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Anisotropic Minkowski Content and Application to minimizers of free discontinuity problems

David Vicente

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Abstract

This paper deals with a generalization of a result of Geometric Measure Theory: if a rectifiable set is closed, then its Minkowski content is equal to its Hausdorff measure. This result is generalized to an anisotropic and continuous perturbation of the Hausdorff measure. So, an adapted Minkowski content is introduced and the proof that it is equal to the perturbed Hausdorff measure under the same hypothesis of closure and rectifiability is given. This result is applied to the jump set of a minimizer of a free boundary problem for the case where the Hausdorff measure of the boundary is perturbed by an anisotropic and continuous function and a rigorous setting is given for the approximation of an anisotropic version for the Mumford-Shah functional.

Introduction

For $S \subset \mathbb{R}^n$, the $(n-1)$-dimensional upper and lower Minkowski contents of S are defined by

$$\mathcal{M}^\star(S) = \limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x\colon \operatorname{dist}(x,S) < \rho\})}{2\rho}, \quad \mathcal{M}_\star(S) = \liminf_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x\colon \operatorname{dist}(x,S) < \rho\})}{2\rho},$$

where $\mathcal{L}^n$ is the Lebesgue measure. If $\mathcal{M}^\star(S) = \mathcal{M}_\star(S)$, their common value, denoted by $\mathcal{M}(S)$, is called the $(n-1)$-dimensional Minkowski content. In [1] (Theorem 3.2.39), the following result is given.

Theorem 0.1 (Federer)**.** *If S is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$, then*

$$\mathcal{M}(S) = \mathcal{H}^{n-1}(S),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

In this article, we focus on a class of anisotropic perturbation of $\mathcal{H}^{n-1}$ defined on $(n-1)$-rectifiable surfaces $S \subset \mathbb{R}^n$ by

$$\int_S \langle \mathbf{M}\nu, \nu \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}, \tag{0.1}$$

where $\mathbf{M} : \mathbb{R}^n \to \mathrm{S}_n^+(\mathbb{R})$ is a continuous field of symmetric definite positive matrices, ν is an unitary vector orthogonal to the tangent plane of S. As S is rectifiable, then ν is well defined $\mathcal{H}^{n-1}$-almost everywhere in S. In order to get an approximation of (0.1) which involves the Lebesgue measure, as in Theorem 0.1, for any $(x, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$, we consider the following Riemannian metric:

$$\phi(x, \mathbf{v}) = \langle \mathbf{M}^{-1}(x)\mathbf{v}, \mathbf{v} \rangle^{1/2}$$

and for any $(x, y) \in (\mathbb{R}^n)^2$ its associated integrated distance:

$$\begin{aligned} \operatorname{dist}_\phi(x,y) &= \inf \left\{ \int_0^1 \phi\left(\gamma, \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) \mathrm{d}t \colon \begin{array}{c} \gamma \in W^{1,1}([0;1];\mathbb{R}^n), \\ \gamma(0) = x, \gamma(1) = y \end{array} \right\}, \\ \operatorname{dist}_\phi(x,S) &= \inf \left\{ \operatorname{dist}_\phi(x,y) \colon y \in S \right\}. \end{aligned} \tag{0.2}$$

Then, the associated anisotropic Minkowski $(n-1)$-dimensional upper and lower contents are defined by the limits

$$\mathcal{M}^\star_{\mathbf{M}}(S) = \limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x\colon \operatorname{dist}_\phi(x,S) < \rho\})}{2\rho}, \quad \mathcal{M}_{\star\mathbf{M}}(S) = \liminf_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x\colon \operatorname{dist}_\phi(x,S) < \rho\})}{2\rho}.$$

If $\mathcal{M}^\star_{\mathbf{M}}(S) = \mathcal{M}_{\star\mathbf{M}}(S)$, we call their common value the $(n-1)$-dimensional anisotropic Minkowski content $\mathcal{M}_{\mathbf{M}}(S)$. In this paper, we prove the following

Theorem 0.2. *If S is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$ and $\mathbf{M} : \mathbb{R}^n \to S^+_n(\mathbb{R})$ is continuous, then we have*

$$\mathcal{M}_{\mathbf{M}}(S) = \int_S \langle \mathbf{M}\nu, \nu\rangle^{1/2} d\mathcal{H}^{n-1},$$

where ν is an unitary vector, normal to S.

A comparable result is given in [2] (Theorem 6.1) for $S = \partial E$ and E is a set of finite perimeter. In [3], the study focuses on anisotropic outer Minkowski content for the same class of sets. In our case, S is not necessary the boundary of a set. This generalization is of interest for the case where S is the jump set J_u of a function u with bounded variation as we will see in last section.

Our result allows us to extend a regularity result for free boundary problems to a larger class of energies which involve a continuous perturbation of the Hausdorff measure. More precisely, as it has been introduced in [4], $u \in \mathrm{SBV}(\Omega)$ is an *almost quasi-minimizer of a free discontinuity problem* if there exist $\sigma \geq 1$, $\alpha > 0$ and $c_\alpha > 0$ such that: if $v \in \mathrm{SBV}(\Omega)$, $\overline{B(x,r)} \subset \Omega$ and $[w \neq v] \subset B(x,r)$, then we have

$$\int_{B(x,r)} |\nabla w|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_w \cap \overline{B(x,r)}) \leq \int_{B(x,r)} |\nabla v|^2 \mathrm{d}x + \sigma\mathcal{H}^{n-1}(J_v \cap \overline{B(x,r)}) + c_\alpha r^{n-1+\alpha},$$

where J_u is the jump set of u. The difference of this definition with respect to the classical notion of *quasi-minimizer* consists in the coefficient $\sigma \geq 1$ which allows bounded perturbation of $\mathcal{H}^{n-1}$. We prove the following

Theorem 0.3. *Let $\mathbf{M} : \mathbb{R}^n \to S^+_n(\mathbb{R})$ be continuous and $u \in SBV(\Omega)$ be an almost quasi-minimizer of a free discontinuity problem, then we have*

$$\mathcal{M}_{\mathbf{M}}(J_u) = \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} d\mathcal{H}^{n-1}.$$

This result is applied to the Mumford-Shah model in an anisotropic setting. Indeed, a minimizer this model is an almost quasi-minimizer of a free boundary problem and the approximation result of Theorem 3.1 is satisfied. In a joint paper [5], this is the key tool to perform a Γ-convergence approximation of our model.

In section 1, we recall some classical results of Geometric Measure Theory. Section 2 is devoted to the proof of Theorem 0.2. In section 3, we give the proof of Theorem 0.3 and we apply this result to give a rigorous setting for the approximation of an anisotropic Mumford-Shah model.

1 Geometric Measure Theory Framework

We adopt the notations:

- Ω for an open and bounded subset of $\mathbb{R}^n$,
- $\langle \mathbf{v}_1, \mathbf{v}_2\rangle \in \mathbb{R}$ for the canonical scalar product of $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$,
- $\bigwedge_{i=1}^{n-1} \mathbf{v}_i \in \mathbb{R}^n$ for the canonical vectorial product of $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \mathbb{R}^n$,
- $|\mathbf{v}|$ for the euclidean norm of $\mathbf{v} \in \mathbb{R}^n$,
- $B(x,r)$ for the open ball in $\mathbb{R}^n$ with radius r and center x,
- $\mathrm{M}_n(\mathbb{R})$ for the space of $n \times n$ real matrices,
- $\|\mathbf{M}\|$ for an induced norm of $\mathbf{M} \in \mathrm{M}_n(\mathbb{R})$,
- $\mathrm{S}^+_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ for the subset of symmetric definite positive matrices,
- $\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ for the subset of invertible matrices,
- $\mathrm{O}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ for the subgroup of orthogonal matrices,
- $\mathcal{B}(\Omega)$ the class of Borelian subsets of Ω,
- $\mathcal{L}^n$ for the Lebesgue measure in $\mathbb{R}^n$,
- $\mathcal{H}^k$ for the k-dimensional Hausdorff measure.

The geometric measure theory framework is mainly extracted from [1] and [2].

Definition 1.1 (Federer, 3.2.1)**.** *Let f maps a subset of $\mathbb{R}^{n-1}$ onto $\mathbb{R}^n$, the $(n-1)$-dimensional Jacobian is defined by*

$$\boldsymbol{J}_{n-1}(f)(a) = \left| \bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(a) \right|$$

whenever f is differentiable at a.

Theorem 1.1 (Area Formula, Federer, 3.2.3)**.** *Suppose $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is Lipshitzian. If u is an $\mathcal{L}^{n-1}$-integrable function, then*

$$\int_{\mathbb{R}^{n-1}} u(x) \boldsymbol{J}_{n-1}(f)(x) d\mathcal{L}^{n-1}(x) = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}\{y\}} u(x) d\mathcal{H}^{n-1}(y).$$

We are interested in the class of subset of $\mathbb{R}^n$ which are $(n-1)$-rectifiable.

Definition 1.2 (Federer, 3.2.14)**.** *The set $E \subset \mathbb{R}^n$ is $(n-1)$-rectifiable if there exists a Lipshitzian function $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ mapping some bounded subset of $\mathbb{R}^n$ onto E.*

The following result is useful to get an univalent parametrization of a rectifiable set.

