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► **To cite this version:**

Fiorin Lucio, Gilles Pagès, Abass Sagna. Markovian and product quantization of an R^d -valued Euler scheme of a diffusion process with applications to finance. *Methodology and Computing in Applied Probability*, 2019, 21, pp.1087-1118. hal-01222936v3

HAL Id: hal-01222936

<https://hal.science/hal-01222936v3>

Submitted on 24 Mar 2017

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Product Markovian quantization of an \mathbb{R}^d -valued Euler scheme of a diffusion process with applications to finance

LUCIO FIORIN ^{*} GILLES PAGÈS [†] ABASS SAGNA [‡] §

Abstract

We introduce a new approach to quantize the Euler scheme of an \mathbb{R}^d -valued diffusion process. This method is based on a Markovian and componentwise product quantization and allows us, from a numerical point of view, to speak of *fast online quantization* in dimension greater than one since the product quantization of the Euler scheme of the diffusion process and its companion weights and transition probabilities may be computed quite instantaneously. We show that the resulting quantization process is a Markov chain, then, we compute the associated companion weights and transition probabilities from (semi-) closed formulas. From the analytical point of view, we show that the induced quantization errors at the k -th discretization step t_k is a cumulative of the marginal quantization error up to time t_k . Numerical experiments are performed for the pricing of a Basket call option, for the pricing of a European call option in a Heston model and for the approximation of the solution of backward stochastic differential equations to show the performances of the method.

1 Introduction

In [19] is proposed and analyzed a Markovian (fast) quantization of an \mathbb{R}^d -valued Euler scheme of a diffusion process. However, in practice, this approach allows to speak of fast quantization only in dimension one since, as soon as $d \geq 2$, one has to use recursive stochastic zero search algorithm (known to be very time consuming, compared to deterministic procedures like the Newton-Raphson algorithm, see [18]) to compute optimal quantizers, their associated weights and transition probabilities. In order to overcome this limitation, we propose in this work another approach to quantize an \mathbb{R}^d -valued Euler scheme of a diffusion process. This method is based on a Markovian and componentwise product quantization. It allows again to speak of fast quantization in high dimension since the product quantization of the Euler scheme of the diffusion process and its transition probabilities can be computed almost instantaneously still using deterministic recursive zero search algorithms.

In a general setting, the stochastic process $(X_t)_{t \in [0, T]}$ of interest is defined as the (strong) solution to the following stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \quad (1)$$

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[§]The first author benefited from the support of the Fondazione Aldo Gini of the University of Padova. The second author benefited from the support of the Chaire “Risques financiers”, a joint initiative of École Polytechnique, ENPC-ParisTech and UPMC, under the aegis of the Fondation du Risque. The third author benefited from the support of the Chaire “Markets in Transition”, under the aegis of Louis Bachelier Laboratory, a joint initiative of École polytechnique, Université d'Evry Val d'Essonne and Fédération Bancaire Française.

where W is a standard q -dimensional Brownian motion, independent from the \mathbb{R}^d -valued random vector X_0 , both defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and the volatility coefficient $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times q}$ are Borel measurable functions satisfying appropriate Lipschitz continuity and linear growth conditions (specified further on) which ensure the existence of a unique strong solution of the stochastic differential equation. In corporate finance, these processes are used to model the dynamics of assets for several quantities of interest involving the pricing and the hedging of derivatives. These quantities are usually of the form

$$\mathbb{E}[f(X_T)], \quad T > 0, \quad (2)$$

or

$$\mathbb{E}[f(X_t)|X_s = x], \quad 0 < s < t, \quad (3)$$

for a given Borel function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. For illustrative purposes, let us consider the following three examples which may be reduced to the computation of regular expectations like (2) or (3). First, let us consider the price of a Basket call option with maturity T and strike K , based on two stocks whose prices S^1 and S^2 evolve following the dynamics

$$\begin{cases} dS_t^1 = rS_t^1 + \rho\sigma_1S_t^1dW_t^1 + \sqrt{1-\rho^2}\sigma_1S_t^1dW_t^2 \\ dS_t^2 = rS_t^2dt + \sigma_2S_t^2dW_t^1 \end{cases} \quad (4)$$

where r is the interest rate, $\sigma_1, \sigma_2 > 0$, $\rho \in (-1, 1)$ is a correlation term and W^1 and W^2 are two independent Brownian motions. We know that the no arbitrage price at time $t = 0$ in a complete market reads

$$e^{-rT}\mathbb{E}[(w_1S_T^1 + w_2S_T^2 - K)_+] = e^{-rT}\mathbb{E}F(X_T), \quad X = (S^1, S^2), \quad (5)$$

where the weights w_1 and w_2 are usually assumed to be positive with a sum equal to one and where the function F is defined, for every $x = (s^1, s^2) \in \mathbb{R}^2$, by $F(x) = (w_1s^1 + w_2s^2 - K)_+$. Keep in mind that $x_+ = \max(x, 0)$, $x \in \mathbb{R}$.

The second example concerns the pricing of a call option with maturity T and strike K , in the Heston model of [13] where the stock price S and its stochastic variance V evolve following the (correlated) dynamics

$$\begin{cases} dS_t = rS_tdt + \rho\sqrt{V_t}S_tdW_t^1 + \sqrt{1-\rho^2}\sqrt{V_t}S_tdW_t^2 \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^1, \quad t \in [0, T]. \end{cases} \quad (6)$$

In the previous equation, the parameter r is still the interest rate; $\kappa > 0$ is the rate at which V reverts to the long run average variance $\theta > 0$; the parameter $\sigma > 0$ is the volatility of the variance and $\rho \in [-1, 1]$ is the correlation term. In this case, the no arbitrage price at time $t = 0$ in a complete market reads under this risk neutral probability

$$e^{-rT}\mathbb{E}[(S_T - K)_+] = e^{-rT}\mathbb{E}H(X_T), \quad X = (S, V), \quad (7)$$

where $H(x) = (x^1 - K)_+$, for $x = (x^1, x^2) \in \mathbb{R}^2$.

Note that both price expressions (5) and (7) are of type (2).

The last example concerns the approximation of the following Backward Stochastic Differential Equation (BSDE),

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T], \quad (8)$$

where terminal condition of the form $\xi = h(X_T)$ for a given Borel function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ and where X is a strong solution to (1). The process $(Z_t)_{t \in [0, T]}$ is a square integrable progressively measurable

process taking values in \mathbb{R}^q and $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^q \rightarrow \mathbb{R}$ is a Borel function. We will see further on that the approximation of the solution of the BSDE (8) involves the computation of expressions of the type (3).

In the general setting (in particular, in both previous examples (4)-(7)) the stochastic differential equation (1) has no explicit solution. Therefore, both quantities (2) and (3) have to be approximated, for example, by

$$\mathbb{E}[f(\bar{X}_T)] \quad (9)$$

and

$$\mathbb{E}[f(\bar{X}_{t_{k+1}}) | \bar{X}_{t_k} = x] \quad \text{where } t = t_{k+1}, s = t_k, \quad (10)$$

and where $(\bar{X}_{t_k})_{k=0, \dots, n}$ is a discretization scheme of the process $(X_t)_{t \geq 0}$ on $[0, T]$, for a given discretization mesh $t_k = k\Delta$, $k = 0, \dots, n$, $\Delta = T/n$. The Euler scheme $(\bar{X}_{t_k})_{k=0, \dots, n}$ associated to $(X_t)_{t \in [0, T]}$ is recursively defined by

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(t_k, \bar{X}_{t_k})\Delta + \sigma(t_k, \bar{X}_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad \bar{X}_0 = X_0.$$

In the sequel, when no confusion may occur, we will identify the value Y_{t_k} at time t_k of any process $(Y_{t_k})_{0 \leq k \leq n}$ by Y_k , $k = 0, \dots, n$.

At this stage, the quantities (9) and (10) still have no closed formulas in the general setting so is the case when for example dealing with a general local volatility model or a stochastic volatility model like the Heston model. Consequently, we have to make a spacial approximation of the expectation or the conditional expectation. This may be done by Monte Carlo simulation techniques or by optimal quantization method in particular, by the Markovian (fast) quantization method.

The fast Markovian quantization of the Euler scheme of an \mathbb{R}^d -valued diffusion process has been introduced in [19]. It consists of a sequence of quantizations $(\hat{X}_k^{\Gamma_k})_{k=0, \dots, N}$ of the Euler scheme $(\bar{X}_k)_{k=0, \dots, N}$ defined recursively as follows:

$$\begin{aligned} \hat{X}_0 &= \bar{X}_0, \\ \hat{X}_k^{\Gamma_k} &= \text{Proj}_{\Gamma_k}(\bar{X}_k) \quad \text{and} \quad \bar{X}_{k+1} = \mathcal{E}_k(\hat{X}_k^{\Gamma_k}, Z_{k+1}), \quad k = 0, \dots, n-1, \end{aligned}$$

where $(Z_k)_{k=1, \dots, n}$ is an i.i.d. sequence of $\mathcal{N}(0; I_q)$ -distributed random vectors, independent of \bar{X}_0 and

$$\mathcal{E}_k(x, z) = x + \Delta b(t_k, x) + \sqrt{\Delta} \sigma(t_k, x) z, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^q, \quad k = 0, \dots, n-1.$$

The sequence of quantizers satisfies for every $k \in \{0, \dots, n\}$,

$$\Gamma_k \in \text{argmin}\{\tilde{D}_k(\Gamma), \Gamma \subset \mathbb{R}^d, \text{card}(\Gamma) \leq N_k\},$$

where for every grid $\Gamma \subset \mathbb{R}^d$, $\tilde{D}_{k+1}(\Gamma) := \mathbb{E}[\text{dist}(\bar{X}_{t_{k+1}}, \Gamma)^2]$. However, this quantization method is fast from the numerical point of view only in one dimension.

The aim of this work is to present another approach to quantize the Euler scheme of an \mathbb{R}^d -valued diffusion process in order to speak of fast quantization in dimension greater than one. We propose a Markovian and product quantization method. It allows us to compute instantaneously the optimal product quantizers and their transition probabilities (and its companion weights) when the size of the quantizations are chosen reasonably.

The method is based on a Markovian and componentwise product quantization of the process $(\bar{X}_k)_{0 \leq k \leq n}$. To be more precise, let us denote by Γ_k^ℓ an N_k^ℓ -quantizer of the ℓ -th component \bar{X}_k^ℓ of the vector \bar{X}_k and let \hat{X}_k^i be the quantization of \bar{X}_k^i of size N_k^i , on the grid Γ_k^i . Let us define the product quantizer $\Gamma_k = \bigotimes_{i=1}^d \Gamma_k^i$ of size $N_k = N_k^1 \times \dots \times N_k^d$ of the vector \bar{X}_k as

$$\Gamma_k = \{(x_k^{1, i_1}, \dots, x_k^{d, i_d}), \quad i_\ell \in \{1, \dots, N_k^\ell\}, \ell \in \{1, \dots, d\}\}.$$

Then, assuming that \bar{X}_0 is already quantized as \hat{X}_0 , we define the product quantization $(\hat{X}_{t_k})_{0 \leq k \leq n}$ of the process $(\bar{X}_{t_k})_{0 \leq k \leq n}$ from the following recursion:

$$\begin{cases} \bar{X}_0 = \hat{X}_0, & \hat{X}_k^\ell = \text{Proj}_{\Gamma_k^\ell}(\bar{X}_k^i), \quad i = 1, \dots, d \\ \hat{X}_k = (\hat{X}_k^1, \dots, \hat{X}_k^d) & \text{and } \hat{X}_{k+1}^\ell = \mathcal{E}_k^\ell(\hat{X}_k, Z_{k+1}), \quad i = 1, \dots, d \\ \mathcal{E}_k^\ell(x, z) = x^\ell + \Delta b^\ell(t_k, x) + \sqrt{\Delta}(\sigma^{\ell \bullet}(t_k, x)|z), & z = (z^1, \dots, z^q) \in \mathbb{R}^q \\ x = (x^1, \dots, x^d), \quad b = (b^1, \dots, b^d) & \text{and } (\sigma^{\ell \bullet}(t_k, x)|z) = \sum_{m=1}^q \sigma^{\ell m}(t_k, x) z^m \end{cases} \quad (11)$$

where for $a \in \mathcal{M}(d, q)$, $a^{\ell \bullet} = [a_{\ell j}]_{j=1, \dots, q}$.

It is easy to see that the sequence of quantizers $(\hat{X}_k)_{k \geq 0}$ is a Markov chain (see Remark 3.1). Then, the challenging question is to know how to compute its (set) values, *i.e.* the product quantizer $\Gamma_k = \bigotimes_{i=1}^d \Gamma_k^\ell$, $k = 0, \dots, n$, and the associated transition probabilities. Using the fact that the conditional distribution of the Euler scheme is a multivariate Gaussian distribution and that each component of a Gaussian vector remains a scalar Gaussian random variable, we propose a way to quantize every component \bar{X}_k^ℓ of the vector \bar{X}_k , for $k = 0, \dots, n$. We then define the product quantization $(\hat{X}_k)_{0 \leq k \leq n}$ of $(\bar{X}_k)_{0 \leq k \leq n}$ from the recursive procedure (11). Then, we show how to compute, for every $k \geq 1$, the companion transition probabilities (and the companion weights) associated to each component of the vector \bar{X}_k , for every $k \geq 1$ and to the vector \hat{X}_k itself.

When the components of the vector \bar{X}_k are independent for every $k = 0, \dots, n$, the method boils down to the usual product quantization of the vector \bar{X}_k , where each component is quantized from the Markovian recursive quantization method (see [19]). In this case, the transition probability weight associated to the vector \hat{X}_k is the product of the transition probability weights associated to its components.

The main difficulties arise when the components of \bar{X}_k are not independent. In this work, we propose a closed formula even in this case by relying on a domain decomposition technique.

