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# The ergodic problem for some subelliptic operators with unbounded coefficients

PAOLA MANNUCCI, CLAUDIO MARCHI, NICOLETTA TCHOU  
Università degli Studi di Padova, Université de Rennes 1

## Abstract

We study existence and uniqueness of the invariant measure for a stochastic process with degenerate diffusion, whose infinitesimal generator is a linear subelliptic operator in the whole space  $\mathbb{R}^N$  with coefficients that may be unbounded. Such a measure together with a Liouville-type theorem will play a crucial role in the characterization of the ergodic constant defined through stationary problems with vanishing discount.

**Keywords:** Subelliptic equations, Heisenberg group, invariant measure, viscosity solutions, degenerate elliptic equations.

## 1 Introduction

This paper is devoted to study with pde's methods, the existence and uniqueness of the invariant measure of stochastic processes with degenerate diffusion, whose infinitesimal generators are linear subelliptic operators in the whole space  $\mathbb{R}^N$  with coefficients that may be unbounded. The invariant measures play a crucial role in ergodicity, homogenization and large time behaviour of the value function associated to the process. These methods, based on optimal control theory and pde's arguments, were introduced in the 80's by Bensoussan and developed until nowadays (see the monograph [7] by Bensoussan and references therein).

We shall first tackle the case of the Heisenberg group as model problem; after we shall extend our techniques to other subelliptic operators. In the

Heisenberg case, we consider the stochastic dynamics

$$(1.1) \quad dY_t = b(Y_t)dt + \sqrt{2}\sigma(Y_t)dW_t \quad \text{for } t \in (0, +\infty), \quad Y_0 = \bar{y} \in \mathbb{R}^3$$

where  $y = (y_1, y_2, z) \in \mathbb{R}^3$ , the matrix  $\sigma(y)$  has the form

$$(1.2) \quad \sigma(y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2y_2 & -2y_1 \end{bmatrix}$$

(in other words, the columns of  $\sigma$  are vectors generating the Heisenberg group) while  $W_t$  is a 3-dimensional Brownian motion.

Our principal aim is to prove, under suitable assumptions on the drift  $b$ , the existence and uniqueness of the invariant measure  $m$  associated to the process (1.1).

Let us recall from [7] that a probability measure  $m$  on  $\mathbb{R}^3$  is an *invariant measure* for process (1.1) if, for each  $u_0 \in \mathbb{L}^\infty(\mathbb{R}^3)$ , it satisfies

$$\int_{\mathbb{R}^3} u(y, t)m(y) dy = \int_{\mathbb{R}^3} u_0(y)m(y) dy$$

where  $u(y, t) = \mathbb{E}_y(u_0(Y_t))$  is the solution to the parabolic Cauchy problem

$$\begin{cases} \partial_t u + \mathcal{L}u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^3 \\ u(0, y) = u_0(y) & \text{on } \mathbb{R}^3 \end{cases}$$

and

$$(1.3) \quad -\mathcal{L}u := \text{tr}(\sigma(y)\sigma^T(y)D^2u(y)) + b(y) \cdot Du(y)$$

is the *infinitesimal generator* of process (1.1).

It is well known (see [7, Sect. II.4 and II.5]) that the density of the probability  $m$  (which, with a slight abuse of notation, we still denote by  $m$ ) solves

$$\mathcal{L}^*m = 0, \quad \int_{\mathbb{R}^3} m dy = 1 \quad \text{and} \quad m \geq 0,$$

where  $\mathcal{L}^*m$  is the adjoint operator

$$\mathcal{L}^*m = - \sum_{i,j} \partial_{ij}((\sigma\sigma^T)_{ij}m) + \sum_i \partial_i(b_i u).$$

In the framework of locally strongly elliptic operators, Has'minskiĭ [19, Sect. IV.4] (see also [28, Sect.8.2]) established the existence of an invariant

measure provided that there exists a bounded set  $U$  with smooth boundary such that

$$(1.4) \quad \begin{cases} \text{for any } \bar{y} \in \mathbb{R}^N \setminus U, \text{ the mean time } \tau \text{ at which the path (1.1) issuing} \\ \text{from } \bar{y} \text{ reaches } U \text{ is finite and } \mathbb{E}_{\bar{y}}\tau \text{ is locally finite.} \end{cases}$$

In our case this result does not apply because the matrix  $A := \sigma\sigma^T$  with  $\sigma$  given by (1.2) is

$$(1.5) \quad A(y) = \begin{bmatrix} 1 & 0 & 2y_2 \\ 0 & 1 & -2y_1 \\ 2y_2 & -2y_1 & 4(y_1^2 + y_2^2) \end{bmatrix}$$

and it is only positive semidefinite.

It is worth noticing (see [5, 14, 27]) that a sufficient condition for property (1.4) is the existence of a *Lyapunov*-like function  $w \in C(\mathbb{R}^3)$  which satisfies, for some positive constants  $k$  and  $R_0$

$$(1.6) \quad \mathcal{L}w \geq k \quad \text{for } |y| \geq R_0 \quad \text{and} \quad w(y) \rightarrow +\infty \quad \text{as } |y| \rightarrow +\infty.$$

As one can easily check the presence of the first order term is somehow 'crucial' for the existence of such a function. We will prove the existence of such Lyapunov function under suitable assumptions on the drift  $b$  that include also the Ornstein-Uhlenbeck case (see [28] and Remark 2.3 below) where the operator is of the following type

$$-\mathcal{L}u := \text{tr}(\sigma(y)\sigma^T(y)D^2u(y)) + \alpha y \cdot Du(y).$$

For ergodicity results based on probabilistic methods we refer to [23] and [24] and the references therein. The existence of a Lyapunov function is reminiscent of similar conditions (for instance, see: [28, Sect. 8.2] and [30, 31, 32]) called "recurrence condition" in the probabilistic jargon.

Ichihara and Kunita [25] (see also [24]) proved the existence and the uniqueness of an invariant measure for hypoelliptic processes as (1.1) which are constrained in a compact set. It is worth to recall that, in unbounded set the existence of an invariant measure may fail as it can be easily seen for (1.1) with  $b = 0$  and  $\sigma = I$ .

In this paper we want to establish existence and uniqueness of an invariant measure for process (1.1), namely for a process with the following features: it lies in an unbounded set and its infinitesimal generator is simultaneously degenerate and with unbounded coefficients. To this end we

shall use only pure analytical arguments.

It is important to stress that the principal part of  $\mathcal{L}u$  can be written as  $\sum_{i=1}^2 X_i^2 u$  where  $X_1, X_2$ , are the vector fields given by the columns of  $\sigma$  and that they satisfy Hörmander condition:  $X_1, X_2$ , and their commutators of any order span  $\mathbb{R}^3$  at each point  $(x_1, x_2, z) \in \mathbb{R}^3$ . In this case we have that  $[X_1, X_2] = -4\partial_z$ .

This property will play a crucial role in this paper since, as for the uniformly elliptic case, we have regularity, comparison and maximum principle ([11]).

We shall adapt the techniques introduced by P.L. Lions and Musiela [27] and developed in P.L. Lions courses [26], formed by two approximations (see also the paper of Cirant [14] for similar arguments). Actually, we shall consider the process

$$(1.7) \quad dY_t^\rho = b(Y_t^\rho)dt + \sqrt{2}\sigma_\rho(Y_t^\rho)dW_t,$$

where  $\sigma_\rho$  is the approximating matrix of  $\sigma$  in (1.2):

$$\sigma_\rho(y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2y_2 & -2y_1 & \rho \end{bmatrix}$$

such that  $A_\rho = \sigma_\rho \sigma_\rho^T$  is locally strictly positive, constrained in a bounded set  $O_n$  suitably chosen.

