# Rotating states in driven clock- and XY-models

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**Abstract**

We consider 3D active plane rotators, where the interaction between the spins is of XY-type and where each spin is driven to rotate. For the clock-model, when the spins take $N \gg 1$ possible values, we conjecture that there are two low-temperature regimes. At very low temperatures and for small enough drift the phase diagram is a small perturbation of the equilibrium case. At larger temperatures the massless modes appear and the spins start to rotate synchronously for arbitrary small drift. For the driven XY-model we prove that there is essentially a unique translation-invariant and stationary distribution despite the fact that the dynamics is not ergodic.

**Keywords:** soft modes, nonequilibrium dynamics

# 1 Introduction

Understanding nonequilibrium phase transitions is a major challenge of statistical physics, bearing many different aspects. Today not much of a systematic theory exists, with few experimental and even fewer mathematical results. One important question of this multifaceted subject is to describe the changes to the equilibrium phase diagram when a steady nonequilibrium driving is added. The best known examples are driven diffusive lattice gases – like boundary driven and asymmetric exclusion processes, see e.g. the books [19, 5], including many results of computer simulations. The nonequilibrium there originates from installing differences in chemical potentials or from adding nonconservative external fields, and follows the prescription of local detailed balance, [15]. In the present paper a uniform nonequilibrium force drives the *internal* degrees of freedom, the planar rotating spins. The spins take values on the unit circle and are placed on the sites of a regular lattice. When mutually uncoupled, all spins undergo the same non-reversible Markov evolution with a bias in the direction of rotation. The interaction couples nearest neighbors, $x \sim y$, with energy following the XY-model,

$$H(\varphi) = -\sum_{x\sim y} \cos\left(\varphi_x - \varphi_y\right) \tag{1}$$

for "angles" $\varphi_x, x \in \mathbb{Z}^D$. The possible values of the angles determine the nature of the spins. A first choice is to take $\varphi_x = 2\pi k/N, k = 1, 2, \ldots, N$ on the discrete circle with $N$ possible values. For $N = 2$ that is the Ising model; for $N = 3$ it is equivalent with the $q = 3$ Potts model. The second possible choice is formally obtained in the $N \uparrow +\infty$ limit, and has a continuum of values $\varphi_x \in [0, 2\pi]$. That truly corresponds to the XY-model where the spins are plane rotators having unit length.

The purpose of the paper is to discuss the modification of the phase diagram when a nonequilibrium driving is inserted that induces biased rotation of the spins over the circle. By doing so the following phenomena can be addressed:
a) the uniqueness of the stationary distribution accompanied by *breakdown of ergodicity* – in the sense that some initial data do not relax;
b) the presence of macroscopic dynamical coherence;
c) the stability of equilibrium phases against small nonequilibrium driving;
We first briefly introduce each of these points, to realize them more concretely in later sections.

## 1.1 Unique stationary distribution without ergodicity

Are there stochastic dynamics with a unique stationary distribution, which are not ergodic? Without any further restrictions this question is easy and not very interesting, with the answer being 'yes'. For discrete time a simple example is given by the two-state Markov chain

$$+1 \to -1 \to +1 \to \dots, \tag{2}$$

flipping deterministically at each time. Similarly, in continuous time we can take the rotation over the circle with a constant angular speed $v$:

$$\theta(t) = \theta(0) + vt \mod 2\pi \tag{3}$$

One wonders whether one can construct non-degenerate random processes, which exhibit the above prototypical behavior. Of course, we must then consider infinite-volume interacting particle systems and infinite probabilistic cellular automata, since finite state non-degenerate Markov processes are always ergodic.

For the case of probabilistic cellular automata (discrete time parallel updating of spins), a construction, mimicking the behavior (2) is presented in a recent paper by [3]. However, the example of [3] still has some degeneracy, because for every time $T$ one can present two local events, $A$ and $B$, such that the transition probability $p_T(A|B)$ in $T$ steps vanishes. Thus, we feel that a truly non-degenerate discrete time example is still missing. We believe that the discrete time version of our 3D driven clock model gives such non-degenerate example.

