Markovian and product quantization of an $\mathbb{R}^d$-valued Euler scheme of a diffusion process with applications to finance

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Abstract

We introduce a new approach to quantize the Euler scheme of an $\mathbb{R}^d$-valued diffusion process. This method is based on a Markovian and componentwise product quantization and allows us, from a numerical point of view, to speak of fast quantization in dimension greater than one since the product quantization of the Euler scheme of the diffusion process and its companion weights and transition probabilities may be computed quite instantaneously using a Newton-Raphson algorithm. We show that the resulting quantization process is a Markov chain, then, we compute the associated companion weights and transition probabilities (for the quantized process and for its components) using closed formulas. From the analytical point of view, we show that the induced quantization errors at the k-th discretization step t_k is a cumulative of the marginal quantization error up to time t_k. Numerical experiments are performed for the pricing of a Basket call option and a European call option in a Heston model to show the performances of the method.

1 Introduction

In [8] is proposed and analyzed a Markovian (fast) quantization of an $\mathbb{R}^d$-valued Euler scheme of a diffusion process. However, in practice, their approach allows to speak of fast quantization only in dimension one since, as soon as $d \geq 2$, one has to use recursive zero search stochastic algorithms (known to be very time consuming, compared to deterministic procedures like the Newton-Raphson algorithm, see [7]) to compute optimal quantizers and theirs associated weights and transition probabilities. In order to overcome this limitation, we propose in this work another approach to quantize an $\mathbb{R}^d$-valued Euler scheme of a diffusion process. This method is based on a Markovian and componentwise product quantization. It allows again to speak of fast quantization in hight dimension since the product quantization of the Euler scheme of the diffusion process and its transition probabilities can almost be computed quite instantaneously using deterministic zero search algorithms.

In a general setting, the stochastic process $(X_t)_{t\in[0,T]}$ of interest is defined as a (strong) solution to the following stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{1}$$

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§The seond author benefited from the support of the Chaire "Risques financiers", a joint initiative of École Polytechnique, ENPC-ParisTech and UPMC, under the aegis of the Fondation du Risque. The third author benefited from the support of the Chaire "Markets in Transition", under the aegis of Louis Bachelier Laboratory, a joint initiative of École polytechnique, Université d'Évry Val d'Essonne and Fédération Bancaire Française.

where W is a standard q-dimensional Brownian motion, independent from the $\mathbb{R}^d$-valued random vector X_0, both defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The drift coefficient $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the volatility coefficient $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times q}$ are Borel measurable functions satisfying appropriate Lipschitz continuity and linear growth conditions (specified further on) which ensure the existence of a unique strong solution of the stochastic differential equation. In corporate finance, these processes are used to model the dynamics of assets for several quantities of interest involved the pricing and the hedging of derivatives. These quantities are usually of the form

$$\mathbb{E}\big[f(X_T)\big], \quad T > 0, \tag{2}$$

or

$$\mathbb{E}\big[f(X_t)|X_s = x\big], \quad 0 < s < t, \tag{3}$$

for a given Borel function $f : \mathbb{R}^d \to \mathbb{R}$. For illustrative purposes, let us consider the following two pricing examples which may be reduced to the computation of regular expectations like (2). First, consider the price of a Basket call option with maturity T and strike K, based on two stocks which prices X^1 and X^2 evolve following the dynamics

$$\begin{cases} dX_t^1 = rX_t^1 dt + \sigma_1 X_t^1 dW_t^1 \\ dX_t^2 = rX_t^2 + \rho\,\sigma_2 X_t^2 dW_t^1 + \sqrt{1-\rho^2}\,\sigma_2 X_t^2 dW_t^2, \qquad t \in [0,T], \end{cases} \tag{4}$$

where r is the interest rate, $\sigma_1, \sigma_2 > 0$, $\rho \in (-1,1)$ is a correlation term and W^1 and W^2 are two independent Brownian motions. We know that the no arbitrage price at time $t = 0$ in a complete market reads

$$e^{-rT}\mathbb{E}\big[(w_1 X_T^1 + w_2 X_T^2 - K)_+\big] = e^{-rT}\mathbb{E}F(X_T), \quad X = (X^1, X^2), \tag{5}$$

where the weights w_1 and w_2 are usually assumed to be positive and their sum is equal to one and where the function F is defined, for every $x = (x^1, x^2) \in \mathbb{R}^2$, by $F(x) = (w_1 x^1 + w_2 x^2 - K)_+$. Keep in mind that $x_+ = \max(x, 0)$, $\forall x \in \mathbb{R}$.

The second example concerns the pricing of a call option with maturity T and strike K, in a Heston model where the stock price S and its stochastic volatility V evolve following the (correlated) dynamics

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^1 \\ dV_t = \kappa(\theta - V_t)dt + \rho\,\sigma\sqrt{V_t}dW_t^1 + \sqrt{1-\rho^2}\,\sigma\sqrt{V_t}dW_t^2, \quad t \in [0,T]. \end{cases} \tag{6}$$

In the previous equation, the parameter r is still the interest rate; $\kappa > 0$ is the rate at which V reverts to the long run average variance $\theta > 0$; the parameter $\sigma > 0$ is the volatility of the variance and $\rho \in [-1, 1]$ is the correlation term. In this case, the no arbitrage price at time $t = 0$ in a complete market reads

$$e^{-rT}\mathbb{E}\big[(S_T - K)_+\big] = e^{-rT}\mathbb{E}H(X_T), \quad X = (S, V), \tag{7}$$

where $H(x) = (x^1 - K)_+$, for $x = (x^1, x^2) \in \mathbb{R}^2$.

In the general setting (in particular, in both previous examples (4)-(7)) the stochastic differential equation (1) has no explicit solution. Therefore, both quantities (2) and (3) (in particular, the price expressions (5) and (7), which are problems of the first kind (8)) have to be approximated, for example, by

$$\mathbb{E}\big[f(\bar{X}_T)\big] \tag{8}$$

and

$$\mathbb{E}\big[f(\bar{X}_{t_{k+1}})|\bar{X}_{t_k} = x\big] \quad \text{when } t = t_{k+1} \text{ and } s = t_k \tag{9}$$

where $(\bar{X}_{t_k})_{k=0,\ldots,n}$ is a discretization scheme of the process $(X_t)_{t\geq 0}$ on $[0,T]$, for a given discretization mesh $(t_k)_{0\leq k\leq n}$. The Euler scheme is usually used. Given the (regular) time discretization mesh

$t_k = k\Delta$, $k = 0,\ldots,n$, $\Delta = T/n$, the Euler scheme $(\bar{X}_{t_k})_{k=0,\ldots,n}$, associated to $(X_t)_{t\in[0,T]}$ is recursively defined by

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(t_k,\bar{X}_{t_k})\Delta + \sigma(t_k,\bar{X}_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad \bar{X}_0 = X_0.$$

In the sequel, when no confusion may occur, we will identify the value Y_{t_k} at time t_k of any process $(Y_{t_k})_{0\le k\le n}$ by Y_k, $k = 0,\ldots,n$.

At this stage, the quantities (8) and (9) still have no closed formulas in the general setting (for example when dealing with a general local volatility model or stochastic volatility models as the Heston model), so that we have to make a spacial approximation of the expectation or the conditional expectation. This may be done by Monte Carlo simulation techniques or by optimal quantization method (in particular, by the Markovian (fast) quantization method).

The Markovian (fast) quantization of the Euler scheme of an $\mathbb{R}^d$-valued diffusion process has been introduced in [8]. It consists of a sequence of quantizations $(\widehat{X}_k^{\Gamma_k})_{k=0,\ldots,N}$ of the Euler scheme $(\bar{X}_k)_{k=0,\ldots,N}$ defined recursively as follows:

$$\begin{aligned}
\widetilde{X}_0 &= \bar{X}_0,\\
\widehat{X}_k^{\Gamma_k} &= \mathrm{Proj}_{\Gamma_k}(\widetilde{X}_k) \quad \text{and} \quad \widetilde{X}_{k+1} = \mathcal{E}_k(\widehat{X}_k^{\Gamma_k}, Z_{k+1}),\ k = 0,\ldots,n-1,
\end{aligned}$$

where $(Z_k)_{k=1,\ldots,n}$ is an i.i.d. sequence of $\mathcal{N}(0;I_q)$-distributed random vectors, independent of $\bar{X}_0$ and

$$\mathcal{E}_k(x,z) = x + \Delta b(t_k,x) + \sqrt{\Delta}\sigma(t_k,x)z, \quad x\in\mathbb{R}^d,\ z\in\mathbb{R}^q,\ k = 0,\ldots,n-1.$$

The sequence of quantizers satisfies for every $k\in\{0,\ldots,n\}$,

$$\Gamma_k \in \mathrm{argmin}\{\widetilde{D}_k(\Gamma),\ \Gamma\subset\mathbb{R}^d,\ \mathrm{card}(\Gamma)\le N_k\},$$

where for every grid $\Gamma\subset\mathbb{R}^d$, $\widetilde{D}_{k+1}(\Gamma) := \mathbb{E}\big[\mathrm{dist}(\widetilde{X}_{t_{k+1}},\Gamma)^2\big]$. However, this method allows fast quantization (from the numerical point of view) only in dimension one. Otherwise, we are led to use time consuming recursive stochastic zero search algorithms.

The aim of this work is to present another approach to quantize the Euler scheme of an $\mathbb{R}^d$-valued diffusion process in order to speak of fast quantization in dimension greater than one. We propose a Markovian and product quantization method. It allows us to compute instantaneously the optimal product quantizers and their transition probabilities (and its companion weights) when the size of the quantizations are chosen reasonably.

The method is based on a Markovian and componentwise product quantization of the process $(\bar{X}_k)_{0\le k\le n}$. To be more precise, let us denote by Γ_k^ℓ an N_k^ℓ-quantizer of the ℓ-th component $\bar{X}_k^\ell$ of the vector $\bar{X}_k$ and let $\widehat{x}_k^i$ be the quantization of $\bar{X}_k^\ell$ of size N_k^ℓ, on the grid Γ_k^ℓ. Let us define the product quantizer $\Gamma_k = \bigotimes_{i=1}^d \Gamma_k^\ell$ of size $N_k = N_k^1\times\ldots\times N_k^d$ of the vector $\bar{X}_k$ as

$$\Gamma_k \;=\; \big\{(x_k^{1,i_1},\ldots,x_k^{d,i_d}),\quad i_\ell\in\{1,\ldots,N_k^\ell\},\ \ell\in\{1,\ldots,d\}\big\}.$$

Then, assuming that $\bar{X}_0$ is already quantized as $\widehat{X}_0$, we define the product quantization $(\widehat{X}_{t_k})_{0\le k\le n}$ of the process $(\bar{X}_{t_k})_{0\le k\le n}$ from the following recursion:

$$\begin{cases}
\widetilde{X}_0 = \widehat{X}_0, \quad \widehat{X}_k^\ell = \mathrm{Proj}_{\Gamma_k^\ell}(\widetilde{x}_k^i),\ i = 1,\ldots,d\\
\widehat{X}_k = (\widehat{X}_k^1,\ldots,\widehat{X}_k^d) \quad \text{and} \quad \widetilde{X}_{k+1}^\ell = \mathcal{E}_k^\ell(\widehat{X}_k, Z_{k+1}),\ i = 1,\ldots,d\\
\mathcal{E}_k^\ell(x,z) = x^\ell + \Delta b^\ell(t_k,x) + \sqrt{\Delta}(\sigma^{\ell\bullet}(t_k,x)|z),\ z = (z^1,\ldots,z^q)\in\mathbb{R}^q\\
x = (x^1,\ldots,x^d),\ b = (b^1,\ldots,b^d) \text{ and } (\sigma^{\ell\bullet}(t_k,x)|z) = \sum_{m=1}^q \sigma^{\ell m}(t_k,x)z^m.
\end{cases} \tag{10}$$

First we will check that the sequence of quantizers $(\widehat{X}_k)_{k\geq 0}$ is a Markov chain (see Proposition 3.1). Then, the challenging question is to know how to compute its set values and their associated transition probabilities. Using the fact that the conditional distribution of the Euler scheme is a multivariate Gaussian distribution and that each component of a Gaussian vector remains a scalar Gaussian random variable, we propose a way to quantize every component $\bar{X}_k^\ell$ of the vector $\bar{X}_k$, for $k = 0, \ldots, n$. We then define the product quantization $(\widehat{X}_k)_{0\leq k\leq n}$ of $(\bar{X}_k)_{0\leq k\leq n}$ from the recursive procedure (10). Then, we show how to compute, for every $k \geq 1$, the transition probabilities (and the companion weights) associated to each component of the vector $\widehat{X}_k$, for every $k \geq 1$ and to the vector $\widehat{X}_k$ itself.

