Duality with Linear-Feedback Schemes for the Scalar Gaussian MAC and BC

Selma Belhadj Amor, Yossef Steinberg, Michèle Wigger

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Selma Belhadj Amor
Telecom ParisTech
Paris, France
belhadjamor@telecom-paristech.fr

Yossef Steinberg
Technion—Israel Inst. of Technology
Haifa, Israel
ysteinbe@ee.technion.ac.il

Michèle Wigger
Telecom ParisTech
Paris, France
wigger@telecom-paristech.fr

***Abstract*—We show that with perfect feedback and when restricting to linear-feedback schemes, the regions achieved over the two-user scalar Gaussian memoryless MAC and over the two-user scalar Gaussian memoryless BC coincide, if the MAC and the BC have equal channel coefficients and if the same (sum-)power constraint P is imposed on their inputs. Since the achievable region for the MAC is well known (it equals Ozarow's perfect-feedback capacity region under a sum-power constraint), we can characterize the region that is achievable over the scalar Gaussian BC with linear-feedback schemes.**

I. Introduction

Feedback is known to increase the capacity of multi-user channels such as the multi-access channel (MAC) and the broadcast channel (BC). But for most multi-user channels the exact capacity region is open even with perfect output-feedback. A notable exception is the two-user memoryless Gaussian MAC whose capacity was derived by Ozarow [1]. Ozarow's capacity-achieving scheme is a *linear-feedback scheme*, i.e., a scheme where at each time the transmitters send linear combinations of the past feedback signals and of code symbols that only depend on their own message. Kramer extended this scheme to $K \geq 3$ users [2]. Under symmetric power constraints P for all users, Kramer's scheme achieves the largest sum-rate among all *linear-feedback schemes* [3], and it achieves the sum-capacity when P is sufficiently large [2]. (It is yet unknown whether the scheme achieves the sum-capacity also when P is small.)

For the memoryless Gaussian BC the capacity region with perfect feedback is unknown even with two receivers. Achievable regions based on linear-feedback schemes have been proposed in [2], [4], [5], [6], [7], [8]. Non-linear feedback schemes have been proposed in [14], [9], [10]. The best known achievable regions are due to linear-feedback schemes.

The linear-feedback schemes in [5], [6], [7] are designed based on control-theoretic considerations. For some setups, e.g., uncorrelated noises of equal variance [7], these schemes achieve the same sum-rate over the Gaussian BC under power constraint P as Ozarow's scheme achieves over the Gaussian MAC under a *sum-power* constraint P. Thus, there is a duality in terms of sum-rate between the BC-schemes in [5], [6], [7] and Ozarow's MAC-scheme [1].

In this paper, we prove a duality between *arbitrary* linear-feedback schemes over the two-user scalar Gaussian MAC and BC, similar to the MIMO (nofeedback) MAC-BC duality in [11], [15]. Specifically, we show that the regions achieved by linear-feedback schemes over the two-user scalar Gaussian MAC under sum-power constraint P and over the two-user scalar Gaussian BC with uncorrelated noises under the same power constraint P coincide, if the scalar channel coefficients of the MAC and the BC are equal. Since the set of achievable regions over the Gaussian MAC using linear-feedback schemes is known—it equals Ozarow's achievable region under a sum-power constraint—our result allows to characterize the region that is achievable with linear-feedback schemes over the two-user scalar Gaussian BC with uncorrelated noises. We can also identify the optimal linear-feedback schemes over the scalar Gaussian BC and show that for equal noise-variances the control-theoretic schemes in [5], [6], [7] achieve the largest sum-rate among all linear-feedback schemes.

II. Gaussian MAC with Feedback

Consider the two-user memoryless scalar Gaussian MAC with perfect output-feedback in Figure 1. At each time $t \in \mathbb{N}$, if $x_{1,t}$ and $x_{2,t}$ denote the real symbols sent by Transmitters 1 and 2, the receiver observes the real channel output

$$Y_t = h_1 x_{1,t} + h_2 x_{2,t} + Z_t, \tag{1}$$

where h_1 and h_2 are constant nonzero channel coefficients and $\{Z_t\}$ is a sequence of independent and identically distributed (i.i.d.) zero-mean unit-variance[1] Gaussian random variables.