Let us stress that, in our argument, it is not enough to approximate the matrix  $A$  with any non-degenerate matrix  $A_\rho$  but we also need that  $A_\rho$  can be written as  $\sigma_\rho \sigma_\rho^T$ , where  $\sigma_\rho$  is the diffusion matrix of a new underlying optimal control problem. This issue motivates the fact that in (1.7) a new Brownian motion appears.

Let us recall from [7] that the invariant measure  $m_\rho^n$  of this process solves

$$\mathcal{L}_\rho^* m_\rho^n = 0 \quad \text{in } O_n$$

coupled with a boundary condition of Neumann type, where

$$-\mathcal{L}_\rho(u) = \text{tr}(\sigma_\rho(y)\sigma_\rho^T(y)D^2u(y)) + b(y) \cdot Du(y)$$

is an uniformly elliptic operator in  $O_n$ . Letting  $n \rightarrow +\infty$ , we obtain an invariant measure  $m_\rho$  for the process (1.7) in the whole space; letting  $\rho \rightarrow 0^+$ , we get the desired invariant measure for (1.1). The Lyapunov function will play a crucial role in these limits: it will be used in order to prove that

all the  $m_\rho$ 's and  $m$  are really measures (in other words, that the  $m_\rho^n$  and the  $m_\rho$  do not "disperse at infinity").

Moreover in this paper we also establish a Liouville type result. Similar result for semilinear operator without the drift term can be founded in the papers [8, 9, 13] and references therein; in all these papers the nonlinear zeroth order term is the key ingredient whereas, in our setting, the crucial contribution is due to the drift.

As an interesting application, we shall use the invariant measure and the Liouville property to study the ergodic problem. We consider the family of problems

$$(1.8) \quad \delta u_\delta - \text{tr}(\sigma(x)\sigma^T(x)D^2u_\delta) - b(x)Du_\delta = F(x) \quad \text{in } \mathbb{R}^3,$$

where  $\delta > 0$  and we shall prove that, as  $\delta \rightarrow 0$ ,  $\delta u_\delta$  converges to a constant  $\lambda$ , called "ergodic" constant.

Let us stress that the differential operator in the ergodic problem coincides with the infinitesimal generator  $\mathcal{L}$  of process (1.1).

We recall that the study of ergodic problems for equations with periodic, uniformly elliptic, operators has been addressed in [1, 6] while, for periodic, possibly degenerate (still satisfying the Hörmander's condition) operators, we refer the reader to the papers [2, 3].

The main difficulties in our problem are the lack of periodicity and the degeneracy of the operator. We shall overcome these issues using some techniques introduced by [5] for an elliptic operator on the whole space. Moreover, we shall give an explicit formula for the ergodic constant  $\lambda$  in terms of the invariant measure for (1.1).

Finally, we shall show how to extend our previous results to other degenerate operators satisfying Hörmander condition with possibly unbounded coefficients.

Our future purpose is to use the ergodic problem to study the homogenization problem

$$(1.9) \quad -\epsilon \text{tr}(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon) - b(\frac{x}{\epsilon}) \cdot Du_\epsilon + f(x, \frac{x}{\epsilon}) + \delta u_\epsilon = 0 \text{ in } \mathbb{R}^3,$$

where  $\sigma$  has the form (1.2). In this case the approximated cell problem formally coincides with the problem (1.8).

For the study of homogenization problems for periodic, possibly nonlinear, degenerate (still satisfying the Hörmander's condition) operators, we refer the reader to the papers [2, 12, 29].

This paper is organized as follows: Section 2 contains the main result of the paper: we find conditions on the drift  $b$  such that a Lyapunov function does exist and by means of this function we prove the existence and uniqueness of an invariant measure associated to our process. In Section 3, we establish a Liouville type result assuming the existence of a Lyapunov-like function. Section 4 is devoted to study the ergodic problem through stationary problems with vanishing discount. In Section 5 we generalize the previous results to a more general class of subelliptic operators, encompassing *e.g.* the Grushin one. The Appendix contains a condition equivalent to (1.6) which will be useful to manage the Lyapunov function founded in Section 2.

## 2 Existence and uniqueness of the invariant measure associated to the process

This section is devoted to the invariant measure for process (1.1). Let us recall (see [27] or Proposition 2.1 below) that, when the matrix associated to the infinitesimal generator  $\mathcal{L}$  is a strictly definite positive matrix, a sufficient condition for the existence of an invariant measure is given by: there exists a *Lyapunov-like* function such that

$$(2.1) \quad \begin{aligned} w &\in C^2(B_0^C) \cap C^0(\mathbb{R}^3) \\ \mathcal{L}w &\geq 1, \quad \text{in } B_0^C \\ w &\geq 0 \text{ in } B_0^C, \quad w = 0 \text{ on } \partial B_0, \end{aligned}$$

where  $B_0$  is a ball centered in 0 with suitable radius. (For less regular functions  $w$ , we refer to ([27])).

In our case, the matrix  $A = \sigma\sigma^T$  in (1.5) is degenerate in any point, and the rank of the matrix is 2. In order to overcome this issue, for  $\rho > 0$ , we introduce the approximating operators

$$(2.2) \quad \mathcal{L}_\rho w := -\text{tr}(A_\rho(\xi)D^2w) - b(\xi)Dw,$$

where

$$(2.3) \quad A_\rho(\xi) = \begin{bmatrix} 1 & 0 & 2\xi_2 \\ 0 & 1 & -2\xi_1 \\ 2\xi_2 & -2\xi_1 & 4(\xi_1^2 + \xi_2^2) + \rho^2 \end{bmatrix} = \sigma(\xi)\sigma^T(\xi) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho^2 \end{bmatrix}.$$

In the following Lemma we collect some useful properties of  $\mathcal{L}_\rho$ .

**Lemma 2.1** *The matrix  $A_\rho(\xi)$  is locally strictly positive definite (namely, for any compact  $K \subset \mathbb{R}^3$ , there holds  $\lambda A_\rho(\xi) \lambda^T \geq \nu(\xi) |\lambda|^2$  for any  $\xi \in K$ , with  $\nu(\xi) \geq a(K, \rho) > 0$ ) and it is positive definite in  $\mathbb{R}^3$ .*

*Moreover, there exists a  $3 \times 3$  matrix  $\sigma_\rho(\xi)$  with linear coefficients such that*

$$(2.4) \quad A_\rho(\xi) = \sigma_\rho(\xi) \sigma_\rho^T(\xi).$$

*Proof.* Set  $\alpha = 4(\xi_1^2 + \xi_2^2) + \rho^2 + 1$ . The eigenvalues of  $A_\rho$  are

$$\lambda_1 = 1, \quad \lambda_{2,3} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\rho^2}}{2}.$$

It is easy to remark that  $\lambda_2 \geq \frac{1}{2}$ .

The last eigenvalue is  $\lambda_3 = \frac{2\rho^2}{\alpha + \sqrt{\alpha^2 - 4\rho^2}} > \frac{\rho^2}{\alpha}$  hence, for any fixed  $R > 0$ , if

$$\xi_1^2 + \xi_2^2 \leq R^2, \quad \alpha \leq 4R^2 + \rho^2 + 1 \quad \text{and} \quad \lambda_3 > \frac{\rho^2}{4R^2 + \rho^2 + 1} > 0.$$

The matrix

$$(2.5) \quad \sigma_\rho(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\xi_2 & -2\xi_1 & \rho \end{bmatrix}.$$

verifies (2.4) □

**Remark 2.1** From (2.4), beside being uniformly elliptic, the operator  $-\mathcal{L}_\rho$  is also the infinitesimal generator of the stochastic process

$$(2.6) \quad dY_t^\rho = b(Y_t^\rho) dt + \sqrt{2} \sigma_\rho(Y_t^\rho) dW_t,$$

where  $\sigma_\rho$  is defined in (2.5) and  $W_t = (W_{1t}, W_{2t}, W_{3t})$  and  $W_{1t}, W_{2t}, W_{3t}$  are three independent Brownian motions whereas our starting process (1.1) only contains two independent Brownian motions.