Theorem 1.2 (Federer, 3.2.4)**.** *For every $\mathcal{L}^{n-1}$ measurable set A, there exists a Borel set*

$$B \subset A \cap \{x \colon \boldsymbol{J}_{n-1}(f)(x) > 0\}$$

such that $f|_B$ is univalent and $\mathcal{H}^{n-1}(f(A) \setminus f(B)) = 0$.

We say that $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ is a Finsler metric if φ is continuous and if there exists $\lambda, \Lambda > 0$ such that, for any $(x, \mathbf{v}, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}$, it satisfies

$$\phi(x, t\mathbf{v}) = |t|\phi(x, \mathbf{v}), \qquad \lambda|\mathbf{v}| \leq \phi(x, \mathbf{v}) \leq \Lambda|\mathbf{v}|.$$

We define the integrated distance associated to φ as

$$\forall (x, y) \in (\mathbb{R}^n)^2, \quad \mathrm{dist}_\varphi(x, y) = \inf \left\{ \int_0^1 \varphi\left(\gamma, \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) \mathrm{d}t \colon \begin{array}{c} \gamma \in W^{1,1}([0;1]; \mathbb{R}^n), \\ \gamma(0) = x, \gamma(1) = y \end{array} \right\}.$$

For any $E \subset \mathbb{R}^n$, we define the associated diameter as

$$\mathrm{diam}_\varphi(E) = \sup\{\mathrm{dist}_\varphi(x, y) \colon x, y \in E\}$$

and the associated k-dimensional Hausdorff measure by

$$\mathcal{H}^k_\varphi(E) = \lim_{\delta \to 0^+} \inf \left\{ \frac{\omega_k}{2^k} \sum_{i \in I} (\mathrm{diam}_\varphi(A_i))^k \colon E \subset \bigcup_{i \in I} A_i, \forall i \in I, \mathrm{diam}_\varphi(A_i) \leq \delta \right\}$$

where ω_k is the volume of the unit ball in $\mathbb{R}^k$.

Theorem 1.3 (Bellettini)**.** *Let φ be a Finsler metric, then, for all Borel set $E \subset \mathbb{R}^n$, we have*

$$\mathcal{H}^n_\varphi(E) = \int_E \frac{\mathcal{H}^n(B(0,1))}{\mathcal{H}^n(B_\varphi(x,1))} dx,$$

where $B_\varphi(x, 1)$ is the unit ball centered at x for the metric φ.

2 Proof of Theorem 0.2

The proof is divided in two steps. First, we assume in section 2.1 that the metric does not depend on x. Then, in section 2.2, we remove this assumption and we prove the Theorem 0.2 in the general setting.

2.1 Homogeneous Case

In this section, we assume that the metric is homogeneous. So, we denote by $\mathbf{M}_0$ its common value and, for any $\mathbf{v} \in \mathbb{R}^n$, we set

$$\phi_0(\mathbf{v}) = \langle \mathbf{M}_0^{-1}\mathbf{v}, \mathbf{v}\rangle^{1/2}, \tag{2.1}$$

so that $(\mathbb{R}^n, \phi_0)$ is an Euclidean space. We prove the following

Theorem 2.1. *If S is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$ and $\mathbf{M}_0 \in S_n^+(\mathbb{R})$. Then we have*

$$\mathcal{M}_{\mathbf{M}_0}(S) = \int_S \langle \mathbf{M}_0\nu, \nu\rangle^{1/2} d\mathcal{H}^{n-1},$$

where ν is an unitary and normal vector to S.

We decompose the proof in three Lemmas. As $\mathbf{M}_0^{-1}$ is symmetric positive definite, $\mathbb{R}^n$ may be viewed as an euclidean space according to the scalar product $\langle \mathbf{M}_0^{-1}\cdot, \cdot\rangle$. So, Theorem 0.1 may be directly applied. The three following Lemmas consist in the change of variable from $\mathbb{R}^n$ endowed with the scalar product $\langle \mathbf{M}_0^{-1}\cdot, \cdot\rangle$ to $\mathbb{R}^n$ endowed with the canonical scalar product.

Lemma 2.1. *For $\mathbf{M}_0 \in S_n^+(\mathbb{R})$, ϕ_0 defined by (2.1) and for any $E \in \mathcal{B}(\mathbb{R}^n)$, we have*

$$\mathcal{L}^n(E) = \sqrt{det(\mathbf{M}_0)}\mathcal{H}^n_{\phi_0}(E).$$

Proof. As $\mathbf{M}_0$ is a symmetric positive definite matrix, there exists $P \in \mathrm{O}_n(\mathbb{R})$ and $(\lambda_i)_{i=1\dots n}$ such that $\lambda_i > 0$ for any $i \in \{1, \dots, n\}$ and

$$\mathbf{M}_0 = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} P^{-1}. \tag{2.2}$$

We denote by $(c_i)_{i=1\dots n}$ the canonical basis of $\mathbb{R}^n$, then $(\sqrt{\lambda_i}Pc_i)_{i=1\dots n}$ is an orthonormal basis for the scalar product $\langle \mathbf{M}_0^{-1}\cdot, \cdot\rangle$. So, the linear application $L : \mathbb{R}^n \to \mathbb{R}^n$ characterized by $L(c_i) = \sqrt{\lambda_i}Pc_i$ for any $i \in \{1, \dots, n\}$, is an isomorphism which satisfies $L(B(0,1)) = B_{\phi_0}(0,1)$ and $\det(L) = \sqrt{\det(\mathbf{M}_0)}$. It gives

$$\begin{aligned} \mathcal{L}^n(B_{\phi_0}(0,1)) &= \int_{B(0,1)} |\det(L)| \mathrm{d}x, \\ &= \sqrt{\det(\mathbf{M}_0)}\mathcal{L}^n(B(0,1)) \end{aligned}$$

According to isodiametric inequality (see 2.10.35 in [1]), we have $\mathcal{H}^n = \mathcal{L}^n$ and then, for any $E \in \mathcal{B}(\mathbb{R}^n)$, Theorem 1.3 yields

$$\mathcal{H}^n_{\phi_0}(E) = \frac{\mathcal{L}^n(E)}{\sqrt{\det(\mathbf{M}_0)}}.$$

which concludes the proof of Lemma 2.1. □

Setting $e_i = \sqrt{\lambda_i}Pc_i$, then $(e_i)_{i=1,\dots,n}$ is an orthonormal basis of $\mathbb{R}^n$ for the scalar product $\langle \mathbf{M}_0^{-1}\cdot, \cdot\rangle$. In the Euclidean space $\mathbb{R}^n, \phi_0)$, the vectorial product of $(\mathbf{v}_i)_{i=1,\dots,n-1}$, denoted by $\bigwedge_{\phi_0, i=1}^{n-1}\mathbf{v}_i$, is characterized by the following equality

$$\begin{vmatrix} \langle \mathbf{M}_0^{-1}\mathbf{v}_1, e_1\rangle & \dots & \langle \mathbf{M}_0^{-1}\mathbf{v}_{n-1}, e_1\rangle & \langle \mathbf{M}_0^{-1}\mathbf{w}, e_1\rangle \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \langle \mathbf{M}_0^{-1}\mathbf{v}_1, e_n\rangle & \dots & \langle \mathbf{M}_0^{-1}\mathbf{v}_{n-1}, e_n\rangle & \langle \mathbf{M}_0^{-1}\mathbf{w}, e_n\rangle \end{vmatrix} = \left\langle \mathbf{M}_0^{-1}\left(\bigwedge_{\phi_0, i=1}^{n-1}\mathbf{v}_i\right), \mathbf{w}\right\rangle, \tag{2.3}$$

which must be true for any $\mathbf{w} \in \mathbb{R}^n$.

Lemma 2.2. *If $\mathbf{M}_0 \in S_n^+(\mathbb{R})$ and ϕ_0 defined by (2.1), then we have*

$$\bigwedge_{\phi_0, i=1}^{n-1}\mathbf{v}_i = \frac{\mathbf{M}_0(\wedge_{i=1}^{n-1}\mathbf{v}_i)}{\sqrt{det(\mathbf{M}_0)}}.$$

Proof. As $P \in \mathrm{O}_n(\mathbb{R})$, then $(Pc_i)_{i=1}^n$ is an orthonormal basis for the usual scalar product and, for any $\mathbf{w} \in \mathbb{R}^n$, we have

$$\begin{vmatrix} \langle \mathbf{v}_1, Pc_1 \rangle & \dots & \langle \mathbf{v}_{n-1}, Pc_1 \rangle & \langle \mathbf{w}, Pc_1 \rangle \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \langle \mathbf{v}_1, Pc_n \rangle & \dots & \langle \mathbf{v}_{n-1}, Pc_n \rangle & \langle \mathbf{w}, Pc_n \rangle \end{vmatrix} = \left\langle \wedge_{i=1}^{n-1} \mathbf{v}_i, \mathbf{w} \right\rangle.$$

According to (2.2), for any $\mathbf{v} \in \mathbb{R}^n$, we have

$$\langle \mathbf{v}, Pc_i \rangle = \sqrt{\lambda_i} \langle \mathbf{M}_0^{-1} \mathbf{v}, \sqrt{\lambda_i} Pc_i \rangle$$

and then

$$\sqrt{\lambda_1 \dots \lambda_n} \begin{vmatrix} \langle \mathbf{M}_0^{-1} \mathbf{v}_1, e_1 \rangle & \dots & \langle \mathbf{M}_0^{-1} \mathbf{v}_{n-1}, e_1 \rangle & \langle \mathbf{M}_0^{-1} \mathbf{w}, e_1 \rangle \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \langle \mathbf{M}_0^{-1} \mathbf{v}_1, e_n \rangle & \dots & \langle \mathbf{M}_0^{-1} \mathbf{v}_{n-1}, e_n \rangle & \langle \mathbf{M}_0^{-1} \mathbf{w}, e_n \rangle \end{vmatrix} = \left\langle \mathbf{M}_0^{-1} \left(\mathbf{M}_0 \wedge_{i=1}^{n-1} \mathbf{v}_i \right), \mathbf{w} \right\rangle.$$

According to (2.3), it concludes the proof of Lemma 2.2.

