mathbb{T}_I/\mathfrak{m} \simeq \overline{\mathbb{F}}_l \text{ such that } P_{\mathfrak{m},w}(\lambda) = 0\}$ . For a maximal ideal  $\tilde{\mathfrak{m}}$  of  $\mathbb{T}_I \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ , we also have the multi-set of Satake parameters

$$S_{\tilde{\mathfrak{m}}}(w) := \{\lambda \in \mathbb{T}_I \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l/\tilde{\mathfrak{m}} \simeq \overline{\mathbb{Q}}_l \text{ such that } P_{\tilde{\mathfrak{m}},w}(\lambda) = 0\}.$$

**2.1.3. Notation.** — Let  $\Pi$  be an irreducible automorphic representation of  $G(\mathbb{A})$  of level  $I$  which means here, that  $\Pi$  has non trivial invariants under  $I$  and for every  $x \in \text{Unr}(I)$ , then  $\Pi_x$  is unramified. Then  $\Pi$  defines

- a maximal ideal  $\tilde{\mathfrak{m}}(\Pi)$  of  $\mathbb{T}_I \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ , or
- a minimal prime ideal  $\tilde{\mathfrak{m}}(\Pi)$  of  $\mathbb{T}_I$  contained in a maximal ideal  $\mathfrak{m}(\Pi)$  of  $\mathbb{T}_I$  which corresponds to its the modulo  $l$  Satake parameters.

A minimal prime ideal  $\tilde{\mathfrak{m}}$  of  $\mathbb{T}_I$  is said cohomological if there exists a cohomological automorphic  $\overline{\mathbb{Q}}_l$ -representation  $\Pi$  of  $G(\mathbb{A})$  of level  $I$  with  $\tilde{\mathfrak{m}} = \tilde{\mathfrak{m}}(\Pi)$ . Such a  $\Pi$  is not unique but  $\tilde{\mathfrak{m}}$  defines a unique near equivalence class in the sense of [15], we denote it  $\Pi_{\tilde{\mathfrak{m}}}$ . Let then

$$\rho_{\tilde{\mathfrak{m}},\overline{\mathbb{Q}}_l} : \text{Gal}(\overline{F}/F) \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

be the galoisian representation associated to such a  $\Pi$  thanks to [13] and [15], which by Chebotarev theorem, can be defined over some finite extension  $K_{\tilde{\mathfrak{m}}}$ , i.e.  $\rho_{\tilde{\mathfrak{m}},\overline{\mathbb{Q}}_l} \simeq \rho_{\tilde{\mathfrak{m}}} \otimes_{K_{\tilde{\mathfrak{m}}}} \overline{\mathbb{Q}}_l$ .

It's well known that  $\rho_{\tilde{\mathfrak{m}}}$  has stable lattices and the semi-simplification of its modulo  $l$  reduction is independent of the chosen stable lattice. Moreover it depends only of the maximal ideal  $\mathfrak{m}$ , we denote

$$\overline{\rho}_{\mathfrak{m}} : G_F \longrightarrow GL_d(\overline{\mathbb{F}}_l),$$

<sup>(3)</sup>A multi-set is a set with multiplicities.

its extension to  $\overline{\mathbb{F}}_l$ . For every  $w \in \text{Spl}(I)$ , recall that the multiset of eigenvalues of  $\overline{\rho}_{\mathfrak{m}}(\text{Frob}_w)$  is  $S_{\mathfrak{m}}(w)$  obtained from  $S_{\mathfrak{m}}(w)$  by taking modulo  $l$  reduction.

**2.2. Freeness of the cohomology.** — From now on we fix such a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}_I$  verifying one of the following two conditions:

- (1) There exists  $w_1 \in \text{Spl}(I)$  such that  $S_{\mathfrak{m}}(w_1)$  does not contain any sub-multi-set of the shape  $\{\alpha, q_{w_1}\alpha\} \circ q_{w_1}$ ;
- (2) With  $l \geq d+2$  we ask  $\mathfrak{m}$  to verify one of the two following hypothesis
  - either  $\overline{\rho}_{\mathfrak{m}}$  is induced from a character of  $G_K$  for a cyclic galoisian extension  $K/F$ ;
  - either  $SL_n(k) \subset \overline{\rho}_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_l^\times GL_n(k)$  for a sub-field  $k \subset \overline{\mathbb{F}}_l$ .

**2.2.1. Theorem.** — (cf. [7]) For  $\mathfrak{m}$  as above, the localized cohomology groups  $H^i(X_{I,\overline{\eta}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$  are free.

As  $X_I \rightarrow \text{Spec } \mathcal{O}_v$  is proper, we have a  $G(\mathbb{A}^\infty) \times W_v$ -equivariant isomorphism  $H^{d-1+i}(X_{\mathcal{I},\overline{\eta}_v}, \overline{\mathbb{Z}}_l) \simeq H^i(X_{I,\overline{s}_v}, \Psi_{\mathcal{I}})$ . Using the previous filtration of  $\Psi_{\mathcal{I}}$ , we can compute  $H^{p+q}(X_{I,\overline{s}_v}, \Psi_{\mathcal{I},\varrho})_{\mathfrak{m}}$  through a spectral sequence whose entries  $E_1^{p,q}$  are the  $H^j(X_{\mathcal{I},\overline{s}_v}, \mathfrak{P}(\pi_v, t)(\frac{1-s_i(\varrho)+2k}{2}))_{\mathfrak{m}}$  for  $\pi_v \in \text{Scusp}_i(\varrho)$  with  $i \geq -1$ , and  $k = 0, \dots, s_i(\varrho) - 1$ . Over  $\overline{\mathbb{Q}}_l$ , it follows from [3], thanks to the hypothesis (1) above on  $\mathfrak{m}$ , that all these cohomology groups are concentrated in degree 0 so that this  $\overline{\mathbb{Q}}_l$ -spectral sequence degenerates in  $E_1$ . In this section, under (H3), we want to prove the same result on  $\overline{\mathbb{F}}_l$  which is equivalent to the freeness of the  $H^j(X_{\mathcal{I},\overline{s}_v}, \mathfrak{P}(\pi_v, t)(\frac{1-s_i(\varrho)+2k}{2}))_{\mathfrak{m}}$ .