When the components of the vector $\bar{X}_k$ are independent for every $k = 0, \ldots, n$, the method boils down to the usual product quantization of the vector $\bar{X}_k$, where each component is quantized from the Markovian recursive quantization method (see [8]). In this case, the transition probability weight associated to the vector $\widehat{X}_k$ is the product of the transition probability weights associated to its components.

The main problem arises when the components of $\bar{X}_k$ are not independent. In this work, we propose a closed formula even in this case by relying to a domain decomposition technique.

To be more precise, set, for every $k \in \{0, \ldots, n\}$,

$$\mathscr{I}_k = \big\{(i_1, \ldots, i_d),\ i_\ell \in \{1, \ldots, N_k^\ell\}\big\} \tag{11}$$

and for $i := (i_1, \ldots, i_d) \in \mathscr{I}_k$, set

$$x_k^i := (x_k^{1,i_1}, \ldots, x_k^{d,i_d}). \tag{12}$$

We will show in Proposition 3.2 that the transition probabilities of the Markov chain $(\widehat{X}_k)_{k\geq 0}$ reads, for any multi-indices $\ell \in \mathscr{I}_k$ and $j =\in \mathscr{I}_{k+1}$,

$$\mathbb{P}\big(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^i\big) \;=\; \mathbb{E} \prod_{i\in \mathbb{J}_k^0(x_k^i)} \mathbb{1}_{\{\zeta \in \mathbb{J}^0_{k,j_\ell}(x_k^i)\}} \big(\Phi_0(\beta_j(x_k^i,\zeta)) - \Phi_0(\alpha_j(x_k^i,\zeta))\big)_+ \tag{13}$$

where $\zeta \sim \mathcal{N}(0; I_{q-1})$. For every $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^{q-1}$,

$$\begin{aligned} \alpha_j(x,z) &= \max\big(\sup_{i\in\mathbb{J}_k^+(x)} x_{k+1}^{i,j_\ell -}(x,z),\ \sup_{i\in\mathbb{J}_k^-(x)} x_{k+1}^{i,j_\ell +}(x,z)\big) \\ \text{and} \quad \beta_j(x,z) &= \min\big(\inf_{i\in\mathbb{J}_k^+(x)} x_{k+1}^{i,j_\ell +}(x,z),\ \inf_{i\in\mathbb{J}_k^-(x)} x_{k+1}^{i,j_\ell -}(x,z)\big), \end{aligned}$$

with the convention that $\prod_{i\in\emptyset} = 1$, and,

$$\begin{aligned} \mathbb{J}_k^0(x) &= \big\{\ell \in \{1, \ldots, d\},\quad \sigma^{\ell 1}(t_k, x) = 0\big\} \\ \mathbb{J}_k^-(x) &= \big\{\ell\{1, \ldots, d\},\quad \sigma^{\ell 1}(t_k, x) < 0\big\} \\ \mathbb{J}_k^+(x) &= \big\{\ell\{1, \ldots, d\},\quad \sigma^{\ell 1}(t_k, x) > 0\big\}. \end{aligned}$$

The quantities $x_{k+1}^{\ell,j_\ell -}(x,z)$ and $x_{k+1}^{\ell,j_\ell +}(x,z)$ are precisely defined in Section 3.2. Although if this formula looks complicated, it is very important in practice. In fact, keeping in mind that the optimal quantization grids associated to multivariate Gaussian random vectors (up to dimension $d = 10$) can be downloaded on the website `www.quantize.maths-fi.com`, it is clear that (14) can be computed instantaneously using these optimal grids of multivariate normal vectors. Furthermore, Equation (14) allows us to deduce the weights associated to the product quantization $\widehat{X}_{k+1}$, $k = 0, \ldots, n-1$, since for every $j \in \mathscr{I}_{k+1}$,

$$\mathbb{P}\big(\widehat{X}_{k+1} = x_{k+1}^j\big) = \sum_{i\in\mathscr{I}_k} \mathbb{P}\big(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^i\big)\mathbb{P}\big(\widehat{X}_k = x_k^i\big). \tag{14}$$

Formulas (13)-(14) are useful when, for example, we deal with the price of a Basket call like Equation (5). In this situation, given a time discretization mesh $t_0 = 0, \ldots, t_n = T$, the price of the Basket call option will be approximated by the cubature

$$e^{-rT} \sum_{j \in \mathscr{I}_n} F(x_n^j)\, \mathbb{P}(\widehat{X}_n = x_n^j), \tag{15}$$

where $\mathbb{P}(\widehat{X}_n = x_n^j)$ is computed in a recursive way, using equations (13) and (14).

When the correlation coefficient $\rho = 0$ in (4), the probabilities $\mathbb{P}(\widehat{X}_n = x_n^j)$ in the formula (15) will be computed in a simplified way. In fact, when the components of the vectors $\bar{X}_k$ are independent, $k = 0, \ldots, n$, we show in Proposition 3.3 that the formula (13) reads in the simplified form

$$\begin{aligned} \mathbb{P}\big(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^i\big) &= \prod_{\ell=1}^d \mathbb{P}\big(\widehat{X}_{k+1}^\ell = x_{k+1}^{j_\ell} | \widehat{X}_k = x_k^i\big) \\ &= \prod_{\ell=1}^d \big(\Phi_0\big(x_{k+1}^{\ell, j_\ell +}(x_k^i, 0)\big) - \Phi_0\big(x_{k+1}^{\ell, j_\ell -}(x_k^i, 0)\big)\big), \end{aligned}$$

for $i \in \mathscr{I}_k, j \in \mathscr{I}_{k+1}$, where $\Phi_0(\cdot)$ stands for the cumulative distribution of the scalar Gaussian random variable.

We also compute the (transition) distribution of each component of the product quantizations. Indeed, we show in Proposition 3.4 that for any $\ell \in \{1, \ldots, d\}$ and for every $j_\ell \in \{1, \ldots, N_{k+1}^\ell\}$, the transition probability $\mathbb{P}(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell) | \widehat{X}_k = x_k^i)$ is given by

$$\mathbb{P}\big(\widetilde{X}_{k+1}^\ell = x_{k+1}^{\ell j_\ell} | \widehat{X}_k = x_k^i\big) = \Phi_0\left(\frac{x_{k+1}^{\ell, j_\ell + 1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta}\, |\sigma_k^{\ell \bullet}(x_k^i)|_2}\right) - \Phi_0\left(\frac{x_{k+1}^{\ell, j_\ell - 1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta}\, |\sigma_k^{\ell \bullet}(x_k^i)|_2}\right), \tag{16}$$

where $m_k^\ell(x) = x + b(t_k, x)\Delta$ and $|\sigma_k^{\ell\bullet}(x)|_2$ is the Euclidean norm of the ℓ-th row of the volatility matrix $\sigma(t_k, x)$, for $x \in \mathbb{R}^d$. We deduce immediately the formulas for the probabilities $\mathbb{P}(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell))$, $k = 0, \ldots, n-1$, $j_\ell \in \{1, \ldots, N_{k+1}^\ell\}$ using (16) (and (14)).

Equation (16) allows us to approximate the price of the call in the Heston model by

$$e^{-rT} \sum_{j_1=1}^{N_n^1} H(x_n^{1j_1})\, \mathbb{P}(\widehat{X}_n^1 = x_n^{1j_1}). \tag{17}$$

Another important issue form the analytical point of view is to compute the quantization error bound associated to the Markovian quantization process. Using some results from [8], we show (in particular, when $N_k^\ell = N_k$, for avery $\ell = 1, \ldots, d$) that for any sequence $(\widehat{X}_k^{\Gamma_k})_{0 \le k \le n}$ of (quadratic) Markovian product quantization of $(\widetilde{X}_k)_{0 \le k \le n}$, the quantization error $\|\bar{X}_k - \widehat{X}_k^{\Gamma_k}\|_2$, at step k of the recursion, is bounded by the cumulative quantization errors $\sqrt{d}\|\widetilde{X}_p - \widehat{X}_p^{\Gamma_p}\|_2$, for $p = 0, \ldots, k$. More precisely, one shows that for every $k = 0, \ldots, n$, for any $\eta \in (0, 1]$,

$$\|\bar{X}_k - \widehat{X}_k^{\Gamma_k}\|_2 \le \sqrt{d} \sum_{p=0}^k a_p(b, \sigma, \Delta, x_0, \eta) N_p^{-1/d},$$

where $a_p(b, \sigma, \Delta, x_0, \eta)$ is a positive real constant depending on b, σ, Δ, x_0, η (see Theorem 3.6 further on for a more general statement).

The paper is organized as follows: in Section 2 we recall some basic results on optimal quantization. Section 3 is the main part of this paper. We present the algorithm and show the Markov property of the product quantization of the Euler scheme of a diffusion process. Then, we show how to compute the weights and transition probabilities associated to the product quantizers and to its components. We also show how to compute the optimal quantizers associated to each component of the Euler scheme (keep in mind that this is the foundation of our method). Finally, we provide some a priori error bounds for the quantization error associated to the Markovian product quantization and show that, at every step discretization step t_k, this error is a cumulated (weighted) sum of the regular quantization errors, up to time t_k. In Section 4, we present some numerical results for the pricing of a European call Basket option and a European call option in the Heston model.

NOTATIONS. We denote by $\mathcal{M}(d,q,\mathbb{R})$, the set of $d\times q$ real value matrices. If $A=[a_{ij}]\in\mathcal{M}(d,q,\mathbb{R})$, $A^\star$ denotes its transpose and we define the norm $\|A\|:=\sqrt{\mathrm{Tr}(AA^\star)}=(\sum_{i,j}a_{ij}^2)^{1/2}$, where $\mathrm{Tr}(M)$ stands for the trace of M, for $M\in\mathcal{M}(d,d,\mathbb{R})$. For every $f:\mathbb{R}^d\to\mathcal{M}(d,q,\mathbb{R})$, we will set $[f]_{\mathrm{Lip}}=\sup_{x\neq y}\frac{\|f(x)-f(y)\|}{|x-y|}$. For $x,y\in\mathbb{R}$, $x\vee y=\max(x,y)$. If $x\in\mathbb{R}^d$, $|x|_2$ will stand for the Euclidean norm on $\mathbb{R}^d$. For every vectors x,y, the notation $(x|y)$ denotes the dot product of the vectors x and y. For a given vector $z\in\mathbb{R}^q$ and a matrix $M\in\mathcal{M}(d,q,\mathbb{R})$, z^i denotes the component i of z, $z^{(j:k)}$ the vector made up from the component j to the component k of the vector z and $M^{(i,j:k)}$ is the vector made up from the component j to the component k of the i-th row of the matrix M and M^{ij} for the component (i,j) of the matrix M. The notation $M^{i\bullet}$ stands for the i-th row of M. The function Φ_0 will denote the cumulative distribution function of the standard real valued Normal distribution and Φ_0' will stand for its density function.

