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# RIESZ TRANSFORM FOR $1 \leq p \leq 2$ WITHOUT GAUSSIAN HEAT KERNEL BOUND

LI CHEN, THIERRY COULHON, JOSEPH FENEUIL, AND EMMANUEL RUSS

ABSTRACT. We study the  $L^p$  boundedness of Riesz transform as well as the reverse inequality on Riemannian manifolds and graphs under the volume doubling property and a sub-Gaussian heat kernel upper bound. We prove that the Riesz transform is then bounded on  $L^p$  for  $1 < p < 2$ , which shows that Gaussian estimates of the heat kernel are not a necessary condition for this. In the particular case of Vicsek manifolds and graphs, we show that the reverse inequality does not hold for  $1 < p < 2$ . This yields a full picture of the ranges of  $p \in (1, +\infty)$  for which respectively the Riesz transform is  $L^p$ -bounded and the reverse inequality holds on  $L^p$  on such manifolds and graphs. This picture is strikingly different from the Euclidean one.

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## 1. INTRODUCTION

Let  $M$  be a complete connected non-compact Riemannian manifold. Let  $d$  be the geodesic distance and  $\mu$  be the Riemannian measure. Denote by  $B(x, r)$  the open ball of center  $x$  and of geodesic radius  $r$ . We write  $V(x, r)$  for  $\mu(B(x, r))$ . Let  $\nabla$  be the Riemannian gradient and  $\Delta$  be the non-negative Laplace-Beltrami operator on  $M$ . Denote by  $(e^{-t\Delta})_{t>0}$  the heat semigroup associated with  $\Delta$  and  $h_t(x, y)$  the associated heat kernel. It is well-known that  $h_t(x, y)$  is everywhere positive, symmetric in  $x, y \in M$  and smooth in  $t > 0, x, y \in M$  (see for instance [27, Theorem 7.13]). For  $f \in C_0^\infty(M)$ , denote by  $|\nabla f|$  the Riemannian length of  $\nabla f$ . From the definition and spectral theory, it holds that for all  $f \in C_0^\infty(M)$ ,

$$(1.1) \quad \|\nabla f\|_2^2 = (\Delta f, f) = \|\Delta^{1/2} f\|_2^2.$$

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In the case where  $M = \mathbb{R}^n$  endowed with the Euclidean metric, it is well-known that this equality extends to  $L^p$ ,  $1 < p < +\infty$ , in the form of an equivalence of seminorms

$$(E_p) \quad \|\|\nabla f\|\|_p \simeq \|\Delta^{1/2} f\|_p,$$

for  $f \in C_0^\infty(\mathbb{R}^n)$ . Here and in the sequel we use the notation  $u \simeq v$  if  $u \lesssim v$  and  $v \lesssim u$ , where  $u \lesssim v$  means that there exists a constant  $C$  (independent of the important parameters) such that  $u \leq Cv$ .

It was asked by Strichartz [37] in 1983 on which non-compact Riemannian manifolds  $M$ , and for which  $p$ ,  $1 < p < +\infty$ , the equivalence of seminorms  $(E_p)$  still holds on  $C_0^\infty(M)$ . Let us distinguish between the two inequalities involved in  $(E_p)$  and introduce a specific notation.

The first one

$$(R_p) \quad \|\|\nabla f\|\|_p \lesssim \|\Delta^{1/2} f\|_p, \quad \forall f \in C_0^\infty(M),$$

can be reformulated by saying that the Riesz transform  $\nabla\Delta^{-1/2}$  is  $L^p$  bounded on  $M$ . We will refer to the second one

$$(RR_p) \quad \|\Delta^{1/2} f\|_p \lesssim \|\|\nabla f\|\|_p, \quad \forall f \in C_0^\infty(M),$$

as the reverse inequality.

It is well-known (see for example [15, Proposition 2.1]) that by duality  $(R_p)$  implies  $(RR_{p'})$ , where  $p'$  is the conjugate exponent of  $p$ .

We say that  $M$  satisfies the volume doubling property if, for any  $x \in M$  and  $r > 0$ ,

$$(D) \quad V(x, 2r) \lesssim V(x, r).$$

The key result on  $(R_p)$  in the case  $1 < p < 2$  is the following one.

**Theorem 1.1.** [14, Theorem 1.1] *Let  $M$  be a complete non-compact Riemannian manifold satisfying the volume doubling property  $(D)$  and the heat kernel upper estimate*

$$(UE) \quad h_t(x, y) \lesssim \frac{1}{V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right), \quad \forall x, y \in M, t > 0,$$

for some  $c > 0$ . Then  $(R_p)$  holds for  $1 < p < 2$ .

Note that one does not know any complete Riemannian manifold where  $(R_p)$  does not hold for some  $p \in (1, 2)$ . It is therefore very natural to ask whether the above assumptions are necessary or not. It is clear that  $(D)$  is not. For instance, as soon as the manifold has a spectral gap (and satisfies local volume doubling and small time gaussian estimates), it is easy to see that  $(R_p)$  holds (see [14, Theorem 1.3]); the presence of a spectral gap is clearly incompatible with global volume doubling. Moreover it is known that  $(R_p)$  holds for all  $p \in (1, 2)$  on some specific Lie groups of exponential growth (see for instance [33, 34]). Yet, if  $(D)$  is assumed, are Gaussian estimates of the heat kernel necessary for the Riesz transform to be  $L^p$ -bounded for  $1 < p < 2$ ? As a matter of fact, no examples were known so far where  $(R_p)$  holds for  $1 < p < 2$  and  $(D)$  holds without  $(UE)$  holding as well.

The main goal of the present paper is to give a negative answer to the above question. We will deduce this from a positive result: we will in fact establish that  $(R_p)$  also holds for all  $p \in (1, 2)$  if we replace  $(UE)$  in the assumptions of Theorem 1.1 by a so-called sub-Gaussian heat kernel upper estimate. The conclusion follows because there exist manifolds that satisfy such sub-Gaussian upper estimates as well as the matching lower estimates and the latter are clearly incompatible with  $(UE)$  (see the Appendix below for both facts). As a by-product, this provides new classes of examples where  $(R_p)$  does hold for  $p \in (1, 2)$ .

Note that the question remains whether any heat kernel estimate at all is necessary. A bold conjecture has been made in this direction in [15, Conjecture 1.1].

Let  $m > 2$ . One says that  $M$  satisfies the sub-Gaussian heat kernel upper estimate with exponent  $m$  if for any  $x, y \in M$ ,

$$(UE_m) \quad h_t(x, y) \lesssim \begin{cases} \frac{1}{V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right), & 0 < t < 1, \\ \frac{1}{V(x, t^{1/m})} \exp\left(-c \left(\frac{d^m(x, y)}{t}\right)^{1/(m-1)}\right), & t \geq 1, \end{cases}$$

for some  $c > 0$ .

We refer to [30, Section 5] for a detailed introduction of a class of non-classical heat kernel estimates that includes this one.

Fractal manifolds are typical examples that satisfy sub-Gaussian heat kernel estimates; they are built from graphs with a self-similar structure at infinity by replacing the edges of the graph with tubes of length 1 and then gluing the tubes together smoothly at the vertices (see [6]). It has been proved in [3] that for any  $D, m \in \mathbb{R}$  such that  $D > 1$  and  $2 < m \leq D + 1$ , there exists an infinite connected locally finite graph with polynomial growth of exponent  $D$  satisfying the discrete analogue of  $(UE_m)$  (see Section 4 for a precise definition). It follows that for any  $D, m \in \mathbb{R}$  such that  $D > 1$  and  $2 < m \leq D + 1$ , there exist complete connected Riemannian manifolds satisfying  $V(x, r) \simeq r^D$  for  $r \geq 1$  and  $(UE_m)$  (see the Appendix below).

The article [24] by the third author is the main source of inspiration for the present paper. It treats the analogous problem in a discrete setting: [24, Corollary 1.30] states that the discrete Riesz transform is  $L^p$  bounded for  $1 < p < 2$  on graphs satisfying the volume doubling property and a sub-Gaussian upper estimate for the Markov kernel. The author introduces a Hardy space theory for functions and 1-forms on graphs and obtains the  $H^1$ -boundedness of the Riesz transform. Then its  $L^p$ -boundedness for  $1 < p < 2$  follows by interpolation between  $H^1$  and  $L^2$ . We rely on a crucial idea from [24], namely a new way to use a well-known and very powerful trick by Stein ([36, Chapter 2, Lemma 2]; for other utilizations of the argument in our subject see [16, Theorem 1.2] and [10, Proposition 1.8]). Instead of introducing  $H^1$  spaces as in [24], we follow a more direct route and prove as in [14] that the Riesz transform is weak  $(1, 1)$  by adapting the method to the sub-Gaussian case as in [9]. This proof also works in the discrete case and gives a simpler proof of the case  $1 < p < 2$  in [24, Corollary 1.30].