The goal of communication is that Transmitters 1 and 2 convey their independent messages M_1 and M_2 to the common receiver. The messages M_1 and M_2 are independent of the noises $\{Z_t\}$ and uniformly distributed over the sets $\mathcal{M}_1 \triangleq \{1, \ldots, \lfloor 2^{nR_1} \rfloor\}$ and $\mathcal{M}_2 \triangleq \{1, \ldots, \lfloor 2^{nR_2} \rfloor\}$, where R_1 and R_2 denote the rates of transmission and n the blocklength.

The two transmitters observe perfect feedback from the channel outputs. Thus, the time-t channel input at Transmitter $i \in \{1, 2\}$ can depend on all previous channel outputs Y^{t-1} and its message M_i:

$$X_{i,t} = f_{i,t}^{(n)}(M_i, Y^{t-1}), \quad t \in \{1, \ldots, n\}, \tag{2}$$

[1] Assuming unit-variance entails no loss in generality because otherwise the receiver can simply scale its outputs appropriately.

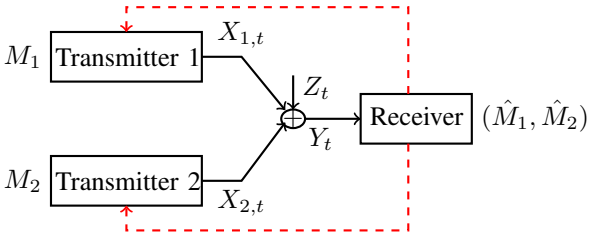

Fig. 1. Two-user Gaussian MAC with feedback.

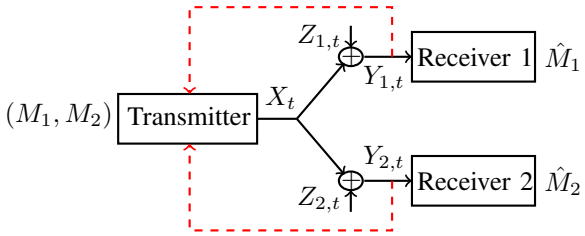

Fig. 2. Two-user Gaussian BC with feedback.

for some encoding function $f_{i,t}^{(n)}\colon \mathcal{M}_i \times \mathbb{R}^{t-1} \to \mathbb{R}$. The channel inputs $\{X_{1,t}\}_{t=1}^n$ and $\{X_{2,t}\}_{t=1}^n$ have to satisfy an expected average *sum-power constraint*

$$\frac{1}{n}\sum_{t=1}^{n}\big(\mathbf{E}[X_{1,t}^2]+\mathbf{E}[X_{2,t}^2]\big) \le P, \tag{3}$$

where the expectation is over the messages and the realizations of the channel. The receiver produces a guess

$$(\hat{M}_1^{(n)}, \hat{M}_2^{(n)}) = \Phi^{(n)}(Y^n)$$

by means of a decoding function $\Phi^{(n)}\colon \mathbb{R}^n \to \mathcal{M}_1 \times \mathcal{M}_2$.

The average probability of error is

$$P_{\text{e,MAC}}^{(n)} \triangleq \Pr\big[(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)\big]. \tag{4}$$

We say that a rate-pair (R_1, R_2) is achievable over the Gaussian MAC with feedback under a sum-power constraint P, if there exists a sequence of encoding and decoding functions $\big\{\{f_{1,t}^{(n)}\}_{t=1}^n, \{f_{2,t}^{(n)}\}_{t=1}^n, \Phi^{(n)}\big\}_{n=1}^{\infty}$ as described above, satisfying (3) and such that the average probability of error $P_{\text{e,MAC}}^{(n)}$ tends to zero as the blocklength n tends to infinity. The closure of the union of all achievable regions is called *capacity region*.