2 Brief background on optimal quantization

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space and let $X:(\Omega,\mathcal{A},\mathbb{P})\longrightarrow\mathbb{R}^d$ be a random variable with distribution $\mathbb{P}_X$. The L^r-optimal quantization problem at level N for the random vector X (or for the distribution $\mathbb{P}_X$) consists in finding the best approximation of X by a Borel function $\pi(X)$ of X taking at most N values. Assuming that $X\in L^r(\mathbb{P})$, we associate to every Borel function $\pi(X)$ taking at most N values, the L^r-mean error $\|X-\pi(X)\|_r$ measuring the distance between the two random vectors X and $\pi(X)$ w.r.t. the mean L^r-norm, where $\|X\|_r:=(\mathbb{E}|X|^r)^{1/r}$ and $|\cdot|$ denotes an arbitrary norm on $\mathbb{R}^d$. Then finding the best approximation of X by a Borel function of X taking at most N values turns out to solve the following minimization problem:

$$e_{N,r}(X)=\inf\{\|X-\pi(X)\|_r,\pi:\mathbb{R}^d\to\Gamma,\Gamma\subset\mathbb{R}^d,|\Gamma|\leq N\},$$

where $|A|$ stands for the cardinality of A, for $A\subset\mathbb{R}^d$. Now, let $\Gamma=\{x_1,\cdots,x_N\}\subset\mathbb{R}^d$ be a codebook of size N (also called an N-quantizer or a grid of size N) and define a Voronoi partition $C_i(\Gamma)_{i=1,\cdots,N}$ of $\mathbb{R}^d$, which is a Borel partition of $\mathbb{R}^d$ satisfying for every $i\in\{1,\cdots,N\}$,

$$C_i(\Gamma)\subset\big\{x\in\mathbb{R}^d:|x-x_i|=\min_{j=1,\cdots,N}|x-x_j|\big\}.$$

Consider the Voronoi quantization of X (simply called quantization of X) by the N-quantizer Γ defined by

$$\widehat{X}^\Gamma=\sum_{i=1}^N x_i\mathbb{1}_{\{X\in C_i(\Gamma)\}}.$$

Then, for any Borel function $\pi:\mathbb{R}^d\to\Gamma=\{x_1,\cdots,x_N\}$ we have

$$|X-\pi(X)|\geq\min_{i=1,\cdots,N}d(X,x_i)=d(X,\Gamma)=|X-\widehat{X}^\Gamma|\quad\mathbb{P}\text{ a.s}$$

so that the optimal L^r-mean quantization error $e_{N,r}(X)$ reads

$$\begin{aligned} e_{N,r}(X) &= \inf\{\|X-\widehat{X}^\Gamma\|_r, \Gamma\subset\mathbb{R}^d, |\Gamma|\le N\} \\ &= \inf_{\substack{\Gamma\subset\mathbb{R}^d\\ |\Gamma|\le N}} \left(\int_{\mathbb{R}^d} d(z,\Gamma)^r d\mathbb{P}_X(z)\right)^{1/r}. \end{aligned} \tag{18}$$

Recall that for every $N\ge 1$, the infimum in (18) is attained at least one codebook. Any N-quantizer realizing this infimum is called an L^r-optimal N-quantizer. Moreover, when $|\mathrm{supp}(\mathbb{P}_X))|\ge N$ then any L^r-mean optimal N-quantizer has exactly size N (see [2] or [6]). On the other hand, the quantization error, $e_{N,r}(X)$, decreases to zero as the grid size N goes to infinity and its rate of convergence is ruled by the so-called Zador Theorem recalled below. There also is a non-asymptotic upper bound for optimal quantizers. It is called Pierce Lemma (we recall it below for the quadratic case) and will allows us to put a finishing touches to the proof of the main result of the paper, stated in Theorem 3.6.

Theorem 2.1. *(a) (**Zador**, see [2, 10]). Let X be an $\mathbb{R}^d$-valued random vector such that $\mathbb{E}|X|^{r+\eta}<+\infty$ for some $\eta>0$ and let $\mathbb{P}_X=f\cdot\lambda_d+P_s$ be the Lebesgue decomposition of $\mathbb{P}_X$ with respect to the Lebesgue measure λ_d and P_s denotes its singular part. Then*

$$\lim_{N\to+\infty} N^{\frac{1}{d}} e_{N,r}(P)=\widetilde{Q}_r(\mathbb{P}_X) \tag{19}$$

with

$$\widetilde{Q}_r(\mathbb{P}_X)=\widetilde{J}_{r,d}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}d\lambda_d\right)^{\frac{1}{r}+\frac{1}{d}}=\widetilde{J}_{r,d}\,\|f\|^{1/r}_{\frac{d}{d+r}}\ \in[0,+\infty),$$

$$\widetilde{J}_{r,d}=\inf_{N\ge1} N^{\frac{1}{d}} e_{N,r}(U([0,1]^d))\in(0,+\infty)$$

where $U([0,1]^d)$ denotes the uniform distribution over the hypercube $[0,1]^d$.
*(b) (**Pierce**, see [2, 5]). Let $\eta>0$. There exists a universal constant $K_{2,d,\eta}$ such that for every random vector $X:(\Omega,\mathcal{A},\mathbb{P})\to\mathbb{R}^d$,*

$$\inf_{|\Gamma|\le N}\|X-\widehat{X}^\Gamma\|_2\le K_{2,d,\eta}\,\sigma_{2,\eta}(X)N^{-\frac{1}{d}} \tag{20}$$

where

$$\sigma_{2,\eta}(X)=\inf_{\zeta\in\mathbb{R}^d}\|X-\zeta\|_{2+\eta}\le+\infty.$$

From the Numerical Probability point of view, finding an optimal N-quantizer Γ may be a challenging task. In practice (we will only consider the quadratic case, i.e. $r=2$ for numerical implementations) we are sometimes led to find some "good" quantizations $\widehat{X}^\Gamma$ which are close to X in distribution, so that for every continuous function $f:\mathbb{R}^d\to\mathbb{R}$, we can approximate $\mathbb{E}f(X)$ by

$$\mathbb{E}f\big(\widehat{X}^\Gamma\big)=\sum_{i=1}^N p_i f(x_i), \tag{21}$$

where $p_i=\mathbb{P}(\widehat{X}^\Gamma=x_i)$. When we approximate $\mathbb{E}f(X)$ by (21), this induced an error which bound depends on the regularity of the function f (see e.g. [7] for more details).

We recall below the stationarity property for a quantizer.

Definition 2.1. A quantizer $\Gamma = \{x_1, \ldots, x_N\}$ of size N inducing the Voronoi quantization $\widehat{X}^\Gamma$ of X is stationary if $\mathbb{P}\left(X \in \cup_i \partial C_i(\Gamma)\right) = 0$, $\mathbb{P}\left(X \in C_i(\Gamma)\right) > 0$, $i = 1, \ldots, N$, and

$$\mathbb{E}\big(X|\widehat{X}^\Gamma\big) = \widehat{X}^\Gamma \ \mathbb{P}\text{-}a.s. \quad \Longleftrightarrow \quad x_i = \frac{\mathbb{E}(X \mathbb{1}_{\{X \in C_i(\Gamma)\}})}{\mathbb{P}(X \in C_i(\Gamma))}, \ i = 1, \ldots, N. \tag{22}$$

The notion of stationarity is related to the critical point of the so-called *distortion* function defined on $(\mathbb{R}^d)^N$ by

$$D_{N,2}(x) = \mathbb{E}\big(\min_{1 \leq i \leq N} |X - x_i|^2\big) = \int_{\mathbb{R}^d} |\xi - x_i|^2 \mathbb{P}_X(d\xi), \quad x = (x_1, \ldots, x_{_N}) \in (\mathbb{R}^d)^N. \tag{23}$$

As any grid of size at most N can be "represented" by some N-tuples (by repeating, if necessary, some of its elements), we will often put grids of all size N as an argument of the distortion function $D_{2,N}$ as well as for its gradient and Hessian matrix when its Voronoi boundary is negligible. It is also clear, from the definition of the quantization error, that

$$e^2_{N,2}(X) = \inf_{(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N} D_{N,2}(x_1, \ldots, x_N).$$

Furthermore, the function $D_{N,2}$ is continuous and differentiable at any N-tuple having pairwise distinct components with a $\mathbb{P}$-negligible Voronoi partition boundary and the following result makes this more precise.

Proposition 2.2. *(see [2, 6])* (a) *The function $D_{N,2}$ is differentiable at any N-tuple $(x_1, \ldots, x_{_N}) \in (\mathbb{R}^d)^N$ having pairwise distinct components and such that $\mathbb{P}\left(X \in \cup_i \partial C_i(\Gamma)\right) = 0$. Furthermore, we have*

$$\begin{aligned} \nabla D_{N,2}(x_1, \ldots, x_{_N}) \ &= \ 2\Big(\int_{C_i(\Gamma)} (x_i - x) d\mathbb{P}_X(x)\Big)_{i=1,\ldots,N} & (24)\\ &= \ 2\Big(\mathbb{P}(X \in C_i(\Gamma)) x_i - \mathbb{E}(X \mathbb{1}_{\{X \in C_i(\Gamma)\}})\Big)_{i=1,\ldots,N}. & (25) \end{aligned}$$

(b) *A grid $\Gamma = \{x_1, \ldots, x_{_N}\}$ of full size N is stationary if and only if*

$$\mathbb{P}\left(X \in \cup_i \partial C_i(\Gamma)\right) = 0 \quad \textit{and} \quad \nabla D_{N,2}(\Gamma) = 0. \tag{26}$$

(c) *If the support of $\mathbb{P}_{_X}$ has at least N elements, any L^2-optimal quantizer at level N has full size and a $\mathbb{P}$-negligible Voronoi boundary. Hence it is a stationary N-quantizer.*

For numerical implementations, the search of stationary quantizers is based on zero search recursive procedures like Newton-Raphson algorithm for real valued random variables, and some algorithms like Lloyd's I algorithms (see e.g. [1, 9]), the Competitive Learning Vector Quantization (CLVQ) algorithm (see [1]) or stochastic algorithms (see [7]) in the multidimensional framework. Optimal quantization grids associated to multivariate Gaussian random vectors can be downloaded on the website `www.quantize.math-fi.com`.

3 Markovian product quantization of an $\mathbb{R}^d$-valued Euler process

Let $(X_t)_{t \geq 0}$ be a stochastic process taking values in a d-dimensional Euclidean space $\mathbb{R}^d$ and solution to the stochastic differential equation:

$$X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad x_0 \in \mathbb{R}^d, \tag{27}$$

where W is a standard q-dimensional Brownian motion starting at 0 and where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the matrix diffusion coefficient function $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}(d,q,\mathbb{R})$ are measurable and satisfy the global Lipschitz and linear growth conditions: for every $t \in [0,T]$,

$$|b(t,x) - b(t,y)| \le [b]_{\mathrm{Lip}}|x-y| \tag{28}$$
$$\|\sigma(t,x) - \sigma(t,y)\| \le [\sigma]_{\mathrm{Lip}}|x-y| \tag{29}$$
$$|b(t,x)| \le L(1+|x|) \;\text{ and }\; \|\sigma(t,x)\| \le L(1+|x|). \tag{30}$$

$L > 0$. This guarantees the existence and pathwise uniqueness of a strong solution of (27), adapted to the (augmented) filtration of W. We also suppose that the matrix σ is positive definite. Throughout the paper we will suppose that $\mathbb{R}^d$ is equipped with the canonical Euclidean norm.

3.1 The algorithm and the Markov property of the quantized process

Recall that the Euler scheme of the stochastic process $(X_t)_{t\ge 0}$ is defined recursively from the following procedure:

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + \Delta b(t_k, \bar{X}_{t_k}) + \sigma(t_k, \bar{X}_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad \bar{X}_0 \in \mathbb{R}^d,$$

where $\Delta = \Delta_n = \frac{T}{n}$ and $t_k = \frac{kT}{n}$, for every $k \in \{0,\cdots,n\}$. To simplify notations, we will often set $X_k := X_{t_k}$ to denote the process X evaluated at time t_k. We also set $b_k(x) := b(t_k,x)$ and $\sigma_k(x) = \sigma(t_k,x)$ for $x \in \mathbb{R}^d$. Recall also that the operator associated to the conditional distribution of $\bar{X}_{k+1}$ given $\bar{X}_k = x$ is defined by

$$\mathcal{E}_k(x,z) := x + \Delta b(t_k,x) + \sqrt{\Delta}\sigma(t_k,x)z, \quad x\in\mathbb{R}^d,\ z\in\mathbb{R}^q$$

and that if Γ_{k+1} is an N_{k+1}-quantizer for $\bar{X}_{k+1}$, the distortion function $\bar{D}_{k+1}$ associated to $\bar{X}_{k+1}$ may be written for every $k = 0,\cdots,n-1$, as

$$\begin{aligned}\bar{D}_{k+1}(\Gamma_{k+1}) &= \mathbb{E}\big((\mathrm{dist}(\bar{X}_{k+1},\Gamma_{k+1})^2\big)\\ &= \mathbb{E}\big[\mathrm{dist}(\mathcal{E}_k(\bar{X}_k, Z_{k+1}),\Gamma_{k+1})^2\big]\end{aligned}$$

where $Z_{k+1} \sim \mathcal{N}(0;I_q)$ is independent from $\bar{X}_k$. The previous way to write the distortion function has been used in [8] to propose a fast recursive (and Markovian) quantization of the Euler process (using the Newton-Raphson algorithm for the numerical computation of the optimal grids) when $d=1$. The proposed algorithm extends to the dimension d. However, when $d \ge 2$, it makes use of stochastic or Lloyd (fixed point) like algorithms. These algorithms are very time consuming so that we cannot speak of fast quantization of the Euler process $(\bar{X}_k)_{k\ge 0}$ when $d \ge 2$. Our aim is to propose a faster way of getting the optimal quantizers when $d \ge 2$.