To be more precise, set, for every $k \in \{0, \dots, n\}$,

$$I_k = \{(i_1, \dots, i_d), \quad i_\ell \in \{1, \dots, N_k^\ell\}\} \quad (12)$$

and for $i := (i_1, \dots, i_d) \in I_k$, set

$$x_k^i := (x_k^{1, i_1}, \dots, x_k^{d, i_d}). \quad (13)$$

We will show in Proposition 3.1 that the transition probabilities of the Markov chain $(\hat{X}_k)_{k \geq 0}$ reads, for any multi-indices $i \in I_k$ and $j \in I_{k+1}$,

$$\mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i) = \mathbb{E} \left[\prod_{\ell \in \mathbb{J}_k^0(x_k^i)} \mathbf{1}_{\{\zeta \in \mathbb{J}_{k, j_\ell}^0(x_k^i)\}} \max(\Phi_0(\beta_j(x_k^i, \zeta)) - \Phi_0(\alpha_j(x_k^i, \zeta)), 0) \right] \quad (14)$$

where $\zeta \sim \mathcal{N}(0; I_{q-1})$, with the convention that $\prod_{\ell \in \emptyset} (\cdot) = 1$. The function Φ_0 stands for the cumulative distribution function of the $\mathcal{N}(0, 1)$ and the functions α_j and β_j are defined in Proposition 3.1. Although this formula looks quite involved, it turns out to be very simple to use in practice. In fact, keeping in mind that the optimal quantization grids associated to multivariate Gaussian random vectors (up to dimension $d = 10$) can be downloaded on the website www.quantize.maths-fi.com, it is clear that (14) can be computed instantaneously using these optimal grids of multivariate normal vectors. Furthermore, Equation (14) allows us to deduce the weights associated to the product quantization \hat{X}_{k+1} , $k = 0, \dots, n-1$, since for every $j \in I_{k+1}$,

$$\mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j) = \sum_{i \in I_k} \mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i) \mathbb{P}(\hat{X}_k = x_k^i). \quad (15)$$

Formulas (14)-(15) are useful when, for example, we deal with the approximation of the solution of BSDEs or when we deal with the pricing of a Basket call like Equation (5). In this last situation, given a time discretization mesh $t_0 = 0, \dots, t_n = T$, the price of the Basket call option will be approximated by the cubature formula

$$e^{-rT} \sum_{j \in I_n} F(x_n^j) \mathbb{P}(\hat{X}_n = x_n^j), \quad (16)$$

where the probability weights $\mathbb{P}(\hat{X}_n = x_n^j)$ are computed recursively in a forward way using equations (14) and (15).

When the correlation coefficient $\rho = 0$ in (4), the probabilities $\mathbb{P}(\hat{X}_n = x_n^j)$ in (16) will be computed in a simplified since the components of the vectors \bar{X}_k are independent. We show in Proposition 3.2 that the formula (14) reads in the simplified form

$$\begin{aligned} \mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i) &= \prod_{\ell=1}^d \mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{j_\ell} | \hat{X}_k = x_k^i) \\ &= \prod_{\ell=1}^d (\Phi_0(x_{k+1}^{\ell, j_\ell+}(x_k^i, 0)) - \Phi_0(x_{k+1}^{\ell, j_\ell-}(x_k^i, 0))), \end{aligned}$$

for $i \in I_k, j \in I_{k+1}$.

For the approximation of the solution of the BSDE (8), many numerical schemes and approximating methods have been proposed (see e.g. [1, 3, 7, 11, 14, 10, 2]). In this paper, we just aim to test the numerical performances of our method using the (quantization) numerical scheme proposed in [20]. Setting $\hat{Y}_k = \hat{y}_k(\hat{X}_k)$ (where (\hat{X}_k) is the quantization of the Euler scheme (\bar{X}_{t_k})), for every $k \in \{0, \dots, n\}$, this quantized BSDE scheme reads (see [20]) as

$$\begin{cases} \hat{y}_n(x_n^i) = h(x_n^i) & x_n^i \in \Gamma_n \\ \hat{y}_k(x_k^i) = \hat{\alpha}_k(x_k^i) + \Delta_n f(t_k, x_k^i, \hat{\alpha}_k(x_k^i), \hat{\beta}_k(x_k^i)) & x_k^i \in \Gamma_k \end{cases}$$

where for $k = 0, \dots, n-1$, for $i \in I_k$,

$$\hat{\alpha}_k(x_k^i) = \sum_{j \in I_{k+1}} \hat{y}_{k+1}(x_{k+1}^j) p_k^{ij} \quad \text{and} \quad \hat{\beta}_k(x_k^i) = \frac{1}{\sqrt{\Delta_n}} \sum_{j \in I_{k+1}} \hat{y}_{k+1}(x_{k+1}^j) \Lambda_k^{ij}, \quad (17)$$

with

$$p_k^{ij} = \mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i) \quad \text{and} \quad \Lambda_k^{ij} = \mathbb{E}(Z_{k+1} |_{\{\hat{X}_{k+1} = x_{k+1}^j\}} | \hat{X}_k = x_k^i).$$

We will show further on how to compute the previous quantities from the Markovian product quantization method and using (semi)-closed formulas.

We also compute the (transition) distribution of each component of the product quantizations. Indeed, we show in Proposition 3.3 that for every $\ell \in \{1, \dots, d\}$ and for every $j_\ell \in \{1, \dots, N_{k+1}^\ell\}$, the transition probability $\mathbb{P}(\tilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell) | \hat{X}_k = x_k^i)$ is given by

$$\mathbb{P}(\tilde{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell} | \hat{X}_k = x_k^i) = \Phi_0\left(\frac{x_{k+1}^{\ell, j_\ell+1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta} |\sigma_k^{\ell \bullet}(x_k^i)|_2}\right) - \Phi_0\left(\frac{x_{k+1}^{\ell, j_\ell-1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta} |\sigma_k^{\ell \bullet}(x_k^i)|_2}\right), \quad (18)$$

where $m_k^\ell(x) = x + b(t_k, x)\Delta$ and $|\sigma_k^{\ell \bullet}(x)|_2$ is the Euclidean norm of the ℓ -th row of the volatility matrix $\sigma(t_k, x)$, for $x \in \mathbb{R}^d$. We deduce immediately the formulas for the probabilities $\mathbb{P}(\tilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell))$, $k = 0, \dots, n-1, j_\ell \in \{1, \dots, N_{k+1}^\ell\}$ using (18) (and (15)).

Equation (18) allows us to approximate the price of the call in the Heston model by

$$e^{-rT} \sum_{j_1=1}^{N_n^1} H(x_n^{1j_1}) \mathbb{P}(\hat{X}_n^1 = x_n^{1j_1}). \quad (19)$$

Another important issue from the analytical point of view is to compute the quantization error bound associated to the Markovian quantization process. Using some results from [19], we show (in particular, when $N_k^\ell = N_k$, for every $\ell = 1, \dots, d$) that for any sequence $(\hat{X}_k^{\Gamma_k})_{0 \leq k \leq n}$ of (quadratic) Markovian product quantization of $(\tilde{X}_k)_{0 \leq k \leq n}$, the quantization error $\|\bar{X}_k - \hat{X}_k^{\Gamma_k}\|_2$, at step k of the recursion, is bounded by the cumulative quantization errors $\|\tilde{X}_{k'} - \hat{X}_{k'}^{\Gamma_{k'}}\|_2$, for $k' = 0, \dots, k$. More precisely, one shows that for every $k = 0, \dots, n$, for any $\eta \in (0, 1]$,

$$\|\bar{X}_k - \hat{X}_k^{\Gamma_k}\|_2 \leq \sum_{k'=0}^k a_{k'}(b, \sigma, k, d, \Delta, x_0, \eta) N_{k'}^{-1/d},$$

where $a_{k'}(b, \sigma, k, d, \Delta, x_0, \eta)$ is a positive real constant depending on $b, \sigma, \Delta, x_0, \eta$ (see Theorem 3.4 further on for a more precise statement).

The paper is organized as follows: in Section 2 we recall some basic results on optimal quantization. Section 3 is the main part of this paper. We present the algorithm and show the Markov property of the product quantization of the Euler scheme of a diffusion process. Then, we show how to compute the weights and transition probabilities associated to the product quantizers and to its components. We also show how to compute the optimal quantizers associated to each component of the Euler scheme (keep in mind that this is the foundation of our method). Finally, we provide in Theorem 3.4 some a priori error bounds for the quantization error associated to the Markovian product quantization and show that, at every step discretization step t_k , this error is a cumulated (weighted) sum of the regular quantization errors, up to time t_k . In Section 4, we present some numerical results for the pricing of a European call Basket option and a European call option in the Heston model, as well for the approximation of BSDEs.

NOTATIONS. We denote by $\mathcal{M}(d, q, \mathbb{R})$, the set of $d \times q$ real value matrices. If $A = [a_{ij}] \in \mathcal{M}(d, q, \mathbb{R})$, A^* denotes its transpose and we define the norm $\|A\| := \sqrt{\text{Tr}(AA^*)} = (\sum_{i,j} a_{ij}^2)^{1/2}$, where $\text{Tr}(M)$ stands for the trace of M , for $M \in \mathcal{M}(d, d, \mathbb{R})$. For every $f : \mathbb{R}^d \rightarrow \mathcal{M}(d, q, \mathbb{R})$, we will set $\|f\|_{\text{Lip}} = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{|x - y|}$. For $x, y \in \mathbb{R}$, $x \vee y = \max(x, y)$. If $x \in \mathbb{R}^d$, $|x|_2$ will stand for the Euclidean norm on \mathbb{R}^d . For every vectors x, y , the notation $(x|y)$ denotes the dot product of the vectors x and y . For a given vector $z \in \mathbb{R}^q$ and a matrix $M \in \mathcal{M}(d, q, \mathbb{R})$, z^i denotes the component i of z , $z^{(j:k)}$ the vector made up from the component j to the component k of the vector z and $M^{(i,j:k)}$ is the vector made up from the component j to the component k of the i -th row of the matrix M and M^{ij} for the component (i, j) of the matrix M . The notation $M^{i\bullet}$ stands for the i -th row of M . The function Φ_0 will denote the cumulative distribution function of the standard real valued Normal distribution and its derivative Φ'_0 will stand for its density function.

2 Brief background on optimal quantization

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}^d$ be a random variable with distribution \mathbb{P}_X . The L^r -optimal quantization problem at level N for the random vector X (or for the distribution \mathbb{P}_X) consists in finding the best approximation of X by a Borel function $\pi(X)$ of X taking at most N values. Assuming that $X \in L^r(\mathbb{P})$, we associate to every Borel function $\pi(X)$ taking at

most N values, the L^r -mean error $\|X - \pi(X)\|_r$ measuring the distance between the two random vectors X and $\pi(X)$ w.r.t. the mean L^r -norm, where $\|X\|_r := (\mathbb{E}|X|^r)^{1/r}$ and $|\cdot|$ denotes an Euclidean norm on \mathbb{R}^d . Then finding the best approximation of X by a Borel function of X taking at most N values turns out to solve the following minimization problem:

$$e_{N,r}(X) = \inf \{ \|X - \pi(X)\|_r, \pi : \mathbb{R}^d \rightarrow \Gamma, \Gamma \subset \mathbb{R}^d, |\Gamma| \leq N \},$$

where $|A|$ stands for the cardinality of A , for $A \subset \mathbb{R}^d$. Now, let $\Gamma = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$ be a codebook of size N (also called an N -quantizer or a grid of size N) and define a Voronoi partition $C_i(\Gamma)_{i=1, \dots, N}$ of \mathbb{R}^d , which is a Borel partition of \mathbb{R}^d satisfying for every $i \in \{1, \dots, N\}$,

$$C_i(\Gamma) \subset \{x \in \mathbb{R}^d : |x - x_i| = \min_{j=1, \dots, N} |x - x_j|\}.$$

Consider the Voronoi quantization of X (or simply quantization of X) by the N -quantizer Γ defined by

$$\hat{X}^\Gamma = \sum_{i=1}^N x_i \mathbb{1}_{\{X \in C_i(\Gamma)\}}.$$

Then, for any Borel function $\pi : \mathbb{R}^d \rightarrow \Gamma = \{x_1, \dots, x_N\}$ we have

$$|X - \pi(X)| \geq \min_{i=1, \dots, N} d(X, x_i) = d(X, \Gamma) = |X - \hat{X}^\Gamma| \quad \mathbb{P} \text{ a.s.}$$

so that the optimal L^r -mean quantization error $e_{N,r}(X)$ reads

$$\begin{aligned} e_{N,r}(X) &= \inf \{ \|X - \hat{X}^\Gamma\|_r, \Gamma \subset \mathbb{R}^d, |\Gamma| \leq N \} \\ &= \inf_{\substack{\Gamma \subset \mathbb{R}^d \\ |\Gamma| \leq N}} \left(\int_{\mathbb{R}^d} d(z, \Gamma)^r d\mathbb{P}_X(z) \right)^{1/r}. \end{aligned} \quad (20)$$

Recall that for every $N \geq 1$, the infimum in (20) is attained at least one codebook. Any N -quantizer realizing this infimum is called an L^r -optimal N -quantizer. Moreover, when $|\text{supp}(\mathbb{P}_X)| \geq N$ then any L^r -mean optimal N -quantizer has exactly size N (see [12] or [17]). On the other hand, the quantization error, $e_{N,r}(X)$, decreases to zero as the grid size N goes to infinity and its rate of convergence is ruled by the so-called Zador Theorem recalled below. There also is a non-asymptotic upper bound for optimal quantizers. It is called Pierce Lemma (we recall it below for the quadratic case) and will allows us to put a finishing touches to the proof of the main result of the paper, stated in Theorem 3.4.

Theorem 2.1. (a) *Sharp asymptotic rate (Zador Theorem, see [12, 22]). Let X be an \mathbb{R}^d -valued random vector such that $\mathbb{E}|X|^{r+\eta} < +\infty$ for some $\eta > 0$ and let $\mathbb{P}_X = f \cdot \lambda_d + P_s$ be the Lebesgue decomposition of \mathbb{P}_X with respect to the Lebesgue measure λ_d and P_s denotes its singular part. Then*

$$\lim_{N \rightarrow +\infty} N^{\frac{1}{d}} e_{N,r}(P) = \tilde{Q}_r(\mathbb{P}_X) \quad (21)$$

where

$$\begin{aligned} \tilde{Q}_r(\mathbb{P}_X) &= \tilde{J}_{r,d} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d \right)^{\frac{1}{r} + \frac{1}{d}} = \tilde{J}_{r,d} \|f\|_{\frac{d}{d+r}}^{1/r} \in [0, +\infty) \\ \tilde{J}_{r,d} &= \inf_{N \geq 1} N^{\frac{1}{d}} e_{N,r}(U([0, 1]^d)) \in (0, +\infty) \end{aligned}$$

with $U([0, 1]^d)$ denotes the uniform distribution over the hypercube $[0, 1]^d$.