In the present paper we focus on *linear-feedback schemes for the MAC*, where the channel inputs can be written as

$$\mathbf{X}_i = \mathbf{V}_i + \mathsf{C}_i\mathbf{Y}, \qquad i \in \{1,2\}, \tag{5}$$

where $\mathbf{Y} \triangleq \big(Y_1, \ldots, Y_n\big)^{\mathsf{T}}$ is the channel output vector, C_1 and C_2 are n-by-n strictly lower-triangular matrices and $\mathbf{V}_i$ is an n-dimensional information-carrying vector $\mathbf{V}_i = \varphi_i(M_i)$. The mapping $\varphi_i\colon \mathcal{M}_i \to \mathbb{R}^n$ as well as the decoder mapping $\Phi^{(n)}$ can be arbitrary (also non-linear). The strict-lower-triangularity of the matrices C_1 and C_2 ensures that the feedback is used in a strictly causal way. (Notice that any nofeedback scheme is of the form in (5) with $\mathsf{C}_1 = \mathsf{C}_2 = 0$.)

The set of all rate-pairs achieved by linear-feedback schemes is called *linear-feedback capacity region* and is denoted $\mathcal{C}_{\text{MAC}}^{\text{linfb}}(h_1, h_2; P)$. The largest sum-rate achieved by a linear-feedback scheme is called *linear-feedback sum-capacity* and is denoted $C_{\text{MAC},\Sigma}^{\text{linfb}}(h_1, h_2, P)$. Since Ozarow's capacity-achieving scheme [1] is a linear-feedback scheme,[2] the general feedback capacity region and the linear-feedback capacity region coincide. They are both given by

$$\mathcal{C}_{\text{MAC}}^{\text{linfb}}(h_1, h_2; P) = \bigcup_{\substack{P_1, P_2 \ge 0:\\ P_1+P_2=P}} \bigcup_{\rho\in[0,1]} \mathcal{R}_{\text{Oz}}^{\rho}(h_1, h_2; P_1, P_2) \tag{6}$$

[2]Notice that also Ozarow's rate-splitting scheme has the form in (5) because the feedback signals are combined linearly with code-symbols.

where $\mathcal{R}_{\text{Oz}}^{\rho}(h_1, h_2; P_1, P_2)$ is the set of all nonnegative rate-pairs (R_1, R_2) satisfying

$$R_i \le \frac{1}{2}\log\big(1 + h_i^2 P_i(1-\rho^2)\big), \quad i \in \{1,2\}, \tag{7a}$$

$$R_1 + R_2 \le \frac{1}{2}\log\big(1 + h_1^2P_1 + h_2^2P_2 + 2\sqrt{h_1^2h_2^2P_1P_2}\rho\big). \tag{7b}$$

The linear-feedback sum-capacity is given in Equation (8) on top of the next page, where $\rho^\star(h_1, h_2; P_1, P_2)$ is the unique solution in $[0,1]$ to the following quartic equation in ρ

$$(1 + h_1^2P_1 + h_2^2P_2 + 2\sqrt{h_1^2h_2^2P_1P_2}\rho) = (1 + h_1^2P_1(1-\rho^2))(1 + h_2^2P_2(1-\rho^2)). \tag{9}$$

III. Gaussian BC with Feedback

Consider the two-user scalar Gaussian BC with perfect output-feedback in Figure 2. We now have one transmitter and two receivers. At each time $t \in \mathbb{N}$, if $x_t \in \mathbb{R}$ denotes the transmitter's channel input, Receiver $i \in \{1,2\}$ observes the real channel output

$$Y_{i,t} = h_i x_t + Z_{i,t}, \tag{10}$$

where h_1 and h_2 are constant non-zero channel coefficients and $\{Z_{1,t}\}_{t=1}^n$ and $\{Z_{2,t}\}_{t=1}^n$ model the additive noise at Receivers 1 and 2. The noise sequences $\{Z_{1,t}\}_{t=1}^n$ and $\{Z_{2,t}\}_{t=1}^n$ are independent and each consists of i.i.d. centered Gaussian random variables of unit variance[3].