Our main result is:

**Theorem 1.2.** *Let  $M$  be a complete non-compact Riemannian manifold satisfying the doubling property  $(D)$  and the sub-Gaussian heat kernel upper bound  $(UE_m)$  for some  $m > 2$ . Then the Riesz transform is weak  $(1, 1)$  and  $L^p$  bounded for  $1 < p \leq 2$ .*

Recall that a (sub)-linear operator defined on  $L^1$  is said to be weak  $(1, 1)$  if, for all  $\lambda > 0$ ,

$$\mu(\{x \in M; |Tf(x)| > \lambda\}) \lesssim \frac{1}{\lambda} \|f\|_1.$$

The plan of the paper is as follows. Section 2 is devoted to a crucial integral estimate for the gradient of the heat kernel. In Section 3, we prove our main result by using the Duong-McIntosh singular integral method from [21] as in [14]. As a consequence, under the doubling property and the sub-Gaussian heat kernel estimate, the Riesz transform is weak  $(1, 1)$  hence  $L^p$  bounded for  $p \in (1, 2)$  by interpolation with (1.1). Section 4 is the counterpart of Sections 2 and 3 on graphs. Finally, in Section 5, we prove that on Vicsek manifolds, for all  $p \in (1, 2)$ ,  $(RR_p)$  does not hold. It follows that on such manifolds, in the range  $1 < p < +\infty$ ,  $(R_p)$  holds if and only if  $1 < p \leq 2$  and  $(RR_p)$  holds if and only if  $2 \leq p < +\infty$ .

## 2. INTEGRATED ESTIMATE FOR THE GRADIENT OF THE HEAT KERNEL

In the Gaussian case, the estimates from this section are due to Grigor'yan in [26]. They go through a version of Lemma 2.2 for  $q = 2$ , which follows easily from the Gaussian estimate of the heat kernel by integration by parts (see also [14, Lemma 2.3] for a simpler version). In the subgaussian case, the corresponding argument yields a factor  $t^{-\alpha}$  with  $0 < \alpha < 1/2$  in the right hand side of the crucial estimate (2.10) below, which is not enough to prove the  $L^p$ -boundedness of the Riesz transform, see [9, Lemma 3.2]. To obtain the stronger estimate we need, namely (2.11), we will choose  $1 < q < 2$ , which will enable us to use a powerful argument by Stein.

Let us first derive some easy consequences of the doubling property and the sub-Gaussian heat kernel estimate.

A classical consequence of (D) is that there exists  $\nu > 0$  such that

$$(2.1) \quad \frac{V(x, r)}{V(x, s)} \lesssim \left(\frac{r}{s}\right)^\nu, \quad \forall x \in M, r \geq s > 0.$$

Fix now once and for all  $m > 2$ .

**Lemma 2.1.** *Let  $M$  be a complete Riemannian manifold satisfying the volume doubling property (D) and the sub-Gaussian heat kernel upper bound (UE<sub>m</sub>). For any  $q > 1$ , there holds, for all  $y \in M$ ,*

$$(2.2) \quad \int_M h_t^q(x, y) d\mu(x) \lesssim \begin{cases} \frac{1}{[V(y, \sqrt{t})]^{q-1}}, & 0 < t < 1, \\ \frac{1}{[V(y, t^{1/m})]^{q-1}}, & t \geq 1, \end{cases}$$

and

$$(2.3) \quad \int_M |\Delta h_t(x, y)|^q d\mu(x) \lesssim \begin{cases} \frac{1}{t^q [V(y, \sqrt{t})]^{q-1}}, & 0 < t < 1, \\ \frac{1}{t^q [V(y, t^{1/m})]^{q-1}}, & t \geq 1. \end{cases}$$

*Proof.* It follows from (D) that for any  $r \geq 0$  and any  $c > 0$  (see for example Step 1 of the proof of Lemma 3.2 in [9]),

$$(2.4) \quad \int_{d(x,y)>r} e^{-c\frac{d^2(x,y)}{t}} d\mu(x) \lesssim e^{-\frac{c}{2}\frac{r^2}{t}} V(y, t^{1/2}),$$

and

$$(2.5) \quad \int_{d(x,y)>r} e^{-c\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}} d\mu(x) \lesssim e^{-\frac{c}{2}\left(\frac{r^m}{t}\right)^{1/(m-1)}} V(y, t^{1/m}).$$

Thus (UE<sub>m</sub>) yields that for  $t \geq 1$ ,

$$\begin{aligned} \int_M h_t^q(x, y) d\mu(x) &\lesssim \frac{1}{[V(y, t^{1/m})]^q} \int_M e^{-cq\left(\frac{d^m(x,y)}{t}\right)^{1/(m-1)}} d\mu(x) \\ &\lesssim \frac{1}{[V(y, t^{1/m})]^{q-1}}, \end{aligned}$$

with an analogous argument for  $0 < t < 1$ , hence (2.2).

Now, by the analyticity of the heat semigroup on  $L^q(M, \mu)$  for  $1 < q < +\infty$ , for  $t \geq 2$ ,

$$\int_M |\Delta h_t(x, y)|^q d\mu(x) \leq \left\| \Delta e^{-\frac{t}{2}\Delta} \right\|_{q \rightarrow q}^q \int_M |h_{t/2}(x, y)|^q d\mu(x)$$

$$\lesssim \frac{1}{t^q [V(y, t^{1/m})]^{q-1}}.$$

Similarly for  $0 < t < 2$ , we repeat the proof above and obtain that

$$\int_M |\Delta h_t(x, y)|^q d\mu(x) \lesssim \frac{1}{t^q [V(y, t^{1/2})]^{q-1}}.$$

Since, for  $1 < t < 2$ , it holds that  $V(y, t^{1/2}) \simeq V(y, t^{1/m})$ , we obtain (2.3).  $\square$

**Lemma 2.2.** *Let  $M$  be a complete Riemannian manifold satisfying the volume doubling property (D) and the sub-Gaussian heat kernel upper bound ( $UE_m$ ). For  $q \in (1, 2)$ , there exists  $c > 0$  such that, for all  $y \in M$  and all  $0 < t < 1$ , one has*

$$\left\| |\nabla h_t(\cdot, y)| \exp\left(c \frac{d^2(\cdot, y)}{t}\right) \right\|_q \lesssim \frac{1}{\sqrt{t} [V(y, t^{1/2})]^{1-\frac{1}{q}}},$$

and for all  $y \in M$  and all  $t \geq 1$ , one has

$$\left\| |\nabla h_t(\cdot, y)| \exp\left(c \left(\frac{d^m(\cdot, y)}{t}\right)^{1/(m-1)}\right) \right\|_q \lesssim \frac{1}{\sqrt{t} [V(y, t^{1/m})]^{1-\frac{1}{q}}}.$$

*Proof.* Let  $q \in (1, 2)$ . Fix  $y \in M$ . Define, for all  $x \in M$  and all  $t > 0$ ,  $u(x, t) := h_t(x, y)$ . Since  $M$  is connected,  $u(x, t)$  is positive. A straightforward computation (see [36, Lemma 1, page 86], [35, Lemma 2, page 49] in different settings) shows that

$$\left(\frac{\partial}{\partial t} + \Delta\right) u^q(x, t) = qu^{q-1}(x, t) \left(\frac{\partial}{\partial t} + \Delta\right) u(x, t) - q(q-1)u^{q-2}(x, t) |\nabla_x u(x, t)|^2.$$

Since  $u$  is a solution to the heat equation,

$$\left(\frac{\partial}{\partial t} + \Delta\right) u^q(x, t) = -q(q-1)u^{q-2}(x, t) |\nabla_x u(x, t)|^2,$$

hence

$$(2.6) \quad |\nabla_x u(x, t)|^2 = \frac{1}{q(q-1)} u^{2-q}(x, t) J(x, t),$$

where  $J(x, t) = -\left(\frac{\partial}{\partial t} + \Delta\right) u^q(x, t)$ . Note in particular that  $J(x, t) \geq 0$  for all  $x \in M$  and all  $t > 0$ .

Since  $u(\cdot, t) \in L^q$ , then  $u^q(\cdot, t) \in L^1$  and  $\int_M \Delta u^q(x, t) d\mu(x) = 0$ . Together with Hölder inequality, we have

$$\begin{aligned} \int_M J(x, t) d\mu(x) &= -\int_M \frac{\partial}{\partial t} u^q(x, t) d\mu(x) \leq \int_M |qu^{q-1}(x, t) \Delta u(x, t)| d\mu(x) \\ &\lesssim \left(\int_M u^q(x, t) d\mu(x)\right)^{\frac{q-1}{q}} \left(\int_M |\Delta u(x, t)|^q d\mu(x)\right)^{1/q}. \end{aligned}$$

From now on, let us assume that  $t \geq 1$ . The proof for  $0 < t < 1$  is similar and we omit it here. It follows easily from Lemma 2.1 that

$$(2.7) \quad \int_M J(x, t) d\mu(x) \lesssim \frac{1}{t [V(y, t^{1/m})]^{q-1}}.$$

According to (2.6), for all  $x \in M$  and  $t > 0$ ,

$$|\nabla_x u(x, t)| \exp\left(c \left(\frac{d^m(x, y)}{t}\right)^{1/(m-1)}\right) = C_q [u(x, t)]^{1-\frac{q}{2}} J^{1/2}(x, t) \exp\left(c \left(\frac{d^m(\cdot, y)}{t}\right)^{1/(m-1)}\right),$$

so that

$$\begin{aligned} & \left\| |\nabla u(\cdot, t)| \exp \left( c \left( \frac{d^m(\cdot, y)}{t} \right)^{1/(m-1)} \right) \right\|_q^q \\ &= C_q^q \int_M [u(x, t)]^{q(1-\frac{q}{2})} J^{q/2}(x, t) \exp \left( cq \left( \frac{d^m(x, y)}{t} \right)^{1/(m-1)} \right) d\mu(x). \end{aligned}$$

