Ergodic Infinite Permutations of Minimal Complexity

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Abstract. An *infinite permutation* can be defined as a linear ordering of the set of natural numbers. Similarly to infinite words, a *complexity* $p(n)$ of an infinite permutation is defined as a function counting the number of its factors of length n. For infinite words, a classical result of Morse and Hedlind, 1940, states that if the complexity of an infinite word satisfies $p(n) \leq n$ for some n, then the word is ultimately periodic. Hence minimal complexity of aperiodic words is equal to $n + 1$, and words with such complexity are called *Sturmian*. For infinite permutations this does not hold: There exist aperiodic permutations with complexity functions of arbitrarily slow growth, and hence there are no permutations of minimal complexity.

In the paper we introduce a new notion of *ergodic permutation*, i.e., a permutation which can be defined by a sequence of numbers from $[0, 1]$, such that the frequency of its elements in any interval is equal to the length of that interval. We show that the minimal complexity of an ergodic permutation is $p(n) = n$, and that the class of ergodic permutations of minimal complexity coincides with the class of so-called Sturmian permutations, directly related to Sturmian words.

1 Introduction

In this paper, we continue the study of combinatorial properties of infinite permutations analogous to those of words. In this approach, infinite permutations are interpreted as equivalence classes of real sequences with distinct elements, such that only the order of elements is taken into account. In other words, an infinite permutation is a linear order in $\mathbb{N}$. We consider it as an object close to an infinite word, but instead of symbols, we have transitive relations $<$ or $>$ between each pair of elements.

Infinite permutations in the considered sense were introduced in [10]; see also a very similar approach coming from dynamics [6] and summarised in [3]. Since then, they were studied in two main directions: first, permutations directly constructed with the use of words are studied to reveal new properties of words used for their construction [8, 16–18, 20–22]. In the other approach, properties of infinite permutations are compared with those of infinite words, showing some resemblance and some difference.

In particular, both for words and permutations, the (factor) complexity is bounded if and only if the word or the permutation is ultimately periodic [10, 19]. However, the minimal complexity of an aperiodic word is $n+1$, and the words of this complexity are well-studied Sturmian words; as for the permutations, there is no "minimal" complexity function for the aperiodic case. By contrary, if we modify the definition to consider the *maximal pattern complexity* [13, 14], the result for permutations is

more classifying than that for words: in both cases, there is a minimal complexity for aperiodic objects, but for permutations, unlike for words, the cases of minimal complexity are characterised [4]. All the permutations of lowest maximal pattern complexity are closely related to Sturmian words, whereas words may have lowest maximal pattern complexity even if they have another structure [14].

Other results on the comparison of words and permutations include an attempt to define automatic permutations [12] analogously to automatic words [1], a discussion [11] of the Fine and Wilf theorem and a study of square-free permutations [5].

In this paper, we return to the initial definition of factor complexity and prove a result on permutations of minimal complexity analogous to that for words. To do it, we restrict ourselves to *ergodic* permutations. This new notion means that a permutation can be defined by a sequence of numbers from $[0, 1]$ such that the frequency of its elements in any interval is equal to the length of than interval. We show that this class of permutations is natural and wide, and the ergodic permutations of minimal complexity are exactly Sturmian permutations in the sense of Makarov [18].

The paper is organized as follows. After general basic definitions and a necessary section on the properties of Sturmian words (and permutations), we introduce ergodic permutations and study their basic properties. The main result of the paper, Theorem 5.2, is proved in Section 5.

2   Basic definitions

In what concerns words, in this paper we mostly follow the terminology and notation from [15]. We consider finite and infinite words over a finite alphabet Σ; here we consider $\Sigma = \{0, 1\}$. A *factor* of an infinite word is any sequence of its consecutive letters. The factor $u[i] \cdots u[j]$ of an infinite word $u = u[0]u[1] \cdots u[n] \cdots$, with $u_k \in \Sigma$, is denoted by $u[i..j]$; *prefixes* of a finite or an infinite word are as usual defined as starting factors. A factor s of a right infinite word u is called *right* (resp., *left*) *special* if sa, sb (resp., as, bs) are both factors of u for distinct letters $a, b \in \Sigma$. A word which is both left and right special is called *bispecial*.

The length of a finite word s is denoted by $|s|$. An infinite word $u = vwwww \cdots = vw^\omega$ for some non-empty word w is called *ultimately ($|w|$-)periodic*; otherwise it is called *aperiodic*.

The *complexity* $p_u(n)$ of an infinite word u is a function counting the number of its factors of length n; see [7] for a survey. Due to a classical result of Morse and Hedlund [19], if the complexity $p_u(n)$ of an infinite word u satisfies $p_u(n) \le n$ for some n, then u is ultimately periodic. Therefore, the minimal complexity of aperiodic words is $n + 1$; such words are called *Sturmian* and are discussed in the next section.

When considering words on the binary alphabet $\{0, 1\}$, we refer to the *order* on finite and infinite words meaning lexicographic (partial) order: $0 < 1$, $u < v$ if $u[0..i] = v[0..i]$ and $u[i + 1] < v[i + 1]$ for some i. For words such that one of them is the prefix of the other the order is not defined.

A *conjugate* of a finite word w is any word of the form vu, where $w = uv$. Clearly, conjugacy is an equivalence, and in particular, all the words from the same conjugate class have the same number of occurrences of each symbol.