The goal of the communication is that the transmitter conveys Message M_1 to Receiver 1 and Message M_2 to Receiver 2. The transmitter observes perfect output-feedback from both receivers. Thus, the time-t channel input X_t can depend on all previous channel outputs Y_1^{t-1} and Y_2^{t-1} and the messages M_1 and M_2:

$$X_t = g_t^{(n)}(M_1, M_2, Y_1^{t-1}, Y_2^{t-1}), \quad t \in \{1, \ldots, n\}, \tag{11}$$

for some encoding function $g_t^{(n)}\colon \mathcal{M}_1 \times \mathcal{M}_2 \times \mathbb{R}^{t-1} \times \mathbb{R}^{t-1} \to \mathbb{R}$. We impose an *expected average block-power constraint*

$$\frac{1}{n}\sum_{t=1}^{n}\mathbf{E}[X_t^2] \le P, \tag{12}$$

where the expectation is over the messages and the realizations of the channel. Each Receiver $i \in \{1,2\}$ produces the guess

$$\hat{M}_i^{(n)} = \phi_i^{(n)}(Y_i^n)$$

[3]As for the MAC, assuming $Z_{1,t}$ and $Z_{2,t}$ have unit variance entails no loss in generality.

$$C^{\text{linfb}}_{\text{MAC},\Sigma}(h_1,h_2;P)=\sup_{\substack{P_1,P_2\geq 0:\\ P_1+P_2=P}}\frac{1}{2}\log\left(1+h_1^2P_1+h_2^2P_2+2\sqrt{h_1^2h_2^2P_1P_2}\cdot\rho^\star(h_1,h_2;P_1,P_2)\right) \tag{8}$$

for some decoding function $\phi_i^{(n)}\colon \mathbb{R}^n\to\mathcal{M}_i$.

The average probability of error is

$$P_{\text{e,BC}}^{(n)}\triangleq \Pr\big[(\hat{M}_1\neq M_1) \text{ or } (\hat{M}_2\neq M_2)\big]. \tag{13}$$

A rate-pair (R_1,R_2) is achievable over the Gaussian BC with feedback and power constraint P, if there exists a sequence of encoding and decoding functions $\big\{\{g_t^{(n)}\}_{t=1}^n,\phi_1^{(n)},\phi_2^{(n)}\big\}_{n=1}^\infty$ as described above, satisfying the power constraint (12) and such that the average probability of error $P_{\text{e,BC}}^{(n)}$ tends to zero as n tends to infinity.

Also here we restrict attention to *linear-feedback schemes for the BC* where the transmitter's channel input vector $\mathbf{X}\triangleq\big(X_1,\ldots,X_n\big)^\mathsf{T}$ can be written as:

$$\mathbf{X}=\mathbf{W}+\mathsf{B}_1\mathbf{Z}_1+\mathsf{B}_2\mathbf{Z}_2, \tag{14}$$

where $\mathbf{Z}_i\triangleq\big(Z_{i,1},\ldots,Z_{i,n}\big)^\mathsf{T}$ represents the noise vector at Receiver i, B_1 and B_2 are strictly lower-triangular matrices, and $\mathbf{W}$ is an n-dimensional information-carrying vector

$$\mathbf{W}=\xi(M_1,M_2) \tag{15}$$

The mapping $\xi\colon\mathcal{M}_1\times\mathcal{M}_2\to\mathbb{R}^n$ and the decoding operations $\phi_1^{(n)}$ and $\phi_2^{(n)}$ can be arbitrary.

Taking a linear combination of the information-carrying vector $\mathbf{W}$ and the past noise vectors $\mathbf{Z}_1$ and $\mathbf{Z}_2$ is equivalent to taking a (different) linear combination of $\mathbf{W}$ and the past outputs $\mathbf{Y}_1$ and $\mathbf{Y}_2$. Thus, the strict lower-triangularity of B_1 and B_2 ensures that the feedback is used strictly causally.

Linear-feedback capacity and linear-feedback sum-capacity for the BC are defined analogously as for the MAC. We denote them by $\mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1,h_2;P)$ and $C^{\text{linfb}}_{\text{BC},\Sigma}(h_1,h_2;P)$.

IV. Main Results

We first present multi-letter expressions for $\mathcal{C}^{\text{linfb}}_{\text{MAC}}(h_1,h_2;P)$ and $\mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1,h_2;P)$. They are used to prove Theorem 1 ahead.