By Hölder, since  $q < 2$ , the latter quantity is bounded from above by

$$\left( \int_M u^q(x, t) \exp \left( c \frac{q}{1-\frac{q}{2}} \left( \frac{d^m(x, y)}{t} \right)^{1/(m-1)} \right) d\mu(x) \right)^{1-\frac{q}{2}} \left( \int_M J(x, t) d\mu(x) \right)^{\frac{q}{2}}.$$

The second factor is bounded from above by (2.7). As for the first one, one uses (UE<sub>m</sub>), so that if  $c$  is small enough (only depending on  $q$ ), there exists  $c' > 0$  such that

$$\begin{aligned} (2.8) \quad & \left( \int_M u^q(x, t) \exp \left( c \frac{q}{1-\frac{q}{2}} \left( \frac{d^m(x, y)}{t} \right)^{1/(m-1)} \right) d\mu(x) \right)^{1-\frac{q}{2}} \\ & \lesssim \frac{1}{[V(y, t^{1/m})]^{q(1-\frac{q}{2})}} \left( \int_M \exp \left( -c' \left( \frac{d^m(x, y)}{t} \right)^{1/(m-1)} \right) d\mu(x) \right)^{1-\frac{q}{2}} \\ & \lesssim \frac{1}{[V(y, t^{1/m})]^{(q-1)(1-\frac{q}{2})}}, \end{aligned}$$

where the last line is due to (2.5) with  $r = 0$ . It follows that

$$\left\| |\nabla u(\cdot, t)| \exp \left( c \left( \frac{d^m(\cdot, y)}{t} \right)^{1/(m-1)} \right) \right\|_q^q \lesssim \frac{1}{t^{q/2} [V(y, t^{1/m})]^{q-1}},$$

which yields the conclusion.  $\square$

**Corollary 2.3.** *Let  $M$  be a complete Riemannian manifold satisfying (D) and (UE<sub>m</sub>). Then there exists  $c > 0$  such that for all  $y \in M$ , all  $0 < t < 1$  and all  $r > 0$ ,*

$$(2.9) \quad \int_{M \setminus B(y, r)} |\nabla_x h_t(x, y)| d\mu(x) \lesssim \frac{1}{\sqrt{t}} \exp \left( -c \frac{r^2}{t} \right),$$

and for all  $y \in M$ , all  $t \geq 1$  and all  $r > 0$ ,

$$(2.10) \quad \int_{M \setminus B(y, r)} |\nabla_x h_t(x, y)| d\mu(x) \lesssim \frac{1}{\sqrt{t}} \exp \left( -c \left( \frac{r^m}{t} \right)^{1/(m-1)} \right).$$

*Proof.* Fix  $q \in (1, 2)$  and let  $c > 0$  be given by Lemma 2.2. Then, by Hölder,

$$\begin{aligned} \int_{M \setminus B(y, r)} |\nabla_x h_t(x, y)| d\mu(x) & \leq \left\| |\nabla h_t(\cdot, y)| \exp \left( c \left( \frac{d^m(\cdot, y)}{t} \right)^{1/(m-1)} \right) \right\|_q \\ & \quad \left( \int_{M \setminus B(y, r)} \exp \left( -c' \left( \frac{d^m(x, y)}{t} \right)^{1/(m-1)} \right) d\mu(x) \right)^{1-\frac{1}{q}}, \end{aligned}$$

and by (2.5) and Lemma 2.2, for  $t \geq 1$ ,

$$\int_{M \setminus B(y, r)} |\nabla_x h_t(x, y)| d\mu(x) \lesssim \frac{1}{\sqrt{t} [V(y, t^{1/m})]^{1-\frac{1}{q}}} \exp \left( -c'' \left( \frac{r^m}{t} \right)^{1/(m-1)} \right) [V(y, t^{1/m})]^{1-\frac{1}{q}}.$$

which gives (2.10).

For  $0 < t < 1$ , (2.9) is obtained in the same way.  $\square$

**Corollary 2.4.** *Let  $M$  be a complete Riemannian manifold satisfying (D) and  $(UE_m)$ . Then for all  $y \in M$ , all  $r, t > 0$ ,*

$$(2.11) \quad \int_{M \setminus B(y,r)} |\nabla_x h_t(x, y)| d\mu(x) \lesssim \frac{1}{\sqrt{t}} \exp\left(-c \left(\frac{s}{t}\right)^{\frac{1}{m-1}}\right),$$

where  $s = r^2$  if  $r < 1$  and  $s = r^m$  if  $r \geq 1$ .

*Proof.* Set

$$I(t, r) := \sqrt{t} \int_{M \setminus B(y,r)} |\nabla_x h_t(x, y)| d\mu(x).$$

- Let  $r, t < 1$ , then (2.9) yields

$$I(t, s) \lesssim e^{-c \frac{r^2}{t}} = e^{-c \frac{s}{t}}.$$

- Let  $t < 1 \leq r$ , then (2.9) yields  $I(t, s) \lesssim e^{-c \frac{r^2}{t}}$ . Yet, since  $t < r$ , we have  $\frac{r^2}{t} \geq \left(\frac{r^m}{t}\right)^{1/(m-1)}$ . Therefore

$$I(t, s) \lesssim e^{-c \left(\frac{s}{t}\right)^{1/(m-1)}}.$$

- Let  $r < 1 \leq t$ , then (2.10) yields  $I(t, s) \lesssim e^{-c \left(\frac{r^m}{t}\right)^{1/(m-1)}}$ . Yet, since  $r < t$ , we have  $\left(\frac{r^m}{t}\right)^{1/(m-1)} \geq \frac{r^2}{t}$ . Therefore

$$I(t, s) \lesssim e^{-c \frac{s}{t}}.$$

- Let  $r, t \geq 1$ , then (2.10) yields

$$I(t, s) \lesssim e^{-c \left(\frac{r^m}{t}\right)^{1/(m-1)}} = e^{-c \left(\frac{s}{t}\right)^{1/(m-1)}}.$$

The conclusion follows from the fact that, for all  $x > 0$ ,

$$e^{-cx^\gamma} \lesssim e^{-cx^\beta}$$

if  $0 < \beta < \gamma$ .  $\square$

### 3. WEAK (1,1) BOUNDEDNESS OF THE RIESZ TRANSFORM

We will prove the weak (1, 1) boundedness of the Riesz transform by adapting the argument of the second author and Duong ([14]), or rather the part of the argument that is directly inspired from [21], to the sub-Gaussian case as in [10]. This adaptation is straightforward, which confirms that the main novelty of the present paper is the use of Stein's argument in Section 2 to derive (2.11). We give it however for the sake of completeness.

*Remark 3.1.* One could also use a theorem on Calderón-Zygmund operators without kernel as written in [8] and improved in [1, Theorem 1.1]. However, since [1, Theorem 1.1] is not exactly written in the form we need, we choose to follow the original method of the second author and Duong.

First recall the Calderón-Zygmund decomposition (see for example [11, Corollaire 2.3]):

**Theorem 3.2.** *Let  $(M, d, \mu)$  be a metric measured space satisfying (D). Then there exists  $C > 0$  such that for any given function  $f \in L^1(M) \cap L^2(M)$  and  $\lambda > 0$ , there exists a decomposition of  $f$ ,  $f = g + b = g + \sum_{i \in I} b_i$  so that*

- (1)  $|g(x)| \leq C\lambda$  for almost all  $x \in M$ ;



(2) *There exists a sequence of balls  $B_i = B(x_i, r_i)$  so that, for all  $i \in I$ ,  $b_i \in L^1(M) \cap L^2(M)$  is supported in  $B_i$  and*

$$\int |b_i(x)| d\mu(x) \leq C\lambda\mu(B_i) \text{ and } \int b_i(x) d\mu(x) = 0;$$

$$(3) \sum_{i \in I} \mu(B_i) \leq \frac{C}{\lambda} \int |f(x)| d\mu(x);$$

(4) *There exists  $k \in \mathbb{N}^*$  such that each  $x \in M$  is contained in at most  $k$  balls  $B_i$ .*

Note that it follows from (2) and (3) that  $\|b\|_1 \leq C\|f\|_1$  and  $\|g\|_1 \leq (1+C)\|f\|_1$ .

*Proof of Theorem 1.2:*

Denote  $T = \nabla\Delta^{-1/2}$ . Since  $T$  is always  $L^2$  bounded, it is enough to show that  $T$  is weak  $(1, 1)$ . Then, for  $1 < p < 2$ , the  $L^p$  boundedness of  $T$  follows from the Marcinkiewicz interpolation theorem.

We aim to show that

$$\mu(\{x : |Tf(x)| > \lambda\}) \leq C\lambda^{-1}\|f\|_1,$$

for all  $\lambda > 0$  and all  $f \in L^1(M) \cap L^2(M)$ .