We need first some notations from [3] §1.2. For all  $t \geq 0$ , we denote

$$\Gamma^t := \left\{ (a_1, \dots, a_r, \epsilon_1, \dots, \epsilon_r) \in \mathbb{N}^r \times \{\pm\}^r : r \geq 1, \sum_{i=1}^r a_i = t \right\}.$$

A element of  $\Gamma^t$  will be denoted  $(\overleftarrow{a}_1, \dots, \overrightarrow{a}_r)$  where for the arrow above each integer  $a_i$  is  $\overleftarrow{a}_i$  (resp.  $\overrightarrow{a}_i$ ) if  $\epsilon_i = -$  (resp.  $\epsilon_i = +$ ). We then consider on  $\Gamma^t$  the equivalence relation induced by

$$(\dots, \overleftarrow{a}, \overleftarrow{b}, \dots) = (\dots, \overleftarrow{a+b}, \dots), \quad (\dots, \overrightarrow{a}, \overrightarrow{b}, \dots) = (\dots, \overrightarrow{a+b}, \dots),$$

and  $(\dots, \overleftarrow{0}, \dots) = (\dots, \overrightarrow{0}, \dots)$ . We denote  $\overline{\Gamma}^t$  the set of these equivalence classes whose elements are denoted  $[\overleftarrow{a}_1, \dots, \overrightarrow{a}_r]$ .

*Remark:* in each class  $[\vec{a}_1, \dots, \vec{a}_k] \in \vec{\Gamma}^t$ , there exists a unique reduced element  $(b_1, \dots, b_r, \epsilon_1, \dots, \epsilon_r) \in \Gamma^t$  such that for all  $1 \leq i \leq r$   $b_i > 0$  and for  $1 \leq i < r$ ,  $\epsilon_i \epsilon_{i+1} = -$ .

**2.2.2. Definition.** — Let  $(b_1, \dots, b_r, \epsilon_1, \dots, \epsilon_r)$  be the reduced element in  $[\vec{a}_1, \dots, \vec{a}_k] \in \vec{\Gamma}^t$ . We then define

$$\mathcal{S}([\vec{a}_1, \dots, \vec{a}_k])$$

as the subset of permutations  $\sigma$  of  $\{0, \dots, t-1\}$  such that for all  $1 \leq i \leq r$  with  $\epsilon_i = +$  (resp.  $\epsilon_i = -$ ) and for all  $b_1 + \dots + b_{i-1} \leq k < k' \leq b_1 + \dots + b_i$  then  $\sigma^{-1}(k) < \sigma^{-1}(k')$  (resp.  $\sigma^{-1}(k) > \sigma^{-1}(k')$ ).

We also introduce  $\mathcal{S}^{op}([\vec{a}_1, \dots, \vec{a}_k])$ , by imposing under the same conditions,  $\sigma^{-1}(k) > \sigma^{-1}(k')$  (resp.  $\sigma^{-1}(k) < \sigma^{-1}(k')$ ).

**2.2.3. Proposition.** — (cf. [17] §2) Let  $g$  be a divisor of  $d = sg$  and  $\pi$  an irreducible cuspidal representation of  $GL_g(F_v)$ . There exists a bijection

$$[\vec{a}_1, \dots, \vec{a}_r] \in \vec{\Gamma}^{s-1} \mapsto [\vec{a}_1, \dots, \vec{a}_r]_\pi$$

into the set of irreducible sub-quotients of the induced representation

$$\pi\left\{\frac{1-s}{2}\right\} \times \pi\left\{\frac{3-s}{2}\right\} \times \dots \times \pi\left\{\frac{s-1}{2}\right\}$$

characterized by the following property

$$J_{P_{g,2g,\dots,sg}}([\vec{a}_1, \dots, \vec{a}_r]_\pi) = \sum_{\sigma \in \mathcal{S}([\vec{a}_1, \dots, \vec{a}_r])} \pi\left\{\frac{1-s}{2} + \sigma(0)\right\} \otimes \dots \otimes \pi\left\{\frac{1-s}{2} + \sigma(s-1)\right\},$$

or equivalently by

$$J_{P_{g,2g,\dots,sg}^{op}}([\vec{a}_1, \dots, \vec{a}_r]_\pi) = \sum_{\sigma \in \mathcal{S}^{op}([\vec{a}_1, \dots, \vec{a}_r])} \pi\left\{\frac{1-s}{2} + \sigma(0)\right\} \otimes \dots \otimes \pi\left\{\frac{1-s}{2} + \sigma(s-1)\right\}.$$

*Remark:* with these notations  $\text{St}_s(\pi)$  (resp.  $\text{Speh}_s(\pi)$ ) is  $[\overleftarrow{s-1}]_\pi$  (resp.  $[\overrightarrow{s-1}]_\pi$ ).

**2.2.4. Lemma.** — Let  $\pi$  be an irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representation of  $GL_g(F_v)$  such that its modulo  $l$  reduction  $\varrho$  is supercuspidal. Suppose the cardinal of the Zelevinsky line of  $\varrho$  is greater or equal to  $s$ . Then the

irreducible sub-quotients of the modulo  $l$  reduction of  $[\overleftarrow{a}_1, \dots, \overrightarrow{a}_r]_\pi$  for  $[\overleftarrow{a}_1, \dots, \overrightarrow{a}_r]$  describing  $\overrightarrow{\Gamma}^{s-1}$ , are pairwise distinct.

*Proof.* — By hypothesis on the cardinal of the Zelevinsky line, all these irreducible  $\overline{\mathbb{F}}_l$ -sub-quotient have a non trivial image under  $J_{P_{g,2g,\dots,sg}}$ . The result then follows directly

- from the commutation of Jacquet functors with the modulo  $l$  reduction and
- from the fact that the  $r_l(\pi)\{\frac{1-s}{2}+k\}$  are pairwise distinct for  $0 \leq k < s$  so that the image under  $J_{P_{g,2g,\dots,sg}}$  of  $\pi\{\frac{1-s}{2}\} \times \pi\{\frac{3-s}{2}\} \times \dots \times \pi\{\frac{s-1}{2}\}$  is multiplicity free.