Definition 1. *Given $\eta\in\mathbb{N}$ and η-by-η matrices D_1 and D_2, let Q_1 and Q_2 be the positive square roots of the (positive-definite) η-by-η matrices*

$$\mathsf{M}_1\triangleq(\mathsf{I}_\eta+h_1\mathsf{D}_1)^\mathsf{T}(\mathsf{I}_\eta+h_1\mathsf{D}_1)+h_1^2\mathsf{D}_2^\mathsf{T}\mathsf{D}_2 \tag{16a}$$

$$\mathsf{M}_2\triangleq h_2^2\mathsf{D}_1^\mathsf{T}\mathsf{D}_1+(\mathsf{I}_\eta+h_2\mathsf{D}_2)^\mathsf{T}(\mathsf{I}_\eta+h_2\mathsf{D}_2) \tag{16b}$$

and let $\mathcal{R}_{\text{MAC}}(\eta,\mathsf{D}_1,\mathsf{D}_2,h_1,h_2;P)$ denote the (private messages) nofeedback capacity [12] of the MIMO MAC

$$\tilde{\mathbf{Y}}^{\text{MAC}}\triangleq h_1\mathsf{Q}_1^{-1}\mathbf{V}_1+h_2\mathsf{Q}_2^{-1}\mathbf{V}_2+\tilde{\mathbf{Z}} \tag{17}$$

when the η-by-1 input vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ have to satisfy

$$\text{tr}(\mathsf{K}_{\mathbf{V}_1}+\mathsf{K}_{\mathbf{V}_2})\leq\max\{0,\eta P-\text{tr}(\mathsf{D}_1\mathsf{D}_1^\mathsf{T})-\text{tr}(\mathsf{D}_2\mathsf{D}_2^\mathsf{T})\}, \tag{18}$$

where $\mathsf{K}_{\mathbf{V}_i}$ denotes the covariance matrix of $\mathbf{V}_i$ and in (17) $\tilde{\mathbf{Z}}$ is a centered Gaussian vector of identity covariance matrix I_η.

Proposition 1. *The linear-feedback capacity of the scalar Gaussian MAC under sum-power constraint P satisfies*

$$\mathcal{C}^{\text{linfb}}_{\text{MAC}}(h_1,h_2;P)=\text{cl}\left(\bigcup_{\eta,\mathsf{D}_1,\mathsf{D}_2}\frac{1}{\eta}\mathcal{R}_{\text{MAC}}(\eta,\mathsf{D}_1,\mathsf{D}_2,h_1,h_2;P)\right) \tag{19}$$

where the union is over all positive integers η and strictly lower-triangular η-by-η matrices D_1 and D_2.

Observe that [1] and Proposition 1 imply that the right-hand sides (RHSs) of (19) and (6) coincide.

Proof: For fixed η, D_1, and D_2, the rate region $\frac{1}{\eta}\mathcal{R}_{\text{MAC}}(\eta,\mathsf{D}_1,\mathsf{D}_2,h_1,h_2;P)$ is achieved by coding over blocks of η channel uses. Choose the η channel inputs of Transmitter i for a given block as

$$\mathbf{X}_i=\mathsf{Q}_i^{-1}\mathbf{V}_i+\mathsf{C}_i\mathbf{Y},\quad i\in\{1,2\}, \tag{20}$$

where

$$\mathsf{C}_i\triangleq\mathsf{D}_i(\mathsf{I}_\eta+h_1\mathsf{D}_1+h_2\mathsf{D}_2)^{-1}, \tag{21}$$

and where $\mathbf{Y}$ denotes the η-by-1 vector consisting of the channel outputs in this block and $\mathbf{V}_i$ is an η-by-1 vector that depends on Message M_i, and over which we can code. The corresponding output vector is

$$\mathbf{Y}=(\mathsf{I}_\eta-h_1\mathsf{C}_1-h_2\mathsf{C}_2)^{-1}(h_1\mathsf{Q}_1^{-1}\mathbf{V}_1+h_2\mathsf{Q}_2^{-1}\mathbf{V}_2+\mathbf{Z}). \tag{22}$$

By coding over the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ of the different blocks, we can achieve the capacity of the MIMO MAC in (22), which equals the capacity of the MIMO MAC in (17). Algebraic manipulations show that the inputs in a given block (20) are sum-power constrained to P, if (18) holds and if $\eta P-\text{tr}(\mathsf{D}_1\mathsf{D}_1^\mathsf{T})-\text{tr}(\mathsf{D}_2\mathsf{D}_2^\mathsf{T})$ is positive.