□

As for the vectorial product, we may define an anisotropic $(n-1)$-dimensional Jacobian by

$$\mathbf{J}_{n-1}^{\phi_0}(f)(a) = \left| \bigwedge_{\phi_0, i=1}^{n-1} \frac{\partial f}{\partial x_i}(a) \right|$$

whenever f is differentiable at a. The following result is a straightforward consequence of Lemma 2.2.

Lemma 2.3. *Let $\mathbf{M}_0 \in S_n^+(\mathbb{R})$ be fixed and ϕ_0 defined by (2.1), then we have*

$$\boldsymbol{J}_{n-1}^{\phi_0}(f)(a) = \frac{1}{\sqrt{det(\mathbf{M}_0)}} \left\langle \mathbf{M}_0 \left(\bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(a) \right), \bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(a) \right\rangle^{1/2}.$$

Now we give the proof of Theorem 2.1.

Proof. Let S be a $(n-1)$-rectifiable closed set. As L, introduced in the proof of Lemma 2.1, is an automorphism, then S is also $(n-1)$-rectifiable and closed for $\mathbb{R}^n$ endowed with the scalar product $\langle \mathbf{M}_0^{-1} \cdot, \cdot \rangle$. According to Lemma 2.1, we have

$$\frac{\mathcal{L}^n(\{x \colon \mathrm{dist}_{\phi_0}(x, S) < \rho\})}{2\rho} = \sqrt{\det(\mathbf{M}_0)} \frac{\mathcal{H}_{\phi_0}^n(\{x \colon \mathrm{dist}_{\phi_0}(x, S) < \rho\})}{2\rho}. \tag{2.4}$$

We may apply Theorem 0.1 in the euclidean space $(\mathbb{R}^n, \langle \mathbf{M}_0^{-1} \cdot, \cdot \rangle)$, it gives

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}_{\phi_0}^n(\{x \colon \mathrm{dist}_{\phi_0}(x, S) < \rho\})}{2\rho} = \mathcal{H}_{\phi_0}^{n-1}(S). \tag{2.5}$$

As S is rectifiable, there exists a Lipschitzian function $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ and a bounded subset $A \subset \mathbb{R}^{n-1}$ such that $S \subset f(A)$. According to Corollary 1.2, there exists a Borel set

$$B \subset A \cap \{x \colon \mathbf{J}_{n-1}^{\phi_0}(f)(x) > 0\}$$

such that $f|_B$ is univalent and $\mathcal{H}^{n-1}(f(A) \setminus f(B)) = 0$. We denote by $C = f^{-1}(S) \cap B$ and then $\mathcal{H}_{\phi_0}^{n-1}(f(C)) = \mathcal{H}_{\phi_0}^{n-1}(S)$. As $f|_C$ is univalent, according to Area formula 1.1 with $u = \mathbf{1}_C$, we have

$$\mathcal{H}_{\phi_0}^{n-1}(S) = \int_C \mathbf{J}_{n-1}^{\phi_0}(f)(x) \mathrm{d}x. \tag{2.6}$$

For any $x \in C$ we have $\mathbf{J}^{\phi_0}_{n-1}(f)(x) \neq 0$ which implies that $\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_{n-1}}(x)$ is free and then $\mathbf{J}_{n-1}(f)(x) \neq 0$. According to Lemma 2.3, we may write

$$\begin{aligned}
\mathbf{J}^{\phi_0}_{n-1}(f)(x) &= \frac{\mathbf{J}^{\phi_0}_{n-1}(f)(x)}{\mathbf{J}_{n-1}(f)(x)}\mathbf{J}_{n-1}(f)(x), \\
&= \frac{1}{\sqrt{\det(\mathbf{M}_0)}}\left\langle \mathbf{M}_0\left(\frac{\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x)}{\left|\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x)\right|}\right), \frac{\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x)}{\left|\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x)\right|}\right\rangle^{1/2}\mathbf{J}_{n-1}(f)(x).
\end{aligned}$$

We set

$$u(x) = \left\langle \mathbf{M}_0\left(\frac{\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x)}{\left|\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x)\right|}\right), \frac{\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x)}{\left|\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x)\right|}\right\rangle^{1/2}.$$

According to Lemma 3.2.25 in [1], we have $u(x) = \langle \mathbf{M}_0\nu(f(x)), \nu(f(x))\rangle^{1/2}$ where $\nu(f(x))$ is an unitary and orthogonal vector to $f(C)$ at $f(x)$. Applying Area formula 1.1, this time in $\mathbb{R}^n$ endowed with its canonical euclidean structure, gives

$$\begin{aligned}
\int_C \mathbf{J}^{\phi_0}_{n-1}(f)(x)\mathrm{d}x &= \frac{1}{\sqrt{\det(\mathbf{M}_0)}}\int_C u(x)\mathbf{J}_{n-1}(f)(x)\mathrm{d}x, \\
&= \frac{1}{\sqrt{\det(\mathbf{M}_0)}}\int_{f(C)} \langle \mathbf{M}_0\nu, \nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}, \\
&= \frac{1}{\sqrt{\det(\mathbf{M}_0)}}\int_S \langle \mathbf{M}_0\nu, \nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}$$

According to (2.4), (2.5) and (2.6) we may conclude

$$\lim_{\rho\to 0^+}\frac{\mathcal{L}^n(\{x\colon \mathrm{dist}_{\phi_0}(x,S) < \rho\})}{2\rho} = \int_S \langle \mathbf{M}_0\nu, \nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

□

2.2 Inhomogeneous setting

In this section, we remove the homogeneity assumption and, using a piecewise constant approximation of $\mathbf{M}$, we prove Theorem 0.2. The proof is divided in three Propositions.

Proposition 2.1. *Let $\mathbf{M}$ be as in Theorem 0.2. Then, for any compact neighborhood C of S, there exists $\lambda, \Lambda > 0$, such that the following ellipticity condition is satisfied for any $(x, \mathbf{v}) \in C \times \mathbb{R}^n$*

$$\lambda|\mathbf{v}|^2 \leq \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle \leq \Lambda|\mathbf{v}|^2.$$

Proof. As $\mathbf{M}(x)$ is symmetric positive definite for any $x \in C$, there exists $\lambda(x), \Lambda(x) > 0$ such that, for any $\mathbf{v} \in \mathbb{R}^n$, we have

$$\lambda(x)|\mathbf{v}|^2 \leq \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle \leq \Lambda(x)|\mathbf{v}|^2.$$

As $\mathbf{M}$ is continuous, the previous inequalities remain true in any compact C with (λ, Λ) only depending on C. □

Proposition 2.2. *Let S and $\mathbf{M}$ be as in Theorem 0.2. Then, we have*

$$\mathcal{M}^\star_{\mathbf{M}}(S) \leq \int_S \langle \mathbf{M}\nu, \nu\rangle^{1/2} d\mathcal{H}^{n-1}.$$

Proof. We may assume that $\int_S \langle \mathbf{M}\nu, \nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}$ is finite, otherwise the result is ensured. Let C be a compact neighborhood of S. Let $\delta > 0$ be fixed. As $\mathbf{M}$ is continuous, there exists $\eta > 0$ such that

$$|x - y| \leq \eta \quad \Rightarrow \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \leq \delta \tag{2.7}$$

for any $(x,y)\in C^2$. According to Proposition 2.1, there exists $\lambda>0$ such that

$$\lambda^{1/2}\mathcal{H}^{n-1}(S)\leq\int_S\langle\mathbf{M}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}$$

and then $\mathcal{H}^{n-1}(S)$ is also finite. For $t\in\mathbb{R}$ and $i\in\{1,\dots,n\}$, we set $\Pi_t^i=\{x\in C\colon\langle x,e_i\rangle=t\}$. Thus, for $k\in\mathbb{N}$ fixed, $\left\{t\in\mathbb{R}\colon\mathcal{H}^{n-1}(S\cap\Pi_t^i)>\frac{1}{k}\right\}$ is finite and then, for any $i\in\{1,\dots,n\}$, the set $\left\{t\in\mathbb{R}\colon\mathcal{H}^{n-1}(S\cap\Pi_t^i)>0\right\}$ is at most countable. So, if we consider the cubes with its faces orthogonal to the vectors of the orthogonal basis, there exists a partition $\mathcal{K}$ of C by cubes with diameter less than η, such that for any $K\in\mathcal{K}$, we have

$$\mathcal{H}^{n-1}(S\cap\partial K)=0. \tag{2.8}$$

For any $K\in\mathcal{K}$, we fix $a_K\in K$ and, for any $(x,\mathbf{v})\in K\times\mathbb{R}^n$, we set

$$\begin{cases}\widetilde{\mathbf{M}}(x)=\mathbf{M}(a_K),\\ \widetilde{\phi}(x,\mathbf{v})=\langle(\mathbf{M}(a_K))^{-1}\mathbf{v},\mathbf{v}\rangle.\end{cases} \tag{2.9}$$

For $K\in\mathcal{K}$ and $r>0$, we set $K_r=\{x\colon\mathrm{dist}(x,\partial K)\leq r\}$. We have the following decomposition

$$\mathcal{M}^\star_{\mathbf{M}}(S)\leq\sum_{K\in\mathcal{K}}\mathcal{M}^\star_{\mathbf{M}}(S\cap K_r)+\sum_{K\in\mathcal{K}}\mathcal{M}^\star_{\mathbf{M}}(S\cap(K\setminus K_r)). \tag{2.10}$$

In *Claim 1* and *Claim 2*, we will determine an upper bound for the two previous sums.