Our constructions below present the case of rotating interacting spins in continuous time. The rotation speed $v$ in (3) will be induced by the nonequilibrium driving, and the angle $\theta$ should be thought of as the order parameter, or collective phase, of the model. The dynamics will be nondegenerate, as local fluctuations in the phase are allowed. See Theorem 7 for the precise result.

## 1.2 Macroscopic coherence

Not surprisingly, the mechanism above connects with the old but still not completely resolved question of whether macroscopic dynamical coherence or pattern formation in spatially extended systems can be obtained by local

translation-invariant and non-degenerate updating of spins and whether that is even possible for a continuous time (sequential) dynamics and for dynamics that satisfy detailed balance, [4, 5]. In fact, an example with that flavor was recently described in [18]. There an infinite queuing network was considered, with several types of clients and with exponential service times. In the high load regime the system exhibits coherent behavior. That means that if the initial state of the network is close to the 'coherent' one, characterized by a given value of the 'phase' observable (which takes values on the circle), then in the process of evolution this phase evolves with a constant speed, is never 'forgotten', and the initial synchronization is never broken. Still the system has a unique stationary distribution, with the phase being uniformly distributed over the circle. In the language of queuing networks it is an example of violation of the Poisson hypothesis. Yet, this example lives not on a lattice with short range interactions, but on a mean-field graph (which is, in some sense, an infinite complete graph).
There is also a vast literature on the emergence of synchronized rotators using variants of the so called Kuramoto model; for a review, see [1]. Recently, a mean field analysis for active rotator models was carried out in [12], which for some choice of the drift is the mean field version of the model we consider later in (7). We believe that rotating states emerge in low temperature uniformly driven $N$-clock models, if $N$ is sufficiently large. See the conjectures in Section 3.

## 1.3 Stability of equilibrium phases

When the equilibrium model has finitely many macroscopic phases in some regime of its parameters, then we expect that these remain in place for small driving. Below a critical driving, the changes in basin of attraction and in macroscopic appearance will be small. To understand the nature of that critical driving, one must realize that the stability of equilibrium phases requires some sort of free energy barriers, or, in particle language, the excitations must be massive. Therefore, the presence of soft modes, or Goldstone bosons, can break stability. In the context above, that means that the critical driving (the minimum we need to truly disturb the phase diagram) will go down as $N \uparrow +\infty$. See conjectures 3–4 for more precise speculations.

The following Section 2 contains the details of the nonequilibrium model – driven $N$-clock models – together with a summary of the situation in equi-

librium. Section 3 is devoted to what we believe happens for finite $N$; these are mostly a collection of conjectures in which we firmly believe but where the proofs are missing. Some of this is remedied in Section 4 for the driven XY-model where the picture is more complete. The main result is that the 3D driven XY-model shows nonergodicity, while having a unique stationary translation invariant distribution, at (almost) all low temperatures.

# 2 The $N$-Clock Model

The $N$-clock model is an interacting particle system that lives on $\mathbb{Z}^3$. At each site $x \in \mathbb{Z}^3$ there is a spin $\sigma_x \in \mathbb{Z}_N$, where $\mathbb{Z}_N \subset \mathbb{S}^1 \subset \mathbb{C}^1$ is the group of $N$-th roots of unity. Each spin $\sigma_x$ has its clock, and when the clock rings, the spin jumps to one of the two 'nearest' values: $\sigma_x \to \sigma_x^{\pm} = \exp\left\{\pm\frac{2\pi i}{N}\right\}\sigma_x$. That is equivalent to introducing the angles $\varphi_x$ with $\sigma_x = \exp i\varphi_x$, $\varphi_x = 2\pi k/N$, $k = 1, 2, \ldots, N$, and moves $\varphi_x \to \varphi_x \pm 2\pi/N$. In what follows we will use the notation $\zeta(\varphi)$ for $\exp\{i\varphi\}$, $\zeta(\varphi) \in \mathbb{S}^1 \subset \mathbb{C}^1$, so in particular