Now, we want to prove that, for some classes of drifts  $b$ , there exists a function  $w$  satisfying (2.1) with  $\mathcal{L}$  replaced by  $\mathcal{L}_\rho$ . To this end, we consider a continuous drift  $b = (b_1, b_2, b_3)$  such that

$$(2.7) \quad b_i(\xi) = b_i(\xi_i), \quad \begin{cases} b_i(\xi_i) \leq -\frac{C_i}{|\xi_i|^{1-\alpha}} & \text{for } \xi_i \geq R \\ b_i(\xi_i) \geq \frac{C_i}{|\xi_i|^{1-\alpha}} & \text{for } \xi_i \leq -R \end{cases}$$

for some constants  $\alpha \in [0, 1)$ ,  $R > 0$  and  $C_i > 0$  ( $i = 1, 2, 3$ ).

Note that Lemma 2.2 here below, holds also for  $\rho = 0$  then we have a Lyapunov-like function  $w$  ( i.e. satisfying condition (2.1)) also for the degenerate starting problem where  $\mathcal{L}$  is given by (3.1).

Similar conditions to (2.7) was obtained in [27] with  $\sigma = I$  the identity matrix.

**Lemma 2.2** *Assume  $\sigma$  as in (1.2). Assume that  $b$  is a continuous function verifying (2.7) with*

- i) either  $\alpha > 0$ ,*
- ii) or  $\alpha = 0$  and sufficiently large  $C_i$ .*

*Then, there exists a  $R_0$  and a  $C^\infty$  function  $w$  which satisfies*

$$(2.8) \quad \mathcal{L}_\rho w \geq 1 \quad \text{in } \overline{B(0, R_0)}^C, \quad w \geq 0 \quad \text{in } \overline{B(0, R_0)}^C, \quad \lim_{|\xi| \rightarrow \infty} w = \infty$$

*for  $\rho$  sufficiently small.*

*Proof.* We set

$$w := \frac{(\xi_1^4 + \xi_2^4)}{12} + \frac{\xi_3^2}{2}.$$

Then, there holds

$$\mathcal{L}_\rho w = -5(\xi_1^2 + \xi_2^2) - \rho^2 - \frac{1}{3}(b_1 \xi_1^3 + b_2 \xi_2^3) - b_3 \xi_3.$$

We denote  $K_i := \max_{\xi_i \in [-R, R]} |b_i(\xi_i)|$ .

*Case (i).* Assume  $\alpha > 0$ . We want to prove that there exists  $R_0$  such that  $\mathcal{L}_\rho w > 1$  in  $\overline{B(0, R_0)}^C$  for  $\rho$  sufficiently small. To this end, we split the arguments in several cases.

(I). If  $|\xi_i| \geq R$  for any  $i \in \{1, 2, 3\}$ , then

$$\mathcal{L}_\rho w \geq \xi_1^2(-5 + C_1|\xi_1|^\alpha/3) + \xi_2^2(-5 + C_2|\xi_2|^\alpha/3) + C_3|\xi_3|^\alpha - \rho^2.$$

Hence, for  $|\xi_1|, |\xi_2| > R_1 := \max\{(15/C_1)^{1/\alpha}, (15/C_2)^{1/\alpha}, R\}$ ,  $|\xi_3| \geq R_3 := \max\{C_3^{-1/\alpha}, R\}$ , we get:  $\mathcal{L}_\rho w \geq 1$  for  $\rho$  sufficiently small.

(II). If  $|\xi_1|, |\xi_2| \leq R_1$  and  $|\xi_3| \geq R$ , then

$$\mathcal{L}_\rho w \geq -10R_1^2 - \rho^2 - R^3(K_1 + K_2)/3 + C_3|\xi_3|^\alpha$$

(here, we used the relation:  $-b_i \xi_i^3 \geq 0$  for  $|\xi_i| \in [R, R_1]$ ,  $i = 1, 2$ ). Hence, for  $|\xi_3| \geq \tilde{R}_3$  with  $\tilde{R}_3$  sufficiently large, taking  $\rho$  sufficiently small, we get  $\mathcal{L}_\rho w \geq 1$ .

(III). If  $|\xi_1| \leq R_1$ ,  $|\xi_2| \geq R_1$  and  $|\xi_3| \geq R$  (and similarly, for  $|\xi_1| \geq R_1$ ,  $|\xi_2| \leq R_1$  and  $|\xi_3| \geq R$ ), then

$$\mathcal{L}_\rho w \geq -5R_1^2 + \xi_2^2(-5 + |\xi_2|^\alpha/3) - \rho^2 - K_1 R^3/3 + C_3 |\xi_3|^\alpha.$$

Hence, for  $|\xi_3| \geq \tilde{R}_3$ , we get  $\mathcal{L}_\rho w \geq 1$  for  $\rho$  sufficiently small.

(IV). If  $|\xi_1| \leq R_1$ ,  $|\xi_2| > R$ ,  $|\xi_3| < \tilde{R}_3$  (and similarly for  $|\xi_1| > R$ ,  $|\xi_2| \leq R_1$ ,  $|\xi_3| < \tilde{R}_3$ ), then

$$\mathcal{L}_\rho w \geq |\xi_2|^2(-5 + C_2 |\xi_2|^\alpha/3) - 5R_1^2 - \rho^2 - K_1 R^2/3 - K_3 R.$$

Hence, for  $|\xi_2| > R_1$ , we get  $\mathcal{L}_\rho w \geq 1$  for  $\rho$  sufficiently small.

In conclusion, gluing together all these cases, we accomplish the proof for  $\alpha > 0$ .

*Case (ii).* Assume  $\alpha = 0$ ; we want to prove that there exist some constants  $C_i$  and a radius  $R_0$  such that  $\mathcal{L}_\rho w \geq 1$  in  $\overline{B(0, R_0)}^C$ .

(I). If  $|\xi_i| > R$  for any  $i = 1, 2, 3$ , then  $\mathcal{L}_\rho w \geq \xi_1^2(-5 + \frac{1}{3}C_1) + \xi_2^2(-5 + \frac{1}{3}C_1) + C_3 - \rho^2$ ; hence, for  $C_1, C_2 > 15$ ,  $C_3 > 1$ , we have  $\mathcal{L}_\rho w > 1$  for  $\rho$  sufficiently small.

(II). If  $|\xi_i| < R$  for  $i = 1, 2$  and  $|\xi_3| > R$ , then  $\mathcal{L}_\rho w \geq -10R^2 - R^3(K_1 + K_2)/3 - \rho^2 + C_3$ ; hence, for  $C_3 > 10R^2 + R^3(K_1 + K_2)/3 + 1$ , we have  $\mathcal{L}_\rho w > 1$  for  $\rho$  sufficiently small.

(III). If  $|\xi_1| < R$ ,  $|\xi_2| > R$  and  $|\xi_3| > R$  (and similarly, for  $|\xi_1| \geq R$ ,  $|\xi_2| \leq R$  and  $|\xi_3| \geq R$ ), then  $\mathcal{L}_\rho w \geq -5R^2 + \xi_2^2(-5 + C_2/3) - R^2 K_1/3 - \rho^2 + C_3$ , hence, for  $C_2 > 15$ ,  $C_3$  sufficiently large and  $\rho$  sufficiently small, we have  $\mathcal{L}_\rho w > 1$ .

