The inclusion of configuration spaces of surfaces in Cartesian products, its induced homomorphism, and the virtual cohomological dimension of the braid groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$

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Abstract

Let S be a surface, perhaps with boundary, and either compact, or with a finite number of points removed from the interior of the surface. We consider the inclusion $\iota\colon\ F_n(S) \longrightarrow \prod_1^n S$ of the n^{th} configuration space $F_n(S)$ of S into the n-fold Cartesian product of S, as well as the induced homomorphism $\iota_\#\colon\ P_n(S) \longrightarrow \prod_1^n \pi_1(S)$, where $P_n(S)$ is the n-string pure braid group of S. Both ι and $\iota_\#$ were studied initially by J. Birman who conjectured that Ker $(\iota_\#)$ is equal to the normal closure of the Artin pure braid group P_n in $P_n(S)$. The conjecture was later proved by C. Goldberg for compact surfaces without boundary different from the 2-sphere $\mathbb{S}^2$ and the projective plane $\mathbb{R}P^2$. In this paper, we prove the conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$. In the case of $\mathbb{R}P^2$, we prove that Ker $(\iota_\#)$ is equal to the commutator subgroup of $P_n(\mathbb{R}P^2)$, we show that it may be decomposed in a manner similar to that of $P_n(\mathbb{S}^2)$ as a direct sum of a torsion-free subgroup L_n and the finite cyclic group generated by the full twist braid, and we prove that L_n may be written as an iterated semi-direct product of free groups. Finally, we show that the groups $B_n(\mathbb{S}^2)$ and $P_n(\mathbb{S}^2)$ (resp. $B_n(\mathbb{R}P^2)$ and $P_n(\mathbb{R}P^2)$) have finite virtual cohomological dimension equal to $n-3$ (resp. $n-2$), where $B_n(S)$ denotes the full n-string braid group of S. This allows us to determine the virtual cohomological

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dimension of the mapping class groups of the mapping class groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$ with marked points, which in the case of $\mathbb{S}^2$, reproves a result due to J. Harer.

1 Introduction

Let S be a connected surface, perhaps with boundary, and either compact, or with a finite number of points removed from the interior of the surface. The n^{th} *configuration space* of S is defined by:

$$F_n(S) = \left\{(x_1, \ldots, x_n) \in S^n \;\middle|\; x_i \neq x_j \text{ if } i \neq j\right\}.$$

It is well known that $\pi_1(F_n(S)) \cong P_n(S)$, the *pure braid group* of S on n strings, and that $\pi_1(F_n(S)/S_n) \cong B_n(S)$, the *braid group* of S on n strings, where $F_n(S)/S_n$ is the quotient space of $F_n(S)$ by the free action of the symmetric group S_n given by permuting coordinates [FaN, FoN]. If S is the 2-disc $\mathbb{D}^2$ then $B_n(\mathbb{D}^2)$ (resp. $P_n(\mathbb{D}^2)$) is the Artin braid group B_n (resp. the Artin pure braid group P_n). The canonical projection $F_n(S) \longrightarrow F_n(S)/S_n$ is a regular $n!$-fold covering map, and thus gives rise to the following short exact sequence:

$$1 \longrightarrow P_n(S) \longrightarrow B_n(S) \longrightarrow S_n \longrightarrow 1. \tag{1}$$

If $\mathbb{D}^2$ is a topological disc lying in the interior of S and that contains the basepoints of the braids then the inclusion $j\colon\ \mathbb{D}^2 \longrightarrow S$ induces a group homomorphism $j_\#\colon\ B_n \longrightarrow B_n(S)$. This homomorphism is injective if S is different from the 2-sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{R}P^2$ [Bi1, G]. Let $j_\#|_{P_n} : P_n \longrightarrow P_n(S)$ denote the restriction of $j_\#$ to the corresponding pure braid groups. If $\beta \in B_n$ then we shall denote its image $j_\#(\beta)$ in $B_n(S)$ simply by β. It is well known that the centre of B_n and of P_n is infinite cyclic, generated by the full twist braid that we denote by Δ_n^2, and that Δ_n^2, considered as an element of $B_n(\mathbb{S}^2)$ or of $B_n(\mathbb{R}P^2)$, is of order 2 and generates the centre. If G is a group then we denote its commutator subgroup by $\Gamma_2(G)$, its Abelianisation by G^{Ab}, and if H is a subgroup of G then we denote its normal closure in G by $\langle\langle H \rangle\rangle_G$.

Let $\prod_1^n S = S \times \cdots \times S$ denote the n-fold Cartesian product of S with itself, let $\iota_n\colon\ F_n(S) \longrightarrow \prod_1^n S$ be the inclusion map, and let $\iota_{n\#}\colon\ \pi_1(F_n(S)) \longrightarrow \pi_1\left(\prod_1^n S\right)$ denote the induced homomorphism on the level of fundamental groups. To simplify the notation, we shall often just write ι and $\iota_\#$ if n is given. The study of $\iota_\#$ was initiated by Birman in 1969 [Bi1]. She had conjectured that $\left\langle\left\langle \mathrm{Im}\left(j_\#|_{P_n}\right)\right\rangle\right\rangle_{P_n(S)} = \mathrm{Ker}\left(\iota_\#\right)$ if S is a compact orientable surface, but states without proof that her conjecture is false if S is of genus greater than or equal to 1 [Bi1, page 45]. However, Goldberg proved the conjecture several years later in both the orientable and non-orientable cases for compact surfaces without boundary different from $\mathbb{S}^2$ and $\mathbb{R}P^2$ [G, Theorem 1]. In connection with the study of Vassiliev invariants of surface braid groups, Gonzlez-Meneses and Paris showed that $\mathrm{Ker}\left(\iota_\#\right)$ is also normal in $B_n(S)$, and that the resulting quotient is isomorphic to the semi-direct product $\pi_1\left(\prod_1^n S\right) \rtimes S_n$, where the action is given by permuting coordinates (their work was within the framework of compact, orientable surfaces without boundary, but their construction is valid for any surface S) [GMP]. In the case of $\mathbb{R}P^2$, this result was reproved using geometric methods [T].

If $S = \mathbb{S}^2$, $\mathrm{Ker}\,(\iota_\#)$ is clearly equal to $P_n(\mathbb{S}^2)$, and so by [GG1, Theorem 4], it may be decomposed as:

$$\mathrm{Ker}\,(\iota_\#) = P_n(\mathbb{S}^2) \cong P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \times \mathbb{Z}_2, \tag{2}$$

where the first factor of the direct product is torsion free, and the $\mathbb{Z}_2$-factor is generated by Δ_n^2.

The aim of this paper is to resolve Birman's conjecture for surfaces without boundary in the remaining cases, namely $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, to determine the cohomological dimension of $B_n(S)$ and $P_n(S)$, where S is one of these two surfaces, and to elucidate the structure of $\mathrm{Ker}\,(\iota_\#)$ in the case of $\mathbb{R}P^2$. In Section 2, we start by considering the case $S = \mathbb{R}P^2$, we study $\mathrm{Ker}\,(\iota_\#)$, which we denote by K_n, and we show that it admits a decomposition similar to that of equation (2).

PROPOSITION 1. *Let $n \in \mathbb{N}$.*
(a) (i) Up to isomorphism, the homomorphism $\iota_\#\colon \pi_1(F_n(\mathbb{R}P^2)) \longrightarrow \pi_1(\Pi_1^n(\mathbb{R}P^2))$ coincides with Abelianisation. In particular, $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$.
(ii) If $n \geqslant 2$ then there exists a torsion-free subgroup L_n of K_n such that K_n is isomorphic to the direct sum of L_n and the subgroup $\langle \Delta_n^2 \rangle$ generated by the full twist that is isomorphic to $\mathbb{Z}_2$.
(b) If $n \geqslant 2$ then any subgroup of $P_n(\mathbb{R}P^2)$ that is normal in $B_n(\mathbb{R}P^2)$ and that properly contains K_n possesses an element of order 4.

Note that if $n = 1$ then $B_1(\mathbb{R}P^2) = P_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ and Δ_1^2 is the trivial element, so parts (a)(ii) and (b) do not hold. Part (a)(i) will be proved in Proposition 8. We shall see later on in Remark 14 that there are precisely $2^{n(n-2)}$ subgroups that satisfy the conclusions of part (a)(ii), and to prove the statement, we shall exhibit an explicit torsion-free subgroup L_n. We then prove Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$, using Proposition 1(a)(i) in the case of $\mathbb{R}P^2$.

THEOREM 2. *Let S be one of $\mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \geqslant 1$. Then $\langle\langle \mathrm{Im}\,(j_\#|_{P_n}) \rangle\rangle_{P_n(S)} = \mathrm{Ker}\,(\iota_\#)$.*

In Section 3, we analyse L_n in more detail, and we show that it may be decomposed as an iterated semi-direct product of free groups.

THEOREM 3. *Let $n \geqslant 3$. Consider the Fadell-Neuwirth short exact sequence:*

$$1 \longrightarrow P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_n(\mathbb{R}P^2) \xrightarrow{q_{2\#}} P_2(\mathbb{R}P^2) \longrightarrow 1, \tag{3}$$

where $q_{2\#}$ is given geometrically by forgetting the last $n-2$ strings. Then L_n may be identified with the kernel of the composition

$$P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_n(\mathbb{R}P^2) \xrightarrow{\iota_\#} \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ copies}},$$

where the first homomorphism is that appearing in equation (3). The image of this composition is the product of the last $n-2$ copies of $\mathbb{Z}_2$. In particular, L_n is of index 2^{n-2} in $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Further, L_n is isomorphic to an iterated semi-direct product of free groups of the form $\mathbb{F}_{2n-3} \rtimes (\mathbb{F}_{2n-5} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3) \cdots))$, where for all $m \in \mathbb{N}$, $\mathbb{F}_m$ denotes the free group of rank m.

In the semi-direct product decomposition of L_n, note that every factor acts on each of the preceding factors. This is also the case for $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ (see equation (12)), and as we shall see in Remarks 13(a), this implies an Artin combing-type result for this group. Analysing these semi-direct products in more detail, we obtain the following results.

PROPOSITION 4. *If $n \geqslant 3$ then:*
(a) $\big(P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})\big)^{Ab} \cong \mathbb{Z}^{2(n-2)}$.
(b) $(L_n)^{Ab} \cong \mathbb{Z}^{n(n-2)}$.

In two papers in preparation, we shall analyse the homotopy fibre of ι, as well as the induced homomorphism $\iota_\#$ when $S = \mathbb{S}^2$ or $\mathbb{R}P^2$ [GG9], and when S is a space form manifold of dimension different from two [GGG]. In the first of these papers, we shall also see that L_n is closely related to the fundamental group of an orbit configuration space of the open cylinder.

In Section 4, we study the virtual cohomological dimension of the braid groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$. Recall from [Br, page 226] that if a group Γ is virtually torsion-free then all finite index torsion-free subgroups of Γ have the same cohomological dimension by Serre's theorem, and this dimension is defined to be the *virtual cohomological dimension* of Γ. Using equations (2) and (3), we prove the following result, namely that if $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, the groups $B_n(S)$ and $P_n(S)$ have finite virtual cohomological dimension, and we compute these dimensions.

THEOREM 5.
(a) Let $n \geqslant 4$. Then the virtual cohomological dimension of both $B_n(\mathbb{S}^2)$ and $P_n(\mathbb{S}^2)$ is equal to the cohomological dimension of the group $P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$. Furthermore, for all $m \geqslant 1$, the cohomological dimension of the group $P_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ is equal to m.
(b) Let $n \geqslant 3$. Then the virtual cohomological dimension of both $B_n(\mathbb{R}P^2)$ and $P_n(\mathbb{R}P^2)$ is equal to the cohomological dimension of the group $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Furthermore, for all $m \geqslant 1$, the cohomological dimension of the group $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ is equal to m.