Keep in mind that the conditional distribution of the discrete Euler process $\bar{X}$ is Gaussian and that one of the properties of a Gaussian vector is that any sub-component of the vector remains a Gaussian random vector. So, a natural alternative way to quantize the vector $\bar{X}_k \in \mathbb{R}^d$ is to quantize each component $\bar{X}_k^\ell$ by a grid Γ_k^ℓ of size N_k^ℓ, for $\ell = 1,\ldots,d$, and then to define its product quantization $\widehat{X}_k$ associated with the product quantizer $\Gamma_k = \bigotimes_{\ell=1}^d \Gamma_k^\ell$ of size $N_k = N_k^1 \times \ldots \times N_k^d$, as $\widehat{X}_k = (\widehat{X}_k^1,\ldots,\widehat{X}_k^d)$.

The question is now to know how to quantize the $\bar{X}_k^i$'s. On the other hand, since the components of the vector $\bar{X}_k$ are not independent it is also a challenging question to know how to compute (from closed formula) the companions weights and transition probabilities associated with the quantizations of the $\bar{X}_k^i$'s and the vector $\bar{X}_k$. We describe below the componentwise recursive Markovian quantization of the process $\{\bar{X}_k,\ k = 0,\ldots,n\}$.

It is clear that for every $\ell = 1, \ldots, d$, and for every $k = 0, \ldots, n-1$, the transition operator $\mathcal{E}_k^\ell(x, z)$ associated with the distribution of $\bar{X}_{k+1}^\ell$ given $\bar{X}_k = x$ reads as

$$\mathcal{E}_k^\ell(x, z) := m_k^\ell(x) + \sqrt{\Delta}\big(\sigma_k^{\ell\bullet}(x)|z\big), \quad x \in \mathbb{R}^d,\ z \in \mathbb{R}^q,$$

where

$$m_k^\ell(x) := x^\ell + \Delta b_k(x).$$

For every $k = 0, \ldots, n$, for every given $\ell \in \{1, \ldots, d\}$, we denote by $\widehat{X}_k^\ell$ the quantization of $\bar{X}_k^\ell$ on the grid $\Gamma_k^\ell = \{x_k^{\ell, i_\ell}, i_\ell = 1, \ldots, N_k^\ell\}$. We propose in what follows a recursive and componentwise product quantization of the process $\{\bar{X}_k,\ k = 0, \ldots, n\}$. In fact, for every $\ell = 1, \ldots, d$, we denote by Γ_k^ℓ an N_k^ℓ-quantizer (we suppose that we have access to it) of the ℓ-th component $\bar{X}_k^\ell$ of the vector $\bar{X}_k$ and by $\widehat{X}_k^\ell$, the resulting quantization of $\bar{X}_k^\ell$. Then, we define a componentwise recursive product quantizer $\Gamma_k = \bigotimes_{\ell=1}^d \Gamma_k^\ell$ of size $N_k = N_k^1 \times \ldots \times N_k^d$ of the vector $\bar{X}_k = (\bar{X}_k^\ell)_{\ell=1,\ldots,d}$ by

$$\Gamma_k \ = \ \big\{(x_k^{1,i_1}, \ldots, x_k^{d,i_d}), \quad i_\ell \in \{1, \ldots, N_k^\ell\},\ \ell \in \{1, \ldots, d\}\big\}.$$

To define the Markovian product quantization, suppose that $\bar{X}_k$ has already been quantized and that we have access to the companion weights $\mathbb{P}(\widehat{X}_k = x_k^i)$, $i \in \mathscr{I}_k$, where $\mathscr{I}_k$ and x_k^i are defined by Equations (11) and (12). Setting $\widetilde{X}_k^\ell = \mathcal{E}_k^\ell(\widehat{X}_k, Z_{k+1})$, we may approximate the distortion function $\bar{D}_{k+1}^\ell$ associated to the ℓ-th component of the vector $\bar{X}_{k+1}^\ell$ by

$$\begin{aligned}
\widetilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell) \ &:= \ \mathbb{E}\big[\mathrm{dist}(\widetilde{X}_{k+1}^\ell, \Gamma_{k+1}^\ell)^2\big] \\
&= \ \mathbb{E}\big[\mathrm{dist}(\mathcal{E}_k^\ell(\widehat{X}_k, Z_{k+1}), \Gamma_{k+1}^\ell)^2\big] \\
&= \ \sum_{i \in \mathscr{I}_k} \mathbb{E}\big[\mathrm{dist}(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}), \Gamma_{k+1})^2\big] \mathbb{P}\big(\widehat{X}_k = x_k^i\big).
\end{aligned}$$

This allows us to consider the sequence of product recursive quantizations of $(\widehat{X}_k)_{k=0,\cdots,n}$, defined for every $k = 0, \ldots, n-1$, by the following recursion:

$$\begin{cases}
\widetilde{X}_0 = \widehat{X}_0, \quad \widehat{X}_k^\ell = \mathrm{Proj}_{\Gamma_k^\ell}(\widetilde{X}_k^\ell),\ \ell = 1, \ldots, d, \\
\widehat{X}_k = (\widehat{X}_k^1, \ldots, \widehat{X}_k^d) \quad \text{and} \quad \widetilde{X}_{k+1}^\ell = \mathcal{E}_k^\ell(\widehat{X}_k, Z_{k+1}),\ \ell = 1, \ldots, d, \\
\mathcal{E}_k^\ell(x, z) = m_k^\ell(x) + \sqrt{\Delta}(\sigma^{\ell\bullet}(t_k, x)|z),\ z = (z^1, \ldots, z^q) \in \mathbb{R}^q, \\
x = (x^1, \ldots, x^d),\ b = (b^1, \ldots, b^d) \text{ and } (\sigma^{\ell\bullet}(t_k, x)|z) = \sum_{m=1}^q \sigma^{\ell m}(t_k, x) z^m.
\end{cases} \tag{31}$$

where $(Z_k)_{k=1,\cdots,n}$ is $i.i.d.$, $\mathcal{N}(0; I_q)$-distributed, independent of $\bar{X}_0$.

In the following result, we show that the sequence $(\widehat{X}_k)_{k \geq 0}$ of Markovian and product quantizations is in fact a Markov chain. Its transition probabilities will be computed further on.

Proposition 3.1. *The process $(\widehat{X}_k)_{k \geq 0}$ is a Markov chain on $\mathbb{R}^d$.*

Proof. Set $\mathcal{E}_k(\widehat{X}_k, Z_{k+1}) = (\mathcal{E}_k^1(\widehat{X}_k, Z_{k+1}), \ldots, \mathcal{E}_k^d(\widehat{X}_k, Z_{k+1}))$, for every $k \geq 0$. Then, it follows from the definition of $\widehat{X}_{k+1}$ that

$$\widehat{X}_{k+1} = \sum_{j \in \mathscr{I}_{k+1}} x_{k+1}^j \mathbb{1}_{\{\mathcal{E}_k(\widehat{X}_k, Z_{k+1}) \in \prod_{\ell=1}^d C_{j_\ell}(\Gamma_{k+1}^\ell)\}}.$$

For any bounded function $f:\mathbb{R}^d\to\mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}(f(\widehat{X}_{k+1})|\widehat{X}_k) &= \sum_{j\in\mathscr{I}_{k+1}} \mathbb{E}\left(f(x_{k+1}^j)\mathbb{1}_{\{\widehat{X}_{k+1}=x_{k+1}^j\}}|\widehat{X}_k\right)\\ &= \sum_{j\in\mathscr{I}_{k+1}} f(x_{k+1}^j)\mathbb{E}\left(\mathbb{1}_{\{\mathcal{E}_k(\widehat{X}_k,Z_{k+1})\in\prod_{\ell=1}^d C_{j_\ell}(\Gamma_{k+1}^\ell)\}}|\widehat{X}_k\right)\\ &= \sum_{j\in\mathscr{I}_{k+1}} f(x_{k+1}^j)h_j(\widehat{X}_k)\end{aligned}$$

where for every $x\in\mathbb{R}^d$,

$$h_j(x)=\mathbb{P}\big(\mathcal{E}_k(x,Z_{k+1})\in\prod_{\ell=1}^d C_{j_\ell}(\Gamma_{k+1}^\ell)\big).$$

As a consequence, $\mathbb{E}(f(\widehat{X}_{k+1})|\widehat{X}_k)=\varphi(\widehat{X}_k)$, so that $(\widehat{X}_k)_{k\geq 0}$ is a Markov chain. □

Now, for a given componentwise (quadratic) optimal quantizers $\Gamma_k=\bigotimes_{\ell=1}^d\Gamma_k^\ell$, let us explain how to compute the companion transition probability weights associated with the quantizations of the $\bar{X}_k^\ell$'s and the whole vector $\bar{X}_k$. We write all the quantities of interest as an expectation of a function of a standard $\mathbb{R}^{q-1}$-valued Normal distribution. These transformations are the key step of this work. In fact, since the optimal quantization grids associated to standard Normal random vectors (up to dimension 10) and their companion weights are available on `www.quantize.maths-fi.com`, these quantities of interest may be computed instantaneously using a cubature formula.

3.2 Computing the companion weights and transition probabilities of the marginal quantizations

First of all we define the following quantities which will be needed in the sequel. For every $k\in\{0,\ldots,n-1\}$ and for every $j\in\mathscr{I}_{k+1}$ we set

$$x_{k+1}^{\ell,j_\ell-1/2}=\frac{x_{k+1}^{\ell,j_\ell}+x_{k+1}^{\ell,j_\ell-1}}{2},\ x_{k+1}^{\ell,j_\ell+1/2}=\frac{x_{k+1}^{\ell,j_\ell}+x_{k+1}^{\ell,j_\ell+1}}{2},\ \text{ with } x_{k+1}^{\ell,1/2}=-\infty, x_{k+1}^{\ell,N_{k+1}^\ell+1/2}=+\infty,$$

and if $Z_k^{(2:q)}=z\in\mathbb{R}^{q-1}$ and $x\in\mathbb{R}^d$, we set (if $\sigma_k^{\ell 1}(x)\neq 0$)

$$\begin{aligned} x_{k+1}^{\ell,j_\ell-}(x,z) &:= \frac{x_{k+1}^{\ell,j_\ell-1/2}-m_k^\ell(x)-\sqrt{\Delta}\big(\sigma_k^{(\ell,2:q)}(x)|z\big)}{\sqrt{\Delta}\sigma_k^{\ell 1}(x)}\\ \text{and}\quad x_{k+1}^{\ell,j_\ell+}(x,z) &:= \frac{x_{k+1}^{\ell,j_\ell+1/2}-m_k^\ell(x)-\sqrt{\Delta}\big(\sigma_k^{(\ell,2:q)}(x)|z\big)}{\sqrt{\Delta}\sigma_k^{\ell 1}(x)}.\end{aligned}$$

We also define the hyper-bounds

$$\mathbb{J}_{k,j_\ell}^0(x)=\Big\{z\in\mathbb{R}^{q-1},\quad \sqrt{\Delta}\big(\sigma_k^{(\ell,2:q)}(x)\big|z\big)\in\big(x_{k+1}^{\ell,j_\ell-1/2}-m_k^\ell(x),x_{k+1}^{\ell,j_\ell+1/2}-m_k^\ell(x)\big)\Big\}$$

and

$$\begin{aligned}\mathbb{J}_k^0(x) &= \big\{\ell\in\{1,\ldots,d\},\quad \sigma_k^{\ell 1}(x)=0\big\}\\ \mathbb{J}_k^-(x) &= \big\{\ell\in\{1,\ldots,d\},\quad \sigma_k^{\ell 1}(x)<0\big\}\\ \mathbb{J}_k^+(x) &= \big\{\ell\in\{1,\ldots,d\},\quad \sigma_k^{\ell 1}(x)>0\big\}.\end{aligned}$$

The following result allows us to compute (the weights and) the transition probability weights associated to the quantizations $\widehat{X}_k$, $k=0,\ldots,n$.

Proposition 3.2. *Let* $\{\widehat{X}_k, k=0,\ldots,n\}$ *be the sequence defined from the algorithm* (31).

1. *The transition probability* $\mathbb{P}(\widehat{X}_{k+1}=x^j_{k+1}|\widehat{X}_k=x^i_k)$, $i\in\mathscr{I}_k$, $j\in\mathscr{I}_{k+1}$, *is given by*

$$\mathbb{P}\big(\widehat{X}_{k+1}=x^j_{k+1}|\widehat{X}_k=x^i_k\big) \;=\; \mathbb{E}\prod_{\ell\in\mathbb{J}^0_k(x^i_k)}\mathbb{1}_{\{\zeta\in\mathbb{J}^0_{k,j_\ell}(x^i_k)\}}\big(\Phi_0(\beta_j(x^i_k,\zeta))-\Phi_0(\alpha_j(x^i_k,\zeta))\big)_+ \quad (32)$$

where $\zeta\sim\mathcal{N}(0;I_{q-1})$ *and where for every* $x\in\mathbb{R}^d$ *and* $z\in\mathbb{R}^{q-1}$,

$$\alpha_j(x,z)=\max\big(\sup_{\ell\in\mathbb{J}^+_k(x)} x^{\ell,j_\ell-}_{k+1}(x,z),\ \sup_{\ell\in\mathbb{J}^-_k(x)} x^{\ell,j_\ell+}_{k+1}(x,z)\big)$$
$$\textit{and}\quad \beta_j(x,z)=\min\big(\inf_{\ell\in\mathbb{J}^+_k(x)} x^{\ell,j_\ell+}_{k+1}(x,z),\ \inf_{\ell\in\mathbb{J}^-_k(x)} x^{\ell,j_\ell-}_{k+1}(x,z)\big),$$

with the convention that $\prod_{\ell\in\emptyset}=1$.