(IV). If  $|\xi_1| \leq R$ ,  $|\xi_2| > R$ ,  $|\xi_3| < R$  (and similarly for  $|\xi_1| > R$ ,  $|\xi_2| \leq R$ ,  $|\xi_3| < R$ ), then  $\mathcal{L}_\rho w \geq -5R^2 + \xi_2^2(-5 + C_2/3) - K_1 R^2 - \rho^2 - K_3 R$ ; hence, for  $C_2 > 15$ ,  $|\xi_2|$  sufficiently large and  $\rho$  sufficiently small, we have  $\mathcal{L}_\rho w > 1$ .  $\square$

**Remark 2.2** Stronger sufficient condition on  $b_i$  for the existence of a Lyapunov-like function  $w$  satisfying condition (2.1) could be found using  $w(x) := \log((\xi_1^2 + \xi_2^2)^2 + \xi_3^2)$ .

**Remark 2.3** The drifts of the Ornstein-Uhlenbeck operator (i.e.,  $b(y) = -\alpha y$  for  $\alpha > 0$ ) satisfy assumption (i) of Lemma 2.2. For further properties of this operator we refer the reader to the monograph [28].

In the next proposition we will establish the existence of an invariant measure  $m_\rho$  of the approximating process (2.6). This measure will be used in the main theorem of this paper when the invariant measure for the process (1.1) will be obtained as the limit of  $m_\rho$  as  $\rho \rightarrow 0$ .

**Proposition 2.1** *Let  $\sigma_\rho(y)$  defined by (2.5) and  $b(y)$  be a continuous function satisfying (2.7) either with  $\alpha > 0$  or  $\alpha = 0$  and  $C_i$  sufficiently large. There exists a unique invariant probability measure  $m_\rho$  on  $\mathbb{R}^3$  for the process (2.6).*

*Proof.* As proved in Lemma 2.1 the operator  $\mathcal{L}_\rho$  is uniformly elliptic in each bounded set (but the ellipticity constant degenerates in the whole  $\mathbb{R}^3$ ). We adapt some techniques introduced by [27] (see also [14] for similar arguments), by considering approximate problems in domains  $O_n$  such that  $O_n \nearrow \mathbb{R}^3$  if  $n \rightarrow +\infty$ .

Let us recall (see [26] or Lemma A.1 in the Appendix) that condition (2.8) is equivalent to the following one:

$$(2.9) \quad \text{there exists a function } \bar{w} \in C^\infty(\mathbb{R}^3) \text{ such that} \\ \mathcal{L}_\rho \bar{w} + \chi \bar{w} = \phi \quad \text{in } \mathbb{R}^3, \quad \lim_{|\xi| \rightarrow +\infty} \bar{w} = \infty$$

where  $\chi \in C_0^\infty$  and  $\phi \in C^\infty$  are suitable functions such that,  $\chi > 0$  on  $B_0$ ,  $\text{supp} \chi = \bar{B}_0$  ( $B_0$  is a suitable open set) and  $\lim_{|\xi| \rightarrow \infty} \phi = \infty$ . (As a matter of facts, this condition is satisfied by the function  $w$  chosen in the proof of Lemma 2.2-(i)).

We define  $O_n := \{\xi \in \mathbb{R}^3 \mid \bar{w}(\xi) < M_n\}$  where  $M_n \rightarrow +\infty$  if  $n \rightarrow +\infty$  and  $M_n$  is not a critical value of  $\bar{w}$ . Since  $\bar{w} \rightarrow +\infty$  if  $|\xi| \rightarrow +\infty$  then  $O_n$  are bounded and smooth and  $O_n \nearrow \mathbb{R}^3$ .

Fix  $\rho > 0$  and  $n$ , the results by Bensoussan [7, Section 4] ensure that there exists an unique invariant measure  $m_\rho^n$  associated to the diffusion process  $Y_t^\rho$  in  $O_n$  with reflecting boundary whose infinitesimal generator is  $L_\rho$  in  $O_n$  with boundary conditions

$$\sum_{i,j} (a_\rho)_{ij} \frac{\partial u}{\partial \nu_j} = 0 \quad \text{on } \partial O_n$$

where  $\nu$  denotes the unit outward normal to  $\partial O_n$  and the matrix  $A_\rho = (a_\rho)_{ij} = \sigma_\rho \sigma_\rho^T$  as in Lemma 2.1.

The invariant measure  $m_\rho^n$  satisfies the problem

$$(2.10) \quad \mathcal{L}_\rho^* m_\rho^n := - \sum_{i,j} \frac{\partial^2((a_\rho)_{ij} m_\rho^n)}{\partial x_i \partial x_j} + \sum_i \frac{\partial(b_i m_\rho^n)}{\partial x_i} = 0 \quad \text{in } O_n,$$

$$(2.11) \quad \sum_{ij} \nu_i \left( \frac{\partial((a_\rho)_{ij} m_\rho^n)}{\partial x_j} - b_i m_\rho^n \right) = 0 \quad \text{on } \partial O_n$$

$$\int_{O_n} m_\rho^n = 1, \quad m_\rho^n > 0.$$

We have to prove that, as  $n \rightarrow +\infty$ ,  $m_\rho^n$  converges in some sense to  $m_\rho$  invariant measure to the process with generator  $\mathcal{L}_\rho$ , i.e.  $m_\rho$  solves

$$(2.12) \quad \mathcal{L}_\rho^* m_\rho := - \sum_{i,j} \frac{\partial^2((a_\rho)_{ij} m_\rho)}{\partial x_i \partial x_j} + \sum_i \frac{\partial(b_i m_\rho)}{\partial x_i} = 0 \quad \text{in } \mathbb{R}^3$$

$$\int_{\mathbb{R}^3} m_\rho = 1, \quad m_\rho \geq 0.$$

From Prohorov Theorem and the fact that  $\int_{O_n} m_\rho^n = 1$  we know that  $m_\rho^n \rightarrow m_\rho$  as  $n \rightarrow +\infty$  (possibly passing to a subsequence).

We prove now that  $\int_{\mathbb{R}^3} m_\rho = 1$ . Multiplying equation (2.10) by  $\bar{w}$  defined in (2.9), integrating on  $O^n$  and taking into account (2.11) we obtain

$$0 = \int_{O^n} \mathcal{L}_\rho^* m_\rho^n \bar{w} = \int_{O^n} m_\rho^n \mathcal{L}_\rho \bar{w} + \int_{\partial O^n} m_\rho^n \sum_{i,j} (a_\rho)_{ij} \frac{\partial \bar{w}}{\partial x_i} \nu_j.$$

Since  $\bar{w} = M_n$  on  $\partial O^n$  and  $\bar{w} < M_n$  on  $O^n$ , we have  $\frac{\partial \bar{w}}{\partial x_i} = \frac{\partial w}{\partial \nu} \nu_i$  and  $\frac{\partial \bar{w}}{\partial \nu} \geq 0$  on  $\partial O^n$ . Then, there holds

$$0 = \int_{O^n} m_\rho^n \mathcal{L}_\rho \bar{w} + \frac{\partial \bar{w}}{\partial \nu} \int_{\partial O^n} m_\rho^n \sum_{i,j} (a_\rho)_{ij} \nu_i \nu_j,$$

and since  $\sum_{i,j} (a_\rho)_{ij} \nu_i \nu_j \geq 0$ , we obtain  $\int_{O^n} m_\rho^n \mathcal{L}_\rho \bar{w} \leq 0$ . Hence

$$\int_{O^n} m_\rho^n \mathcal{L}_\rho \bar{w} = \int_{O^n} (\phi - \chi \bar{w}) m_\rho^n \leq 0,$$

and

$$\int_{O^n} \phi m_\rho^n \leq \int_{\text{supp} \chi} \chi \bar{w} m_\rho^n \leq C$$

where  $C$  is a positive constant independent of  $n$ . Let us extend  $m_\rho^n$  by zero outside  $O^n$ , and call it again  $m_\rho^n$ , then

$$(2.13) \quad \int_{\mathbf{R}^3} \phi m_\rho^n \leq C$$

where  $C$  is a positive constant independent of  $n$ .