The methods of the proof of Theorem 5 have recently been applied to compute the cohomological dimension of the braid groups of all other compact surfaces (orientable and non orientable) without boundary [GGM]. Theorem 5 also allows us to deduce the virtual cohomological dimension of the punctured mapping class groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$. If $n \geqslant 0$, let $\mathcal{MCG}(S, n)$ denote the mapping class group of a connected, compact surface S relative to an n-point set. If S is orientable then Harer determined the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ [H, Theorem 4.1]. In the case of $\mathbb{S}^2$ and $\mathbb{D}^2$, he obtained the following results:

(a) if $n \geqslant 3$, the virtual cohomological dimension of $\mathcal{MCG}(\mathbb{S}^2, n)$ is equal to $n - 3$.
(b) if $n \geqslant 2$, the cohomological dimension of $\mathcal{MCG}(\mathbb{D}^2, n)$ is equal to $n - 1$ (recall that $\mathcal{MCG}(\mathbb{D}^2, n)$ is isomorphic to B_n [Bi2]).

As a consequence of Theorem 5, we are able to compute the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ for $S = \mathbb{S}^2$ and $\mathbb{R}P^2$.

COROLLARY 6. *Let $n \geqslant 4$ (resp. $n \geqslant 3$). Then the virtual cohomological dimension of $\mathcal{MCG}(\mathbb{S}^2, n)$ (resp. $\mathcal{MCG}(\mathbb{R}P^2, n)$) is finite, and is equal to $n - 3$ (resp. $n - 2$).*

If $S = \mathbb{S}^2$ or $\mathbb{R}P^2$ then for the values of n given by Theorem 5 and Corollary 6, the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ is equal to that of $B_n(S)$. If $S = \mathbb{S}^2$, we thus recover the corresponding result of Harer.

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2 The structure of K_n, and Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$

Let $n \in \mathbb{N}$. As we mentioned in the introduction, if S is a surface different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, the kernel of the homomorphism $\iota_\#\colon P_n(S) \longrightarrow \pi_1\left(\prod_1^n S\right)$ was studied in [Bi1, G], and that if $S = \mathbb{S}^2$ then $\mathrm{Ker}\,(\iota_\#) = P_n(\mathbb{S}^2)$. In the first part of this section, we recall a presentation of $P_n(\mathbb{R}P^2)$, and we prove Proposition 1(a)(i). The second part of this section is devoted to proving the rest of Proposition 1 and Theorem 2, the latter being Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$.

Consider the model of $\mathbb{R}P^2$ given by identifying antipodal boundary points of $\mathbb{D}^2$. We equip $F_n(\mathbb{R}P^2)$ with a basepoint $(x_1, \ldots, x_n)$. For $1 \leqslant i < j \leqslant n$ (resp. $1 \leqslant k \leqslant n$), we define the element $A_{i,j}$ (resp. τ_k, ρ_k) of $P_n(\mathbb{R}P^2)$ by the geometric braids depicted in Figure 1. Note that the arcs represent the projections of the strings onto $\mathbb{R}P^2$, so that

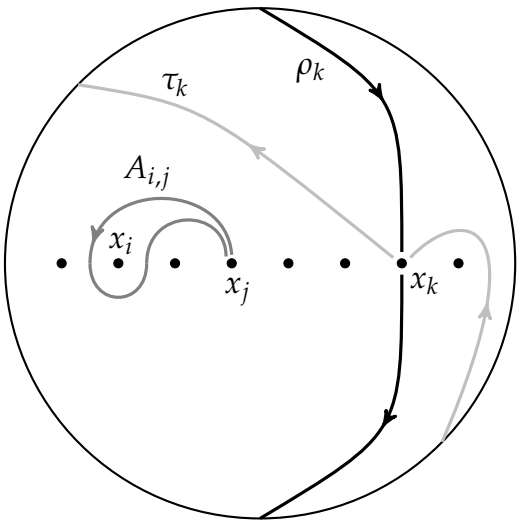

Figure 1: The elements $A_{i,j}$, τ_k and ρ_k of $P_n(\mathbb{R}P^2)$.

all of the strings of the given braid are vertical, with the exception of the j^{th} (resp. k^{th}) string that is based at the point x_j (resp. x_k).

THEOREM 7 ([GG4, Theorem 4]). *Let $n \in \mathbb{N}$. The following constitutes a presentation of the pure braid group $P_n(\mathbb{R}P^2)$:*

generators: *$A_{i,j}$, $1 \leqslant i < j \leqslant n$, and τ_k, $1 \leqslant k \leqslant n$.*

relations:

(a) the Artin relations between the $A_{i,j}$ emanating from those of P_n:

$$A_{r,s}A_{i,j}A_{r,s}^{-1} = \begin{cases} A_{i,j} & \text{if } i < r < s < j \text{ or } r < s < i < j \\ A_{i,j}^{-1}A_{r,j}^{-1}A_{i,j}A_{r,j}A_{i,j} & \text{if } r < i = s < j \\ A_{s,j}^{-1}A_{i,j}A_{s,j} & \text{if } i = r < s < j \\ A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{i,j}A_{r,j}^{-1}A_{s,j}^{-1}A_{r,j}A_{s,j} & \text{if } r < i < s < j. \end{cases} \tag{4}$$

(b) for all $1 \leqslant i < j \leqslant n$, $\tau_i\tau_j\tau_i^{-1} = \tau_j^{-1}A_{i,j}^{-1}\tau_j^2$.

(c) for all $1 \leqslant i \leqslant n$, $\tau_i^2 = A_{1,i}\cdots A_{i-1,i}A_{i,i+1}\cdots A_{i,n}$.

(d) for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant k \leqslant n$, $k \neq j$,

$$\tau_k A_{i,j}\tau_k^{-1} = \begin{cases} A_{i,j} & \text{if } j < k \text{ or } k < i \\ \tau_j^{-1}A_{i,j}^{-1}\tau_j & \text{if } k = i \\ \tau_j^{-1}A_{k,j}^{-1}\tau_j A_{k,j}^{-1}A_{i,j}A_{k,j}\tau_j^{-1}A_{k,j}\tau_j & \text{if } i < k < j. \end{cases}$$

This enables us to prove that $\iota_\#$ is in fact Abelianisation, which is part (a)(i) of Proposition 1.

PROPOSITION 8. *Let $n \in \mathbb{N}$. The homomorphism $\iota_\#\colon P_n(\mathbb{R}P^2) \longrightarrow \pi_1(\prod_1^n \mathbb{R}P^2)$ is defined on the generators of Theorem 7 by $\iota_\#(A_{i,j}) = (\overline{0},\ldots,\overline{0})$ for all $1 \leqslant i < j \leqslant n$, and $\iota_\#(\tau_k) = (\overline{0},\ldots,\overline{0},\underbrace{\overline{1}}_{k^{th}\text{ position}},\overline{0},\ldots,\overline{0})$ for all $1 \leqslant k \leqslant n$. Further, $\iota_\#$ is Abelianisation, and $Ker\,(\iota_\#) = K_n = \Gamma_2(P_n(\mathbb{R}P^2))$.*

Proof. For $1 \leqslant k \leqslant n$, let $p_k\colon F_n(\mathbb{R}P^2) \longrightarrow \mathbb{R}P^2$ denote projection onto the k^{th} coordinate. Observe that $\iota_\# = p_{1\#} \times \cdots \times p_{n\#}$, where $p_{k\#}\colon P_n(\mathbb{R}P^2) \longrightarrow \pi_1(\mathbb{R}P^2)$ is the induced homomorphism on the level of fundamental groups. Identifying $\pi_1(\mathbb{R}P^2)$ with $\mathbb{Z}_2$ and using the geometric realisation of Figure 1 of the generators of the presentation of $P_n(\mathbb{R}P^2)$ given by Theorem 7, it is straightforward to check that for all $1 \leqslant k,l \leqslant n$ and $1 \leqslant i < j \leqslant n$, $p_{k\#}(A_{i,j}) = \overline{0}$, $p_{k\#}(\tau_l) = \overline{0}$ if $l \neq k$ and $p_{k\#}(\tau_k) = \overline{1}$, and this yields the first part of the proposition. The second part follows easily from the presentation of the Abelianisation $(P_n(\mathbb{R}P^2))^{\text{Ab}}$ of $P_n(\mathbb{R}P^2)$ obtained from Theorem 7. More precisely, if we denote the Abelianisation of an element $x \in P_n(\mathbb{R}P^2)$ by $\overline{x}$, relations (b) and (c) imply respectively that for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant k \leqslant n$, $\overline{A_{i,j}}$ and $\overline{\tau_k}^2$ represent the trivial element of $(P_n(\mathbb{R}P^2))^{\text{Ab}}$. Since the remaining relations give no other information under Abelianisation, it follows that $(P_n(\mathbb{R}P^2))^{\text{Ab}} \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, where $\overline{\tau_k} = (\overline{0},\ldots,\overline{0},\underbrace{\overline{1}}_{k^{\text{th}}\text{ position}},\overline{0},\ldots,\overline{0})$ and $\overline{A_{i,j}} = (\overline{0},\ldots,\overline{0})$ via this isomorphism, and the Abelianisation homomorphism indeed coincides with $\iota_\#$ on $P_n(\mathbb{R}P^2)$. □

REMARKS 9.

(a) Since $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$, it follows immediately that K_n is normal in $B_n(\mathbb{R}P^2)$, since $\Gamma_2(P_n(\mathbb{R}P^2))$ is characteristic in $P_n(\mathbb{R}P^2)$, and $P_n(\mathbb{R}P^2)$ is normal in $B_n(\mathbb{R}P^2)$.
(b) A presentation of K_n may be obtained by a long but routine computation using the Reidemeister-Schreier method, although it is not clear how to simplify the presentation. In Theorem 3, we will provide an alternative description of K_n using algebraic methods.
(c) In what follows, we shall use Van Buskirk's presentation of $B_n(\mathbb{R}P^2)$ [VB, page 83] whose generating set consists of the standard braid generators $\sigma_1, \ldots, \sigma_{n-1}$ emanating from the 2-disc, as well as the surface generators $\rho_1, \ldots, \rho_n$ depicted in Figure 1. We have the following relation between the elements τ_k and ρ_k:

$$\tau_k = \rho_k^{-1} A_{k,k+1} \cdots A_{k,n} \quad \text{for all } 1 \leqslant k \leqslant n,$$

where for $1 \leqslant i < j \leqslant n$, $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$. In particular, it follows from Proposition 8 that:

$$\iota_\#(\rho_k) = \iota_\#(\tau_k) = (\overline{0}, \ldots, \overline{0}, \underbrace{\overline{1}}_{k^{\text{th}} \text{ position}}, \overline{0}, \ldots, \overline{0}) \text{ for all } 1 \leqslant k \leqslant n. \tag{5}$$

If $n \geqslant 2$, the full twist braid Δ_n^2, which may be defined by $\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n$, is of order 2 [VB, page 95], it generates the centre of $B_n(\mathbb{R}P^2)$ [M, Proposition 6.1], and is the unique element of $B_n(\mathbb{R}P^2)$ of order 2 [GG2, Proposition 23]. Since $\Delta_n^2 \in P_n(\mathbb{R}P^2)$, it thus belongs to the centre of $P_n(\mathbb{R}P^2)$, and just as for the Artin braid groups and the braid groups of $\mathbb{S}^2$, it generates the centre of $P_n(\mathbb{R}P^2)$:

PROPOSITION 10. *Let $n \geqslant 2$. Then the centre $Z(P_n(\mathbb{R}P^2))$ of $P_n(\mathbb{R}P^2)$ is generated by Δ_n^2.*

Proof. We prove the result by induction on n. If $n = 2$ then $P_2(\mathbb{R}P^2) \cong \mathcal{Q}_8$ [VB, page 87], the quaternion group of order 8, and the result follows since Δ_2^2 is the element of $P_2(\mathbb{R}P^2)$ of order 2. So suppose that $n \geqslant 3$. From the preceding remarks, $\langle \Delta_n^2 \rangle \subset Z(P_n(\mathbb{R}P^2))$. Conversely, let $x \in Z(P_n(\mathbb{R}P^2))$, and consider the following Fadell-Neuwirth short exact sequence:

$$1 \longrightarrow \pi_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_{n-1}\}) \longrightarrow P_n(\mathbb{R}P^2) \xrightarrow{q_{(n-1)\#}} P_{n-1}(\mathbb{R}P^2) \longrightarrow 1,$$

where $q_{(n-1)\#}$ is the surjective homomorphism induced on the level of fundamental groups by the projection $q_{n-1}\colon F_n(\mathbb{R}P^2) \longrightarrow F_{n-1}(\mathbb{R}P^2)$ onto the first $n-1$ coordinates. Now $q_{(n-1)\#}(x) \in Z(P_{n-1}(\mathbb{R}P^2))$ by surjectivity, and thus $q_{(n-1)\#}(x) = \Delta_{n-1}^{2\varepsilon}$ for some $\varepsilon \in \{0, 1\}$ by the induction hypothesis. Further, $q_{(n-1)\#}(\Delta_n^2) = \Delta_{n-1}^2$, hence

$$\Delta_n^{-2\varepsilon} x \in \operatorname{Ker}(q_{(n-1)\#}) \cap Z(P_n(\mathbb{R}P^2)),$$

and thus $\Delta_n^{-2\varepsilon} x \in Z(\operatorname{Ker}(q_{(n-1)\#}))$. But $Z(\operatorname{Ker}(q_{(n-1)\#}))$ is trivial because $\operatorname{Ker}(q_{(n-1)\#})$ is a free group of rank $n-1$. This implies that $x \in \langle \Delta_n^2 \rangle$ as required. □

We now prove Proposition 1.