2. *For every* $j\in\mathscr{I}_{k+1}$,

$$\mathbb{P}\big(\widehat{X}_{k+1}=x^j_{k+1}\big) \;=\; \sum_{i\in\mathscr{I}_k}\mathbb{P}\big(\widehat{X}_{k+1}=x^j_{k+1}|\widehat{X}_k=x^i_k\big)\mathbb{P}\big(\widehat{X}_k=x^i_k\big), \qquad (33)$$

where the conditional probabilities are computed using the formula (32).

Remark when $d=1$, the results of Proposition 3.2 coincide with the results of Proposition 4.1. in [8].

Proof. We will only show the first assertion. The second assertion is a consequence of the first one. Let us set $v^{\ell,j_\ell+}:=x^{\ell,j_\ell+1/2}_{k+1}$ and $v^{\ell,j_\ell-}=x^{\ell,j_\ell-1/2}_{k+1}$. We have

$$\begin{aligned}
\mathbb{P}(\widehat{X}_{k+1}=x^j_{k+1}|\widehat{X}_k=x^i_k) \;&=\; \mathbb{P}\Big(\bigcap_{\ell=1}^d\big\{\widetilde{X}^\ell_{k+1}\in\big(v^{\ell,j_\ell-},v^{\ell,j_\ell+}\big)\big\}\big|\widehat{X}_k=x^i_k\Big)\\
&=\; \mathbb{P}\Big(\bigcap_{\ell=1}^d\big\{\mathcal{E}^\ell_k(x^i_k,Z_{k+1})\in\big(v^{\ell,j_\ell-},v^{\ell,j_\ell+}\big)\big\}\Big)\\
&=\; \mathbb{E}\Big(\mathbb{E}\Big(\mathbb{1}_{\bigcap_{\ell=1}^d\big\{\mathcal{E}^\ell_k(x^i_k,Z_{k+1})\in\big(v^{\ell,j_\ell-},v^{\ell,j_\ell+}\big)\big\}}\Big)\big|Z^{(2:q)}_{k+1}\Big)\\
&=\; \mathbb{E}\big(\Psi(x^i_k,Z^{(2:q)}_{k+1})\big),
\end{aligned}$$

where for every $u\in\mathbb{R}^{q-1}$,

$$\Psi(x,u) \;=\; \mathbb{P}\Big(\bigcap_{\ell=1}^d\big\{m^\ell_k(x)+\sqrt{\Delta}\sigma^{\ell1}_k(x)Z^1_{k+1}+\sqrt{\Delta}\big(\sigma^{(\ell,2:q)}_k(x)|u\big)\in\big(v^{\ell,j_\ell-},v^{\ell,j_\ell+}\big)\big\}\Big).$$

Let us set

$$A_{\ell,k}=\big\{m^\ell_k(x)+\sqrt{\Delta}\sigma^{\ell1}_k(x)Z^1_{k+1}+\sqrt{\Delta}\big(\sigma^{(\ell,2:q)}_k(x)|u\big)\in\big(v^{\ell,j_\ell-},v^{\ell,j_\ell+}\big)\big\}.$$

We know that if $\ell\in\mathbb{J}^0_k(x)$ then $A_{\ell,k}=\{u\in\mathbb{J}^0_{k,j_\ell}(x)\}$ and we deduce that

$$\Psi(x,u)=\prod_{\ell_0\in\mathbb{J}^0_k(x)}\mathbb{1}_{\{u\in\mathbb{J}^0_{k,j_{\ell_0}}(x)\}}\mathbb{P}\Bigg(\big(\bigcap_{\ell_-\in\mathbb{J}^-_k(x)}A_{\ell_-,k}\big)\cap\big(\bigcap_{\ell_+\in\mathbb{J}^+_k(x)}A_{\ell_+,k}\big)\Bigg).$$

Notice that $\Psi(x,u)=0$ if $A_{\ell_-,k}$ or $A_{\ell_+,k}$ is empty. Furthermore, we note that if $\ell_+\in\mathbb{J}_k^+(x)$, then

$$A_{\ell_+,k}=\{Z^1_{k+1}\in(x^{\ell,j_\ell-}_{k+1}(x,u),x^{\ell,j_\ell+}_{k+1}(x,u))\}$$

and if $\ell_-\in\mathbb{J}_k^-(x)$, then

$$A_{\ell_-,k}=\{Z^1_{k+1}\in(x^{\ell,j_\ell+}_{k+1}(x,u),x^{\ell,j_\ell-}_{k+1}(x,u))\}.$$

It follows that (having in mind that the sets $\mathbb{J}_k^-(x)$ or $\mathbb{J}_k^+(x)$ may be empty)

$$\begin{aligned}\mathbb{P}\Bigg(\Big(\bigcap_{\ell_-\in\mathbb{J}_k^-(x)}A_{\ell_-,k}\Big)\cap\Big(\bigcap_{\ell_+\in\mathbb{J}_k^+(x)}A_{\ell_+,k}\Big)\Bigg) \;=\; &\mathbb{P}\Big(Z^1_{k+1}\in\Big(\sup_{\ell\in\mathbb{J}_k^+(x)}x^{\ell,j_\ell-}_{k+1}(x,u),\inf_{\ell\in\mathbb{J}_k^+(x)}x^{\ell,j_\ell+}_{k+1}(x,u)\Big)\\ &\cap\Big(\sup_{\ell\in\mathbb{J}_k^-(x)}x^{\ell,j_\ell+}_{k+1}(x,u),\inf_{\ell\in\mathbb{J}_k^-(x)}x^{\ell,j_\ell-}_{k+1}(x,u)\Big)\Big).\end{aligned}$$

This completes the proof since $Z^{(2:q)}_{k+1}\sim\mathcal{N}(0;I_{q-1})$. □

Now, we focus on in the particular case where the matrix $\sigma(t,x)$, for $(t,x)\in[0,T]\times\mathbb{R}^d$, is diagonal with positive diagonal entries $\sigma^{\ell\ell}(t,x)$, $\ell=1,\dots,d$. The following result says how to compute the transition probability weights of the $\widehat{X}_k$'s. Let us set for every $x\in\mathbb{R}^d$, every $\ell\in\{1,\dots,d\}$ and $j_\ell\in\{1,\dots,N^\ell_{k+1}\}$,

$$x^{\ell,j_\ell-}_{k+1}(x,0):=\frac{x^{\ell,j_\ell-1/2}_{k+1}-m^\ell_k(x)}{\sqrt{\Delta}\sigma^{\ell\ell}_k(x)}\quad\text{and}\quad x^{\ell,j_\ell+}_{k+1}(x,0):=\frac{x^{\ell,j_\ell+1/2}_{k+1}-m^\ell_k(x)}{\sqrt{\Delta}\sigma^{\ell\ell}_k(x)}.$$

Proposition 3.3. *Let $\{\widehat{X}_k,k=0,\dots,n\}$ be the sequence of quantizers defined by the algorithm* (31) *and associated with the solution (X_t) of* (27)*. Suppose that the volatility matrix $\sigma(t,x)$ of $(X_t)_{t\ge0}$ is diagonal with positive diagonal entries $\sigma^{\ell\ell}(t,x)$, $\ell=1,\dots,d$. Then, the transition probability weights $\mathbb{P}(\widehat{X}_{k+1}=x^j_{k+1}|\widehat{X}_k=x^i_k)$, $i\in\mathscr{I}_k,j\in\mathscr{I}_{k+1}$, are given by*

$$\mathbb{P}\big(\widehat{X}_{k+1}=x^j_{k+1}|\widehat{X}_k=x^i_k\big)\;=\;\prod_{\ell=1}^d\mathbb{P}\big(\widehat{X}^\ell_{k+1}=x^{j_\ell}_{k+1}|\widehat{X}_k=x^i_k\big)\tag{34}$$

$$\;=\;\prod_{\ell=1}^d\big[\Phi_0\big(x^{\ell,j_\ell+}_{k+1}(x^i_k,0)\big)-\Phi_0\big(x^{\ell,j_\ell-}_{k+1}(x^i_k,0)\big)\big],\tag{35}$$

and the companions probabilities $\mathbb{P}\big(\widehat{X}_{k+1}=x^j_{k+1}\big)$ are given for every $k=0,\dots,n-1$ and every $j\in\mathscr{I}_{k+1}$ by

$$\mathbb{P}\big(\widehat{X}_{k+1}=x^j_{k+1}\big)\;=\;\sum_{i\in\mathscr{I}_k}\prod_{\ell=1}^d\big[\Phi_0\big(x^{\ell,j_\ell+}_{k+1}(x^i_k,0)\big)-\Phi_0\big(x^{\ell,j_\ell-}_{k+1}(x^i_k,0)\big)\big]\mathbb{P}(\widehat{X}_k=x^i_k).\tag{36}$$

Proof. 1. Set $v^{\ell,j_\ell+}:=x^{\ell,j_\ell+1/2}_{k+1}$ and $v^{\ell,j_\ell-}=x^{\ell,j_\ell-1/2}_{k+1}$, for $j\in\mathscr{I}_{k+1}$ and $\ell=1,\dots,d$. We have

$$\begin{aligned}\mathbb{P}(\widehat{X}_{k+1}=x^j_{k+1}|\widehat{X}_k=x^i_k)\;&=\;\mathbb{P}\Big(\bigcap_{\ell=1}^d\big\{\widetilde{X}^\ell_{k+1}\in\big(v^{\ell,j_\ell-},v^{\ell,j_\ell+}\big)\big\}\big|\widehat{X}_k=x^i_k\Big)\\&=\;\mathbb{P}\Big(\bigcap_{\ell=1}^d\big\{\mathcal{E}^\ell_k(x^i_k,Z_{k+1})\in\big(v^{\ell,j_\ell-},v^{\ell,j_\ell+}\big)\big\}\Big).\end{aligned}$$

Since for every $k = 0, \ldots, n-1$, $\sigma(t_k, x)$ is a diagonal matrix, it follows that the operators $\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) = \mathcal{E}_k^\ell(x_k^i, Z_{k+1}^\ell)$, for $\ell = 1, \ldots, d$, are independent, so that

$$\begin{aligned}\mathbb{P}(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^i) &= \prod_{\ell=1}^d \mathbb{P}\Big(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}^\ell) \in \big(v^{\ell,j_\ell-}, v^{\ell,j_\ell+}\big)\Big)\\ &= \prod_{\ell=1}^d \big[\Phi_0\big(x_{k+1}^{\ell,j_\ell+}(x_k^i)\big) - \Phi_0\big(x_{k+1}^{\ell,j_\ell-}(x_k^i)\big)\big].\end{aligned}$$

The second assertion immediately follows. □

The following result is useful in the situation where we need to approximate the expectation of a function of one component of the vector $\bar{X}_k$ as for example in the pricing of European options in the Heston model.