Since  $\lim_{|\xi| \rightarrow +\infty} \phi(\xi) = +\infty$ , for any  $N$  there exists a  $R_N$  such that  $\phi(\xi) > N$  on  $B_{R_N}^C$ . Hence, from (2.13)

$$(2.14) \quad \int_{B_{R_N}^C} m_\rho^n \leq \frac{C}{N}.$$

Since  $\int_{\mathbf{R}^3} m_\rho^n = 1$  then from (2.14)

$$\int_{B_{R_N}} m_\rho^n \geq 1 - \frac{C}{N}$$

and from the weak convergence of  $m_\rho^n$  to  $m_\rho$ , we have

$$\int_{B_{R_N}} m_\rho \geq 1 - \frac{C}{N},$$

hence letting  $N \rightarrow +\infty$  we obtain that  $\int_{\mathbf{R}^3} m_\rho = 1$ .

Moreover from the local regularity  $W^{2,p}$  for any  $p > 1$  of  $m_\rho^n$  since  $m_\rho^n$  solves equation (2.10), passing to the limit we easily obtain that  $m_\rho$  solves equation (2.12). □

**Remark 2.4** The condition of strict ellipticity in the compact subsets of  $\mathbb{R}^N$  is sufficient to deduce from (2.8) the existence of the invariant measure  $m_\rho$  (see [19, Theorem IV.4.1] under their assumption B.1). Nevertheless we gave the proof of Proposition 2.1 because it is purely analytic and for the sake of completeness.

Now we want to prove that  $m_\rho$  converges in some sense to  $m$ , invariant measure to the process (1.1), solving

$$(2.15) \quad \mathcal{L}^* m = 0, \quad \int_{\mathbf{R}^3} m = 1 \quad \text{and} \quad m \geq 0.$$

**Theorem 2.1** *Let  $\sigma$  be defined by (1.2) and  $b(y)$  be a continuous function satisfying (2.7) either with  $\alpha > 0$  or  $\alpha = 0$  and  $C_i$  sufficiently large. Then there exists a unique invariant probability measure  $m$  on  $\mathbb{R}^3$  for the process (1.1).*

*Proof.* The uniqueness of the measure  $m$  comes from the results of Arnold, Klieman [4], or Ichihara, Kunita [25].

The existence of the invariant measure it is obtained proving that the invariant measure  $m_\rho$  of Proposition 2.1 converges, if  $\rho$  tends to 0, to the measure  $m$  associated to the process (1.1). We proceed analogously to Proposition 2.1. The measure  $m_\rho$  satisfies the following conditions:

$$(2.16) \quad \mathcal{L}_\rho^* m_\rho = 0 \quad \text{in } \mathbb{R}^3, \quad \int_{\mathbb{R}^N} m_\rho = 1, \quad m_\rho \geq 0.$$

We know that  $m_\rho \rightarrow m$  as  $\rho \rightarrow 0$  (at least for a subsequence) where  $m$  is a measure. We have to prove that  $m$  is an invariant measure to the process (1.1) i.e. that  $m$  solves (2.15).

From condition (2.8) and the equivalent conditions (2.9), we know that there exists smooth functions  $\chi$  and  $\phi$  such that  $\bar{w}$  satisfies  $\mathcal{L}_\rho \bar{w} + \chi \bar{w} = \phi$ , in  $\mathbb{R}^3$ ,  $\bar{w}$  and  $\phi$  such that  $\rightarrow +\infty$  if  $|\xi| \rightarrow +\infty$  and  $\chi$  has compact support.

Multiplying equation (2.12) by such  $\bar{w}$  and integrating on  $\mathbb{R}^3$  we obtain

$$0 = \int_{\mathbb{R}^3} \mathcal{L}_\rho^* m_\rho \bar{w} = \int_{\mathbb{R}^3} \mathcal{L}_\rho \bar{w} m_\rho = \int_{\mathbb{R}^3} (\phi - \chi \bar{w}) m_\rho,$$

hence

$$(2.17) \quad \int_{\mathbb{R}^3} \phi m_\rho = \int_{\text{supp } \chi} \chi \bar{w} m_\rho \leq C,$$

where  $C$  is a positive constant independent of  $\rho$ . From (2.17), since

$$1 = \int_{B_{R_N}} m_\rho + \int_{B_{R_N}^c} m_\rho,$$

then

$$\int_{B_{R_N}} m_\rho \geq 1 - \frac{C}{N}$$

and from the convergence of  $m_\rho$

$$\int_{B_{R_N}} m \geq 1 - \frac{C}{N},$$

hence letting  $N \rightarrow +\infty$  we obtain  $\int_{\mathbb{R}^3} m = 1$ .

To prove that  $\mathcal{L}^*m = 0$  we write, for any  $\psi$  smooth,

$$0 = \int_{\mathbb{R}^3} \mathcal{L}_\rho^* m_\rho \psi = \int_{\mathbb{R}^3} \mathcal{L}_\rho \psi m_\rho \rightarrow \int_{\mathbb{R}^3} \mathcal{L} \psi m = \int_{\mathbb{R}^3} \mathcal{L}^* m \psi.$$

Taking account that  $\mathcal{L}_\rho \psi \rightarrow \mathcal{L} \psi$  strongly and  $m_\rho \rightharpoonup m$  weakly in  $L^1$ .  $\square$

### 3 A Liouville type theorem

In this section, we establish a Liouville type result, which holds true not only in the Heisenberg setting but also for  $\sigma$  whose columns satisfy the general Hörmander condition.

Let us first recall from [20] the definition of Hörmander condition.

**Definition 3.1** *The vector fields  $X_j$ ,  $j = 1, \dots, m$ , satisfy the Hörmander condition if  $X_1, \dots, X_m$  and their commutators of any order span  $\mathbb{R}^N$  at each point of  $\mathbb{R}^N$ .*

**Proposition 3.1** *Consider the problem*

$$(3.1) \quad \mathcal{L}V = -\text{tr}(\sigma(y)\sigma^T(y)D^2V) - b(y) \cdot DV = 0, \quad y \in \mathbb{R}^N$$

where  $\sigma$  and  $b$  are Lipschitz continuous functions and the vector fields  $X_j = \sigma^j \cdot \nabla$ ,  $j = 1, \dots, m$  satisfy the Hörmander condition as in definition (3.1). Assume that there exists  $w(y) \in C(\mathbb{R}^N)$  and  $R_0 > 0$  such that in the sense of viscosity

$$(3.2) \quad \mathcal{L}w \geq 0 \quad \text{in } \overline{B(0, R_0)}^C, \quad w(y) \rightarrow +\infty \quad \text{as } |y| \rightarrow +\infty.$$

Then:

- i) every viscosity subsolution  $V$  to (3.1) such that  $\lim_{|y| \rightarrow +\infty} \frac{V}{w} \leq 0$  is constant;
- ii) every viscosity supersolution  $V$  to (3.1) such that  $\lim_{|y| \rightarrow +\infty} \frac{V}{w} \geq 0$  is constant.

*Proof.* The proof uses the same arguments as in [27] (see also [5, Lemma 4.1 and remark 4.1]). For the sake of completeness, we shall give the proof of case (i); being similar, the proof of case (ii) is omitted.