Proof of Proposition 1. Let $n \geqslant 3$.

(a) Recall that part (a)(i) of Proposition 1 was proved in Proposition 8, so let us prove part (ii). The projection $q_2 \colon F_n(\mathbb{R}P^2) \longrightarrow F_2(\mathbb{R}P^2)$ onto the first two coordinates gives rise to the Fadell-Neuwirth short exact sequence (3). Since $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$ by Proposition 8, the image of the restriction $q_{2\#}|_{K_n}$ of $q_{2\#}$ to K_n is the subgroup $\Gamma_2(P_2(\mathbb{R}P^2)) = \left\langle \Delta_2^2 \right\rangle$, and so we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_n \cap P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) & \longrightarrow & K_n & \xrightarrow{q_{2\#}|_{K_n}} & \left\langle \Delta_2^2 \right\rangle & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) & \longrightarrow & P_n(\mathbb{R}P^2) & \xrightarrow{q_{2\#}} & P_2(\mathbb{R}P^2) & \longrightarrow & 1, \end{array} \tag{6}$$

where the vertical arrows are inclusions. Now $\left\langle \Delta_2^2 \right\rangle \cong \mathbb{Z}_2$, so K_n is an extension of the group $\mathrm{Ker}\,(q_{2\#}|_{K_n}) = K_n \cap P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ by $\mathbb{Z}_2$. The fact that $q_{2\#}(\Delta_n^2) = \Delta_2^2$ implies that the upper short exact sequence splits, a section being defined by the correspondence $\Delta_2^2 \longmapsto \Delta_n^2$, and since $\Delta_n^2 \in Z(P_n(\mathbb{R}P^2))$, the action by conjugation on $\mathrm{Ker}\,(q_{2\#}|_{K_n})$ is trivial. Part (a) of the proposition follows by taking $L_n = \mathrm{Ker}\,(q_{2\#}|_{K_n})$ and by noting that $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ is torsion free.

(b) Recall first that any torsion element in $P_n(\mathbb{R}P^2) \setminus \left\langle \Delta_n^2 \right\rangle$ is of order 4 [GG2, Corollary 19 and Proposition 23], and is conjugate in $B_n(\mathbb{R}P^2)$ to one of a^n or b^{n-1}, where $a = \rho_n \sigma_{n-1} \cdots \sigma_1$ and $b = \rho_{n-1}\sigma_{n-2} \cdots \sigma_1$ satisfy:

$$a^n = \rho_n \cdots \rho_1 \text{ and } b^{n-1} = \rho_{n-1} \cdots \rho_1 \tag{7}$$

by [GG7, Proposition 10]. Let N be a normal subgroup of $B_n(\mathbb{R}P^2)$ that satisfies $K_n \subsetneqq N \subset P_n(\mathbb{R}P^2)$. We claim that for all $u \in \pi_1(\prod_1^n \mathbb{R}P^2)$ (which we identify henceforth with $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$), exactly one of the following two conditions holds:

(i) $N \cap \iota_\#^{-1}(\{u\})$ is empty.

(ii) $\iota_\#^{-1}(\{u\})$ is contained in N.

To prove the claim, suppose that $x \in N \cap \iota_\#^{-1}(\{u\}) \neq \varnothing$, and let $y \in \iota_\#^{-1}(\{u\})$. Now $\iota_\#(x) = \iota_\#(y) = u$, so there exists $k \in K_n$ such that $x^{-1}y = k$. Since $K_n \subset N$, it follows that $y = xk \in N$, which proves the claim. Further, $\iota_\#(a^n) = (\overline{1}, \ldots, \overline{1})$ and $\iota_\#(b^{n-1}) = (\overline{1}, \ldots, \overline{1}, \overline{0})$ by Proposition 8 and equations (5) and (7), so by the claim it suffices to prove that there exists $z \in N$ such that $\iota_\#(z) \in \left\{(\overline{1}, \ldots, \overline{1}), (\overline{1}, \ldots, \overline{1}, \overline{0})\right\}$, for then we are in case (ii) above, and it follows that one of a^n and b^{n-1} belongs to N.

It thus remains to prove the existence of such a z. Let $x \in N \setminus K_n$. Then $\iota_\#(x)$ contains an entry equal to $\overline{1}$ because $K_n = \mathrm{Ker}\,(\iota_\#)$. If $\iota_\#(x) = (\overline{1}, \ldots, \overline{1})$ then we are done. So assume that $\iota_\#(x)$ also contains an entry that is equal to $\overline{0}$. By equation (5), there exist $1 \leqslant r < n$ and $1 \leqslant i_1 < \cdots < i_r \leqslant n$ such that $\iota_\#(\rho_{i_1} \cdots \rho_{i_r}) = \iota_\#(x)$. It follows from the claim and the fact that $x \in N$ that $\rho_{i_1} \cdots \rho_{i_r} \in N$ also, and so without loss of generality, we may suppose that $x = \rho_{i_1} \cdots \rho_{i_r}$. Further, since $\iota_\#(x)$ contains both a $\overline{0}$ and a $\overline{1}$, there exists $1 \leqslant j \leqslant r$ such that $p_{i_j\#}(x) = \overline{1}$ and $p_{(i_j+1)\#}(x) = \overline{0}$, the homomorphisms $p_{k\#}$ being those defined in the proof of Proposition 8. Note that we consider the indices modulo n, so if $i_j = n$ (so $j = r$) then we set $i_j + 1 = 1$. By [GG2, page 777], conjugation by

a^{-1} permutes cyclically the elements $\rho_1, \ldots, \rho_n, \rho_1^{-1}, \ldots, \rho_n^{-1}$ of $P_n(\mathbb{R}P^2)$, so the $(n-1)^{\text{th}}$ (resp. n^{th}) entry of $x' = a^{-(n-1-i_j)} x a^{(n-1-i_j)}$ is equal to $\overline{1}$ (resp. $\overline{0}$), and $x' \in N$ because N is normal in $B_n(\mathbb{R}P^2)$. Using the relation $b = \sigma_{n-1} a$, we determine the conjugates of the ρ_i by b^{-1}:

$$\begin{aligned} b^{-1}\rho_i b &= a^{-1}\sigma_{n-1}^{-1}\rho_i\sigma_{n-1}a = a^{-1}\rho_i a = \rho_{i+1} \quad \text{for all } 1 \leqslant i \leqslant n-2 \\ b^{-1}\rho_{n-1}b &= a^{-1}\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}a = a^{-1}\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}^{-1}.\,\sigma_{n-1}^2 a \\ &= a^{-1}\rho_n a.\,a^{-1}\sigma_{n-1}^2 a = \rho_1^{-1}.\,a^{-1}\sigma_{n-1}^2 a, \end{aligned}$$

where we have used the relations $\rho_i\sigma_{n-1} = \sigma_{n-1}\rho_i$ if $1 \leqslant i \leqslant n-2$ and $\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}^{-1} = \rho_n$ of Van Buskirk's presentation of $B_n(\mathbb{R}P^2)$, as well as the effect of conjugation by a^{-1} on the ρ_j. Now $\sigma_{n-1}^2 = A_{n-1,n} \in K_n$ by Proposition 8, so $a^{-1}\sigma_{n-1}^2 a \in K_n$ by Remarks 9(a), and hence $\iota_\#(b^{-1}\rho_{n-1}b) = (\overline{1}, \overline{0}, \ldots, \overline{0})$. It then follows that $\iota_\#(a^{-1}x'a)$ and $\iota_\#(b^{-1}x'b)$ have the same entries except in the first and last positions, so if $x'' = a^{-1}x'a.\,b^{-1}x'b$, we have $\iota_\#(x'') = (\overline{1}, \overline{0}, \ldots, \overline{0}, \overline{1})$. Further, $x'' \in N$ since N is normal in $B_n(\mathbb{R}P^2)$. Let $n = 2m + \varepsilon$, where $m \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. Then setting

$$z = a^{-\varepsilon} x'' a^{\varepsilon}.\,a^{-(2+\varepsilon)} x'' a^{2+\varepsilon} \cdots a^{-(2(m-1)+\varepsilon)} x'' a^{2(m-1)+\varepsilon},$$

we see once more that $z \in N$, and $\iota_\#(z) = (\overline{1}, \ldots, \overline{1})$ if n is even and $\iota_\#(z) = (\overline{1}, \ldots, \overline{1}, \overline{0})$ if n is odd, which completes the proof of the existence of z, and thus that of Proposition 1(b). □

We end this section by proving Theorem 2.

Proof of Theorem 2. Let $S = \mathbb{S}^2$ or $\mathbb{R}P^2$. If $n = 1$ then $\iota_\#$ is an isomorphism and $\mathrm{Im}\,(j_\#|_{P_n})$ is trivial so the result holds. If $n = 2$ and $S = \mathbb{S}^2$ then $P_n(\mathbb{S}^2)$ is trivial, and there is nothing to prove. Now suppose that $S = \mathbb{S}^2$ and $n \geqslant 3$. As we mentioned in the introduction, $\mathrm{Ker}\,(\iota_\#) = P_n(\mathbb{S}^2)$. Let $(A_{i,j})_{1 \leqslant i < j \leqslant n}$ be the generating set of P_n, where $A_{i,j}$ has a geometric representative similar to that given in Figure 1. It is well known that the image of this set by $j_\#$ yields a generating set for $P_n(\mathbb{S}^2)$ (*cf.* [S, page 616]), so $j_\#|_{P_n}$ is surjective, and the statement of the theorem follows. Finally, assume that $S = \mathbb{R}P^2$ and $n \geqslant 2$. Once more, $\mathrm{Im}\,(j_\#|_{P_n}) = \langle A_{i,j} \mid 1 \leqslant i < j \leqslant n \rangle$, and since $A_{i,j} \in \mathrm{Ker}\,(\iota_\#)$ by Proposition 8, we conclude that $\langle\!\langle \mathrm{Im}\,(j_\#|_{P_n}) \rangle\!\rangle_{P_n(S)} \subset \mathrm{Ker}\,(\iota_\#)$. To prove the converse, first recall from Proposition 8 that $\mathrm{Ker}\,(\iota_\#) = \Gamma_2(P_n(\mathbb{R}P^2))$. Using the standard commutator identities $[x, yz] = [x, y][y, [x, z]][x, z]$ and $[xy, z] = [x, [y, z]][y, z][x, z]$, $\Gamma_2(P_n(\mathbb{R}P^2))$ is equal to the normal closure in $P_n(\mathbb{R}P^2)$ of $\{[x, y] \mid x, y \in \{A_{i,j}, \rho_k \mid 1 \leqslant i < j \leqslant n \text{ and } 1 \leqslant k \leqslant n\}\}$. It then follows using the relations of Theorem 7 that the commutators $[x, y]$ belonging to this set also belong to $\langle\!\langle A_{i,j} \mid 1 \leqslant i < j \leqslant n \rangle\!\rangle_{P_n(\mathbb{R}P^2)}$, which is nothing other than $\langle\!\langle \mathrm{Im}\,(j_\#|_{P_n}) \rangle\!\rangle_{P_n(S)}$. We conclude by normality that $\mathrm{Ker}\,(\iota_\#) \subset \langle\!\langle \mathrm{Im}\,(j_\#|_{P_n}) \rangle\!\rangle_{P_n(S)}$, and this completes the proof of the theorem. □

3 Some properties of the subgroup L_n

Let $S = \mathbb{S}^2$ or $S = \mathbb{R}P^2$, and for all $m, n \geqslant 1$, let $\Gamma_{m,n}(S) = P_m(S \setminus \{x_1, \ldots, x_n\})$ denote the m-string pure braid group of S with n points removed. In this section, we study

$P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, which is $\Gamma_{n-2,2}(\mathbb{R}P^2)$, in more detail, and we prove Theorem 3 and Proposition 4 that enable us to understand better the structure of the subgroup L_n defined in the proof of Proposition 1(a)(ii).