(b) *Non-asymptotic bound (Pierce Lemma, see [12, 16]).* Let $\eta > 0$. There exists a universal constant $K_{2,d,\eta}$ such that for every random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}^d$,

$$\inf_{|\Gamma| \leq N} \|X - \hat{X}^\Gamma\|_2 \leq K_{2,d,\eta} \sigma_{2,\eta}(X) N^{-\frac{1}{d}} \quad (22)$$

where

$$\sigma_{2,\eta}(X) = \inf_{\zeta \in \mathbb{R}^d} \|X - \zeta\|_{2+\eta} \leq +\infty.$$

From the Numerical Probability point of view, finding an optimal N -quantizer Γ may be a challenging task. In practice (we will only consider the quadratic case, i.e. $r = 2$ for numerical implementations) we are sometimes led to find some “good” quantizations \hat{X}^Γ which are close to X in distribution, so that for every continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we can approximate $\mathbb{E}f(X)$ by

$$\mathbb{E}f(\hat{X}^\Gamma) = \sum_{i=1}^N p_i f(x_i) \quad (23)$$

where $p_i = \mathbb{P}(\hat{X}^\Gamma = x_i)$. When we approximate $\mathbb{E}f(X)$ by (23), this induced an error which bound depends on the regularity of the function f (see e.g. [18] for more details).

We recall below the stationarity property for a quantizer.

Definition 2.1. A quantizer $\Gamma = \{x_1, \dots, x_N\}$ of size N inducing the Voronoi quantization \hat{X}^Γ of X is stationary if $\mathbb{P}(X \in \cup_i \partial C_i(\Gamma)) = 0$, $\mathbb{P}(X \in C_i(\Gamma)) > 0$, $i = 1, \dots, N$, and

$$\mathbb{E}(X | \hat{X}^\Gamma) = \hat{X}^\Gamma \text{ P-a.s.} \iff x_i = \frac{\mathbb{E}(X \mathbb{1}_{\{X \in C_i(\Gamma)\}})}{\mathbb{P}(X \in C_i(\Gamma))}, \quad i = 1, \dots, N. \quad (24)$$

The notion of stationarity is related to the critical point of the so-called *distortion* function defined on $(\mathbb{R}^d)^N$ by

$$D_{N,2}(x) = \mathbb{E}\left(\min_{1 \leq i \leq N} |X - x_i|^2\right) = \int_{\mathbb{R}^d} |\xi - x_i|^2 \mathbb{P}_X(d\xi), \quad x = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N. \quad (25)$$

As any grid of size at most N can be “represented” by some N -tuples (by repeating, if necessary, some of its elements), we will often put grids of all size N as an argument of the distortion function $D_{2,N}$ as well as for its gradient and Hessian matrix when its Voronoi boundary is negligible. It is also clear, from the definition of the quantization error, that

$$e_{N,2}^2(X) = \inf_{(x_1, \dots, x_N) \in (\mathbb{R}^d)^N} D_{N,2}(x_1, \dots, x_N).$$

Furthermore, the function $D_{N,2}$ is continuous and differentiable at any N -tuple having pairwise distinct components with a \mathbb{P} -negligible Voronoi partition boundary and the following result makes this more precise.

Proposition 2.2. (see [12, 17]) (a) *The function $D_{N,2}$ is differentiable at any N -tuple $(x_1, \dots, x_N) \in (\mathbb{R}^d)^N$ having pairwise distinct components and such that $\mathbb{P}(X \in \cup_i \partial C_i(\Gamma)) = 0$. Furthermore, we have*

$$\nabla D_{N,2}(x_1, \dots, x_N) = 2 \left(\int_{C_i(\Gamma)} (x_i - x) d\mathbb{P}_X(x) \right)_{i=1, \dots, N} \quad (26)$$

$$= 2 \left(\mathbb{P}(X \in C_i(\Gamma)) x_i - \mathbb{E}(X \mathbb{1}_{\{X \in C_i(\Gamma)\}}) \right)_{i=1, \dots, N}. \quad (27)$$

(b) A grid $\Gamma = \{x_1, \dots, x_N\}$ of full size N is stationary if and only if

$$\mathbb{P}(X \in \cup_i \partial C_i(\Gamma)) = 0 \quad \text{and} \quad \nabla D_{N,2}(\Gamma) = 0. \quad (28)$$

(c) If the support of \mathbb{P}_X has at least N elements, any L^2 -optimal quantizer at level N has full size and a \mathbb{P} -negligible Voronoi boundary. Hence it is a stationary N -quantizer.

For numerical implementations, the search of stationary quantizers is based on zero search recursive procedures like Newton-Raphson algorithm for real valued random variables, and some algorithms like Lloyd's I algorithms (see e.g. [9, 21]), the Competitive Learning Vector Quantization (CLVQ) algorithm (see [9]) or stochastic algorithms (see [18]) in the multidimensional framework. Optimal quantization grids associated to multivariate Gaussian random vectors can be downloaded on the website www.quantize.math-fi.com.

3 Markovian product quantization of an \mathbb{R}^d -valued Euler process

Let $(X_t)_{t \geq 0}$ be a stochastic process taking values in a d -dimensional Euclidean space \mathbb{R}^d and solution to the stochastic differential equation:

$$X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad x_0 \in \mathbb{R}^d, \quad (29)$$

where W is a standard q -dimensional Brownian motion starting at 0 and where $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and the matrix diffusion coefficient function $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{M}(d, q, \mathbb{R})$ are measurable and satisfy the global Lipschitz continuity and linear growth conditions: for every $t \in [0, T]$,

$$|b(t, x) - b(t, y)| \leq [b]_{\text{Lip}} |x - y| \quad (30)$$

$$\|\sigma(t, x) - \sigma(t, y)\| \leq [\sigma]_{\text{Lip}} |x - y| \quad (31)$$

$$|b(t, x)| \leq L(1 + |x|) \quad \text{and} \quad \|\sigma(t, x)\| \leq L(1 + |x|) \quad (32)$$

for some $L > 0$. This guarantees the existence and pathwise uniqueness of a strong solution of (29), adapted to the (augmented) filtration of W . We also suppose that the matrix σ is positive definite. Throughout the paper we will suppose that \mathbb{R}^d is equipped with the canonical Euclidean norm.

3.1 The algorithm and the Markov property of the quantized process

Recall that the Euler scheme of the stochastic process $(X_t)_{t \geq 0}$ is defined recursively from the following procedure:

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + \Delta b(t_k, \bar{X}_{t_k}) + \sigma(t_k, \bar{X}_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad \bar{X}_0 \in \mathbb{R}^d,$$

where $\Delta = \Delta_n = \frac{T}{n}$ and $t_k = \frac{kT}{n}$, for every $k \in \{0, \dots, n\}$. To simplify notations, we will often set $X_k := X_{t_k}$ to denote the process X evaluated at time t_k . We also set $b_k(x) := b(t_k, x)$ and $\sigma_k(x) = \sigma(t_k, x)$ for $x \in \mathbb{R}^d$. Recall also that the Euler operator associated to the conditional distribution of \bar{X}_{k+1} given $\bar{X}_k = x$ is defined by

$$\mathcal{E}_k(x, z) := x + \Delta b(t_k, x) + \sqrt{\Delta} \sigma(t_k, x) z, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^q$$

and that if Γ_{k+1} is an N_{k+1} -quantizer for \bar{X}_{k+1} , the distortion function \bar{D}_{k+1} associated to \bar{X}_{k+1} may be written for every $k = 0, \dots, n-1$, as

$$\begin{aligned} \bar{D}_{k+1}(\Gamma_{k+1}) &= \mathbb{E}((\text{dist}(\bar{X}_{k+1}, \Gamma_{k+1}))^2) \\ &= \mathbb{E}[\text{dist}(\mathcal{E}_k(\bar{X}_k, Z_{k+1}), \Gamma_{k+1})^2] \end{aligned}$$

where $Z_{k+1} \sim \mathcal{N}(0; I_q)$ is independent from \bar{X}_k . The previous way to write the distortion function has been used in [19] to propose a fast recursive (and Markovian) quantization of the Euler process (using the Newton-Raphson algorithm for the numerical computation of the optimal grids) when $d = 1$.

Keep in mind that the conditional distribution of the discrete Euler process \bar{X} is Gaussian and that one of the properties of a Gaussian vector is that any sub-component of the vector remains a Gaussian random vector. So, a natural alternative way to quantize the vector $\bar{X}_k \in \mathbb{R}^d$ is to quantize each component \bar{X}_k^ℓ by a grid Γ_k^ℓ of size N_k^ℓ , for $\ell = 1, \dots, d$, and then to define its product quantization \hat{X}_k associated with the product quantizer $\Gamma_k = \bigotimes_{\ell=1}^d \Gamma_k^\ell$ of size $N_k = N_k^1 \times \dots \times N_k^d$, as $\hat{X}_k = (\hat{X}_k^1, \dots, \hat{X}_k^d)$.

The question is now to know how to quantize the \bar{X}_k^i 's. On the other hand, since the components of the vector \bar{X}_k are not independent it is also a challenging question to know how to compute (from closed formula) the companion weights and transition probabilities associated with the quantizations of the \bar{X}_k^i 's and the vector \bar{X}_k . We describe below the componentwise recursive Markovian quantization of the process $\{\bar{X}_k, k = 0, \dots, n\}$.

It is clear that for every $\ell = 1, \dots, d$, and for every $k = 0, \dots, n-1$, the transition operator $\mathcal{E}_k^\ell(x, z)$ associated with the distribution of \bar{X}_{k+1}^ℓ given $\bar{X}_k = x$ reads as

$$\mathcal{E}_k^\ell(x, z) := m_k^\ell(x) + \sqrt{\Delta}(\sigma_k^{\ell \bullet}(x)|z), \quad x \in \mathbb{R}^d, z \in \mathbb{R}^q,$$

where

$$m_k^\ell(x) := x^\ell + \Delta b_k(x).$$

For every $k = 0, \dots, n$, for every given $\ell \in \{1, \dots, d\}$, we denote by \hat{X}_k^ℓ the quantization of \bar{X}_k^ℓ on the grid $\Gamma_k^\ell = \{x_k^{\ell, i_\ell}, i_\ell = 1, \dots, N_k^\ell\}$. We propose in what follows a recursive and componentwise product quantization of the process $\{\bar{X}_k, k = 0, \dots, n\}$. In fact, for every $\ell = 1, \dots, d$, we denote by Γ_k^ℓ an N_k^ℓ -quantizer (we suppose that we have access to it) of the ℓ -th component \bar{X}_k^ℓ of the vector \bar{X}_k and by \hat{X}_k^ℓ , the resulting quantization of \bar{X}_k^ℓ . Then, we define a componentwise recursive product quantizer $\Gamma_k = \bigotimes_{\ell=1}^d \Gamma_k^\ell$ of size $N_k = N_k^1 \times \dots \times N_k^d$ of the vector $\bar{X}_k = (\bar{X}_k^\ell)_{\ell=1, \dots, d}$ by

$$\Gamma_k = \{(x_k^{1, i_1}, \dots, x_k^{d, i_d}), \quad i_\ell \in \{1, \dots, N_k^\ell\}, \ell \in \{1, \dots, d\}\}.$$

To define the Markovian product quantization, suppose that \bar{X}_k has already been quantized and that we have access to the companion weights $\mathbb{P}(\hat{X}_k = x_k^i)$, $i \in I_k$, where I_k and x_k^i are defined by Equations (12) and (13). Setting $\tilde{X}_k^\ell = \mathcal{E}_k^\ell(\hat{X}_k, Z_{k+1})$, we may approximate the distortion function \bar{D}_{k+1}^ℓ associated to the ℓ -th component of the vector \bar{X}_{k+1} by

$$\begin{aligned} \bar{D}_{k+1}^\ell(\Gamma_{k+1}^\ell) &:= \mathbb{E}[\text{dist}(\tilde{X}_{k+1}^\ell, \Gamma_{k+1}^\ell)^2] \\ &= \mathbb{E}[\text{dist}(\mathcal{E}_k^\ell(\hat{X}_k, Z_{k+1}), \Gamma_{k+1}^\ell)^2] \\ &= \sum_{i \in I_k} \mathbb{E}[\text{dist}(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}), \Gamma_{k+1}^\ell)^2] \mathbb{P}(\hat{X}_k = x_k^i). \end{aligned}$$

This allows us to consider the sequence of product recursive quantizations of $(\hat{X}_k)_{k=0, \dots, n}$, defined for every $k = 0, \dots, n-1$, by the following recursion:

$$\begin{cases} \tilde{X}_0 = \hat{X}_0, & \hat{X}_k^\ell = \text{Proj}_{\Gamma_k^\ell}(\tilde{X}_k^\ell), \ell = 1, \dots, d, \\ \hat{X}_k = (\hat{X}_k^1, \dots, \hat{X}_k^d) & \text{and } \tilde{X}_{k+1}^\ell = \mathcal{E}_k^\ell(\hat{X}_k, Z_{k+1}), \ell = 1, \dots, d, \\ \mathcal{E}_k^\ell(x, z) = m_k^\ell(x) + \sqrt{\Delta}(\sigma_k^{\ell \bullet}(t_k, x)|z), & z = (z^1, \dots, z^q) \in \mathbb{R}^q, \\ x = (x^1, \dots, x^d), b = (b^1, \dots, b^d) & \text{and } (\sigma_k^{\ell \bullet}(t_k, x)|z) = \sum_{m=1}^q \sigma_k^{\ell m}(t_k, x) z^m. \end{cases} \quad (33)$$

where $(Z_k)_{k=1, \dots, n}$ is *i.i.d.*, $\mathcal{N}(0; I_q)$ -distributed, independent of \bar{X}_0 .