$$\sigma_x \to \sigma_x^{\pm} = \zeta\left(\pm\frac{2\pi}{N}\right)\sigma_x.$$

The particles are interacting, with the energy given by (1). We define the rates $c(x, \sigma, \pm)$ of the jumps $\sigma_x \to \sigma_x^{\pm}$ of the spin $\sigma_x$ in the environment $\sigma$ at inverse temperature $\beta$ by

$$c(x, \sigma, \pm) = p_{\pm} \exp\left\{\frac{\beta}{2}\sum_{y\sim x}\left[\cos\left(\varphi_x - \varphi_y \pm \frac{2\pi}{N}\right) - \cos\left(\varphi_x - \varphi_y\right)\right]\right\} \quad (4)$$

(sum over nearest neighbors $y$ of $x \in \mathbb{Z}^3$), where the numbers $p_+ \geq p_- > 0$ are two extra parameters. Their difference is measured by $d := \log\left(p_+/p_-\right) \geq 0$ and is called the drift. One can imagine it as the coupling of the planar rotator with a magnetic field that acts perpendicular to the plane. We call the above model the '$N$-Clock model' with a drift. (Of course, the drift does not make sense for $N = 2$ (Ising model).)

We first describe the properties of the symmetric Clock model, when the drift $d = 0$. Note that in this case the evolution defined above satisfies detailed balance. That is what we call the equilibrium or symmetric Clock model. Then, the Gibbs measures for (1) are reversible stationary measures. We can thus use the results of [9], theorem 4.6:

**Theorem 1** *(symmetric Clock model) There exists a value $\beta_0$ of the inverse temperature, such that for every $\beta > \beta_0$ and every $N \geq 2$, the symmetric $N$-Clock model has at least $N$ different extremal stationary distributions, $\langle\cdot\rangle_{\zeta_k,\beta}$, $\zeta_k = \zeta\left(\frac{2\pi k}{N}\right), k = 1, ..., N$. These states are translation-invariant, exhibit long-range order and are magnetized:*

$$\langle\sigma_x\rangle_{\zeta_k,\beta} = m_N(\beta)\, e^{\frac{2\pi i k}{N}}, \text{ with } m_N(\beta) > 0 \text{ for } \beta > \beta_0.$$

*(Here we interpret the spins $\sigma_x$ as elements of $\mathbb{C}^1$.)*

Observe that the finite $\beta_0$ above remains the same for all $N \geq 2$, which makes the Theorem unreachable for the standard low-temperature analysis based on the Peierls condition. For example, the Pirogov-Sinai theory [17, 21] would establish the stability of the $N$ ground states only for $\beta \geq \beta_N^{PS}$, where $\beta_N^{PS} \to \infty$ as $N \to \infty$. The reason for that is not purely technical: indeed, in the domain of validity of the Pirogov-Sinai theory one necessarily has additional properties of the pure phases, such as the exponential decay of the truncated correlation functions. However, we believe that in reality such exponential decay holds only for low enough temperatures, $\beta > \beta_N^G$, with $\beta_N^G \to \infty$ as $N \to \infty$, so the PS-method can not be improved to reach $\beta_0$. Moreover, we think that the following is true:

**Conjecture 2** *There exists a value $\bar{N}$, such that for each $N \geq \bar{N}$ the 3D symmetric $N$-clock model undergoes two phase transitions. Namely, for all $\beta > \beta_N^G$ it has $N$ pure magnetized phases, with exponential decay of truncated correlations, with $\beta_N^G \to \infty$ as $N \to \infty$. For smaller intermediate values of $\beta$, $\beta_N^G > \beta > \beta_N^{cr}$ it also has at least $N$ pure phases with non-zero magnetization; however, the correlation decay in these phases is only algebraic. (A stronger recent conjecture of [6] even talks about a continuum of pure phases, $\langle\cdot\rangle_{\zeta,\beta}$, $\zeta \in \mathbb{S}^1$.) Finally, for $\beta < \beta_N^{cr}$ the model has only one Gibbs state, again with exponential decay of correlations.*

We know from [2] that there is no intermediate phase for the Ising model, hence $\bar{N} > 2$. The conjectured behavior is somewhat similar to the one for 2D Clock models; it was proven in [11] that these indeed undergo a Berezinskii-Kosterlitz-Thouless phase transition. There, massless Goldstone modes appear (hence our notation $\beta_N^G$).