We start by exhibiting a presentation of the group $\Gamma_{m,n}(\mathbb{R}P^2)$ in terms of the generators of $P_{m+n}(\mathbb{R}P^2)$ given by Theorem 7. A presentation for $\Gamma_{m,n}(\mathbb{S}^2)$ is given in [GG3, Proposition 7] and will be recalled later in Proposition 15, when we come to proving Theorem 5. For $1 \leqslant i < j \leqslant m+n$, let

$$C_{i,j} = A_{j-1,j}^{-1} \cdots A_{i+1,j}^{-1} A_{i,j} A_{i+1,j} \cdots A_{j-1,j}. \tag{8}$$

Geometrically, in terms of Figure 1, $C_{i,j}$ is the image of $A_{i,j}^{-1}$ under the reflection about the straight line segment that passes through the points $x_1, \ldots, x_{m+n}$. The proof of the following proposition, which we leave to the reader, is similar in nature to that for $\mathbb{S}^2$, but is a little more involved due to the presence of extra generators that emanate from the fundamental group of $\mathbb{R}P^2$.

PROPOSITION 11. *Let $n, m \geqslant 1$. The following constitutes a presentation of the group $\Gamma_{m,n}(\mathbb{R}P^2)$:*

generators: *$A_{i,j}$, ρ_j, where $1 \leqslant i < j$ and $n+1 \leqslant j \leqslant m+n$.*
relations:

(I) the Artin relations described by equation (4) among the generators $A_{i,j}$ of $\Gamma_{m,n}(\mathbb{R}P^2)$.
(II) for all $1 \leqslant i < j$ and $n+1 \leqslant j < k \leqslant m+n$, $A_{i,j}\rho_k A_{i,j}^{-1} = \rho_k$.
(III) for all $1 \leqslant i < j$ and $n+1 \leqslant k < j \leqslant m+n$,

$$\rho_k A_{i,j} \rho_k^{-1} = \begin{cases} A_{i,j} & \text{if } k < i \\ \rho_j^{-1} C_{i,j}^{-1} \rho_j & \text{if } k = i \\ \rho_j^{-1} C_{k,j}^{-1} \rho_j A_{i,j} \rho_j^{-1} C_{k,j} \rho_j & \text{if } k > i. \end{cases}$$

(IV) for all $n+1 \leqslant k < j \leqslant m+n$, $\rho_k \rho_j \rho_k^{-1} = C_{k,j}\rho_j$.
(V) for all $n+1 \leqslant j \leqslant m+n$,

$$\rho_j \left(\prod_{i=1}^{j-1} A_{i,j} \right) \rho_j = \left(\prod_{l=j+1}^{m+n} A_{j,l} \right).$$

The elements $C_{i,j}$ and $C_{k,j}$ appearing in relations (III) and (IV) should be rewritten using equation (8).

In the rest of this section, we shall assume that $n = 2$, and we shall focus our attention on the groups $\Gamma_{m,2}(\mathbb{R}P^2)$, where $m \geqslant 1$, that we interpret as subgroups of $P_{m+2}(\mathbb{R}P^2)$ via the short exact sequence (3). Before proving Theorem 3 and Proposition 4, we introduce some notation that will be used to study the subgroups K_n and L_n. Let $m \geqslant 2$, and consider the following Fadell-Neuwirth short exact sequence:

$$1 \longrightarrow \Omega_{m+1} \longrightarrow P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \xrightarrow{r_{m+1}} P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow 1, \tag{9}$$

where r_{m+1} is given geometrically by forgetting the last string, and where $\Omega_{m+1} = \pi_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_{m+1}\}, x_{m+2})$. From the Fadell-Neuwirth short exact sequences of the

form of equation (3), r_{m+1} is the restriction of $q_{(m+1)\#}\colon\ P_{m+2}(\mathbb{R}P^2) \longrightarrow P_{m+1}(\mathbb{R}P^2)$ to Ker $(q_{2\#})$. The kernel Ω_{m+1} of r_{m+1} is a free group of rank $m+1$ with a basis $\mathcal{B}_{m+1}$ being given by:

$$\mathcal{B}_{m+1} = \left\{ A_{k,m+2}, \rho_{m+2} \,\middle|\, 1 \leqslant k \leqslant m \right\}. \tag{10}$$

The group Ω_{m+1} may also be described as the subgroup of $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ generated by $\{A_{1,m+2}, \ldots, A_{m+1,m+2}, \rho_{m+2}\}$ subject to the relation:

$$A_{m+1,m+2} = A_{m,m+2}^{-1} \cdots A_{1,m+2}^{-1} \rho_{m+2}^{-2}, \tag{11}$$

obtained from relation (V) of Proposition 11. Equations (8) and (11) imply notably that $A_{l,m+2}$ and $C_{l,m+2}$ belong to Ω_{m+1} for all $1 \leqslant l \leqslant m+1$. Using geometric methods, for $m \geqslant 2$, we proved the existence of a section

$$s_{m+1}\colon\ P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$$

for r_{m+1} in [GG6, Theorem 2(a)]. Applying induction to equation (9), it follows that for all $m \geqslant 1$:

$$P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \cong \Omega_{m+1} \rtimes (\Omega_m \rtimes (\cdots \rtimes (\Omega_3 \rtimes \Omega_2) \cdots)). \tag{12}$$

So $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \cong \mathbb{F}_{m+1} \rtimes (\mathbb{F}_m \rtimes (\cdots \rtimes (\mathbb{F}_3 \rtimes \mathbb{F}_2) \cdots))$, which may be interpreted as the Artin combing operation for $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. It follows from this and equation (10) that $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ admits $\mathcal{X}_{m+2}$ as a generating set, where:

$$\mathcal{X}_{m+2} = \left\{ A_{i,j},\ \rho_j \,\middle|\, 3 \leqslant j \leqslant m+2,\ 1 \leqslant i \leqslant j-2 \right\}. \tag{13}$$

REMARK 12. For what follows, we will need to know an explicit section s_{m+1} for r_{m+1}. Such a section may be obtained as follows: for $m \geqslant 2$, consider the homomorphism $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ given by forgetting the string based at x_3. By [GG6, Theorem 2(a)]), a geometric section is obtained by doubling the second (vertical) string, so that there is a new third string, and renumbering the following strings, which gives rise to an algebraic section for the given homomorphism of the form:

$$A_{i,j} \longmapsto \begin{cases} A_{1,j+1} & \text{if } i = 1 \\ A_{2,j+1} A_{3,j+1} & \text{if } i = 2 \\ A_{i+1,j+1} & \text{if } 3 \leqslant i < j \end{cases}$$

$$\rho_j \longmapsto \rho_{j+1},$$

for all $3 \leqslant j \leqslant m+1$. However, in view of the nature of r_{m+1}, we would like this new string to be in the $(m+2)^{\text{th}}$ position. We achieve this by composing the above algebraic section with conjugation by $\sigma_{m+1} \cdots \sigma_3$, which gives rise to a section

$$s_{m+1}\colon\ P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$$

for r_{m+1} that is defined by:

$$\begin{cases} s_{m+1}(A_{i,j}) = \begin{cases} A_{j,m+2} A_{1,j} A_{j,m+2}^{-1} & \text{if } i = 1 \\ A_{j,m+2} A_{2,j} & \text{if } i = 2 \\ A_{i,j} & \text{if } 3 \leqslant i < j \end{cases} \\ s_{m+1}(\rho_j) = \rho_j A_{j,m+2}^{-1}. \end{cases} \tag{14}$$

for all $1 \leqslant i < j$ and $3 \leqslant j \leqslant m+1$. A long but straightforward calculation using the presentation of $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ given by Proposition 11 shows that s_{m+1} does indeed define a section for r_{m+1}.

We now prove Theorem 3, which allows us to give a more explicit description of L_n.

Proof of Theorem 3. Let $n \geqslant 3$. By the commutative diagram (6) of short exact sequences, the restriction of the homomorphism $q_{2\#}\colon P_n(\mathbb{R}P^2) \longrightarrow P_2(\mathbb{R}P^2)$ to K_n factors through the inclusion $\langle \Delta_2^2 \rangle \longrightarrow P_2(\mathbb{R}P^2)$, and the kernel L_n of $q_{2\#}|_{K_n}$ is contained in the group $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. We may then add a third row to this diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & L_n & \longrightarrow & K_n & \xrightarrow{q_{2\#}|_{K_n}} & \langle \Delta_2^2 \rangle & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) & \longrightarrow & P_n(\mathbb{R}P^2) & \xrightarrow{q_{2\#}} & P_2(\mathbb{R}P^2) & \longrightarrow & 1 \\
 & & \downarrow \widehat{\iota}_{n-2} & & \downarrow \iota_{n\#} & & \downarrow \iota_{2\#} & & \\
1 & \longrightarrow & \mathbb{Z}_2^{n-2} & \xrightarrow{j} & \mathbb{Z}_2^n & \xrightarrow{\widehat{q}_2} & \mathbb{Z}_2^2 & \longrightarrow & 1, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{15}$$

where $\widehat{q}_2\colon \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2^2$ is projection onto the first two factors, and $j\colon \mathbb{Z}_2^{n-2} \longrightarrow \mathbb{Z}_2^n$ is the monomorphism defined by $j(\overline{\varepsilon_1}, \ldots, \overline{\varepsilon_{n-2}}) = (\overline{0}, \overline{0}, \overline{\varepsilon_1}, \ldots, \overline{\varepsilon_{n-2}})$. The commutativity of diagram (15) thus induces a homomorphism $\widehat{\iota}_{n-2}\colon P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow \mathbb{Z}_2^{n-2}$ that is the restriction of $\iota_{n\#}$ to $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ that makes the bottom left-hand square commute. To see that $\widehat{\iota}_{n-2}$ is surjective, notice that if $x \in \mathbb{Z}_2^{n-2}$ then the first two entries of $j(x)$ are equal to $\overline{0}$, and using equation (5), it follows that there exist $3 \leqslant i_1 < \cdots < i_r \leqslant n$ such that $\iota_{n\#}(\rho_{i_1} \cdots \rho_{i_r}) = j(x)$. Furthermore, $\rho_{i_1} \cdots \rho_{i_r} \in \operatorname{Ker}(q_{2\#})$, and by commutativity of the diagram, we also have $\iota_{n\#}(\rho_{i_1} \cdots \rho_{i_r}) = j \circ \widehat{\iota}_{n-2}(\rho_{i_1} \cdots \rho_{i_r})$, whence $x = \widehat{\iota}_{n-2}(\rho_{i_1} \cdots \rho_{i_r})$ by injectivity of j. It remains to prove exactness of the first column. The fact that $L_n \subset \operatorname{Ker}(\widehat{\iota}_{n-2})$ follows easily. Conversely, if $x \in \operatorname{Ker}(\widehat{\iota}_{n-2})$ then $x \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, and $x \in K_n$ by commutativity of the diagram, so $x \in L_n$. This proves the first two assertions of the theorem.