Let us first observe that if  $\psi \in \mathcal{C}^2(A)$  ( $A$  is any open set  $A \subset \overline{B(0, R_0)}^C$ ) is a classical supersolution in  $A$ , i.e.

$$\mathcal{L}\psi \geq 0 \text{ in } A$$

then  $w + \psi$  is a viscosity supersolution in  $A$ , i.e.

$$\mathcal{L}(w + \psi) \geq 0 \text{ in } A.$$

Define for each  $\eta > 0$ :

$$V_\eta := V(y) - \eta w(y).$$

We claim that  $V_\eta$  is a viscosity subsolution in  $\overline{B(0, R_0)}^C$  i.e.

$$(3.3) \quad \mathcal{L}(V_\eta) \leq 0 \text{ in } \overline{B(0, R_0)}^C.$$

Indeed, let us assume by contradiction that there exists  $\psi \in \mathcal{C}^2(\overline{B(0, R_0)}^C)$  such that  $V_\eta - \psi$  attains a strict maximum in some point  $\bar{y} \in \overline{B(0, R_0)}^C$ ,  $V(\bar{y}) = \eta w(\bar{y}) + \psi(\bar{y})$ , and that there holds

$$\mathcal{L}(\psi)(\bar{y}) > 0.$$

By the the continuity of the coefficients of  $L$ , and the regularity of  $\psi$  there exists a  $r_0 > 0$  such that

$$(3.4) \quad \mathcal{L}(\psi)(y) > 0, \text{ in } B(\bar{y}, r_0) \subset \overline{B(0, R_0)}^C.$$

As remarked above  $\eta w + \psi$  is a supersolution in  $B(\bar{y}, r_0)$ . Moreover there exists  $\alpha > 0$  such that  $V(y) < \eta w(y) + \psi(y) - \alpha$  for any  $y \in \partial B(\bar{y}, r_0)$ . Then by a local comparison principle (Bony [11]),  $V(y) \leq \eta w(y) + \psi(y) - \alpha$  in  $B(\bar{y}, r_0)$  and for  $y = \bar{y}$  we get a contradiction; our claim (3.3) is completely proved.

Thanks to  $V_\eta \rightarrow -\infty$  as  $|y| \rightarrow +\infty$ , there exist  $R_1(\eta) = R_1 > R_0$  such that

$$V_\eta(y) \leq \sup_{|z|=R_0} V_\eta(z), \quad \forall |y| \geq R_1$$

then, using the weak maximum principle applied to  $V_\eta$ ,

$$\max_{B(0, R_1) \setminus \overline{B(0, R_0)}} V_\eta = \max_{\partial B(0, R_0)} V_\eta$$

and this implies that

$$V_\eta(y) \leq \max_{\partial B(0, R_0)} V_\eta, \quad \forall y \in \overline{B(0, R_0)}^C.$$

Letting  $\eta \rightarrow 0$  in the preceding inequality:

$$V(y) \leq \max_{\partial B(0, R_0)} V, \quad \forall y \in \overline{B(0, R_0)}^C.$$

Therefore  $V$  attains its global maximum so it is a constant by the strong maximum principle established by Bony [11].  $\square$

**Remark 3.1** Note that, in the Heisenberg group, the function  $w$  introduced in Lemma 2.2 satisfies assumptions (3.2).

**Remark 3.2** Let us stress that the above arguments work for any linear operator  $\mathcal{L}$  with continuous coefficients, satisfying a local comparison principle and a strong maximum principle.

**Remark 3.3** Note that conditions on the sub and super solutions in i) and ii) imply the boundedness of the sub and super solutions.

## 4 The ergodic problem

In this section we tackle the following ergodic problem. We consider the family of problems

$$(4.1) \quad \delta u_\delta(x) - \text{tr}(\sigma(x)\sigma^T(x)D^2u_\delta) - b(x)Du_\delta = f(x) \quad \text{in } \mathbb{R}^3,$$

where  $\delta > 0$  and we investigate about the convergence as  $\delta \rightarrow 0$  of  $\delta u_\delta$  to a constant  $\lambda$  called the ergodic constant. Throughout this section, we assume

- $A_1$ ) the diffusion matrix  $\sigma$  has the form (1.2);
- $A_2$ ) the drift  $b$  is  $C^\infty$  in  $\mathbb{R}^3$  and satisfies the hypotheses of Lemma 2.2;
- $A_3$ ) the function  $f(x)$  is  $C^\infty$  and is bounded in  $\mathbb{R}^3$ .

The next two Lemma contain several properties of  $u_\delta$  which will be used later on.

**Lemma 4.1** *Under our assumptions, there exists a unique smooth viscosity solution  $u_\delta$  of the approximating problem (4.1) such that*

$$(4.2) \quad |u_\delta(x)| \leq \frac{C}{\delta}, \quad \forall x \in \mathbb{R}^3,$$

for some positive constant  $C$  independent of  $\delta$ .

*Proof.* The uniqueness follows from the comparison principle proved in Da Lio-Ley [16] (which also applies to the elliptic case: see [5]).

By assumption  $(A_3)$  it is easy to see that  $w^\pm = \pm \frac{C}{\delta}$  with  $C$  sufficiently large is respectively a supersolution and a subsolution for problem (4.1). In conclusion, applying Perron's method, we infer the existence of a solution to (4.1) verifying (4.2).  $\square$

**Remark 4.1** It is worth noticing that Lemma 4.1 still holds true for  $f$  such that  $|f(x)| \leq C(1 + |x|)$ . Actually in this case, it suffices to consider  $w^\pm = \frac{\pm C_1(1+|x|) \pm C_2}{\delta}$  with  $C_1$  and  $C_2$  sufficiently large.

**Lemma 4.2** *Under assumptions  $(A_1)$ - $(A_3)$ , the functions  $v_\delta := \delta u_\delta$ , where  $u_\delta$  is the solution of problem (4.1), are locally uniformly Hölder continuous. Namely, there exists  $\alpha \in (0, 1)$  such that for every compact  $K \subset \mathbb{R}^3$  there exists a constant  $N$  such that*

$$(4.3) \quad |v_\delta(x_1) - v_\delta(x_2)| \leq N|x_1 - x_2|^\alpha, \quad \forall x_1, x_2 \in K, \quad \forall \delta \in (0, 1).$$

The constant  $N$  only depends on  $K$  and on the data of the problem (in particular is independent of  $\delta$ ).

*Proof.* The statement is a direct consequence of the result of Krylov [21]. For the sake of completeness let us sketch how to apply Krylov's result in our case. The function  $v_\delta$  solves the equation

$$(4.4) \quad \delta v_\delta - \text{tr}(\sigma(x)\sigma^T(x)D^2v_\delta) - b(x)Dv_\delta = \delta f(x) \quad \text{in } \mathbb{R}^3.$$

From Lemma 4.1 we know that there exists a smooth solution  $v_\delta$ . We observe that equation (4.4) can be written in the form

$$(4.5) \quad -L_0v_\delta + v_\delta := -\sigma^{ik}\partial_{x_i}(\sigma^{jk}\partial_{x_j}v_\delta) - BDv_\delta + v_\delta = \delta f + (1 - \delta)v_\delta$$

where  $B_j = b_j - \sum_{j,k} \sigma^{jk} \partial_{x_j} \sigma^{jk}$ .  
For  $\delta$  fixed, consider the problem

$$\begin{cases} -L_0 v_{\delta n} + v_{\delta n} = \delta f + (1 - \delta)v_\delta & \text{in } B(0, n) \\ v_{\delta n} = 0 & \text{on } \partial B(0, n). \end{cases}$$

We observe that  $\{v_{\delta n}\}$  is a equibounded family (by the same arguments of Lemma 4.1). [21, Theorem 2.1] of Krylov ensures that there exists  $\alpha \in (0, 1)$  such that for every compact  $K \subset \mathbb{R}^3$  there exists a constant  $N_1$  (independent of  $\delta, n$ ) such that

$$(4.6) \quad |v_{\delta n}(x_1) - v_{\delta n}(x_2)| \leq N_1 |x_1 - x_2|^\alpha, \quad \forall x_1, x_2 \in K.$$

By Ascoli-Arzelà Theorem, letting  $n \rightarrow +\infty$  (possibly passing to a subsequence) we get that  $v_{\delta n}$  converges locally uniformly to a function  $V$ . By the stability and uniqueness results we infer  $V = v_\delta$ . Moreover, passing to the limit in  $n$  in (4.6), we get (4.3).  $\square$

In the next result we prove that  $\delta u_\delta$  converges to a constant which will be characterize in terms of the invariant measure of the process (1.1).