In the following result, we show that the sequence $(\hat{X}_k)_{k \geq 0}$ of Markovian and product quantizations is in fact a Markov chain. Its transition probabilities will be computed further on.

Remark 3.1. We may remark that the process $(\hat{X}_k)_{k \geq 0}$ is a Markov chain on \mathbb{R}^d .

In fact, setting $\mathcal{F}_k^{\hat{X}} = \sigma(\hat{X}_0, \dots, \hat{X}_k)$, we have for any bounded function $f : \mathbb{R}^d \rightarrow \mathbb{R}$

$$\begin{aligned} \mathbb{E}(f(\hat{X}_{k+1}) | \mathcal{F}_k^{\hat{X}}) &= \sum_{j \in I_{k+1}} \mathbb{E} \left(f(x_{k+1}^j) \mathbb{1}_{\{\hat{X}_{k+1} = x_{k+1}^j\}} | \mathcal{F}_k^{\hat{X}} \right) \\ &= \sum_{j \in I_{k+1}} f(x_{k+1}^j) \mathbb{E} \left(\mathbb{1}_{\{\mathcal{E}_k(\hat{X}_k, Z_{k+1}) \in \prod_{\ell=1}^d C_{j_\ell}(\Gamma_{k+1}^\ell)\}} | \mathcal{F}_k^{\hat{X}} \right), \end{aligned}$$

where $\mathcal{E}_k(\hat{X}_k, Z_{k+1}) = (\mathcal{E}_k^1(\hat{X}_k, Z_{k+1}), \dots, \mathcal{E}_k^d(\hat{X}_k, Z_{k+1}))$. It follows that

$$\mathbb{E}(f(\hat{X}_{k+1}) | \mathcal{F}_k^{\hat{X}}) = \sum_{j \in I_{k+1}} f(x_{k+1}^j) h_j(\hat{X}_k),$$

where for every $x \in \mathbb{R}^d$,

$$h_j(x) = \mathbb{P}(\mathcal{E}_k(x, Z_{k+1}) \in \prod_{\ell=1}^d C_{j_\ell}(\Gamma_{k+1}^\ell)).$$

As a consequence, $\mathbb{E}(f(\hat{X}_{k+1}) | \mathcal{F}_k^{\hat{X}}) = \varphi(\hat{X}_k)$, so that $(\hat{X}_k)_{k \geq 0}$ is a Markov chain.

Now, for a given componentwise (quadratic) optimal quantizers $\Gamma_k = \bigotimes_{\ell=1}^d \Gamma_k^\ell$, let us explain how to compute the companion transition probability weights associated with the quantizations of the \bar{X}_k^ℓ 's and the whole vector \bar{X}_k . We write all the quantities of interest as an expectation of a function of a standard \mathbb{R}^{q-1} -valued Normal distribution. These transformations are the key step of this work. In fact, since the optimal quantization grids associated to standard Normal random vectors (up to dimension 10) and their companion weights are available on www.quantize.maths-fi.com, these quantities of interest may be computed instantaneously using a cubature formula.

3.2 Computing the companion weights and transition probabilities of the marginal quantizations

First of all we define the following quantities which will be needed in the sequel. For every $k \in \{0, \dots, n-1\}$ and for every $j \in I_{k+1}$ we set

$$x_{k+1}^{i, j_i - 1/2} = \frac{x_{k+1}^{i, j_i} + x_{k+1}^{i, j_i - 1}}{2}, \quad x_{k+1}^{i, j_i + 1/2} = \frac{x_{k+1}^{i, j_i} + x_{k+1}^{i, j_i + 1}}{2}, \quad \text{with } x_{k+1}^{i, 1/2} = -\infty, x_{k+1}^{i, N_{k+1}^i + 1/2} = +\infty,$$

and if $Z_k^{(2;q)} = z \in \mathbb{R}^{q-1}$ and $x \in \mathbb{R}^d$, we set (if $\sigma_k^{i1}(x) > 0$)

$$\begin{aligned} x_{k+1}^{i, j_i -}(x, z) &:= \frac{x_{k+1}^{i, j_i - 1/2} - m_k^i(x) - \sqrt{\Delta}(\sigma_k^{(i, 2;q)}(x)|z)}{\sqrt{\Delta} \sigma_k^{i1}(x)} \\ \text{and } x_{k+1}^{i, j_i +}(x, z) &:= \frac{x_{k+1}^{i, j_i + 1/2} - m_k^i(x) - \sqrt{\Delta}(\sigma_k^{(i, 2;q)}(x)|z)}{\sqrt{\Delta} \sigma_k^{i1}(x)}. \end{aligned}$$

We also set

$$\mathbb{J}_{k, j_i}^0(x) = \left\{ z \in \mathbb{R}^{q-1}, \quad \sqrt{\Delta}(\sigma_k^{(i, 2;q)}(x)|z) \in (x_{k+1}^{i, j_i - 1/2} - m_k^i(x), x_{k+1}^{i, j_i + 1/2} - m_k^i(x)) \right\}$$

and

$$\begin{aligned}\mathbb{J}_k^0(x) &= \{i \in \{1, \dots, d\}, \sigma_k^{i1}(x) = 0\} \\ \mathbb{J}_k^-(x) &= \{i \in \{1, \dots, d\}, \sigma_k^{i1}(x) < 0\} \\ \mathbb{J}_k^+(x) &= \{i \in \{1, \dots, d\}, \sigma_k^{i1}(x) > 0\}.\end{aligned}$$

The following result allows us to compute the weights and the transition probabilities associated to the quantizations $\widehat{X}_k, k = 0, \dots, n$.

Proposition 3.1. *Let $\{\widehat{X}_k, k = 0, \dots, n\}$ be the sequence defined from the algorithm (33).*

The transition probability $\mathbb{P}(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^\ell), i \in I_k, j \in I_{k+1}$, is given by

$$\mathbb{P}(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^i) = \mathbb{E} \prod_{\ell \in \mathbb{J}_k^0(x_k^i)} \mathbf{1}_{\{\zeta \in \mathbb{J}_{k,j}^\ell(x_k^i)\}} \max(\Phi_0(\beta_j(x_k^i, \zeta)) - \Phi_0(\alpha_j(x_k^i, \zeta)), 0) \quad (34)$$

where $\zeta \sim \mathcal{N}(0; I_{q-1})$ and, for every $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^{q-1}$,

$$\begin{aligned}\alpha_j(x, z) &= \max\left(\sup_{\ell \in \mathbb{J}_k^+(x)} x_{k+1}^{\ell, j\ell^-}(x, z), \sup_{\ell \in \mathbb{J}_k^-(x)} x_{k+1}^{\ell, j\ell^+}(x, z)\right) \\ \text{and } \beta_j(x, z) &= \min\left(\inf_{\ell \in \mathbb{J}_k^+(x)} x_{k+1}^{\ell, j\ell^+}(x, z), \inf_{\ell \in \mathbb{J}_k^-(x)} x_{k+1}^{\ell, j\ell^-}(x, z)\right).\end{aligned}$$

Before proving this result, remark that we may deduce the probability weights associated to the quantizations (\widehat{X}_{k+1}) by

$$\mathbb{P}(\widehat{X}_{k+1} = x_{k+1}^j) = \sum_{i \in I_k} \mathbb{P}(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^i) \mathbb{P}(\widehat{X}_k = x_k^i) \quad (35)$$

where the conditional probabilities are computed using the formula (34).

Proof. We have

$$\begin{aligned}\mathbb{P}(\widehat{X}_{k+1} = x_{k+1}^j | \widehat{X}_k = x_k^\ell) &= \mathbb{P}\left(\bigcap_{i=1}^d \{\widetilde{X}_{k+1}^i \in (v^{i, j_i^-}, v^{i, j_i^+})\} | \widehat{X}_k = x_k^\ell\right) \\ &= \mathbb{P}\left(\bigcap_{i=1}^d \{\mathcal{E}_k^i(x_k^\ell, Z_{k+1}) \in (v^{i, j_i^-}, v^{i, j_i^+})\}\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\bigcap_{i=1}^d \{\mathcal{E}_k^i(x_k^\ell, Z_{k+1}) \in (v^{i, j_i^-}, v^{i, j_i^+})\}}\right) | Z_{k+1}^{(2:d)}\right) \\ &= \mathbb{E}(\Psi(x_k^\ell, Z_{k+1}^{(2:d)}))\end{aligned}$$

where, for every $u \in \mathbb{R}^{q-1}$,

$$\Psi(x, u) = \mathbb{P}\left(\bigcap_{i=1}^d \{m_k^i(x_k^\ell) + \sqrt{\Delta} \sigma_k^{i1}(x) Z_{k+1}^1 + \sqrt{\Delta} (\sigma_k^{(i, 2:q)}(x) | u) \in (v^{i, j_i^-}, v^{i, j_i^+})\}\right).$$

Let us set

$$A_{i,k} = \left\{m_k^i(x_k^\ell) + \sqrt{\Delta} \sigma_k^{i1}(x) Z_{k+1}^1 + \sqrt{\Delta} (\sigma_k^{(i, 2:q)}(x) | u) \in (v^{i, j_i^-}, v^{i, j_i^+})\right\}.$$

We know that if $i \in \mathbb{J}_k^0(x)$ then $A_{i,k} = \{u \in \mathbb{J}_{k,j_i}^0(x)\}$ and we deduce that

$$\Psi(x, u) = \prod_{i_0 \in \mathbb{J}_k^0(x)} \mathbf{1}_{\{u \in \mathbb{J}_{k,j_i}^0(x)\}} \mathbb{P} \left(\left(\bigcap_{i_- \in \mathbb{J}_k^-(x)} A_{i_-,k} \right) \cap \left(\bigcap_{i_+ \in \mathbb{J}_k^+(x)} A_{i_+,k} \right) \right).$$

Furthermore, notice that if $i_+ \in \mathbb{J}_k^+(x)$ then

$$A_{i_+,k} = \{Z_{k+1}^1 \in (x_{k+1}^{i,j_i^-}(x, u), x_{k+1}^{i,j_i^+}(x, u))\}$$

and $i_- \in \mathbb{J}_k^-(x)$ then

$$A_{i_-,k} = \{Z_{k+1}^1 \in (x_{k+1}^{i,j_i^+}(x, u), x_{k+1}^{i,j_i^-}(x, u))\}.$$

It follows that (remark that the sets $\mathbb{J}_k^-(x)$ or $\mathbb{J}_k^+(x)$ may be empty)

$$\begin{aligned} \mathbb{P} \left(\left(\bigcap_{i_- \in \mathbb{J}_k^-(x)} A_{i_-,k} \right) \cap \left(\bigcap_{i_+ \in \mathbb{J}_k^+(x)} A_{i_+,k} \right) \right) &= \mathbb{P} \left(Z_{k+1}^1 \in \left(\sup_{i \in \mathbb{J}_k^+(x)} x_{k+1}^{i,j_i^-}(x, u), \inf_{i \in \mathbb{J}_k^+(x)} x_{k+1}^{i,j_i^+}(x, u) \right) \right. \\ &\quad \left. \cap \left(\sup_{i \in \mathbb{J}_k^-(x)} x_{k+1}^{i,j_i^+}(x, u), \inf_{i \in \mathbb{J}_k^-(x)} x_{k+1}^{i,j_i^-}(x, u) \right) \right). \end{aligned}$$

This completes the proof since $Z_{k+1}^{(2;d)} \sim \mathcal{N}(0; I_{q-1})$. \square

Now, we focus on in the particular case where the matrix $\sigma(t, x)$, for $(t, x) \in [0, T] \times \mathbb{R}^d$, is diagonal with positive diagonal entries $\sigma^{\ell\ell}(t, x)$, $\ell = 1, \dots, d$. The following result says how to compute the transition probability weights of the \hat{X}_k 's. Let us set for every $x \in \mathbb{R}^d$, every $\ell \in \{1, \dots, d\}$ and $j\ell \in \{1, \dots, N_{k+1}^\ell\}$,

$$x_{k+1}^{\ell, j\ell^-}(x, 0) := \frac{x_{k+1}^{\ell, j\ell^-1/2} - m_k^\ell(x)}{\sqrt{\Delta} \sigma_k^{\ell\ell}(x)} \quad \text{and} \quad x_{k+1}^{\ell, j\ell^+}(x, 0) := \frac{x_{k+1}^{\ell, j\ell^+1/2} - m_k^\ell(x)}{\sqrt{\Delta} \sigma_k^{\ell\ell}(x)}.$$

Proposition 3.2. *Let $\{\hat{X}_k, k = 0, \dots, n\}$ be the sequence of quantizers defined by the algorithm (33) and associated with the solution (X_t) of (29). Suppose that the volatility matrix $\sigma(t, x)$ of $(X_t)_{t \geq 0}$ is diagonal with positive diagonal entries $\sigma^{\ell\ell}(t, x)$, $\ell = 1, \dots, d$. Then, the transition probability weights $\mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i)$, $i \in I_k, j \in I_{k+1}$, are given by*

$$\mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i) = \prod_{\ell=1}^d \mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{j\ell} | \hat{X}_k = x_k^i) \quad (36)$$

$$= \prod_{\ell=1}^d [\Phi_0(x_{k+1}^{\ell, j\ell^+}(x_k^i, 0)) - \Phi_0(x_{k+1}^{\ell, j\ell^-}(x_k^i, 0))], \quad (37)$$

and the companion probability weights $\mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j)$ are given for every $k = 0, \dots, n-1$ and every $j \in I_{k+1}$ by

$$\mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j) = \sum_{i \in I_k} \prod_{\ell=1}^d [\Phi_0(x_{k+1}^{\ell, j\ell^+}(x_k^i, 0)) - \Phi_0(x_{k+1}^{\ell, j\ell^-}(x_k^i, 0))] \mathbb{P}(\hat{X}_k = x_k^i). \quad (38)$$

Proof. 1. Set $v^{\ell, j_{\ell}+} := x_{k+1}^{\ell, j_{\ell}+1/2}$ and $v^{\ell, j_{\ell}-} = x_{k+1}^{\ell, j_{\ell}-1/2}$, for $j \in I_{k+1}$ and $\ell = 1, \dots, d$. We have

$$\begin{aligned} \mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i) &= \mathbb{P}\left(\bigcap_{\ell=1}^d \{\tilde{X}_{k+1}^\ell \in (v^{\ell, j_{\ell}-}, v^{\ell, j_{\ell}+})\} | \hat{X}_k = x_k^i\right) \\ &= \mathbb{P}\left(\bigcap_{\ell=1}^d \{\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \in (v^{\ell, j_{\ell}-}, v^{\ell, j_{\ell}+})\}\right). \end{aligned}$$

Since for every $k = 0, \dots, n-1$, $\sigma(t_k, x)$ is a diagonal matrix, it follows that the operators $\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) = \mathcal{E}_k^\ell(x_k^i, Z_{k+1}^\ell)$, for $\ell = 1, \dots, d$, are independent, so that

$$\begin{aligned} \mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i) &= \prod_{\ell=1}^d \mathbb{P}\left(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}^\ell) \in (v^{\ell, j_{\ell}-}, v^{\ell, j_{\ell}+})\right) \\ &= \prod_{\ell=1}^d [\Phi_0(x_{k+1}^{\ell, j_{\ell}+}(x_k^i)) - \Phi_0(x_{k+1}^{\ell, j_{\ell}-}(x_k^i))]. \end{aligned}$$

The second assertion immediately follows. \square

The following result is useful in the situation where we need to approximate the expectation of a function of one component of the vector \tilde{X}_k as for example in the pricing of European options in the Heston model.