Analogously to a factor of a word, for a sequence $(a[n])_{n=0}^\infty$ of real numbers, we denote by $a[i..j]$ and call a *factor* of $(a[n])$ the finite sequence of numbers $a[i], a[i + 1], \ldots, a[j]$. We need sequences of numbers to correctly define an *infinite permutation* α as an equivalence class of real infinite sequences with pairwise distinct elements under the following equivalence $\sim$: we have $(a[n])_{n=0}^\infty \sim (b[n])_{n=0}^\infty$ if and only if for

all i, j the conditions $a[i] < a[j]$ and $b[i] < b[j]$ are equivalent. Since we consider only sequences of pairwise distinct real numbers, the same condition can be defined by substituting $(<)$ by $(>)$: $a[i] > a[j]$ if and only if $b[i] > b[j]$. So, an infinite permutation is a linear ordering of the set $\mathbb{N}_0 = \{0, \ldots, n, \ldots\}$. We denote it by $\alpha = (\alpha[n])_{n=0}^\infty$, where $\alpha[i]$ are abstract elements equipped by an order: $\alpha[i] < \alpha[j]$ if and only if $a[i] < a[j]$ or, which is the same, $b[i] < b[j]$ of every *representative* sequence $(a[n])$ or $(b[n])$ of α. So, one of the simplest ways to define an infinite permutation is by a representative, which can be any sequence of distinct real numbers.

Example 2.1. Both sequences $(a[n]) = (1, -1/2, 1/4, \ldots)$ with $a[n] = (-1/2)^n$ and $(b[n])$ with $b[n] = 1000 + (-1/3)^n$ are representatives of the same permutation $\alpha = \alpha[0], \alpha[1], \ldots$ defined by

$$\alpha[2n] > \alpha[2n+2] > \alpha[2k+3] > \alpha[2k+1]$$

for all $n, k \geq 0$.

A *factor* $\alpha[i..j]$ of an infinite permutation α is a finite sequence $(\alpha[i], \alpha[i+1], \ldots, \alpha[j])$ of abstract elements equipped by the same order than in α. Note that a factor of an infinite permutation can be naturally interpreted as a finite permutation: for example, if in a representative $(a[n])$ we have a factor $(2.5, 2, 7, 1.6)$, that is, the 4th element is the smallest, followed by the 2nd, 1st and 3rd, then in the permutation, it will correspond to a factor $\begin{pmatrix} 1\ 2\ 3\ 4 \\ 3\ 2\ 4\ 1 \end{pmatrix}$, which we will denote simply as (3241). Note that in general, we number elements of infinite objects (words, sequences or permutations) starting with 0 and elements of finite objects starting with 1.

A factor of a sequence (permutation) should not be confused with its subsequence $a[n_0], a[n_1], \ldots$ (subpermutation $\alpha[n_0], \alpha[n_1], \ldots$) which is defined as indexed with a growing subsequence (n_i) of indices.

Note, however, that in general, an infinite permutation cannot be defined as a permutation of $\mathbb{N}_0$. For instance, the permutation from Example 2.1 has all its elements between the first two ones.

Analogously to words, the *complexity* $p_\alpha(n)$ of an infinite permutation α is the number of its distinct factors of length n. As for words, this is a non-decreasing function, but it was proved in [10] that contrary to words, we cannot distinguish permutations of "minimal" complexity: for each unbounded non-decreasing function $f(n)$, we can find a permutation α on $\mathbb{N}_0$ such that ultimately, $p_\alpha(n) < f(n)$. The needed permutation can be defined by the inequalities $\alpha[2n-1] < \alpha[2n+1]$ and $\alpha[2n] < \alpha[2n+2]$ for all $n \geq 1$, and $\alpha[2n_k - 2] < \alpha[2k-1] < \alpha[2n_k]$ for a sequence $\{n_k\}_{k=1}^\infty$ which grows sufficiently fast (see [10] for a picture and a discussion).

In this paper, we prove that by contrary, as soon as we restrict ourselves to *ergodic* permutations defined below, the minimal complexity of an ergodic permutation is n. The ergodic permutations of complexity n are directly related to Sturmian words which we discuss in the next section.

3   Sturmian words and Sturmian permutations

Definition 3.1. An aperiodic infinite word u is called *Sturmian* if its factor complexity satisfies $p_u(n) = n + 1$ for all $n \in \mathbb{N}$.

Sturmian words are by definition binary and they have the lowest possible factor complexity among aperiodic infinite words. This extremely popular class of words admits various types of characterizations of geometric and combinatorial nature (see, e. g., Chapter 2 of [15]). In this paper, we need their characterization via irrational rotations on the unit circle found already in the seminal paper [19].

Definition 3.2. The *rotation* by slope σ is the mapping R_σ from $[0,1)$ (identified with the unit circle) to itself defined by $R_\sigma(x) = \{x + \sigma\}$, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

Considering a partition of $[0,1)$ into $I_0 = [0, 1-\sigma)$, $I_1 = [1-\sigma, 1)$, define an infinite word $s_{\sigma,\rho}$ by

$$s_{\sigma,\rho}[n] = \begin{cases} 0 & \text{if } R_\sigma^n(\rho) = \{\rho + n\sigma\} \in I_0, \\ 1 & \text{if } R_\sigma^n(\rho) = \{\rho + n\sigma\} \in I_1. \end{cases}$$

We can also define $I_0' = (0, 1-\sigma]$, $I_1' = (1-\sigma, 1]$ and denote the corresponding word by $s_{\sigma,\rho}'$. As if was proved by Morse and Hedlund, Sturmian words on $\{0,1\}$ are exactly words $s_{\sigma,\rho}$ or $s_{\sigma,\rho}'$.