Claim 1: We have

$$\sum_{K\in\mathcal{K}}\mathcal{M}^\star_{\mathbf{M}}(S\cap K_r)\leq\lambda^{-\frac{n}{2}}\Lambda^{1/2}\sum_{K\in\mathcal{K}}\mathcal{H}^{n-1}(S\cap K_r).$$

According to Proposition 2.1, there exists $\lambda>0$ and $\Lambda>0$ such that, for any $(x,\mathbf{v})\in C\times\mathbb{R}^n$, we have

$$\Lambda^{-\frac{1}{2}}|\mathbf{v}|\leq\langle\mathbf{M}(x)^{-1}\mathbf{v},\mathbf{v}\rangle^{1/2}\leq\lambda^{-\frac{1}{2}}|\mathbf{v}|. \tag{2.11}$$

Let $(x,y)\in C^2$ and $\gamma\in\mathrm{W}^{1,1}([0;1];\mathbb{R}^n)$ such that $\gamma(0)=x$ and $\gamma(1)=y$, then we have

$$\Lambda^{-\frac{1}{2}}\int_0^1|\dot\gamma|\mathrm{d}t\leq\int_0^1\phi(\gamma,\dot\gamma)\mathrm{d}t\leq\lambda^{-\frac{1}{2}}\int_0^1|\dot\gamma|\mathrm{d}t.$$

According to (0.2) and (2.11), for any $(x,y)\in C^2$, it gives

$$\Lambda^{-\frac{1}{2}}\mathrm{dist}(x,y)\leq\mathrm{dist}_\phi(x,y)\leq\lambda^{-\frac{1}{2}}\mathrm{dist}(x,y) \tag{2.12}$$

and then, for any $x\in C$, the following inclusions are satisfied

$$B(x,\lambda^{1/2})\subset B_\phi(x,1)\subset B(x,\Lambda^{1/2}).$$

For any $x\in C$, , as $\mathcal{H}^n(B(x,t))=t^n\mathcal{H}^n(B(0,1))$, we get

$$\Lambda^{-\frac{n}{2}}\leq\frac{\mathcal{H}^n(B(0,1))}{\mathcal{H}^n(B_\phi(x,1))}\leq\lambda^{-\frac{n}{2}}$$

and with Theorem 1.3, for any $E\in\mathcal{B}(C)$, it yields

$$\Lambda^{-\frac{n}{2}}\mathcal{H}^n(E)\leq\mathcal{H}^n_\phi(E)\leq\lambda^{-\frac{n}{2}}\mathcal{H}^n(E).$$

Moreover, (2.12) implies $\{x\colon\mathrm{dist}_\phi(x,S\cap K_r)<\rho\}\subset\{x\colon\mathrm{dist}(x,S\cap K_r)<\Lambda^{1/2}\rho\}$ and then we have

$$\begin{aligned}\mathcal{M}^\star_{\mathbf{M}}(S\cap K_r)&=\limsup_{\rho\to0^+}\frac{\mathcal{H}^n_\phi(\{x\colon\mathrm{dist}_\phi(x,S\cap K_r)<\rho\})}{2\rho},\\&\leq\lambda^{-\frac{n}{2}}\limsup_{\rho\to0^+}\frac{\mathcal{H}^n(\{x\colon\mathrm{dist}(x,S\cap K_r)<\Lambda^{1/2}\rho\})}{2\rho},\\&\leq\lambda^{-\frac{n}{2}}\Lambda^{1/2}\mathcal{M}^\star(S\cap K_r).\end{aligned}$$

As S is closed and $(n-1)$-rectifiable, so is $S\cap K_r$. We may apply Theorem 0.1 to get $\mathcal{M}^\star(S\cap K_r) = \mathcal{H}^{n-1}(S\cap K_r)$ and *Claim 1* is proved.

Claim 2: We have

$$\sum_{K\in\mathcal{K}} \mathcal{M}^\star_{\mathbf{M}}(S\cap(K\setminus K_r)) \le (1+\theta'\delta)\int_S \langle \mathbf{M}\nu,\nu\rangle^{1/2} d\mathcal{H}^{n-1},$$

where θ' is a constant which depends only on the restriction of $\mathbf{M}$ in C.

Let C be a compact neighborhood of S. As $\mathbf{M}$ is continuous, then there exists a constant $m>0$ which depends only on C such that, for any $(x,y)\in C^2$, we have

$$\|\mathbf{M}^{-1}(x)-\mathbf{M}^{-1}(y)\| \le m\|\mathbf{M}(x)-\mathbf{M}(y)\|.$$

Let $(x,y)\in C^2$ and $\gamma\in \mathrm{W}^{1,1}([0;1];C)$ such that $\gamma(0)=x$ and $\gamma(1)=y$, according to (2.7), we have the inequalities

$$\begin{aligned}
\left|\int_0^1 \langle \mathbf{M}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}\mathrm{d}t - \int_0^1 \langle \widetilde{\mathbf{M}}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}\mathrm{d}t\right| &\le \int_0^1 \frac{|\langle(\mathbf{M}^{-1}(\gamma)-\widetilde{\mathbf{M}}^{-1}(\gamma))\dot\gamma,\dot\gamma\rangle|}{\langle \mathbf{M}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}+\langle \mathbf{M}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}}\mathrm{d}t,\\
&\le \int_0^1 \frac{\|\mathbf{M}^{-1}(\gamma)-\widetilde{\mathbf{M}}^{-1}(\gamma))\|\langle\dot\gamma,\dot\gamma\rangle}{\langle \mathbf{M}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}+\langle \mathbf{M}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}}\mathrm{d}t,\\
&\le \frac{m\delta}{2\Lambda^{-1}}\int_0^1 \langle\dot\gamma,\dot\gamma\rangle^{1/2}\mathrm{d}t,\\
&\le \frac{m\delta}{2\Lambda^{-\frac{3}{2}}}\int_0^1 \langle \widetilde{\mathbf{M}}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}\mathrm{d}t.
\end{aligned}$$

We denote $\frac{m}{2\Lambda^{-\frac{3}{2}}}$ by θ', it gives

$$\left|\int_0^1 \langle \mathbf{M}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}\mathrm{d}t\right| \le (1+\theta'\delta)\int_0^1 \langle \widetilde{\mathbf{M}}^{-1}(\gamma)\dot\gamma,\dot\gamma\rangle^{1/2}\mathrm{d}t,$$

so that

$$\{x\colon \mathrm{dist}_\phi(x,S\cap(K\setminus K_r))<\rho\} \subset \left\{x\colon \mathrm{dist}_{\widetilde{\phi}}(x,S\cap(K\setminus K_r))<(1+\theta'\delta)\,\rho\right\}, \tag{2.13}$$

and then

$$\mathcal{M}^\star_{\mathbf{M}}(S\cap(K\setminus K_r)) \le (1+\theta'\delta)\mathcal{M}^\star_{\widetilde{\mathbf{M}}}(S\cap(K\setminus K_r)). \tag{2.14}$$

As $\widetilde{\phi}$ is an homogeneous metric in a neighborhood of $K\setminus K_r$ as in Section 2.1, we may apply Theorem 2.1 to obtain

$$\mathcal{M}^\star_{\widetilde{\mathbf{M}}}(S\cap(K\setminus K_r)) = \int_{S\cap\overline{(K\setminus K_r)}} \langle \mathbf{M}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}$$

and

$$\mathcal{M}^\star_{\widetilde{\mathbf{M}}}(S\cap(K\setminus K_r)) \le \int_{S\cap K} \langle \mathbf{M}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

According to (2.14), we get

$$\sum_{K\in\mathcal{K}} \mathcal{M}^\star_{\mathbf{M}}(S\cap(K\setminus K_r)) \le (1+\theta'\delta)\int_S \langle \mathbf{M}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1},$$

so it concludes the proof of *Claim 2*.