□

**2.2.1. Notation.** — We will denote  $[\overleftarrow{a}_1, \dots, \overrightarrow{a}_r]_\varrho$  any irreducible sub-quotient of the modulo  $l$  reduction of  $[\overleftarrow{a}_1, \dots, \overrightarrow{a}_r]_\pi$ .

*Remark:* if moreover the cardinal of the Zelevinsky line of  $\varrho$  is strictly greater than  $s$ , then, cf. [4], the modulo  $l$  reduction of  $[\overleftarrow{s-1}]_\pi$  is irreducible and non degenerate, i.e.  $[\overleftarrow{s-1}]_\varrho$  is well determinate and non degenerate.

**2.2.5. Theorem.** — Consider a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}_S$  such that for all  $i$ , the  $\overline{\mathbb{Z}}_l$ -module  $H^i(X_{U,\overline{\eta}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$  is free. We suppose moreover (H3) that the image of  $\overline{\rho}_{\mathfrak{m},w_0}$  in the Grothendieck group is multiplicity free. Then for all  $\overline{\mathbb{Z}}_l$ -Harris-Taylor local system  $HT(\pi_{w_0}, t)$ , the  $H^i(X_{U,\overline{s}_{w_0}}, {}^p j_{1*}^{t_g} HT(\pi_{w_0}, t))_{\mathfrak{m}}$  are free.

*Remark:* in the previous statement, we just need the multiplicity free part of (H3) as it is used in the previous lemma.

*Proof.* — First denote  $\text{Scusp}_{w_0}(\mathfrak{m})$  the set of inertial equivalence classes of irreducible supercuspidal  $\overline{\mathbb{F}}_l$ -representations belonging to the supercuspidal support of the modulo  $l$  reduction of the local component at  $w_0$  of a representation  $\Pi$  in the near equivalence class  $\Pi_{\mathfrak{m}}$  associated to  $\mathfrak{m}$ .

We then consider the vanishing cycle spectral sequence at  $w_0$ , localized at  $\mathfrak{m}$

$$H^i(X_{U,\overline{\eta}_{w_0}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}} \simeq \bigoplus_{\varrho \in \text{Scusp}_{w_0}(\mathfrak{m})} H^i(X_{U,\overline{s}_{w_0}}, \Psi_{\mathcal{I},\varrho})_{\mathfrak{m}}.$$

Then for every  $\varrho \in \text{Scusp}_{w_0}(\mathfrak{m})$ , the  $H^i(X_{U, \bar{s}_{w_0}}, \Psi_{\mathcal{I}, \varrho})_{\mathfrak{m}}$  are free. For  $\pi_v$  of type  $\varrho$ , the strategy to prove the freeness of the  $H^i(X_{U, \bar{s}_{w_0}}, {}^p j_{!*}^{=tg} HT(\pi_{w_0}, t))_{\mathfrak{m}}$  is then to argue by absurdity and to produce some torsion cohomology class in one of the  $H^i(X_{U, \bar{s}_{w_0}}, \Psi_{\mathcal{I}, \varrho})_{\mathfrak{m}}$ . Let then  $t$  be minimal such that there exists  $i \neq 0$  with

$$H^i(X_{\mathcal{I}, \bar{s}_{w_0}}, \mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \neq (0)$$

for  $0 \leq k < t$ , and where  $\mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2})$  is a graduate of some filtration of  $\Psi_{\varrho}$ .

Consider for example the filtration constructed before using the adjunction  $j_!^{1 \leq h} j^{1 \leq h, *}$   $\longrightarrow$  Id. As remarked before, even at taking  $\Psi_{\varrho^\vee}$  and its filtration constructed using Id  $\longrightarrow j_*^{1 \leq h} j^{1 \leq h, *}$ , we can suppose that such  $i$  is strictly negative and we denote  $i_0$  such a minimal  $i$ .

By lemma 1.3.2, an irreducible  $GL_d(F_{w_0}) \times W_{w_0}$ -equivariant sub-quotient of  $H^{i_0}(X_{\mathcal{I}, \bar{s}_{w_0}}, \mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  is one of the modulo  $l$  reduction of a representation we can write in the following shape

$$\left( [\overleftarrow{a}_1, \dots, \overrightarrow{a}_{i-1}, \overleftarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overrightarrow{a}_{i+1}, \dots, \overleftarrow{a}_r]_{\pi} \times \Upsilon_{w_0} \right) \otimes \mathbb{L}(\pi\{\frac{\delta}{2}\})$$

where the  $a_j$  are some integers,  $\Upsilon_{w_0}$  is an irreducible entire  $\overline{\mathbb{Q}}_l$ -representation whose modulo  $l$  reduction have a supercuspidal support away from those of the previous segment and where

- the symbol  $\overleftarrow{1}$  before (resp. after) the  $\overleftarrow{t-1}$  can be  $\overleftarrow{1}$  or  $\overrightarrow{1}$  if  $\sum_{j=1}^{i-1} a_j > 0$  (resp.  $\sum_{j=i+1}^r a_j > 0$ ). We will denote moreover  $a_i = t + 1$ .
- Let denote  $\left\{ \pi\{\frac{\alpha}{2}\}, \pi\{\frac{\alpha}{2} + 1\}, \dots, \pi\{\frac{\alpha}{2} + t - 1\} \right\}$  the supercuspidal support of  $\overleftarrow{t-1}$  inside  $[\overleftarrow{a}_1, \dots, \overleftarrow{t-1}, \dots, \overrightarrow{a}_r]_{\pi}$ . The  $\frac{\delta}{2} = \frac{\alpha}{2} + k$  where  $k$  is the integer in  $\mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2})$ .

*Remark:* in particular we can suppose that the previous  $k$  is equal to 0.

Consider then such an irreducible sub-quotient  $\tau \times \psi_{w_0} \otimes \sigma$  where

- $\psi_{w_0}$  (resp.  $\sigma$ ) is any irreducible sub-quotient of the modulo  $l$  reduction of  $\Upsilon_{w_0}$  (resp.  $\mathbb{L}(\pi\{\frac{\delta}{2}\})$ ), and
- $\tau$  is an irreducible sub-quotient of the modulo  $l$  reduction of some

$$[\overleftarrow{a}_1, \dots, \overrightarrow{a}_{i-1}, \overrightarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overrightarrow{a}_{i+1}, \dots, \overrightarrow{a}_r]_{\pi}.$$

By the previous lemma, we can recover the  $a_i$  from  $\tau$ .