**Theorem 4.1** *Under assumptions  $(A_1)$ - $(A_3)$ , the solution  $u_\delta$  of problem (4.1) given in Lemma 4.1 satisfies*

$$(4.7) \quad \lim_{\delta \rightarrow 0} \delta u_\delta = \int_{\mathbb{R}^3} f(x) d\mu(x), \text{ locally uniformly,}$$

where  $\mu$  is the invariant measure of the diffusion process in  $\mathbb{R}^3$  with infinitesimal generator the operator  $\mathcal{L}$  introduced in (1.3).

*Proof.* We shall proceed following some arguments of [5]. The functions  $v_\delta := \delta u_\delta$  solve (4.4) and, from estimate (4.2), satisfy

$$(4.8) \quad |v_\delta| \leq C, \quad \text{in } \mathbb{R}^3,$$

with  $C$  independent of  $\delta$ , hence they are uniformly bounded in  $\mathbb{R}^3$ . From Lemma 4.2  $v_\delta$  are also uniformly Hölder continuous in any compact set of  $\mathbb{R}^3$ . Then by the Ascoli-Arzelà theorem there is a sequence  $\delta_n \rightarrow 0$  such that  $v_{\delta_n} \rightarrow v$  locally uniformly and  $v$  is a viscosity solution of

$$(4.9) \quad -tr(\sigma(x)\sigma^T(x)D^2v) - b(x)Dv = 0, \quad x \in \mathbb{R}^3.$$

Then by Proposition 3.1,  $v$  is constant.

In conclusion, we have that, possibly passing to a subsequence,  $\{\delta u_\delta\}_\delta$  converges locally uniformly to a constant. Now, it remains to prove that this constant is independent of the subsequence chosen and that it has the form (4.7). By standard arguments of optimal control theory (see [17]), the function  $u_\delta$  can be written as

$$u_\delta(x) = \mathbb{E}_x \int_0^{+\infty} f(X_t) e^{-\delta t} dt$$

where  $X_t$  is the process in (1.1) with initial data  $X_0 = x$  while  $\mathbb{E}$  denotes the expectation. Integrating both sides with respect to the invariant measure, we infer

$$\begin{aligned} \int_{\mathbb{R}^3} u_\delta(x) d\mu(x) &= \int_0^{+\infty} \left( \mathbb{E}_x \int_{\mathbb{R}^3} f(X_t) d\mu(x) \right) e^{-\delta t} dt \\ &= \int_0^{+\infty} \left( \int_{\mathbb{R}^3} f(x) d\mu(x) \right) e^{-\delta t} dt \\ &= \frac{1}{\delta} \int_{\mathbb{R}^3} f(x) d\mu(x) \end{aligned}$$

where the second inequality is due to the definition of invariant measure. Taking into account that every convergent subsequence of  $\{\delta u_\delta\}_\delta$  must converge to a constant, we conclude that all the sequence  $\{\delta u_\delta\}_\delta$  converges to  $\int_{\mathbb{R}^3} f d\mu$ .  $\square$

## 5 The general case

In this section we address to the process (1.1) under the following assumptions:

$$(5.1) \quad \begin{cases} \sigma(y) \in C^\infty(\mathbb{R}^N), \|\sigma(y)\| \leq C(|y| + 1), & \text{for some } C > 0; \\ \text{the columns of } \sigma \text{ satisfy Hörmander condition.} \end{cases}$$

$$(5.2) \quad b(y), f(y) \in C^\infty(\mathbb{R}^N), \|b(y)\|, |f(y)| \leq C(|y| + 1), \quad C > 0.$$

$$(5.3) \quad \begin{cases} \text{For } A := \sigma\sigma^T, \text{ there exists } \{A_\rho(y)\}_{\rho \in (0,1)} \text{ with } A_\rho = \sigma_\rho\sigma_\rho^T \\ A_\rho \rightarrow A \text{ in } L^\infty \text{ and } A_\rho \text{ is locally definite positive.} \end{cases}$$

$$(5.4) \quad \text{There exists a function } w \text{ verifying (2.8), for } \rho \text{ sufficiently small.}$$

**Theorem 5.1** *Under assumptions (5.1)-(5.4) there exists an invariant probability measure  $m$  associated to the diffusion process (1.1).*

*Proof.* We observe that, by assumptions (5.4), Proposition 2.1 still holds true. Then we accomplish the proof arguing as in the proof of Theorem 2.1.  $\square$

**Remark 5.1** Under assumptions (5.1), (5.2), Proposition 3.1 still holds true. In fact, the results of Bony [11] on comparison principle and strong maximum principle hold also in this setting.

**Corollary 5.1** *Under assumptions (5.1)-(5.4) Theorem 4.1 holds true.*

**Example 5.1** *The Grushin operator*

For  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ , consider the diffusion matrix

$$(5.5) \quad \sigma(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & \xi_1 \end{pmatrix}$$

and observe that  $\sigma$  satisfies (5.1) since  $X_1 = (1, 0)$  and  $[X_1, X_2] = (0, 1)$  span all  $\mathbb{R}^2$ . In this case the infinitesimal generator is

$$\mathcal{L}V = -V_{\xi_1\xi_1} - \xi_1^2 V_{\xi_2\xi_2} - b(\xi)DV.$$

We take  $f$  and  $b(\xi) = (b_1(\xi_1), b_2(\xi_2))$  with

$$(5.6) \quad \begin{cases} -b_1\xi_1 \geq 6, & \text{if } |\xi_1| > 1, \\ -b_2\xi_2 \geq 1, & \text{if } |\xi_2| > 1, \\ b_i \text{ are bounded in } [-1, 1], & i = 1, 2. \end{cases}$$

satisfying (5.2). Under these assumptions it is easy to check that the matrix

$$\sigma_\rho(\xi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi_1 & \rho \end{pmatrix}$$

satisfies (5.3) and that the function  $W(\xi) = \frac{1}{12}\xi_1^4 + \frac{1}{2}\xi_2^2$  satisfies (5.4). In conclusion, since all the hypotheses (5.1)-(5.4) are satisfied, Theorem 5.1 apply.

**Remark 5.2** Lions-Musiela in [27] have considered a similar degenerate case but in their paper the elements of the matrix are bounded in  $\mathbb{R}^2$  in this way

$$(5.7) \quad \sigma(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\xi_1}{\sqrt{1+\xi_1^2}} \end{pmatrix}.$$

## A Appendix

In the following Lemma we state the equivalence between conditions (2.8) and (2.9). This property has already been established by P.L. Lions [26]; however, for the sake of completeness we shall provide the proof.