Proposition 3.3. *Let Γ_{k+1}^ℓ be an optimal quantizer for the random variable \tilde{X}_{k+1}^ℓ . Suppose that the optimal product quantizer Γ_k for \tilde{X}_k and its companion weights $\mathbb{P}(\hat{X}_k = x_k^i)$, $i \in I_k$, are computed.*

For any $\ell \in \{1, \dots, d\}$ and any $j_\ell \in \{1, \dots, N_{k+1}^\ell\}$, the transition probability weights $\mathbb{P}(\tilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell) | \hat{X}_k = x_k^i)$ are given by

$$\mathbb{P}(\tilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell) | \hat{X}_k = x_k^i) = \Phi_0(x_{k+1}^{\ell, j_\ell+}(x_k^i)) - \Phi_0(x_{k+1}^{\ell, j_\ell-}(x_k^i)). \quad (39)$$

Proof. 1. For every $k \in \{1, \dots, n-1\}$, for every $\ell = 1, \dots, d$ and for every $j_\ell = 1, \dots, N_{k+1}^\ell$, we have

$$\begin{aligned} \mathbb{P}(\tilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell) | \hat{X}_k = x_k^i) &= \mathbb{P}(\tilde{X}_{k+1}^\ell \leq x_{k+1}^{\ell, j_\ell+1/2} | \hat{X}_k = x_k^i) - \mathbb{P}(\tilde{X}_{k+1}^\ell \leq x_{k+1}^{\ell, j_\ell-1/2} | \hat{X}_k = x_k^i) \\ &= \mathbb{P}(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \leq x_{k+1}^{\ell, j_\ell+1/2}) - \mathbb{P}(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \leq x_{k+1}^{\ell, j_\ell-1/2}). \end{aligned}$$

To complete the proof we just have to remark that $\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \sim \mathcal{N}(m_k^\ell(x_k^i); \Delta |\sigma_k^{\ell \bullet}(x_k^i)|_2^2)$. \square

Notice that the companion probability $\mathbb{P}(\tilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell))$ is given, for every $\ell \in \{1, \dots, d\}$ and for every $j_\ell \in \{1, \dots, N_{k+1}^\ell\}$, by

$$\begin{aligned} \mathbb{P}(\tilde{X}_{k+1}^\ell \in C_{j_\ell}(\Gamma_{k+1}^\ell)) &= \sum_{i \in I_k} \left[\Phi_0\left(\frac{x_{k+1}^{\ell, j_\ell+1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta} |\sigma_k^{\ell \bullet}(x_k^i)|_2}\right) \right. \\ &\quad \left. - \Phi_0\left(\frac{x_{k+1}^{\ell, j_\ell-1/2} - m_k^\ell(x_k^i)}{\sqrt{\Delta} |\sigma_k^{\ell \bullet}(x_k^i)|_2}\right) \right] \mathbb{P}(\hat{X}_k = x_k^i). \end{aligned} \quad (40)$$

We may note that the ℓ -th component process $(\tilde{X}_k^\ell)_{k \geq 0}$ is not a Markov chain. We may however compute the transition probabilities

$$\mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell} | \hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}}), \quad \ell, \ell' \in \{1, \dots, d\}, \quad j_\ell \in \{1, \dots, N_{k+1}^\ell\}, \quad j_{\ell'} \in \{1, \dots, N_k^{\ell'}\}.$$

This is the aim of the following remark which follows from *Bayes* formula.

Remark 3.2. For $\ell, \ell' \in \{1, \dots, d\}$, $j_\ell \in \{1, \dots, N_{k+1}^\ell\}$ and $j_{\ell'} \in \{1, \dots, N_k^{\ell'}\}$, we have

$$\mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell} | \hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}}) = \sum_{i \in I_k} \mathbb{1}_{\{i_{\ell'} = j_{\ell'}\}} \frac{\mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell} | \hat{X}_k = x_k^i)}{\mathbb{P}(\hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}})} \mathbb{P}(\hat{X}_k = x_k^i) \quad (41)$$

where the terms $\mathbb{P}(\hat{X}_k = x_k^i)$, $\mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell} | \hat{X}_k = x_k^i)$ and $\mathbb{P}(\hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}})$ are computed from (35), (39) and (40), respectively.

As a matter of fact, applying *Bayes* formula and summing over $i \in I_k$ yields:

$$\begin{aligned} \mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell} | \hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}}) &= \sum_{i \in I_k} \frac{\mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell}, \hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}}, \hat{X}_k = x_k^i)}{\mathbb{P}(\hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}})} \\ &= \sum_{i \in I_k} \mathbb{1}_{\{i_{\ell'} = j_{\ell'}\}} \frac{\mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell}, \hat{X}_k = x_k^i)}{\mathbb{P}(\hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}})} \\ &= \sum_{i \in I_k} \mathbb{1}_{\{i_{\ell'} = j_{\ell'}\}} \frac{\mathbb{P}(\hat{X}_{k+1}^\ell = x_{k+1}^{\ell, j_\ell} | \hat{X}_k = x_k^i)}{\mathbb{P}(\hat{X}_k^{\ell'} = x_k^{\ell', j_{\ell'}})} \mathbb{P}(\hat{X}_k = x_k^i). \end{aligned}$$

In the foregoing, we assume that we have access to the N_k^ℓ -quantizers Γ_k^ℓ of the ℓ -th component \bar{X}_k^ℓ of the vector \bar{X}_k , for every $\ell = 1, \dots, d$. We show how to compute the distortion functions associated with every component of the vector \bar{X}_{k+1} , $k = 0, \dots, n-1$. From the numerical point of view, this will allow us to use the Newton-Raphson algorithm to compute the optimal quantizers associated to each component \tilde{X}_{k+1}^ℓ , $\ell = 1, \dots, d$, of the vector \tilde{X}_{k+1} , for $k = 0, \dots, n-1$. Then, the quantization \hat{X}_{k+1} of \tilde{X}_{k+1} is defined as the product quantization $\hat{X}_k = (\hat{X}_k^1, \dots, \hat{X}_k^d)$, where $\hat{X}_k^\ell = \text{Proj}_{\Gamma_{k+1}^\ell}(\tilde{X}_{k+1}^\ell)$.

3.3 Computing the distortion, the gradient and the Hessian matrix associated to a componentwise quantizer

Our aim, for numerical computation of the componentwise optimal quantizations, is to use the Newton-Raphson's algorithm in \mathbb{R}^{N_k} which involves the gradient and the Hessian matrix of the distortion functions \tilde{D}_k^ℓ , $k = 0, \dots, n$; $\ell = 1, \dots, d$. In the following, we give useful expressions for the distortion functions \tilde{D}_k^ℓ , their gradient vectors $\nabla \tilde{D}_k^\ell$ and their Hessian matrices $\nabla^2 \tilde{D}_k^\ell$. We state these results in the next proposition.

Above all, recall that for every $\ell = 1, \dots, d$, for every $k = 0, \dots, n-1$,

$$\tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell) = \sum_{i \in I_k} \mathbb{E}[d(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}), \Gamma_{k+1}^\ell)^2] \mathbb{P}(\hat{X}_k = x_k^i)$$

and notice that using Proposition 2.2, the distortion function $\tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)$ is continuously differentiable as a function of the N_{k+1} -quantizer $\Gamma_{k+1}^\ell = \{x_{k+1}^{\ell, j_\ell}, j_\ell = 1, \dots, N_{k+1}^\ell\}$ (having pairwise distinct components so that it can be viewed as an N_{k+1}^ℓ -tuple) and its gradient vector reads

$$\nabla \tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell) = 2 \left[\sum_{i \in I_k} \mathbb{E} \left(\mathbb{1}_{\{\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \in C_{j_\ell}(\Gamma_{k+1}^\ell)\}} (x_{k+1}^{\ell, j_\ell} - \mathcal{E}_k^\ell(x_k^i, Z_{k+1})) \right) \mathbb{P}(\hat{X}_k = x_k^i) \right]_{j_\ell=1, \dots, N_{k+1}^\ell}.$$

We recall that key point of our method is to deal with the product quantization of the components of the process $(\bar{X}_k)_{0 \leq k \leq n}$. From a numerical point of view, each component will be quantized using

the Newton-Raphson algorithm. To this end, we have to compute (explicitly) the distortion function $\tilde{D}_{k+1}^\ell(\cdot)$, the components of its gradient vector and the components its Hessian matrix. This is the purpose of the following remark. Its proof relies on tedious though elementary computation. Therefore, we have deliberately omitted the proof.

Remark 3.3. Recall that for every $\ell \in \{1, \dots, d\}$, $\vartheta_k^\ell(x)^2 = \sum_{p=1}^q \Delta(\sigma_k^{\ell p}(x))^2$.

a) *Distortion.* We have for every $\ell = 1, \dots, d$ and every $k = 0, \dots, n-1$,

$$\tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell) = \sum_{j_\ell=1}^{N_{k+1}^\ell} \sum_{i \in I_k} \Psi_{\ell, j_\ell}(x_k^i) p_k^i = \sum_{j_\ell=1}^{N_{k+1}^\ell} \mathbb{E} \Psi_{\ell, j_\ell}(\hat{X}_k), \quad (42)$$

where for every $x \in \mathbb{R}^d$,

$$\begin{aligned} \Psi_{\ell, j_\ell}(x) = & \left((m_k^\ell(x) - x_{k+1}^{\ell, j_\ell})^2 + \vartheta_k^\ell(x)^2 \right) \left(\Phi_0(x_{k+1}^{\ell, j_\ell+}(x)) - \Phi_0(x_{k+1}^{\ell, j_\ell-}(x)) \right) \\ & + 2\vartheta_k^\ell(x) \left(x_{k+1}^{\ell, j_\ell} - m_k^\ell(x) \right) \left(\Phi_0'(x_{k+1}^{\ell, j_\ell+}(x)) - \Phi_0'(x_{k+1}^{\ell, j_\ell-}(x)) \right) \\ & - \vartheta_k^\ell(x)^2 \left(x_{k+1}^{\ell, j_\ell+}(x) \Phi_0'(x_{k+1}^{\ell, j_\ell+}(x)) - x_{k+1}^{\ell, j_\ell-}(x) \Phi_0'(x_{k+1}^{\ell, j_\ell-}(x)) \right). \end{aligned}$$

b) *Gradient.* The components of the gradient $\nabla \tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)$ are given for every $j_\ell = 1, \dots, N_{k+1}^\ell$ by

$$\frac{\partial \tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)}{\partial x_{k+1}^{\ell, j_\ell}} = \sum_{i \in I_k} \Psi'_{j_\ell}(x_k^i) p_k^i = \mathbb{E} \Psi'_{j_\ell}(\hat{X}_k) \quad (43)$$

where for every $x \in \mathbb{R}^d$,

$$\begin{aligned} \Psi'_{j_\ell}(x) = & (x_{k+1}^{\ell, j_\ell} - m_k^\ell(x)) \left(\Phi_0'(x_{k+1}^{\ell, j_\ell+}(x)) - \Phi_0'(x_{k+1}^{\ell, j_\ell-}(x)) \right) \\ & + \vartheta_k^\ell(x) \left(\Phi_0''(x_{k+1}^{\ell, j_\ell+}(x)) - \Phi_0''(x_{k+1}^{\ell, j_\ell-}(x)) \right). \end{aligned}$$

c) *Hessian.* The sub-diagonal, the super-diagonals and the diagonal terms of the Hessian matrix are given respectively by

$$\begin{aligned} \frac{\partial^2 \tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)}{\partial x_{k+1}^{\ell, j_\ell} \partial x_{k+1}^{\ell, j_\ell-1}} &= \sum_{i \in I_k} \Psi''_{j_\ell, j_\ell-1}(x_k^i) p_k^i = \mathbb{E} \Psi''_{j_\ell, j_\ell-1}(\hat{X}_k), \\ \frac{\partial^2 \tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)}{\partial x_{k+1}^{\ell, j_\ell} \partial x_{k+1}^{\ell, j_\ell+1}} &= \sum_{i \in I_k} \Psi''_{j_\ell, j_\ell+1}(x_k^i) p_k^i = \mathbb{E} \Psi''_{j_\ell, j_\ell+1}(\hat{X}_k) \\ \text{and} \quad \frac{\partial^2 \tilde{D}_{k+1}^\ell(\Gamma_{k+1}^\ell)}{\partial^2 x_{k+1}^{\ell, j_\ell}} &= \sum_{i \in I_k} \Psi''_{j_\ell, j_\ell}(x_k^i) p_k^i = \mathbb{E} \Psi''_{j_\ell, j_\ell}(\hat{X}_k) \end{aligned}$$

where, for every $x \in \mathbb{R}^d$,

$$\begin{aligned} \Psi''_{j_\ell, j_\ell-1}(x) &= -\frac{1}{4} \frac{1}{\vartheta_k^\ell(x)} (x_{k+1}^{\ell, j_\ell} - x_{k+1}^{\ell, j_\ell-1}) \Phi_0'(x_{k+1}^{\ell, j_\ell-}(x)), \\ \Psi''_{j_\ell, j_\ell+1}(x) &= -\frac{1}{4} \frac{1}{\vartheta_k^\ell(x)} (x_{k+1}^{\ell, j_\ell+1} - x_{k+1}^{\ell, j_\ell}) \Phi_0'(x_{k+1}^{\ell, j_\ell+}(x)), \\ \Psi''_{j_\ell, j_\ell}(x) &= \Phi_0(x_{k+1}^{\ell, j_\ell+}(x)) - \Phi_0(x_{k+1}^{\ell, j_\ell-}(x)) + \Psi''_{j_\ell, j_\ell-1}(x) + \Psi''_{j_\ell, j_\ell+1}(x). \end{aligned}$$