To prove the last part of the statement of the theorem, let $m \geqslant 1$, and consider equation (9). Since $\widehat{\iota}_m$ is the restriction of $\iota_{(m+2)\#}$ to $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, we have $\widehat{\iota}_m(\rho_j) = (\overline{0}, \ldots, \overline{0}, \underbrace{\overline{1}}_{(j-2)^{\text{nd}}\text{ position}}, \overline{0}, \ldots, \overline{0})$ and $\widehat{\iota}_m(A_{i,j}) = (\overline{0}, \ldots, \overline{0})$ for all $1 \leqslant i < j$ and $3 \leqslant j \leqslant m+2$. So for each $2 \leqslant l \leqslant m+1$, $\widehat{\iota}_m$ restricts to a surjective homomorphism $\widehat{\iota}_m|_{\Omega_l}\colon \Omega_l \longrightarrow \mathbb{Z}_2$ of each of the factors of equation (12), $\mathbb{Z}_2$ being the $(l-1)^{\text{st}}$ factor of $\mathbb{Z}_2^m$, and using equation (10), we see that $\operatorname{Ker}(\widehat{\iota}_m|_{\Omega_l})$ is a free group of rank $2l-1$ with basis $\widehat{\mathcal{B}}_l$ given by:

$$\widehat{\mathcal{B}}_l = \left\{ A_{k,l+1}, \rho_{l+1} A_{k,l+1} \rho_{l+1}^{-1}, \rho_{l+1}^2 \;\middle|\; 1 \leqslant k \leqslant l-1 \right\}. \tag{16}$$

As we shall now explain, for all $m \geqslant 2$, the short exact sequence (9) may be extended to a commutative diagram of short exact sequences as follows:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Ker}\left(\widehat{\imath}_m\big|_{\Omega_{m+1}}\right) & \longrightarrow & L_{m+2} & \xrightarrow{r_{m+1}\big|_{L_{m+2}}} & L_{m+1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Omega_{m+1} & \longrightarrow & P_m(\mathbb{R}P^2\setminus\{x_1,x_2\}) & \underset{s_{m+1}}{\overset{r_{m+1}}{\rightleftarrows}} & P_{m-1}(\mathbb{R}P^2\setminus\{x_1,x_2\}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \widehat{\imath}_m|_{\Omega_{m+1}}} & & \downarrow{\scriptstyle \widehat{\imath}_m} & & \downarrow{\scriptstyle \widehat{\imath}_{m-1}} & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2^m & \longrightarrow & \mathbb{Z}_2^{m-1} & \longrightarrow & 1. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{17}$$

To obtain this diagram, we start with the commutative diagram that consists of the second and third rows and the three columns (so *a priori*, the arrows of the first row are missing). The commutativity implies that r_{m+1} restricts to the homomorphism $r_{m+1}\big|_{L_{m+2}} : L_{m+2} \longrightarrow L_{m+1}$, which is surjective, since if $w \in L_{m+1}$ is written in terms of the elements of $\mathcal{X}_{m+1}$ then the same word w, considered as an element of the group $P_m(\mathbb{R}P^2\setminus\{x_1,x_2\})$, belongs to L_{m+2}, and satisfies $r_{m+1}(w) = w$. Then the kernel of $r_{m+1}\big|_{L_{m+2}}$, which is also the kernel of $\widehat{\imath}_m\big|_{\Omega_{m+1}}$, is equal to $L_{m+2} \cap \Omega_{m+1}$. This establishes the existence of the complete commutative diagram (17) of short exact sequences. By induction, it follows from (16) and (17) that for all $m \geqslant 1$, L_{m+2} is generated by

$$\widehat{\mathcal{X}}_{m+2} = \bigcup_{j=3}^{m+2} \widehat{\mathcal{B}}_{j-1} = \left\{ A_{i,j},\ \rho_j A_{i,j} \rho_j^{-1},\ \rho_j^2 \;\middle|\; 3 \leqslant j \leqslant m+2,\ 1 \leqslant i \leqslant j-2 \right\}. \tag{18}$$

Using the section s_{m+1} defined by equation (14), we see that $s_{m+1}(x) \in L_{m+2}$ for all $x \in \widehat{\mathcal{X}}_{m+1}$, and thus s_{m+1} restricts to a section $s_{m+1}\big|_{L_{m+1}} : L_{m+1} \longrightarrow L_{m+2}$ for $r_{m+1}\big|_{L_{m+2}}$. We conclude by induction on the first row of (17) that:

$$\begin{aligned}
L_{m+2} &\cong \mathrm{Ker}\left(\widehat{\imath}_m\big|_{\Omega_{m+1}}\right) \rtimes L_{m+1} && (19)\\
&\cong \mathrm{Ker}\left(\widehat{\imath}_m\big|_{\Omega_{m+1}}\right) \rtimes \left(\mathrm{Ker}\left(\widehat{\imath}_m|_{\Omega_m}\right) \rtimes \left(\cdots \rtimes \left(\mathrm{Ker}\left(\widehat{\imath}_m\big|_{\Omega_3}\right) \rtimes \mathrm{Ker}\left(\widehat{\imath}_m\big|_{\Omega_2}\right)\right)\cdots\right)\right), && (20)
\end{aligned}$$

the actions being induced by those of equation (12), so by equation (16), L_{m+2} is isomorphic to a repeated semi-direct product of the form $\mathbb{F}_{2m+1} \rtimes (\mathbb{F}_{2m-1} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3)\cdots))$. The last part of the statement of Theorem 3 follows by taking $m = n-2$. □

A finer analysis of the actions that appear in equations (12) and (20) now allows us to determine the Abelianisations of $P_{n-2}(\mathbb{R}P^2\setminus\{x_1,x_2\})$ and L_n.

Proof of Proposition 4. If $n = 3$ then the two assertions are clear. So assume by induction that they hold for some $n \geqslant 3$. From the split short exact sequence (9) and equation (19) with $m = n-1$, we have:

$$\begin{cases} P_{n-1}(\mathbb{R}P^2\setminus\{x_1,x_2\}) \cong \Omega_n \rtimes_\psi P_{n-2}(\mathbb{R}P^2\setminus\{x_1,x_2\}) \quad \text{and} \\ L_{n+1} \cong \mathrm{Ker}\left(\widehat{\imath}_{n-1}|_{\Omega_n}\right) \rtimes_\psi L_n, \end{cases} \tag{21}$$

where ψ denotes the action given by the section s_n, and the action induced by the restriction of the section s_n to L_n respectively.

Before going any further, we recall some general considerations from the paper [GG5, pages 3387–88] concerning the Abelianisation of semi-direct products. If H and K are groups, and if $\varphi\colon H \longrightarrow \mathrm{Aut}(K)$ is an action of H on K then one may deduce easily from [GG5, Proposition 3.3] that:

$$(K \rtimes_\varphi H)^{\mathrm{Ab}} \cong \Delta(K) \oplus H^{\mathrm{Ab}}, \tag{22}$$

where:

$$\Delta(K) = K/\left\langle \Gamma_2(K) \cup \widehat{K} \right\rangle \quad \text{and} \quad \widehat{K} = \left\langle \varphi(h)(k) \cdot k^{-1} \;\middle|\; h \in H \text{ and } k \in K \right\rangle.$$

Recall that $\widehat{K}$ is normal in K (*cf.* [GG5, lines 1–4, page 3388]), so $\langle \Gamma_2(K) \cup \widehat{K} \rangle$ is normal in K. If $k \in K$, let $\overline{k}$ denote its image under the canonical projection $K \longrightarrow \Delta(K)$. For all $k, k' \in K$ and $h, h' \in H$, we have:

$$\begin{aligned} \varphi(hh')(k) \cdot k^{-1} &= \varphi(h)(\varphi(h')(k)) \cdot \varphi(h')(k^{-1}) \cdot \varphi(h')(k) \cdot k^{-1} \\ &= \varphi(h)(k'') \cdot k''^{-1} \cdot \varphi(h')(k) \cdot k^{-1} \qquad (23) \\ \varphi(h)(kk') \cdot (kk')^{-1} &= \big(\varphi(h)(k) \cdot k^{-1}\big) \cdot k\big(\varphi(h)(k') \cdot k'^{-1}\big)k^{-1}. \qquad (24) \end{aligned}$$

where $k'' = \varphi(h')(k)$ belongs to K. Let $\mathcal{H}$ and $\mathcal{K}$ be generating sets for H and K respectively. By induction on word length relative to the elements of $\mathcal{H}$, equation (23) implies that $\widehat{K}$ is generated by elements of the form $\varphi(h)(k) \cdot k^{-1}$, where $h \in \mathcal{H}$ and $k \in K$. A second induction on word length relative to the elements of $\mathcal{K}$ and equation (24) implies that $\widehat{K}$ is normally generated by the elements of the form $\varphi(h)(k) \cdot k^{-1}$, where $h \in \mathcal{H}$ and $k \in \mathcal{K}$. By standard arguments involving group presentations, since $\Gamma_2(K) \subset \langle \Gamma_2(K) \cup \widehat{K} \rangle$, $\Delta(K)$ is Abelian, and a presentation of $\Delta(K)$ may be obtained by Abelianising a given presentation of K, and by adjoining the relators of the form $\overline{\varphi(h)(k) \cdot k^{-1}}$, where $h \in \mathcal{H}$ and $k \in \mathcal{K}$.

We now take $K = \Omega_n$ (resp. $K = \mathrm{Ker}\,(\widehat{\imath}_{n-1}|_{\Omega_n})$), $H = P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ (resp. $H = L_n$) and $\varphi = \psi$. Applying the induction hypothesis and equation (22) to equation (21), to prove parts (a) and (b), it thus suffices to show that:

$$\Delta(\Omega_n) \cong \mathbb{Z}^2, \quad \text{and that} \tag{25}$$

$$\Delta\left(\mathrm{Ker}\,(\widehat{\imath}_{n-1}|_{\Omega_n})\right) \cong \mathbb{Z}^{2n-1} \tag{26}$$

respectively. We first establish the isomorphism (25). As we saw previously, $\Delta(\Omega_n)$ is Abelian, and to obtain a presentation of $\Delta(\Omega_n)$, we add the relators of the form $\overline{\psi(\tau)(\omega) \cdot \omega^{-1}}$ to a presentation of $(\Omega_n)^{\mathrm{Ab}}$, where $\tau \in \mathcal{X}_n$ and $\omega \in \mathcal{B}_n$. In $\Delta(\Omega_n)$, such relators may be written as:

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}\omega^{-1}} = \overline{s_n(\tau)\omega(s_n(\tau))^{-1}}\;\overline{\omega^{-1}}. \tag{27}$$

We claim that it is not necessary to know explicitly the section s_n in order to determine these relators. Indeed, for all $\tau \in \mathcal{X}_n$, we have $p_{n+1}(\tau) = \tau$; note that we abuse notation

here by letting τ also denote the corresponding element of $\mathcal{X}_{n+1}$ in $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Thus $s_n(\tau)\tau^{-1} \in \mathrm{Ker}\,(p_{n+1})$, and hence there exists $\omega_\tau \in \Omega_n$ such that $s_n(\tau) = \omega_\tau \tau$. Since $\Delta(\Omega_n)$ is Abelian, it follows that:

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}} = \overline{\omega_\tau \tau \omega \tau^{-1} \omega_\tau^{-1}} = \overline{\omega_\tau}\ \overline{\tau\omega\tau^{-1}}\ \overline{\omega_\tau^{-1}} = \overline{\tau\omega\tau^{-1}},$$

and thus the relators of equation (27) become:

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}\omega^{-1}} = \overline{\tau\omega\tau^{-1}}\ \overline{\omega^{-1}}. \tag{28}$$

This proves the claim. In what follows, the relations (I)–(V) refer to those of the presentation of $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ given by Proposition 11. Using this presentation and the fact that $\Delta(\Omega_n)$ is Abelian, we see immediately that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$ for all $\tau \in \mathcal{X}_n$ and $\omega \in \mathcal{B}_n$, with the following exceptions:

(i) $\tau = \rho_j$ and $\omega = A_{j,n+1}$ for all $3 \leqslant j \leqslant n-1$. Then $\overline{\rho_j A_{j,n+1} \rho_j^{-1}} = \overline{C_{j,n+1}^{-1}} = \overline{A_{j,n+1}^{-1}}$, using relation (III) and equation (8), which yields the relator $\left(\overline{A_{j,n+1}}\right)^2$ in $\Delta(\Omega_n)$.