The same irrational rotation R_σ can be used to define Sturmian permutations:

Definition 3.3. *A* Sturmian permutation $\beta = \beta(\sigma, \rho)$ *is defined by its representative* $(b[n])$, *where* $b[n] = R_\sigma^n(\rho) = \{\rho + n\sigma\}$.

These permutations are obviously related to Sturmian words: indeed, $\beta[i+1] > \beta[i]$ if and only if $s[i] = 0$, where $s = s_{\sigma,\rho}$. Strictly speaking, the case of s' corresponds to a permutation β' defined with the upper fractional part.

Sturmian permutations have been studied in [18]; in particular, it is known that their complexity is $p_\beta(n) \equiv n$.

To continue, we now need a series of properties of a Sturmian word $s = s(\sigma, \rho)$. They are either trivial or classical, and the latter can be found, in particular, in [15].

1. The frequency of ones in s is equal to the slope σ.
2. In any factor of s of length n, the number of ones is either $\lfloor n\sigma \rfloor$, or $\lceil n\sigma \rceil$. In the first case, we say that the factor is *light*, in the second case, it is *heavy*.
3. The factors of s from the same conjugate class are all light or all heavy.
4. Let the continued fraction expansion of σ be $\sigma = [0, 1 + d_1, d_2, \ldots]$. Consider the sequence of *standard* finite words s_n defined by

$$s_{-1} = 1, s_0 = 0, s_n = s_{n-1}^{d_n} s_{n-2} \text{ for } n > 0.$$

   Then
   - The set of bispecial factors of s coincides with the set of words obtained by erasing the last two symbols from the words $s_n^k s_{n-1}$, where $0 < k \leq d_{n+1}$.
   - For each n, we can decompose s as a concatenation

$$s = p \prod_{i=1}^{\infty} s_n^{k_i} s_{n-1}, \tag{1}$$

     where $k_i = d_{n+1}$ or $k_i = d_{n+1} + 1$ for all i, and p is a suffix of $s_n^{d_{n+1}+1} s_{n-1}$.
   - For all $n \geq 0$, if s_n is light, then all the words $s_n^k s_{n-1}$ for $0 < k \leq d_{n+1}$ (including s_{n+1}) are heavy, and vice versa.
5. A *Christoffel word* can be defined as a word of the form $0b1$ or $1b0$, where b is a bispecial factor of a Sturmian word s. For a given b, both Christoffel words are also factors of s and are conjugate of each other. Moreover, they are conjugates of all but one factors of s of that length.
6. The lengths of Christoffel words in s are exactly the lengths of words $s_n^k s_{n-1}$, where $0 < k \leq d_{n+1}$. Such a word is also conjugate of both Christoffel words of the respective length obtained from one of them by sending the first symbol to the end of the word.

The following statement will be needed for our result.

Proposition 3.4. *Let n be such that $\{n\alpha\} < \{i\alpha\}$ for all $0 < i < n$. Then the word $s_{\alpha,0}[0..n-1]$ is a Christoffel word. The same assertion holds if $\{n\alpha\} > \{i\alpha\}$ for all $0 < i < n$.*

PROOF. We will prove the statement for the inequality $\{n\alpha\} < \{i\alpha\}$; the other case is symmetric. First notice that there are no elements $\{i\alpha\}$ in the interval $[1-\alpha, 1-\alpha+\{n\alpha\})$ for $0 \le i < n$. Indeed, assuming that for some i we have $1-\alpha \le \{i\alpha\} < 1-\alpha+\{n\alpha\}$, we get that $0 \le \{(i+1)\alpha\} < \{n\alpha\}$, which contradicts the conditions of the claim.

Next, consider a word $s_{\alpha,1-\varepsilon}[0..n-1]$ for $0 < \varepsilon < \{n\alpha\}$, i.e., the word obtained from the previous one by rotating by ε clockwise. Clearly, all the elements except for $s[0]$ stay in the same interval, so the only element which changes is $s[0]$: $s_{\alpha,0}[0] = 0$, $s_{\alpha,1-\varepsilon}[0] = 1$, $s_{\alpha,0}[1..n-1] = s_{\alpha,1-\varepsilon}[1..n-1]$. This means that the factor $s_{\alpha,0}[1..n-1]$ is left special.

Now consider a word $s_{\alpha,1-\varepsilon'}[0..n-1]$ for $\{n\alpha\} < \varepsilon' < \min_{i\in\{0<i<n\}}\{i\alpha\}$, i.e., the word obtained from $s_{\alpha,0}[0..n-1]$ by rotating by ε' (i.e., we rotate a bit more). Clearly, all the elements except for $s[0]$ and $s[n-1]$ stay in the same interval, so the only elements which change are $s[0]$ and $s[n-1]$: $s_{\alpha,0}[0] = 0$, $s_{\alpha,1-\varepsilon'}[0] = 1$, $s_{\alpha,0}[n-1] = 1$, $s_{\alpha,1-\varepsilon'}[n-1] = 0$, $s_{\alpha,0}[1..n-2] = s_{\alpha,1-\varepsilon'}[1..n-2]$. This means that the factor $s_{\alpha,0}[1..n-2]$ is right special.