Proposition 3.4. *Let Γ_{k+1}^ℓ be an optimal quantizer for the random variable $\widetilde{X}_{k+1}^\ell$. Suppose that the optimal product quantizer Γ_k for $\widetilde{X}_k$ and its companion weights $\mathbb{P}(\widehat{X}_k = x_k^i)$, $i \in \mathscr{I}_k$, are computed.*

1. *For any $\ell \in \{1, \ldots, d\}$ and any $j_\ell \in \{1, \ldots, N_{k+1}^\ell\}$, the transition probability weights $\mathbb{P}(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell) | \widehat{X}_k = x_k^i)$ are given by*

$$\mathbb{P}\big(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell) | \widehat{X}_k = x_k^i\big) = \Phi_0\left(\frac{x_{k+1}^{\ell,j_\ell+1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta}|\sigma_k^{\ell\bullet}(x_k^i)|_2}\right) - \Phi_0\left(\frac{x_{k+1}^{\ell,j_\ell-1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta}|\sigma_k^{\ell\bullet}(x_k^i)|_2}\right). \quad (37)$$

2. *The companion probability $\mathbb{P}(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}))$ is given, for every $\ell \in \{1, \ldots, d\}$ and for every $j_\ell \in \{1, \cdots, N_{k+1}^\ell\}$, by*

$$\begin{aligned}\mathbb{P}\big(\tilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell)\big) = \sum_{i \in \mathscr{I}_k} &\left[\Phi_0\left(\frac{x_{k+1}^{\ell,j_\ell+1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta}|\sigma_k^{\ell\bullet}(x_k^i)|_2}\right)\right.\\ &\left. - \Phi_0\left(\frac{x_{k+1}^{\ell,j_\ell-1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta}|\sigma_k^{\ell\bullet}(x_k^i)|_2}\right)\right]\mathbb{P}\big(\widehat{X}_k = x_k^i\big). \quad (38)\end{aligned}$$

Proof. 1. For every $k \in \{1, \ldots, n-1\}$, for every $\ell = 1, \ldots, d$ and for every $j_\ell = 1, \ldots, N_{k+1}^\ell$, we have

$$\begin{aligned}\mathbb{P}\big(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell) | \widehat{X}_k = x_k^i\big) &= \mathbb{P}\big(\widetilde{X}_{k+1}^\ell \leq x_{k+1}^{\ell,j_\ell+1/2} | \widehat{X}_k = x_k^i\big) - \mathbb{P}\big(\widetilde{X}_{k+1}^\ell \leq x_{k+1}^{\ell,j_\ell-1/2} | \widehat{X}_k = x_k^i\big)\\ &= \mathbb{P}\big(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \leq x_{k+1}^{\ell,j_\ell+1/2}\big) - \mathbb{P}\big(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \leq x_{k+1}^{\ell,j_\ell-1/2}\big).\end{aligned}$$

Now, since $Z_{k+1} \sim \mathcal{N}(0; I_q)$, we have for every $x \in \mathbb{R}$,

$$\mathbb{P}\big(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \leq x\big) = \mathbb{E}\left(\mathbb{1}_{\{\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \leq x\}} | Z_{k+1}^{(2:d)}\right) = \mathbb{E}\big(\Psi(x, Z)\big)$$

where $Z \sim \mathcal{N}(0; I_{q-1})$ and, for every $u \in \mathbb{R}^{q-1}$ and $x \in \mathbb{R}$,

$$\begin{aligned}\Psi(x, u) &= \mathbb{P}\big(m_k^\ell(x_k^i) + \sqrt{\Delta}\sigma_k^{\ell 1}(x_k^i) Z_{k+1}^1 + \sqrt{\Delta}\big(\sigma_k^{(\ell,2:q)}(x_k^i)|u\big) \leq x\big)\\ &= \Phi_0\left(\frac{x - m_k^\ell(x_k^i) - \sqrt{\Delta}\big(\sigma_k^{(\ell,2:q)}(x_k^i)|u\big)}{\sqrt{\Delta}\sigma_k^{\ell 1}(x_k^i)}\right).\end{aligned}$$

Temporarily set

$$a_{\ell p} = \sqrt{\Delta}\sigma^{\ell p}(t_k, x_k^i), \quad \sigma^2 = \sum_{p=2}^{q} a_{\ell p}^2 \quad \text{and} \quad b(x) = x - m_k^\ell(x_k^i).$$

Note that, if $Z \sim \mathcal{N}(0; I_{q-1})$, then $-\sqrt{\Delta}\big(\sigma_k^{(\ell,2:q)}(x_k^i)|Z\big) \sim \mathcal{N}\big(0; \sum_{p=2}^q a_{\ell p}^2\big)$. Consequently, making a change of variable and using Fubini's Theorem yields, for every $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\Psi(x,Z) &= \frac{1}{\sigma\sqrt{2\pi}} \int \Phi_0\Big(\frac{b(x)+v}{a_{\ell 1}}\Big) e^{-\frac{v^2}{2\sigma^2}} dv \\
&= \frac{1}{\sqrt{2\pi}} \int \Phi_0\Big(\frac{b(x)+\sigma v}{a_{\ell 1}}\Big) e^{-\frac{1}{2}v^2} dv \\
&= \frac{1}{2\pi} \int \int_{-\infty}^{\frac{b(x)+\sigma v}{a_{\ell 1}}} e^{-\frac{1}{2}(v^2+w^2)} dv dw \\
&= \mathbb{E}\big(\mathbb{1}_{\{V \le \frac{b(x)+\sigma W}{a_{\ell 1}}\}}\big) \\
&= \mathbb{P}\big(a_{\ell 1}V - \sigma W \le b(x)\big)
\end{aligned}$$

where V and W are two independent and standard scalar normal random variables. On the other hand, since $a_{\ell 1}V - \sigma W \sim \mathcal{N}\big(0; \Delta|\sigma_k^{\ell\bullet}(x_k^i)|_2^2\big)$, it is clear that

$$\mathbb{P}\big(a_{\ell 1}V - \sigma W \le b(x)\big) = \Phi_0\Big(\frac{b(x)}{\sqrt{\Delta}|\sigma_k^{\ell\bullet}(x_k^i)|_2}\Big).$$

This completes the proof of the first statement.

2. We have for every $k \in \{1, \cdots, n-1\}$, every $i = 1, \ldots, d$, and every $j_\ell = 1, \cdots, N_{k+1}$,

$$\begin{aligned}
\mathbb{P}\big(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell)\big) &= \mathbb{E}\big[\mathbb{P}\big(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1})|\widehat{X}_k\big)\big] \\
&= \sum_{i \in \mathscr{I}_k} \mathbb{P}\big(\widetilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell)|\widehat{X}_k = x_k^i\big)\mathbb{P}(\widehat{X}_k = x_k^i).
\end{aligned}$$

We complete the proof using the arguments of the first statement. $\square$

Remark that, although the process $(\widehat{X}_k)_{k\ge 0}$ is a Markov chain, its ℓ-th component process $(\widehat{X}_k^\ell)_{k\ge 0}$ is not. We may however compute the transition probabilities

$$\mathbb{P}(\widehat{X}_{k+1}^\ell = x_{k+1}^{\ell,j_\ell}|\widehat{X}_k^{\ell'} = x_k^{\ell',j_{\ell'}}), \quad \ell,\ell' \in \{1,\ldots,d\},\ j_\ell \in \{1,\ldots,N_{k+1}^\ell\},\ j_{\ell'} \in \{1,\ldots,N_k^{\ell'}\}.$$

This is the aim of the following remark which follows from *Bayes* formula.

Remark 3.1. For $\ell,\ell' \in \{1,\ldots,d\}$, $j_\ell \in \{1,\ldots,N_{k+1}^\ell\}$ and $j_{\ell'} \in \{1,\ldots,N_k^{\ell'}\}$, we have

$$\mathbb{P}\big(\widehat{X}_{k+1}^\ell = x_{k+1}^{\ell,j_\ell}|\widehat{X}_k^{\ell'} = x_k^{\ell',j_{\ell'}}\big) = \sum_{i\in\mathscr{I}_k} \mathbb{1}_{\{i_{\ell'}=j_{\ell'}\}} \frac{\mathbb{P}(\widehat{X}_{k+1}^\ell = x_{k+1}^{\ell,j_\ell}|\widehat{X}_k = x_k^i)}{\mathbb{P}(\widehat{X}_k^{\ell'} = x_k^{\ell',j_{\ell'}})}\mathbb{P}(\widehat{X}_k = x_k^i) \tag{39}$$

where the terms $\mathbb{P}(\widehat{X}_k = x_k^i)$, $\mathbb{P}(\widehat{X}_{k+1}^\ell = x_{k+1}^{\ell,j_\ell}|\widehat{X}_k = x_k^i)$ and $\mathbb{P}(\widehat{X}_k^{\ell'} = x_k^{\ell',j_{\ell'}})$ are computed from (33), (37) and (38), respectively.

As a matter of fact, applying *Bayes* formula and summing over $i \in \mathscr{I}_k$ yields:

$$\begin{aligned}
\mathbb{P}\big(\widehat{X}^{\ell}_{k+1} = x^{\ell,j_\ell}_{k+1} | \widehat{X}^{\ell'}_k = x^{\ell',j_{\ell'}}_k\big) &= \sum_{i \in \mathscr{I}_k} \frac{\mathbb{P}(\widehat{X}^{\ell}_{k+1} = x^{\ell,j_\ell}_{k+1}, \widehat{X}^{\ell'}_k = x^{\ell',j_{\ell'}}_k, \widehat{X}_k = x^i_k)}{\mathbb{P}(\widehat{X}^{\ell'}_k = x^{\ell',j_{\ell'}}_k)} \\
&= \sum_{i \in \mathscr{I}_k} \mathbb{1}_{\{i_{\ell'} = j_{\ell'}\}} \frac{\mathbb{P}(\widehat{X}^{\ell}_{k+1} = x^{\ell,j_\ell}_{k+1}, \widehat{X}_k = x^i_k)}{\mathbb{P}(\widehat{X}^{\ell'}_k = x^{\ell',j_{\ell'}}_k)} \\
&= \sum_{i \in \mathscr{I}_k} \mathbb{1}_{\{i_{\ell'} = j_{\ell'}\}} \frac{\mathbb{P}(\widehat{X}^{\ell}_{k+1} = x^{\ell,j_\ell}_{k+1} | \widehat{X}_k = x^i_k)}{\mathbb{P}(\widehat{X}^{\ell'}_k = x^{\ell',j_{\ell'}}_k)} \mathbb{P}(\widehat{X}_k = x^i_k)
\end{aligned}$$

In the foregoing, we assume that we have access to the N^{ℓ}_k-quantizers Γ^{ℓ}_k of the ℓ-th component $\bar{X}^{\ell}_k$ of the vector $\bar{X}_k$, for every $\ell = 1, \ldots, d$. We show how to compute the distortion functions associated with every component of the vector $\widetilde{X}_{k+1}$, $k = 0, \ldots, n-1$. From the numerical point of view, this will allow us to use the Newton-Raphson algorithm to compute the optimal quantizers associated to each component $\widetilde{X}^{\ell}_{k+1}$, $\ell = 1, \ldots, d$, of the vector $\widetilde{X}_{k+1}$, for $k = 0, \ldots, n-1$. Then, the quantization $\widehat{X}_{k+1}$ of $\widetilde{X}_{k+1}$ is defined as the product quantization $\widehat{X}_k = (\widehat{X}^1_k, \ldots, \widehat{X}^d_k)$, where $\widehat{X}^{\ell}_k = \mathrm{Proj}_{\Gamma^{\ell}_{k+1}}(\widetilde{X}^{\ell}_{k+1})$.

3.3 Computing the distortion, the gradient and the Hessian matrix associated to a componentwise quantizer

Our aim, for numerical computation of the componentwise optimal quantizations, is to use the Newton-Raphson's algorithm in $\mathbb{R}^{N_k}$ which involves the gradient and the Hessian matrix of the distortion functions $\widetilde{D}^{\ell}_k$, $k = 0, \ldots, n$; $\ell = 1, \ldots, n$. In the following, we give useful expressions for the distortion functions $\widetilde{D}^{\ell}_k$, for their gradient vectors $\nabla \widetilde{D}^{\ell}_k$ and their Hessian matrices $\nabla^2 \widetilde{D}^{\ell}_k$. We state these results in the next proposition.

Above all, recall that for every $\ell = 1, \ldots, d$, for every $k = 0, \ldots, n-1$,

$$\widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1}) = \sum_{i \in \mathscr{I}_k} \mathbb{E}\big[d(\mathcal{E}^{\ell}_k(x^i_k, Z_{k+1}), \Gamma^{\ell}_{k+1})^2\big] \mathbb{P}\big(\widehat{X}_k = x^i_k\big)$$

and notice that using Proposition 2.2, the distortion function $\widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1})$ is continuously differentiable as a function of the N_{k+1}-quantizer $\Gamma^{\ell}_{k+1} = \{x^{\ell,j_\ell}_{k+1},\ j_\ell = 1, \ldots N^{\ell}_{k+1}\}$ (having pairwise distinct components so that it can be viewed as an N^{ℓ}_{k+1}-tuple) and its gradient vector reads

$$\nabla \widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1}) = 2 \left[\sum_{i \in \mathscr{I}_k} \mathbb{E}\Big(\mathbb{1}_{\{\mathcal{E}^{\ell}_k(x^i_k, Z_{k+1}) \in C_{j_\ell}(\Gamma^{\ell}_{k+1})\}} \big(x^{\ell,j_\ell}_{k+1} - \mathcal{E}^{\ell}_k(x^i_k, Z_{k+1})\big)\Big) \mathbb{P}(\widehat{X}_k = x^{\ell}_k) \right]_{j_\ell = 1, \cdots, N^{\ell}_{k+1}}.$$

We recall that key point of our method is to deal with the product quantization of the components of the process $(\bar{X}_k)_{0 \le k \le n}$. From a numerical point of view, each component will be quantized using the Newton-Raphson algorithm. To this end, we have to compute (explicitly) the distortion function $\widetilde{D}^{\ell}_{k+1}(\cdot)$, the components of its gradient vector and the components its Hessian matrix. This is the purpose of the following proposition. Its proof relies on tedious though easy computation. Therefore, we have deliberately omitted the proof.