More details and the converse are omitted for brevity. ■

Definition 2. *Given $\eta\in\mathbb{N}$ and η-by-η matrices B_1 and B_2, let S_1 and S_2 be the positive square roots of the η-by-η matrices*

$$\mathsf{A}_1\triangleq(\mathsf{I}_\eta+h_1\mathsf{B}_1)(\mathsf{I}_\eta+h_1\mathsf{B}_1)^\mathsf{T}+h_1^2\mathsf{B}_2\mathsf{B}_2^\mathsf{T} \tag{23a}$$

$$\mathsf{A}_2\triangleq h_2^2\mathsf{B}_1\mathsf{B}_1^\mathsf{T}+(\mathsf{I}_\eta+h_2\mathsf{B}_2)(\mathsf{I}_\eta+h_2\mathsf{B}_2)^\mathsf{T} \tag{23b}$$

and let $\mathcal{R}_{\text{BC}}(\eta,\mathsf{B}_1,\mathsf{B}_2,h_1,h_2;P)$ denote the (private-messages) nofeedback capacity of the MIMO BC [13]

$$\tilde{\mathbf{Y}}_i^{\text{BC}}\triangleq h_i\mathsf{S}_i^{-1}\mathbf{W}+\tilde{\mathbf{Z}}_i,\quad i\in\{1,2\}, \tag{24}$$

when the η-by-1 input vector $\mathbf{W}$ has to satisfy

$$\text{tr}(\mathsf{K}_{\mathbf{W}})\leq\max\{0,\eta P-\text{tr}(\mathsf{B}_1\mathsf{B}_1^\mathsf{T})-\text{tr}(\mathsf{B}_2\mathsf{B}_2^\mathsf{T})\}, \tag{25}$$

where $\mathsf{K}_{\mathbf{W}}$ denotes the covariance matrix of $\mathbf{W}$ and where in (24) $\tilde{\mathbf{Z}}_1$ and $\tilde{\mathbf{Z}}_2$ denote independent centered Gaussian vectors of identity covariance matrix I_η.

Proposition 2. *The linear-feedback capacity region of the Gaussian BC with feedback is:*

$$\mathcal{C}^{\text{linfb}}_{\text{BC}}(h_1,h_2;P)=\text{cl}\left(\bigcup_{\eta,\mathsf{B}_1,\mathsf{B}_2}\frac{1}{\eta}\mathcal{R}_{\text{BC}}(\eta,\mathsf{B}_1,\mathsf{B}_2,h_1,h_2;P)\right) \tag{26}$$

where the union is over all positive integers η and strictly lower-triangular η-by-η matrices B_1 and B_2.

Proof: For fixed η, B_1, and B_2, the rate region $\frac{1}{\eta}\mathcal{R}_{\text{BC}}(\eta, \mathsf{B}_1, \mathsf{B}_2, h_1, h_2; P)$ is achieved by coding over blocks of η channel uses, if the channel inputs in a block are

$$\mathbf{X} = \mathbf{W} + \mathsf{B}_1\mathbf{Z}_1 + \mathsf{B}_2\mathbf{Z}_2, \tag{27}$$

where $\mathbf{Z}_1$ and $\mathbf{Z}_2$ denote the block's η-by-1 noise vectors at Receivers 1 and 2 and $\mathbf{W}$ is an η-by-1 input vector that depends on the messages (M_1, M_2). Receiver i's outputs $\mathbf{Y}_i$ in a block are then given by

$$\mathbf{Y}_i = h_i\mathbf{W} + h_i\mathsf{B}_1\mathbf{Z}_1 + h_i\mathsf{B}_2\mathbf{Z}_2 + \mathbf{Z}_i, \quad i \in \{1, 2\}. \tag{28}$$

By coding over the inputs $\mathbf{W}$ of the different blocks, we can achieve the capacity of the MIMO BC in (28), which coincides with the capacity of the MIMO BC in (24).