Let show now that this  $\tau \times \psi_{w_0} \otimes \sigma$  is also a sub-quotient of  $H^{i_0}(X_{\mathcal{I}, \bar{s}_{w_0}}, \Psi_{\mathcal{I}, \varrho})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ , which contradicts our hypothesis on  $\mathfrak{m}$ . Let denote  $\text{Fil}^{k-1} \subset \text{Fil}^k \subset \Psi_{\mathcal{I}, \varrho}$  such that  $\text{gr}^k = \text{Fil}^k / \text{Fil}^{k-1} \simeq \mathcal{P}(\pi_{w_0}, t)(\frac{1-t}{2})$ . By the hypothesis (H3), all the irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representations  $\pi'_{w_0} \in \text{Scusp}_{\overline{\mathbb{F}}_l}(\varrho)$  such that one of the  $H^i(X_{\mathcal{I}, \bar{s}_{w_0}}, \mathcal{P}(\pi'_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \neq (0)$  is necessarily of  $\varrho$ -type  $-1$ . Then in particular all the Harris-Taylor perverse sheaves  $\mathcal{P}(\pi'_{w_0}, t')$  which are sub-quotient of  $\text{Fil}^{k-1}$ , must verify  $t' > t$ . The spectral sequence which computes  $H^{i_0+1}(X_{\mathcal{I}, \bar{s}_{w_0}}, \text{Fil}^{k-1})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  thanks to a filtration of  $\text{Fil}^{k-1}$ , allows to describe it as extensions between irreducible sub-quotients of the modulo  $l$  reduction of some

$$\left( [\overleftarrow{a}_1, \dots, \overleftarrow{a}_{i-1}, \overleftarrow{1}, \overleftarrow{t'-1}, \overleftarrow{1}, \overrightarrow{a}_{i+1}, \dots, \overrightarrow{a}_r]_{\pi'} \times \psi_{w_0} \right) \otimes \mathbb{L}(\pi' \{ \frac{\delta'}{2} \})$$

with  $t' > t$  and where  $\pi' \{ \frac{\delta'}{2} \}$  belongs to the supercuspidal support of  $\overleftarrow{t'-1}$  in the previous writing. But using the inequality  $t' > t$ , we see that  $\tau$  can't be a sub-quotient of the modulo  $l$  reduction of any  $[\overleftarrow{a}_1, \dots, \overleftarrow{a}_{i-1}, \overleftarrow{1}, \overleftarrow{t-1}, \overleftarrow{1}, \overrightarrow{a}_{i+1}, \dots, \overrightarrow{a}_r]_{\pi}$ .

Now using the filtration  $\text{Fil}^{k-1} \subset \text{Fil}^k \subset \Psi_{\varrho}$ , to conclude it suffices to look at  $H^{i_0-1}(X_{\mathcal{I}, \bar{s}_{w_0}}, \Psi_{\mathcal{I}, \varrho} / \text{Fil}^k)_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ . For the Harris-Taylor perverse sheaves  $\mathcal{P}(\pi'_{w_0}, t)(\frac{1-t+2k}{2})$  with  $t' > t$  we argue as before, and for the others we invoke the minimality of  $t$  and  $i_0$ .  $\square$

**2.3. From Ihara's lemma to the cohomology.** — Recall first that

$$X_{\mathcal{I}, \bar{s}_{v_0}}^{=d} = \coprod_{i \in \text{Ker}^1(\mathbb{Q}, G)} X_{\mathcal{I}, \bar{s}_{v_0}, i}^{=d}$$

and that for a  $G(\mathbb{A}^\infty)$ -equivariant sheaf  $\mathcal{F}_{\mathcal{I}, i}$  on  $X_{\mathcal{I}, \bar{s}_{v_0}, i}^{=d}$ , its fiber at some compatible system  $z_{i, \mathcal{I}}$  of supersingular points, has an action of  $\overline{G}(\mathbb{Q}) \times GL_d(F_{v_0})^0$  where  $GL_d(F_{v_0})^0$  is the kernel of the valuation of the determinant so that, cf. [2] proposition 5.1.1, as a  $G(\mathbb{A}^\infty) \simeq \overline{G}(\mathbb{A}^{\infty, v_0}) \times GL_d(F_{v_0})$ -module, we have

$$H^0(\overline{X}_{\mathcal{I}, \bar{s}_{v_0}, i}^{=d}, \mathcal{F}_{\mathcal{I}, i}) \simeq \text{ind}_{\overline{G}(\mathbb{Q})}^{\overline{G}(\mathbb{A}^{\infty, v_0}) \times \mathbb{Z}} z_i^* \mathcal{F}_{\mathcal{I}, i}$$

with  $\delta \in \overline{G}(\mathbb{Q}) \mapsto (\delta^{\infty, v_0}, \text{val} \circ \text{rn}(\delta_{v_0})) \in \overline{G}(\mathbb{A}^{\infty, v_0}) \times \mathbb{Z}$  and where the action of  $g_{v_0} \in GL_d(F_{v_0})$  is given by those of  $(g_0^{-\text{val det } g_{v_0}} g_{v_0}, \text{val det } g_{v_0}) \in GL_d(F_{v_0})^0 \times \mathbb{Z}$  where  $g_0 \in GL_d(F_{v_0})$  is any fixed element with  $\text{val det } g_0 = 1$ . Moreover, cf. [2] corollaire 5.1.2, if  $z_i^* \mathcal{F}_{\mathcal{I}, i}$  is provided with an action of the kernel  $(D_{v_0, d}^\times)^0$  of the valuation of the reduced norm, action compatible with those of  $\overline{G}(\mathbb{Q}) \hookrightarrow D_{v_0, d}^\times$ , then as a  $G(\mathbb{A}^\infty)$ -module, we have

$$H^0(X_{\mathcal{I}, \overline{s}_{v_0}, i}^=d, \mathcal{F}_{\mathcal{I}, i}) \simeq \mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}^\infty), \Lambda) \otimes_{D_{v_0, d}^\times} \text{ind}_{(D_{v_0, d}^\times)^0}^{D_{v_0, d}^\times} z_i^* \mathcal{F}_{\mathcal{I}, i} \quad (2.3.1)$$