So, the factor $s_{\alpha,0}[1..n-2]$ is both left and right special and hence bispecial. By the construction, $s_{\alpha,0}[0..n-1]$ is a Christoffel word.

The proof is illustrated by Fig. 1, where all the numbers on the circle are denoted modulo 1.   □
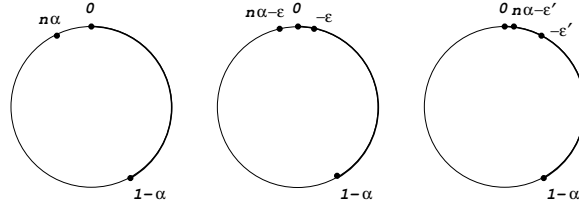

Fig. 1. Intervals for a bispecial word

Note also that in the Sturmian permutation $\beta = \beta(\sigma, \rho)$, we have $\beta[i] < \beta[j]$ for $i < j$ if and only if the respective factor $s[i..j-1]$ of s is light (and, symmetrically, $\beta[i] > \beta[j]$ if and only if the factor $s[i..j-1]$ is heavy).

4   Ergodic permutations

In this section, we define a new notion of an ergodic permutation.

Let $(a[i])_{i=1}^{\infty}$ be a sequence of real numbers from the interval $[0,1]$, representing an infinite permutation, a and p also be real numbers from $[0,1]$. We say that the *probability* for any element $a[j]$ to be less than a exists and is equal to p if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ \forall j \in \mathbb{N} \ \left| \frac{\#\{a[j+k]|0 \le k < n, a[j+k] < a\}}{n} - p \right| < \varepsilon.$$

In other words, if we substitute all the elements from $(a[i])$ which are smaller than a by 1, and those which are bigger by 0, the above condition means that

the uniform frequency of the letter 1 exists and equals p. So, the probability to be smaller than a is the uniform frequency of the elements which are less than a. For more on uniform frequencies of letters in words we refer to [9].

We note that this is not exactly probability on the classical sense, since we do not have a random sequence. But we are interested in permutations where this "probability" behaves in certain sense like probability of a random sequence uniformly distributed on $[0, 1]$:

Definition 4.1. A sequence $(a[i])_{i=1}^{\infty}$ of real numbers is *canonical* if and only if

- all the numbers are pairwise distinct;
- for all i we have $0 \le a[i] \le 1$;
- and for all a, the probability for any element $a[i]$ to be less than a is well-defined and equal to a.

More formally, the last condition should be rephrased as

$$\forall a \in [0, 1], \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N, j \in \mathbb{N} \; \left| \frac{\#\{a[j + k] | 0 \le k < n, a[j + k] < a\}}{n} - a \right| < \varepsilon.$$

Remark 4.2. The set $\{a[i] | i \in \mathbb{N}\}$ for a canonical sequence $(a[i])$ is dense on $[0, 1]$.

Remark 4.3. In a canonical sequence, the frequency of elements which fall into any interval $(t_1, t_2) \subseteq [0, 1]$ exists and is equal to $t_2 - t_1$.

Remark 4.4. Symmetrically to the condition "the probability to be less than a is a" we can consider the equivalent condition "the probability to be greater than a is $1 - a$".

Definition 4.5. An infinite permutation $\alpha = (\alpha_i)_{i=1}^{\infty}$ is called *ergodic* if it has a canonical representative.

Example 4.6. Since for any irrational σ and for any ρ the sequence of fractional parts $\{\rho + n\sigma\}$ is uniformly distributed in $[0, 1)$, a Sturmian permutation $\beta_{\sigma,\rho}$ is ergodic.

Example 4.7. Consider the sequence

$$\frac{1}{2}, 1, \frac{3}{4}, \frac{1}{4}, \frac{5}{8}, \frac{1}{8}, \frac{3}{8}, \frac{7}{8}, \dots$$

defined as the fixed point of the morphism

$$\varphi_{tm} : [0, 1] \mapsto [0, 1]^2, \varphi_{tm}(x) = \begin{cases} \frac{x}{2} + \frac{1}{4}, \frac{x}{2} + \frac{3}{4}, & \text{if } 0 \le x \le \frac{1}{2}, \\ \frac{x}{2} + \frac{1}{4}, \frac{x}{2} - \frac{1}{4}, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

If can be proved that this sequence is canonical and thus the respective permutation is ergodic. In fact, this permutation, as well as its construction [17], are closely related to the famous Thue-Morse word [2], and thus it is reasonable to call it the *Thue-Morse permutation*.

Proposition 4.8. *The canonical representative* $(a[n])$ *of an ergodic permutation* α *is unique.*

PROOF. Given α, for each i we define

$$a[i] = \lim_{n \to \infty} \frac{\#\{\alpha[k] | 0 \le k < n, \alpha[k] < \alpha[i]\}}{n}$$

and see that, first, this limit must exist since α is ergodic, and second, $a[i]$ is the only possible value of an element of a canonical representative of α. □

Note, however, that even if all the limits exist, it does not imply the existence of the canonical representative. Indeed, there is another condition to fulfill: for different i the limits must be different.

Consider a growing sequence $(n_i)_{i=1}^{\infty}$, $n_i \in \mathbb{N}$, $n_{i+1} > n_i$. The respective sub-permutation $(\alpha[n_i])$ will be called N-growing (N-decreasing) if $n_{i+1} - n_i \le N$ and $\alpha[n_{i+1}] > \alpha[n_i]$ ($\alpha[n_{i+1}] < \alpha[n_i]$) for all i. A subpermutation which is N-growing or N-decreasing is called N-monotone.