(ii) $\tau = \rho_j$ and $\omega = \rho_{n+1}$ for all $3 \leqslant j \leqslant n$. Then $\overline{\rho_j \rho_{n+1} \rho_j^{-1}} = \overline{C_{j,n+1}\rho_{n+1}} = \overline{A_{j,n+1}}\ \overline{\rho_{n+1}}$ by relation (IV) and equation (8), which yields the relator $\overline{A_{j,n+1}}$ in $\Delta(\Omega_n)$.

The relators of (ii) above clearly give rise to those of (i). To obtain a presentation of $\Delta(\Omega_n)$, which by equation (10) is an Abelian group with generating set

$$\left\{ \overline{A_{l,n+1}}, \overline{\rho_{n+1}} \;\middle|\; 1 \leqslant l \leqslant n-1 \right\},$$

we must add the relators $\overline{A_{j,n+1}}$ for all $3 \leqslant j \leqslant n$. Thus for $j = 3, \ldots, n-1$, the elements $\overline{A_{j,n+1}}$ of this generating set are trivial. Further, $\overline{A_{n,n+1}}$ is also trivial, but by relation (11), one of the remaining generators $\overline{A_{j,n+1}}$ may be deleted, $\overline{A_{2,n+1}}$ say, from which we see that $\Delta(\Omega_n)$ is a free Abelian group of rank two with $\left\{\overline{A_{1,n+1}}, \overline{\rho_{n+1}}\right\}$ as a basis. This establishes the isomorphism (25), and so proves part (a).

We now prove part (b). As we mentioned previously, it suffices to establish the isomorphism (26). Since $\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n})$ is a free group of rank $2n-1$, we must thus show that $\Delta(\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n})) = (\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n}))^{\mathrm{Ab}}$. We take $K = \mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n})$ (resp. $H = L_{n-2}$) to be equipped with the basis $\widehat{\mathcal{B}}_n$ (resp. the generating set $\widehat{\mathcal{X}}_n$) of equation (16) (resp. of equation (18)). The fact that $\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n})$ is normal in Ω_n implies that $A_{l,n+1}$, $\rho_{n+1}A_{l,n+1}\rho_{n+1}^{-1}$, $C_{l,n+1}$ and $\rho_{n+1}C_{l,n+1}\rho_{n+1}^{-1}$ belong to $\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n})$ for all $1 \leqslant l \leqslant n$ by equations (8) and (11). Repeating the argument given between equations (27) and (28), we see that equation (28) holds for all $\tau \in \widehat{\mathcal{X}}_n$ and $\omega \in \widehat{\mathcal{B}}_n$, where $\overline{\omega}$ now denotes the image of ω under the canonical projection $\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n}) \longrightarrow \Delta(\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n}))$. For $\alpha \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, let c_α denote conjugation in $\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n})$ by α (which we consider to be an element of $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$). The automorphism c_α is well defined because $\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n}) = \Omega_n \cap L_{n-1}$, so that $\mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n})$ is normal in $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. We claim that $\left\langle \Gamma_2(K) \cup \widehat{K} \right\rangle$ is invariant under c_α. To see this, note first that $\Gamma_2(K)$ is clearly invariant since it is a characteristic subgroup of K. On the other hand, suppose that $\omega \in \mathrm{Ker}\,(\widehat{\iota}_{n-1}|_{\Omega_n})$, $\tau \in L_{n-2}$ and $\alpha \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Since $s_n(\tau) \in L_{n-1}$, L_{n-1} is normal in $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ and L_{n-2} is normal in $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, we have $\alpha s_n(\tau)\alpha^{-1} \in$

$L_{n-1}, \tau' = p_{n+1}(\alpha s_n(\tau)\alpha^{-1}) = \alpha\tau\alpha^{-1} \in L_{n-2}$, and thus $s_n(\tau'^{-1})(\alpha s_n(\tau)\alpha^{-1}) \in \mathrm{Ker}\,(\widehat{\imath}_{n-1}|_{\Omega_n})$. Hence there exists $\omega_{\tau'} \in \mathrm{Ker}\,(\widehat{\imath}_{n-1}|_{\Omega_n})$ such that $\alpha s_n(\tau)\alpha^{-1} = s_n(\tau')\omega_{\tau'}$. Now $\mathrm{Ker}\,(\widehat{\imath}_{n-1}|_{\Omega_n})$ is normal in $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, so $\omega' = \alpha\omega\alpha^{-1} \in \mathrm{Ker}\,(\widehat{\imath}_{n-1}|_{\Omega_n})$, and therefore:

$$\begin{aligned} c_\alpha\big(s_n(\tau)\omega s_n(\tau^{-1})\omega^{-1}\big) &= \alpha\big(s_n(\tau)\omega s_n(\tau^{-1})\omega^{-1}\big)\alpha^{-1} = s_n(\tau')\omega_{\tau'}\omega'\omega_{\tau'}^{-1}s_n(\tau'^{-1})\omega'^{-1} \\ &= s_n(\tau')(\omega_{\tau'}\omega'\omega_{\tau'}^{-1})s_n(\tau'^{-1})(\omega_{\tau'}\omega'^{-1}\omega_{\tau'}^{-1}) \cdot \omega_{\tau'}\omega'\omega_{\tau'}^{-1}\omega'^{-1}, \end{aligned}$$

which belongs to $\big\langle \Gamma_2(K) \cup \widehat{K} \big\rangle$ because $s(\tau')(\omega_{\tau'}\omega'\omega_{\tau'}^{-1})s(\tau'^{-1})(\omega_{\tau'}\omega'^{-1}\omega_{\tau'}^{-1}) \in \widehat{K}$ and $\omega_{\tau'}\omega'\omega_{\tau'}^{-1}\omega'^{-1} \in \Gamma_2(K)$. This proves the claim, and implies that c_α induces an endomorphism $\widehat{c}_\alpha$ (an automorphism in fact, whose inverse is $\widehat{c}_{\alpha^{-1}}$) of $\Delta(K)$, in particular, if $\alpha, \alpha' \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ and $\omega \in \mathrm{Ker}\,(\widehat{\imath}_{n-1}|_{\Omega_n})$ then $\overline{\alpha\alpha'\omega\alpha'^{-1}\alpha^{-1}} = \overline{c_{\alpha\alpha'}(\omega)} = \widehat{c}_\alpha(\widehat{c}_{\alpha'}(\overline{\omega}))$.

We next compute the elements $\overline{\tau\omega\tau^{-1}}$ of $\Delta(\mathrm{Ker}\,(\widehat{\imath}_{n-1}|_{\Omega_n}))$ in the case where $\tau = A_{i,j}$, $3 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant j-2$, and $\omega \in \widehat{\mathcal{B}}_n$:

(i) Let $\omega = A_{l,n+1}$, for $1 \leqslant l \leqslant n-1$. Then

$$\tau\omega\tau^{-1} = \begin{cases} A_{l,n+1} & \text{if } j < l \text{ or if } l < i \\ A_{l,n+1}^{-1}A_{i,n+1}^{-1}A_{l,n+1}A_{i,n+1}A_{l,n+1} & \text{if } j = l \\ A_{j,n+1}^{-1}A_{l,n+1}A_{j,n+1} & \text{if } i = l \\ A_{j,n+1}^{-1}A_{i,n+1}^{-1}A_{j,n+1}A_{i,n+1}A_{l,n+1}A_{i,n+1}^{-1}A_{j,n+1}^{-1}A_{i,n+1}A_{j,n+1} & \text{if } i < l < j \end{cases}$$

by the Artin relations. We thus conclude that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$ in this case.
(ii) If $\omega = \rho_{n+1}A_{l,n+1}\rho_{n+1}^{-1}$, where $1 \leqslant l \leqslant n-1$ then $\tau\omega\tau^{-1} = \rho_{n+1}(A_{i,j}A_{l,n+1}A_{i,j}^{-1})\rho_{n+1}^{-1}$, and from case (i), we deduce also that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$.
(iii) Let $\omega = \rho_{n+1}^2$. Then $\tau\omega\tau^{-1} = \omega$, hence $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$.

So if $\tau = A_{i,j}$ then the relators given by equation (28) are trivial for all $\omega \in \widehat{\mathcal{B}}_n$, and $\widehat{c}_{A_{i,j}} = \mathrm{Id}_{\Delta(\mathrm{Ker}(\widehat{\imath}_{n-1}|_{\Omega_n}))}$.

Now suppose that $\tau = \rho_j A_{i,j}\rho_j^{-1}$, where $3 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant j-2$. Then for all $\omega \in \widehat{\mathcal{B}}_n$, we have:

$$\overline{\tau\omega\tau^{-1}} = \overline{c_\tau(\omega)} = \widehat{c}_{\rho_j} \circ \widehat{c}_{A_{i,j}} \circ \widehat{c}_{\rho_j^{-1}}(\overline{\omega}) = \overline{\omega},$$

since $\widehat{c}_{A_{i,j}} = \mathrm{Id}_{\Delta(\mathrm{Ker}(\widehat{\imath}_{n-1}|_{\Omega_n}))}$, so $\widehat{c}_{\rho_j A_{i,j}\rho_{j-1}} = \mathrm{Id}_{\Delta(\mathrm{Ker}(\widehat{\imath}_{n-1}|_{\Omega_n}))}$.

By equation (18), it remains to study the elements of the form $\overline{\tau\omega\tau^{-1}}$, where $\tau = \rho_j^2$, $3 \leqslant j \leqslant n$, and $\omega \in \widehat{\mathcal{B}}_n$. Since $\overline{\rho_j^2\omega\rho_j^{-2}} = \widehat{c}_{\rho_j}^2(\overline{\omega})$, we first analyse $\widehat{c}_{\rho_j}$.