Let  $\lambda > 0$  and  $f \in L^1(M) \cap L^2(M)$ . Consider the Calderón-Zygmund decomposition of  $f$  at the level  $\lambda$  (we shall use the notation of Theorem 3.2 in the sequel of the argument). Then

$$\mu(\{x : |Tf(x)| > \lambda\}) \leq \mu(\{x : |Tg(x)| > \lambda/2\}) + \mu(\{x : |Tb(x)| > \lambda/2\}).$$

Since  $T$  is  $L^2$  bounded and  $|g(x)| \leq C\lambda$ , it follows that

$$\mu(\{x : |Tg(x)| > \lambda/2\}) \leq C\lambda^{-2}\|g\|_2^2 \leq C\lambda^{-1}\|g\|_1 \leq C\lambda^{-1}\|f\|_1.$$

It remains to prove

$$(3.1) \quad \mu\left(\left\{x : \left|T\left(\sum_{i \in I} b_i\right)(x)\right| > \lambda/2\right\}\right) \leq C\lambda^{-1}\|f\|_1.$$

Define for convenience

$$(3.2) \quad \rho(r) = \begin{cases} r^2, & 0 < r < 1, \\ r^m, & r \geq 1. \end{cases}$$

For each  $i \in I$ , write

$$(3.3) \quad Tb_i = Te^{-t_i\Delta}b_i + T(I - e^{-t_i\Delta})b_i,$$

where  $t_i = \rho(r_i)$ .

We have then

$$\begin{aligned} \mu\left(\left\{x : \left|T\left(\sum_{i \in I} b_i\right)(x)\right| > \lambda/2\right\}\right) &\leq \mu\left(\left\{x : \left|T\left(\sum_{i \in I} e^{-t_i\Delta}b_i\right)(x)\right| > \lambda/4\right\}\right) \\ &\quad + \mu\left(\left\{x : \left|T\left(\sum_{i \in I} (I - e^{-t_i\Delta})b_i\right)(x)\right| > \lambda/4\right\}\right). \end{aligned}$$

We begin with the estimate of the first term. Since  $T$  is  $L^2$  bounded, then

$$\mu\left(\left\{x : \left|T\left(\sum_{i \in I} e^{-t_i\Delta}b_i\right)(x)\right| > \lambda/4\right\}\right) \leq \frac{C}{\lambda^2} \left\| \sum_{i \in I} e^{-t_i\Delta}b_i \right\|_2^2.$$

By a duality argument,

$$\left\| \sum_{i \in I} e^{-t_i\Delta}b_i \right\|_2 = \sup_{\|\phi\|_2=1} \left| \sum_{i \in I} e^{-t_i\Delta}b_i, \phi \right|$$

$$\begin{aligned}
&= \sup_{\|\phi\|_2=1} \left| \sum_{i \in I} \langle b_i, e^{-t_i \Delta} \phi \rangle \right| \\
&\leq \sup_{\|\phi\|_2=1} \sum_{i \in I} \left( \sup_{y \in B_i} e^{-t_i \Delta} |\phi|(y) \right) \int_M |b_i| d\mu.
\end{aligned}$$

We claim that for all  $r, t > 0$  such that  $t = \rho(r)$  and every measurable function  $\varphi \geq 0$

$$(3.4) \quad \sup_{y \in B(x,r)} e^{-t \Delta} \varphi(y) \leq C \inf_{z \in B(x,r)} \mathcal{M}\varphi(z),$$

where  $\mathcal{M}$  denotes the uncentered Littlewood-Paley maximal operator:

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

Estimates of the type (3.4) are classical (see for instance [22, Proposition 2.4]). Let us give a proof of this particular instance for the sake of completeness. We use the notation  $C_j(B)$  for  $2^{j+1}B \setminus 2^jB$  when  $j \geq 2$  and  $C_1(B) = 4B$ . If  $B = B(x, r)$  and  $t = \rho(r)$ , for any  $y \in B$  and any  $z \in C_j(B)$ , we have with (UE<sub>m</sub>)

$$h_t(y, z) \lesssim \frac{1}{V(y, r)} e^{-c2^j}.$$

Therefore

$$\begin{aligned}
e^{-t \Delta} \varphi(y) &= \int_M h_t(y, z) \varphi(z) d\mu(z) = \sum_{j=1}^{\infty} \int_{C_j(B)} h_t(y, z) \varphi(z) d\mu(z) \\
&\lesssim \sum_{j=1}^{\infty} \frac{V(x, 2^{j+1}r)}{V(y, r)} e^{-c2^j} \frac{1}{V(x, 2^{j+1}r)} \int_{B(x, 2^{j+1}r)} \varphi(z) d\mu(z) \\
&\lesssim \sum_{j=1}^{\infty} \frac{V(y, 2^{j+2}r)}{V(y, r)} e^{-c2^j} \frac{1}{V(2^{j+1}B)} \int_{2^{j+1}B} \varphi(z) d\mu(z) \\
&\lesssim \inf_{z \in B} \mathcal{M}\varphi(z),
\end{aligned}$$

which is exactly (3.4).

Then, if  $\|\phi\|_2 = 1$ ,

$$\begin{aligned}
\sum_{i \in I} \left( \sup_{y \in B_i} e^{-t_i \Delta} |\phi|(y) \right) \int_M |b_i| d\mu &\lesssim \sum_{i \in I} \inf_{y \in B(x, t^{1/m})} \mathcal{M}\phi(y) \int_M |b_i| d\mu \\
&\lesssim \lambda \sum_{i \in I} \int_{B_i} \mathcal{M}\phi(y) d\mu(y) \\
&= \lambda \int_M \sum_{i \in I} \mathbf{1}_{B_i}(y) \mathcal{M}\phi(y) d\mu(y) \\
&\lesssim \lambda \int_{\cup_{i \in I} B_i} \mathcal{M}\phi(y) d\mu(y) \\
&\lesssim \lambda \left[ \mu \left( \bigcup_{i \in I} B_i \right) \right]^{1/2} \|\mathcal{M}\phi\|_2 \\
&\lesssim \lambda^{1/2} \|f\|_1^{1/2}.
\end{aligned}$$

The second line follows from property (2) of the Calderón-Zygmund decomposition. The fourth line is due to the finite overlapping of  $\{B_i\}$ . The last but one is a consequence of Hölder inequality and the last one is due to the  $L^2$ -boundedness of the Hardy-Littlewood maximal function and property (3) of the Calderón-Zygmund decomposition.

Hence

$$\left\| \sum_{i \in I} e^{-t_i \Delta} b_i \right\|_2 \lesssim \lambda^{1/2} \|f\|_1^{1/2},$$

and

$$\mu \left( \left\{ x : \left| T \left( \sum_{i \in I} e^{-t_i \Delta} b_i \right) (x) \right| > \lambda/4 \right\} \right) \lesssim \lambda^{-1} \|f\|_1.$$

It remains to show that  $\mu \left( \left\{ x : \left| T \left( \sum_{i \in I} (I - e^{-t_i \Delta}) b_i \right) (x) \right| > \lambda/4 \right\} \right) \leq C \lambda^{-1} \|f\|_1$ . Write

$$\begin{aligned} & \mu \left( \left\{ x : \left| T \left( \sum_{i \in I} (I - e^{-t_i \Delta}) b_i \right) (x) \right| > \lambda/4 \right\} \right) \\ & \leq \mu \left( \left\{ x \in \bigcup_{i \in I} 2B_i : \left| T \left( \sum_{i \in I} (I - e^{-t_i \Delta}) b_i \right) (x) \right| > \lambda/4 \right\} \right) \\ & \quad + \mu \left( \left\{ x \in M \setminus \bigcup_{i \in I} 2B_i : \left| T \left( \sum_{i \in I} (I - e^{-t_i \Delta}) b_i \right) (x) \right| > \lambda/4 \right\} \right) \\ & \leq \sum_{i \in I} \mu(2B_i) + \frac{4}{\lambda} \sum_{i \in I} \int_{M \setminus 2B_i} |T(I - e^{-t_i \Delta}) b_i(x)| d\mu(x). \end{aligned}$$

We claim that for every  $i \in I$ , it holds

$$(3.5) \quad \int_{M \setminus 2B_i} |T(I - e^{-t_i \Delta}) b_i(x)| d\mu(x) \leq C \|b_i\|_1.$$

Therefore by the doubling property and the Calderón-Zygmund decomposition,

$$\mu \left( \left\{ x : \left| T \left( \sum_{i \in I} (I - e^{-t_i \Delta}) b_i \right) (x) \right| > \lambda/4 \right\} \right) \lesssim \sum_{i \in I} \mu(B_i) + \frac{1}{\lambda} \sum_{i \in I} \|b_i\|_1 \lesssim \lambda^{-1} \|f\|_1,$$

which concludes the proof.