Proposition 4.9. *If a permutation has a N-monotone subpermutation for some N, then it is not ergodic.*

PROOF. Suppose the opposite and consider a subsequence $(a[n_i])$ of the canonical representative a corresponding to the N-monotone (say, N-growing) subpermutation $(\alpha[n_i])$. Consider $b = \lim_{i \to \infty} a[n_i]$ (which exists) and an $\varepsilon < 1/N$. Let M be the number such that $a[n_m] > b - \varepsilon$ for $m \ge M$. Then the probability for an element $a[i]$ to be in the interval $[a[n_M], b]$ must be equal to $b - a[n_M] < \varepsilon$ due to Remark 4.3. On the other hand, since all $a[n_m]$ for $m > M$ are in this interval, and $n_{i+1} - n_i \le N$, this probability is at least $1/N > \varepsilon$. A contradiction. □

An element $\alpha[i]$, $i > N$, of a permutation α is called N-*maximal* (resp., N-*minimal*) if $\alpha[i]$ is greater (resp., less) than all the elements at the distance at most N from it: $\alpha[i] > \alpha[j]$ (resp., $\alpha[i] < \alpha[j]$) for all $j = i - N, i - N + 1, \ldots, i - 1, i + 1, \ldots, i + N$.

Proposition 4.10. *In an ergodic permutation α, for each N there exists an N-maximal and an N-minimal element.*

PROOF. Consider a permutation α without N-maximal elements and prove that it is not ergodic. Suppose first that there exists an element $\alpha[n_1]$, $n_1 > N$, in α which is greater than any of its N left neighbours: $\alpha[n_1] > \alpha[n_1 - i]$ for all i from 1 to N. Since $\alpha[n_1]$ is not N-maximal, there exist some $i \in \{1, \ldots, N\}$ such that $\alpha[n_1 + i] > \alpha[n_1]$. If such i are several, we take the maximal $\alpha[n_1 + i]$ and denote $n_2 = n_1 + i$. By the construction, $\alpha[n_2]$ is also greater than any of its N left neighbours, and we can continue the sequence of elements $\alpha[n_1] < \alpha[n_2] < \cdots < \alpha[n_k] < \cdots$. Since for all k we have $n_{k+1} - n_k \le N$, it is an N-growing subpermutation, and due to the previous proposition, α is not ergodic.

Now suppose that there are no elements in α which are greater than all their N left neighbours:

For all $n > N$, there exists some $i \in \{1, \ldots, N\}$ such that $\alpha[n - i] > \alpha[n]$.   (2)

We take $\alpha[n_1]$ to be the greatest of the first N elements of α and $\alpha[n_2]$ to be the greatest among the elements $\alpha[n_1 + 1], \ldots, \alpha[n_1 + N]$. Then due to (2) applied to n_2, $\alpha[n_1] > \alpha[n_2]$. Moreover, $n_2 - n_1 \le N$ and for all $n_1 < k < n_2$ we have $\alpha[k] < \alpha[n_2]$.

Now we take n_3 such that $\alpha[n_3]$ is the maximal element among $\alpha[n_2 + 1], \ldots, \alpha[n_2 + N]$, and so on. Suppose that we have chosen $n_1, \ldots, n_i$ such that $\alpha[n_1] > \alpha[n_2] > \cdots > \alpha[n_i]$, and

For all $j \le i$ and for all k such that $n_{j-1} < k < n_j$, we have $\alpha[k] < \alpha[n_j]$.   (3)

For each new $\alpha[n_{i+1}]$ chosen as the maximal element among $\alpha[n_i+1], \ldots, \alpha[n_i+N]$, we have $n_{i+1} - n_i \leq N$. Due to (2) applied to n_{i+1} and by the construction, $\alpha[n_{i+1}] < \alpha[l]$ for some l from $n_{i+1} - N$ to n_i. Because of (3), without loss of generality we can take $l = n_j$ for some $j \leq i$. Moreover, we cannot have $\alpha[n_i] < \alpha[n_{i+1}]$ and thus $j < i$: otherwise n_{i+1} would have been chosen as n_{j+1} since it fits the condition of maximality better.

So, we see that $\alpha[n_i] > \alpha[n_{i+1}]$, (3) holds for $i+1$ as well as for i, and thus by induction the subpermutation $\alpha[n_1] > \cdots > \alpha[n_i] > \cdots$ is N-decreasing. Again, due to the previous proposition, α is not ergodic. □

5   Minimal complexity of ergodic permutations

Proposition 5.1. *For any ergodic permutation α, we have $p_\alpha(n) \geq n$.*

PROOF. Due to Proposition 4.10, there exists an n-maximal element α_i, $i > n$. All the n factors of α of length n containing it are different: in each of them, the maximal element is at a different position. □

The complexity of Sturmian permutations considered in Section 3 is known to be $p_\alpha(n) = n$ [18]. In what follows, we are going to prove that these are the only ergodic examples of this minimal complexity, and thus the Sturmian construction remains a natural "simplest" example if we restrict ourselves to ergodic permutations. So, the rest of the section is devoted to the proof of

Theorem 5.2. *The minimal complexity of an ergodic permutation α is $p_\alpha(n) \equiv n$. The set of ergodic permutations of minimal complexity coincides with the set of Sturmian permutations.*

Since the complexity of ergodic permutations satisfies $p_\alpha(n) \geq n$ due to Proposition 5.1; and the complexity of Sturmian permutations is $p_\alpha(n) \equiv n$, it remains to prove just that if $p_\alpha(n) \equiv n$ for an ergodic permutation α, then α is Sturmian.