(a) If $\omega = A_{l,n+1}$, for $1 \leqslant l \leqslant n-1$ then by relation (III) and equations (8) and (11), we

have:

$$\widehat{c}_{\rho_j}(\overline{\omega}) = \widehat{c}_{\rho_j}(\overline{A_{l,n+1}}) = \overline{\rho_j A_{l,n+1}\rho_j^{-1}}$$
$$= \begin{cases} \overline{A_{l,n+1}} & \text{if } j < l \\ \overline{\rho_{n+1}^{-2}\cdot\rho_{n+1}C_{l,n+1}^{-1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^{2}} & \text{if } j = l \\ \overline{\rho_{n+1}^{-2}\cdot\rho_{n+1}C_{j,n+1}^{-1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^{2}\cdot A_{l,n+1}\cdot\rho_{n+1}^{-2}\cdot\rho_{n+1}C_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^{2}} & \text{if } j > l \end{cases}$$
$$= \begin{cases} \overline{A_{l,n+1}} & \text{if } j \neq l \\ \overline{\rho_{n+1}C_{j,n+1}^{-1}\rho_{n+1}^{-1}} = \left(\overline{\rho_{n+1}A_{j,n+1}\rho_{n+1}^{-1}}\right)^{-1} & \text{if } j = l. \end{cases}$$

(b) Let $\omega = \rho_{n+1}A_{l,n+1}\rho_{n+1}^{-1}$, where $1 \leqslant l \leqslant n-1$. Relation (IV) implies that $\rho_j\rho_{n+1}\rho_j^{-1} = C_{j,n+1}\rho_{n+1}$, and so by case (a) above, we have:

$$\widehat{c}_{\rho_j}(\overline{\omega}) = \widehat{c}_{\rho_j}\left(\overline{\rho_{n+1}A_{l,n+1}\rho_{n+1}^{-1}}\right) = \begin{cases} \overline{\rho_{n+1}A_{l,n+1}\rho_{n+1}^{-1}} & \text{if } j \neq l \\ \overline{C_{j,n+1}^{-1}} = \left(\overline{A_{j,n+1}}\right)^{-1} & \text{if } j = l. \end{cases}$$

(c) Let $\omega = \rho_{n+1}^2$. By relation (IV) and equations (8) and (11), we have:

$$\widehat{c}_{\rho_j}(\overline{\omega}) = \widehat{c}_{\rho_j}(\overline{\rho_{n+1}^2}) = \overline{(\rho_j\rho_{n+1}\rho_j^{-1})^2} = \overline{\rho_{n+1}C_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^2\cdot C_{j,n+1}}$$
$$= \overline{\rho_{n+1}A_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^2\cdot A_{j,n+1}},$$

from which we obtain:

$$\widehat{c}_{\rho_j^2}\left(\overline{\rho_{n+1}^2}\right) = \widehat{c}_{\rho_j}\left(\overline{\rho_{n+1}A_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^2\cdot A_{j,n+1}}\right)$$
$$= \overline{A_{j,n+1}^{-1}\cdot\rho_{n+1}A_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^2\cdot A_{j,n+1}\cdot\left(\rho_{n+1}A_{j,n+1}\rho_{n+1}^{-1}\right)^{-1}} = \overline{\rho_{n+1}^2}.$$

So by equation (16), we also have $\widehat{c}_{\rho_j^2} = \mathrm{Id}_{\Delta\left(\mathrm{Ker}\left(\widehat{\iota}_{n-1}|_{\Omega_n}\right)\right)}$. Hence for all $\tau \in L_{n-2}$ and $\omega \in \mathrm{Ker}\left(\widehat{\iota}_{n-1}|_{\Omega_n}\right)$, it follows that $\widehat{c}_\tau(\overline{\omega}) = \overline{\omega}$, and thus the relators $\overline{\psi(\tau)(\omega)\cdot\omega^{-1}}$ are all trivial. Since a presentation for $\Delta\left(\mathrm{Ker}\left(\widehat{\iota}_{n-1}|_{\Omega_n}\right)\right)$ is obtained by Abelianising a given presentation of $\mathrm{Ker}\left(\widehat{\iota}_{n-1}|_{\Omega_n}\right)$ and adjoining these relators, we conclude that $\Delta\left(\mathrm{Ker}\left(\widehat{\iota}_{n-1}|_{\Omega_n}\right)\right) = \left(\mathrm{Ker}\left(\widehat{\iota}_{n-1}|_{\Omega_n}\right)\right)^{\mathrm{Ab}}$. In particular, the fact that $\mathrm{Ker}\left(\widehat{\iota}_{n-1}|_{\Omega_n}\right)$ is a free group of rank $2n-1$ gives rise to the isomorphism (26). This completes the proof of the proposition. □

REMARKS 13.

(a) An alternative description of $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, similar to that of equation (12), but with the parentheses in the opposite order, may be obtained as follows. Let $m \geqslant 2$ and $q \geqslant 1$, and consider the following Fadell-Neuwirth short exact sequence:

$$1 \longrightarrow P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_{q+1}\}) \longrightarrow P_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_q\}) \longrightarrow$$
$$P_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_q\}) \longrightarrow 1, \quad (29)$$

given geometrically by forgetting the last $m-1$ strings. Since the quotient is a free group $\mathbb{F}_q$ of rank q, the above short exact sequence splits, and so

$$P_m(\mathbb{R}P^2\setminus\{x_1,\dots,x_q\}) \cong P_{m-1}(\mathbb{R}P^2\setminus\{x_1,\dots,x_{q+1}\}) \rtimes \mathbb{F}_q,$$

and thus

$$P_{n-2}(\mathbb{R}P^2\setminus\{x_1,x_2\})) \cong (\cdots((\mathbb{F}_{n-1}\rtimes\mathbb{F}_{n-2})\rtimes\mathbb{F}_{n-3})\rtimes\cdots\rtimes\mathbb{F}_3)\rtimes\mathbb{F}_2. \tag{30}$$

by induction. The ordering of the parentheses thus occurs from the left, in contrast with that of equation (12). The decomposition given by equation (12) is in some sense stronger than that of (30), since in the first case, every factor acts on each of the preceding factors, which is not necessarily the case in equation (30), so equation (12) engenders a decomposition of the form (30). This is a manifestation of the fact that the splitting of the corresponding Fadell-Neuwirth sequence (9) is non trivial, while that of (29) is obvious.

(b) Note that L_4, which is the kernel of the homomorphism $\widehat{\imath}_2\colon\ P_2(\mathbb{R}P^2\setminus\{x_1,x_2\}) \longrightarrow \mathbb{Z}_2^2$, is also the subgroup of index 4 of the group $(B_4(\mathbb{R}P^2))^{(3)}$ that appears in [GG8, Theorem 3(d)]. Indeed, by [GG8, equation (127)], this subgroup of index 4 is isomorphic to the semi-direct product:

$$\mathbb{F}_5(A_{1,4}, A_{2,4}, \rho_4^2, \rho_4A_{1,4}\rho_4^{-1}, \rho_4A_{2,4}\rho_4^{-1}) \rtimes \mathbb{F}_3(A_{2,3}, \rho_3^2, \rho_3A_{2,3}\rho_3^{-1}),$$

the action being given by [GG8, equations (129)–(131)] (the element $B_{i,j}$ of [GG8] is the element $A_{i,j}$ of this paper).

REMARK 14. Using the ideas of the last paragraph of the proof of Proposition 1(b), one may show that L_n is not normal in $B_n(\mathbb{R}P^2)$. Although the subgroup L_n is not unique with respect to the properties of the statement of Proposition 1(a)(ii), there are only a finite number of subgroups, $2^{n(n-2)}$ to be precise, that satisfy these properties. To prove this, we claim that the set of torsion-free subgroups L'_n of K_n such that $K_n = L'_n \oplus \langle\Delta_n^2\rangle$ is in bijection with the set $\{\mathrm{Ker}\,(f)\mid f\in \mathrm{Hom}(L_n,\mathbb{Z}_2)\}$. To prove the claim, let $K=K_n$, $L=L_n$, let $q\colon\ K\longrightarrow K/L$ be the canonical surjection, and set

$$\Delta = \left\{ L' \;\middle|\; L'<K,\ L' \text{ is torsion free, and } K = L'\oplus\langle\Delta_n^2\rangle \right\}.$$

Clearly $L\in\Delta$, so $\Delta\neq\varnothing$. Consider the map $\varphi\colon\ \Delta\longrightarrow\{\mathrm{Ker}\,(f)\mid f\in\mathrm{Hom}(L,\mathbb{Z}_2)\}$ defined by $\varphi(L')=L\cap L'$. This map is well defined, since if $L'=L$ then $\varphi(L')=L$ is the kernel of the trivial homomorphism of $\mathrm{Hom}(L,\mathbb{Z}_2)$, and if $L'\neq L$ then $L'\nsubseteq L$ since $[K:L']=[K:L]=2$, and so $q\,|_{L'}$ is surjective as $K/L\cong\mathbb{Z}_2$. Thus $\mathrm{Ker}\,(q\,|_{L'})=\varphi(L')$ is of index 2 in L, in particular, $\varphi(L')$ is the kernel of some non-trivial element of $\mathrm{Hom}(L,\mathbb{Z}_2)$.

We now prove that φ is surjective. Let $f\in\mathrm{Hom}(L,\mathbb{Z}_2)$, and set $L''=\mathrm{Ker}\,(f)$. If $f=0$ then $L''=L$, and $\varphi(L)=L''$. So suppose that $f\neq 0$. Then f is surjective, and $L''=\mathrm{Ker}\,(f)$ is of index 2 in L. Let $x\in L\setminus L''$. Then

$$L = L''\amalg xL'', \tag{31}$$

where $\amalg$ denotes the disjoint union. Since $K = L \amalg \Delta_n^2 L$, it follows that

$$K = L'' \amalg xL'' \amalg \Delta_n^2 L'' \amalg x\Delta_n^2 L'', \tag{32}$$

where $\amalg$ denotes the disjoint union. Set $L' = L'' \amalg x\Delta_n^2 L''$. By equation (31), $x^2\Delta_n^2 L'' = \Delta_n^2 x^2 L'' = \Delta_n^2 L''$ because Δ_n^2 is central and of order 2, and hence $K = L' \amalg xL'$. Using once more equation (31), we see that L' is a group, and so the equality $K = L' \amalg xL'$ implies that $[K : L'] = 2$. Further, since the only non-trivial torsion element of K is Δ_n^2, L' is torsion free by equation (32), and so the short exact sequence $1 \longrightarrow L' \longrightarrow K \longrightarrow \mathbb{Z}_2 \longrightarrow 1$ splits. Thus $L' \in \Delta$, and $\varphi(L') = L''$ using equations (31) and (32).

It remains to prove that φ is injective. Let $L_1', L_2' \in \Delta$ be such that $L_1' \cap L = \varphi(L_1') = \varphi(L_2') = L_2' \cap L$. If one of the L_i', L_1' say, is equal to L then we must also have $L_2' = L$ because $L \subset L_2'$ and L and L_2' have the same index in K. So suppose that $L_i' \neq L$ for all $i \in \{1,2\}$. If $i \in \{1,2\}$ then $L'' = \varphi(L_i') = L \cap L_i' = \mathrm{Ker}\,(f_i)$ for some non-trivial $f_i \in \mathrm{Hom}(L, \mathbb{Z}_2)$, and thus $[L : L''] = 2$. Let us show that $L_1' \subset L_2'$. Let $x \in L_1'$. If $x \in L$ then $x \in L''$, so $x \in L_2'$, and we are done. So assume that $x \notin L$, and suppose that $x \notin L_2'$. Then $q(x)$ is equal to the non-trivial element of K/L, and since $K/L \cong \mathbb{Z}_2$ and $\Delta_n^2 \notin L$, we see that $x\Delta_n^2 \in L$. Further, $K = L_2' \amalg xL_2'$ since $[K : L_2'] = 2$, and so $x\Delta_n^2 \in L_2'$ (for otherwise $x\Delta_n^2 \in xL_2'$, which implies that $\Delta_n^2 \in L_2'$, which is impossible because L_2' is torsion free). But then $x\Delta_n^2 \in L \cap L_2' = L''$, and hence $x\Delta_n^2 \in L_1'$. But this would imply that $\Delta_n^2 \in L_1'$, which contradicts the fact that L_1' is torsion free. We conclude that $L_1' \subset L_2'$, and exchanging the rles of L_1' and L_2', we see that $L_1' = L_2'$, which proves that φ is injective, so is bijective, which proves the claim. Therefore the cardinality of Δ is equal to the order of the group $H^1(L, \mathbb{Z}_2)$, which is equal in turn to that of $H_1(L, \mathbb{Z}_2)$. By Proposition 4(b), we have $L^{\mathrm{Ab}} = H_1(L, \mathbb{Z}) \cong \mathbb{Z}^{n(n-2)}$, so $H_1(L, \mathbb{Z}_2) \cong H_1(L, \mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2^{n(n-2)}$, and the number of subgroups of K that satisfy the properties of Proposition 1(a) is equal to $2^{n(n-2)}$ as asserted.