# 3 Conjectures for the driven Clock-model

We now discuss the situation with non-zero drift $d$, (where we do not have detailed balance). Let us run our $N$-Clock model with a drift for a time duration $T$, starting in one of the equilibrium phases $\langle\cdot\rangle_{\zeta_k,\beta}$. Let us denote the resulting state by $\langle\cdot\rangle_{\beta,d}^{k,T}$.

**Conjecture 3** *For every $N \geq \bar{N}$, $\beta > \beta_N^{cr}$ there exists a critical value $d_{cr}(\beta, N)$ of the drift, $0 < d_{cr}(\beta, N)$, such that the following holds:*

1. *if $|d| < d_{cr}(\beta, N)$, then the state $\langle\cdot\rangle_{\beta,d}^{k,T}$ approaches the state $\langle\cdot\rangle_{\beta,d}^{k}$, as $T \to \infty$, which is magnetized:*

$$\langle\sigma_0\rangle_{\beta,d}^{k} \neq 0$$

*and is close to $\langle\cdot\rangle_{\zeta_k,\beta}$ for small $d$, in the sense of expectations of local observables;*

2. *if $d > d_{cr}(\beta, N)$, then the state $\langle\cdot\rangle_{\beta,d}^{k,T}$ is a 'rotating' state as $T \to \infty$ (in particular, it has no limit as $T \to \infty$). Namely, there exist two periodic functions: $m(T) = m(T; \beta, N, d) > 0$ and $\Phi(T) = \Phi(T; \beta, N, d)$, i.e.*

$$m(T + \omega) = m(T), \ \Phi(T + \omega) = \Phi(T),$$

*with period $\omega$ being the mean angular velocity, $\omega = \omega(\beta, N, d)$, and a phase shift $\phi_k = \phi_k(\beta, N, d)$, such that*

$$\left|\langle\sigma_x\rangle_{\beta,d}^{k,T} - m(T)\, e^{i(\Phi(T)+\phi_k)}\right| \to 0 \text{ as } T \to \infty. \tag{5}$$

*(Here we again are treating the spin $\sigma_x$ as belonging to $\mathbb{C}^1$.)*

It is interesting to compare the curve of the states $\langle\cdot\rangle_{\beta,d}^{k,T}$, $T \geq 0$, with the conjectured ([6]) pure phases of the symmetric clock-model, $\langle\cdot\rangle_{\zeta,\beta}$, $\zeta \in \mathbb{S}^1$. One cannot be stopped from guessing that perhaps for every $T$ we have $\langle\cdot\rangle_{\beta,d}^{k,T} = \langle\cdot\rangle_{\zeta(T),\beta}$, for some $\zeta(T) \equiv \zeta(T, k, d, \beta) \in \mathbb{S}^1$.

The next conjecture deals with the behavior of the critical drift $d_{cr}$ introduced above.

**Conjecture 4** *The critical drift is positive at low temperatures. It decreases to become zero at $\beta_N^G$:*

$$d_{cr}(\beta, N) = 0 \text{ for } \beta_N^{cr} < \beta < \beta_N^G.$$

The rationale behind this conjecture is that at temperatures above $\left(\beta_N^G\right)^{-1}$ the $N$-Clock model enters into the spin-wave phase or Goldstone modes regime and so qualitatively should behave like the $XY$-model, which is in the rotating phase for any non-zero value of drift, see below. This similarity of the intermediate phases with the $XY$-model is the basis of all our speculations. Hence, we believe in the following