Definition 5.3. *Given an infinite permutation $\alpha = \alpha[1] \cdots \alpha[n] \cdots$, consider its underlying infinite word $s = s[1] \cdots s[n] \cdots$ over the alphabet $\{0,1\}$ defined by*

$$s[i] = \begin{cases} 0, & \text{if } \alpha[i] < \alpha[i+1], \\ 1, & \text{otherwise.} \end{cases}$$

Note that in some papers the word s was denoted by γ and considered directly as a word over the alphabet $\{<, >\}$.

It is not difficult to see that a factor $s[i+1..i+n-1]$ of s contains only a part of information on the factor $\alpha[i+1..i+n]$ of α, i.e., does not define it uniquely. Different factors of length $n-1$ of s correspond to different factors of length n of α. So,

$$p_\alpha(n) \geq p_s(n-1).$$

Together with the above mentioned result of Morse and Hedlund [19], it gives the following

Proposition 5.4. *If $p_\alpha(n) = n$, then the underlying sequence s of α is either ultimately periodic or Sturmian.*

Now we consider different cases separately.

Proposition 5.5. *If $p_\alpha(n) \equiv n$ for an ergodic permutation α, then its underlying sequence s is aperiodic.*

PROOF. Suppose the converse and let p be the minimal period of s. If $p = 1$, then the permutation α is monotone, increasing or decreasing, so that its complexity is always 1, a contradiction. So, $p \geq 2$. There are exactly p factors of s of length $p - 1$: each residue modulo p corresponds to such a factor and thus to a factor of α of length p. The factor $\alpha[kp+i..(k+1)p+i-1]$, where $i \in \{1, \dots, p\}$, does not depend on k, but for all the p values of i, these factors are different.

Now let us fix i from 1 to p and consider the subpermutation $\alpha[i], \alpha[p+i], \dots, \alpha[kp+i], \dots$. It cannot be monotone due to Proposition 4.9, so, there exist k_1 and k_2 such that $\alpha[k_1 p + i] < \alpha[(k_1 + 1)p + i]$ and $\alpha[k_2 p + i] > \alpha[(k_2 + 1)p + i]$. So, $\alpha[k_1 p + i..(k_1 + 1)p + i] \neq \alpha[k_2 p + i..(k_2 + 1)p + i]$. We see that each of p factors of α of length p, uniquely defined by the residue i, can be extended to the right to a factor of length $p + 1$ in two different ways, and thus $p_\alpha(p+1) \geq 2p$. Since $p > 1$ and thus $2p > p + 1$, it is a contradiction. $\qquad\square$

So, Propositions 5.4 and 5.5 imply that the underlying word s of an ergodic permutation α of complexity n is Sturmian. Let $s = s(\sigma, \rho)$, that is,

$$s_n = \lfloor \sigma(n+1) + \rho \rfloor - \lfloor \sigma n + \rho \rfloor.$$

In the proofs we will only consider $s(\sigma, \rho)$, since for $s'(\sigma, \rho)$ the proofs are symmetric.

It follows directly from the definitions that the Sturmian permutation $\beta = \beta(\sigma, \rho)$ defined by its canonical representative b with $b[n] = \{\sigma n + \rho\}$ has s as the underlying word.

Suppose that α is a permutation whose underlying word is s and whose complexity is n. We shall prove the following statement concluding the proof of Theorem 5.2:

Lemma 5.6. *Let α be a permutation of complexity $p_\alpha(n) \equiv n$ whose underlying word is $s(\sigma, \rho)$. If α is ergodic, then $\alpha = \beta(\sigma, \rho)$.*

PROOF. Assume the converse, i.e., that α is not equal to β. We will prove that hence α is not ergodic, which is a contradiction.

Recall that in general, $p_\alpha(n) \geq p_s(n-1)$, but here we have the equality since $p_\alpha(n) \equiv n$ and $p_s(n) \equiv n+1$. It means that a factor u of s of length $n-1$ uniquely defines a factor of α of length n which we denote by α^u. Similarly, there is a unique factor β^u of β.

Clearly, if u is of length 1, we have $\alpha^u = \beta^u$: if $u = 0$, then $\alpha^0 = \beta^0 = (12)$, and if $u = 1$, then $\alpha^1 = \beta^1 = (21)$. Suppose now that $\alpha^u = \beta^u$ for all u of length up to $n-1$, but there exists a word v of length n such that $\alpha^v \neq \beta^v$.

Since for any factor $v' \neq v$ of v we have $\alpha^{v'} = \beta^{v'}$, the only difference between α^v and β^v is the relation between the first and last element: $\alpha^v[1] < \alpha^v[n+1]$ and $\beta^v[1] > \beta^v[n+1]$, or vice versa. (Note that we number elements of infinite objects starting with 0 and elements of finite objects starting with 1.)

Consider the factor b^v of the canonical representative b of β corresponding to an occurrence of β^v. We have $b^v = (\{\tau\}, \{\tau + \sigma\}, \dots, \{\tau + n\sigma\})$ for some τ.