**2.3.2. Lemma.** — *Let  $\overline{\pi}$  be an irreducible sub- $\overline{\mathbb{F}}_l$ -representation of  $\mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A})/U^{v_0}, \overline{\mathbb{F}}_l)$ . Denote its local component  $\overline{\pi}_{v_0}$  at  $v_0$  as  $\pi_{v_0}[s]_D$  with  $\pi_{v_0}$  an irreducible cuspidal representation of  $GL_d(F_{v_0})$  with  $d = sg$ . Then  $\overline{\pi}^{v_0}$  is a sub-representation of  $H^0(X_{U^{v_0}, \overline{s}_{v_0}}^=d, HT(\pi_{v_0}^\vee, s)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ .*

*Proof.* — Clearly we have  $\overline{\pi}^{v_0} \subset \mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A})/U^{v_0}, \overline{\mathbb{F}}_l) \otimes \overline{\pi}_{v_0}^\vee$ . The result then follows from (2.3.1) and the definition of the Harris-Taylor local system  $HT(\pi_{v_0}^\vee, s)$  with support on the supersingular stratum.  $\square$

**2.3.3. Proposition.** — *Let  $\mathfrak{m}$  be a maximal ideal of  $\mathbb{T}_S$  verifying the same hypothesis than in theorem 2.2.5 and let  $\overline{\pi}$  be an irreducible sub- $\overline{\mathbb{F}}_l$ -representation of  $\mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A})/U^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ . Then  $\overline{\pi}^{\infty, v}$  is a sub- $\overline{\mathbb{F}}_l$ -representation of  $H^{d-1}(X_{U, \overline{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ .*

*Proof.* — Write  $\overline{\pi}_{v_0} = \pi_{v_0}[s]_D$  with  $\pi_{v_0}$  an irreducible cuspidal representation of  $GL_d(F_{v_0})$ . Thanks to (H3), the modulo  $l$  reduction of  $\pi_{v_0}^\vee$  is supercuspidal, we denote it  $\varrho$ . Recall that  $H^i(X_{U, \overline{s}_{v_0}}, \Psi_\varrho)_{\mathfrak{m}}$  is a direct factor of  $H^{d-1}(X_{U, \overline{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ , so that it suffices to prove that  $\overline{\pi}^{\infty, v}$  is a sub- $\overline{\mathbb{F}}_l$ -representation of  $H^i(X_{U, \overline{s}_{v_0}}, \Psi_\varrho)_{\mathfrak{m}}$ .

Consider then the filtration of  $\Psi_\varrho$  introduced before so that its graduates are some Harris-Taylor perverse sheaves of type  $\varrho$  and whose  $\mathfrak{m}$ -localized cohomology groups are free concentrated in degree 0, cf. theorem 2.2.5. Note in particular that  $\mathcal{P}(\pi_{v_0}^\vee, s)(\frac{s-1}{2})$  is its first graduate so that, using the spectral sequence computing  $H^i(X_{U, \overline{s}_{v_0}}, \Psi_\varrho)_{\mathfrak{m}}$  with  $E_1^{p, q}$  given by the  $H^i(X_{U, \overline{s}_{v_0}}, \mathcal{P}(\pi_{v_0}^\vee, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}}$ , we see that

$$H^i(X_{U, \overline{s}_{v_0}}, \mathcal{P}(\pi_{v_0}^\vee, s)(\frac{s-1}{2}))_{\mathfrak{m}} \hookrightarrow H^i(X_{U, \overline{s}_{v_0}}, \Psi_\varrho)_{\mathfrak{m}}$$



with free cokernel, so that  $H^0(X_{U^{=d}, \bar{s}_{v_0}}, \mathcal{P}(\pi_{v_0}^\vee, s)(\frac{s-1}{2}))_{\mathfrak{m}}$  is a subspace of  $H^{d-1}(X_{U, \bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ . The result then follows from the previous lemma.  $\square$

The strategy now to prove Ihara's lemma, under our restrictive hypothesis on  $\mathfrak{m}$ , is then to prove the same statement on  $H^{d-1}(X_{U, \bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ , i.e. if  $\pi^{\infty, v_0}$  is a subspace of it, then its local component  $\pi_{w_0}^{\infty, v_0}$  at the place  $w_0$ , is generic. Finally our Ihara's lemma statement will follow from proposition 3.2.2.

### 3. Non degeneracy property for global cohomology

**3.1. Global lattices are tensorial product.** — From now on we suppose that the hypothesis (H1) of the introduction is verified, i.e. there exists a place  $v_1 \in \text{Spl}$  such that  $\bar{\rho}_{\mathfrak{m}, v_1}$  is irreducible so that for any  $\Pi \in \Pi_{\mathfrak{m}}$ , its local component  $\Pi_{v_1}$  is cuspidal and its modulo  $l$  reduction  $\varrho$  is supercuspidal. We then have

$$H^{d-1+i}(X_{U, \bar{\eta}_{v_1}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}} \simeq H^i(X_{U, \bar{s}_{v_1}}, \Psi_{\varrho})_{\mathfrak{m}},$$

where  $\Psi_{\varrho}$  is a local system on the supersingular stratum with, for  $z$  any supersingular point

$$\text{ind}_{(D_{v_1, d}^\times)^0}^{D_{v_1, d}^\times} z^* \Psi_{\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_{v_1} \in \text{Scusp}_{-1}(\rho)} \pi_{v_1} \otimes \pi_{v_1}[1]_D \otimes L_d(\pi_{v_1}^\vee).$$

**3.1.1. Proposition.** — *Let  $\Pi^{\infty, U} \otimes L_g(\Pi_{v_1}^\vee)$  be a direct factor of  $H^{d-1}(X_{U, \bar{\eta}_{v_1}}, \overline{\mathbb{Q}}_l)_{\mathfrak{m}}$ , and consider its lattice given by the  $\overline{\mathbb{Z}}_l$ -cohomology. Then this lattice is a tensorial product  $\Gamma_G \otimes \Gamma_W$  of a stable lattice  $\Gamma_G$  (resp.  $\Gamma_W$ ) of  $\Pi^{\infty, U}$  (resp. of  $L_d(\Pi_{v_1}^\vee)$ ).*