4 The virtual cohomological dimension of $B_n(S)$ and $P_n(S)$ for $S = \mathbb{S}^2, \mathbb{R}P^2$

Let $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$), and for all $m, n \geqslant 1$, let $\Gamma_{n,m}(S) = P_n(S \setminus \{x_1, \ldots, x_m\})$ denote the n-string pure braid group of S with m points removed. In order to study various cohomological properties of the braid groups of S and prove Theorem 5, we shall study $\Gamma_{n,m}(S)$. To prove Theorem 5 in the case $S = \mathbb{S}^2$, by equation (2), it will suffice to compute the cohomological dimension of $P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$. We recall the following presentation of $\Gamma_{n,m}(\mathbb{S}^2)$ from [GG3]. The result was stated for $m \geqslant 3$, but it also holds for $m \leqslant 2$.

PROPOSITION 15 ([GG3, Proposition 7]). *Let $n, m \geqslant 1$. The following constitutes a presentation of the group $\Gamma_{n,m}(\mathbb{S}^2)$:*

generators: *$A_{i,j}$, where $1 \leqslant i < j$ and $m+1 \leqslant j \leqslant m+n$.*

relations:

(i) the Artin relations described by equation (4) among the generators $A_{i,j}$ of $\Gamma_{n,m}(\mathbb{S}^2)$.

(ii) for all $m+1 \leqslant j \leqslant m+n$, $\left(\prod_{i=1}^{j-1} A_{i,j}\right)\left(\prod_{k=j+1}^{m+n} A_{j,k}\right) = 1$.

Let N denote the kernel of the homomorphism $\Gamma_{n,m}(S) \longrightarrow \Gamma_{n-1,m}(S)$ obtained geometrically by forgetting the last string. If $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$) then N is a free group of rank $m+n-2$ (resp. $m+n-1$) and is equal to $\langle A_{1,m+n}, A_{2,m+n}, \ldots, A_{m+n-1,m+n}\rangle$ (resp. $\langle A_{1,m+n}, A_{2,m+n}, \ldots, A_{m+n-1,m+n}, \rho_{m+n}\rangle$). Clearly N is normal in $\Gamma_{n,m}(S)$. Further, it follows from relations (i) of Proposition 15 (resp. relations (III) and (IV) of Proposition 11) that the action by conjugation of $\Gamma_{n,m}(S)$ on N induces (resp. does not induce) the trivial action on the Abelianisation of N. In order to determine the virtual cohomological dimension of the braid groups of S and prove Theorem 5, we shall compute the cohomological dimension of a torsion-free finite-index subgroup. In the case of $\mathbb{S}^2$ (resp. $\mathbb{R}P^2$), we choose the subgroup $\Gamma_{n-3,3}(\mathbb{S}^2)$ that appears in the decomposition given in equation (2) (resp. the subgroup $\Gamma_{n-2,2}(\mathbb{R}P^2)$ that appears in equation (3)).

Proof of Theorem 5. Let $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$), let $n > 3$ and $k = 3$ (resp. $n > 2$ and $k = 2$), and let $k \leqslant m < n$. Then by equation (2) (resp. equation (3)) and equation (1), $\Gamma_{n-m,m}(S)$ is a subgroup of finite index of both $P_n(S)$ and $B_n(S)$. Further, since $F_{n-m}(S\backslash\{x_1,\ldots,x_m\})$ is a finite-dimensional CW-complex and an Eilenberg-Mac Lane space of type $K(\pi,1)$ [FaN], the cohomological dimension of $\Gamma_{n-m,m}(S)$ is finite, and the first part follows by taking $m = k$.

We now prove the second part, namely that the cohomological dimension of $\Gamma_{n-k,k}(S)$ is equal to $n-k$ for all $n > k$. We first claim that $\mathrm{cd}(\Gamma_{m,l}(S)) \leqslant m$ for all $m \geqslant 1$ and $l \geqslant k-1$. The result holds if $m = 1$ since $F_1(S\backslash\{x_1,\ldots,x_l\})$ has the homotopy type of a bouquet of circles, therefore $H^i(F_1(S\backslash\{x_1,\ldots,x_l\}), A)$ is trivial for all $i \geqslant 2$ and for any local coefficients A, and $H^1(F_1(S\backslash\{x_1,\ldots,x_l\}), \mathbb{Z}) \neq 0$. Suppose by induction that the result holds for some $m \geqslant 1$, and consider the Fadell-Neuwirth short exact sequence:

$$1 \longrightarrow \Gamma_{1,l+m}(S) \longrightarrow \Gamma_{m+1,l}(S) \longrightarrow \Gamma_{m,l}(S) \longrightarrow 1$$

that emanates from the fibration:

$$F_1(S\backslash\{x_1,\ldots,x_l,z_1,\ldots,z_m\}) \longrightarrow F_{m+1}(S\backslash\{x_1,\ldots,x_l\}) \longrightarrow F_m(S\backslash\{x_1,\ldots,x_l\}) \qquad (33)$$

obtained by forgetting the last coordinate. By [Br, Chapter VIII], it follows that:

$$\mathrm{cd}(\Gamma_{m+1,l}(S)) \leqslant \mathrm{cd}(\Gamma_{m,l}(S)) + \mathrm{cd}(\Gamma_{1,l+m}(S)) \leqslant m+1.$$

which proves the claim. In particular, taking $l = k$, we have $\mathrm{cd}(\Gamma_{m,k}(S)) \leqslant m$.

To conclude the proof of the theorem, it remains to show that for each $m \geqslant 1$ there are local coefficients A such that $H^m(\Gamma_{m,l}(S), A) \neq 0$ for all $l \geqslant k$. We will show that this is the case for $A = \mathbb{Z}$. Again by induction suppose that $H^m(\Gamma_{m,l}(S), \mathbb{Z}) \neq 0$ for all $l \geqslant k-1$ and for some $m \geqslant 1$ (we saw above that this is true for $m = 1$). Consider the Serre spectral sequence with integral coefficients associated to the fibration (33). Then we have that

$$E_2^{p,q} = H^p\big(\Gamma_{m,l}(S), H^q(F_1(S\backslash\{x_1,\ldots,x_l,z_1,\ldots,z_m\}), \mathbb{Z})\big).$$

Since $\mathrm{cd}(\Gamma_{m,l}(S)) \leqslant m$ and $\mathrm{cd}(F_1(S\backslash\{x_1,\ldots,x_l,z_1,\ldots,z_m\}) \leqslant 1$ from above, it follows that this spectral sequence has two horizontal lines whose possible non-vanishing terms

occur for $0 \leqslant p \leqslant m$ and $0 \leqslant q \leqslant 1$. We claim that the group $E_2^{m,1}$ is non trivial. To see this, first note that $H^1(F_1(S\backslash\{x_1,\dots,x_l,z_1,\dots,z_m\}),\mathbb{Z})$ is isomorphic to the free Abelian group of rank $r = m+l-k+2$, so $r \geqslant m+2$, and hence $E_2^{m,1} = H^m\big(\Gamma_{m,l}(S),\mathbb{Z}^r\big)$, where we identify $\mathbb{Z}^r$ with (the dual of) N^{Ab}. The action of $\Gamma_{m,l}(S)$ on N by conjugation induces an action of $\Gamma_{m,l}(S)$ on N^{Ab}. Let H be the subgroup of N^{Ab} generated by the elements of the form $\alpha(x)-x$, where $\alpha \in \Gamma_{m,l}(S)$, $x \in N^{\mathrm{Ab}}$, and $\alpha(x)$ represents the action of α on x. Then we obtain a short exact sequence $0 \longrightarrow H \longrightarrow N^{\mathrm{Ab}} \longrightarrow N^{\mathrm{Ab}}/H \longrightarrow 0$ of Abelian groups, and the long exact sequence in cohomology applied to $\Gamma_{m,l}(S)$ yields:

$$\cdots \longrightarrow H^m(\Gamma_{m,l}(S),N^{\mathrm{Ab}}) \longrightarrow H^m(\Gamma_{m,l}(S),N^{\mathrm{Ab}}/H) \longrightarrow H^{m+1}(\Gamma_{m,l}(S),H) \longrightarrow \cdots . \quad (34)$$

The last term is zero since $\mathrm{cd}(\Gamma_{m,l}(S)) \leqslant m$, and so the map between the two remaining terms is surjective. Let us determine N^{Ab}/H. If $S = \mathbb{S}^2$ then from the comments following Proposition 15, the action of $\Gamma_{m,l}(S)$ on N^{Ab} is trivial, so H is trivial, and $N^{\mathrm{Ab}}/H \cong \mathbb{Z}^r$. So suppose that $S = \mathbb{R}P^2$. Choosing the basis

$$\big\{A_{1,m+l+1}, A_{2,m+l+1}, \dots, A_{m+l-1,m+l+1}, \rho_{m+l+1}\big\}$$

of N and using Proposition 11, one sees that the action by conjugation of the generators of $\Gamma_{m,l}(S)$ on the corresponding basis elements of N^{Ab} is trivial, with the exception of that of ρ_i on $A_{i,m+l+1}$ for $l+1 \leqslant i \leqslant m+l-1$, which yields elements $A_{i,m+l+1}^2 \in H$ (by abuse of notation, we denote the elements of N^{Ab} in the same way as those of N), and that of ρ_i on ρ_{m+l+1}, where $l+1 \leqslant i \leqslant m+l$, which yields elements $A_{i,m+l+1} \in H$. In the quotient N^{Ab}/H the basis elements $A_{l+1,m+l+1}, \dots, A_{m+l-1,m+l+1}$ thus become zero, and additionally, we have also that $A_{m+l,m+l+1}$ (which is not in the given basis) becomes zero. Hence the relation $\prod_{i=1}^{m+l} A_{i,m+l+1} = \rho_{m+l+1}^{-2}$ is sent to the relation $\prod_{i=1}^{l} A_{i,m+l+1} = \rho_{m+l+1}^{-2}$, and so N^{Ab}/H is generated by (the images of) the elements $A_{1,m+l+1}, \dots, A_{l,m+l+1}, \rho_{m+l+1}$, subject to this relation (as well as the fact that the elements commute pairwise). It thus follows that $N^{\mathrm{Ab}}/H \cong \mathbb{Z}^l$. Since the induced action of $\Gamma_{m,l}(S)$ on N^{Ab}/H is trivial, we conclude that

$$H^m(\Gamma_{m,l}(S),N^{\mathrm{Ab}}/H) = \big(H^m(\Gamma_{m,l}(S),\mathbb{Z})\big)^s,$$

where $s = m+l$ if $S = \mathbb{S}^2$ and $s = l$ if $S = \mathbb{R}P^2$. It then follows from equation (34) that $E_2^{m,1} = H^m(\Gamma_{m,l}(S),N^{\mathrm{Ab}}) \neq 0$. Since $E_2^{p,q} = 0$ for all $p > m$ and $q > 1$, we have $E_2^{m,1} = E_\infty^{m,1}$, thus $E_\infty^{m,1}$ is non trivial, and hence $H^{m+1}(\Gamma_{m+1,l}(S),\mathbb{Z}) \neq 0$. This concludes the proof of the theorem. □

We end this paper with a proof of Corollary 6.

Proof of Corollary 6. Let $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$). If $n \geqslant 3$ (resp. $n \geqslant 2$) then $B_n(S)$ and $\mathcal{MCG}(S,n)$ are closely related by the following short exact sequence [S]:

$$1 \longrightarrow \big\langle \Delta_n^2 \big\rangle \longrightarrow B_n(S) \xrightarrow{\beta} \mathcal{MCG}(S,n) \longrightarrow 1,$$

where the kernel is isomorphic to $\mathbb{Z}_2$. Now assume that $n \geqslant 4$ (resp. $n \geqslant 3$), so that $B_n(S)$ is infinite. If Γ is a torsion-free subgroup of $B_n(S)$ of finite index then $\beta(\Gamma)$, which is isomorphic to Γ, is a torsion-free subgroup of $\mathcal{MCG}(S,n)$ of finite index, and hence the virtual cohomological dimension of $\mathcal{MCG}(S,n)$ is equal to that of $B_n(S)$. The result then follows by Theorem 5. □

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