Proposition 5.7. *All the numbers $\{\tau + i\sigma\}$ for $0 < i < n$ are situated outside of the interval whose ends are $\{\tau\}$ and $\{\tau + n\sigma\}$.*

PROOF. Consider the case of $\beta^v[1] < \beta^v[n+1]$ (meaning $\{\tau\} < \{\tau + n\sigma\}$) and $\alpha^v[1] > \alpha^v[n+1]$; the other case is symmetric. Suppose by contrary that there is an element $\{\tau + i\sigma\}$ such that $\{\tau\} < \{\tau + i\sigma\} < \{\tau + n\sigma\}$ for some i. It means that $\beta^v[1] < \beta^v[i] < \beta^v[n+1]$. But the relations between the 1st and the ith elements, as well as between the ith and $(n+1)$st elements, are equal in α^v and in β^v, so, $\alpha^v[1] < \alpha^v[i]$ and $\alpha^v[i] < \alpha^v[n+1]$. Thus, $\alpha^v[1] < \alpha^v[n+1]$, a contradiction. $\qquad\square$

Proposition 5.8. *The word v belongs to the conjugate class of a Christoffel factor of s, or, which is the same, of a factor of the form $s_n^k s_{n-1}$ for $0 < k \leq d_{n+1}$.*

PROOF. The condition "For all $0 < i < n$, the number $\{\tau + i\sigma\}$ is not situated between $\{\tau\}$ and $\{\tau + n\sigma\}$" is equivalent to the condition "$\{n\alpha\} < \{i\alpha\}$ for all $0 < i < n$" considered in Proposition 3.4 and corresponding to a Christoffel word of the same length. The set of factors of s of length n is exactly the set $\{s_{\alpha,\tau}[0..n-1] | \tau \in [0,1]\}$. These words are n conjugates of the Christoffel word plus one singular factor corresponding to $\{\tau\}$ and $\{\tau + n\sigma\}$ situated in the opposite ends of the interval $[0,1]$ ("close" to 0 and "close" to 1), so that all the other points $\{\tau + i\sigma\}$ are between them.

Example 5.9. Consider a Sturmian word s of the slope $\sigma \in (1/3, 2/5)$. Then the factors of s of length 5 are 01001, 10010, 00101, 01010, 10100, 00100. Fig. 2 depicts permutations of length 6 with their underlying words. In the picture the elements of the permutations are denoted by points; the order between two elements is defined by which element is "higher" on the picture. We see that in the first five cases, the relation between the first and the last elements can be changed, and in the last case, it cannot since there are other elements between them. Indeed, the first five words are exactly the conjugates of the Christoffel word 1 010 0, where the word 010 is bispecial.
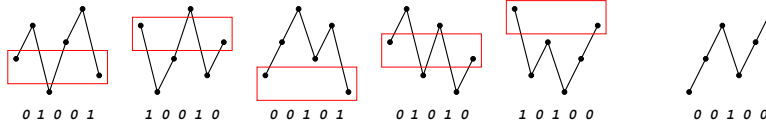

0 1 0 0 1   1 0 0 1 0   0 0 1 0 1   0 1 0 1 0   1 0 1 0 0   0 0 1 0 0

Fig. 2. Five candidates for v and a non-candidate word

Note also that due to Proposition 5.8, the shortest word v such that $\alpha^v \neq \beta^v$ is a conjugate of some $s_n^k s_{n-1}$ for $0 < k \leq d_{n+1}$.

In what follows without loss of generality we suppose that the word s_n is heavy and thus s_{n-1} and $s_n^k s_{n-1}$ for all $0 < k \leq d_{n+1}$ are light.

Consider first the easiest case: $v = s_n^{d_{n+1}} s_{n-1} = s_{n+1}$. This word is light, so, $\beta^{s_{n+1}}[1] < \beta^{s_{n+1}}[|s_{n+1}| + 1]$. Since the first and the last elements of $\alpha^{s_{n+1}}$ must be in the other relation, we have $\alpha^{s_{n+1}}[1] > \alpha^{s_{n+1}}[|s_{n+1}| + 1]$. At the same time, since s_n is shorter than s_{n+1}, we have $\alpha^{s_n} = \beta^{s_n}$ and in particular, since s_n is heavy, $\alpha^{s_n}[1] > \alpha^{s_n}[|s_n| + 1]$.

Due to (1), the word s after a finite prefix can be represented as an infinite concatenation of occurrences of s_{n+1} and s_n: $s = p \prod_{i=1}^{\infty} s_n^{t_i} s_{n+1}$, where $t_i = k_i - d_{n+1} = 0$ or 1. But both α^{s_n} and $\alpha^{s_{n+1}}$ are permutations with the last elements less than the first ones. Moreover, if we have a concatenation uw of factors u and w of s, we see that the first symbol of α^w is the last symbol of α^u: $\alpha^u[|u| + 1] = \alpha^w[1]$. So, an infinite sequence of factors s_n and s_{n+1} of s gives us a chain of the first elements of respective factors of the permutation α, and each next elements is less than the previous one. This chain is a $|s_{n+1}|$-monotone subpermutation, and thus α is not ergodic.

Now let us consider the general case: v is from the conjugate class of $s_n^t s_{n-1}$, where $0 < t \leq d_{n+1}$. We consider two cases: the word $s_n^t s_{n-1}$ can be cut either in one of the occurrences of s_n, or in the suffix occurrence of s_{n-1}.