Once we have access to the gradient vector and the Hessian matrix associated with \tilde{X}_{k+1}^ℓ and to the optimal grids and companions weights associated with the $\hat{X}_{k'}$'s, $k' = 0, \dots, k$, it is possible to write down (at least formally) a Newton-Raphson zero search procedure to compute the optimal quantizer Γ_{k+1}^ℓ . The Newton-Raphson algorithm is in fact indexed by $p \geq 0$, where a current grid $\Gamma_{k+1}^{\ell,p}$ is updated as follows:

$$\Gamma_{k+1}^{\ell,p+1} = \Gamma_{k+1}^{\ell,p} - (\nabla^2 \tilde{D}_{k+1}^\ell(\Gamma_{k+1}^{\ell,p}))^{-1} \nabla \tilde{D}_{k+1}^\ell(\Gamma_{k+1}^{\ell,p}), \quad p \geq 1, \quad (44)$$

starting from a $\Gamma_{k+1}^{\ell,0} \in \mathbb{R}^{N_{k+1}^\ell}$ (with increasing components).

Remark 3.4. (*Stationarity property*) If Γ_{k+1}^ℓ is an optimal Markovian product quantizer for \tilde{X}_{k+1}^ℓ and if \hat{X}_{k+1}^ℓ denotes the quantization of \tilde{X}_{k+1}^ℓ by the grid Γ_{k+1}^ℓ , then Γ_{k+1}^ℓ is a stationary quantizer for \tilde{X}_{k+1}^ℓ , means, $\mathbb{E}(\tilde{X}_{k+1}^\ell | \hat{X}_{k+1}^\ell) = \hat{X}_{k+1}^\ell$. Equivalently, this means that if $\Gamma_{k+1}^\ell = \{x_{k+1}^{\ell,j_\ell}, j_\ell = 1, \dots, N_{k+1}^\ell\}$ with

$$x_{k+1}^{\ell,j_\ell} = \frac{\sum_{i \in I_k} \mathbb{E}(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \mathbb{1}_{\{\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \in C_{j_\ell}(\Gamma_{k+1}^\ell)\}}) \mathbb{P}(\hat{X}_k = x_k^i)}{p_{k+1}^{j_\ell}} \quad (45)$$

$$\text{and } p_{k+1}^{j_\ell} = \sum_{i \in I_k} \mathbb{P}(\mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \in C_{j_\ell}(\Gamma_{k+1}^\ell)) \mathbb{P}(\hat{X}_k = x_k^i), \quad j_\ell = 1, \dots, N_{k+1}^\ell. \quad (46)$$

A straightforward computation leads to the following result: if $\Gamma_{k+1}^\ell = \{x_{k+1}^{\ell,j_\ell}, j_\ell = 1, \dots, N_{k+1}^\ell\}$ is a stationary quantizer for \tilde{X}_{k+1}^ℓ then, for every $\ell \in \{1, \dots, d\}$ and for every $j_\ell \in \{1, \dots, N_{k+1}^\ell\}$,

$$x_{k+1}^{\ell,j_\ell} = \frac{\sum_{i \in I_k} [m_k^\ell(x_k^i) \gamma_{\ell,k}(x_k^i) - \vartheta_k^\ell(x_k^i) \gamma'_{\ell,k}(x_k^i)] p_k^i}{\sum_{i \in I_k} \gamma_{\ell,k}(x_k^i) p_k^i} \quad (47)$$

where for every $x \in \mathbb{R}^d$,

$$\gamma_{\ell,k}(x) = \Phi_0(x_{k+1}^{\ell,j_\ell+}(x)) - \Phi_0(x_{k+1}^{\ell,j_\ell-}(x)) \quad \text{and} \quad \gamma'_{\ell,k}(x) = \Phi'_0(x_{k+1}^{\ell,j_\ell+}(x)) - \Phi'_0(x_{k+1}^{\ell,j_\ell-}(x)). \quad (48)$$

3.4 The error analysis

Our aim is now to compute the quadratic quantization error bound $\|\bar{X}_T - \hat{X}_T\|_2 := \|\bar{X}_n - \hat{X}_n^{\Gamma_n}\|_2$. The analysis of this error bound is the subject of the following theorem. We suppose that $x_0 = X_0 = \tilde{X}_0$. We consider here a regular time discretization $(t_k)_{0 \leq k \leq n}$ with step $\Delta = T/n$: $t_k = \frac{kT}{n}$, $k = 0, \dots, n$.

Theorem 3.4. *Assume the coefficients b, σ satisfy the classical Lipschitz assumptions (30), (31) and (32). Let, for every $k = 0, \dots, n$, Γ_k be a Markovian product quantizer for \tilde{X}_k at level N_k . Then, for every $k = 0, \dots, n$, for any $\eta \in]0, 1]$,*

$$\|\bar{X}_k - \hat{X}_k^{\Gamma_k}\|_2 \leq K_{2,\eta} \sum_{k'=1}^k e^{(k-k')\Delta C_{b,\sigma}} a_{k'}(b, \sigma, d, k, \Delta, x_0, L, 2 + \eta) N_{k'}^{-1/d} \quad (49)$$

where for every $p \in (2, 3]$,

$$a_{k'}(b, \sigma, d, k, \Delta, x_0, L, \theta) := e^{\Delta C_{b,\sigma} \frac{(k-k')}{p}} \left[e^{(\kappa_p + K_p)t_{k'}} |x_0|^p + \frac{d^{k(\frac{p}{2}-1)} e^{\kappa_p \Delta L} + K_p}{d^{\frac{p}{2}-1} (\kappa_p + K_p)} (e^{(\kappa_p + K_p)t_{k'}} - 1) \right]^{\frac{1}{p}},$$

with $C_{b,\sigma} = [b]_{\text{Lip}} + \frac{1}{2}[\sigma]_{\text{Lip}}^2$, $K_{2,\eta} := K_{2,1,\eta}$ is a universal constant defined in Equation (22);

$$\kappa_\theta := \left(\frac{(\theta+1)(\theta-2)}{2} + 2\theta L \right) \quad \text{and} \quad K_\theta := 2^{\theta-1} L^\theta \left(1 + \theta + \frac{p(p-1)}{2} \Delta^{\frac{\theta}{2}-1} \right) \mathbb{E}|Z|^\theta, \quad Z \sim \mathcal{N}(0; I_d).$$

Proof. (of Theorem 3.4). Recall that for every $k \geq 0$, $\hat{X}_k = (\hat{X}_k^1, \dots, \hat{X}_k^d)$, where \hat{X}_k^ℓ is the quantization of the ℓ -th component \tilde{X}_k^ℓ of the vector \tilde{X}_k . Therefore, following step by step the proof of Lemma 3.2. in [19], we obtain for every $k \geq 1$:

$$\|\tilde{X}_k - \hat{X}_k\|_2 \leq \sum_{k'=1}^k e^{(t_k - t_{k'})C_{b,\sigma}} \|\tilde{X}_{k'} - \hat{X}_{k'}^{\Gamma}\|_2,$$

where $C_{b,\sigma} = [b]_{\text{Lip}} + \frac{1}{2}[\sigma]_{\text{Lip}}^2$. Using the definition of \hat{X}_k combined with Pierce's Lemma (see Theorem 2.1(b)) yields for every $k = 1, \dots, n$, for any $\eta \in (0, 1]$,

$$\|\tilde{X}_k - \hat{X}_k\|_2 \leq K_{2,\eta} \sum_{k'=1}^k e^{(k-k')\Delta C_{b,\sigma}} \|\tilde{X}_{k'}^\ell\|_{2+\eta} (N_{k'}^\ell)^{-1/d}.$$

Recall that each component \hat{X}_k^ℓ of the vector \hat{X}_k is defined as

$$\hat{X}_k^\ell = \widehat{\tilde{X}_k^\ell}, \quad \ell = 1, \dots, d,$$

where $\widehat{\tilde{X}_k^\ell}$ is an optimal quadratic quantization of \tilde{X}_k^ℓ . Hence each component of \hat{X}_k is stationary with respect to \tilde{X}_k^ℓ , that is $\hat{X}_k = \mathbb{E}(\tilde{X}_k^\ell | \widehat{\tilde{X}_k^\ell}) = \widehat{\tilde{X}_k^\ell}$. We deduce that

$$\mathbb{E}|\hat{X}_k|^p \leq d^{\frac{p}{2}-1} \mathbb{E}|\tilde{X}_k|^p.$$

Following the lines of the proof of Lemma 3.2. in [19], we easily show that for every $\ell \in \{1, \dots, d\}$, $\|\tilde{X}_{k'}^\ell\|_{2+\eta} \leq a_{k'}(b, \sigma, k, d, \Delta, x_0, L, 2 + \eta)$. This completes the proof. \square

4 Numerical examples

First of all, keep in mind that the computations for all numerical examples have been performed on a CPU 2.7 GHz and 4 Go memory computer.

4.1 Pricing of a Basket European option

We consider a European Basket option with maturity T and strike K , based on two stocks with prices S^1 and S^2 with associated weights w_1 and w_2 . We suppose that S^1 and S^2 evolve following the dynamics

$$\begin{cases} dS_t^1 = rS_t^1 + \rho \sigma_1 S_t^1 dW_t^1 + \sqrt{1 - \rho^2} \sigma_1 S_t^1 dW_t^2 \\ dS_t^2 = rS_t^2 dt + \sigma_2 S_t^2 dW_t^1 \end{cases} \quad (50)$$

where W^1 and W^2 are two independent Brownian motion, r is the interest rate and $\rho \in [-1, 1]$, is the correlation term.

We know that in this case, the price at time $t = 0$ of the call option reads

$$e^{-rT} \mathbb{E}[\max(w_1 S_T^1 + w_2 S_T^2 - K, 0)] = e^{-rT} \mathbb{E}F(X_T), \quad X = (S^1, S^2), \quad (51)$$

where the function F is defined, for every $x = (s^1, s^2) \in \mathbb{R}^2$, by $F(x) = \max(w_1 s^1 + w_2 s^2 - K, 0)$. Using the Markovian product quantization, the price of the Basket European option is approximated by

$$e^{-rT} \sum_{j \in I_n} F(x_n^j) \mathbb{P}(\hat{X}_n = x_n^j). \quad (52)$$

For the numerical exercises we will use the following set of parameters:

$$r = 0.04, \quad \sigma_1 = 0.3, \quad \sigma_2 = 0.4, \quad \rho = 0.5, \quad w_1 = w_2 = 0.5, \quad S_0^1 = 100, \quad S_0^2 = 100, \quad T = 1.$$

The benchmark price is given by the algorithm developed in [15]. The results are given in Table 1 and Table 2. For both tables, we consider Call prices for the strikes $K \in \{80, 85, 90, 95, 100\}$ and Put prices for strikes K lying to $\{100, 105, 110, 115, 120\}$. We also depict in Table 1 the associated relative error ($\text{Err.} = \text{Err.} = \frac{|\text{Price} - \text{MQ}_N^n|}{\text{Price}}$) between the benchmark prices (Price) and the prices obtained from the Markovian and product quantization method of size $N = N_1 = N_2$ with n discretization steps (denoted by MQ_N^n) and the computation time (in seconds) for the Markovian and product quantization method.

In Table 1, we set the number of time steps $n = 10$ and make the sizes N_1 et N_2 of the marginal quantizers varying whereas, for the results of Table 2, we set $N_1 = N_2 = 30$ and make varying the number n of time steps.

We verify, as expected, that increasing the size of the marginal quantizers lead to more precise results (see Theorem 3.4). However, this increases the computation time. On the other hand, it is also clear from Theorem 3.4 that fixing the marginal quantization size, the number n of the time steps increases the global quantization error. From the numerical results, the choice $N_1 = N_2 = 20$ and $n = 10$ seems to be a good compromise.

K	Price	MQ_{10}^{10}	Err. (%)	MQ_{20}^{10}	Err. (%)	MQ_{30}^{10}	Err. (%)
80	25.9491	25.4427	1.9516	25.8721	0.2966	25.9656	0.0636
85	22.4481	21.9007	2.4384	22.3543	0.4177	22.4532	0.0229
90	19.2736	18.6934	3.0101	19.1596	0.5915	19.2612	0.0645
95	16.4323	15.8139	3.7633	16.2935	0.8450	16.3964	0.2183
100	13.9197	13.2858	4.5541	13.7537	1.1929	13.8566	0.4535
100	9.9987	9.3727	6.2602	9.8406	1.5810	9.9435	0.5515
105	12.6050	11.9509	5.1895	12.4218	1.4536	12.5218	0.6603
110	15.5060	14.8264	4.3828	15.2981	1.3408	15.3965	0.7062
115	18.6768	18.0111	3.5646	18.4441	1.2461	18.5422	0.7204
120	22.0904	21.4207	3.0317	21.8432	1.1189	21.9345	0.7055
Time (s)		0.49		8.41		41.82	

Table 1: Prices of a Basket option. The prices correspond to Call prices for strikes $K \in \{80, 85, 90, 95, 100\}$ and to Put prices for strikes $K \in \{100, 105, 110, 115, 120\}$. The number of time steps $n = 10$. Size of the grids $N_1 = N_2 = N$. The error $\text{Err.} = \frac{|\text{Price} - \text{MQ}_N^n|}{\text{Price}}$ corresponds to the relative error between the benchmark price (Price) and Product Markovian quantization price MQ_N^n of size N with n discretization steps.