*Proof.* — Let  $\Gamma_{GDW}$  be the lattice of  $\Pi_{v_1} \otimes \Pi_{v_1}[1]_D \otimes L_d(\Pi_{v_1}^\vee)$  given by  $\text{ind}_{(D_{v_1, d}^\times)^0}^{D_{v_1, d}^\times} z^* \Psi_{\varrho}$ . As the modulo  $l$  reduction of  $\Pi_{v_1} \otimes \Pi_{v_1}[1]_D \otimes L_d(\Pi_{v_1}^\vee)$  is irreducible, all its stable lattices are homothetic and so  $\Gamma_{GDW} \simeq \Gamma_G \otimes \Gamma_D \otimes \Gamma_W$  where  $\Gamma_G$  (resp.  $\Gamma_D$ , resp.  $\Gamma_W$ ) is a stable lattice of  $\Pi_{v_1}$  (resp.  $\Pi_{v_1}[1]_D$ , resp.  $L_d(\Pi_{v_1}^\vee)$ ). We then have

$$H^{d-1+i}(X_{U, \bar{\eta}_{v_1}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}} \simeq H^i(X_{U, \bar{s}_{v_1}}, \Psi_{\varrho}) \simeq H^0(X_{\mathcal{L}, \bar{s}_{v_1}, i}^{=d}, \Psi_{\varrho})$$

which by (2.3.1) is isomorphic to

$$\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}^\infty), \overline{\mathbb{Z}}_l) \otimes_{D_{v_1, d}^\times} \text{ind}_{(D_{v_1, d}^\times)^0}^{D_{v_1, d}^\times} z^* \Psi_{\mathcal{I}, \rho}.$$

The lattice of the statement is then given by

$$\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}^\infty), \overline{\mathbb{Z}}_l) \otimes_{D_{v_1, d}^\times} (\Gamma_G \otimes \Gamma_D \otimes \Gamma_W),$$

and is so a tensorial product.  $\square$

**3.2. Proof of the main result.** — Let  $\mathcal{S}(\mathfrak{m})$  be the supercuspidal support of the modulo  $l$  reduction of any  $\Pi_{\tilde{m}, w_0}$  in the near equivalence class associated to a minimal prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ . Recall that  $\mathcal{S}(\mathfrak{m})$  depends only on  $\mathfrak{m}$  and by (H3) it's multiplicity free, we decompose it according to the inertial equivalence classes  $\varrho$  of irreducible  $\overline{\mathbb{F}}_l$ -representations of some  $GL_{g(\varrho)}(F_{w_0})$  with  $1 \leq g(\varrho) \leq d$ :

$$\mathcal{S}(\mathfrak{m}) = \coprod_{\varrho} \mathcal{S}_{\varrho}(\mathfrak{m}).$$

For each such  $\varrho$  we denote  $l_1(\varrho) \geq \dots \geq l_{r(\varrho)}(\varrho) > 0$  integers so that  $\mathcal{S}_{\varrho}(\mathfrak{m})$  can be written as a disjoint union of  $r(\varrho)$  unlinked Zelevinsky segments

$$[\varrho\{\delta_i\}, \varrho\{\delta_i + l_i(\varrho)\}] = \{\varrho\{\delta_i\}, \varrho\{\delta_i + 1\}, \dots, \varrho\{\delta_i + l_i(\varrho)\}\}.$$

An irreducible  $\overline{\mathbb{F}}_l$ -representation  $\tau_{w_0}$  of  $GL_d(F_{w_0})$  whose supercuspidal support is equal to  $\mathcal{S}(\mathfrak{m})$ , can be written as a full induced  $\tau_{w_0} \simeq \times_{\varrho} \tau_{\varrho}$  where each  $\tau_{\varrho}$  is also a full induced representation

$$\tau_{\varrho} \simeq \times_{i=1}^{r(\varrho)} \tau_{\varrho, i}$$

with  $\tau_{\varrho, i}$  of supercuspidal support equals to those of  $[\varrho\{\delta_i\}, \varrho\{\delta_i + l_i(\varrho)\}]$ . With the notations of 2.2.1, each of these  $\tau_{\varrho, i}$  can be written as

$$\tau_{\varrho, i} \simeq [\overleftarrow{a_1(\varrho)}, \overrightarrow{a_2(\varrho)}, \dots, \overrightarrow{a_{t_i(\varrho)}(\varrho)}]_{\varrho}.$$

**3.2.1. Definition.** — We say, cf. the remark following notation 2.2.1, that  $\tau_{w_0}$  is non degenerate if for all  $\varrho$  and for all  $1 \leq i \leq r(\varrho)$ , then  $\tau_{\varrho, i} \simeq [\overleftarrow{a_i(\varrho)}]_{\varrho}$ .

**3.2.2. Proposition.** — Let  $\tau_{w_0}$  be an irreducible representation of  $GL_d(F_{w_0})$  which is a subspace of

$$H^{d-1}(X_{U^{w_0}(\infty), \bar{\eta}_{w_0}}, \bar{\mathbb{F}}_l)_m := \varinjlim_n H^{d-1}(X_{U^{w_0}(n), \bar{\eta}_{w_0}}, \bar{\mathbb{F}}_l)_m.$$

Then  $\pi_{w_0}$  is non degenerate.

*Proof.* — Note first that the supercuspidal support of  $\tau_{w_0}$  must be  $\mathcal{S}(\mathfrak{m})$ . The exhaustive filtration of  $\Psi_{\varrho_0}$ , cf. §1.5, whose graduates are Harris-Taylor perverse sheaves, gives a filtration of  $H^{d-1}(X_{U^{w_0}(\infty), \bar{\eta}_{w_0}}, \bar{\mathbb{F}}_l)_m$ , whose graduate are, thanks to theorem 2.2.5, the

$$H^0(X_{U^{w_0}(\infty), \bar{\eta}_{w_0}}, \mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_m \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$$

for  $\pi_{w_0} \in \text{Scusp}_{-1}(\varrho)$  with  $\varrho$  such that  $S_{\varrho}(\mathfrak{m})$  is non empty. Then  $\tau_{w_0}$  must be a subspace of one of these graduates. We argue by absurdity using the following lemma.