Applying *Claim 1* and *Claim 2* to the decomposition (2.10), we have

$$\mathcal{M}^\star_{\mathbf{M}}(S) \le \lambda^{-\frac{n}{2}}\Lambda^{1/2}\sum_{K\in\mathcal{K}} \mathcal{H}^{n-1}(S\cap K_r) + (1+\theta'\delta)\int_S \langle \mathbf{M}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}$$

Taking the limit $r\to 0^+$ gives

$$\mathcal{M}^\star_{\mathbf{M}}(S) \leq \lambda^{-\frac{n}{2}}\Lambda^{1/2}\sum_{K\in\mathcal{K}}\mathcal{H}^{n-1}(S\cap\partial K) + (1+\theta'\delta)\int_S \langle \mathbf{M}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

Applying (2.8), and then taking the limit as $\delta \to 0^+$ concludes the proof of Proposition 2.2

$$\mathcal{M}^\star_{\mathbf{M}}(S) \leq \int_S \langle \mathbf{M}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

□

Proposition 2.3. *Let S and $\mathbf{M}$ as in Theorem 0.2. Then, we have*

$$\mathcal{M}_{\star\mathbf{M}}(S) \geq \int_S \langle \mathbf{M}\nu,\nu\rangle^{1/2}d\mathcal{H}^{n-1}.$$

Proof. Let $\delta > 0$, $\eta > 0$ and $\mathcal{K}$ as in (2.8) and (2.9). For $r > 0$ and $K \in \mathcal{K}$, we still set $K_r = \{x\colon \mathrm{dist}(x,\partial K) \leq r\}$. As $\mathcal{K}$ is finite and

$$(K,L)\in\mathcal{K}^2, K\neq L \quad\Rightarrow\quad \mathrm{dist}(K_r,L_r) > 0,$$

then we have

$$\mathcal{M}_{\star\mathbf{M}}(S) \geq \sum_{K\in\mathcal{K}}\mathcal{M}_{\star\mathbf{M}}(S\cap(K\setminus K_r)) \tag{2.15}$$

With the same proof as for (2.13), we have

$$\{x\colon \mathrm{dist}_{\widetilde{\phi}}(x,S\cap(K\setminus K_r)) < \rho\} \subset \{x\colon \mathrm{dist}_{\phi}(x,S\cap(K\setminus K_r)) < (1+\theta'\delta)\rho\}$$

and then

$$\frac{\mathcal{L}^n(\{x\colon \mathrm{dist}_{\widetilde{\phi}}(x,S\cap(K\setminus K_r)) < \rho\})}{2\rho} \leq \frac{\mathcal{L}^n(\{x\colon \mathrm{dist}_{\phi}(x,S\cap(K\setminus K_r)) < (1+\theta'\delta)\rho\})}{2\rho}.$$

Passing to the lim inf with $\rho \to 0^+$ gives

$$\mathcal{M}_{\star\widetilde{\mathbf{M}}}(S\cap(K\setminus K_r)) \leq (1+\theta'\delta)\mathcal{M}_{\star\mathbf{M}}(S\cap(K\setminus K_r)).$$

As $\widetilde{\mathbf{M}}$ is constant in a neighborhood of $K\setminus K_r$ as in Section 2.1, we may apply Theorem 2.1 to obtain

$$\mathcal{M}_{\star\widetilde{\mathbf{M}}}(S\cap(K\setminus K_r)) = \int_{S\cap\overline{(K\setminus K_r)}}\langle\widetilde{\mathbf{M}}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

According to (2.15), we get

$$\begin{aligned}\mathcal{M}_{\star\mathbf{M}}(S) &\geq (1+\theta'\delta)^{-1}\sum_{K\in\mathcal{K}}\int_{S\cap\overline{(K\setminus K_r)}}\langle\widetilde{\mathbf{M}}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1},\\ &\geq (1+\theta'\delta)^{-1}\int_{S\cap C_r}\langle\widetilde{\mathbf{M}}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

where we have set $C_r = \bigcup_{K\in\mathcal{K}}\overline{(K\setminus K_r)}$. Remark that

$$r_1 < r_2 \quad\Rightarrow\quad C_{r_2}\subset C_{r_1},\quad \bigcup_{r>0}C_r = C\setminus\left(\bigcup_{K\in\mathcal{K}}\partial K\right)$$

so, we have

$$\lim_{r\to 0^+}\int_{S\cap C_r}\langle\widetilde{\mathbf{M}}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1} = \int_{S\setminus\cup\partial K}\langle\widetilde{\mathbf{M}}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

As $\mathcal{H}^{n-1}(\partial K) = 0$ for any $K\in\mathcal{K}$ (2.8), then

$$\lim_{r\to 0^+}\int_{S\cap C_r}\langle\widetilde{\mathbf{M}}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1} = \int_{S}\langle\widetilde{\mathbf{M}}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

We deduce that

$$\mathcal{M}_{\star\mathbf{M}}(S) \geq (1+\theta'\delta)^{-1}\int_S\langle\widetilde{\mathbf{M}}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}$$

and passing to the limit for $\eta\to 0^+$ and then for $\delta\to 0^+$ concludes the proof.

□

3 Regularity result for almost quasi-minimizers of a free boundary problem

The standard functional framework is the theory of functions with bounded variation and in particular the *special set* of functions with bounded variations SBV that may be found in [6].

3.1 Definition and main result

In [7], it is proved that if $u \in \mathrm{SBV}(\Omega)$ is a minimizer of the well known Mumford-Shah functional

$$E(u) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_u),$$

where J_u is the jump set of u, then we have $\mathcal{H}^{n-1}(J_u) = \mathcal{M}(J_u)$. We want to get the same result when $\mathcal{H}^{n-1}$ is perturbed by an anisotropic and continuous function. So, for $u \in \mathrm{SBV}(\Omega)$ and $\overline{B(x,r)} \subset \Omega$, we set

$$F(u,\sigma,B(x,r)) = \int_{B(x,r)} |\nabla u|^2 \mathrm{d}x + \sigma \mathcal{H}^{n-1}(J_u \cap \overline{B(x,r)}),$$

In [4], the following definition is introduced.

Definition 3.1. *For $\sigma \geq 1$, $\alpha > 0$ and $c_\alpha > 0$, we say that $w \in SBV(\Omega)$ is a (σ,α,c_α)-almost quasi-minimizer of a free discontinuity problem, if there exists $\alpha > 0$ and $c_\alpha \geq 0$ such that for any $v \in SBV(\Omega)$, we have*

$$[w \neq v] \subset B(x,r),\ \overline{B(x,r)} \subset \Omega \quad \Rightarrow \quad F(w,1,B(x,r)) \leq F(v,\sigma,B(x,r)) + c_\alpha r^{n-1+\alpha}.$$

The main result we introduce in this section is the following.

Theorem 3.1. *Let $\mathbf{M} : \Omega \to S_n^+(\mathbb{R})$ be continuous and $u \in SBV(\Omega)$ be an almost quasi-minimizer of a free discontinuity problem (3.1), then we have*

$$\mathcal{M}_{\mathbf{M}}(J_u) = \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1}.$$

3.2 Proof of Theorem 3.1

To prove Thorem 3.1, we need the two following regularity results for the jump set of almost quasi-minimizers which are extracted from [4].

Theorem 3.2. *Let u be an almost quasi-minimizer of a free discontinuity problem, then*

$$\mathcal{H}^{n-1}\left(\overline{J_u} \setminus J_u\right) = 0.$$

Theorem 3.3. *There exist constants β, ρ_0 such that for every (σ,α,c_α) almost quasi-minimizer u, for every $x \in J_u$ and for every $0 < \rho < \rho_0$ such that $B(x,\rho) \subset \Omega$, we have*

$$\mathcal{H}^{n-1}(J_u \cap B(x,\rho)) \geq \beta \rho^{d-1}.$$

First, we prove the following

Lemma 3.1. *Let u be an almost quasi-minimizer of a free discontinuity problem (3.1), then for any compact set $K \subset \overline{J_u} \cap \Omega$, we have*

$$\mathcal{M}^\star_{\mathbf{M}}(K) \leq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1}.$$

Proof. We may assume that $\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$ is finite, otherwise the result is ensured. According to Proposition 2.1, we have

$$\lambda^{1/2}\mathcal{H}^{n-1}(J_u) \leq \int_{J_u \cap K} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$$

so, $\mathcal{H}^{n-1}(J_u)$ is finite too.