More details and the converse are omitted for brevity. ■

Theorem 1. *The linear-feedback capacity regions of the scalar Gaussian BC under power constraint P and of the scalar Gaussian MAC under sum-power constraint P coincide:*

$$\mathcal{C}_{\text{MAC}}^{\text{linfb}}(h_1, h_2; P) = \mathcal{C}_{\text{BC}}^{\text{linfb}}(h_1, h_2; P). \tag{29}$$

Corollary 1.

$$C_{\text{MAC},\Sigma}^{\text{linfb}}(h_1, h_2; P) = C_{\text{BC},\Sigma}^{\text{linfb}}(h_1, h_2; P). \tag{30}$$

Corollary 2. *If $h_1 = h_2 = h$, then*

$$C_{\text{BC},\Sigma}^{\text{linfb}}(h, h; P) = \frac{1}{2}\log\left(1 + h^2P + h^2P \cdot \rho^\star(h, h; P, P)\right) \tag{31}$$

and thus the control-theoretic scheme in [7] achieves the linear-feedback sum-capacity.

Proof: By a symmetry argument. Omitted. ■

Proof of Theorem 1: We show that

$$\mathcal{R}_{\text{MAC}}(\eta, \mathsf{D}_1, \mathsf{D}_2, h_1, h_2; P) = \mathcal{R}_{\text{BC}}(\eta, \mathsf{B}_1, \mathsf{B}_2, h_1, h_2; P), \tag{32}$$

coincide if

$$\mathsf{B}_i = \bar{D}_i, \quad i \in \{1, 2\}, \tag{33}$$

where for a matrix A, $\bar{\mathsf{A}} \triangleq \mathsf{E}_\eta\mathsf{A}^\intercal\mathsf{E}_\eta$ is called its *reversed image* and E_η is the *exchange matrix* which is 0 everywhere except on the counter-diagonal where it is 1. The theorem then follows by Propositions 1 and 2, and since the mapping in (33) is one-to-one over the set of strictly lower-triangular matrices.

Under (33), the RHSs of the power constraints (18) and (25) coincide. Moreover, under power constraint (18), the MIMO MAC in (17) has the same capacity as the MIMO MAC[4]

$$\bar{\mathbf{Y}}^{\text{MAC}} \triangleq \mathsf{E}_\eta\tilde{\mathbf{Y}}^{\text{MAC}} = h_1\bar{\mathsf{Q}}_1^{-1}\bar{\mathbf{V}}_1 + h_2\bar{\mathsf{Q}}_2^{-1}\bar{\mathbf{V}}_2 + \bar{\mathbf{Z}}, \tag{34}$$

where $\bar{\mathbf{Z}} \triangleq \mathsf{E}_\eta\tilde{\mathbf{Z}}$ and where $\bar{\mathbf{V}}_i \triangleq \mathsf{E}_\eta\mathbf{V}_i$ has to satisfy the power constraint (18) when $\bar{\mathbf{V}}_i$ replaces $\mathbf{V}_i$. Now, Equality (32) follows by the MIMO MAC-BC (nofeedback) duality in [11], [15] and because under (33) the MAC $\bar{\mathbf{Y}}^{\text{MAC}}$ is dual to the BC in (24). In fact, under (33), $h_1\bar{\mathsf{Q}}_1^{-1} = h_1\mathsf{S}_1^{-\intercal}$ and $h_2\bar{\mathsf{Q}}_2^{-1} = h_2\mathsf{S}_2^{-\intercal}$. ■

[4]Multiplying $\tilde{\mathbf{Y}}^{\text{MAC}}$ by E_η from the left only reverses the order of receiving antennas.

Remark 1. *The optimal MAC scheme is described in [1]. From this we can deduce the optimal MAC-parameters $\mathbf{V}_1$, $\mathbf{V}_2$, C_1, and C_2 describing the block inputs in* (20)*. (A different set of parameters is required to approach each point on the boundary of the capacity region.) Now, by* (32) *and comparing* (21) *and* (33)*, from these parameters we can deduce the optimal BC-parameters B_1 and B_2 describing the block inputs in* (27)*. Finally, the results in [13] tell us how to code and decode over the resulting MIMO BC in* (28)*.*

Remark 2. *Theorem 1 extends to the scalar Gaussian MAC and BC with $K \geq 3$ users.*

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