So it remains to prove (3.5). First, the spectral theorem gives us that  $\Delta^{-1/2} f = c \int_0^\infty e^{-s\Delta} f \frac{ds}{\sqrt{s}}$ . Therefore, if  $f \in L^2(M)$ ,

$$\begin{aligned} \Delta^{-1/2} (I - e^{-t\Delta}) f &= c \int_0^\infty (e^{-s\Delta} - e^{-(s+t)\Delta}) f \frac{ds}{\sqrt{s}} \\ &= c \int_0^\infty \left( \frac{1}{\sqrt{s}} - \frac{\mathbf{1}_{\{s>t\}}}{\sqrt{s-t}} \right) e^{-s\Delta} f ds. \end{aligned}$$

and thus for all  $x \in M$

$$|T(I - e^{-t\Delta}) f(x)| \leq C \int_M |f(y)| d\mu(y) \int_0^\infty \left| \frac{1}{\sqrt{s}} - \frac{\mathbf{1}_{\{s>t\}}}{\sqrt{s-t}} \right| |\nabla_x h_s(x, y)| ds.$$

Set

$$k_t(x, y) = \int_0^\infty \left| \frac{1}{\sqrt{s}} - \frac{\mathbf{1}_{\{s>t\}}}{\sqrt{s-t}} \right| |\nabla_x h_s(x, y)| ds.$$

Since  $b_i \in L^2(M)$ , we have

$$\begin{aligned} \int_{M \setminus 2B_i} |T(I - e^{-t_i \Delta}) b_i(x)| d\mu(x) &\lesssim \int_{M \setminus 2B_i} \int_{B_i} k_{t_i}(x, y) |b_i(y)| d\mu(y) d\mu(x) \\ &\lesssim \int_M |b_i(y)| \int_{d(x, y) \geq r_i} k_{t_i}(x, y) d\mu(x) d\mu(y). \end{aligned}$$

The claim (3.5) will be proven if  $\int_{d(x,y) \geq r} k_t(x,y) d\mu(x)$  is uniformly bounded for  $t = \rho(r) > 0$ . By Corollary 2.4, we have

$$\begin{aligned}
& \int_{d(x,y) \geq r} k_t(x,y) d\mu(x) \\
&= \int_{d(x,y) \geq r} \int_0^\infty \left| \frac{1}{\sqrt{s}} - \frac{1_{\{s>t\}}}{\sqrt{s-t}} \right| |\nabla_x h_s(x,y)| ds d\mu(x) \\
&= \int_0^\infty \left| \frac{1}{\sqrt{s}} - \frac{1_{\{s>t\}}}{\sqrt{s-t}} \right| \cdot \int_{d(x,y) \geq r} |\nabla_x h_s(x,y)| d\mu(x) ds \\
&\lesssim \int_0^\infty \left| \frac{1}{\sqrt{s}} - \frac{1_{\{s>t\}}}{\sqrt{s-t}} \right| e^{-c(\frac{t}{s})^{\frac{1}{m-1}}} \frac{ds}{\sqrt{s}} \\
&= \int_0^t e^{-c(\frac{t}{s})^{\frac{1}{m-1}}} \frac{ds}{s} + \int_t^\infty \left| \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s-t}} \right| e^{-c(\frac{t}{s})^{\frac{1}{m-1}}} \frac{ds}{\sqrt{s}} \\
&:= K_1 + K_2.
\end{aligned}$$

It is easily checked that  $K_1, K_2$  are bounded uniformly in  $t > 0$ . Indeed,

$$K_1 = \int_0^1 e^{-c(\frac{1}{u})^{\frac{1}{m-1}}} \frac{du}{u} < +\infty$$

and

$$\begin{aligned}
K_2 &\leq \int_t^\infty \left| \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s-t}} \right| \frac{ds}{\sqrt{s}} \\
&= \int_0^\infty \left| \frac{1}{\sqrt{u+1}} - \frac{1}{\sqrt{u}} \right| \frac{du}{\sqrt{u+1}} < +\infty.
\end{aligned}$$

Note that we get the second line by the change of variable  $u = \frac{s}{t} - 1$ .

#### 4. THE CASE OF GRAPHS

In this section, we give the counterpart of Theorem 1.2 on graphs. Let us begin with some definitions.

A (weighted unoriented) graph is defined by a couple  $(\Gamma, \mu)$  where  $\Gamma$  is an infinite countable set (the set of vertices) and  $\mu \geq 0$  is a symmetric weight on  $\Gamma \times \Gamma$ . We define the set of edges as  $E = \{(x,y) \in \Gamma \times \Gamma, \mu_{xy} > 0\}$  and we write  $x \sim y$  (we say that  $x$  and  $y$  are neighbours) whenever  $(x,y) \in E$ .

We assume that the graph is connected and locally uniformly finite. A graph is connected if for all  $x, y \in \Gamma$ , there exists a path joining  $x$  and  $y$ , that is a sequence  $x = x_0, \dots, x_n = y$  such that for all  $i \in \llbracket 1, n \rrbracket$ ,  $x_{i-1} \sim x_i$  (the length of such a path being  $n$ ). A graph is locally uniformly finite if there exists  $M_0 \in \mathbb{N}$  such that, for all  $x \in \Gamma$ ,  $\#\{y \in \Gamma, y \sim x\} \leq M_0$ .

The graph is endowed with its natural metric  $d$ , where, for all  $x, y \in \Gamma$ ,  $d(x,y)$  is the length of the shortest path joining  $x$  and  $y$ .

We define the weight  $m(x)$  of a vertex  $\Gamma$  by  $m(x) = \sum_{y \sim x} \mu_{xy}$ . More generally, the measure of a subset  $E \subset \Gamma$  is defined as  $m(E) := \sum_{x \in E} m(x)$ . For  $x \in \Gamma$  and  $r > 0$ , we denote by  $B(x, r)$  the open ball of center  $x$  and radius  $r$  and by  $V(x, r)$  the measure (or volume) of  $B(x, r)$ .

In this situation, we recall the volume doubling property. The graph  $(\Gamma, \mu)$  satisfies  $(D_\Gamma)$  if, for any  $x \in \Gamma$  and any  $r > 0$ ,

$$(D_\Gamma) \quad V(x, 2r) \lesssim V(x, r).$$

For all  $x, y \in \Gamma$ , the reversible Markov kernel  $p(x, y)$  is defined as  $p(x, y) = \frac{\mu_{xy}}{m(x)}$ . The kernel  $p_k(x, y)$  is then defined recursively for all  $k \in \mathbb{N}$  by

$$\begin{cases} p_0(x, y) = \delta(x, y) \\ p_{k+1}(x, y) = \sum_{z \in \Gamma} p(x, z)p_k(z, y). \end{cases}$$

Note that, for all  $x \in \Gamma$  and all  $k \geq 0$ ,  $\sum_{y \in \Gamma} p_k(x, y) = 1$ . Moreover, one has  $p(x, y)m(x) = p(y, x)m(y)$  for all  $x, y \in \Gamma$ . The operator  $P$  has kernel  $p$ , meaning that the following formula holds for all  $x \in \Gamma$ :  $Pf(x) = \sum_{y \in \Gamma} p(x, y)f(y)$ . For all  $k \geq 1$ ,  $P^k$  has kernel  $p_k$ .

A second assumption will be used. We say that  $\Gamma$  satisfies **(LB)** if there exists  $\varepsilon > 0$  such that, for all  $x \in \Gamma$ ,

$$(LB) \quad p(x, x) > \varepsilon.$$

The assumption **(LB)** implies the  $L^2$ -analyticity of  $P$ , that is  $\|(I - P)P^n\|_{2 \rightarrow 2} \leq \frac{c}{n}$ , for all  $n \in \mathbb{N}^*$ , but the converse is false. We refer to [25, Theorem 1.9] for a way to recover **(LB)** from the  $L^2$ -analyticity of  $P$ .

We say that  $(\Gamma, \mu)$  satisfies the (Markov kernel) estimates **(UE<sub>m,Γ</sub>)** with  $m \geq 2$  if for all  $x, y \in \Gamma$  and  $k \in \mathbb{N}^*$

$$(UE_{m,\Gamma}) \quad p_{k-1}(x, y) \lesssim \frac{m(y)}{V(y, k^{1/m})} \exp\left(-c \left(\frac{d^m(x, y)}{k}\right)^{\frac{1}{m-1}}\right).$$

The shift between  $k$  and  $k - 1$  allows **(UE<sub>m,Γ</sub>)** to cover the obvious estimate of  $p_0(x, y)$ . When  $m = 2$ , the above estimates are called Gaussian estimates. When  $m > 2$ , they are called sub-Gaussian estimates.

We use the random walk  $P$  to define a positive Laplacian and a length of the gradient by

$$\Delta_\Gamma := I - P$$

and

$$\nabla_\Gamma g(x) = \left( \frac{1}{2} \sum_{y \in \Gamma} p(x, y) |g(y) - g(x)|^2 \right)^{1/2},$$

say for  $g \in c_0(\Gamma)$ , where  $c_0(\Gamma)$  denotes the space of finitely supported functions on  $\Gamma$ . In this context, the Riesz transform is the sublinear operator  $\nabla_\Gamma \Delta_\Gamma^{-1/2}$ . As in the case of Riemannian manifolds, we can study the  $L^p$  boundedness of the Riesz transform, that is we can wonder when

$$(R_{p,\Gamma}) \quad \|\nabla_\Gamma g\|_p \lesssim \|\Delta_\Gamma^{1/2} g\|_p, \quad \forall g \in c_0(\Gamma).$$

The counterpart on graphs of Theorem 1.1 is a result of the fourth author:

**Theorem 4.1.** [32, Theorem 1] *Let  $(\Gamma, \mu)$  be a weighted graph satisfying **(LB)**, **(D<sub>Γ</sub>)** and **(UE<sub>2,Γ</sub>)**. Then the Riesz transform  $\nabla_\Gamma \Delta_\Gamma^{-1/2}$  is weak  $(1, 1)$  and  $L^p$  bounded for all  $p \in (1, 2]$ .*

As in the case of Riemannian manifolds, we can extend the  $L^p$ -boundedness of the Riesz transform for  $1 < p \leq 2$  when  $(\Gamma, \mu)$  satisfies sub-Gaussian estimates.