**Lemma A.1** *Consider a linear operator*

$$\mathcal{G}(u) := -\text{tr}(\bar{a}\bar{a}^T D^2 u) - \bar{b} \cdot Du$$

where  $\bar{a}$  verifies the Hörmander condition (3.1),  $\bar{a}$  and  $\bar{b}$  are Lipschitz continuous functions with

$$|\bar{a}(\xi)|, |\bar{b}(\xi)| \leq C(1 + |\xi|) \quad \forall \xi \in \mathbb{R}^N.$$

Then, conditions (2.8) and (2.9) are equivalent; namely the following properties are equivalent:

*i) there exists  $w \in C^\infty(\mathbb{R}^N)$  such that*

$$\mathcal{G}(w) \geq 1 \quad \text{in } \overline{B(0, R_0)}^C, \quad w \geq 0 \quad \text{in } \overline{B(0, R_0)}^C, \quad \lim_{|\xi| \rightarrow +\infty} w = +\infty$$

*for some constant  $R_0 > 0$ ;*

*ii) there exists  $\bar{w} \in C^\infty(\mathbb{R}^N)$  such that*

$$\mathcal{G}(\bar{w}) + \chi \bar{w} = \phi \quad \text{in } \mathbb{R}^N, \quad \lim_{|\xi| \rightarrow +\infty} \bar{w} = +\infty$$

*for some  $C^\infty$  functions  $\chi$  and  $\phi$  with  $\lim_{|\xi| \rightarrow +\infty} \phi = +\infty$ ,  $\chi \geq 0$  and  $\text{supp}\chi$  compact.*

*Proof.* For completeness, we report the arguments of [26]. As one can easily check, property (ii) obviously implies property (i) (possibly adding a constant).

Now, assuming (i), we want to prove (ii). We denote  $K := \frac{\max_{B(0, R_0)} |w|}{\max_{B(0, R_0)} |\mathcal{G}(w)|}$ . We fix  $\chi \in C_0^\infty(\mathbb{R}^N)$  such that  $\chi \geq 0$ ,  $\chi = 1$  in  $B(0, R_0)$  and  $\text{supp}\chi \subset B(0, 2R_0)$ . We claim that the function  $w^b(\xi) := w(\xi) + K + K_G + 1$  satisfies

$$(A.1) \quad \mathcal{G}(w^b) + \chi w^b =: f^*(\xi) \geq 1 \quad \text{in } \mathbb{R}^N, \quad \lim_{|\xi| \rightarrow +\infty} w^b = +\infty.$$

Indeed, the latter property is an immediate consequence of (i). Moreover, for  $|\xi| \leq R_0$ , we have

$$\mathcal{G}(w^b) + \chi w^b \geq -K_{\mathcal{G}} + \chi(w + K + K_{\mathcal{G}} + 1) \geq 1$$

while, for  $|\xi| \geq R_0$ , we have

$$\mathcal{G}(w^b) + \chi w^b \geq \mathcal{G}(w) \geq 1;$$

hence, our claim (A.1) is proved. Let us now consider a regular partition of unity  $\{\phi_i\}_{i \in \mathbb{N}}$  such that  $\phi_i \geq 0$ ,  $\sum_{i=1}^{\infty} \phi_i(\xi) = 1$ ,  $\text{supp } \phi_i \subset B(0, i+1) \setminus B(0, i-1)$ ,  $\phi_i = 1$  on  $B(0, i + \frac{1}{2}) \setminus B(0, i - \frac{1}{2})$ .

We claim that there exists a regular solution to

$$(A.2) \quad \mathcal{G}(W_n) + \chi W_n = \sum_{i=1}^n \phi_i \quad \text{in } \mathbb{R}^N, \quad 0 \leq W_n \leq w^b.$$

In order to prove this existence, it is expedient to introduce, for  $m \geq n+1$ , the following boundary value problem

$$(A.3) \quad \begin{cases} \mathcal{G}(W_{nm}) + \chi W_{nm} = \sum_{i=1}^n \phi_i & \text{in } B(0, m) \\ W_{nm} = 0 & \text{on } \partial B(0, m). \end{cases}$$

We observe that the Hörmander's condition guarantees the comparison principle; since 0 and  $w^b$  are respectively a sub- and a supersolution, the Perron's method entails that there exists a (unique) solution to (A.3) such that  $0 \leq W_{nm} \leq w^b$  in  $B(0, m)$ . Furthermore, for  $m_1 > m$ , still by comparison principle, we infer  $W_{nm_1} \geq W_{nm}$ , namely, the sequence  $\{W_{nm}\}_m$  is nondecreasing and locally bounded. Passing to the limit, we accomplish the proof of our claim (A.2).

By (A.2), the functions  $w_i(\xi) := W_i(\xi) - W_{i-1}(\xi)$  solve

$$\mathcal{G}(w_i) + \chi w_i = \phi_i \quad \text{in } \mathbb{R}^N$$

and verify:  $\sum_{i=1}^{\infty} w_i(\xi) < \infty$  in  $\mathbb{R}^N$ .

Let us now state an elementary result whose proof we refer the reader to exercise 3.5 page 139 of [18].

**Lemma A.2** *Let  $a_i \geq 0$  be such that  $\sum_{i=1}^{\infty} a_i < +\infty$ . Then, there exists a sequence  $\{\lambda_i\}_i$  such that  $\lim_{i \rightarrow +\infty} \lambda_i = +\infty$  and  $\sum_{i=1}^{\infty} \lambda_i a_i < +\infty$ .*

By Lemma A.2, there exists a sequence  $\{\lambda_i\}_i$  such that  $\lim_{i \rightarrow +\infty} \lambda_i = +\infty$  and  $\sum_{i=1}^{\infty} \lambda_i w_i(0) = K < +\infty$ .

Let  $n_0 \in \mathbf{N}$  be fixed. Let us denote by  $w_n^\sharp(\xi) := \sum_{i=1}^n \lambda_i w_i(\xi)$ . In  $B(0, n_0)$   $w_n^\sharp(\xi)$  satisfies:

$$(A.4) \quad \mathcal{G}(w_n^\sharp) + \chi w_n^\sharp = \sum_{i=1}^{n_0+1} \lambda_i \phi_i \quad w_n^\sharp \geq 0$$

and by Harnack inequality there exists a constant  $C_{n_0}$  independent of  $n$  such that

$$\begin{aligned} \sup_{B(0, \frac{n_0}{2})} w_n^\sharp &\leq C_{n_0} \left( \inf_{B(0, \frac{n_0}{2})} w_n^\sharp + \sup_{B(0, \frac{n_0}{2})} \sum_{i=1}^{n_0+1} \lambda_i \phi_i \right) \\ &\leq C_{n_0} \left( K + \sup_{B(0, \frac{n_0}{2})} \sum_{i=1}^{n_0+1} \lambda_i \phi_i \right) = C_{n_0}^* \end{aligned}$$

This implies that in any bounded set  $w^\sharp$  is well defined, i.e.  $w^\sharp(\xi) := \sum_{i=1}^{\infty} \lambda_i w_i(\xi) < \infty$  for every  $\xi \in \mathbb{R}^N$ . Moreover the function  $w^\sharp$  satisfies

$$(A.5) \quad \mathcal{G}(w^\sharp) + \chi w^\sharp = \sum_{i=1}^{\infty} \lambda_i \phi_i =: \phi, \quad w^\sharp \geq 0$$

with  $\lim_{\xi \rightarrow \infty} \phi(\xi) = +\infty$ .

In conclusion, by (A.1) and (A.5), the function  $\bar{w} := w^b + w^\sharp$  satisfies (ii).  $\square$

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**Address of the authors**

Paola Mannucci,

Dipartimento di Matematica, Università degli Studi di Padova,

Via Trieste 63, 35131, Padova, Italy,

Claudio Marchi,

Dipartimento di Ingegneria dell'Informazione, Università degli Studi di

Padova,

Via Gradenigo 6, 35131, Padova, Italy,  
Nicoletta Tchou, IRMAR,  
Università de Rennes 1  
Campus de Beaulieu, 35042 Rennes Cedex, France  
mannucci@math.unipd.it, claudio.marchi@unipd.it, nicoletta.tchou@univ-  
rennes1.fr