Proposition 3.5. *Set* $v^{\ell,1}_k(x) = \sqrt{\Delta}\sigma^{\ell 1}_k(x)$, $v^{(\ell,2:q)}_k(x) = \sqrt{\Delta}\sigma^{(\ell,2:q)}_k(x)$, $x \in \mathbb{R}^d$.

$a)$ Distortion. *We have for every* $\ell = 1, \ldots, d$ *and every* $k = 0, \ldots, n-1$,

$$\widetilde{D}^{\ell}_{k+1}(\Gamma^{\ell}_{k+1}) = \sum_{i \in \mathscr{I}_k} \sum_{j_\ell = 1}^{N^{\ell}_{k+1}} \mathbb{E}\big[\Psi_{\ell, j_\ell}(x^i_k, \zeta)\big] \mathbb{P}\big(\widehat{X}_k = x^i_k\big), \tag{40}$$

where $\zeta \sim \mathcal{N}(0; I_{q-1})$ *and, for every* $x \in \mathbb{R}^d$ *and* $u \in \mathbb{R}^{q-1}$,

$$\begin{aligned}\Psi_{\ell,j_\ell}(x,u) = &\Big(\big(m_k^\ell(x) + \big(v_k^{(\ell,2:q)}(x)|u\big) - x_{k+1}^{\ell,j_\ell}\big)^2 + v_k^{\ell,1}(x)^2\Big)\Big(\Phi_0\big(x_{k+1}^{\ell,j_\ell+}(x,u)\big) - \Phi_0\big(x_{k+1}^{\ell,j_\ell-}(x,u)\big)\Big)\\ &+ 2v_k^{\ell,1}(x)\Big(x_{k+1}^{\ell,j_\ell} - m_k^\ell(x) - \big(v_k^{(\ell,2:q)}(x)|u\big)\Big)\Big(\Phi_0'\big(x_{k+1}^{\ell,j_\ell+}(x,u)\big) - \Phi_0'\big(x_{k+1}^{\ell,j_\ell-}(x,u)\big)\Big)\\ &- v_k^{\ell,1}(x)^2\Big(x_{k+1}^{\ell,j_\ell+}(x,u)^2\Phi_0'\big(x_{k+1}^{\ell,j_\ell+}(x,u)\big) - x_{k+1}^{\ell,j_\ell-}(x,u)^2\Phi_0'\big(x_{k+1}^{\ell,j_\ell-}(x,u)\big)\Big).\end{aligned}$$

$b)$ Gradient. *The components of the gradient* $\nabla \widetilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)$ *are given for every* $j_\ell = 1, \ldots, N_{k+1}^\ell$ *by*

$$\frac{\partial \widetilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)}{\partial x_{k+1}^{\ell,j_\ell}} = \sum_{i \in \mathscr{I}_k} \mathbb{E}\big[\Psi_{j_\ell}'(x_k^i,\zeta)\big]\mathbb{P}\big(\widehat{X}_k = x_k^i\big), \tag{41}$$

where $\zeta \sim \mathcal{N}(0; I_{q-1})$ *and, for every* $x \in \mathbb{R}^d$, *for every* $u \in \mathbb{R}^{q-1}$,

$$\begin{aligned}\Psi_{j_\ell}'(x,u) = &\big(x_{k+1}^{\ell,j_\ell} - m_k^\ell(x)\big)\Big(\Phi_0\big(x_{k+1}^{\ell,j_\ell+}(x,u)\big) - \Phi_0\big(x_{k+1}^{\ell,j_\ell-}(x,u)\big)\Big)\\ &+ v_k^{\ell,1}(x)\Big(\Phi_0'\big(x_{k+1}^{\ell,j_\ell+}(x,u)\big) - \Phi_0'\big(x_{k+1}^{\ell,j_\ell-}(x,u)\big)\Big).\end{aligned}$$

$c)$ Hessian. *The sub-diagonal, the super-diagonals and the diagonal terms of the Hessian matrix are given respectively by*

$$\begin{aligned}\frac{\partial^2 \widetilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)}{\partial x_{k+1}^{\ell,j_\ell}\partial x_{k+1}^{\ell,j_\ell-1}} &= \sum_{i \in \mathscr{I}_k} \mathbb{E}\big[\Psi_{j_\ell,j_\ell-1}''(x_k^i,\zeta)\big]\mathbb{P}\big(\widehat{X}_k = x_k^i\big),\\ \frac{\partial^2 \widetilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)}{\partial x_{k+1}^{\ell,j_\ell}\partial x_{k+1}^{\ell,j_\ell+1}} &= \sum_{i \in \mathscr{I}_k} \mathbb{E}\big[\Psi_{j_\ell,j_\ell+1}''(x_k^i,\zeta)\big]\mathbb{P}\big(\widehat{X}_k = x_k^i\big),\\ \textit{and } \frac{\partial^2 \widetilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)}{\partial^2 x_{k+1}^{\ell,j_\ell}} &= \sum_{i \in \mathscr{I}_k} \mathbb{E}\big[\Psi_{j_\ell,j_\ell}''(x_k^i,\zeta)\big]\mathbb{P}\big(\widehat{X}_k = x_k^i\big),\end{aligned}$$

where $\zeta \sim \mathcal{N}(0; I_{q-1})$ *and, for every* $x \in \mathbb{R}^d$, *for every* $u \in \mathbb{R}^{q-1}$,

$$\begin{aligned}\Psi_{j_\ell,j_\ell-1}''(x,u) &= -\frac{1}{4}\frac{1}{v_k^{\ell,1}(x)}\big(x_{k+1}^{\ell,j_\ell} - x_{k+1}^{\ell,j_\ell-1}\big)\Phi_0'\big(x_{k+1}^{\ell,j_\ell-}(x,u)\big),\\ \Psi_{j_\ell,j_\ell+1}''(x,u) &= -\frac{1}{4}\frac{1}{v_k^{\ell,1}(x)}\big(x_{k+1}^{\ell,j_\ell+1} - x_{k+1}^{\ell,j_\ell}\big)\Phi_0'\big(x_{k+1}^{\ell,j_\ell+}(x,u)\big),\\ \Psi_{j_\ell,j_\ell}''(x,u) &= \Phi_0\big(x_{k+1}^{\ell,j_\ell+}(x,u)\big) - \Phi_0\big(x_{k+1}^{\ell,j_\ell-}(x,u)\big) + \Psi_{j_\ell,j_\ell-1}''(x,u) + \Psi_{j_\ell,j_\ell+1}''(x,u).\end{aligned}$$

Once we have access to the gradient vector and the Hessian matrix associated with $\widetilde{X}_{k+1}^\ell$ and to the optimal grids and companions weights associated with the $\widehat{X}_p$'s, $p = 0, \ldots, k$, it is possible to write down (at least formally) a Newton-Raphson zero search procedure to compute the optimal quantizer Γ_{k+1}^ℓ. The Newton-Raphson algorithm is in fact indexed by $p \geq 0$, where a current grid $\Gamma_{k+1}^{\ell,p}$ is updated as follows:

$$\Gamma_{k+1}^{\ell,p+1} = \Gamma_{k+1}^{\ell,p} - \big(\nabla^2 \widetilde{D}_{k+1}^\ell(\Gamma_{k+1}^{\ell,p})\big)^{-1}\nabla \widetilde{D}_{k+1}^\ell(\Gamma_{k+1}^{\ell,p}), \quad p \geq 1, \tag{42}$$

starting from a $\Gamma_{k+1}^{\ell,0} \in \mathbb{R}^{N_{k+1}^\ell}$ (with increasing components).

Remark 3.2. (*Stationarity property*) If Γ^ℓ_{k+1} is a quantizer for $\widetilde{X}^\ell_{k+1}$ and if $\widehat{X}^\ell_{k+1}$ denotes the quantization of $\widetilde{X}^\ell_{k+1}$ by the grid Γ^ℓ_{k+1}, then Γ^ℓ_{k+1} is a stationary quantizer for $\widetilde{X}^\ell_{k+1}$, means, $\mathbb{E}\Big(\widetilde{X}^\ell_{k+1}\big|\widehat{X}^\ell_{k+1}\Big) = \widehat{X}^\ell_{k+1}$. Equivalently, this means that if $\Gamma^\ell_{k+1} = \{x^{\ell,j_\ell}_{k+1},\ j_\ell = 1,\dots,N^\ell_{k+1}\}$ with

$$x^{\ell,j_\ell}_{k+1} = \frac{\sum_{i\in\mathscr{I}_k}\mathbb{E}\big(\mathcal{E}^\ell_k(x^i_k,Z_{k+1})\mathbb{1}_{\{\mathcal{E}^\ell_k(x^i_k,Z_{k+1})\in C_{j_\ell}(\Gamma^\ell_{k+1})\}}\big)\mathbb{P}(\widehat{X}_k = x^i_k)}{p^{j_\ell}_{k+1}} \tag{43}$$

$$\text{and}\quad p^{j_\ell}_{k+1} = \sum_{i\in\mathscr{I}_k}\mathbb{P}\big(\mathcal{E}^\ell_k(x^i_k,Z_{k+1})\in C_{j_\ell}(\Gamma^\ell_{k+1})\big)\mathbb{P}(\widehat{X}_k = x^i_k),\ j_\ell = 1,\dots,N^\ell_{k+1}. \tag{44}$$

3.4 The error analysis

Our aim is now to compute the quadratic quantization error bound $\|\bar{X}_T - \widehat{X}_T\|_2 := \|\bar{X}_n - \widehat{X}^{\Gamma_n}_n\|_2$. The analysis of this error bound is the subject of the following theorem, which is the main theoretical result of the paper. We suppose that $x_0 = X_0 = \widetilde{X}_0$.

Theorem 3.6. *Assume the coefficients b, σ satisfy the classical Lipschitz assumptions* (28), (29) *and* (30)*. Let, for every $k = 0,\dots,n$, Γ_k be a Markovian product quantizer for $\widetilde{X}_k$ at level N_k. Then, for every $k = 0,\cdots,n$, for any $\eta\in]0,1]$,*

$$\|\bar{X}_k - \widehat{X}^{\Gamma_k}_k\|_2 \le K_{2,eta}\sum_{k'=1}^{k} e^{(k-k')\frac{T}{n}C_{b,\sigma}}a_{k'}\big(b,\sigma,t_k,T/n,x_0,L,2+\eta\big)\Big(\sum_{\ell=1}^{d}(N^\ell_{k'})^{-2/d}\Big)^{1/2} \tag{45}$$

where for every $p\in(2,3]$,

$$a_{k'}(b,\sigma,t_k,T/n,x_0,L,\theta) := e^{C_{b,\sigma}\frac{(t_k-t_{k'})}{p}}\Big[e^{(\kappa_p+K_p)t_{k'}}|x_0|^p + \frac{e^{\kappa_p\Delta}L+K_p}{\kappa_p+K_p}\big(e^{(\kappa_p+K_p)t_{k'}}-1\big)\Big]^{\frac{1}{p}},$$

with $C_{b,\sigma} = [b]_{\rm Lip} + \frac{1}{2}[\sigma]^2_{\rm Lip}$, $K_{2,\eta} := K_{2,1,\eta}$ is a universal constant defined in Equation (20)*;*

$$\kappa_\theta := \Big(\frac{(\theta+1)(\theta-2)}{2} + 2\theta L\Big)\ \text{and}\ K_\theta := 2^{\theta-1}L^\theta\Big(1+\theta+\Delta^{\frac{\theta}{2}-1}\Big)\mathbb{E}|Z|^\theta,\ Z\sim\mathcal{N}(0;I_d).$$

Before dealing with the proof let us notice that if we take the same grid size $N^\ell_k = N_k$, for every $\ell\in\{1,\dots,d\}$, the error bound (45) becomes

$$\|\bar{X}_k - \widehat{X}^{\Gamma_k}_k\|_2 \le K_{2,\eta}\sqrt{d}\sum_{k'=1}^{k}a_{k'}(b,\sigma,t_k,T/n,x_0,L,2+\eta)N_{k'}^{-1/d}. \tag{46}$$