**Theorem 4.2.** *Let  $(\Gamma, \mu)$  be a weighted graph satisfying **(LB)**, **(D<sub>Γ</sub>)** and **(UE<sub>m,Γ</sub>)** for some  $m > 2$ . Then the Riesz transform  $\nabla_\Gamma \Delta_\Gamma^{-1/2}$  is weak  $(1, 1)$  and  $L^p$  bounded for all  $p \in (1, 2]$ .*

Recall that the statement on  $L^p$  in Theorem 4.2 also follows from [24, Corollary 1.30]. The proof of Theorem 4.2 is similar to the one of Theorem 1.2 and we will only prove the counterpart of Lemma 2.2, where the main difficulty lies. Indeed, the ‘‘Hardy-Stein’’ identity (2.6) doesn’t hold on graphs and a trick of Dungey (in [20]), improved by the third author in [23, Section 4], is needed.

**Lemma 4.3.** *Let  $(\Gamma, \mu)$  be a weighted graph satisfying (LB),  $(D_\Gamma)$  and  $(UE_{m,\Gamma})$  for some  $m > 2$ . Then, for any  $q \in (1, 2)$ , there exists  $c > 0$  such that for all  $y \in \Gamma$  and all  $k \in \mathbb{N}^*$ ,*

$$\left\| \nabla_\Gamma p_{k-1}(\cdot, y) e^{c \left( \frac{d^m(\cdot, y)}{k} \right)^{\frac{1}{m-1}}} \right\|_q \lesssim \frac{m(y)}{\sqrt{k} V(y, k^{1/m})^{1-\frac{1}{q}}}.$$

*Proof.* Let  $q \in (1, 2)$  and  $y \in \Gamma$ . In the sequel, we write  $u_k(x)$  for  $p_{k-1}(x, y)$ . We may and do assume that  $k \geq 2$ .

Step 1: Estimate of the gradient by a pseudo-gradient.

Let us define the pseudo-gradient

$$N_q u_k(x) = -u_k^{2-q}(x) [\partial_k + \Delta_\Gamma] u_k^q(x)$$

where  $\partial_k$  denotes the discrete time differentiation defined by  $\partial_k u_k = u_{k+1} - u_k$ . We also need the ‘‘averaging’’ operator

$$A g(x) = \sum_{z \sim x} g(z).$$

Propositions 4.6 and 4.7 in [23] yield that for all  $k \in \mathbb{N}^*$  and all  $x \in \Gamma$ ,

$$|\nabla_\Gamma u_k(x)|^2 \lesssim A[N_q u_k](x).$$

Therefore,

$$\begin{aligned} \left\| \nabla_\Gamma u_k e^{c \left( \frac{d^m(\cdot, y)}{k} \right)^{\frac{1}{m-1}}} \right\|_q &\lesssim \left\| A[N_q u_k] e^{2c \left( \frac{d^m(\cdot, y)}{k} \right)^{\frac{1}{m-1}}} \right\|_{q/2}^{1/2} \\ &\lesssim \left\| A \left[ N_q u_k e^{2c \left( \frac{[d(\cdot, y)+1]^m}{k} \right)^{\frac{1}{m-1}}} \right] \right\|_{q/2}^{1/2} \\ &\lesssim \left\| N_q u_k e^{2c \left( \frac{[d(\cdot, y)+1]^m}{k} \right)^{\frac{1}{m-1}}} \right\|_{q/2}^{1/2}, \end{aligned}$$

where the last line holds because of the boundedness of the operator  $A$  on  $L^r$  for  $r \in (0, 1]$  (see for example [20, Proposition 3.1]). Remark now that  $[d+1]^{m/(m-1)} \leq 2^{1/(m-1)} [d^{m/(m-1)} + 1]$ . Thus, if  $c' = 2^{\frac{m}{m-1}} c$ , we have

$$(4.1) \quad \left\| \nabla_\Gamma u_k e^{c \left( \frac{d^m(\cdot, y)}{k} \right)^{\frac{1}{m-1}}} \right\|_q \lesssim \left\| N_q u_k e^{c' \left( \frac{d(\cdot, y)^m}{k} \right)^{\frac{1}{m-1}}} \right\|_{q/2}^{1/2}.$$

Step 2: A Hardy-Stein type identity.

Let  $J_k = -[\partial_k + \Delta_\Gamma] u_k^q$ . Proposition 4.7 in [23] yields that  $N_q u_k$  is non-negative, thus so is  $J_k$ .

Since  $u_k \in L^q(\Gamma)$ , that is  $u_k^q \in L^1(\Gamma)$ , we have  $\sum_{x \in \Gamma} \Delta_\Gamma u_k^q(x) m(x) = 0$ . Therefore, with Hölder inequality,

$$\begin{aligned} \sum_{x \in \Gamma} J_k(x) m(x) &= - \sum_{x \in \Gamma} \partial_k(u_k^q)(x) m(x) \\ &\leq -q \sum_{x \in \Gamma} u_k^{q-1}(x) \partial_k u_k(x) m(x) \\ &\lesssim \|u_k\|_q^{q-1} \|\partial_k u_k\|_q = \|u_k\|_q^{q-1} \|\Delta_\Gamma u_k\|_q, \end{aligned}$$

where the second line is a consequence of Young's inequality. As a consequence, (4.1) becomes

$$\begin{aligned}
\left\| \nabla_{\Gamma} u_k e^{c \left( \frac{d^m(\cdot, y)}{k} \right)^{\frac{1}{m-1}}} \right\|_q^q &\lesssim \left\| N_q u_k e^{c' \left( \frac{d(\cdot, y)^m}{k} \right)^{\frac{1}{m-1}}} \right\|_{q/2}^{q/2} \\
&= \sum_{x \in \Gamma} u_k(x)^{q(1-\frac{q}{2})} e^{\frac{c'q}{2} \left( \frac{d(x,y)^m}{k} \right)^{\frac{1}{m-1}}} J_k(x)^{q/2} m(x) \\
(4.2) \quad &\leq \left( \sum_{x \in \Gamma} u_k(x)^q e^{\frac{c'q}{2-q} \left( \frac{d(x,y)^m}{k} \right)^{\frac{1}{m-1}}} m(x) \right)^{1-\frac{q}{2}} \left( \sum_{x \in \Gamma} J_k(x) m(x) \right)^{q/2} \\
&\lesssim \left( \sum_{x \in \Gamma} u_k(x)^q e^{\frac{c'q}{2-q} \left( \frac{d(x,y)^m}{k} \right)^{\frac{1}{m-1}}} m(x) \right)^{1-\frac{q}{2}} \|u_k\|_q^{q(q-1)/2} \|\Delta_{\Gamma} u_k\|_q^{q/2}.
\end{aligned}$$

Step 3: Conclusion.

The use of  $(UE_{m,\Gamma})$ , as well as the discrete version of (2.5), yields, if  $c$  (and thus  $c'$ ) is small enough,

$$\begin{aligned}
\sum_{x \in \Gamma} u_k(x)^q e^{\frac{c'q}{2-q} \left( \frac{d(x,y)^m}{k} \right)^{\frac{1}{m-1}}} m(x) &\lesssim \frac{m^q(y)}{V(y, k^{1/m})^q} \sum_{x \in \Gamma} e^{-c' \left( \frac{d(x,y)^m}{k} \right)^{\frac{1}{m-1}}} m(x) \\
&\lesssim \frac{m^q(y)}{V(y, k^{1/m})^{q-1}}.
\end{aligned}$$

As a consequence,

$$\|u_k\|_q^q \lesssim \frac{m^q(y)}{V(y, k^{1/m})^{q-1}}.$$

Finally, let  $l \simeq n \simeq \frac{k}{2}$  (recall that  $k \geq 2$ ). Since  $(LB)$  implies the  $L^q$  analyticity of  $\Delta_{\Gamma}$  (see [17, p. 426]), one has

$$\|\Delta_{\Gamma} u_k\|_q^q \lesssim \frac{1}{l^q} \|u_n\|_q^q \lesssim \frac{m^q(y)}{l^q V(y, n^{1/m})^{q-1}} \lesssim \frac{m^q(y)}{k^q V(y, k^{1/m})^{q-1}},$$

where the last inequality holds thank to the doubling property  $(D_{\Gamma})$ .