**Conjecture 5** *For every $\beta$ large enough there exists $N = N(\beta)$, such that for any $N \geq N(\beta)$ and for all $d > 0$ the $N$-Clock model with a drift $d$ has a continuum of different rotating states.*

One might wonder whether there is a difference between the structure of the stationary states in the Pirogov-Sinai regime and in the Goldstone modes regime, when $d > d_{cr}(\beta, N)$, i.e., when we are in the regime of rotating states. We expect the answer to be positive:

**Conjecture 6** *1.* ***Rotating soft modes.*** *In the regime $\beta \in \left(\beta_N^{cr}, \beta_N^G\right)$, $d > 0$ there is a unique stationary distribution, $\langle\cdot\rangle_{\beta,d}^{st}$. It is translation invariant and has zero magnetization. It is given by the limit*

$$\langle\cdot\rangle_{\beta,d}^{st} = \lim_{T\to\infty} \int_T^{T+\omega} \langle\cdot\rangle_{\beta,d}^{k,T} \; dT \tag{6}$$

*(which does not depend on $k$).*

2. ***Rotating PS.*** *In the regime $\beta > \beta_N^G$, $d > d_{cr}(\beta, N)$, in addition to the time-stationary translation invariant state (6) there are also time-stationary non-translation invariant states (the 'Dobrushin' states). They are given by the same formula (6), where instead of the states $\langle\cdot\rangle_{\beta,d}^{k,T}$ one should use the states $\langle\cdot\rangle_{\beta,d}^{\pm k,T}$. The latter are obtained by starting the driven $N$-Clock dynamics with the measure $\delta_{\pm k}$ that gives weight 1 to the configuration*

$$\sigma_{x=\{x_1,x_2,x_3\}} = \begin{cases} e^{2\pi k i/N} & \text{for } x_1 \geq 0 \\ e^{2\pi\left(k+\frac{N}{2}\right)i/N} & \text{for } x_1 < 0 \end{cases},$$

*and where we suppose for simplicity that $N$ is even.*

The Dobrushin time-stationary non-translation invariant states have a rigid interface at the level $x_1 = 0$. In a typical configuration drawn from such a state the spins on different sides of the interface are pointing in (approximately) opposite directions, though the direction itself can be arbitrary. One should remember here that no Dobrushin states exist in the 3D XY-model, as shown in [10]; see also the discussion in [20]. In fact, it is argued in [10] that there are no non-translation invariant states at all in the 3D XY-model. This is the basis of our Conjecture 6. More precisely, we believe that in all cases there is a unique translation invariant stationary distribution for $\beta > \beta_N^{cr}, d > d_{cr}(\beta, N)$.

# 4 The three dimensional $XY$-model: main result

The dynamical $XY$-model with a drift – called $dXY$ model below – can be obtained from the energy (1) , and the dynamics (4) by taking the limit $N \to \infty$, in a diffusive rescaling of time by $t \to t/N^2$. Alternatively, it is a 3D model of coupled Brownian motions $\varphi_x, x \in \mathbb{Z}^3$, on circles $\varphi_x \in \mathbb{S}^1 \subset \mathbb{C}$. The Brownian motions $\varphi_x$ have a constant drift, $d$, and they are interacting via the nearest neighbor attraction (1) . The dynamics for $\varphi_x(t) \in [0, 2\pi]$ is then as follows: modulo $2\pi$,

$$\mathrm{d}\varphi_x(t) = d\ \mathrm{d}t - \frac{\partial H}{\partial \varphi_x}\ \mathrm{d}t + \sqrt{\frac{2}{\beta}}\,\mathrm{d}W_x(t), \tag{7}$$

where the $W_x(t)$ are independent standard Wiener processes and

$$\frac{\partial H}{\partial \varphi_x} = \sum_{y:y\sim x} \sin(\varphi_x(t) - \varphi_y(t))$$