4.2 Pricing of a European option in the Heston model

In this example, we consider a European option with maturity T and strike K , in the Heston model, introduced in [13], where the stock price S and its stochastic variance V evolve following the dynamics

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^1 \\ dV_t = \kappa(\theta - V_t) dt + \rho \sigma \sqrt{V_t} dW_t^1 + \sqrt{1 - \rho^2} \sigma \sqrt{V_t} dW_t^2. \end{cases} \quad (53)$$

In the previous equation, the parameter r is the interest rate; $\kappa > 0$, is the rate at which V reverts to the long running average variance $\theta > 0$; the parameter $\sigma > 0$, is the volatility of the variance and $\rho \in [-1, 1]$, is the correlation term. In this case, the price of the call at time $t = 0$ reads

$$e^{-rT} \mathbb{E}[\max(S_T - K, 0)] = e^{-rT} \mathbb{E}H(X_T), \quad X = (S, V), \quad (54)$$

K	Price	MQ_{30}^{20}	Err. (%)	MQ_{30}^{30}	Err. (%)	MQ_{30}^{40}	Err. (%)
80	25.9491	25.8461	0.3969	25.7646	0.7112	25.6937	0.9844
85	22.4481	22.3335	0.5107	22.2473	0.8943	22.1731	1.2252
90	19.2736	19.1460	0.6618	19.0599	1.1086	18.9837	1.5041
95	16.4323	16.2916	0.8561	16.2074	1.3686	16.1310	1.8336
100	13.9197	13.7660	1.1041	13.6861	1.6781	13.6117	2.2130
100	9.9987	9.8490	1.4971	9.7677	2.3095	9.6926	3.0608
105	12.6050	12.4428	1.2869	12.3675	1.8842	12.2954	2.4566
110	15.5060	15.3328	1.1173	15.2642	1.5593	15.1966	1.9956
115	18.6768	18.4954	0.9714	18.4342	1.2990	18.3717	1.6337
120	22.0904	21.9045	0.8413	21.8506	1.0854	21.7931	1.3459
Time (s)		84.13		151.90		194.05	

Table 2: Prices of a Basket option. The prices correspond to Call prices for strikes $K \in \{80, 85, 90, 95, 100\}$ and to Put prices for strikes $K \in \{100, 105, 110, 115, 120\}$. Size of the grids $N_1 = N_2 = 30$ and the number of time steps $n \in 20, 30, 40$. The error $\text{Err.} = \frac{|\text{Price} - \text{MQ}_{30}^n|}{\text{Price}}$ corresponds to the relative error between the benchmark price (Price) and Product Markovian quantization price MQ_{30}^n of size N with n discretization steps.

where $H(x) = \max(x^1 - K, 0)$, for $x = (x^1, x^2) \in \mathbb{R}^2$.

Using the Markovian and product quantization method, the price of a call in the Heston model is approximated as

$$e^{-rT} \sum_{j_1=1}^{N_n^1} \max(s_n^{1j_1} - K, 0) \mathbb{P}(\hat{S}_n^1 = s_n^{1j_1}), \quad (55)$$

where $\mathbb{P}(\hat{S}_n^1 = s_n^{1j_1})$ is computed according to (35).

For the numerical experiments we will use the following parameters, obtained from a calibration on market prices:

$$r = 0.04, \quad \kappa = 2.3924, \quad \theta = 0.0929, \quad \sigma = 0.6903, \quad \rho = -0.82, \quad S_0 = 100, \quad V_0 = 0.0719, \quad T = 1.$$

The pricing of European options under local and stochastic volatility models using recursive quantization techniques has already been studied, see e.g. [19] and [4] for the local volatility case, and [5] for the stochastic volatility case. However, the method we present here is more general and is model free compared to [5] where the method depends on the structure of the model.

The benchmark price is obtained using a Fourier based approach like in [6], since the Heston model is affine and the characteristic function is known in closed form.

Due to the fact that the derivative only depends on the price and not on the variance, it seems reasonable to choose the marginal grid size N_1 greater than N_2 . To guarantee a good balance between precision and computational time, setting $N_1 = 2N_2$ seems to be a good trade-off. As for the previous example, we consider Call prices for the strikes $K \in \{80, 85, 90, 95, 100\}$ and Put prices for strikes K belonging to $\{100, 105, 110, 115, 120\}$ and depict in Table 3 and Table 4 the corresponding prices (by making varying even n or the sizes N_1 and N_2 of the quantizers) and the associated relative errors ($\text{Err.} = \frac{|\text{Price} - \text{MQ}_{N_1, N_2}^n|}{\text{Price}}$) between the Fourier prices (Price) and the prices obtained from the Markovian and product quantization method of size $N = N_1 \times N_2$ with n discretization steps (denoted by MQ_{N_1, N_2}^n) and the computation time (in seconds) for the Markovian and product quantization method.

The best choice from the point of view of accuracy and computational effort, is obtained by taking the size N_1 of the quantization of the price process S equal to 20, by taking $N_2 = 10$ for the variance

process V , and by setting $n = 20$.

K	Price	$MQ_{10,6}^{20}$	Err. (%)	$MQ_{20,10}^{20}$	Err. (%)	$MQ_{30,16}^{20}$	Err. (%)
80	26.3910	25.8790	1.9401	26.2684	0.4645	26.3705	0.0777
85	22.6069	22.0686	2.3813	22.4879	0.5264	22.5804	0.1174
90	19.0506	18.5191	2.7897	18.9360	0.6012	19.0478	0.0148
95	15.7524	15.2279	3.3295	15.6471	0.6681	15.7634	0.0700
100	12.7422	12.2750	3.6668	12.6532	0.6987	12.7560	0.1083
100	8.8212	8.3579	5.2515	8.7361	0.9639	8.8390	0.2017
105	10.9308	10.4676	4.2371	10.8690	0.5655	10.9861	0.5056
110	13.3794	13.0481	2.4763	13.3460	0.2497	13.4429	0.4741
115	16.1828	15.8037	2.3425	16.1796	0.0201	16.2862	0.6389
120	19.3456	19.1549	0.9855	19.3697	0.1250	19.4603	0.5929
Time (s)		0.52		4.24		24.82	

Table 3: Prices in the Heston model. The prices correspond to Call prices for strikes $K \in \{80, 85, 90, 95, 100\}$ and to Put prices for strikes $K \in \{100, 105, 110, 115, 120\}$. The number of time steps $n = 20$. The error $\text{Err.} = \frac{|\text{Price} - MQ_{N_1, N_2}^n|}{\text{Price}}$ corresponds to the relative error between the benchmark price (Price) and Product Markovian quantization price MQ_{N_1, N_2}^n of sizes $N = N_1 \times N_2$ with n discretization steps.

Strike	Price	$MQ_{20,10}^{10}$	Err. (%)	$MQ_{20,10}^{30}$	Err. (%)	$MQ_{20,10}^{40}$	Err. (%)
80	26.3910	26.4632	0.2737	26.1709	0.8339	26.0900	1.1406
85	22.6069	22.7001	0.4121	22.3799	1.0041	22.2960	1.3752
90	19.0506	19.1619	0.5841	18.8229	1.1954	18.7418	1.6207
95	15.7524	15.8851	0.8428	15.5342	1.3851	15.4616	1.8460
100	12.7422	12.9032	1.2635	12.5458	1.5415	12.4863	2.0083
100	8.8212	8.9902	1.9155	8.6274	2.1964	8.5673	2.8784
105	10.9308	11.1520	2.0235	10.7704	1.4670	10.7266	1.8678
110	13.3794	13.6555	2.0633	13.2609	0.8860	13.2347	1.0820
115	16.1828	16.5099	2.0212	16.1090	0.4562	16.0992	0.5165
120	19.3456	19.7143	1.9063	19.3123	0.1717	19.3157	0.1545
Time (s)		2.18		6.69		8.98	

Table 4: Prices in the Heston model. The prices correspond to Call prices for strikes $K \in \{80, 85, 90, 95, 100\}$ and to Put prices for strikes $K \in \{100, 105, 110, 115, 120\}$. The error $\text{Err.} = \frac{|\text{Price} - MQ_{N_1, N_2}^n|}{\text{Price}}$ corresponds to the relative error between the benchmark price (Price) and Product Markovian quantization price MQ_{N_1, N_2}^n of sizes $N_1 = 20, N_2 = 20$ with n discretization steps.

4.3 Approximation of BSDE

In this section, we consider a Markovian BSDE

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T], \quad (56)$$

where W is a q -dimensional Brownian motion, $(Z_t)_{t \in [0, T]}$ is a square integrable progressively measurable process taking values in \mathbb{R}^q , $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^q \rightarrow \mathbb{R}$. The terminal condition is of the form

$\xi = h(X_T)$, for a given Borel function $h : \mathbb{R}^d \rightarrow \mathbb{R}$, where X_T is the value at time T of a Brownian diffusion process $(X_t)_{t \geq 0}$, strong solution to the stochastic differential equation:

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad x \in \mathbb{R}^d. \quad (57)$$

As pointed out in the introduction, many (time) discretization schemes and several (spacial) numerical approximation method of the solution of such as BSDE are proposed in the literature (we refer for example to [1, 3, 7, 11, 14, 10, 2, 20]). Our aim in this section is to test the performances of our method to the numerical scheme proposed in [20]. To this end, we first show that the Markovian product quantization method allows us to compute the term appearing in the numerical schemes proposed in [20] (as well as for several numerical schemes) using (semi)-closed formula. We then test the performance of our method to a BSDE associated to the price of the Call option in the Black-Scholes model and to a multidimensional BSDE.

4.3.1 Explicit numerical scheme for the BSDE

Let us set for $i \in I_k, j \in I_{k+1}$,

$$p_k^i = \mathbb{P}(\hat{X}_k = x_k^i), \quad k = 0, \dots, n$$

and $p_k^{ij} = \mathbb{P}(\hat{X}_{k+1} = x_{k+1}^j | \hat{X}_k = x_k^i), \quad k = 0, \dots, n-1.$

Setting $\hat{Y}_k = \hat{y}_k(\hat{X}_k)$, for every $k \in \{0, \dots, n\}$, the quantized BSDE scheme reads as

$$\begin{cases} \hat{y}_n(x_n^i) = h(x_n^i) & x_n^i \in \Gamma_n \\ \hat{y}_k(x_k^i) = \hat{\alpha}_k(x_k^i) + \Delta_n f(t_k, x_k^i, \hat{\alpha}_k(x_k^i), \hat{\beta}_k(x_k^i)) & x_k^i \in \Gamma_k \end{cases}$$

where for $k = 0, \dots, n-1$,

$$\hat{\alpha}_k(x_k^i) = \sum_{j \in I_{k+1}} \hat{y}_{k+1}(x_{k+1}^j) p_k^{ij} \quad \text{and} \quad \hat{\beta}_k(x_k^i) = \frac{1}{\sqrt{\Delta_n}} \sum_{j \in I_{k+1}} \hat{y}_{k+1}(x_{k+1}^j) \Lambda_k^{ij}, \quad (58)$$

with

$$\Lambda_k^{ij} = \mathbb{E}(Z_{k+1} \mathbb{1}_{\{\hat{X}_{k+1} = x_{k+1}^j\}} | \hat{X}_k = x_k^i).$$

In the following, we give closed formula for the Λ_k^{ij} 's. We will first suppose that the components of \bar{X}_k are independent, for every $k = 0, \dots, n$.

Proposition 4.1. *Suppose that $q = d$ and $\mathcal{E}_k^\ell(x, Z_{k+1}) = \mathcal{E}_k^\ell(x, Z_{k+1}^\ell)$, for every $\ell \in \{1, \dots, d\}$ and $x \in \mathbb{R}^d$. Then*

$$\Lambda_k^{ij, \ell} = (\Phi_0'(x_{k+1}^{\ell, j_{\ell}^-}(x_k^i)) - \Phi_0'(x_{k+1}^{\ell, j_{\ell}^+}(x_k^i))) \prod_{\ell' \neq \ell}^d [\Phi_0(x_{k+1}^{\ell', j_{\ell'}^+}(x_k^i)) - \Phi_0(x_{k+1}^{\ell', j_{\ell'}^-}(x_k^i))] \quad (59)$$

Proof. Let us set $v^{\ell, j_{\ell}^+} := x_{k+1}^{\ell, j_{\ell}^+ + 1/2}$ and $v^{\ell, j_{\ell}^-} = x_{k+1}^{\ell, j_{\ell}^- - 1/2}$. We have

$$\begin{aligned} \Lambda_k^{ij} &= \mathbb{E}(Z_{k+1} \mathbb{1}_{\{\mathcal{E}_k(\hat{X}_k, Z_{k+1}) \in C_j(\Gamma_{k+1})\}} | \hat{X}_k = x_k^i) \\ &= \mathbb{E}(Z_{k+1} \mathbb{1}_{\{\mathcal{E}_k(x_k^i, Z_{k+1}) \in C_j(\Gamma_{k+1})\}}) \\ &= \mathbb{E}\left(Z_{k+1} \prod_{\ell'=1}^d \mathbb{1}_{\{\mathcal{E}_k^{\ell'}(x_k^i, Z_{k+1}^{\ell'}) \in (v^{\ell', j_{\ell'}^-}, v^{\ell', j_{\ell'}^+})\}}\right). \end{aligned}$$