According to [8], Section 5.9, the set J_u is rectifiable up to a $\mathcal{H}^{n-1}$-negligible set $\mathcal{N}$. More precisely, there exists a countable family $(K_i)_{i\in\mathbb{N}}$ of compact $\mathcal{C}^1$-hypersurfaces such that

$$J_u = \mathcal{N} \cup \left(\bigcup_{i\in\mathbb{N}} K_i \right),$$

where $\mathcal{H}^{n-1}(\mathcal{N}) = 0$. Let $K \subset \overline{J_u} \cap \Omega$ be compact, we have the decomposition

$$K = \left[K \cap \left(\overline{J_u} \setminus J_u\right)\right] \cup \left[K \cap \bigcup_{i=1}^{q} K_i\right] \cup \left[K \cap \bigcup_{i=q+1}^{\infty} K_i\right] \cup \left[K \cap \mathcal{N}\right].$$

Theorem 3.2 gives $\mathcal{H}^{n-1}(K \cap \left(\overline{J_u} \setminus J_u\right)) = 0$. Let $\delta > 0$ be fixed. As $\mathcal{H}^{n-1}(J_u)$ is finite, there exists $q \in \mathbb{N}$ such that

$$\mathcal{H}^{n-1}\left(K \cap \bigcup_{i=q+1}^{\infty} K_i\right) \leq \delta. \tag{3.1}$$

In the sequel, we omit the dependance with δ for the sake of simplicity. We set $S = K \cap \bigcup_{i=1}^{q} K_i$ and for $A \subset \mathbb{R}^n$ we adopt the following notation

$$A_\rho := \{x \colon \mathrm{dist}_\phi(x, A) < \rho\}.$$

Let $\tau > 0$ and $\rho > 0$ be fixed, we decompose $K = (K \setminus S_{\tau\rho}) \cup S_{\tau\rho}$ and set $E = K \setminus S_{\tau\rho}$. So, we have

$$K_\rho \subset E_\rho \cup S_{(1+\tau)\rho},$$

and then

$$\frac{\mathcal{L}^n(K_\rho)}{2\rho} \leq \frac{\mathcal{L}^n(E_\rho)}{2\rho} + \frac{\mathcal{L}^n(S_{(1+\tau)\rho})}{2\rho}. \tag{3.2}$$

The rest of the proof consists in computing an upper bound, when $\rho \to 0^+$, for the two terms in the right hand side of (3.2). As S is a closed and rectifiable set, Theorem 0.2 gives

$$\lim_{\rho\to 0^+} \frac{\mathcal{L}^n(S_{(1+\tau)\rho})}{2\rho} = (1+\tau) \int_S \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$$

and then

$$\lim_{\rho\to 0^+} \frac{\mathcal{L}^n(S_{(1+\tau)\rho})}{2\rho} = (1+\tau) \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}. \tag{3.3}$$

In the fourth following *Claims*, we prove that $\limsup \frac{\mathcal{L}^n(E_\rho)}{2\rho}$ converges to 0 when δ converges to 0. The main tool is the regularity result given by Theorem 3.3.

Claim 1: There exists $p \in \mathbb{N}$ and $x_1, \ldots, x_p \in E \cap J_u$ such that $E \subset \bigcup_{i=1}^{p} \overline{B_\phi(x_i, \tau\rho)}$ and

$$(i,j) \in \{1,\ldots,p\}^2, i \neq j \quad \Rightarrow \quad dist_\phi(x_i, x_j) \geq \tau\rho. \tag{3.4}$$

We construct $(x_i)_i$ by an iterative way and we show that the number of iterations is finite. If $E = \emptyset$, then the result is obvious. Otherwise, there exists $\tilde{x}_1 \in E$. As $E \subset \overline{J_u}$, there exists $(y_k)_k \subset J_u$ converging to $\tilde{x}_1$. As $\mathrm{dist}_\phi(\tilde{x_1}, S) > \tau\rho$, there exists $k_0 \in \mathbb{N}$ such that $\mathrm{dist}_\phi(y_{k_0}, S) > \tau\rho$ and we set $x_1 = y_{k_0}$.

Let us assume that there exits $x_1, \ldots, x_p \in E \cap J_u$ which satisfy (3.4). If $E \subset \bigcup_{i=1}^{p} \overline{B_\phi(x_i, \tau\rho)}$, then the iterative process stops. Otherwise, there exists $\tilde{x}_{p+1} \in E \setminus \bigcup_{i=1}^{p} \overline{B_\phi(x_i, \tau\rho)}$. As $E \subset \overline{J_u}$, there exists $(y_k)_k \subset J_u$ converging to $\tilde{x}_{p+1}$. As $\mathrm{dist}_\phi(\tilde{x}_{p+1}, S) > \tau\rho$ and $\mathrm{dist}_\phi(\tilde{x}_{p+1}, \tilde{x}_i) > \tau\rho$ for any $i \in \{1,\ldots,p\}$, there exists $k_p \in \mathbb{N}$ such that $\mathrm{dist}_\phi(y_{k_p}, S) > \tau\rho$ and $\mathrm{dist}_\phi(y_{k_p}, \tilde{x}_i) > \tau\rho$. We set $x_p = y_{k_p}$.

If the iterative process does not finish, then there exists a sequence $(x_i)_{i\in\mathbb{N}} \subset \Omega$ which satisfies (3.4). According to Proposition 2.1, we have

$$(i,j) \in \{1,\dots,p\}^2, i \neq j \quad \Rightarrow \quad |x_i - x_j| \geq \frac{\tau\rho}{\sqrt{\Lambda}}.$$

As Ω is bounded there exists a converging subsequence which is a contradiction. So, the iterative process is finite.

Claim 2: There exists a constant $c = c(n,\lambda,\Lambda)$ which only depends on the dimension n and the ellipticity coefficients λ, Λ such that, for any $x \in \Omega$, we have

$$\#\left\{i \in \{1,\dots,p\} \colon x \in B_\phi(x_i, \tau\rho)\right\} \leq c(n,\lambda,\Lambda).$$

We set

$$c(n,\lambda,\Lambda) = \sup\left\{q \in \mathbb{N} \colon \exists y_1,\dots,y_q \in \mathbb{R}^n, \quad \forall i, |y_i| \leq \frac{1}{\sqrt{\lambda}}, \quad i \neq j \Rightarrow |y_i - y_j| \geq \frac{1}{\sqrt{\Lambda}}\right\}.$$

We consider a finite partition of $B\left(0, \frac{1}{\sqrt{\lambda}}\right)$ of parallelepipeds whose diameter is less than $\frac{1}{\sqrt{\Lambda}}$. Then, $c(n,\lambda,\Lambda)$ is finite and less than the cardinality of such partition. For $x \in \Omega$, we set $y_i = \frac{x_i - x}{\tau\rho}$. According to Proposition 2.1, we have

$$x \in B_\phi(x_i, \tau\rho) \quad \Rightarrow \quad y_i \in B\left(0, \frac{1}{\sqrt{\lambda}}\right), \qquad i \neq j \quad \Rightarrow \quad |y_i - y_j| \geq \frac{1}{\sqrt{\Lambda}}.$$

and then $\#\left\{i \in \{1,\dots,p\} \colon x \in B_\phi(x_i, \tau\rho)\right\} \leq c(n,\lambda,\Lambda)$.

Claim 3: We still denote by c a generic constant depending on (n,λ,Λ). We have

$$p\beta\rho^{n-1} \leq \frac{c}{\tau^{n-1}} \mathcal{H}^{n-1}(J_u \cap E_{\tau\rho}).$$

According to *Claim 2*, we have

$$\sum_{i=1}^{p} \mathbf{1}_{J_u \cap B_\phi(x_i,\tau\rho)} \leq c\mathbf{1}_{J_u \cap E_{\tau\rho}}.$$

Integrating with respect to $\mathcal{H}^{n-1}$ gives

$$\sum_{i=1}^{p} \mathcal{H}^{n-1}(J_u \cap B_\phi(x_i, \tau\rho)) \leq c\mathcal{H}^{n-1}(J_u \cap E_{\tau\rho}).$$

Proposition 2.1 condition gives $B(x_i, \lambda^{1/2}\tau\rho)) \subset B_\phi(x_i, \tau\rho))$ and then

$$\sum_{i=1}^{p} \mathcal{H}^{n-1}(J_u \cap B(x_i, \lambda^{1/2}\tau\rho)) \leq c\mathcal{H}^{n-1}(J_u \cap E_{\tau\rho}). \tag{3.5}$$

According to Theorem 3.3, there exists $\beta > 0$ such that, for any $i \in \{1,\dots,p\}$, we have

$$\mathcal{H}^{n-1}(J_u \cap B(x_i, \lambda^{1/2}\tau\rho)) \geq \beta\lambda^{\frac{n-1}{2}}\tau^{n-1}\rho^{n-1}. \tag{3.6}$$

Inequalities (3.5) and (3.6) conclude the proof of *Claim 3*.

Claim 4: We have

$$\limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(E_\rho)}{2\rho} \leq \frac{(1+\tau)^n c}{\tau^{n-1}}\delta$$

where δ is given by (3.1), β by Theorem 3.3 and ω_n is the volume of the unit ball of $\mathbb{R}^n$.