The formal generator of this process, acting on local smooth functions $f$ is

$$\begin{aligned} Lf &= L_0 f + d \sum_x \frac{\partial f}{\partial \varphi_x} \\ L_0 f &= \sum_x \big[ - \frac{\partial H}{\partial \varphi_x}\, \frac{\partial f}{\partial \varphi_x} + \frac{1}{\beta}\, \frac{\partial^2 f}{\partial \varphi_x^2} \big] \end{aligned} \tag{8}$$

Observe that $L_0$ commutes with the new generator for the driven model: $[L, L_0] = 0$ because for all $x$,

$$\sum_y \frac{\partial^2 H}{\partial \varphi_x \partial \varphi_y} = 0$$

Of course $L - L_0$ generates independent rotation on each angle with angular speed $d$ and thus commutes with the generator $L_0$ of the (undriven) $XY-$model. In other words, the dynamics of the $XY-$model can be interchanged with uniform rotation of all spins.

Much of the equilibrium structure of the $XY$-model is known. At low temperatures $\beta^{-1}$ the 3D $XY$-model has a continuum of translation-invariant Gibbs states. They can be obtained as thermodynamic limits $\langle\cdot\rangle_{\zeta,\beta}$ of the finite-volume Gibbs states with coherent boundary conditions $\varphi_x \equiv \zeta \in \mathbb{S}^1$. outside the volume. These translation invariant states have non-zero spontaneous magnetization,

$$\langle\varphi_0\rangle_{\zeta,\beta} = m(\beta)\,\zeta, \text{ with } m(\beta) > 0,$$

see [8].

Here is our main result. Consider the set $\mathcal{S}$ of stationary and translation-invariant distributions for the $dXY-$model.

**Theorem 7** *The set $\mathcal{S}$ is a singleton for almost all temperatures, while at sufficiently low temperatures there exist rotating states as in Conjecture 3.2: $\langle\cdot\rangle_{\beta,d}^{\zeta,T} = \langle\cdot\rangle_{\zeta+dT,\beta}$.*

**Proof.** We start by repeating that the evolution of the random variables $\psi_x(t) = \varphi_x(t) - d\,t$ is that of the $XY$-model with zero drift. If therefore $\mu$ is a stationary distribution for the $dXY-$model, then $\mu$ is periodically repeated under the symmetric $XY$-dynamics. Suppose now that $\mu$ is translation-invariant. Then, Holley's argument shows that $\mu$ is in fact a translation invariant Gibbs measure for the XY-model, [14]. Moreover, $\mu$ must then be rotation invariant ($\mathbb{S}^1$-invariant) since stationary states of the $dXY-$model have zero magnetization. From [16] it then follows that $\mu$ is unique for almost all temperatures.

On the other hand, at low temperatures $\beta^{-1}$ the 3D $XY$-model has a continuum of translation-invariant Gibbs states. These phases of the $XY$-model correspond in an evident way to rotating states of the $dXY-$model. These

rotating states then do not converge to a stationary state of the $dXY$ model.
■

**Remarks**:
1. The Gibbs field $\langle\cdot\rangle_\beta^{st}$ , defined by

$$\langle\cdot\rangle_\beta^{st} = \int_{\mathbb{S}^1} \langle\cdot\rangle_{\zeta,\beta}\, d\zeta$$

is $\mathbb{S}^1$-invariant. We believe that for all $\beta$ the state $\langle\cdot\rangle_\beta^{st}$ is the only translation-invariant Gibbs state of the $XY$-model, which is $\mathbb{S}^1$-invariant, which would remove the "almost all". (This statement is proven to hold for the $XY$-model for almost all values of $\beta$, [10].)

2. Note, however, that by changing the interaction from $\cos(\varphi_x - \varphi_y)$ to $\left(\frac{1+\cos(\varphi_x-\varphi_y)}{2}\right)^p$ – i.e. by passing to the so-called 'very-nonlinear $\sigma-$model' – we obtain an example of a system which at some temperatures has at least two translation-invariant $\mathbb{S}^1$-invariant Gibbs states (for $p$ large enough), see [7].