Using the last three estimates in (4.2), one has

$$\left\| \nabla_{\Gamma} u_k e^{c \left( \frac{d^m(\cdot, y)}{k} \right)^{\frac{1}{m-1}}} \right\|_q^q \lesssim \frac{m^q(y)}{k^{q/2} V(y, k^{1/m})^{q-1}},$$

which concludes the proof.  $\square$

## 5. THE REVERSE INEQUALITY

A consequence of Theorems 1.2 and 4.2 through the duality argument recalled in the introduction is that  $(RR_p)$  holds for all  $p \in [2, +\infty)$  under the assumptions of these theorems. In this section, we exhibit a class of manifolds (respectively of graphs) satisfying the assumptions of Theorem 1.2 (respectively of Theorem 4.2) on which  $(RR_p)$  is false whenever  $1 < p < 2$  (therefore, again by duality,  $(R_p)$  is false for  $2 < p < +\infty$ ).

Our model space will be the infinite Vicsek graph built in  $\mathbb{R}^N$  whose first steps of construction are shown in Figure 1. A Vicsek graph has polynomial volume growth of exponent  $D = \log_3(1 + 2^N)$ .

As in [6], we will consider the associated Vicsek manifold built by replacing edges by tubes.

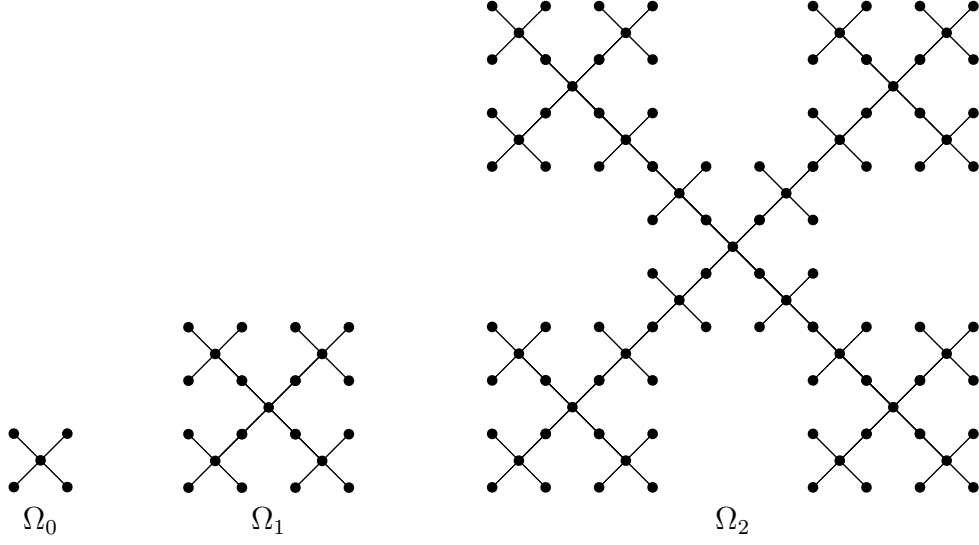


FIGURE 1. The first steps of the Vicsek graph with parameter  $D = \log_3 5$  built in  $\mathbb{R}^2$ .

Such a manifold  $M$  has polynomial volume growth of exponent  $D$  and satisfies the following heat kernel estimate (see [7, Proposition 5.2]):

$$h_t(x, y) \lesssim \frac{1}{t^{\frac{D}{D+1}}} \exp\left(-c\left(\frac{d^{D+1}(x, y)}{t}\right)^{1/D}\right), t \geq 1,$$

in other words, it satisfies  $(UE_m)$  with  $m = D + 1$ .

Also, it satisfies the following non-standard Poincaré inequality

$$(5.1) \quad \int_B |f - f_B|^2 d\mu \lesssim r_B^{D+1} \int_B |\nabla f|^2 d\mu, \quad \forall r_B \geq 1, \quad \forall f \in C_0^\infty(M),$$

where  $f_E$  denotes the average of  $f$  on the set  $E$  ([18, Section 5], [4, Theorem 1.2]). Note that (5.1) is weaker than the  $L^2$  Poincaré inequality:

$$(P) \quad \int_B |f - f_B|^2 d\mu \lesssim r_B^2 \int_B |\nabla f|^2 d\mu, \quad \forall f \in C_0^\infty(M),$$

but it is optimal on  $M$  (see for example [10, Remark 5.2]).

On Vicsek manifolds, Theorem 1.2 tells us that  $(R_p)$  holds for  $1 < p \leq 2$ . The following is a negative result for  $p > 2$ . It comes from [10, Théorème 0.30].

**Theorem 5.1.** *Consider a Vicsek manifold  $M$  with the polynomial volume growth  $V(x, r) \simeq r^D$  for  $r \geq 1$ . Let  $\beta \in (\frac{1}{D+1}, \frac{D}{D+1})$ . Then the inequality*

$$(5.2) \quad \|\Delta^\beta f\|_p \lesssim \|\nabla f\|_p, \quad \forall f \in C_0^\infty(M),$$

is false for  $1 < p < \frac{D-1}{\beta(D+1)-1}$ .

In particular, choosing  $\beta = 1/2$ , for all  $p \in (1, 2)$ ,  $(RR_p)$  does not hold. Consequently, the Riesz transform is not bounded on  $L^p$  for any  $2 < p < +\infty$ .

A similar statement holds for Vicsek graphs.



*Remark 5.2.* (1) The first part of the following proof is taken from [15, Proposition 6.2], where the authors proved that  $(RR_p)$  is false for  $p \in (1, \frac{2D}{D+1})$ . We will rewrite it for the sake of completeness. Note that the conclusion of [15, Proposition 6.2] is weaker than the one of Theorem 5.1.

(2) Recall that Auscher and the second author proved in [2] that under  $(D)$  and  $(P)$  the Riesz transform is bounded on  $L^p$  for  $2 < p < 2 + \varepsilon$ , where  $\varepsilon > 0$ . But of course the Vicsek manifold does not satisfy  $(P)$  (as we already said, the non-standard Poincaré inequality (5.1) is optimal on  $M$ ). By contrast, Theorem 5.1 shows that the conjunction of  $(D)$  and (5.1) does not imply the existence of  $\varepsilon > 0$  such that  $(R_p)$  holds for  $p \in (2, 2 + \varepsilon)$ .

*Proof.* Consider a Vicsek manifold  $M$  modelled on a Vicsek graph  $\Gamma$  built in  $\mathbb{R}^N$  with polynomial volume growth of exponent  $D$ . Recall ([6, Section 6]) that  $M$  satisfies

$$h_t(x, x) \lesssim t^{-\frac{D}{D+1}}, \quad t \geq 1, \quad x \in M.$$

Let  $D' = \frac{2D}{D+1}$ . Therefore, we have for  $p > 1$

$$\|e^{-t\Delta}\|_{1 \rightarrow p} \lesssim t^{-\frac{D'}{2} \left(1 - \frac{1}{p}\right)}, \quad t \geq 1.$$

This implies the following Nash inequality (see [12]): for all  $\beta > 0$ , it holds

$$\|f\|_p^{1 + \frac{2\beta p}{(p-1)D'}} \lesssim \|f\|_1^{\frac{2\beta p}{(p-1)D'}} \|\Delta^\beta f\|_p, \quad \forall f \in C_0^\infty(M) \text{ such that } \frac{\|f\|_p}{\|f\|_1} \leq 1.$$

Let  $\beta \in (\frac{1}{D+1}, \frac{D}{D+1})$  and assume that (5.2) holds. It follows

$$(5.3) \quad \|f\|_p^{1 + \frac{2\beta p}{(p-1)D'}} \lesssim \|f\|_1^{\frac{2\beta p}{(p-1)D'}} \|\nabla f\|_p, \quad \forall f \in C_0^\infty(M) \text{ such that } \frac{\|f\|_p}{\|f\|_1} \leq 1.$$

The Vicsek graph  $\Gamma$  is endowed with the standard weight  $m$  (that is,  $\mu_{xy} = 1$  if  $x, y$  are neighbours, 0 otherwise, hence  $m(x)$  is nothing but the number of neighbours of  $x$ ) out of which  $M$  is constructed by replacing the edges with tubes. For any vertex set  $\Omega \subset \Gamma$ ,  $m(\Omega) \simeq |\Omega|$ , where  $|\Omega|$  is the cardinality of  $\Omega$ . We learn from [19, Section 6] that (5.3) implies the following analogue of (5.3) on  $\Gamma$ :

$$(5.4) \quad \|g\|_p^{1 + \frac{2\beta p}{(p-1)D'}} \lesssim \|g\|_1^{\frac{2\beta p}{(p-1)D'}} \|\nabla_\Gamma g\|_p, \quad \forall g \in c_0(\Gamma).$$

We will show that there exists a family of functions that disproves (5.4) hence (5.2).

Indeed, take the same  $\Omega_n$  and  $g_n$  as in [6, Section 4] (see Figure 2). That is:  $\Omega_n = \Gamma \cap [0, 3^n]^N$ , where  $2^N + 1 = 3^D$ . Then  $|\Omega_n| \simeq 3^{Dn}$ . Denote by  $z_0$  the center of  $\Omega_n$  and by  $z_i, 1 \leq i \leq 2^N$  its corners. Note that  $d(z_0, z_i) = 3^n$ . Define  $g_n$  as follows:  $g_n(z_0) = 1, g_n(z_i) = 0, 1 \leq i \leq 2^N$ , and extend  $g_n$  as a harmonic function in the rest of  $\Omega_n$ . More precisely, if  $z$  belongs to the diagonal linking  $z_0$  and  $z_i$ , then  $g_n(z) = 3^{-n}d(z_i, z)$ . Otherwise,  $g_n(z) = g_n(z')$  where  $z'$  is the only vertex satisfying  $d(z, z') = \inf\{d(z, y), y \text{ belongs to a diagonal of } \Omega_n\}$ .