According to *Claim 1*, we have $E \subset \bigcup_{i=1}^p B_\phi(x_i, \tau\rho)$ and then $E_\rho \subset \bigcup_{i=1}^p B_\phi(x_i, (1+\tau)\rho)$. Proposition 2.1 gives

$$B_\phi(x_i, (1+\tau)\rho) \subset B(x_i, \Lambda^{1/2}(1+\tau)\rho),$$

it yields

$$\begin{aligned}\mathcal{L}^n(E_\rho) &\leq p\mathcal{L}^n(B_\phi(x_i, (1+\tau)\rho)),\\ &\leq p\mathcal{L}^n(B(x_i, \Lambda^{1/2}(1+\tau)\rho)),\\ &\leq p\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n\rho^n.\end{aligned}$$

As $E = K \setminus S_{\tau\rho}$, then we deduce $E_{\tau\rho} \subset K_{\tau\rho} \setminus S$ and *Claim 3* gives

$$\begin{aligned}\mathcal{L}^n(E_\rho) &\leq \frac{\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n c}{\beta\tau^{n-1}}\rho\mathcal{H}^{n-1}(J_u \cap E_{\tau\rho}),\\ &\leq \frac{\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n c}{\beta\tau^{n-1}}\rho\mathcal{H}^{n-1}(J_u \cap (K_{\tau\rho} \setminus S)).\end{aligned}$$

According to (3.1), we deduce

$$\begin{aligned}\limsup_{\rho\to 0^+}\frac{\mathcal{L}^n(E_\rho)}{2\rho} &\leq \frac{\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n c}{2\beta\tau^{n-1}}\mathcal{H}^{n-1}(J_u \cap (K \setminus S)),\\ &\leq \frac{\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n c}{2\beta\tau^{n-1}}\delta.\end{aligned}$$

Conclusion of the proof.

According to (3.2), (3.3) and *Claim 4*, we have

$$\limsup_{\rho\to 0^+}\frac{\mathcal{L}^n(K_\rho)}{2\rho} \leq \frac{(1+\tau)^n c}{\tau^{n-1}}\delta + (1+\tau)\int_{J_u}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

Letting $\delta, \tau \to 0^+$ successively gives

$$\limsup_{\rho\to 0^+}\frac{\mathcal{L}^n(K_\rho)}{2\rho} \leq \int_{J_u}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1},$$

which concludes the proof of Lemma 3.1.

□

Now, we give the proof of Theorem 3.1.

Proof. We divide the proof in two inequalities.

Claim 1: We have

$$\mathcal{M}^\star_{\mathbf{M}}(J_u) \leq \int_{J_u}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}d\mathcal{H}^{n-1}.$$

We assume that $\int_{J_u}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}$ is finite, otherwise the result is ensured. We set $\Omega_r = \{x \in \Omega\colon \mathrm{dist}(x, \partial\Omega) > r\}$. So, the following set is at most countable

$$\Pi = \left\{r > 0\colon \int_{J_u\cap\partial\Omega_r}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1} > 0\right\}.$$

In particular, for any $\delta > 0$, there exists a sequence $(r_i)_i \subset \Pi$ strictly decreasing to 0^+ and such that

$$\int_{J_u\cap(\Omega\setminus\Omega_{r_0})}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1} \leq \delta.$$

We consider the following partition

$$\Omega = \Omega_{r_0} \cup \left(\bigcup_{i=0}^{\infty} \Omega_{r_{i+1}} \setminus \Omega_{r_i} \right),$$

it gives

$$\mathcal{M}^{\star}_{\mathbf{M}}(J_u) \leq \mathcal{M}^{\star}_{\mathbf{M}}(J_u \cap \Omega_{r_0}) + \sum_{i=0}^{\infty} \mathcal{M}^{\star}_{\mathbf{M}}(J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})).$$

As $\mathcal{M}^{\star}_{\mathbf{M}}(E) = \mathcal{M}^{\star}_{\mathbf{M}}(\overline{E})$ for any $E \subset \mathbb{R}^n$ such that $\overline{E} \subset \Omega$, we have

$$\mathcal{M}^{\star}_{\mathbf{M}}(J_u) \leq \mathcal{M}^{\star}_{\mathbf{M}}(\overline{J_u \cap \Omega_{r_0}}) + \sum_{i=0}^{\infty} \mathcal{M}^{\star}_{\mathbf{M}}(\overline{J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})}).$$

We may apply Lemma 3.1, with $K = \overline{J_u \cap \Omega_{r_0}}$ and $K = \overline{J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})}$, it gives

$$\mathcal{M}^{\star}_{\mathbf{M}}(J_u) \leq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \sum_{i=0}^{\infty} \int_{J_u \cap \overline{(\Omega_{r_{i+1}} \setminus \Omega_{r_i})}} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

As $r_i \in \Pi$ for any $i \geq 1$, we have

$$\begin{aligned}
\mathcal{M}^{\star}_{\mathbf{M}}(J_u) &\leq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \sum_{i=0}^{\infty} \int_{J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}, \\
&\leq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \int_{J_u \cap (\Omega \setminus \Omega_{r_0})} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}, \\
&\leq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \delta.
\end{aligned}$$

As $\delta > 0$ is arbitrary, it proves *Claim 1*.

Claim 2: We have

$$\mathcal{M}_{\star\mathbf{M}}(J_u) \geq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1}.$$

As for the proof of Lemma 3.1, there exists a countable family $(K_i)_{i \in \mathbb{N}}$ of compact $\mathcal{C}^1$-hypersurfaces such that

$$J_u = \mathcal{N} \cup \left(\bigcup_{i \in \mathbb{N}} K_i \right),$$

where $\mathcal{H}^{n-1}(\mathcal{N}) = 0$. As $\bigcup_{i=0}^{q} K_i$ is rectifiable and closed, Theorem 0.2 gives

$$\mathcal{M}_{\star\mathbf{M}} \left(\bigcup_{i=0}^{q} K_i \right) = \int_{\bigcup_{i=0}^{q} K_i} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

So, we deduce that

$$\mathcal{M}_{\star\mathbf{M}}(J_u) \geq \int_{\bigcup_{i=0}^{\infty} K_i} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$$

and then, as $\mathcal{H}^{n-1}(\mathcal{N}) = 0$, we conclude that

$$\mathcal{M}_{\star\mathbf{M}}(J_u) \geq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

□

3.3 Application to an anisotropic Mumford-Shah model

This work was initially motivated by the problem of segmentation of thin structures in an image. Let $\Omega \subset \mathbb{R}^n$ be the domain of an image $g \in L^\infty(\Omega)$. In order to favor thin details which are *elongated* in one or two directions (as tubes or thin plates), we introduce a field of symmetric definite positive matrices $\mathbf{M}$ whose principal directions correspond to the elongation of the sets we want to detect. So, we introduce an anisotropic version of the Mumford-Shah model (see [9] for the seminal paper)

$$E(u,\mathbf{M}) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} \mathrm{d}x,$$

where $u \in \mathrm{SBV}(\Omega)$, J_u is its jump set and ν_u is an unitary and normal vector to J_u. The field of matrices $\mathbf{M}$ belongs to $W^{1,n+\alpha}(\Omega)$ and $\alpha > 0$ ensures the continuity of $\mathbf{M}$. Moreover, $\mathbf{M}$ is constrained to take its values in a compact subset $\mathcal{G}$ of $\mathrm{S}_n^+(\mathbb{R})$.

If $\mathbf{M} \equiv \mathrm{Id}_n$, our model corresponds to the classical Mumford-Shah energy. The novelty of our approach consists in introducing this new variable which must capture the local anisotropy of the image. For example, if we want to detect thin tubes in a 3-D image, we fix $\mu > 1$, the *elongation* parameter, and we define $\mathcal{G}$ as the subset of symmetric matrices whose spectra is $\{1;1;\mu\}$. By this way, in order to minimize E, ν_u must be orthogonal to the principal directions of $\mathbf{M}$ and then this energy will favor set elongated in the principal direction of $\mathbf{M}$.

In order to compute the Euler-Lagrange equation associated to E, we need to perform an approximation of $\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$ which involves the Lebesgue measure. That is the reason why we are interested by the previous results on Minkowski contents. We prove the following

Theorem 3.4. *Let $\mathbf{M} : \Omega \to S_n^+(\mathbb{R})$ be continuous, $h \in L^\infty(\Omega)$, $\alpha > 0$ and $u \in SBV(\Omega)$ a minimizer of*

$$\left\{ E^{\alpha,h,\mathbf{M}}(v) = \alpha \int_\Omega (v-h)^2 dx + \int_\Omega |\nabla v|^2 dx + \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v\rangle^{1/2} d\mathcal{H}^{n-1} \colon v \in SBV(\Omega) \right\}.$$

Then, u is an almost quasi-minimizer and we have

$$\mathcal{M}_{\mathbf{M}}(J_u) = \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} d\mathcal{H}^{n-1}.$$

Proof. We prove in *Claim 1* that, up to a scaling, a minimizer is an almost quasi-minimizer. Then, in *Claim 2*, we compute the re-scaled problem.

Claim 1: For $\beta > 0$ and f function defined in Ω we denote by f_β the function defined for any $x \in \beta\Omega$ by

$$f_\beta(x) = f\left(\frac{x}{\beta}\right).$$

Then, there exists $\beta > 0$ such that $\widetilde{v}_\beta \in SBV(\beta\Omega)$ is an almost quasi-minimizer of a free discontinuity problem.