In the first case, $v = r_1 s_n^l s_{n-1} s_n^{t-l-1} r_2$, where $s_n = r_2 r_1$ and $0 \leq l < t$. Then

$$s = p \prod_{i=1}^{\infty} s_n^{k_i} s_{n-1} = p r_2 (r_1 r_2)^{k_1 - l - 1} \prod_{i=2}^{\infty} v(r_1 r_2)^{k_i - t}. \qquad (4)$$

We see that after a finite prefix, the word s is an infinite catenation of words v and $r_1 r_2$. The word $r_1 r_2$ is shorter than v and heavy since it is a conjugate of s_n. So, $\alpha^{r_1 r_2} = \beta^{r_1 r_2}$ and in particular, $\alpha^{r_1 r_2}[1] > \alpha^{r_1 r_2}[|r_1 r_2| + 1]$. The word v is light since it is a conjugate of $s_n^t s_{n-1}$, but the relation between the first and the last elements of α^v is different than between those in β^v, that is, $\alpha^v[1] > \alpha^v[|v| + 1]$. But as above, in a concatenation uw, we have $\alpha^u[|u| + 1] = \alpha^w[1]$, so, we see a $|v|$-decreasing subpermutation in α. So, α is not ergodic.

Analogous arguments work in the second case, when $s_n^t s_{n-1}$ is cut somewhere in the suffix occurrence of s_{n-1}: $v = r_1 s_n^t r_2$, where $s_{n-1} = r_2 r_1$. Note that s_{n-1} is a prefix of s_n, and thus $s_n = r_2 r_3$ for some r_3. In this case,

$$s = p \prod_{i=1}^{\infty} s_n^{k_i} s_{n-1} = p r_2 (r_3 r_2)^{k_1} \prod_{i=2}^{\infty} v (r_3 r_2)^{k_i - t}. \tag{5}$$

As above, we see that after a finite prefix, s is an infinite catenation of the heavy word $r_3 r_2$, a conjugate of s_n, and the word v. For both words, the respective factors of α have the last element less than the first one, which gives a $|v|$-decreasing subpermutation. So, α is not ergodic.

The case when s_n is not heavy but light is considered symmetrically and gives rise to $|v|$-increasing subpermutations. This concludes the proof of Theorem 5.2.

References

1. J.-P. Allouche, J. Shallit, Automatic sequences — theory, applications, generalizations. Cambridge University Press, 2003.
2. J.-P. Allouche, J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, Sequences and their Applications, Discrete Mathematics and Theoretical Computer Science, Springer, London, 1999. P. 1–16.
3. J. Amigó, Permutation Complexity in Dynamical Systems - Ordinal Patterns, Permutation Entropy and All That. Springer Series in Synergetics, 2010
4. S. V. Avgustinovich, A. Frid, T. Kamae, P. Salimov, Infinite permutations of lowest maximal pattern complexity, Theoretical Computer Science 412 (2011) 2911–2921.
5. S. V. Avgustinovich, S. Kitaev, A. Pyatkin, A. Valyuzhenich, On square-free permutations, J. Autom. Lang. Comb. 16 (2011) 1, 3–10.
6. C. Bandt, G. Keller and B. Pompe, Entropy of interval maps via permutations, Nonlinearity 15 (2002), 1595–1602.
7. J. Cassaigne, F. Nicolas. Factor complexity. Combinatorics, automata and number theory, 163–247, Encyclopedia Math. Appl., 135, Cambridge Univ. Press, 2010.
8. S. Elizalde, The number of permutations realized by a shift, SIAM J. Discrete Math. 23 (2009), 765–786.
9. S. Ferenczi and T. Monteil, Infinite words with uniform frequencies, and invariant measures. Combinatorics, automata and number theory, 373–409, Encyclopedia Math. Appl., 135, Cambridge Univ. Press, 2010.
10. D. G. Fon-Der-Flaass, A. E. Frid, On periodicity and low complexity of infinite permutations, European J. Combin. 28 (2007), 2106–2114.
11. A. Frid, Fine and Wilf's theorem for permutations, Sib. Elektron. Mat. Izv. 9 (2012) 377–381.
12. A. Frid, L. Zamboni, On automatic infinite permutations. Theoret. Inf. Appl. 46 (2012) 77–85.
13. T. Kamae, L. Zamboni, Sequence entropy and the maximal pattern complexity of infinite words, Ergodic Theory and Dynamical Systems 22 (2002), 1191–1199.
14. T. Kamae, L. Zamboni, Maximal pattern complexity for discrete systems, Ergodic Theory and Dynamical Systems 22 (2002), 1201–1214.
15. Lothaire, M.: Algebraic combinatorics on words. Cambridge University Press, 2002.
16. M. Makarov, On permutations generated by infinite binary words, Sib. Elektron. Mat. Izv. 3 (2006), 304–311.

17. M. Makarov, On an infinite permutation similar to the Thue–Morse word, Discrete Math. 309 (2009), 6641–6643.
18. M. Makarov, On the permutations generated by Sturmian words. Sib. Math. J. 50 (2009), 674–680.
19. M. Morse and G. Hedlund, *Symbolic dynamics II: Sturmian sequences*, Amer. J. Math. 62 (1940), 1–42.
20. A. Valyuzhenich, On permutation complexity of fixed points of uniform binary morphisms, Discr. Math. Theoret. Comput. Sci. 16 (2014), 95–128.
21. S. Widmer, Permutation complexity of the Thue-Morse word, Adv. Appl. Math. 47 (2011) 309–329.
22. S. Widmer, Permutation complexity related to the letter doubling map, WORDS 2011.