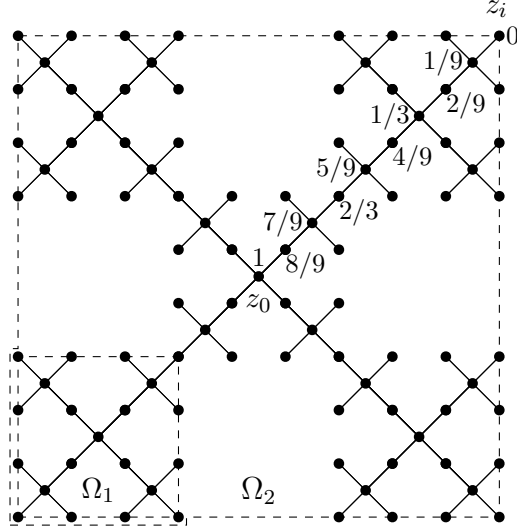
On the one hand, we have

$$\sum_{x \in \Gamma} |g_n(x)|m(x) \leq m(\Omega_n) \simeq |\Omega_n|.$$

On the other hand, for any  $x$  in the  $n - 1$  block with centre  $z_0$ , we have  $g_n(x) \geq \frac{2}{3}$ .

Therefore

$$\sum_{x \in \Gamma} |g_n(x)|^p m(x) \geq (2/3)^p m(\Omega_{n-1}) \simeq 3^{Dn} \simeq |\Omega_n|.$$

FIGURE 2. the function  $g_2$  on the diagonal  $z_0z_i$ 

Since, whenever  $x \sim y$ ,  $|g_n(x) - g_n(y)| = 3^{-n}$  if  $x, y$  belong to the same diagonal connecting  $z_0$  and  $z_i$ , and otherwise  $g_n(x) - g_n(y) = 0$ , we obtain

$$\|\nabla_{\Gamma} g_n\|_p^p \simeq \sum_{x \sim y} |g_n(x) - g_n(y)|^p = \sum_{i=1}^{2^N} 3^{-np} d(z_0, z_i) = 2^N 3^{-n(p-1)} \simeq |\Omega_n|^{-\frac{p-1}{D}}.$$

Thus (5.4) yields

$$(5.5) \quad |\Omega_n|^{\frac{1}{p} \left(1 + \frac{2\beta p}{(p-1)D'}\right)} \lesssim |\Omega_n|^{\frac{2\beta p}{(p-1)D'} - \frac{1}{p'D}}.$$

Obviously  $|\Omega_n|$  tends to infinity as  $n \rightarrow \infty$ , hence (5.5) cannot hold if the RHS exponent is smaller than the LHS one, that is if

$$\begin{aligned} \frac{2\beta p}{(p-1)D'} - \frac{1}{p'D} - \frac{1}{p} \left(1 + \frac{2\beta p}{(p-1)D'}\right) &= \left(1 - \frac{1}{p}\right) \frac{2\beta p}{(p-1)D'} - \frac{1}{p} - \frac{1}{p'D} \\ &= \beta \frac{D+1}{D} - \frac{1}{p} - \frac{1}{p'D} < 0. \end{aligned}$$

In other words, (5.5) is false for  $1 < p < \frac{D-1}{\beta(D+1)-1}$ , where  $\beta \in (\frac{1}{D+1}, \frac{D}{D+1})$ . This contradicts our assumption  $(RR_p)$  on  $\Gamma$  and  $M$ .

In particular, taking  $\beta = 1/2$  shows that (5.5) is false for  $1 < p < 2$ . Thus  $(RR_p)$  is false for  $1 < p < 2$ . This also indicates that the Riesz transform is not bounded on  $L^p$  for  $2 < p < +\infty$ . As we already said, the statement on  $(R_p)$  follows by duality.  $\square$

Combining Theorem 1.2 and Theorem 5.1, we have a full picture for the comparison of  $\|\nabla f\|_p$  and  $\|\Delta^{1/2} f\|_p$  on Vicsek manifolds and graphs. That is

**Theorem 5.3.** *For any Vicsek manifold, and  $p \in (1, +\infty)$ ,  $(R_p)$  holds if and only if  $p \leq 2$  and  $(RR_p)$  holds if and only if  $p \geq 2$ .*

A similar statement holds for Vicsek graphs.

## APPENDIX: ESTIMATES FOR THE HEAT KERNEL IN FRACTAL MANIFOLDS

Let  $(\Gamma, \mu)$  be a graph as in Section 4. For all  $A \subset \Gamma$ , let  $\partial A$  denote the exterior boundary of  $A$ , defined as  $\{x \in \Gamma \setminus A; \text{there exists } y \sim x \text{ with } y \in A\}$ , and let  $\bar{A} := A \cup \partial A$ .

A function  $h : \bar{A} \rightarrow \mathbb{R}$  is said to be harmonic in  $A$  if and only if, for all  $x \in A$ ,  $\Delta h(x) = 0$ .

Say that  $(\Gamma, \mu)$  satisfies a Harnack elliptic inequality if and only if, for all  $x \in \Gamma$ , all  $R \geq 1$  and all nonnegative harmonic functions  $u$  in  $B(x, 2R)$ ,

$$(EHI) \quad \sup_{B(x,R)} u \lesssim \inf_{B(x,R)} u.$$

Let  $D > 0$ . Say that  $\Gamma$  satisfies  $(V_D)$  if and only if, for all  $x \in \Gamma$  and all  $r > 0$ ,

$$(V_D) \quad V(x, r) \sim r^D.$$

Denote by  $(X_n)_{n \geq 1}$  a random walk on  $\Gamma$ , that is a Markov chain with transition probability given by  $p$ . For all  $A \subset \Gamma$ , define

$$T_A := \min \{n \geq 1; X_n \in A\}$$

and, for all  $x \in \Gamma$  and all  $r > 0$ , let

$$\tau_{x,r} := T_{B^c(x,r)}.$$

If  $m > 0$ , say that  $\Gamma$  satisfies  $(E_m)$  if and only if, for all  $x \in \Gamma$  and all  $r > 0$ ,

$$(E_m) \quad \mathbb{E}^x \tau_{x,r} \sim r^m.$$

Theorem 2 in [3] claims that, for all  $m \in [2, D + 1]$ , there exists a graph  $\Gamma$  satisfying  $(V_D)$  (hence the doubling volume property),  $(EHI)$  and  $(E_m)$ . Therefore, Theorem 2.15 in [5] (see also [29, Theorem 3.1]) implies that  $\Gamma$  satisfies the following parabolic Harnack inequality: for all  $x_0 \in \Gamma$ , all  $R \geq 1$  and all non-negative functions  $u : \llbracket 0, 4N \rrbracket \times \overline{B(x_0, 2R)}$  solving  $u_{n+1} - u_n = \Delta u_n$  in  $\llbracket 0, 4N - 1 \rrbracket \times B(x_0, 2R)$ , one has

$$\max_{n \in \llbracket N, 2N-1 \rrbracket, y \in B(x,R)} u_n(y) \lesssim \min_{n \in \llbracket 3N, 4N-1 \rrbracket, y \in B(x,R)} (u_n(y) + u_{n+1}(y)),$$

where  $N$  is an integer satisfying  $N \sim R^m$  and  $N \geq 2R$ .

Consider now the manifold  $M$  built from  $\Gamma$  with a self-similar structure at infinity by replacing the edges of the graph with tubes of length 1 and then gluing the tubes together smoothly at the vertices. Since  $M$  and  $\Gamma$  are roughly isometric, Theorem 2.21 in [5] yields that  $M$  satisfies the following parabolic Harnack inequality: for all  $x_0 \in M$  and all  $R > 0$ , for all non-negative solutions  $u$  of  $\partial_t u = \Delta u$  in  $(0, 4R^m) \times B(x_0, R)$ , one has

$$\sup_{Q_-} u \lesssim \inf_{Q_+} u,$$

where

$$Q_- = (R^m, 2R^m) \times B(x_0, R)$$

and

$$Q_+ = (3R^m, 4R^m) \times B(x_0, R).$$

In turn the parabolic Harnack inequality implies  $(UE_m)$  (see [30, Theorem 5.3]). See also [31, Section 1.2.7].

Finally, let us explain why  $(UE_m)$  for  $m > 2$  is incompatible with  $(UE)$ . It is classical (see for instance [13, Section 3]) that  $(UE_m)$  together with  $(D)$  implies the on-diagonal lower bound

$$h_t(x, x) \gtrsim \frac{1}{V(x, t^{1/m})}, \quad t \geq 1,$$

which is clearly incompatible with

$$h_t(x, x) \lesssim \frac{1}{V(x, t^{1/2})}, \quad t \geq 1$$

because of the so-called reverse volume doubling property (see for instance [28, Proposition 5.2]).

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