Proof. (of Theorem 3.6). Recall that for every $k\ge 0$, $\widehat{X}_k = (\widehat{X}^1_k,\dots,\widehat{X}^d_k)$, where $\widehat{X}^\ell_k$ is the quantization of the ℓ-th component $\bar{X}^\ell_k$ of the vector $\bar{X}_k$. Therefore, following step by step the proof of Lemma 3.2. in [8], we obtain for every $k\ge 1$:

$$\|\bar{X}_k - \widetilde{X}_k\|_2 \le \sum_{k=1}^{k} e^{(t_k-t_{k'})C_{b,\sigma}}\|\widetilde{X}_{k'} - \widehat{X}^{\Gamma_{k'}}_{k'}\|_2,$$

where $C_{b,\sigma} = [b]_{\rm Lip} + \frac{1}{2}[\sigma]^2_{\rm Lip}$. Using the definition of $\widehat{X}_k$ combined with Pierce's Lemma (of Theorem 2.1 (b)) yields for every $k = 1,\dots,n$, for any $\eta\in(0,1]$,

$$\begin{aligned}\|\bar{X}_k - \widehat{X}_k\|_2 &\le e^{(t_k-t_{k'})C_{b,\sigma}}\Big(\sum_{\ell=1}^{d}\|\widetilde{X}^\ell_p - \widehat{X}^\ell_p\|_2^2\Big)^{1/2}\\ &\le K_{2,\eta}\sum_{k'=1}^{k}e^{(t_k-t_{k'})C_{b,\sigma}}\Big(\sum_{\ell=1}^{d}\|\widetilde{X}^\ell_{k'}\|^2_{2+\eta}(N^\ell_{k'})^{-2/d}\Big)^{1/2}.\end{aligned}$$

Following the lines of the proof of Lemma 3.2. in [8], we easily show that for every $\ell \in \{1,\dots,d\}$, $\|\widetilde{X}^{\ell}_{k'}\|_{2+\eta} \le a_{k'}(b,\sigma,t_k,T/n,x_0,L,2+\eta)$. Then, we deduce that

$$\|\bar{X}_k - \widehat{X}_k\|_2 \le K_{2,\eta} \sum_{k'=1}^{k} e^{(t_k - t_{k'})C_{b,\sigma}} a_{k'}(b,\sigma,t_k,T/n,x_0,L,2+\eta) \Big(\sum_{\ell=1}^{d} (N_p^{\ell})^{-2/d} \Big)^{1/2}.$$

This completes the proof. □

4 Numerical examples

In this section, we illustrate the performances of our method through two examples. The first example deals with the approximation of the price of a European Basket call option and the second one consists in the pricing of a European call option in a Heston model. Computations are performed using Matlab on a CPU 2:4 GHz and 8 Gb memory computer.

4.1 Pricing of a Basket call option

We consider a Basket call option with maturity T and with strike K, based on two stocks which prices X^1 and X^2 and associated weights w_1 and w_2. We suppose that X^1 and X^2 evolve following the dynamics

$$\begin{cases} dX_t^1 = rX_t^1 dt + \sigma_1 X_t^1 dW_t^1 \\ dX_t^2 = rX_t^2 + \rho\,\sigma_2 X_t^2 dW_t^1 + \sqrt{1-\rho^2}\,\sigma_2 X_t^2 dW_t^2, \quad t \in [0,T], \end{cases} \tag{47}$$

where W^1 and W^2 are two independent Brownian motions, r is the interest rate and $\rho \in (-1,1)$ is the correlation term. We know that in this case, the non arbitrage price at time $t=0$ in a complete market of the call option is given by

$$e^{-rT}\mathbb{E}F(X_T), \quad X = (X^1, X^2), \tag{48}$$

where the function F is defined, for every $x = (x^1,x^2) \in \mathbb{R}^2$, by $F(x) = (w_1 x^1 + w_2 x^2 - K)_+$. We choose $w_1 = w_2 = 0.5$. Using the Markovian product quantization, the price of the call option in the Basket option is approximated by the cubature formula

$$e^{-rT} \sum_{j \in \mathscr{L}_n} F(x_n^j)\mathbb{P}(\widehat{X}_n = x_n^j). \tag{49}$$

We will consider two cases: the case where $\rho = 0$ and the case where $\rho \neq 0$. When $\rho = 0$, the probabilities $\mathbb{P}(\widehat{X}_n = x_n^j)$ in (49) are computed using the simpler Equation (36), otherwise, we rely on Equation (33). For the numerical experiments, we use the following parameters:

$$r = 0.02, \quad \sigma_1 = 0.4, \quad \sigma_2 = 0.25, \quad X_0^1 = 100, \quad X_0^2 = 100, \quad T = 1,$$

and make the strike K varying. In the correlated case, we choose $\rho = 0.95$. We put the length of the time discretization mesh to $n = 10$ and take $N_1 = N_2 = 50$. We consider that the reference prices are those obtained from Ju's approximation method (see [4]). The numerical results are depicted in Table 1 (for $\rho = 0$) and in Table 2 (for $\rho = 0.95$). When $\rho = 0$, the results are obtained instantaneously (less than 1 second) with associated absolute errors of order between 10^{-1} and 10^{-2}. The absolute error is of the same order when $\rho = 0.95$. However, the computation time increases to 6.8 seconds.

Strike	reference	Quantization	Absolute error
$K = 90$	15.8985	15.9216	2×10^{-2}
$K = 95$	12.9066	12.9120	6×10^{-3}
$K = 100$	10.3510	10.3315	1×10^{-2}
$K = 105$	8.2122	8.1654	4×10^{-2}
$K = 110$	6.4546	6.3797	7×10^{-2}

Table 1: Basket call option prices. Comparison of the reference prices (those obtained via Ju's approximation method) and the Markovian and product quantization method for $\rho = 0$. We chose $N_k^1 = N_k^2 = 50$, $\forall k = 1, \ldots, n, n = 10$. Computational time for the quantization algorithm: less than 1 second.

Strike	reference	Quantization	Absolute error
$K = 90$	18.7361	18.8135	8×10^{-2}
$K = 95$	16.0167	16.0981	8×10^{-2}
$K = 100$	13.6190	13.6767	5×10^{-2}
$K = 105$	11.5249	11.5533	2×10^{-2}
$K = 110$	9.7114	9.7123	1×10^{-3}

Table 2: Basket call option prices. Comparison of the reference prices (those obtained via Ju's approximation method) and the Markovian and product quantization method for $\rho = 0.95$. We chose $N_k^1 = N_k^2 = 50$, $\forall k = 1, \ldots, n, n = 10$. Computational time for the quantization algorithm: 6.8 seconds.

4.2 Pricing of a European call option in a Heston model

In this example, we consider a call option with maturity T and strike K, in a Heston model where the stock price S and its stochastic volatility V evolve following the dynamics

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^1 \\ dV_t = \kappa(\theta - V_t)dt + \rho\,\sigma\sqrt{V_t} dW_t^1 + \sqrt{1-\rho^2}\,\sigma\sqrt{V_t} dW_t^2, \quad t \in [0, T]. \end{cases} \tag{50}$$

In the previous equation, the parameter r still is the interest rate; $\kappa > 0$ is the rate at which V reverts to the long run average variance $\theta > 0$; $\sigma > 0$ is the volatility of the variance and $\rho \in [-1, 1]$ denotes the correlation term. In this case, the price of the call at time $t = 0$ reads

$$e^{-rT}\mathbb{E}H(X_T), \quad X = (S, V), \tag{51}$$

where $H(x) = (x^1 - K)_+$, for $x = (x^1, x^2) \in \mathbb{R}^2$.

Using the Markovian and product quantization method, the price of the call in the Heston model is approximated as

$$e^{-rT} \sum_{j_1=1}^{N_n^1} H(x_n^{1j_1})\,\mathbb{P}(\widehat{X}_n^1 = x_n^{1j_1}), \tag{52}$$

where $\mathbb{P}(\widehat{X}_n^1 = x_n^{1j_1})$ is computed from formula (38).

For the numerical test in the Heston model we will use the following parameters:

$$r = 0.01 \quad \kappa = 4 \quad \theta = 0.09 \quad \sigma = 0.2 \quad \rho = -0.5 \quad S_0 = 100 \quad V_0 = 0.0719 \quad T = 1.$$

The number of time discretization steps n is set at 10. Since the pricing involves only the stock process S and not the volatility V, we will assign more grid points to the quantization of the stock

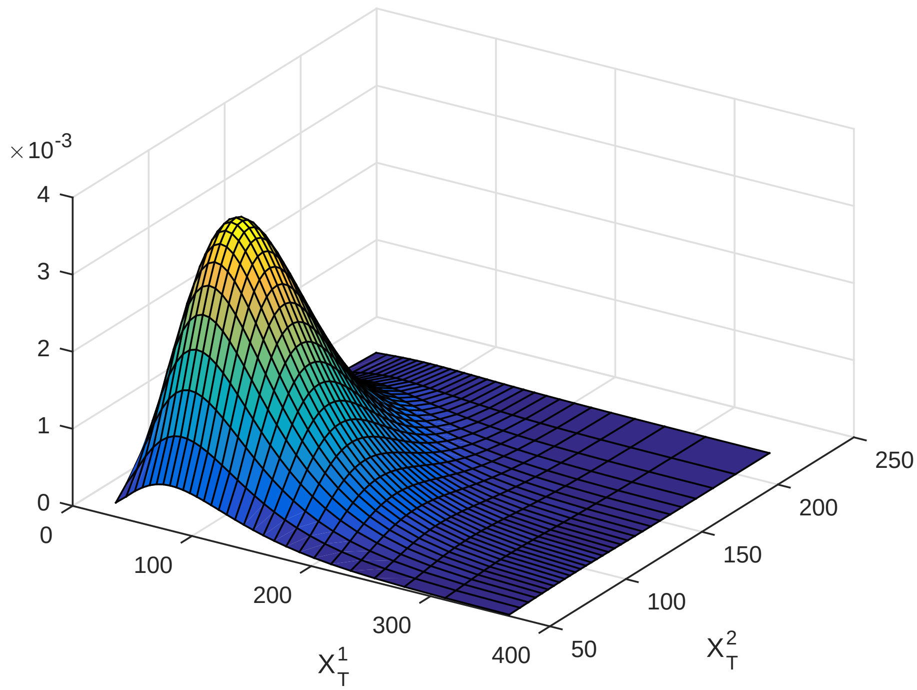

Figure 1: Distribution of the weights in the uncorrelated case at time $t = T$, $N^1 = N^2 = 30$.

process S than to the volatility V. More precisely, we take two grid points for the volatility process: $N_k^2 = 2, k = 1, \dots, n$ and take $N_k^1 = 100$ or $N_k^1 = 400$, $k = 1, \dots, n$, for the sizes of the stock process quantization. The results are depicted in Table 3 and Table 4, where we compare the price obtained by Markovian product quantization with those obtained using closed-form formulas in [3], considered as the reference prices.

When $N_k^1 = 100$, we obtain absolute errors (w.r.t. the reference prices) of magnitude between 10^{-1} and 10^{-2}, with computation times around 2 seconds.

When $N_k^1 = 400$, the computation times are around 19 seconds whereas the absolute errors are of order 10^{-2}.

More generally, we can improve all the accuracy by increasing the size of the grids.

Strike	reference	Quantization	Absolute error
$K = 100$	17.3335	17.2658	6×10^{-2}
$K = 102.5$	15.8694	15.8353	3×10^{-2}
$K = 105$	14.4935	14.4950	1×10^{-3}
$K = 107.5$	13.2053	13.2436	3×10^{-2}
$K = 110$	12.0034	12.0788	7×10^{-2}

Table 3: Call price in a Heston model via closed formulas given in [3] (the reference prices) and the Markovian product quantization method. $N_k^1 = 100$, $\forall k = 1, \dots, n, n = 10$. Computational time for the quantization algorithm: 2.02 seconds.

Strike	reference	Quantization	Absolute error
$K = 100$	17.3335	17.3450	1×10^{-2}
$K = 102.5$	15.8694	15.8721	1×10^{-2}
$K = 105$	14.4935	14.4950	1×10^{-3}
$K = 107.5$	13.2053	13.2131	8×10^{-3}
$K = 110$	12.0034	12.0018	2×10^{-3}

Table 4: Call price in a Heston model via closed formulas given in [3] (the reference prices) and the Markovian product quantization method. $N_k^1 = 400$, $\forall k = 1, \ldots, n, n = 10$. Computational time for the quantization algorithm: 19.13 seconds.

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