**3.2.3. Lemma.** — If  $\rho$  be a subspace of  $[\overleftarrow{t-1}]_{\varrho\{\frac{-\delta}{2}\}} \times \rho'$  then with the notation 2.2.1,

- if  $\delta = s - t$  then  $\rho = [\overleftarrow{t-1}, \overrightarrow{1}, \overbrace{\cdots}^{s-t-1}]_{\varrho}$ ;
- if  $\delta = t - s$  then  $\rho = [\overbrace{\cdots}^{s-t-1}, \overleftarrow{1}, \overleftarrow{t-1}]_{\varrho}$ ;
- otherwise, i.e.  $t - s < \delta < s - t$ , then

$$\rho = [\overbrace{\cdots}^{s-t-\delta-1}, \overleftarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overbrace{\cdots}^{s-t+\delta-1}]_{\varrho}.$$

*Proof.* — The result is well known over  $\bar{\mathbb{Q}}_l$  and we can easily argue in the same way using

- the fact that all the  $\varrho\{\frac{1-s}{2} + k\}$  for  $0 \leq k \leq s-1$  are pairwise distinct
- and the property of commutation between the modulo  $l$  reduction and the Jacquet functors.

Consider for example the case  $t - s < \delta < s - t$ . By Frobenius reciprocity we see that the subspace we look for, is some undetermined irreducible subspace of the modulo  $l$  reduction of  $[\overbrace{\cdots}^{s-t-\delta-1}, \overleftarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overbrace{\cdots}^{s-t+\delta-1}]_{\pi}$ .

By convention, cf. notation 2.2.1, we denote such a subquotient  $[\overbrace{\cdots}^{s-t-\delta-1}, \overleftarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overbrace{\cdots}^{s-t+\delta-1}]_{\varrho}$ .  $\square$

Suppose now, by absurdity, it exists an irreducible supercuspidal  $\overline{\mathbb{F}}_l$ -representation  $\varrho_0$  such that  $\tau_{\varrho_0}$  is degenerate and take  $i$  with

$$\tau_{\varrho_0, i} \simeq [\cdots, \overrightarrow{a}, \cdots]_{\varrho_0},$$

and let  $\beta \in \frac{1}{2}\mathbb{Z}$  such that  $\varrho_0\{\beta\}$  is the supercuspidal corresponding to the end of the arrow  $\overrightarrow{a}$  in the previous notation. From proposition 3.1.1, we see that  $\tau_{w_0} \otimes \overline{\rho}_m$  is a  $\overline{\mathbb{F}}_l[GL_d(F_{w_0}) \times \text{Gal}(\overline{F}/F)]$ -submodule of  $H^{d-1}(X_{U^{w_0}(\infty), \overline{\eta}_{w_0}}, \overline{\mathbb{F}}_l)_m$ . After restricting the Galois action to the Weil group at  $w_0$ , we see that  $\tau_{w_0} \otimes \mathbb{L}(\varrho_0\{\beta\})$  has to be a  $\overline{\mathbb{F}}_l[GL_d(F_{w_0}) \times W_{w_0}]$ -submodule of  $H^{d-1}(X_{U^{w_0}(\infty), \overline{\eta}_{w_0}}, \overline{\mathbb{F}}_l)_m$  and, as before, on one of the

$$H^0(X_{U^{w_0}(\infty), \overline{\eta}_{w_0}}, \mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_m \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$$

for  $\pi_{w_0} \in \text{Scusp}_{-1}(\varrho_0)$ . Recall that this last cohomology group is parabolically induced from

$$H^0(X_{U^{w_0}(\infty), \overline{\eta}_{w_0}, \overline{1}_{tg}}, \mathcal{P}_1(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_m \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$$

where by lemma 1.3.2, every irreducible  $\overline{\mathbb{F}}_l[P_{tg, d}(F_{w_0}) \times W_{w_0}]$ -subquotient of it can be written as  $[\overleftarrow{t-1}]_{\varrho_0\{-\frac{\delta}{2}\}} \otimes \tau \otimes \mathbb{L}(\varrho_0\{\alpha\})$  where  $\tau$  is any irreducible representation of  $GL_{d-tg}(F_{w_0})$  and  $\alpha \in \frac{1}{2}\mathbb{Z}$  is such that  $\varrho_0\{\alpha\}$  belongs to the supercuspidal support of  $[\overleftarrow{t-1}]_{\varrho_0\{-\frac{\delta}{2}\}}$ .

The contradiction then follows from the previous lemma.  $\square$

Finally our restricted version of Ihara's lemma given in the introduction, follows from propositions 2.3.3 and 3.2.2.

*Remark:* note that in the previous proof we used the second part of (H3) to say that the modulo  $l$  reduction of  $[\overleftarrow{s-1}]_{\pi}$  is irreducible and so any of its subspace is non degenerate, cf. the remark just before theorem 2.2.5. Using the main result of [8], we have this last property without any hypothesis so, as this is the only place where we use the second part of (H3), we can remove it.

**3.3. Level rising.** — Before dealing with the general case, consider first the case  $d = 2$  and take  $l \geq 3$  such that the order of  $q_{w_0}$  modulo  $l$  is 2. Suppose then, by absurdity, there exists a maximal ideal  $\mathfrak{m}$  such that

- (a) for every prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ , the local component at  $w_0$  of  $\Pi_{\tilde{\mathfrak{m}}}$  is unramified;
- (b) for such prime ideal, write  $\Pi_{\tilde{\mathfrak{m}}, w_0} \simeq \chi_{w_0, 1} \times \chi_{w_0, 2}$ , and suppose  $\chi_{w_0, 1} \chi_{w_0, 2}^{-1} \equiv \nu \pmod{l}$ .