Since the components of Z_{k+1} are independent, it follows that for $\ell = 1, \dots, d$, the component (i_ℓ, j) of Λ_k^{ij} reads

$$\begin{aligned}\Lambda_k^{ij,\ell} &= \mathbb{E} \left(Z_{k+1}^\ell \prod_{\ell'=1}^d \mathbb{1}_{\{\mathcal{E}_k^{\ell'}(x_k^i, Z_{k+1}^{\ell'}) \in (v^{\ell',j_{\ell'}-}, v^{\ell',j_{\ell'}+})\}} \right) \\ &= \mathbb{E} \left(Z_{k+1}^\ell \mathbb{1}_{\{\mathcal{E}_k^\ell(x_k^i, Z_{k+1}^\ell) \in (v^{\ell,j_\ell-}, v^{\ell,j_\ell+})\}} \right) \times \mathbb{E} \left(\prod_{\ell' \neq \ell}^d \mathbb{1}_{\{\mathcal{E}_k^{\ell'}(x_k^i, Z_{k+1}^{\ell'}) \in (v^{\ell',j_{\ell'}-}, v^{\ell',j_{\ell'}+})\}} \right).\end{aligned}$$

It is clear that

$$\mathbb{E} \left(\prod_{\ell' \neq \ell}^d \mathbb{1}_{\{\mathcal{E}_k^{\ell'}(x_k^i, Z_{k+1}^{\ell'}) \in (v^{\ell',j_{\ell'}-}, v^{\ell',j_{\ell'}+})\}} \right) = \prod_{\ell' \neq \ell}^d [\Phi_0(x_{k+1}^{\ell',j_{\ell'}+}(x_k^i, 0)) - \Phi_0(x_{k+1}^{\ell',j_{\ell'}-}(x_k^i, 0))].$$

On this other hand,

$$\mathbb{E} \left(Z_{k+1}^\ell \mathbb{1}_{\{\mathcal{E}_k^\ell(x_k^i, Z_{k+1}^\ell) \in (v^{\ell,j_\ell-}, v^{\ell,j_\ell+})\}} \right) = \Phi_0'(x_{k+1}^{\ell,j_\ell-}(x_k^i, 0)) - \Phi_0'(x_{k+1}^{\ell,j_\ell+}(x_k^i, 0)).$$

Combining both previous equalities gives the announced result. \square

In the following, we compute the p -th component $\Lambda_k^{ij,p}$ of Λ_k^{ij} in a general setting. Let us set

$$\mathbb{J}_{k,j_\ell}^{0,p}(x) = \left\{ z \in \mathbb{R}, \quad \sqrt{\Delta} \sigma_k^{\ell p}(x) z \in (x_{k+1}^{\ell,j_\ell-1/2} - m_k^\ell(x), x_{k+1}^{\ell,j_\ell+1/2} - m_k^\ell(x)) \right\}$$

and

$$\mathbb{L}_k^{0,p}(x) = \left\{ \ell \in \{1, \dots, d\}, \quad \sum_{p' \neq p} (\sigma_k^{\ell p'}(x))^2 = 0 \right\}.$$

We also set

$$x_{k+1}^{\ell,p,j_\ell-}(x, z) = \frac{x_{k+1}^{\ell,j_\ell-1/2} - m_k^\ell(x) - \sqrt{\Delta} \sigma_k^{\ell p}(x) z}{\sqrt{\Delta} \left(\sum_{p' \neq p} (\sigma_k^{\ell p'}(x))^2 \right)^{1/2}}; \quad x_{k+1}^{\ell,p,j_\ell+}(x, z) = \frac{x_{k+1}^{\ell,j_\ell+1/2} - m_k^\ell(x) - \sqrt{\Delta} \sigma_k^{\ell p}(x) z}{\sqrt{\Delta} \left(\sum_{p' \neq p} (\sigma_k^{\ell p'}(x))^2 \right)^{1/2}}.$$

Proposition 4.2. *For every $p \in \{1, \dots, q\}$, the p -th component $\Lambda_k^{ij,p}$ of Λ_k^{ij} reads*

$$\Lambda_k^{ij,p} = \mathbb{E} \zeta \prod_{\ell \in \mathbb{L}_k^{0,p}(x_k^i)} \mathbb{1}_{\{\zeta \in \mathbb{J}_{k,j_\ell}^{0,p}(x_k^i)\}} (\Phi_0(\alpha_j^p(x_k^i, \zeta)) - \Phi_0(\beta_j^p(x_k^i, \zeta)))^+, \quad \zeta \sim \mathcal{N}(0; 1) \quad (60)$$

(convention: $\prod_{\ell \in \emptyset} (\cdot) = 1$) where for every $x \in \mathbb{R}^d$ and $z \in \mathbb{R}$,

$$\alpha_j^p(x, z) = \sup_{\ell \in (\mathbb{L}_k^{0,p}(x))^c} x_{k+1}^{\ell,p,j_\ell-}(x, z) \quad \text{and} \quad \beta_j^p(x, z) = \inf_{\ell \in (\mathbb{L}_k^{0,p}(x))^c} x_{k+1}^{\ell,p,j_\ell+}(x, z). \quad (61)$$

In particular, if $p \in \{1, \dots, q\}$ and if for every $\ell \in \{1, \dots, d\}$ there exists $p' \neq p$ such that $\sigma_k^{\ell p'}(x) \neq 0$, then,

$$\Lambda_k^{ij,p} = \mathbb{E} \zeta (\Phi_0(\alpha_j^p(x_k^i, \zeta)) - \Phi_0(\beta_j^p(x_k^i, \zeta)))^+. \quad (62)$$

Proof. Let us set $v^{\ell,j_\ell+} := x_{k+1}^{\ell,j_\ell+1/2}$ and $v^{\ell,j_\ell-} = x_{k+1}^{\ell,j_\ell-1/2}$. We have

$$\begin{aligned}\Lambda_k^{ij} &= \mathbb{E} (Z_{k+1} \mathbb{1}_{\{\mathcal{E}_k(\hat{X}_k, Z_{k+1}) \in C_j(\Gamma_{k+1})\}} | \hat{X}_k = x_k^i) \\ &= \mathbb{E} (Z_{k+1} \mathbb{1}_{\{\mathcal{E}_k(x_k^i, Z_{k+1}) \in C_j(\Gamma_{k+1})\}}) \\ &= \mathbb{E} (Z_{k+1} \mathbb{1}_{A_j^i})\end{aligned}$$

where

$$\begin{aligned} A_j^i &= \bigcap_{\ell=1}^d \{ \mathcal{E}_k^\ell(x_k^i, Z_{k+1}) \in (v^{\ell, j_{\ell}^-}, v^{\ell, j_{\ell}^+}) \} \\ &= \bigcap_{\ell=1}^d \{ m_k^\ell(x_k^i) + \sqrt{\Delta} \sigma_k^{\ell p}(x_k^i) Z_{k+1}^p + \sum_{p' \neq p} \sqrt{\Delta} \sigma_k^{\ell p'}(x_k^i) Z_{k+1}^{p'} \in (v^{\ell, j_{\ell}^-}, v^{\ell, j_{\ell}^+}) \}. \end{aligned}$$

Then, conditioning by Z_{k+1}^p shows that the component $\Lambda_k^{ij,p}$ of Λ_k^{ij} reads

$$\Lambda_k^{ij,p} = \mathbb{E}(\mathbb{E}(Z_{k+1}^p \mathbf{1}_{A_{j_\ell}^i}) | Z_{k+1}^p) = \mathbb{E}(\Psi(\zeta)), \quad \zeta \sim \mathcal{N}(0, 1),$$

where for every u ,

$$\Psi(u) = u \mathbb{P}\left(\bigcap_{\ell=1}^d A_{\ell,k}^p(u)\right)$$

with

$$A_{\ell,k}^p(u) = \{ m_k^\ell(x_k^i) + \sqrt{\Delta} \sigma_k^{\ell p}(x_k^i) u + \sum_{p' \neq p} \sqrt{\Delta} \sigma_k^{\ell p'}(x_k^i) Z_{k+1}^{p'} \in (v^{\ell, j_{\ell}^-}, v^{\ell, j_{\ell}^+}) \}.$$

Keep in mind that

$$\sum_{p' \neq p} \sqrt{\Delta} \sigma_k^{\ell p'}(x_k^i) Z_{k+1}^{p'} \stackrel{\mathcal{L}}{=} \left(\Delta \sum_{p' \neq p} (\sigma_k^{\ell p'}(x_k^i))^2 \right)^{1/2} Z, \quad Z \sim \mathcal{N}(0; 1).$$

Then, we may write

$$\Psi(u) = u \mathbf{1}_{\{ \bigcap_{\ell_0 \in \mathbb{L}_k^{0,p}(x_k^i)} A_{\ell_0,k}^p(u) \}} \mathbb{P}\left(\bigcap_{\ell_+ \in (\mathbb{L}_k^{0,p}(x_k^i))^c} A_{\ell_+,k}^p(u)\right)$$

with

$$A_{\ell_+,k}^p(u) = \{ Z \in (x_{k+1}^{\ell_+, p, j_{\ell_+}^-}(x_k^i, u), x_{k+1}^{\ell_+, p, j_{\ell_+}^+}(x_k^i, u)) \}, \quad Z \sim \mathcal{N}(0; 1).$$

It follows that

$$\begin{aligned} \Psi(u) &= u \mathbf{1}_{\{ \bigcap_{\ell_0 \in \mathbb{L}_k^{0,p}(x_k^i)} A_{\ell_0,k}^p(u) \}} \mathbb{P}\left(\bigcap_{\ell_+ \in (\mathbb{L}_k^{0,p}(x_k^i))^c} A_{\ell_+,k}^p(u)\right) \\ &= u \prod_{\ell \in \mathbb{L}_k^{0,p}(x_k^i)} \mathbf{1}_{\{ u \in \mathbb{J}_{k, j_\ell}^{0,p}(x_k^i) \}} \mathbb{P}(Z \in \alpha_j^p(x_k^i, u), \beta_j^p(x_k^i, u)). \end{aligned}$$

The result follows immediately. \square

4.3.2 Pricing a risk neutral Black-Scholes Call under the historical probability

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We consider a call option with maturity T and strike K on a stock price $(X_t)_{t \in [0, T]}$ with dynamics

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

Considering a self financing portfolio Y_t with φ_t assets and bonds with risk free return r . We know that (see [8]) the portfolio evolves according to the following dynamics:

$$Y_t = Y_T + \int_t^T f(Y_s, Z_s) ds - \int_t^T Z_s dW_s \quad (63)$$

where the payoff $Y_T = (X_T - K)^+$, the hedging strategy $Z_t = \sigma \varphi_t X_t$ and $f(y, z) = -ry - \frac{\mu-r}{\sigma}z$. It is clear that the function f is linear with respect to y and z and, it is Lipschitz continuous with $[f]_{\text{Lip}} = \max(r, \frac{\mu-r}{\sigma})$. We perform the numerical tests from the algorithm we propose with the following parameters

$$X_0 = 100, \quad r = 0.1, \quad \mu = 0.2, \quad K = 100, \quad T = 0.5$$

and make varying the volatility σ .

σ	$\hat{Y}_0 (n = 20)$	$\hat{Y}_0 (n = 40)$	Y_0	$\hat{Z}_0 (n = 20)$	$\hat{Z}_0 (n = 40)$	Z_0
0.05	04.97	05.01	05.00	04.67	04.58	04.62
0.07	05.23	05.26	05.27	06.04	05.95	05.95
0.10	05.81	05.84	05.85	07.83	07.72	07.71
0.30	10.88	10.89	10.91	19.00	18.91	19.01
0.40	13.56	13.56	13.58	24.91	24.82	24.99
0.50	16.26	16.25	16.26	31.07	30.98	31.24

Table 5: Call price in the BS model: $N_k = 100, \forall k = 1, \dots, n; n \in \{20, 40\}$. Computational time: < 1 second for $n = 20$ and around 1 second for $n = 40$.

4.3.3 Multidimensional example

We consider the following example due to J.-F. Chassagneux: let $t \in [0, T]$. Set

$$e_t = \exp(W_t^1 + \dots + W_t^d + t)$$

where W is a d -dimensional Brownian motion. Consider the following BSDE:

$$dX_t = dW_t, \quad -dY_t = f(t, Y_t, Z_t)dt - Z_t \cdot dW_t, \quad Y_T = \frac{e_T}{1 + e_T},$$

where $f(t, y, z) = (z_1 + \dots + z_d)(y - \frac{2+d}{2d})$. The solution of this BSDE is given by

$$Y_t = \frac{e_t}{1 + e_t}, \quad Z_t = \frac{e_t}{(1 + e_t)^2}. \quad (64)$$

For the numerical experiments, we put the (regular) time discretization mesh to $n = 20$, with discretization step Δ . We use the uniform dispatching grid allocation and define the quantization $(\hat{W}_{t_k})_{0 \leq k \leq n}$ of the Brownian trajectories $(W_{t_k})_{0 \leq k \leq n}$ from the following recursive procedure

$$\hat{W}_{t_{k+1}} = \hat{W}_{t_k} + \sqrt{\Delta} \hat{\varepsilon}, \quad (65)$$

$\hat{W}_0 = 0$ and where $\hat{\varepsilon}$ is the optimal quantization of the d -dimensional standard Gaussian random variable. We choose $t = 0.5, d = 2, 3$, so that $Y_0 = 0.5$ and $Z_0^i = 0.25$, for every $i = 1, \dots, d$.

Using the Markovian product quantization method we get

1. for $d = 2$, with $N_1 = N_2 = 30$: $\hat{Y}_0 = 0.504, \hat{Z}_0^1 = \hat{Z}_0^2 = 0.24$. The computation time is around 4 seconds.
2. for $d = 3$, with $N_1 = N_2 = N_3 = 15$: $\hat{Y}_0 = 0.547, \hat{Z}_0^1 = \hat{Z}_0^2 = \hat{Z}_0^3 = 0.22$. The computation time is around 1 minute.

Remark that the only reason motivating the choice of this example is the fact that the considered backward has an explicit solution. Nevertheless our method works for a general local volatility diffusion process X . We also note that when choosing the same size N_i for all marginal quantizers, the complexity of the algorithm is equal to N_i^d . This prevents us from going beyond the dimension 3 without increasing significantly the computation time. One way of reducing the computation time in dimension $d \geq 4$ may be to use the parallel computing.

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