Let $\tilde{v} \in SBV(\Omega)$ be a minimizer of $E^{\alpha,h}_{\mathbf{M}}$ in $\mathrm{SBV}(\Omega)$. We introduce

$$\psi(t) = \begin{cases} -\|h\|_{L^\infty(\Omega)} & \text{if } t \le -\|h\|_{L^\infty(\Omega)}, \\ t & \text{if } |t| \le \|h\|_{L^\infty(\Omega)}, \\ \|h\|_{L^\infty(\Omega)} & \text{if } t \ge \|h\|_{L^\infty(\Omega)}. \end{cases}$$

According to the chain rule in SBV (see [6], Theorem 3.99), for $u \in \mathrm{SBV}(\Omega)$, we have

$$D(\psi \circ u) = \mathbf{1}_{\{x\colon |u(x)| \le \|h\|_{L^\infty}\}} \nabla u \mathcal{L}^n + ((\psi \circ u)^+ - (\psi \circ u)^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u$$

and then

$$\mathcal{L}^n(\{x \in \Omega \colon |u(x)| > \|h\|_{L^\infty}\}) > 0 \quad \Rightarrow \quad E^{\alpha,h}_{\mathbf{M}}(\psi \circ u) < E^{\alpha,h}_{\mathbf{M}}(u).$$

We deduce that $|\tilde{v}(x)| \le \|h\|_{L^\infty(\Omega)}$ for any $x \in \Omega$. By an homothetic change of variable, $\tilde{v}_\beta$ is a minimizer of the following rescaled problem

$$\left\{ \alpha\beta^2 \int_{\beta\Omega} (v-h_\beta)^2 \mathrm{d}x + \int_{\beta\Omega} |\nabla v|^2 \mathrm{d}x + \beta \int_{J_v} \langle \mathbf{M}_\beta \nu_v, \nu_v\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \colon v \in \mathrm{SBV}(\beta\Omega) \right\}.$$

Assuming that

$$v \in SBV(\beta\Omega), \quad x \in \beta\Omega, \quad \overline{B(x,r)} \subset \beta\Omega, \quad [\tilde{v}_\beta \neq v] \subset B(x,r),$$

then, the minimality assumption on $\tilde{v}_\beta$ gives

$$\begin{aligned} & \int_{B(x,r)} |\nabla \tilde{v}_\beta|^2 \mathrm{d}x + \beta \int_{J_{\tilde{v}_\beta} \cap \overline{B(x,r)}} \langle \mathbf{M}_\beta \nu_{\tilde{v}_\beta}, \nu_{\tilde{v}_\beta} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \\ \leq \quad & \alpha\beta^2 \int_{B(x,r)} (\phi \circ v - h_\beta)^2 \mathrm{d}x + \int_{B(x,r)} |\nabla(\phi \circ v)|^2 \mathrm{d}x + \beta \int_{J_{\phi\circ v} \cap \overline{B(x,r)}} \langle \mathbf{M}_\beta \nu_{\phi\circ v}, \nu_{\phi\circ v} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

We deduce

$$\begin{aligned} & \int_{B(x,r)} |\nabla \tilde{v}_\beta|^2 \mathrm{d}x + \beta \int_{J_{\tilde{v}_\beta} \cap \overline{B(x,r)}} \langle \mathbf{M}_\beta \nu_{\tilde{v}_\beta}, \nu_{\tilde{v}_\beta} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \\ \leq \quad & 4\alpha\beta^2 \|h\|_{L^\infty}^2 \omega_n r^n + \int_{B(x,r)} |\nabla v|^2 \mathrm{d}x + \beta \int_{J_v \cap \overline{B(x,r)}} \langle \mathbf{M}_\beta \nu_v, \nu_v \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

where $\omega_n = \mathcal{L}^n(B(x,1))$. Now, we set $\beta = \lambda^{-1}$ where λ is the ellipticity coefficient introduced in Proposition 2.1, it gives

$$\begin{aligned} \mathcal{H}^{n-1}(J_{\tilde{v}_\beta} \cap \overline{B(x,r)}) &= \beta\lambda \mathcal{H}^{n-1}(J_{\tilde{v}_\beta} \cap \overline{B(x,r)}), \\ &\leq \beta \int_{J_{\tilde{v}_\beta} \cap \overline{B(x,r)}} \langle \mathbf{M}_\beta \nu_{\tilde{v}_\beta}, \nu_{\tilde{v}_\beta} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

so

$$\begin{aligned} & \int_{B(x,r)} |\nabla \tilde{v}_\beta|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{\tilde{v}_\beta} \cap \overline{B(x,r)}) \\ & \leq \int_{B(x,r)} |\nabla v|^2 \mathrm{d}x + \beta \int_{J_v \cap \overline{B(x,r)}} \langle \mathbf{M}_\beta \nu_v, \nu_v \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + 4\alpha\beta^2 \|h\|_{L^\infty}^2 \omega_n r^n. \end{aligned}$$

The right hand side of Proposition 2.1 gives

$$\begin{aligned} & \int_{B(x,r)} |\nabla \tilde{v}_\beta|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{\tilde{v}_\beta} \cap \overline{B(x,r)}) \\ & \leq \int_{B(x,r)} |\nabla v|^2 \mathrm{d}x + \beta\Lambda \mathcal{H}^{n-1}(J_v \cap \overline{B(x,r)}) + 4\alpha\beta^2 \|h\|_{L^\infty}^2 \omega_n r^n. \end{aligned}$$

So, we may conclude that $\tilde{v}_\beta$ satisfies the definition 3.1 and then it is an $\left(\frac{\Lambda}{\lambda}, 1, \frac{4\alpha\|h\|_{L^\infty}^2 \omega_n}{\lambda^2}\right)$-almost quasi-minimizer of a free boundary problem.

Claim 2: We have

$$\mathcal{M}_{\mathbf{M}}(J_{\tilde{v}}) = \int_{J_{\tilde{v}}} \langle \mathbf{M}\nu_{\tilde{v}}, \nu_{\tilde{v}} \rangle^{1/2} d\mathcal{H}^{n-1}.$$

According to *Claim 1*, we may apply Theorem 3.1 to $\tilde{v}_\beta$ and it gives

$$\mathcal{M}_{\mathbf{M}_\beta}(J_{\tilde{v}_\beta}) = \int_{J_{\tilde{v}_\beta}} \langle \mathbf{M}_\beta \nu_{\tilde{v}_\beta}, \nu_{\tilde{v}_\beta} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}. \tag{3.7}$$

We denote by ϕ_β the metric which is defined, for any $(x, \mathbf{v}) \in \Omega \times \mathbb{R}^n$, by

$$\phi_\beta(x, \mathbf{v}) = \phi\left(\frac{x}{\beta}, \mathbf{v}\right)$$

and by d_{ϕ_β} the associated integrated distance. Then, we have

$$\mathcal{M}_{\mathbf{M}_\beta}(J) = \lim_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega \colon \mathrm{d}^J_{\phi_\beta}(x) < \rho\})}{2\rho}.$$

As

$$\mathrm{d}_{\phi_\beta}^{J_{\tilde{v}_\beta}}(x) < \rho \quad \Leftrightarrow \quad \mathrm{d}_{\phi}^{J_{\tilde{v}}}\left(\frac{x}{\beta}\right) < \beta\rho,$$

it gives

$$\mathcal{L}^n(\{x \in \Omega \colon \mathrm{d}_{\phi_\beta}^{J_{\tilde{v}_\beta}}(x) < \rho\}) = \beta^n \mathcal{L}^n(\{x \in \Omega \colon J_{\tilde{v}}(x) < \beta\rho\})$$

and then

$$\mathcal{M}_{\mathbf{M}_\beta}(J_{\tilde{v}_\beta}) = \beta^{n-1}\mathcal{M}_{\mathbf{M}}(J_{\tilde{v}}). \tag{3.8}$$

The following application

$$\begin{array}{rcl} f : J_{\tilde{v}_\beta} & \to & J_{\tilde{v}} \\ x & \to & \frac{x}{\beta} \end{array}$$

for any $x \in J_{\tilde{v}_\beta}$ satisfies

$$J_{n-1}(f)(x) = \frac{1}{\beta^{n-1}},$$

where $J_{n-1}(f)$ corresponds to the $(n-1)$-dimensional Jacobian of f (3.2.1 in [1]). According to the change of variable formula between $(n-1)$-rectifiable subsets (3.2.20 in [1]), we deduce that

$$\frac{1}{\beta^{n-1}} \int_{J_{\tilde{v}_\beta}} \langle \mathbf{M}_\beta \nu_{\tilde{v}_\beta}, \nu_{\tilde{v}_\beta} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} = \int_{J_{\tilde{v}}} \langle \mathbf{M}\nu_{\tilde{v}}, \nu_{\tilde{v}} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}. \tag{3.9}$$

From equations (3.7), (3.8) and (3.9) we conclude that

$$\mathcal{M}_{\mathbf{M}}(J_{\tilde{v}}) = \int_{J_{\tilde{v}}} \langle \mathbf{M}\nu_{\tilde{v}}, \nu_{\tilde{v}} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

□

4 Conclusion

We have proved that the approximation of Hausdorff measure by Minkowski content may be generalized to continuous and anisotropic perturbation of the Hausdorff measure. It has a natural application in free boundary problems. Classical results of regularity of the jump set may also be generalized in this anisotropic setting. Those results are the key tools to perform Γ-convergence approximation. For example, in a forthcoming work, we will prove that the following energy

$$E(u, \mathbf{M}) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} \mathrm{d}x$$

may be approached, in the sense of the Γ-convergence, by

$$E_\varepsilon(u, z, \mathbf{M}) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 (1-z)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon \langle \mathbf{M}\nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon}\right) \mathrm{d}x + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} \mathrm{d}x,$$

when $\varepsilon \to 0^+$.

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