Using (a) and the spectral sequence of vanishing cycles at  $w_0$ , we obtain

$$H^1(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} \simeq H^1(X_{U, \bar{s}_{w_0}}^=1, \Psi(\overline{\mathbb{F}}_l))_{\mathfrak{m}}$$

where  $X_{U, \bar{s}_{w_0}}^=1$  is the ordinary locus of the geometric special fiber of  $X_U$  at  $w_0$ . It's well known that this cohomology group is parabolic induced. Moreover the only non degenerate irreducible representation of  $GL_d(F_{w_0})$  which is a subquotient of the modulo  $l$  reduction of  $\chi_{w_0, 1} \times \chi_{w_0, 1} \nu$  is cuspidal, because of the fact that  $q_{w_0}$  is of order 2 modulo  $l$ , this non degenerate representation can't be a subspace of the induced representation  $H^1(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ . The contradiction is then given by Ihara's lemma

In higher dimension, recall first the notations of the beginning of the previous section. For a minimal prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  and an automorphic representation  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$  in the near equivalence class associated to  $\tilde{\mathfrak{m}}$ , we write its local component at  $w_0$

$$\Pi_{w_0} \simeq \bigotimes_{\varrho} \Pi_{w_0}(\varrho)$$

and  $\Pi_{w_0}(\varrho) \simeq \bigotimes_{i=1}^{r(\varrho)} \Pi_{w_0}(\varrho, i)$  where for each  $1 \leq i \leq r(\varrho)$ , the modulo  $l$  reduction of the supercuspidal support of  $\Pi_{w_0}(\varrho, i)$  is, with the notations of the previous section, those of the Zelevinsky segment  $[\varrho\{\delta_i\}, \varrho\{\delta_i + l_i(\varrho)\}]$ .

**3.3.1. Proposition.** — *Take a maximal ideal  $\mathfrak{m}$  verifying the hypothesis (H1), (H2) and (H3). Let  $\varrho_0$  such that  $\mathcal{S}_{\varrho_0}(\mathfrak{m})$  is non empty and consider  $1 \leq i \leq r(\varrho_0)$ . Then there exists a minimal prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  and an automorphic representation  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$  such that with the previous notation  $\Pi_{w_0}(\varrho_0, i)$  is non degenerate, i.e. isomorphic to  $\text{St}_{l_i(\varrho)}(\pi_{w_0})$  for some irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representation  $\pi_{w_0}$ .*

*Remark:* in particular if  $\mathcal{S}(\mathfrak{m}) = \mathcal{S}_{\varrho_0}(\mathfrak{m})$  and  $r(\varrho_0) = 1$ , i.e. the supercuspidal support of the modulo  $l$  reduction of the local component at  $w_0$  of any  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$  for any  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ , is a Zelevinsky segment, then  $\Pi_{w_0}$  is non degenerate. Otherwise it seems difficult to rise the level simultaneously for all  $1 \leq i \leq r(\varrho_0)$  and even more for all  $\varrho$  together.

*Proof.* — For a minimal prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  and  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$ , we write

$$\Pi_{w_0}(\varrho_0, i) \simeq \text{St}_{s_1}(\pi_{w_0,1}) \times \cdots \times \text{St}_{s_a}(\pi_{w_0,a})$$

where  $s_1 \geq s_2 \geq \cdots \geq s_a \geq 1$  and  $\pi_{w_0,1}, \dots, \pi_{w_0,a}$  irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representations of type  $\varrho_0$  of  $GL_{g_i}(F_{w_0})$ . We then argue by absurdity, i.e. we suppose  $a \geq 2$  for all  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  and we choose such  $\tilde{\mathfrak{m}}$  so that  $s_1$  is maximal. The strategy is then, using lemma 3.2.3, to construct a degenerate  $\overline{\mathbb{F}}_l[GL_d(F_{w_0})]$ -subspace of  $H^{d-1}(X_{U^{w_0}(\infty), \bar{\eta}_{w_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$  which contradicts our version of Ihara's lemma. In [3] §3.6, we prove that for all minimal prime  $\tilde{\mathfrak{m}}' \subset \mathfrak{m}$

$$H^i(X_{U, \bar{s}_{w_0}}, HT_{\overline{\mathbb{Q}}_l}(\pi_{w_0,1}, t))_{\tilde{\mathfrak{m}}'} = (0)$$

either if  $t > s_1$  or for  $t = s_1$ , if  $i \neq 0$ . Consider now the filtration

$$\text{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}) \hookrightarrow \text{Fil}_*^{1-s_1 g_1}(\Psi_{\varrho_0}) \hookrightarrow \Psi_{\varrho_0}$$

and recall that, by construction,

- $\text{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0})$  is supported in  $X_{\mathcal{I}, \bar{s}_{w_0}}^{> s_1 g_1}$
- $\text{gr}_*^{1-s_1 g_1}(\Psi_{\varrho_0}) \simeq \bigoplus_{\pi_{w_0} \in \text{Scusp}_{-1}(\varrho_0)} \mathcal{P}(\pi_{w_0}, s_1) \left(\frac{s_1-1}{2}\right)$ .

By the theorem 2.2.5 we know the cohomology groups of Harris-Taylor perverse sheaves to be free, so

- $H^i(X_{U, \bar{s}_{w_0}}, \text{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}))_{\mathfrak{m}} = (0)$ ;
- $H^i(X_{U, \bar{s}_{w_0}}, \Psi_{\varrho_0} / \text{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}))_{\mathfrak{m}}$  is free.

Recall moreover, cf. [3] §3.6, that  $\Pi_{w_0} \otimes \mathbb{L}(\pi_{w_0,1}) \left(\frac{s_1-1}{2}\right)$  is a direct factor of

$$H^i(X_{U^{w_0}(\infty), \bar{s}_{w_0}}, HT_{\overline{\mathbb{Q}}_l}(\pi_{w_0,1}, s_1) \left(\frac{s_1-1}{2}\right))_{\tilde{\mathfrak{m}}},$$

Moreover the stable lattice given by the  $\overline{\mathbb{Z}}_l$ -cohomology looks like  $(\Gamma(\varrho_0, 1) \times \Gamma^{\varrho_0, 1}) \times \Gamma_W$  where

- $\Gamma(\varrho_0, 1)$  is a stable lattice of  $\text{St}_{s_1}(\pi_{w_0,1})$ ,
- $\Gamma^{\varrho_0, 1}$  is a stable lattice of  $(\times_{\varrho \neq \varrho_0} \Pi_{w_0}(\varrho)) \times (\times_{i=2}^{r(\varrho_0)} \Pi_{w_0}(\varrho_0, i))$ ,
- and  $\Gamma_W$  is a stable lattice of  $\mathbb{L}(\pi_{w_0,1}) \left(\frac{s_1-1}{2}\right)$ .

The result then follows from lemma 3.2.3.

□

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