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# References

[1] J.A. Acebrón, L.L. Bonilla, C.J. Pérez Vicente, F. Ritort and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, Rev. Mod. Phys. **77**, 137–185 (2005).

[2] M. Aizenman, D.J. Barsky and R. Fernández, The phase transition in a general class of Ising-type models is sharp, J. Stat. Phys. **47**, 343–374 (1987).

[3] P. Chassaing and J. Mairesse, A non-ergodic probabilistic cellular automaton with a unique invariant measure, `arXiv:1009.0143v2 [cs.FL]`.

[4] M. Diakonova and R.S. MacKay, Mathematical examples of space-time phases, to appear in Int. J. Bif. Chaos.

[5] R. Dickman and J. Marro, *Nonequilibrium Phase Transitions in Lattice Models*, Cambridge University Press (May 13, 1999).

[6] A.C.D. van Enter, C. Külske and A.A. Opoku, Discrete approximations to vector spin models, `arXiv:1104.4241v1 [math-ph]`.

[7] A.C.D. van Enter, S.B. Shlosman, Provable first-order transitions for liquid crystal and lattice gauge models with continuous symmetries, Comm. Math. Phys. **255**, 21–32 (2005).

[8] J. Fröhlich, B. Simon and T. Spencer, Infrared bounds, phase transitions and continuous symmetry breaking, Comm. Math. Phys. **50**, 79–95 (1976).

[9] J. Fröhlich, R. Israel, E.H. Lieb and B. Simon, Phase transitions and reflection positivity. I. General theory and long range lattice models, Comm. Math. Phys. **62**, 1–34 (1978).

[10] J. Fröhlich and C.-E. Pfister, Spin waves, vortices, and the structure of equilibrium states in the classical XY model, Comm. Math. Phys. **89**, 303–327 (1983).

[11] J. Fröhlich and T. Spencer, The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas, Comm. Math. Phys. **81**, 527–602 (1981).

[12] G. Giacomin, K. Pakdaman, X. Pellegrin and C. Poquet, Transitions in active rotator systems: invariant hyperbolic manifold approach, `arXiv:1106.0758v1 [math-ph]`.

[13] G. Grinstein, D. Mukamel, R. Seidin, and C.H. Bennett, Temporally periodic phases and kinetic roughening, Phys. Rev. Lett. **70**, 3607—3610 (1993).

[14] R. Holley, Free energy in a Markovian model of a lattice spin system, Commun. Math. Phys. **23**, 87–99 (1971).

[15] S. Katz, J.L. Lebowitz and H. Spohn, Phase Transitions in Stationary Non-equilibrium States of Model lattice Systems, Phys. Rev. B **28**, 1655–1658 (1983).

[16] C.-E. Pfister, Translation invariant equilibrium states of ferromagnetic abelian lattice systems, Comm. Math. Phys. **86**, 375–390 (1982).

[17] S.A. Pirogov and Ya.G. Sinai, Phase Diagrams of Classical Lattice Systems, Theor. and Math. Phys. **25**, 358–369, 1185–1192 (1975).

[18] A. Rybko, S. Shlosman and A. Vladimirov, Spontaneous Resonances and the Coherent States of the Queuing Networks, J. Stat. Phys. **134**, 67–104 (2009).

[19] B. Schmittman and R.K.P. Zia, *Statistical Mechanics of Driven Diffusive Systems*, Volume 17 (Phase Transitions and Critical Phenomena), Cyril Domb (Series Editor), R.K.P. Zia (Series Editor), B. Schmittmann (Series Editor), J.L. Lebowitz (Series Editor), Academic Press (August 28, 1995), pp 3–214.

[20] S. Shlosman and Y. Vignaud, Dobrushin Interfaces via Reflection Positivity, Comm. Math. Phys. **276**, 827–861 (2007).

[21] Ya.G. Sinai, *Theory of Phase Transitions*, Budapest: Academia Kiado and London: Pergamon Press (1982).